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FORMAL SOLUTIONS FOR A CLASS OF STOCHASTIC PURSUIT-EVASION GAMES

By

W. W. Willman

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Technical Report No. 575

Division of Engineering and Applied Physics
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ABSTRACT

A class of differential pursuit-evasion games is examined in which the dynamics are linear and perturbed by additive white Gaussian noise, the performance index is quadratic, and both players receive measurements perturbed independently by additive white Gaussian noise. A direct application of the saddle point condition is used formally to characterize linear minimax solutions in terms of a system of implicit integro-differential equations, which appears to be more complicated than the ordinary kind of two point boundary value problem. It is also shown that games of this type possess a "certainty-coincidence" property, meaning that their behavior coincides with that of corresponding deterministic games in the event that all noise values are zero. This property is used to decompose the minimax strategies into sums of a certainty-equivalent term and error terms.

1. BACKGROUND

The theory of two-person zero-sum differential games was first developed by Isaacs in 1954 [1], who used game-theoretic concepts originated earlier by von Neumann and Morgenstern [2] together with the properties of dynamic systems to produce results of computational interest. A convenient summary of this theory is contained in his book [3]. In 1957, Berkowitz and Fleming [4] used variational techniques to investigate a restricted class of differential games, a class later expanded by Berkowitz [5].

A more thorough examination of linear-quadratic pursuit-evasion games was made in 1965 by Ho, Bryson, and Baron [6], who showed that a linear control law is optimal for both players in this type of game. A natural extension of this game is obtained by assuming that the players have imperfect information to the extent that they must act on the basis of measurements which are linear functions of the state corrupted by additive white noise. An important subclass of such games, including the classical interception problem, was solved by Behn and Ho [7] in 1968 for the case in which one player has access to perfect information about the state of the game. A linear control law is optimal for both players in this situation as well, the player with noisy measurements using a certainty-equivalent control, and his opponent adding a term to his certainty-equivalent control proportional to the first player's error. As a result, assuming that the optimal control laws are used in both cases, the time behavior of the state variable coincides with that of the corresponding deterministic game if the player with noisy measurements happens to receive correct measurements and begins with the correct estimate of the initial state. Some interesting results in the area of stochastic differential games are also being developed by Rhodes and Luenberger [8].

The game of interest here is an extension of the linear-quadratic pursuit-evasion game in which both the pursuer and evader have access only to noisy measurements and in which process noise is added to the dynamics. The main difficulty in making this extension stems from the fact that each player has only imperfect knowledge of the information available to his opponent. The optimal control laws in this context are again found to be linear functionals of the players' available measurements, but as an apparent consequence of this imperfect knowledge, attempts to express these control laws in terms of finite-dimensional "estimate vectors" (i.e., summary statistics of the measurements) have been unsuccessful. An additional complication is that the solution to such a game is expressed implicitly in terms of a set of differential equations which is more complicated than the standard type of two point boundary value problem.

It is possible, however, to show that the optimal control laws for this class of games consist of a certainty-equivalent term plus two error terms, both of which are zero in the event that both players' estimates of the state are initially correct and all measurement and process noise values happen to be zero. This means that the trajectory followed by the stochastic differential game under these circumstances coincides with that followed by the corresponding deterministic game, if the optimal control laws are used in both cases. Thus, a "certainty-coincidence" property, which holds for the class of games considered by Behn and Ho [7], extends to this case as well.

The approach adopted here is basically formal, but hopefully is indicative of how a mathematically more substantial development of this topic might proceed in a sufficiently broad context to be of interest.

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2. PROBLEM STATEMENT

A. A Deterministic Game

Considerable attention has focused on pursuit-evasion games with linear dynamics and quadratic payoff, both because much is known about their solutions and because they arise naturally in the perturbation analysis of a more general class of differential games. The specific type of game considered here is a two-person one in which an n-dimensional state vector (of real numbers) x obeys the transition equation

$$\dot{x} = G_p u - G_e v \; ; \; x(0) \text{ given} \tag{1}$$

and the payoff, which one player (the pursuer) wishes to minimize and the other player (the evader) wishes to maximize, is the quadratic functional

$$J = \frac{1}{2}\left[x^T(t_f)S_f x(t_f) + \int_0^{t_f} (u^T Bu - v^T Cv)dt\right] \tag{2}$$

where t_f is some prescribed terminal time, and the initial time is taken to be zero without loss of generality. The m-dimensional vector u(t) is the control chosen by the pursuer at time t and v(t) is the k-dimensional control vector chosen by the evader at that time. G_p, G_e, B, and C are time-varying matrices of the required dimensions, and B and C are symmetric and positive definite. The matrix S_f is symmetric and positive semidefinite. It is also assumed that both players know all the parameters which define the game. It has been shown by

Ho, Bryson, and Baron [6] that a seemingly more general class of linear games can be reduced to this form by using the concepts of reduced state and predicted terminal miss.

A solution to such a game, and also to two-person zero-sum games in general, is a pair of control laws, or strategies, for the two players which is optimal in a sense to be defined shortly. By a strategy is meant a decision rule that specifies the control to be used by a player, as a function of the information available to him, in any situation that might arise. Therefore, a complete description of a game must include the information sets available to each player upon which his control decisions must be based. To avoid confusion, strategies will be labelled by capital letters to distinguish them from the control values that they determine.

A pair of strategies for the two players (U^o, V^o) is called optimal, or minimax, if it satisfies the saddle-point condition, which means that for any strategy pair (U, V) based on the available information set,

$$J(U, V^o) \geq J(U^o, V^o) \geq J(U^o, V). \tag{3}$$

That is, each strategy of a minimax pair optimizes against the other. Such a minimax strategy pair is defined to be a solution of the game.

It may be that there is more than one minimax pair of strategies. If this happens, say for the pairs (U^o, V^o) and $U^1, V^1)$, the saddle point condition can be applied twice to obtain

$$J(U^o, V^1) \leq J(U^o, V^o) \leq J(U^1, V^o) \leq J(U^1, V^1) \leq J(U^o, V^1). \tag{4}$$

Therefore, since all these values must be the same, the same payoff is obtained if either of these pursuit strategies is played against either of the evasion strategies. For this reason, it makes sense to apply the term "minimax" to strategies as well as strategy pairs, since it is equivalent to say that U^o is a minimax strategy if there exists an evasion strategy V^o such that the pair (U^o, V^o) is minimax in the sense defined earlier. Strictly speaking, it may happen in some games that some combination of pursuit and evasion strategies results in a meaningless situation, whereas another combination of the same strategies is meaningful, in which case the preceding reasoning does not apply. In the type of game under consideration here, however, the controls are combined additively, and this problem does not arise. The definition of solution is also sometimes extended to include randomly mixed strategies, but these will not be considered here.

The solution (i. e., the minimax strategies) to the linear-quadratic game posed earlier has been found by Ho, Bryson, and Baron [6] for the case where both players have perfect knowledge of the state x. When a solution exists, it is given by

$$U^o: \quad u = -B^{-1}G_p^T Sx \tag{5}$$

$$V^o: \quad v = -C^{-1}G_e^T Sx \tag{6}$$

where S is the solution to the matrix Riccati equation

$$\dot{S} = S\left[G_p B^{-1} G_p^T - G_e C^{-1} G_e^T\right]S, \qquad S(t_f) = S_f. \tag{7}$$

A solution to the game does not exist if S becomes infinite on the interval $(0, t_f)$ as the differential equation is integrated backwards from t_f. If the solution is bounded on $[0, t_f]$, then the strategies U^o and V^o are minimax.

B. An Effect of Imperfect Information

It is also well known that if the players' information sets in this game are changed so that they both know the initial state exactly but have no direct knowledge of subsequent values of the state vector, the minimax control laws, if they exist, are given by

$$U^1: \quad u = -B^{-1}G_p^T Sy \tag{8}$$

and

$$V^1: \quad v = -C^{-1}G_e^T Sy \tag{9}$$

where S is the same as before, and y is determined by the differential equation

$$\dot{y} = \left[G_e C^{-1} G_e^T - G_p B^{-1} G_p^T\right] Sy \; ; \; y(0) = x(0). \tag{10}$$

Notice that, although the control variable and state variable histories are the same in both cases when the minimax strategies are played (another example of the certainty-coincidence property), the strategies U^1 and V^1 are open-loop (functions of t only), whereas U^o and V^o are closed-loop (functions of both x and t). There are, however, conditions under which a solution exists for the closed-loop problem, but not the open-loop problem. In fact, a solution to the open-loop game exists only when the solution to the differential equation

$$\dot{K} = -KG_e C^{-1} G_e^T K \; ; \; K(t_f) = S_f$$

remains finite on the interval $[0, t_f]$. An intuitive explanation for this phenomenon is that it is easier for the evader to escape under the open-loop conditions because the lack of information hurts the pursuer more than the evader.

C. A Related Stochastic Game

A type of game intermediate between these open-loop and closed-loop extremes can be formulated by altering the players' information sets so that they each have access to noisy measurements of the state, by including process noise in the transition equation, and by modifying the criterion J to be the prior expected value of the payoff. Specifically, the generic game of interest here is that in which the transition equation is

$$\dot{x} = G_p u - G_e v + \xi , \qquad (11)$$

the criterion to be minimaximized is

$$J = \frac{1}{2}\varepsilon \left[x^T(t_f) S_f x(t_f) + \int_0^{t_f} (u^T B u - v^T C v)\, dt \right], \qquad (12)$$

and the measurements available to the pursuer and evader are, respectively,

$$z_p = H_p x + w_p \qquad (13)$$

and

$$z_e = H_e x + w_e , \qquad (14)$$

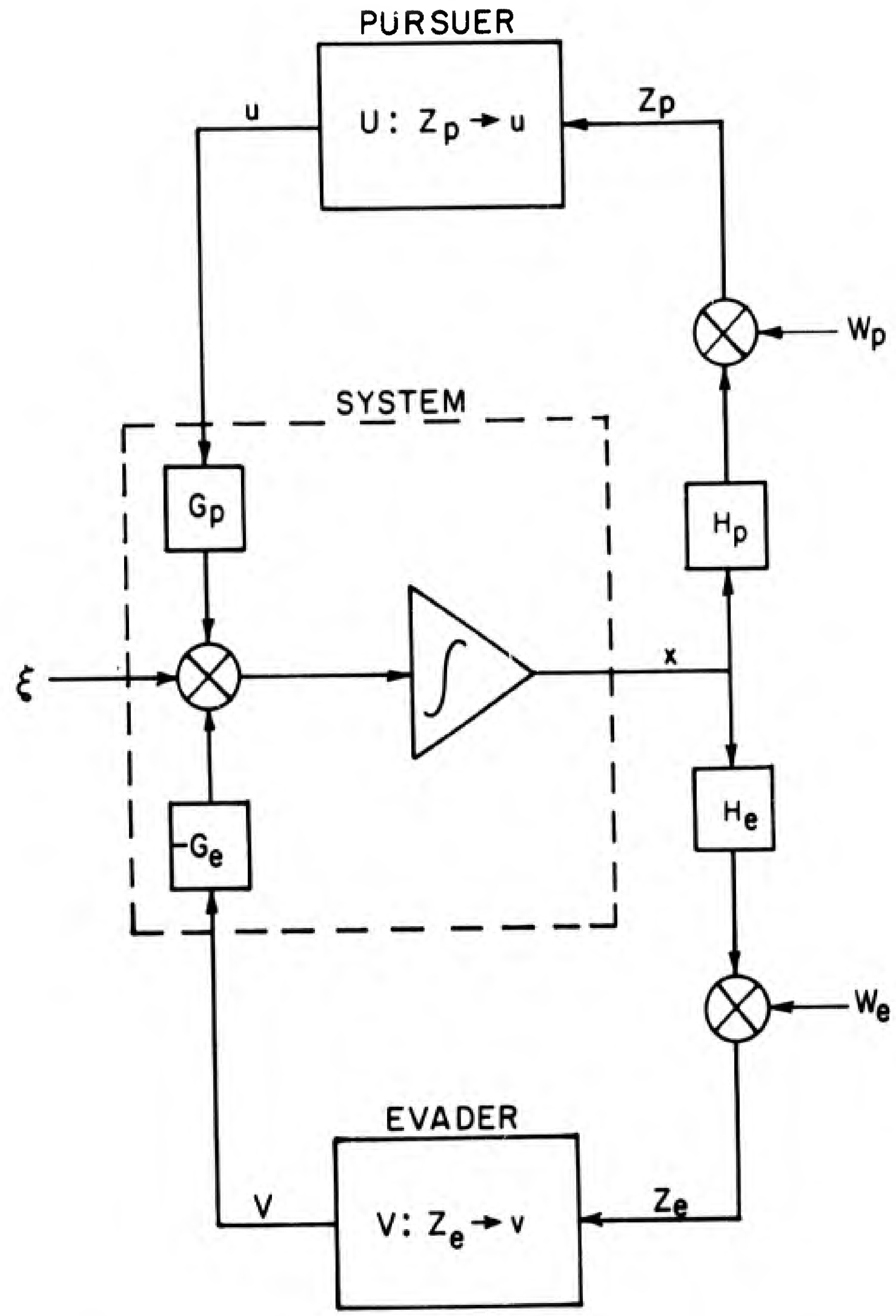
where z_p and z_e are q- and r-dimensional vectors, H_p and H_e are time-varying matrices, and

$$\begin{bmatrix} \xi \\ \hline w_p \\ \hline w_e \end{bmatrix} \text{ is a Gaussian white noise } \left(\begin{bmatrix} 0 \\ \hline 0 \\ \hline 0 \end{bmatrix}, \left[\begin{array}{c|c|c} Q & 0 & 0 \\ \hline 0 & R_p & 0 \\ \hline 0 & 0 & R_e \end{array}\right] \right) \text{ process.}$$

It is also assumed that both players start with a common prior probability assessment of the initial state which is Normal $(\bar{x}_o, P_o)$ and statistically independent of the measurement and process noises. Notice that the expected value operator in equation (12) is unambiguous since both players have the same prior assessment of the initial state.

The assumption that both players know the parameters of the game must in this context be extended to include knowledge of the statistical parameters as well. In particular, this includes their opponents' prior estimates of the initial state, a knowledge of which would be available in a situation where the initial state is determined by an outside random event whose statistics are known to both players.

The dynamics of this stochastic differential game are illustrated in Figure 1.



$Z_p(x) \triangleq \{(Z_p(s), S) \ 0 \leq S \leq t\}; \ Z_e(t) \triangleq \{(Z_e(s), s) : 0 \leq S \leq t\}$

FIG. 1 GAME DYNAMICS

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3. DERIVATION OUTLINE

A. The Basic Method

The method used here to solve the stochastic game is based on the fact that the saddle point condition for a minimax strategy pair (U^o, V^o):

$$J(U, V^o) \geq J(U^o, V^o) \geq J(U^o, V)$$

characterizes these strategies in terms of a pair of associated stochastic optimal control problems. That is, the saddle point condition is equivalent to U^o being a solution to the stochastic optimal control problem

$$J(U^o, V^o) = \min_U J(U, V^o) \tag{15}$$

and V^o being a solution to

$$J(U^o, V^o) = \max_V J(U^o, V) \; . \tag{16}$$

This equivalence is used to characterize minimax strategies as follows:

(i) Guess the functional form of a pair of minimax strategies.

(ii) Substituting the assumed form of the evader's minimax strategy into the transition and criterion equations, solve the resulting stochastic optimal control problem for the pursuer in terms of the parameters of the strategy assumed for the evader.

(iii) Repeat step (ii) with the roles of the pursuer and evader interchanged.

(iv) Obtain conditions on the parameters under which the assumed pair of strategies is minimax by requiring that the results of the optimizations of steps (ii) and (iii) be consistent with the strategies guessed in step (i).

The success of this method clearly depends on making a good guess in step (i), so that the results of steps (ii) and (iii) have the same functional form as the strategies originally assumed. The following simple example, which can be considered a one-stage game, illustrates the application of this method.

Example

$y = x + u - v$ (transition equation)

$J = \frac{1}{2}\, \varepsilon[ay^2 + u^2 - cv^2], \quad c > a > 0$ (criterion)

$z_p = x + w_p$ (pursuer's measurement)

$z_e = x + w_e$ (evader's measurement)

$$\begin{bmatrix} w_p \\ \hline w_e \end{bmatrix} \text{ is Normal} \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} q & 0 \\ 0 & r \end{bmatrix} \right)$$

common prior is Normal(0, p)

Step 1

Assume there exists a minimax strategy pair of the form

U^o: $u = bz_p$ (pursuit strategy)

V^o: $v = dz_e$ (evasion strategy)

Step 2

If the evader uses the generic strategy assumed in step 1, the associated optimal control problem facing the pursuer is described by

$$\begin{cases} y = x + u - d(x+w_e) \\ J = \frac{1}{2}\,\varepsilon\,[ay^2 + u^2 - cd^2(x+w_e)^2] \\ z_p = x + w_p \end{cases}$$

This is an example of the classical "linear-quadratic-Gaussian" control problem, the solution to which is well known. Expressing the solution as a function of z_p gives

$$u = \left[\frac{ap(d-1)}{(a+1)(p+q)}\right] z_p .$$

Step 3

An analogous argument shows that the maximizing evasion strategy against the assumed pursuit strategy is

$$v = \left[\frac{ap(b+1)}{(a-c)(p+r)}\right] z_e .$$

Step 4

Straightforward substitution shows that the results of steps 2 and 3 are consistent with the strategies assumed in step 1 (i. e., the values of u and v are the same for almost all values of z_p and z_e) if

$$\begin{cases} b = \dfrac{ap[(c-a)(p+r) - ap]}{(a+1)(c-a)(p+q)(p+r) + a^2p^2} \\ d = \dfrac{ap[(a+1)(p+q) + ap]}{(a+1)(c-a)(p+q)(p+r) + a^2p^2} \end{cases}$$

Therefore, if b and d satisfy these equations, the strategies assumed in step 1 are minimax because they satisfy the saddle point condition by construction (steps 2 and 3). End of example.

Since white noise processes do not really exist except as a kind of shorthand notation for a sequence of successively better approximating step-function processes, it is misleading and unreasonable to treat them formally as ordinary random processes. For this reason, a direct application of the preceding method to a continuous-time stochastic game requires the use of mathematical concepts beyond the scope of this report. Accordingly, an approach is adopted here in which the solution to the game of interest is obtained in three basic stages. First, the multi-stage analogue of the stochastic differential game described in the previous section is solved by the method outlined above. The (linear) solution is characterized by a system of implicit difference equations. Next, this solution is used to obtain a first order approximation of the solution to a discretized version of the differential game for small discretization intervals. Finally, the asymptotic form of such solutions is found in the limit as the discretization interval becomes infinitesimal. This limiting form of the solutions to successively finer discretizations of the differential game is taken to be the solution to this differential game. Here, the implicit difference equations become a system of implicit integro-differential equations with split boundary conditions. This approach implicitly describes what is really meant by a stochastic differential game of this type and its solution, but a rigorous discussion of these matters is beyond the scope of this report.

B. The Multistage Game

The corresponding multistage game, which is treated in detail in Appendix A, is described as follows, where i is an integer index varying from 0 to N-1 for some fixed integer N:

$$x(i+1) = x(i) + G_p(i)u(i) - G_e(i)v(i) + \xi(i)$$

$$J = \frac{1}{2}\,\varepsilon\left[x^T(N)S_f x(N) + \sum_{i=0}^{N-1}\left(u^T(i)B(i)u(i) - v^T(i)C(i)v(i)\right)\right]$$

$$z_p(i) = H_p(i)x(i) + w_p(i)$$

$$z_e(i) = H_e(i)x(i) + w_e(i)$$

$$\begin{bmatrix} \xi(i) \\ \hline w_p(i) \\ \hline w_e(i) \end{bmatrix} \text{ are independent Normal } \left(\begin{bmatrix} 0 \\ \hline 0 \\ \hline 0 \end{bmatrix}, \begin{bmatrix} Q(i) & 0 & 0 \\ 0 & R_p(i) & 0 \\ 0 & 0 & R_e(i) \end{bmatrix} \right)$$

The common prior is independent of the noises and is Normal $(\bar{x}_o, P_o)$.

A solution to this game is obtained via the method described earlier by assuming minimax strategies of the form

$$u(i) = -a_p(i) - \sum_{j=0}^{i} \Lambda_p(i,j)z_p(j)$$

$$v(i) = a_e(i) + \sum_{j=0}^{i} \Lambda_e(i,j)z_e(j)$$

and characterizing the parameters a and Λ in terms of a set of implicit difference equations. In order to derive the implicit equations, it is expedient to define the "enlarged" state variables $\pi(i)$, $\eta_p(i)$, and $\eta_e(i)$, each of which is a partitioned vector with N+1 partitions, and which are defined for $i = 0, \ldots, N-1$ and $j = 0, \ldots, N$ as

$$\langle \pi(i) \rangle_j = \begin{cases} x(j) \; ; \; j \leq i \\ x(i) \; ; \; j \geq i \end{cases}$$

$$\langle \eta_p(i) \rangle_j = \begin{cases} w_p(j) \; ; \; j \leq i \\ 0 \quad ; \; j > i \end{cases}$$

$$\langle \eta_e(i) \rangle_j = \begin{cases} w_e(j) \; ; \; j \leq i \\ 0 \quad ; \; j > i \end{cases}$$

where $\langle \pi(i) \rangle_j$ denotes the j-th partition of $\pi(i)$, etc. The vector $\pi(i)$ is called "the past of x at time i" since it contains all the information needed to determine all the previous values of the state vector x. For similar reasons $\eta_p(i)$ $[\eta_e(i)]$ is called "the past of w_p $[w_e]$ at time i."

The reason for defining these enlarged state variables is that the associated stochastic optimal control problems for the assumed pair of minimax strategies are of the classical linear-quadratic-Gaussian type when formulated in terms of these variables. This means that the extensive body of results known for this control problem can be applied to obtain the solutions of these associated optimal control problems explicitly in terms of the parameters of the opponents' assumed strategies, leading directly to a set of implicit difference equations for the "enlarged" matrices characterizing the linear solutions (i.e., solutions of this assumed form) to this multistage game.

C. The Basis of the Certainty-Coincidence Property

In the above formulation, moreover, each player's measurement noise appears as process noise in his opponent's associated optimal control problem. For this reason, it can be argued via the certainty-equivalence principle for linear-quadratic-Gaussian control problems that the sample path followed by the controls and state under this minimax strategy pair, for the case in which the initial estimate is correct and all process and measurement noises are zero, is the same path as that followed by the corresponding deterministic multistage game under its minimax strategies when started in the same initial state. This "certainty-coincidence" property is useful in interpreting the solution to the stochastic differential game, which is the topic of ultimate interest.

D. The Discretized Game

The next stage of the derivation is to divide the time interval $[0, t_f]$ into N small subintervals of equal length Δ. If, for the stochastic differential game described earlier, the pursuer and evader are restricted to use constant control values during each subinterval, the transition and criterion equation can be expressed in multistage form. These equations are linear and quadratic, respectively, and for small Δ can be approximated to first order by

$$x[(i+1)\Delta] = x(i\Delta) + \Delta[G_p(i\Delta)u(i\Delta) - G_e(i\Delta)v(i\Delta) + \xi(i\Delta)]$$

$$J = \frac{1}{2}\,\varepsilon\left\{ x^T(N\Delta)S_f x(N\Delta) + \sum_{i=0}^{N-1} [u^T(i\Delta)B(i\Delta)u(i\Delta) - v^T(i\Delta)C(i\Delta)v(i\Delta)]\Delta \right.$$

This game can be posed entirely in the discrete-time format if the players' measurements are appropriately discretized, which is accomplished by allowing them to make measurements of the state only at times $i\Delta$; $i = 0, \ldots, N-1$. That is, the piecewise continuous controls must be based on the measurements

$$z_p(i\Delta) = H_p(i\Delta)x(i\Delta) + w_p(i\Delta) \qquad \text{(pursuer)}$$

$$z_e(i\Delta) = H_e(i\Delta)x(i\Delta) + w_e(i\Delta) \qquad \text{(evader)}$$

In order to approximate the differential game situation properly, it is necessary to adjust the statistics of the measurement noises and the process noise so that they have asymptotically the same effect over each subinterval. This is done by choosing $\Delta\xi(i\Delta)$, $\Delta w_p(i\Delta)$, and $\Delta w_e(i\Delta)$ to have the same respective statistics as the random variables

$$\left[\int_{i\Delta}^{(i+1)\Delta} \xi(t)dt\right], \quad \left[\int_{i\Delta}^{(i+1)\Delta} w_p(t)dt\right], \quad \text{and} \quad \left[\int_{i\Delta}^{(i+1)\Delta} w_e(t)dt\right].$$

That is, in the discretized game, $\begin{bmatrix} \xi(i\Delta) \\ \hline w_p(i\Delta) \\ \hline w_e(i\Delta) \end{bmatrix}$ is defined to be a sequence of independent random variables which are

$$\text{Normal}\left(\begin{bmatrix} 0 \\ \hline 0 \\ \hline 0 \end{bmatrix}, \left[\begin{array}{c|c|c} \frac{1}{\Delta} Q(i\Delta) & 0 & 0 \\ \hline 0 & \frac{1}{\Delta} R_p(i\Delta) & 0 \\ \hline 0 & 0 & \frac{1}{\Delta} R_e(i\Delta) \end{array}\right] \right)$$

The solution to this discretized version of the differential game is then obtained as a special case of the multistage game already solved, which is accomplished by making the following formal substitutions:

$$
\begin{aligned}
i &\rightarrow i\Delta \\
G_p(i) &\rightarrow G_p(i\Delta)\Delta \\
G_e(i) &\rightarrow G_e(i\Delta)\Delta \\
B(i) &\rightarrow B(i\Delta)\Delta \\
C(i) &\rightarrow C(i\Delta)\Delta \\
Q(i) &\rightarrow \frac{1}{\Delta} Q(i\Delta) \\
R_p(i) &\rightarrow \frac{1}{\Delta} R_p(i\Delta) \\
R_e(i) &\rightarrow \frac{1}{\Delta} R_e(i\Delta)
\end{aligned}
$$

The purpose for constructing this discretized game is to determine the asymptotic form of its solution as Δ approaches zero. As a first step in this direction, the first order approximations of the equations characterizing the solution are obtained for small Δ. This step is not altogether straightforward because the dimensions of the enlarged state variables used in the solution vary as $\frac{1}{\Delta}$, so the dimensions of the enlarged matrices involved in the solutions to multi-stage games become larger as Δ becomes smaller. This difficulty is further complicated by the fact that different partitions of the enlarged matrices involved in the solution approach different orders of magnitude as Δ approaches zero. It is possible, however, to obtain through intuition and trial and error the correct orders of magnitude of the various matrix partitions, and to isolate the first order terms in the difference equations characterizing the solution to this problem. These manipulations are carried out in detail in Appendix B. Since the enlarged state variables are partitioned vectors, the enlarged matrices involved in the solution can be partitioned in a natural way. In fact, it is in terms of these matrix partitions that the first order difference equations for the discretized game are obtained.

E. The Differential Game

If the solutions to successively finer discretizations of the stochastic differential game originally posed exist and approach a limiting form as the discretization interval approaches zero, then this asymptotic solution is taken to be the solution to the differential game. Although it is beyond the scope of this report to examine rigorously the significance of the formal solutions resulting from this procedure, it is appropriate to comment briefly here on the justification for such an approach.

First of all, it is reasonable to require that any meaningful solution to this type of stochastic differential game have the property of being the limiting form of the solutions to an approximating sequence of multistage games, especially since most such games arise in practice from situations in which decision making and data processing are actually carried out on a discrete-time basis. Thus, although the definition of such a solution could probably be streamlined by defining some appropriate intermediate concepts, any reasonable solution should agree with the formal solution.

Once a formal solution of this type is obtained, however, there is still the question of whether it has any reasonable meaning. A good definition of "reasonable" here is suggested by the "K-strategy" concept discussed by Isaacs [3]. Although originally defined for deterministic games, the natural extension of this concept to the present stochastic context is essentially the assertion that a player's strategy is minimax (in the sense of K-strategies) if it is the "limiting form" of minimax strategies for this player in games that result from restricting both players to change control values only at specified times between the initial and final times, as the maximum time interval between control changes becomes infinitesimal (but independently so) for both players, if this limit exists. The limit concept referred to here means basically that the control histories produced by the minimax strategies in the "restricted" games (called K-strategies) converge uniformly to that produced by the true minimax strategy for almost all sample paths of the measurements on which this player's control is based. Heuristically, this means that each player, if he processes his data rapidly, is applying approximately the correct control, assuming that his opponent is also processing his data rapidly (but otherwise arbitrarily fast). When consideration is limited to strategies that are linear functionals of the measurements, as is done here, this is equivalent to saying that the piecewise-constant weighting functions corresponding to the "restricted" minimax strategies converge uniformly to the weighting function corresponding to the true minimax strategy.

Although the rigorous details are glossed over, the K-strategy concept is actually the basic motivation behind the analysis used here, in that the solution to the stochastic differential game is treated as the limiting form of the solutions to a sequence of games in which the players are restricted to piecewise-constant controls. This analysis, however, is carried out in such a way that the time interval between control changes is constrained to be the same for both players, even though it becomes infinitesimal. The issue of allowing the maximum time between control changes to go to zero independently for each player is ignored, under the implicit assumption that in most cases of interest this would not affect the asymptotic result. An example of such a situation would be a case in which the evader is allowed to change control twice as often as the pursuer, but with the interval between control changes being infinitesimal for both players. As a partial justification of this omission, however, it should be pointed out that such a situation could be modelled in the present framework by making the pursuer's measurement noise spectral density parameter infinite at every other time step, and halving it the rest of the time. Since the payoff is convex in the control, this would mean that the pursuer would only change control values at every other time step (when more information becomes available), whereas the integrated effect of his measurement noise spectral density parameter in the differential equations, whose solutions characterize the game solution, would remain the same.

As is suggested by the preceding remarks, the problem of determining a solution to the stochastic differential game at hand is essentially reduced to finding the asymptotic form of the matrix difference equations characterizing the solutions to discretized versions of the differential game. Since these equations are available (to first order for small Δ) in partition-by-partition form, where the partitions are indexed (say by j and k) from 0 to N (which is of order $\frac{1}{\Delta}$), it is helpful to make a modification in notation by referring to the j, k-th partition of an enlarged time-varying matrix $A(i\Delta)$ as $A(i\Delta, j\Delta, k\Delta)$. In this way, the partitions of the enlarged matrix partitions become functions of three variables, each taking on values from 0 to t_f ($=N\Delta$). This makes the transition to the continuous limit very natural, by considering the arguments to be continuously varying in the interval $[0, t_f]$. Because of the differences in the orders of magnitude of the various enlarged matrix partitions, it is also convenient to modify some of the variables so that all these partitions are expressed in terms of normalized quantities (and Δ). After these two modifications have been made, it is fairly straightforward to obtain the asymptotic equations characterizing the linear solutions to the differential game. These constitute a system of implicit integro-differential equations. In Appendix C these equations are derived in detail, and are recast into a neater form by a suitable redefinition of parameters.

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4. THE SOLUTION

As a result of applying the derivation described in the preceding section, and carried out in detail in the appendices, it is verified that a solution to this game can be obtained of the form

$$u(t) = -B^{-1}(t)G_p^T(t)\left[A_p(t)\bar{x}_o + \int_0^t \Lambda_p(t,\tau)z_p(\tau)d\tau\right] \tag{17}$$

$$v(t) = -C^{-1}(t)G_e^T(t)\left[A_e(t)\bar{x}_o + \int_0^t \Lambda_e(t,\tau)z_e(\tau)d\tau\right] \tag{18}$$

provided that the parameters A_p, A_e, Λ_p, and Λ_e are "well-behaved" functions of t and τ, and satisfy a certain system of implicit equations. These equations, which were mentioned in the previous section, result basically from the requirement that each player's minimax strategy be a solution to the stochastic optimal control problem which results from substituting his opponent's minimax strategy into the transition equation and the criterion functional.

A. The Two Point Boundary Value Problem

A discussion of the meaning and origin of the parameters involved in this system of equations will be deferred until later. For the present, this system of integro-differential equations should merely be considered operationally as a set of equations which must be solved in order to find a pair of minimax strategies for the stochastic differential game. Adopting this point of view, these equations can be decomposed into a primary equation system, which can be solved independently and which determines the values of the Λ's, and a subsidiary equation system determining the values of the A's, which can be solved given a solution to the primary equation system. Using the symbols T_p, T_e, Q_p, and Q_e to represent, respectively, $G_pB^{-1}G_p^T$, $G_eC^{-1}G_e^T$, $H_p^TR_p^{-1}H_p$, and $H_e^TR_e^{-1}H_e$, and suppressing the "t" argument for functions of t only, these equation systems are:

Primary Equation System

$$\dot{\Omega}_p = Y_pT_p\Omega_p + H_e^T \int_t^{t_f} [T_p(\zeta)\Gamma_p(\zeta, t) - T_e(\zeta)\Lambda_e(\zeta, t)]^T \, \Omega_p(\zeta)d\zeta \, ;$$

$$\Omega_p(t_f) = S_f \tag{19}$$

$$\dot{\Omega}_e = -Y_eT_e\Omega_e - H_p^T \int_t^{t_f} [T_p(\zeta)\Lambda_p(\zeta, t) - T_e(\zeta)\Gamma_e(\zeta, t)]^T \, \Omega_e(\zeta)d\zeta \, ;$$

$$\Omega_e(t_f) = S_f \tag{20}$$

$$\frac{\partial\Gamma_p(t,\tau)}{\partial t} = Y_p[T_p\Gamma_p(t,\tau) - T_e\Lambda_e(t,\tau)] + H_e^T \int_t^{t_f} \Big[[\Gamma_p(\zeta, t) - \Lambda_e(\zeta, t)]^T \, T_e(\zeta) \cdot$$

$$[\Gamma_p(\zeta,\tau) - \Lambda_e(\zeta,\tau)] + \Gamma_p^T(\zeta, t)[T_p(\zeta) - T_e(\zeta)]\Gamma_p(\zeta,\tau)\Big] d\zeta \, ;$$

$$\Gamma_p(t_f,\tau) = 0 \tag{21}$$

$$\frac{\partial\Gamma_e(t,\tau)}{\partial t} = Y_e[T_p\Lambda_p(t,\tau) - T_e\Gamma_e(t,\tau)] + H_p^T \int_t^{t_f} \Big[\Gamma_e^T(\zeta, t)[T_p(\zeta) - T_e(\zeta)]\Gamma_e(\zeta,\tau) -$$

$$[\Lambda_p(\zeta,t) - \Gamma_e(\zeta,t)]^T \, T_p(\zeta)[\Lambda_p(\zeta,\tau) - \Gamma_e(\zeta,\tau)]\Big] d\zeta \, ; \; \Gamma_e(t_f,\tau) = 0 \tag{22}$$

$$\dot{P}_p = T_e\left[\int_0^t \Lambda_e(t,\zeta)M_p^T(t,\zeta)d\zeta\right] + \left[\int_0^t \Lambda_e(t,\zeta)M_p^T(t,\zeta)d\zeta\right]^T T_e - P_pQ_pP_p + Q\,;$$

$$P_p(0) = P_o \tag{23}$$

$$\dot{P}_e = -T_p\left[\int_0^t \Lambda_p(t,\zeta)M_e^T(t,\zeta)d\zeta\right] - \left[\int_0^t \Lambda_p(t,\zeta)M_e^T(t,\zeta)d\zeta\right]^T T_p - P_eQ_eP_e + Q\,;$$

$$P_e(0) = P_o \tag{24}$$

$$\frac{\partial M_p(t,\tau)}{\partial t} = T_e\left[\Lambda_e(t,\tau)R_e(\tau) + \int_0^t \Lambda_e(t,\zeta)N_p^T(t,\tau,\zeta)d\zeta\right] - P_pQ_pM_p(t,\tau)\,;$$

$$M_p(t,t) = P_pH_e^T \tag{25}$$

$$\frac{\partial M_e(t,\tau)}{\partial t} = -T_p\left[\Lambda_p(t,\tau)R_p(\tau) + \int_0^t \Lambda_p(t,\zeta)N_e^T(t,\tau,\zeta)d\zeta\right] - P_eQ_eM_e(t,\tau)\,;$$

$$M_e(t,t) = P_eH_p^T \tag{26}$$

$$\frac{\partial N_p(t,\tau,\sigma)}{\partial t} = -M_p^T(t,\tau)Q_pM_p(t,\sigma)\;;\; N_p(t,\tau,t) = N_p^T(t,t,\tau) = M_p(t,\tau)H_e^T(\tau) \tag{27}$$

$$\frac{\partial N_e(t,\tau,\sigma)}{\partial t} = -M_e^T(t,\tau)Q_eM_e(t,\sigma)\;;\; N_e(t,\tau,t) = N_e^T(t,t,\tau)$$

$$= M_e(t,\tau)H_p^T(\tau) \tag{28}$$

$$\frac{\partial K_p(t,\tau)}{\partial t} = T_e\int_0^t \Lambda_e(t,\zeta)L_p(t,\tau,\zeta)d\zeta - T_p\Lambda_p(t,\tau) - P_pQ_pK_p(t,\tau)\,;$$

$$K_p(t,t) = P_pH_p^TR_p^{-1} \tag{29}$$

$$\frac{\partial K_e(t,\tau)}{\partial t} = T_e\Lambda_e(t,\tau) - T_p\int_0^t \Lambda_p(t,\zeta)L_e(t,\tau,\zeta)d\zeta - P_eQ_eK_e(t,\tau)\,;$$

$$K_e(t,t) = P_eH_e^TR_e^{-1} \tag{30}$$

$$\frac{\partial L_p(t,\tau,\sigma)}{\partial t} = -M_p^T(t,\sigma)Q_pK_p(t,\tau)\;;\; L_p(t,t,\sigma) = M_p^T(t,\sigma)H_pR_p^{-1} \tag{31}$$

$$\frac{\partial L_e(t,\tau,\sigma)}{\partial t} = -M_e^T(t,\sigma)Q_eK_e(t,\tau) \; ; \; L_e(t,t,\sigma) = M_e^T(t,\sigma)H_eR_e^{-1} \quad (32)$$

$$\Lambda_p(t,\tau) = Y_pK_p(t,\tau) + \int_0^t \Gamma_p(t,\zeta)L_p(t,\tau,\zeta)d\zeta \quad (33)$$

$$\Lambda_e(t,\tau) = Y_eK_e(t,\tau) + \int_0^t \Gamma_e(t,\zeta)L_e(t,\tau,\zeta)d\zeta \quad (34)$$

$$Y_p = \Omega_p + \int_t^{t_f} \Gamma_p(t,\zeta)H_e(\zeta)d\zeta \quad (35)$$

$$Y_e = \Omega_e + \int_t^{t_f} \Gamma_e(t,\zeta)H_p(\zeta)d\zeta \quad (36)$$

<u>Subsidiary Equation System</u>

$$\dot{\Pi}_p = Y_p[T_p\Pi_p - T_eA_e] + H^T \int_t^{t_f} [\Gamma_p(\zeta,t) - \Lambda_e(\zeta,t)]^T \, T_e(\zeta)[\Pi_p(\zeta) - A_e(\zeta)]\,d\zeta \; ;$$
$$\Pi_p(t_f) = 0_{n,n} \quad (37)$$

$$\dot{\Pi}_e = Y_e[T_pA_p - T_e\Pi_e] - H_p^T \int_t^{t_f} [\Gamma_e(\zeta,t) - \Lambda_p(\zeta,t)]^T \, T_p(\zeta)[\Pi_e(\zeta) - A_p(\zeta)]\,d\zeta \; ;$$
$$\Pi_e(t_f) = 0_{n,n} \quad (38)$$

$$\dot{D}_p = T_eA_e - T_pA_p - P_pQ_pD_p \; ; \; D_p(0) = I_n \quad (39)$$

$$\dot{D}_e = T_eA_e - T_pA_p - P_eQ_eD_e \; ; \; D_e(0) = I_n \quad (40)$$

$$\frac{\partial \theta_p(t,\tau)}{\partial t} = -M_p^T(t,\tau)Q_pD_p \; ; \; \theta_p(t,t) = H_eD_p \quad (41)$$

$$\frac{\partial \theta_e(t,\tau)}{\partial t} = -M_e^T(t,\tau)Q_eD_e \; ; \; \theta_e(t,t) = H_pD_e \quad (42)$$

$$A_p = \Pi_p + Y_pD_p + \int_0^t \Gamma_p(t,\zeta)\,\theta_p(t,\zeta)d\zeta \quad (43)$$

$$A_e = \Pi_e + Y_eD_e + \int_0^t \Gamma_e(t,\zeta)\,\theta_e(t,\zeta)d\zeta \quad (44)$$

Notice that each of these two equation systems constitutes a two point boundary value problem, in that some of the boundary values are specified at the initial time and some at the final time. However, they are more complicated than the usual type of two point boundary value problem arising in optimal control theory. First, some of the parameters involved in these equations (such as the Γ's and θ's) are functions of more than one variable. Also, the coupling between the differential equations is diffused over time, in that the derivatives of some of the quantities depend not only on the present values of the other quantities involved in the equation system, but on past or future values as well. A qualitative diagram of the couplings among the various equations in these systems is shown in Figure 2.

In summary then, the problem of finding a solution to the stochastic differential game at hand has been reduced to the procedure of:

(a) finding a solution to the two point boundary value problem consisting of the primary equation system,

(b) using this solution to find a solution to the subsidiary equation system (also a two point boundary value problem), and

(c) substituting the values of Λ_p and Λ_e from the solution to the primary equation system, and the values of A_p and A_e from the solution to the subsidiary equation system into equations (17) and (18).

The form in which these minimax strategies are expressed, as obtained from this procedure, will be referred to as "Realization I." It will be shown later how other intuitively more satisfying realizations

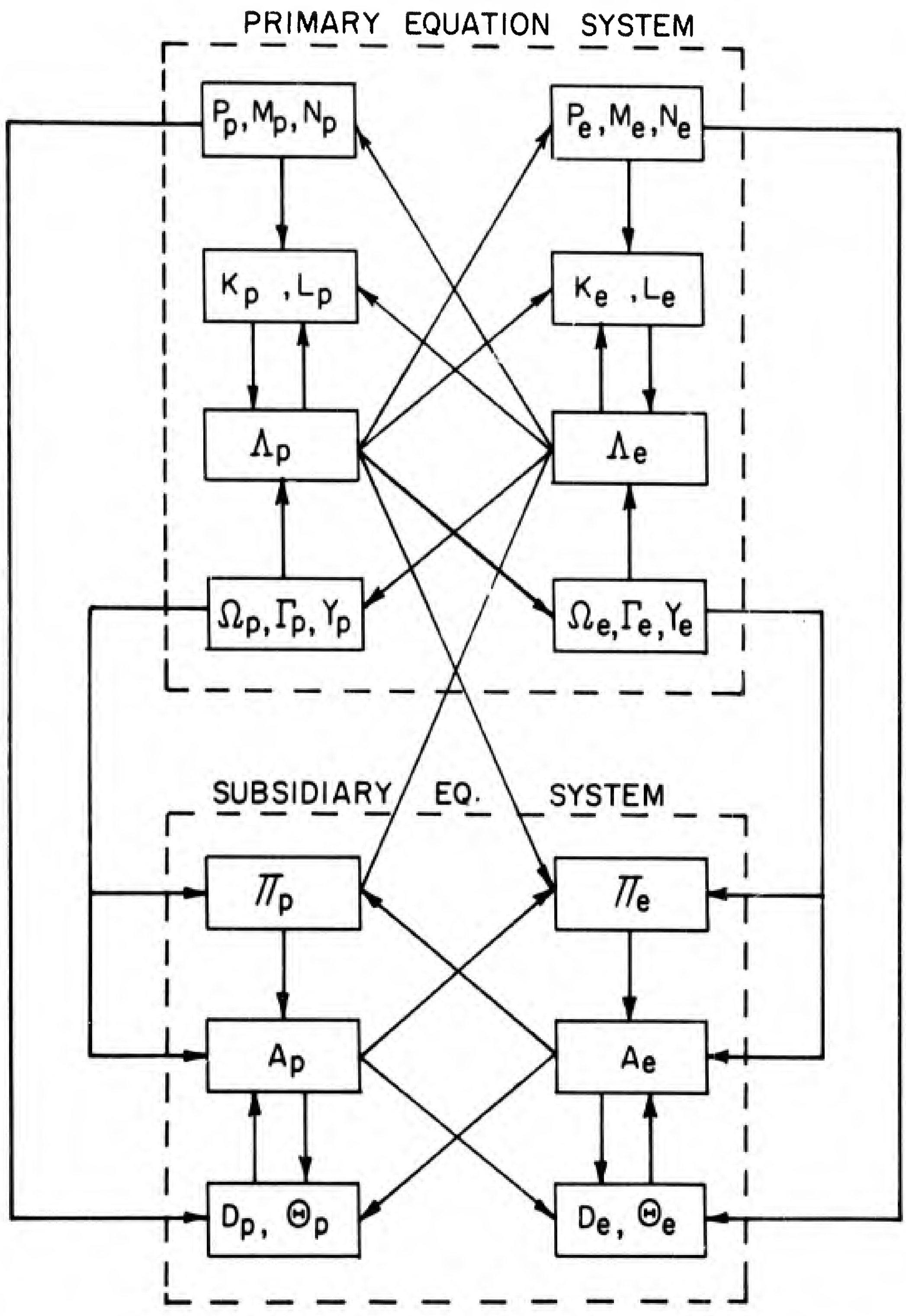

FIG. 2 COUPLING IN THE IMPLICIT EQUATION SYSTEMS

of these same strategies can be obtained. The behavior of Realization I of the pursuer's minimax strategy is depicted schematically in Figure 3.

B. Interpretation of Parameters

Although it is necessary to follow through the details in the derivation of the solution in order to really appreciate the origin and significance of all the parameters involved in the systems of implicit equations characterizing the solution to this game, it is possible to give straightforward interpretations to some of these parameters. Denoting the measurement histories $\{(z_p(\tau), \tau) : \tau \leq t\}$ and $\{(z_e(\tau), \tau) : \tau \leq t\}$ by $Z_p(t)$ and $Z_e(t)$, it is shown in Appendix C that, assuming that both players use their linear minimax strategies,

$$P_p(t) = \text{cov}[x(t), x(t)/Z_p(t)]^* \tag{45}$$

$$P_e(t) = \text{cov}[x(t), x(t)/Z_e(t)] \tag{46}$$

$$\left.\begin{aligned} M_p(t,\tau) &= \text{cov}[x(t), z_e(\tau)/Z_p(t)] \quad (47) \\ M_e(t,\tau) &= \text{cov}[x(t), z_p(\tau)/Z_e(t)] \quad (48) \end{aligned}\right\} \tau \leq t$$

$$\left.\begin{aligned} N_p(t,\tau,\sigma) &= \text{cov}[z_e(\tau), z_e(\sigma)/Z_p(t)] \quad (49) \\ N_e(t,\tau,\sigma) &= \text{cov}[z_p(\tau), z_p(\sigma)/Z_e(t)] \quad (50) \end{aligned}\right\} \tau, \sigma \leq t \ (\tau \neq \sigma)$$

and that, under the same assumption,

$$\hat{x}_p(t) \triangleq \left[D_p \bar{x}_o + \int_0^t K_p(t,\xi) z_p(\xi) d\xi\right] = \varepsilon[x(t)/Z_p(t)] \tag{51}$$

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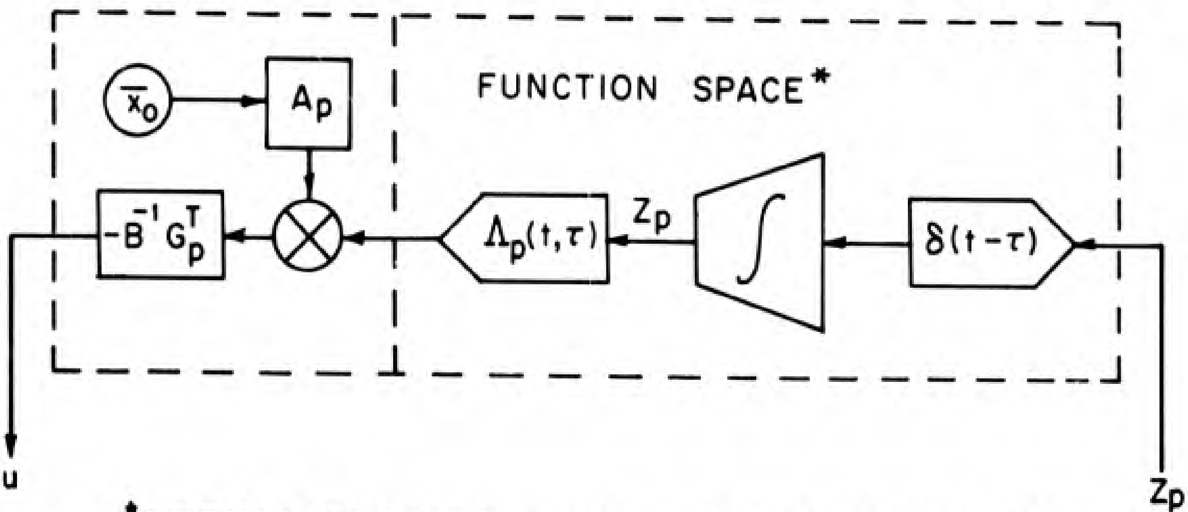
*This notation means $\varepsilon[(x(t)-\bar{x}(t))(x(t)-\bar{x}(t))^T/Z_p(t)]$.



*"FUNCTION SPACE" refers to operations performed on functions of two variables (which are like infinite-dimensional vector functions of time only). The net output of this section of the diagram is just

$$\int_0^t \Lambda_p(t,\upsilon) z_p(\tau) d\tau$$

The reason this operation is represented in this somewhat obscure manner is to illustrate the use of the following function space blocks, which will be used in later figures:

SYMBOL	INPUT	OUTPUT
$X \rightarrow$ [$A(t,\tau)$ $\rangle$] $\rightarrow y$	$x(t,\tau)$	$y(t) = \int_{t_0}^{t} A(t,\tau) x(t,\tau) d\tau$
$x \rightarrow$ [$\langle$ $A(t,\tau)$] $\rightarrow Y$	$x(t)$	$y(t,\tau) = A(t,\tau) x(t)$
$X \rightarrow$ [$\int$] $\rightarrow Y$	$x(t,\tau)$	$y(t,\tau) = \int_{\tau}^{t} x(\zeta,\tau) d\zeta$

FIG. 3 PURSUER'S STRATEGY-REALIZATION I

$$\hat{x}_e(t) \triangleq \left[D_e\bar{x}_o + \int_0^t K_e(t,\zeta)z_e(\zeta)d\zeta\right] = \varepsilon[x(t)/Z_e(t)] \tag{52}$$

$$\left.\begin{aligned}\hat{z}_{ep}(\tau/t) &\triangleq \left[\theta_p(t,\tau)\bar{x}_o + \int_0^t L_p(t,\zeta,\tau)z_p(\zeta)d\zeta\right] = \varepsilon[z_e(\tau)/Z_p(t)] \quad (53)\\ \hat{z}_{pe}(\tau/t) &\triangleq \left[\theta_e(t,\tau)\bar{x}_o + \int_0^t L_e(t,\zeta,\tau)z_e(\zeta)d\zeta\right] = \varepsilon[z_p(\tau)/Z_e(t)] \quad (54)\end{aligned}\right\} \tau \leq t$$

Thus, these parameters may be regarded as being associated with a kind of Kalman filter which produces estimates of the current state $\hat{x}_p$ and $\hat{x}_e$ and the opponents' measurement histories $\hat{z}_{ep}$ and $\hat{z}_{pe}$. These filters, however, are based on the presumption that the opponents are employing their minimax strategies as defined by equations (17) and (18). It is shown in Appendix C, in fact, that the "estimates" defined in equations (51)-(54) can be generated from the available measurements by the following Kalman filter-like differential equations:

$$\dot{\hat{x}}_p = G_pu + T_eA_e\bar{x}_o + T_e\int_0^t \Lambda_e(t,\tau)\hat{z}_{ep}(\tau/t)d\tau + P_pH_p^TR_p^{-1}[z_p - H_p\hat{x}_p]\ ; \quad \hat{x}_p(0) = \bar{x}_o \tag{55}$$

$$\frac{\partial\hat{z}_{ep}(\tau/t)}{\partial t} = M_p^T(t,\tau)H_p^TR_p^{-1}[\hat{z}_p - H_p\hat{x}_p]\ ;\ \hat{z}_{ep}(t/t) = H_e\hat{x}_p \tag{56}$$

and

$$\dot{\hat{x}}_e = -G_ev - T_pA_p\bar{x}_o - T_p\int_0^t \Lambda_p(t,\tau)\hat{z}_{pe}(\tau/t)d\tau + P_eH_e^TR_e^{-1}[z_e - H_e\hat{x}_e]\ ; \quad \hat{x}_e(0) = \bar{x}_o \tag{57}$$

$$\frac{\partial\hat{z}_{pe}(\tau/t)}{\partial t} = M_e^T(t,\tau)H_e^TR_e^{-1}[z_e - H_e\hat{x}_e]\ ;\ \hat{z}_{pe}(t/t) = H_p\hat{x}_e\ , \tag{58}$$

where again the "t" argument has been suppressed for functions of t only. If it is further assumed that each player is using his linear minimax strategy (i. e., those strategies obtained in this report, as opposed to some other possibly existing minimax strategies), the sum of the second and third terms in equation (55) can be interpreted as $G_e\hat{v}$, where $\hat{v}$ is the expected value of v given Z_p. Analogous remarks hold for the second and third terms of equation (57).

The parameters A_p, A_e, Λ_p and Λ_e are obviously (from equations 17 and 18) the weighting functions for the initial state and the measurements in the minimax strategies. Finally, it can be qualitatively said that the remaining parameters (the Ω's, Y's, Γ's, and Π's) are connected with the minimax cost-to-go in the players' associated optimal control problems, a connection which will receive no further comment here.

C. Another Realization

Assuming now that a solution has been found for the primary and subsidiary equation systems, these solutions can be used in Realization I to implement the minimax controls directly as linear functionals of the available measurements. These same strategies can be expressed in a more interesting way, called "Realization II," by substituting equations (33) and (34) for the Λ's and equations (43) and (44) for the A's in equations (17) and (18). These substitutions, followed by a change in the order of integration, show that the minimax controls can be expressed as

$$u = -B^{-1}G_p^T\left\{\Pi_p\bar{x}_o + Y_p\left[D_p\bar{x}_o + \int_0^t K_p(t,\zeta)z_p(\zeta)d\zeta\right] + \int_0^t \Gamma_p(t,\tau)\left[\theta_p(t,\tau)\bar{x}_o + \int_0^t L_p(t,\zeta,\tau)z_p(\zeta)d\zeta\right]d\tau\right\} \tag{59}$$

and

$$v = -C^{-1}G_e^T\left\{\Pi_e\bar{x}_o + Y_e\left[D_e\bar{x}_o + \int_0^t K_e(t,\zeta)z_e(\zeta)d\zeta\right] + \int_0^t \Gamma_e(t,\tau)\left[\theta_e(t,\tau)\bar{x}_o + \int_0^t L_e(t,\zeta,\tau)z_e(\zeta)d\zeta\right]d\tau\right\} \tag{60}$$

Using equations (51)-(54), these optimal control laws can be rewritten as

$$u = -B^{-1}G_p^T\left[\Pi_p\bar{x}_o + Y_p\hat{x}_p + \int_0^t \Gamma_p(t,\tau)\hat{z}_{ep}(\tau/t)d\tau\right] \tag{61}$$

and

$$v = -C^{-1}G_e^T\left[\Pi_e\bar{x}_o + Y_e\hat{x}_e + \int_0^t \Gamma_e(t,\tau)\hat{z}_{pe}(\tau/t)d\tau\right] \tag{62}$$

where the pursuer's estimates are generated by equations (55) and (56), and the evader's by equations (57) and (58). The action of Realization II for the pursuer is displayed in Figure 4. Remember that the estimates produced as a part of this realization can be interpreted as Kalman filter estimates (i. e., conditional mean estimates) only if the assumption is made that the opponents' strategies are the minimax strategies derived here. In actuality, of course, each player's opponent is free to use some other strategy, in which case this interpretation is not valid. By construction, however, such efforts by either player to "confuse" his opponent would only result in an overall detraction from his own expected payoff.

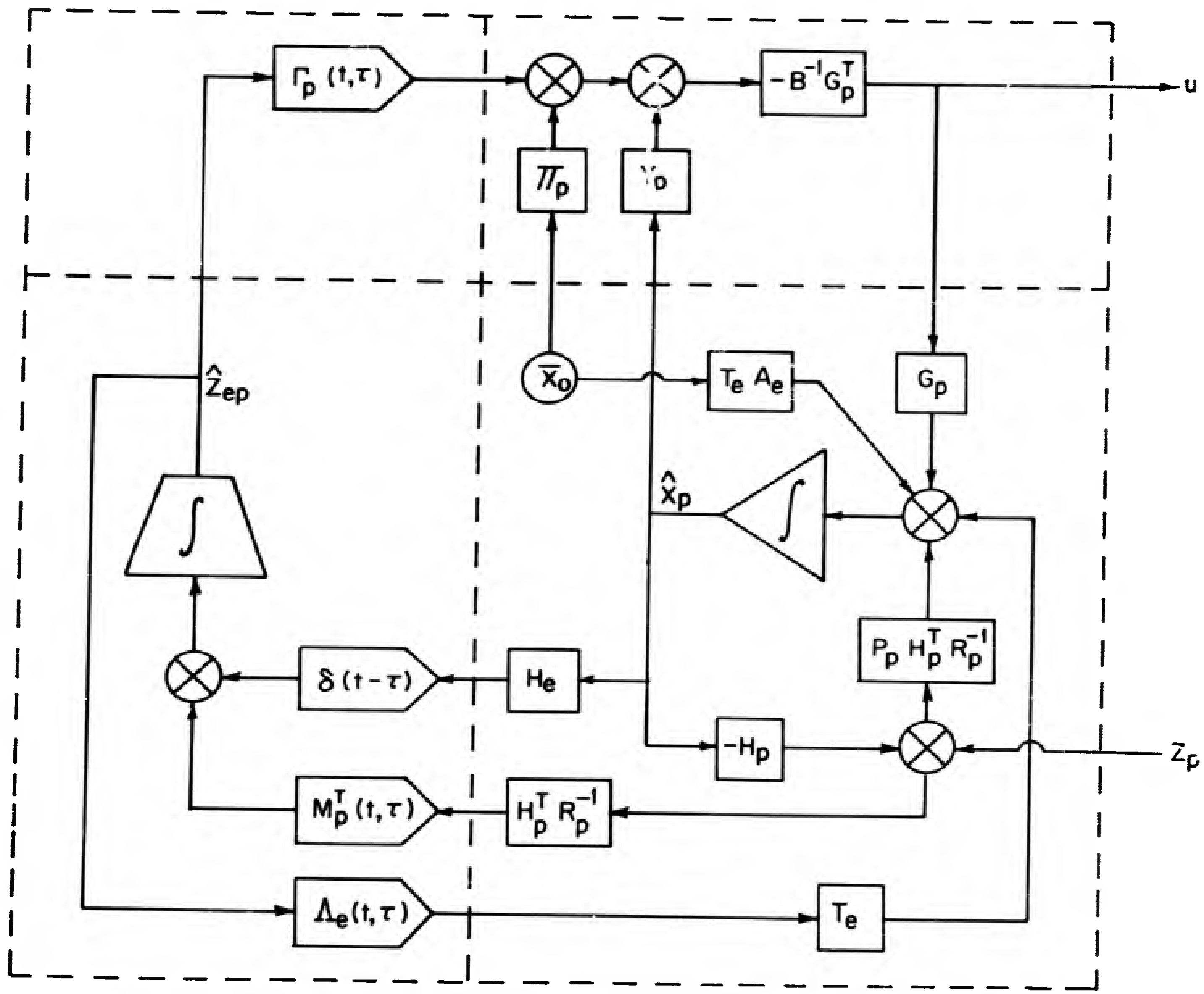


FIG. 4 PURSUER'S STRATEGY REALIZATION II

D. The Certainty-Coincidence Property: Realization III

Realization II expresses the minimax control laws in the more familiar form of linear feedback laws operating on estimates generated by a Kalman-like filter. It should be noted, however, that the estimates of the opponents' measurement histories used in this realization are equivalent to infinite-dimensional vectors, the components of which are indexed by the continuous variable corresponding to the previous times at which measurements were taken. This feature presents a serious difficulty in the implementation of the minimax strategies in this form. The certainty-coincidence property, besides being of theoretical interest in its own right, provides a way around this difficulty to a certain extent. This is accomplished by the construction of another realization of the minimax strategies, designated "Realization III," in which the infinite-dimensional parts affect only a set of correction terms, which are produced by deviations of the game from its deterministic behavior.

The first step in this construction is to notice that the necessary conditions derived in Appendix C for the noiseless sample path (equations C69-C72) uniquely determine this trajectory. The name "noiseless sample path" is used here to denote the trajectory followed by the game in the event that the initial state is actually $\bar{x}_o$ and the process and measurement noises are all identically zero, assuming that the minimax strategies derived here are used by both players. This trajectory, moreover, coincides with the minimax path followed by the corresponding deterministic game, as defined by equations (1) and (5)-(7) with $x(0) = \bar{x}_o$. Hence, the name "certainty-coincidence" is applied to this phenomenon.

Recalling that the matrix $S(t)$ is defined by the differential equation

$$\dot{S} = S[T_p - T_e]S \; ; \quad S(t_f) = S_f$$

and substituting the deterministic minimax strategies into the deterministic transition equation shows that the minimax trajectory of the corresponding deterministic game has the property:

$$\frac{d}{dt}[Sx] = S\dot{x} + \dot{S}x = S[T_e - T_p]Sx + S[T_p - T_e]Sx = 0 \,, \tag{63}$$

which implies that Sx is constant on this path. Therefore, on the noiseless sample path, which coincides with this deterministic trajectory,

$$x(\tau) = S^{-1}(\tau)S(t)x(t) \qquad \text{for} \quad 0 < t, \tau < t_f \,, \tag{64}$$

assuming that S^{-1} exists (which it does unless there are redundant state variables in the problem formulation).

Furthermore, it can be verified either by a direct examination of equations (55)-(58), or by the fact that these equations are the limiting forms of equations for the discrete-time case for which these results are verified in the appendices, that $\hat{x}_p(t) = \hat{x}_e(t) = x(t)$, $\hat{z}_{ep}(\tau/t) = H_e(\tau)x(\tau)$, and $\hat{z}_{pe}(\tau/t) = H_p(\tau)x(\tau)$ for all $0 \leq \tau \leq t \leq t_f$ <u>on the noiseless sample path.</u> Rewriting the minimax strategies from equations (61) and (62) by adding and subtracting identical terms, and writing S_o for $S(0)$, as

$$u = -B^{-1}G_p^T \left\{ \Pi_p[\bar{x}_o - S_o^{-1}S\hat{x}_p] + \int_0^t \Gamma_p(t,\tau)[\hat{z}_{ep}(\tau/t) - H_e(\tau)S^{-1}(\tau)S\hat{x}_p]d\tau + \left[Y_p + \Pi_p S_o^{-1}S + \int_0^t \Gamma_p(t,\tau)H_e(\tau)S^{-1}(\tau)d\tau \right]\hat{x}_p \right\} \tag{65}$$

and

$$v = -C^{-1}G_e^T \{ \Pi_e[\bar{x}_o - S_o^{-1}S\hat{x}_e] + \int_0^t \Gamma_e(t,\tau)[\hat{z}_{pe}(\tau/t) - H_p(\tau)S^{-1}(\tau)S\hat{x}_e]d\tau +$$

$$\left[Y_e + \Pi_e S_o^{-1}S + \int_0^t \Gamma_e(t,\tau)H_p(\tau)S^{-1}(\tau)d\tau\right]\hat{x}_e \} , \qquad (66)$$

this result, together with equation (64), implies that the first two terms in these expressions are zero on the noiseless sample path, so that

$$u = -B^{-1}G_p^T\left[Y_p + \Pi_p S_o^{-1}S + \int_0^t \Gamma_p(t,\tau)H_e(\tau)S^{-1}(\tau)S\,d\tau\right]\hat{x}_p \qquad (67)$$

and

$$v = -C^{-1}G_e^T\left[Y_e + \Pi_e S_o^{-1}S + \int_0^t \Gamma_e(t,\tau)H_p(\tau)S^{-1}(\tau)S\,d\tau\right]\hat{x}_e \qquad (68)$$

there.

But, since $\bar{x}_o$ is arbitrary in the preceding remarks, this means that

$$B^{-1}G_p^T\left[Y_p + \Pi_p S_o^{-1}S + \int_0^t \Gamma_p(t,\tau)H_e(\tau)S^{-1}(\tau)S\,d\tau\right] \equiv B^{-1}G_p^T S \qquad (69)$$

and

$$C^{-1}G_e^T\left[Y_e + \Pi_p S_o^{-1}S + \int_0^t \Gamma_e(t,\tau)H_p(\tau)S^{-1}(\tau)S\,d\tau\right] \equiv C^{-1}G_e^T S \qquad (70)$$

Therefore, defining the <u>discrepancy variables</u>:

$$e_p(t,\tau) \triangleq \hat{z}_{ep}(\tau/t) - H_e(\tau)S^{-1}(\tau)S(t)\hat{x}_p(t) \; ; \quad \tau < t \qquad (71)$$

$$e_e(t,\tau) \triangleq \hat{z}_{pe}(\tau/t) - H_p(\tau)S^{-1}(\tau)S(t)\hat{x}_e(t) \; ; \quad \tau < t \qquad (72)$$

$$h_p(t) \triangleq \bar{x}_o - S_o^{-1}S(t)\hat{x}_p(t) \qquad (73)$$

$$h_e(t) \triangleq \bar{x}_o - S_o^{-1}S(t)\hat{x}_e(t) \qquad (74)$$

the minimax strategies can be expressed as

$$u = -B^{-1}G_p^T\left[S\hat{x}_p + \Pi_p h_p + \int_0^t \Gamma_p(t,\tau)e_p(t,\tau)\,d\tau\right] \tag{75}$$

and

$$v = -C^{-1}G_e^T\left[S\hat{x}_e + \Pi_e h_e + \int_0^t \Gamma_e(t,\tau)e_e(t,\tau)\,d\tau\right]. \tag{76}$$

Notice that the second terms in equations (71)-(74) are, respectively, $z_e(\tau)$, $z_e(\tau)$, $\bar{x}_o$, and $\bar{x}_o$ on the noiseless sample path, so that the discrepancy variables are all zero on the noiseless sample path, and hence represent discrepancies between the deterministic behavior of the game and the behaviors perceived by the two players on the basis of the available measurements.

In view of these results, the action of the minimax strategies can be interpreted as follows. At each time, the players have information equivalent to conditional mean estimates (given that their opponents are using these strategies) of the current state and their opponents' measurement histories. The control that each player applies at this time is determined by the following procedure:

(i) Construct the noiseless sample path (or equivalently, the minimax path of the corresponding deterministic game) that intersects the estimate of the current state.

(ii) Calculate h as the discrepancy between the initial estimate of the initial state, $\bar{x}_o$, and the initial state of the above noiseless sample path (equations 73, 74).

(iii) Calculate the function e as the discrepancy between the estimate of the opponent's measurement history and the measurement history he would have received on this noiseless sample path (equations 71, 72).

(iv) Apply the control which is the certainty-equivalent control based on the estimate of the current state, plus feedback terms on the discrepancy variables h and e.

At the same time, the estimator sections of these strategies update the estimates of the current state and of the opponents' measurement histories.

It is also possible to obtain for each player a self-contained system of differential equations determining the estimate and discrepancy variables as functionals of the available measurements. Concentrating on the pursuer for the moment, equation (55) can be rewritten by adding and subtracting identical terms as

$$\dot{\hat{x}}_p = G_p u + T_e\Big[A_e h_p + A_e S_o^{-1} S\hat{x}_p + \int_0^t \Lambda_e(t,\tau) e_p(t,\tau)d\tau + \int_0^t \Lambda_e(t,\tau)H_e(\tau)S^{-1}(\tau)S\hat{x}_p d\tau\Big] + P_p H_p^T R_p^{-1}[z_p - H_p\hat{x}_p]; \;\; \hat{x}_p(0) = \bar{x}_o . \tag{77}$$

Noting that the noiseless sample path necessary conditions imply that

$$v = -C^{-1}G_e^T Sx ,$$

it follows from equations (18) and (64) that, for any $\bar{x}_o$,

$$C^{-1}G_e^T Sx \equiv C^{-1}G_e^T\Big[A_e S_o^{-1} + \int_0^t \Lambda_e(t,\tau)H_e(\tau)S^{-1}(\tau)d\tau\Big]Sx \tag{78}$$

on the corresponding noiseless sample path, since $z_e = H_e x$ there. Since equation (78) holds for any value of $\bar{x}_o$,

$$C^{-1}G_e^T S \equiv C^{-1}G_e^T\Big[A_e S_o^{-1} + \int_0^t \Lambda_e(t,\tau)H_e(\tau)S^{-1}(\tau)d\tau\Big]S . \tag{79}$$

Using this result, the estimation equation (77) is equivalent to:

$$\dot{\hat{x}}_p = G_p u + T_e\left[S\hat{x}_p + A_e h_p + \int_0^t \Lambda_e(t,\tau)e_p(t,\tau)d\tau\right] + P_p H_p^T R_p^{-1}[z_p - H_p\hat{x}_p] ;$$

$$\hat{x}_p(0) = \bar{x}_o . \qquad (80)$$

Also, from the definitions of the discrepancy variables and equation (75),

$$\dot{h}_p = -S_o^{-1}[\dot{S}\hat{x}_p + S\dot{\hat{x}}_p] = S_o^{-1}S\Big\{[T_p\Pi_p - T_e A_e]h_p + \int_0^t [T_p\Gamma_p(t,\tau) -$$

$$T_e\Lambda_e(t,\tau)]\, e_p(t,\tau)\, d\tau - P_p H_p^T R_p^{-1}[z_p - H_p\hat{x}_p]\Big\} ;$$

$$h_p(0) = 0 \qquad (81)$$

and

$$\frac{\partial e_p(t,\tau)}{\partial t} = \frac{\partial \hat{z}_{ep}(\tau/t)}{\partial t} - H_e(\tau)S^{-1}(\tau)[\dot{S}\hat{x}_p + S\dot{\hat{x}}_p] = H_e(\tau)S^{-1}(\tau)S_o\dot{h}_p(t) +$$

$$M_p^T(t,\tau)H_p^T R_p^{-1}[z_p - H_p\hat{x}_p];\ e_p(t,t) = 0 . \qquad (82)$$

Similarly, the estimation and discrepancy equations for the evader in Realization III can be obtained as:

$$\dot{\hat{x}}_e = -G_e v - T_p\left[S\hat{x}_e + A_p h_e + \int_0^t \Lambda_p(t,\tau)e_e(t,\tau)d\tau\right] + P_e H_e^T R_e^{-1}[z_e - H_e\hat{x}_e];$$

$$\hat{x}_e(0) = \bar{x}_o \qquad (83)$$

$$\dot{h}_e = S_o^{-1}S\Big\{\left[T_p A_p - T_e\Pi_e\right]h_e + \int_0^t \left[T_p\Lambda_p(t,\tau) - T_e\Gamma_e(t,\tau)\right]e_e(t,\tau)\, d\tau -$$

$$P_e H_e^T R_e^{-1}[z_e - H_e\hat{x}_e]\Big\} ;\ h_e(0) = 0 \qquad (84)$$

$$\frac{\partial e_e(t,\tau)}{\partial t} = H_p(\tau)S^{-1}(\tau)S_o\dot{h}_e(t) + M_e^T(t,\tau)H_e^T R_e^{-1}[z_e - H_e\hat{x}_e];\quad e_e(t,t) = 0$$

$$(85)$$

Again, if the further assumption is made that each player's opponent is using his linear minimax strategy, the second terms in the estimation equations (80) and (83) can be interpreted as $-G_e\hat{v}$ and $-G_p\hat{u}$, respectively, where $\hat{v}$ and $\hat{u}$ have the same definitions as before.

To summarize, the pursuer's minimax strategy in Realization III is determined by equations (75) and (80)-(82), and the evader's by equations (76) and (83)-(85). A schematic representation of Realization III for the minimax pursuit strategy is shown in Figure 5.

E. Remarks

The fact that the controls used in Realization III are the certainty-equivalent controls plus discrepancy terms which are zero on the noiseless sample path opens the possibility of constructing quasi-optimal strategies by replacing the infinite-dimensional discrepancy variables e_p and e_e by finite-dimensional "almost-sufficient" statistics. Since such an approximation would only affect the corrections for noise-induced deviations, it is reasonable to expect that only a small decrement in performance would result, although the computational advantage gained thereby would be great, both in determining the feedback and estimator gains and in the implementation of the resulting strategies. Quasi-optimal strategies based on this idea and their performances are under investigation at the present time.

It is appropriate to comment here on the certainty-coincidence concept as compared to the certainty-equivalence principle of optimal control theory, which is instrumental in establishing the former in the present context. Like the certainty-equivalence principle, the

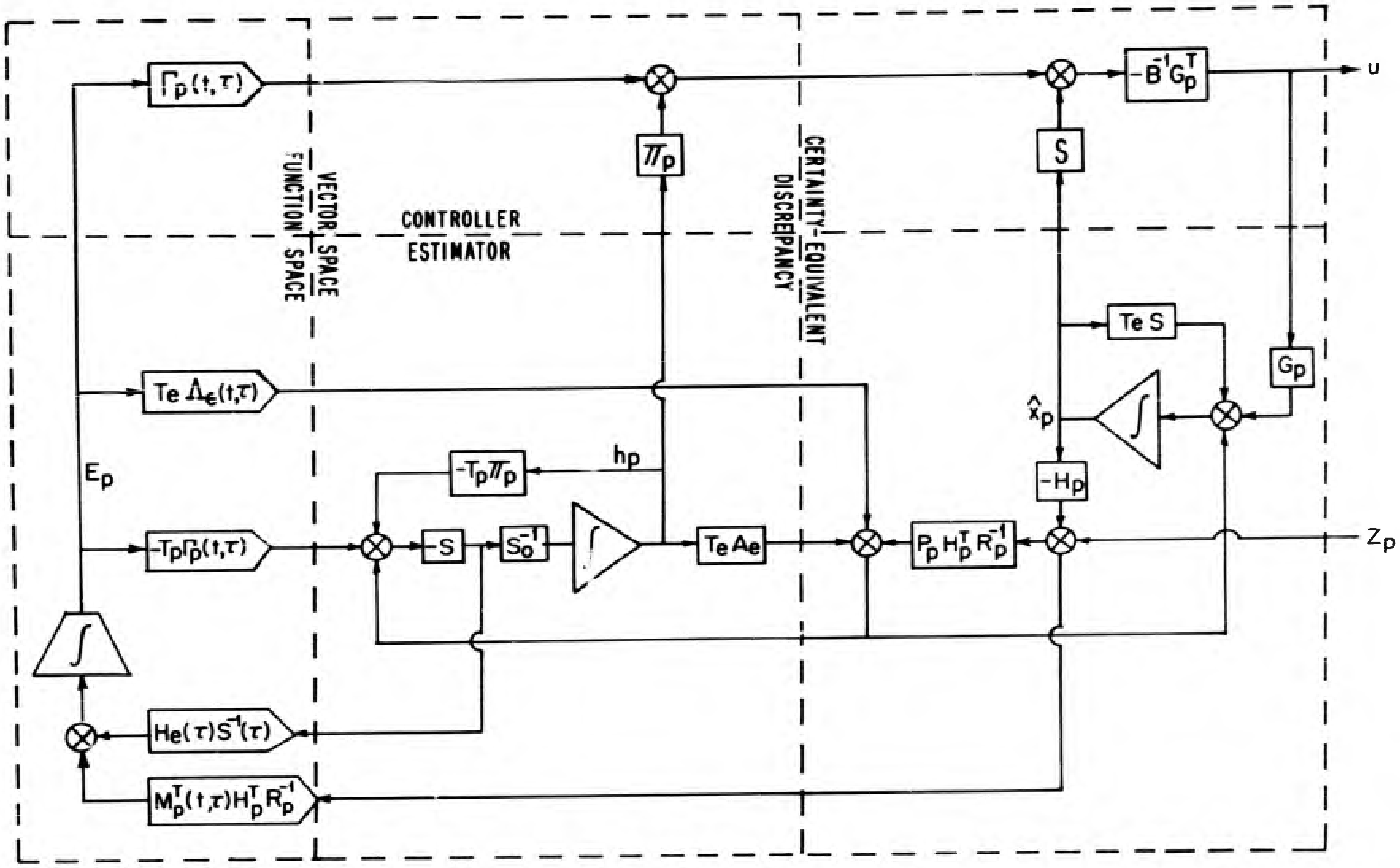


FIG. 5 PURSUER'S STRATEGY – REALIZATION III

certainty-coincidence property implies that the optimal control laws (i. e., minimax strategies) for the stochastic differential game, if they exist, will result in the deterministic minimax state and control variable histories when applied to the corresponding deterministic differential game (with $z_p = H_p x$ and $z_e = H_e x$). Unlike the certainty-equivalence principle, however, it does not say that the minimax control laws for the stochastic game can be obtained by using the minimax strategies from the corresponding deterministic game with the feedback applied to the conditional mean estimates of the current state. In fact, the stochastic game may have no solution. That is, the certainty-coincidence property is like the certainty-equivalence principle going from the stochastic game to the deterministic game, but different going the other way.

Not much is known at present about the existence or computation of solutions to the primary and subsidiary equation systems, in terms of which the minimax strategies obtained here are expressed. Some exploratory numerical calculations have been made, however, for a variety of analogous multistage games (of the type considered in Appendix A). These computations basically consisted of iteratively determining the parameter values for new linear strategy pairs by letting each player optimize against his opponent's linear strategy determined by the parameter values of the previous iteration. Of course, if this procedure converges, the resulting parameter values determine a pair of minimax strategies, practically by definition. The results of these numerical experiments were that this procedure converged, regardless of the starting values, for parameter values that would intuitively give the pursuer a good advantage (such as a high control cost for the evader relative to the pursuer, or a low measurement noise for the pursuer). For parameter values with the opposite characteristics, which correspond to the evader's ability to escape and the non-existence of a minimax solution, this numerical procedure did not converge. Although these computations were only performed for a small set of relatively simple multistage (two-stage, to be exact) games, the indication is that a generalization of this procedure could be applied to solving the primary and subsidiary equation systems, by iteratively calculating new values of A_p, A_e, Λ_p, and Λ_e from old values, solving for the new values (in equations 33, 34, 43, and 44) through an application of these equations to the old values. If the results of the exploratory calculations are valid, then this algorithm would converge if a solution exists, and its failure to converge would mean that the evader could escape. (No solution exists of the form considered here.) The non-existence of such a pair of linear minimax strategies, however, is not known to imply that a non-linear solution does not exist.

5. CONCLUSION

There are two results of major significance concerning the solutions to the class of stochastic differential games examined in this report. First, for conditions under which there are solutions to a certain two point boundary value problem, there exist minimax strategies for both the pursuer and evader which specify the controls as linear functionals of the available measurements. It appears, however, that these strategies are in general infinite-dimensional, meaning that these functionals cannot be represented as the impulse response functions of linear networks described by finite-dimensional state vectors. In view of the construction of Realization II, this difficulty appears to result from the need for each player to continuously estimate his opponent's entire measurement history in order to apply his optimal control.

Also of significance is the certainty-coincidence property which applies in this context. This property, which has been shown to apply for the open-loop game, games in which one player has no measurements (solved by Rhodes [8]), and the class of games investigated by Behn and Ho [7] where one player has perfect measurements, which are all limiting cases (but not special cases) of the class of games considered here, states that the behavior of the stochastic game under the linear minimax strategies coincides with the behavior of the corresponding deterministic (closed-loop) game if the initial estimate happens to be correct and the process and measurement noises happen to be zero. This result makes it possible to construct Realization III, in which the importance of the infinite-dimensional part of the minimax control laws is reduced, thereby opening the way for reasonable approximations to the minimax strategies by finite-dimensional control laws.

Although the class of games considered in this report does not contain the types of games solved by Behn and Ho [7] and Rhodes and Luenberger [8] as special cases, the latter are limiting cases of the type examined here. Figure 6 displays the relationship among these three classes of games, which are differentiated primarily on the basis of the information sets available to the two players. It can be shown that the solution obtained here reduces to that of Rhodes and Luenberger when $Q = 0$ and R_p or R_e approaches infinity. (See Appendix D.) The reduction to the case solved by Behn and Ho is more difficult, however, and has not yet been achieved.

As a final note, it should be mentioned that the analysis used here can be applied almost without modification to the case in which the two controllers of the dynamic system have objectives that are in complete agreement, instead of in direct conflict. That is, the game is a completely cooperative one, or alternatively, a stochastic optimal control problem with a decentralized controller, in this case divided into two non-communicating parts. The only difference in the result for such a situation would be some sign changes in the control laws and the two point boundary value problem. It also seems likely that this approach could be used for linear-quadratic-Gaussian non-zero-sum differential games of the type considered by Starr and Ho [9].

PURSUER \ EVADER	PERFECT MEAS. ($R_p = 0$)	NOISY MEASUREMENTS	NO MEAS. ($R_p = \infty$)
PERFECT MEASUREMENTS ($R_e = 0$)	<u>CLOSED-LOOP GAME:</u> HO, BRYSON, + BARON [6]	BEHN + HO* [7]	
MEASUREMENT NOISE	BEHN + HO* [7]	THIS REPORT	RHODES + LUENBERGER [8]
NO MEASUREMENT ($R_e = \infty$)		RHODES + LUENBERGER [8]	<u>OPEN LOOP GAME</u>

*THIS SOLUTION APPLIES ONLY TO GAMES WITHOUT PROCESS NOISE, AND WITH A FEW OTHER RESTRICTIONS.

FIG. 6 RELATION TO GAMES WITH OTHER INFORMATION SETS

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Appendix A

THE MULTISTAGE GAME

The topic of discussion in this appendix is the multistage pursuit-evasion game defined by the equations

$$x(i+1) = x(i) + G_p(i)u(i) - G_e(i)v(i) + \xi(i); \quad i = 0, \ldots, N \tag{A1}$$

$$J = \frac{1}{2}\,\varepsilon\left\{x^T(N)S_f x(N) + \sum_{i=0}^{N-1}\left[u^T(i)B(i)u(i) - v^T(i)C(i)v(i)\right]\right\} \tag{A2}$$

$$z_p(i) = H_p(i)x(i) + w_p(i) \tag{A3}$$

$$z_e(i) = H_e(i)x(i) + w_e(i) \tag{A4}$$

common prior is Normal$(\bar{x}_o, P_o)$

where S_f, $B(i)$, and $C(i)$ are symmetric and positive definite; and the random vectors $\begin{bmatrix} \xi(i) \\ \hline w_p(i) \\ \hline w_e(i) \end{bmatrix}$ are statistically independent,

$$\text{Normal}\left(\begin{bmatrix} 0 \\ \hline 0 \\ \hline 0 \end{bmatrix}, \begin{bmatrix} Q(i) & 0 & 0 \\ 0 & R_p(i) & 0 \\ 0 & 0 & R_e(i) \end{bmatrix}\right),$$ and independent of the prior.

The objective is to find a pair of pursuit and evasion strategies (U^o, V^o) based on the available information, such that for any other pair of such strategies (U, V),

$$J(U^o, V) \leq J(U^o, V^o) \leq J(U, V^o) .$$

Strictly speaking, the strategies under consideration must be limited to a suitable pair of sets of admissible strategies for which the problem is meaningful. For our purposes here, it is sufficient to restrict the pursuit strategies to those for which u(i) is a (Lebesque) measurable function of $\bar{x}_o$, P_o, $z_p(0)$, ..., $z_p(i)$, and similarly for the evasion strategies.

The derivations in this appendix require extensive use of partitioned matrices. For this reason, it is convenient to introduce the notation

$$\langle A \rangle_{i,j}$$

for the i, j-th partition of a partitioned matrix A. Only one subscript will be used for a partitioned vector, or a vertically or horizontally partitioned matrix. The notation I_r and $0_{k,q}$ will be used to denote the r-dimensional identity matrix and the $k \times q$ - dimensional zero matrix whenever the dimensions of these matrices are not clear from the context.

1. The Solution

Suppose now that the evader uses a strategy consisting of a deterministic term plus a feedback term of the form

$$v(i) = a_e(i) + \sum_{j=0}^{i} \Lambda_e(i, j) z_e(j) . \qquad \text{(A5)}$$

Defining the following sequences of partitioned vectors (all with N+1 partitions of equal dimension):

$\{\pi(i) : i = 0, \ldots, N\}$, called "the past of $x(i)$," such that for each i

$$\langle\pi(i)\rangle_j = \begin{Bmatrix} x(j) \; ; \; j \leq i \\ x(i) \; ; \; j \geq i \end{Bmatrix} ; \quad j = 0, \ldots, N , \tag{A6}$$

and

$\{\eta_e(i) : i = 0, \ldots, N\}$, called "the past of $w_e(i)$," such that for each i

$$\langle\eta_e(i)\rangle_j = \begin{Bmatrix} w_e(j) \; ; \; j \leq i \\ 0 \quad ; \; j > i \end{Bmatrix} ; \quad j = 0, \ldots, N , \tag{A7}$$

it is straightforward but tedious to verify by substitution that, if the evader uses the generic strategy described by equation (A5), the stochastic optimal control problem facing the pursuer can be expressed in terms of these "enlarged" vectors by the transition equation:

$$\begin{bmatrix} \pi(i+1) \\ \hline \eta_e(i+1) \end{bmatrix} = \begin{bmatrix} I - \Phi_e(i) & -\Psi_e(i) \\ \hline 0 & I \end{bmatrix} \begin{bmatrix} \pi(i) \\ \hline \eta_e(i) \end{bmatrix} + \begin{bmatrix} \Gamma_p(i) \\ \hline 0 \end{bmatrix} u(i) - \begin{bmatrix} \Gamma_e(i) \\ \hline 0 \end{bmatrix} a_e(i) + \begin{bmatrix} Y(i) & 0 \\ \hline 0 & \Omega_e(i) \end{bmatrix} \begin{bmatrix} \xi(i) \\ \hline w_e(i) \end{bmatrix} \tag{A8}$$

the "follower" type of criterion (to be minimized):

$$J = \frac{1}{2} \varepsilon \left\{ \left[\pi^T(N) \;\vdots\; \eta_e^T(N) \right] \begin{bmatrix} A_N & 0 \\ \hline 0 & 0 \end{bmatrix} \begin{bmatrix} \pi(N) \\ \hline \eta_e(N) \end{bmatrix} + \sum_{i=0}^{N-1} \left\{ u^T(i)B(i)u(i) - \left(a_e^T(i) + \left[\pi^T(i) \;\vdots\; \eta_e^T(i) \right] \begin{bmatrix} A_e^T(i) \\ \hline D_e^T(i) \end{bmatrix} \right) C(i) \left(\left[A_e(i) \;\vdots\; D(i) \right] \begin{bmatrix} \pi(i) \\ \hline \eta_e(i) \end{bmatrix} + a_e(i) \right) \right\} \right\} \tag{A9}$$

and the measurements (on the basis of which the controls are to be computed):

$$z_p(i) = \left[\Theta_p(i) \;\vdots\; 0 \right] \begin{bmatrix} \pi(i) \\ --- \\ \eta_e(i) \end{bmatrix} + w_p(i) \tag{A10}$$

where, for $i = 0, \ldots, N-1$; and for $j, k = 0, \ldots, N$;

$\Phi_e(i)$ is a partitioned matrix such that

$$\langle \Phi_e(i) \rangle_{j,k} = \begin{Bmatrix} C_e(i)\Lambda_e(i,k)H_e(k) \;;\; j > i \\ 0 \;;\; j \leq i \;\; (\text{or } k > i) \end{Bmatrix} ; \tag{A11}$$

$\Psi_e(i)$ is a partitioned matrix such that

$$\langle \Psi_e(i) \rangle_{j,k} = \begin{Bmatrix} G_e(i)\Lambda_e(i,k) \;;\; j > i \\ 0 \;;\; j \leq i \;\; (\text{or } k > i) \end{Bmatrix} ; \tag{A12}$$

$\Gamma_p(i)$ is a vertically partitioned matrix such that

$$\langle \Gamma_p(i) \rangle_j = \begin{Bmatrix} G_p(i) \;;\; j > i \\ 0 \;;\; j \leq i \end{Bmatrix} ; \tag{A13}$$

$\Theta_p(i)$ is a horizontally partitioned matrix such that

$$\langle \Theta_p(i) \rangle_j = \begin{Bmatrix} H_p(i) \;;\; j = i \\ 0 \;;\; j \neq i \end{Bmatrix} ; \tag{A14}$$

$\Omega_e(i)$ is a vertically partitioned matrix such that

$$\langle \Omega_e(i) \rangle_j = \begin{Bmatrix} I_r \;;\; j = i+1 \\ 0_{r,r} \;;\; j \neq i+1 \end{Bmatrix} ; \tag{A15}$$

$Y(i)$ is a vertically partitioned matrix such that

$$\langle Y(i) \rangle_j = \begin{Bmatrix} I_n \;;\; j \geq i+1 \\ 0_{n,n} \;;\; j \leq i \end{Bmatrix} ; \tag{A16}$$

$A_e(i)$ is a horizontally partitioned matrix such that

$$\langle A_e(i) \rangle_j = \begin{Bmatrix} \Lambda_e(i,j)H_e(j) \;;\; j \leq i \\ 0 \;;\; j > i \end{Bmatrix} ; \tag{A17}$$

$D_e(i)$ is a horizontally partitioned matrix such that

$$\langle D_e(i)\rangle_j = \begin{Bmatrix} \Lambda_e(i,j) \ ; \ j \leq i \\ 0 \quad ; \ j > i \end{Bmatrix} ; \tag{A18}$$

and where A_N is a partitioned matrix such that

$$\langle A_N\rangle_{j,k} = \begin{Bmatrix} S_f \ ; \ j = k = N \\ 0_{nXn} \ \text{otherwise} \end{Bmatrix} , \tag{A19}$$

$\Gamma_e(i)$ is defined in the same way as $\Gamma_p(i)$, except that $G_e(i)$ is used in place of $G_p(i)$. As an aid to comprehension, the forms of these "enlarged" partitioned matrices are displayed in Figure A1 for the case in which $N = 3$.

This formulation of the pursuer's associated optimal control problem (in which J is to be minimized) is useful because it is of the standard linear-quadratic-Gaussian type, the solution to which has been obtained by Joseph and Tou [10], and Gunckel and Franklin [11]. Defining for convenience

$$\bar{\Phi}_e(i) \triangleq \left[\begin{array}{c|c} I - \Phi_e(i) & -\Psi_e(i) \\ \hline 0 & I_{r(N+1)} \end{array}\right] , \tag{A20}$$

$$\bar{\Gamma}_p(i) \triangleq \left[\begin{array}{c} \Gamma_p(i) \\ \hline 0_{r(N+1),m} \end{array}\right] , \tag{A21}$$

$$E_e(i) \triangleq \left[\begin{array}{c} \Gamma_e(i) \\ \hline 0_{r(N+1),k} \end{array}\right] , \qquad \langle \Gamma_e(i)\rangle_j \triangleq \begin{Bmatrix} G_e(i) \ ; \ j > i \\ 0 \quad ; \ j \leq i \end{Bmatrix} \tag{A22}$$

$$\bar{R}_e(i) \triangleq \left[\begin{array}{c|c} Q(i) & 0 \\ \hline 0 & R_e(i+1) \end{array}\right] , \tag{A23}$$

$\underline{\Phi_e(i)}$

$$i=0: \quad \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline G_e(0)\Lambda_e(0,0)H_e(0) & 0 & 0 & 0 \\ \hline G_e(0)\Lambda_e(0,0)H_e(0) & 0 & 0 & 0 \\ \hline G_e(0)\Lambda_e(0,0)H_e(0) & 0 & 0 & 0 \end{array}\right]$$

$$i=1: \quad \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline G_e(1)\Lambda_e(1,0)H_e(0) & G_e(1)\Lambda_e(1,1)H_e(1) & 0 & 0 \\ \hline G_e(1)\Lambda_e(1,0)H_e(0) & G_e(1)\Lambda_e(1,1)H_e(1) & 0 & 0 \end{array}\right]$$

$$i=2 \quad \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline G_e(2)\Lambda_e(2,0)H_e(0) & G_e(2)\Lambda_e(2,1)H_e(1) & G_e(2)\Lambda_e(2,2)H_e(2) & 0 \end{array}\right]$$

FIG. A1 FORMS OF PARTITIONED MATRICES FOR N = 3

$\Psi_e(i)$

$$
i=0:\left[\begin{array}{c|c|c|c}
0 & 0 & 0 & 0 \\ \hline
G_e(0)\Lambda_e(0,0) & 0 & 0 & 0 \\ \hline
G_e(0)\Lambda_e(0,0) & 0 & 0 & 0 \\ \hline
G_e(0)\Lambda_e(0,0) & 0 & 0 & 0
\end{array}\right]
$$

$$
i=0:\left[\begin{array}{c|c|c|c}
0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 \\ \hline
G_e(1)\Lambda_e(1,0) & G_e(1)\Lambda_e(1,1) & 0 & 0 \\ \hline
G_e(1)\Lambda_e(1,0) & G_e(1)\Lambda_e(1,1) & 0 & 0
\end{array}\right]
$$

$$
i=2\left[\begin{array}{c|c|c|c}
0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 \\ \hline
G_e(2)\Lambda_e(2,0) & G_e(2)\Lambda_e(2,1) & G_e(2)\Lambda_e(2,2) & 0
\end{array}\right]
$$

FIGURE A1 (Continued)

	$\Gamma_p(i)$	$\Theta_p(i)$	$\Omega_e(i)$, $Y(i)$
$i=0$:	$\begin{bmatrix} 0 \\ \hline G_p(0) \\ \hline G_p(0) \\ \hline G_p(0) \end{bmatrix}$	$\left[\begin{array}{c\|c\|c\|c} H(0) & 0 & 0 & 0 \end{array}\right]$	$\begin{bmatrix} 0 \\ \hline I \\ \hline 0 \\ \hline 0 \end{bmatrix}$
$i=1$:	$\begin{bmatrix} 0 \\ \hline 0 \\ \hline G_p(1) \\ \hline G_p(1) \end{bmatrix}$	$\left[\begin{array}{c\|c\|c\|c} 0 & H_p(1) & 0 & 0 \end{array}\right]$	$\begin{bmatrix} 0 \\ \hline 0 \\ \hline I \\ \hline 0 \end{bmatrix}$
$i=2$:	$\begin{bmatrix} 0 \\ \hline 0 \\ \hline 0 \\ \hline G_p(2) \end{bmatrix}$	$\left[\begin{array}{c\|c\|c\|c} 0 & 0 & H_p(2) & 0 \end{array}\right]$	$\begin{bmatrix} 0 \\ \hline 0 \\ \hline 0 \\ \hline I \end{bmatrix}$

FIGURE A1 – (Continued)

$\underline{A_e(i)}$

$$i=0:\left[\begin{array}{c|c|c|c} \Lambda_e(0,0)H_e(0) & 0 & 0 & 0 \end{array}\right]$$

$$i=1:\left[\begin{array}{c|c|c|c} \Lambda_e(1,0)H_e(0) & \Lambda_e(1,1)H_e(1) & 0 & 0 \end{array}\right]$$

$$i=2:\left[\begin{array}{c|c|c|c} \Lambda_e(2,0)H_e(0) & \Lambda_e(2,1)H_e(1) & \Lambda_2(2,2)H_2(2) & 0 \end{array}\right]$$

$\underline{D_e(i)}$

$$i=0:\left[\begin{array}{c|c|c|c} \Lambda_e(0,0) & 0 & 0 & 0 \end{array}\right]$$

$$i=1:\left[\begin{array}{c|c|c|c} \Lambda_e(1,0) & \Lambda_e(1,1) & 0 & 0 \end{array}\right]$$

$$i=2:\left[\begin{array}{c|c|c|c} \Lambda_e(2,0) & \Lambda_e(2,1) & \Lambda_e(2,2) & 0 \end{array}\right]$$

FIG. A1 (Continued)

$$\overline{\Omega}_e(i) \triangleq \begin{bmatrix} Y(i) & 0 \\ 0 & \Omega_e(i) \end{bmatrix}, \tag{A24}$$

$$\overline{\Theta}_p(i) \triangleq \begin{bmatrix} \Theta_p(i) & 0_{q,r(N+1)} \end{bmatrix}, \tag{A25}$$

$$\overline{A}_e(i) \triangleq \begin{bmatrix} A_e(i) & D_e(i) \end{bmatrix}, \quad \text{and} \tag{A26}$$

$$\overline{A}_N \triangleq \begin{bmatrix} A_N & 0 \\ 0 & 0_{r(N+1),\, r(N+1)} \end{bmatrix}, \tag{A27}$$

the solution, if it exists, is given by

$$u(i) = -\left[B(i) + \overline{\Gamma}_p^T(i) S_p(i+1) \overline{\Gamma}_p(i)\right]^{-1} \overline{\Gamma}_p^T(i)\Big[\gamma_p(i+1) + S_p(i+1)\left(\overline{\Phi}_e(i)\hat{\sigma}_{ep}(i) - E_e(i)a_e(i)\right)\Big], \tag{A28}$$

where $\hat{\sigma}_{ep}(i)$ is the Kalman filter estimate for $\sigma_e(i) \triangleq \begin{bmatrix} \pi(i) \\ \eta_e(i) \end{bmatrix}$, the state of the pursuer's enlarged system, which is given by

$$\hat{\sigma}_{ep}(i) = \overline{\sigma}_{ep}(i) + P_p(i)\overline{\Theta}_p^T(i) R_p^{-1}(i)[z_p(i) - \overline{\Theta}_p(i)\overline{\sigma}_{ep}(i)] \tag{A29}$$

$$\overline{\sigma}_{ep}(i+1) = \overline{\Phi}_e(i)\hat{\sigma}_{ep}(i) + \overline{\Gamma}_p(i)u(i) - E_e(i)a_e(i)\,; \quad \langle \overline{\sigma}_{ep}(0)\rangle_j = \begin{cases} \overline{x}_o \;; & j \leq N \\ 0 \;; & j > N \end{cases} \tag{A30}$$

where

$$S_p(i) = \overline{\Phi}_e^T(i)\Big[I - S_p(i+1)\overline{\Gamma}_p(i)[B(i) + \overline{\Gamma}_p^T(i) S_p(i+1)\overline{\Gamma}_p(i)]^{-1}\, \overline{\Gamma}_p^T(i)\Big] S_p(i+1)\overline{\Phi}_e(i) - \overline{A}_e^T(i) C(i) \overline{A}_e(i)\,; \qquad \overline{S}_p(N) = \overline{A}_N \tag{A31}$$

$$\gamma_p(i) = \bar{\Phi}_e^T(i)\Big[I - S_p(i+1)\bar{\Gamma}_p(i)[B(i)+\bar{\Gamma}_p^T(i)S_p(i+1)\bar{\Gamma}_p(i)]^{-1}\bar{\Gamma}_p^T(i)\Big]\cdot$$

$$[\gamma_p(i+1) - S_p(i+1)E_e(i)a_e(i)] - \bar{A}_e^T(i)C(i)a_e(i)\ ; \qquad \gamma_p(N) = 0 \tag{A32}$$

$$P_p(i) = M_p(i)\Big[I - \bar{\Theta}_p^T(i)[\bar{\Theta}_p(i)M_p(i)\bar{\Theta}_p^T(i)+R_p(i)]^{-1}\bar{\Theta}_p(i)\Big]M_p(i) \tag{A33}$$

$$M_p(i+1) = \bar{\Phi}_e(i)P_p(i)\bar{\Phi}_e^T(i) + \bar{\Omega}_e(i)\bar{R}_e(i)\bar{\Omega}_e^T(i) \tag{A34}$$

The boundary value $M_p(0)$ is expressed by dividing this matrix into $(2N+2)\cdot(2N+2)$ partitions as follows:

$$\langle M_p(0)\rangle_{j,k} = \begin{Bmatrix} P_o & ;\ j,k = 0,\dots,N \\ R_e(0) & ;\ j = k = N+1 \\ 0_{r,r} & \text{otherwise} \end{Bmatrix} \quad j,k = 0,\ \dots,\ 2N+1 \tag{A35}$$

It has been established by Kalman and Bucy [12] that the estimate $\hat{\sigma}_{ep}(i)$ can be expressed directly in terms of the pursuer's measurements as

$$\hat{\sigma}_{ep}(i) = \delta_p(i) + \sum_{j=0}^{i} K_p(i,j)z_p(j) \tag{A36}$$

where

$$K_p(i+1,j) = \Big[I - P_p(i+1)\bar{\Theta}_p^T(i+1)R_p^{-1}(i+1)\bar{\Theta}_p(i+1)\Big]\cdot$$

$$\Big[I - \bar{\Gamma}_p(i)[B(i)+\bar{\Gamma}_p^T(i)S_p(i+1)\bar{\Gamma}_p(i)]^{-1}\bar{\Gamma}_p^T(i)S_p(i+1)\Big]\bar{\Phi}_e(i)K_p(i,j)\ ;$$

$$K_p(i,i) = P_p(i)\bar{\Theta}_p^T(i)R_p^{-1}(i)\ . \tag{A37}$$

$$\delta_p(i+1) = [I-P_p(i+1)\bar{\Theta}_p^T(i+1)R_p^{-1}(i+1)\bar{\Theta}_p(i+1)] \cdot$$

$$\Big\{\Big[I-\bar{\Gamma}_p(i)[B(i)+\bar{\Gamma}_p^T(i)S_p(i+1)\bar{\Gamma}_p(i)]^{-1}\bar{\Gamma}_p^T(i)S_p(i+1)\Big] \cdot$$

$$[\Phi_e(i)\delta_p(i)-E_e(i)a_e(i)] -$$

$$\bar{\Gamma}_p(i)[B(i)+\bar{\Gamma}_p^T(i)S_p(i+1)\bar{\Gamma}_p(i)]^{-1}\,\bar{\Gamma}_p^T(i)\gamma_p(i+1)\Big\} \qquad \text{(A38)}$$

$$\delta_p(0) = [I-P_p(0)\bar{\Theta}_p^T(0)R_p^{-1}(0)\bar{\Theta}_p(0)]\,\bar{\sigma}_{ep}(0) .$$

In the multistage case, this result is easy to verify by induction.

Substituting equation (A36) into equation (A28), the pursuer's optimal control law in this context can be written as

$$u(i) = -a_p(i) - \sum_{j=0}^{i} \Lambda_p(i,j)Z_p(j) , \quad \text{where} \qquad \text{(A39)}$$

$$a_p(i) = [B(i)+\bar{\Gamma}_p^T(i)S_p(i+1)\bar{\Gamma}_p(i)]^{-1}\,\bar{\Gamma}_p^T(i) \cdot$$

$$\Big[\gamma_p(i+1)+S_p(i+1)[\bar{\Phi}_e(i)\delta_p(i)-E_e(i)a_e(i)]\Big] \qquad \text{(A40)}$$

$$\Lambda_p(i,j) = \begin{cases} [B(i)+\bar{\Gamma}_p^T(i)S_p(i+1)\bar{\Gamma}_p(i)]^{-1}\bar{\Gamma}_p^T(i)S_p(i+1)\bar{\Phi}_e(i)K_p(i,j) ; & j \le i \\ 0_{m,q} ; & N \ge j > i \end{cases}$$

(A41)

Strictly speaking, the equations determining this control law constitute a set of necessary conditions for optimality. If, however, a control law is found which satisfies these necessary conditions and, in addition, satisfies the convexity condition

$$[\bar{\Gamma}_p^T(i)S_p(i+1)\bar{\Gamma}_p(i)+B(i)] \quad \text{positive definite for } i = 0, \ldots, N-1 ;$$

(A42)

then, by a standard result for linear-quadratic-Gaussian control problems, this is sufficient to guarantee that this control law is optimal.

Conversely, if the pursuer employs a strategy of the form

$$u(i) = -a_p(i) - \sum_{j=0}^{i} \Lambda_p(i,j)\, z_p(j) \,, \tag{A43}$$

an analogous construction and argument can be used to show that the optimal opposing evasion strategy is given by

$$v(i) = a_e(i) + \sum_{j=0}^{i} \Lambda_e(i,j)\, z_e(j) \,, \tag{A44}$$

where

$$-\Lambda_e(i,j) = \begin{cases} [\overline{\Gamma}_e^T(i)S_e(i+1)\overline{\Gamma}_e(i)+C(i)]^{-1}\,\overline{\Gamma}_e^T(i)S_e(i+1)\overline{\Phi}_p(i)K_e(i,j)\,; & j \leq i \\ 0_{k,r}\,; & N \geq j > i \end{cases} \,, \tag{A45}$$

$$-a_e(i) = [\overline{\Gamma}_e^T(i)S_e(i+1)\overline{\Gamma}_e(i)+C(i)]^{-1}\,\overline{\Gamma}_e^T(i)\,\cdot$$
$$\left[\gamma_e(i+1)+S_e(i+1)[\overline{\Phi}_p(i)\delta_e(i)-E_p(i)a_p(i)]\right], \tag{A46}$$

and where

$$S_e(i) = \overline{\Phi}_p^T(i)\left[I - S_e(i+1)\overline{\Gamma}_e(i)[C(i)+\overline{\Gamma}_e^T(i)S_e(i+1)\overline{\Gamma}_e(i)]^{-1}\,\overline{\Gamma}_e^T(i)\right]\cdot$$
$$S_e(i+1)\overline{\Phi}_p(i) - \overline{A}_p^T(i)B(i)\overline{A}_p(i) \tag{A47}$$
$$S_e(N) = -\overline{A}_N \,,$$

$$P_e(i) = M_e(i)\left[I - \overline{\Theta}_e^T(i)[\overline{\Theta}_e(i)M_e(i)\overline{\Theta}_e^T(i)+R_e(i)]^{-1}\,\overline{\Theta}_e(i)\right]M_e(i)\,,$$
$$M_e(i+1) = \overline{\Phi}_p(i)P_p(i)\overline{\Phi}_p^T(i) + \overline{\Omega}_p(i)\overline{R}_p(i)\overline{\Omega}_p^T(i) \tag{A48}$$

$$\langle M_e(0)\rangle_{j,k} = \begin{Bmatrix} P_o ; & j,k = 0, \ldots, N \\ R_p(0) ; & j = k = N+1 \\ 0_{q,q} & \text{otherwise} \end{Bmatrix} \quad j,k = 0, \ldots, 2N+1 , \tag{A49}$$

$$K_e(i+1, j) = \left[I - P_e(i+1)\overline{\Theta}_e^T(i+1)R_e^{-1}(i+1)\overline{\Theta}_e(i+1)\right] \cdot$$
$$\left[I - \overline{\Gamma}_e(i)[C(i)+\overline{\Gamma}_e^T(i)S_e(i+1)\overline{\Gamma}_e(i)]^{-1}\overline{\Gamma}_e^T(i)S_e(i+1)\right] \cdot$$
$$\overline{\Phi}_p(i)K_e(i,j) ; \quad K_e(i,i) = P_e(i)\overline{\Theta}_e^T(i)R_e^{-1}(i) , \tag{A50}$$

and where

$$\gamma_e(i) = \overline{\Phi}_p^T(i)\left[I - S_e(i+1)\overline{\Gamma}_e(i)[C(i)+\overline{\Gamma}_e^T(i)S_e(i+1)\overline{\Gamma}_e(i)]^{-1}\ \overline{\Gamma}_e^T(i)\right] \cdot$$
$$\left[\gamma_e(i+1) - S_e(i+1)E_p(i)a_p(i)\right] - \overline{A}_p^T(i)B(i)a_p(i) ; \ \gamma_e(N) = 0 , \tag{A51}$$

$$\delta_e(i+1) = \left[I - P_e(i+1)\overline{\Theta}_e^T(i+1)R_e^{-1}(i+1)\overline{\Theta}_e(i+1)\right] \cdot$$
$$\left[I - \overline{\Gamma}_e(i)[C(i)+\overline{\Gamma}_e^T(i)S_e(i+1)\overline{\Gamma}_e(i)]^{-1}\ \overline{\Gamma}_e^T(i)S_e(i+1)\right] \cdot$$
$$\left[\overline{\Phi}_p(i)\delta_e(i) - E_p(i)a_p(i)\right] - \overline{\Gamma}_e(i)[C(i)+\overline{\Gamma}_e^T(i)S_e(i+1)\overline{\Gamma}_e(i)]^{-1} \cdot$$
$$\overline{\Gamma}_e^T(i)\gamma_e(i+1)$$

$$\delta_e(0) = [I - P_e(0)\overline{\Theta}_e^T(0)R_e^{-1}(0)\overline{\Theta}_e(0)]\overline{\sigma}_{pe}(0) . \tag{A52}$$

Here, the vector $\sigma_p(i)$ is defined as $\begin{bmatrix} \pi(i) \\ --- \\ \eta_p(i) \end{bmatrix}$, where $\eta_p(i)$ is given by

$$\langle \eta_p(i)\rangle_j = \begin{Bmatrix} w_p(i) ; & j \leq i \\ 0 ; & j > i \end{Bmatrix} \quad i = 0, \ldots, N-1 ; \ j = 0, \ldots, N . \tag{A53}$$

The quantities $\hat{\sigma}_{pe}$ (the evader's Kalman filter estimate of σ_p), $\bar{\sigma}_{pe}$, Φ_p, Γ_e, Ψ_p, Θ_e, Ω_p, A_p, D_p, $\bar{\Phi}_p$, $\bar{\Gamma}_e$, E_p, $\bar{R}_p$, $\bar{\Omega}_p$, $\bar{\Theta}_e$, and $\bar{A}_p$ are defined in exactly the same way as their oppositely subscripted counterparts in the pursuer's associated optimal control problem, except that all the "p" and "e" subscripts in the definitions are interchanged, and the dimensions of the zero and identity matrices used in the definitions are altered by interchanging the parameters "m" and "k", and "q" and "r".

Likewise, if any evasion strategy of the form of equation (A44) is found which satisfies equations (A45)-(A52) and the convexity condition

$$C(i) + \bar{\Gamma}_e^T(i) S_e(i+1) \bar{\Gamma}_e(i) \quad \text{positive definite for } i = 0, \ldots, N-1, \tag{A54}$$

then this strategy is optimal (maximizes J) against the generic pursuit strategy assumed in equation (A43).

Therefore, by construction, any pair (U, V) of pursuit and evasion strategies of the form

$$U: \; u(i) = -a_p(i) - \sum_{j=0}^{i} \Lambda_p(i,j) z_p(j) \tag{A55}$$

$$V: \; v(i) = a_e(i) + \sum_{j=0}^{i} \Lambda_e(i,j) z_e(j) \tag{A56}$$

such that the primary system of implicit equations (A31, 33, 34, 35, 37, 41, 45, 47-50), the subsidiary system of implicit equations (A32, 38, 40, 46, 51, 52), and the convexity conditions (A42, 54) are satisfied is a pair of minimax strategies for this stochastic multistage game.

These systems of difference equations, when viewed as sets of implicit equations for determining a minimax strategy pair, each constitute a kind of two point boundary value problem. These are more complicated than the usual kind of two point boundary value problem, however, in that the entire solution to the "M" (or "P") equation schemes is a boundary condition for the "K" equation schemes, although the only externally imposed boundary conditions on the equation system as a whole are at the initial and terminal times. Also, there is the difficulty that the dimensions of the parameter matrices involved in this solution increase in proportion to the number N of stages in the game. This means that the number of variables involved in this set of implicit equations increases approximately as N^3.

Notice that the primary system of implicit difference equations can be solved independently, and that "γ", "a", and "δ" variables can be taken as identically zero to satisfy the subsidiary system of equations in the case where $\bar{x}_o = 0$. Also, the convexity conditions only involve the solution of the primary system of equations. Since B(i) and C(i) are positive definite by assumption, these convexity conditions correspond roughly to the "no conjugate point condition" of optimal control theory. Exploratory numerical calculations have been carried out for a variety of two-stage games, which indicate that linear solutions of the type assumed do indeed exist for intuitively reasonable parameter values, and do not exist for values which would give the evader a good chance to escape. The non-existence of such minimax solutions, however, is not known to imply that some other type of admissible solution does not exist.

The systems of enlarged state variables previously constructed can be looked on merely as abstract transformations that result in solutions of the desired form. It is of interest, however, to use the interpretations developed there to derive some of the properties of the parameters arising in the solution.

Considering for the moment only the system constructed for the pursuer's optimal control problem, and writing $\begin{bmatrix} \hat{\pi}_p(i) \\ --- \\ \hat{\eta}_{ep}(i) \end{bmatrix}$ for $\hat{\sigma}_{ep}(i)$, it is a standard result of the filtering theory developed by Kalman [13] that the vectors $\hat{\pi}(i)$ and $\hat{\eta}_e(i)$ are the conditional expected values of $\pi(i)$ and $\eta_e(i)$ given the measurements $Z_p(i)$. Partitioning these estimate vectors in the natural way, it follows from the definitions of $\pi(i)$ and $\eta_e(i)$ that the last N-i partitions of $\hat{\eta}_e(i)$ are zero, and that the last N-i+1 partitions of $\hat{\pi}(i)$ have the same value, for all plays of the game, because the expectations can be decomposed to give the following results for $j > i$:

$$\langle \hat{\eta}_e(i) \rangle_j = \varepsilon[\langle \eta_e(i) \rangle_j / Z_p(i)] = \varepsilon[0/Z_p(i)] = 0 \tag{A57}$$

and

$$\begin{aligned} \langle \hat{\pi}(i) \rangle_j &= \varepsilon[\langle \pi(i) \rangle_j / Z_p(i)] = \varepsilon\{\varepsilon[\langle \pi(i) \rangle_j / Z_p(i), x(i)] / Z_p(i)\} \\ &= \varepsilon[x(i)/Z_p(i)] = \langle \hat{\pi}(i) \rangle_i \end{aligned} \tag{A58}$$

Notice that estimates are continually being calculated not only for the current values of x and w_e, but for all previous values as well, since these are also included in the π and η_e vectors.

Under this interpretation, it is also clear that the matrix $P_p(i)$ is the covariance matrix of the conditional probability distribution of the random vector $\begin{bmatrix} \pi(i) \\ --- \\ \eta_e(i) \end{bmatrix}$ given the measurements $Z_p(i)$. Since this distribution is Gaussian, $P_p(i)$, $\hat{\pi}(i)$, and $\hat{\eta}_e(i)$ completely characterize it. Again it should be emphasized that the matrix $P_p(i)$ not only contains the covariances of x and w_e at previous times, but also the correlations between their values at <u>different</u> previous times. Partitioning $P_p(i)$ in the obvious way to correspond to the partitions of the $\pi(i)$ and $\eta_e(i)$ vectors, it is a consequence of the definition of these vectors and the fact that $P_p(i)$ is the conditional covariance matrix associated with them that

$$\langle P_p(i)\rangle_{j,k} = 0 \quad \text{for } j \geq N+i+2 \quad \underline{\text{or}} \ k \geq N+i+2 . \tag{A59}$$

Analogous remarks hold for the variables defined for the analysis of the optimal control problem facing the evader.

2. The Certainty-Coincidence Property

The fact that the associated pair of optimal control problems for this game are of the linear-quadratic-Gaussian type when expressed in terms of the enlarged state variables can be used to gain some interesting insights into the nature of the linear minimax strategies. Suppose for the moment that a pair of such minimax strategies (U^*, V^*) has been found that satisfy the implicit equations. Defining

$$\epsilon_e(i) = \sigma_e(i) - \hat{\sigma}_{ep}(i) , \tag{A60}$$

it can be verified from the pursuer's associated optimal control problem solution (equations A28-42) that

$$\epsilon_e(i+1) = \left[I - P_p(i+1)\bar{\Theta}_p^T(i+1)R_p^{-1}(i+1)\bar{\Theta}_p(i+1)\right]\left[\bar{\Phi}_e(i)\epsilon_e(i) - \bar{\Gamma}_p(i)\begin{bmatrix}\xi(i)\\ ---\\ w_e(i)\end{bmatrix}\right]$$

$$+ P_p(i+1)\bar{\Theta}_p^T(i+1)R_p^{-1}(i+1)w_p(i+1) \qquad \text{(A61)}$$

$$\epsilon_e(0) = (x(0)-\bar{x}_o) .$$

Therefore, if this pair of minimax strategies is used and the initial error, the process noise values, and the measurement noise values for both players are all zero, then $\epsilon_e(i)$ is zero for $i = 0, \ldots, N-1$. An analogous argument shows that the error $\epsilon_p(i)$ for the evader's corresponding Kalman filter for $\sigma_p(i)$ (the equations for which have not been listed) is also identically zero under these circumstances, so that $\hat{\sigma}_{ep}(i) = \sigma_e(i)$ and $\hat{\sigma}_{pe}(i) = \sigma_p(i)$.

Furthermore, the certainty-equivalence principle for linear-quadratic-Gaussian optimal control problems, as applied to the formulation of the pursuer's associated optimal control problem in terms of the enlarged state variables, implies that the pursuer's minimax strategy U^* for the stochastic game, with the formal substitution of $\sigma_e(i)$ for $\hat{\sigma}_{ep}(i)$ is optimal in the corresponding deterministic game defined by

$$x(i+1) = x(i) + G_p(i)u(i) - G_e(i)v(i) \; ; \; x(0) = \bar{x}_o \qquad \text{(A62)}$$

$$J_d = \frac{1}{2}\left\{x^T(N)S_f x(N) + \sum_{i=0}^{N-1}\left(u^T(i)B(i)u(i) - v^T(i)C(i)v(i)\right)\right\} \qquad \text{(A63)}$$

against the evader's stochastic minimax strategy V^*, with the substitution of $H_e(i)\,x(i)$ for $z_e(i)$. The certainty-equivalence principle applies here because the evader's measurement noise appears as process noise in the enlarged system constructed for the pursuer's associated optimal control problem. Analogously, V^*, with the formal substitution of $\sigma_p(i)$ for $\hat{\sigma}_{pe}(i)$, optimizes against U^*, with $z_p(i)$ replaced by $H_p(i)x(i)$, in the same deterministic game. Labeling these deterministic strategies as

$$U': \ u(i) = -\left[\bar{\Gamma}_p^T(i)S_p(i+1)\bar{\Gamma}_p(i)+B(i)\right]^{-1} \cdot \bar{\Gamma}_p^T(i)\left[\gamma_p(i+1)+S_p(i+1)[\bar{\Phi}_e(i)\sigma_e(i)-E_e(i)a_e(i)]\right], \tag{A64}$$

$$U'': \ u(i) = -a_p(i) - \sum_{j=0}^{i} \Lambda_p(i,j)H_p(j)x(j), \tag{A65}$$

$$V': \ v(i) = \left[\bar{\Gamma}_e^T(i)S_e(i+1)\bar{\Gamma}_e(i)+C(i)\right]^{-1} \cdot \bar{\Gamma}_e^T\left[\gamma_e(i+1)+S_e(i+1)[\bar{\Phi}_e(i)\sigma_p(i)-E_p(i)a_p(i)]\right], \tag{A66}$$

$$V'': \ v(i) = a_e(i) + \sum_{j=0}^{i} \Lambda_e(i,j)H_e(j)x(j), \tag{A67}$$

this means that in the deterministic game

$$J_d(U', V'') \leq J_d(U, V'') \quad \text{and} \tag{A68}$$

$$J_d(U'', V') \geq J_d(U'', V) \tag{A69}$$

for any other pair of deterministic pursuit and evasion strategies (U, V). The strategies expressed by equations (A64) and (A65) result from applying the certainty-equivalence principle to equations (A28) and (A55), respectively; and similarly for the strategies V' and V'' defined by equations (A66) and (A67).

Moreover, on the noiseless sample path in the stochastic game $\left(x(0)=\bar{x}_o,\ w_p(i)=w_e(i)=\xi(i)=0\right)$, the control histories resulting from playing the stochastic minimax strategy pair (U^*, V^*) can be expressed either as

$$u(i) = -a_p(i) - \sum_{j=0}^{i} \Lambda_p(i,j)\bar{\Theta}_p(j)\sigma_e(j)$$
$$\left(= -a_p(i) - \sum_{j=0}^{i} \Lambda_p(i,j)H_p(j)x(j) \right) \tag{A70}$$

and

$$v(i) = a_e(i) + \sum_{j=0}^{i} \Lambda_e(i,j)\bar{\Theta}_e(j)\sigma_p(j)$$
$$\left(= a_e(i) + \sum_{j=0}^{i} \Lambda_e(i,j)H_e(j)x(j) \right), \tag{A71}$$

or as

$$u(i) = -\left[\bar{\Gamma}_p^T(i)S_p(i+1)\bar{\Gamma}_p(i)+B(i)\right]^{-1} \cdot$$
$$\bar{\Gamma}_p^T(i)\left[\gamma_p(i+1)+S_p(i+1)[\bar{\Phi}_e(i)\sigma_e(i)-E_e(i)a_e(i)]\right] \tag{A72}$$

and

$$v(i) = \left[\bar{\Gamma}_e^T(i)S_e(i+1)\bar{\Gamma}_e(i)+C(i)\right]^{-1} \cdot$$
$$\bar{\Gamma}_e(i)\left[\gamma_e(i+1)+S_e(i+1)[\bar{\Phi}_p(i)\sigma_p(i)-E_p(i)a_p(i)]\right], \tag{A73}$$

since $\hat{\sigma}_{ep}(i) = \sigma_e(i)$ and $\hat{\sigma}_{pe}(i) = \sigma_p(i)$, as was shown earlier in this section, and $z_p(i) = \bar{\Theta}_p(i)\sigma_e(i)$ and $z_e(i) = \bar{\Theta}_e(i)\sigma_p(i)$. Furthermore, the state variable history in the stochastic game is given in terms of these control histories as

$$x(i+1) = x(i) + G_p(i)u(i) - G_e(i)v(i)\,; \qquad x(0) = \bar{x}_o \tag{A74}$$

under these circumstances. Therefore, playing any one of the strategy pairs (U', V'), (U', V''), (U'', V'), or (U'', V'') in the deterministic game results in the same trajectory, namely the noiseless sample path of the stochastic game under the strategy pair (U^*, V^*).

As a result, in the deterministic game,

$$J_d(U', V') = J_d(U', V'') = J_d(U'', V') = J_d(U'', V'') . \qquad \text{(A75)}$$

This result, together with the preceding inequalities, shows that U'' and V'' are minimax strategies for the corresponding deterministic game. Therefore, the noiseless sample path of the stochastic game under the stochastic minimax strategy pair (U^*, V^*) coincides with the deterministic minimax path followed by the corresponding deterministic game under the deterministic minimax strategy pair (U'', V''). Since this noiseless sample path is a deterministic minimax path, it must satisfy necessary conditions that can be derived for minimax paths in the corresponding deterministic game. Such necessary conditions are used later to determine some interesting properties about stochastic minimax strategies, but this procedure is not employed at this stage since the real interest here is in differential games, and since the necessary conditions are very messy for the multistage game.

Appendix B

THE DISCRETIZED GAME

Before deriving the equations for the discretized game discussed in section 3B, it is convenient to divide the enlarged vector and matrix variables involved in the solution to the multistage game into major partitions, in accordance with the partitioning of σ_e and σ_p as $\begin{bmatrix} \pi \\ \hline \eta_e \end{bmatrix}$ and $\begin{bmatrix} \pi \\ \hline \eta_p \end{bmatrix}$. Using the well-known fact that the "S," "P," and "M" matrices are symmetric, this leads to the partitioning of the "S," "P," "M," and "K" matrices as

$$S_p(i) = \left[\begin{array}{c|c} S_{p1}(i) & S_{p2}(i) \\ \hline S_{p2}^T(i) & S_{p3}(i) \end{array}\right], \qquad S_e(i) = \left[\begin{array}{c|c} S_{e1}(i) & S_{e2}(i) \\ \hline S_{e2}^T(i) & S_{e3}(i) \end{array}\right], \tag{B1}$$

$$P_p(i) = \left[\begin{array}{c|c} P_{p1}(i) & P_{p2}(i) \\ \hline P_{p2}^T(i) & P_{p3}(i) \end{array}\right], \qquad P_e(i) = \left[\begin{array}{c|c} P_{e1}(i) & P_{e2}(i) \\ \hline P_{e2}^T(i) & P_{e3}(i) \end{array}\right], \tag{B2}$$

$$M_p(i) = \left[\begin{array}{c|c} M_{p1}(i) & M_{p2}(i) \\ \hline M_{p2}^T(i) & M_{p3}(i) \end{array}\right], \qquad M_e(i) = \left[\begin{array}{c|c} M_{e1}(i) & M_{e2}(i) \\ \hline M_{e2}^T(i) & M_{e3}(i) \end{array}\right], \tag{B3}$$

$$K_p(i) = \left[\begin{array}{c} K_{p1}(i) \\ \hline K_{p2}(i) \end{array}\right], \qquad K_e(i) = \left[\begin{array}{c} K_{e1}(i) \\ \hline K_{e2}(i) \end{array}\right], \tag{B4}$$

and the partitioning of the "γ" and "δ" vectors as

$$\gamma_p(i) = \begin{bmatrix} \gamma_{p1}(i) \\ --- \\ \gamma_{p2}(i) \end{bmatrix}, \qquad \gamma_e(i) = \begin{bmatrix} \gamma_{e1}(i) \\ --- \\ \gamma_{e2}(i) \end{bmatrix}, \tag{B5}$$

$$\delta_p(i) = \begin{bmatrix} \delta_{p1}(i) \\ --- \\ \delta_{p2}(i) \end{bmatrix}, \qquad \delta_e(i) = \begin{bmatrix} \delta_{e1}(i) \\ --- \\ \delta_{e2}(i) \end{bmatrix}, \tag{B6}$$

where

$S_{p1}(i)$, $S_{e1}(i)$, $P_{p1}(i)$, $P_{e1}(i)$, $M_{p1}(i)$, and $M_{e1}(i)$ are $n(N+1) \times n(N+1)$ - dimensional matrices,

$S_{p2}(i)$, $P_{p2}(i)$, and $M_{p2}(i)$ are $n(N(1) \times r(N+1)$ - dimensional matrices,

$S_{e2}(i)$, $P_{e2}(i)$, and $M_{e2}(i)$ are $n(N+1) \times q(N+1)$ - dimensional matrices,

$S_{p3}(i)$, $P_{p3}(i)$, and $M_{p3}(i)$ are $r(N+1) \times r(N+1)$ - dimensional matrices,

$S_{e3}(i)$, $P_{e3}(i)$, and $M_{e3}(i)$ are $q(N+1) \times q(N+1)$ - dimensional matrices,

$K_{p1}(i)$ is an $n(N+1) \times q$-dimensional matrix,

$K_{p2}(i)$ is an $r(N+1) \times q$-dimensional matrix,

$K_{e1}(i)$ is an $n(N+1) \times r$-dimensional matrix,

$K_{e2}(i)$ is a $q(N+1) \times r$-dimensional matrix,

$\gamma_{p1}(i)$, $\delta_{p1}(i)$, $\gamma_{e1}(i)$, and $\delta_{e1}(i)$ are $n(N+1)$-dimensional vectors,

$\gamma_{p2}(i)$ and $\delta_{p2}(i)$ are $r(N+1)$-dimensional vectors, and

$\gamma_{e2}(i)$ and $\delta_{e2}(i)$ are $q(N+1)$-dimensional vectors.

The solution to the multistage game examined in Appendix A can be expressed in terms of the partitions defined by equations (B1) - (B6). Using the equations obtained in Appendix A, it is just a matter of computation to obtain the minimax strategies as

$$u(i) = -a_p(i) - \sum_{j=0}^{i} \Lambda_p(i,j) z_p(j) \qquad \text{(B7)}$$

$$v(i) = a_e(i) + \sum_{j=0}^{i} \Lambda_e(i,j) z_e(j) \qquad \text{(B8)}$$

where the parameters are determined by the primary and subsidiary systems of implicit equations. The primary system of equations, when expressed in terms of these matrix partitions, is:

$$S_{p1}(i) = [I-\Phi_e^T(i)][I-S_{p1}(i+1)\Gamma_p(i)[B(i)+\Gamma_p^T(i)S_{p1}(i+1)\Gamma_p(i)]^{-1}\Gamma_p^T(i)] \cdot$$
$$S_{p1}(i+1)[I-\Phi_e(i)] - A_e^T(i)C(i)A_e(i); \quad S_{p1}(N) = A_N \qquad \text{(B9)}$$

$$S_{p2}(i) = [I-\Phi_e^T(i)][I-S_{p1}(i+1)\Gamma_p(i)[B(i)+\Gamma_p^T(i)S_{p1}(i+1)\Gamma_p(i)]^{-1}\Gamma_p^T(i)] \cdot$$
$$[S_{p2}(i+1)-S_{p1}(i+1)\Psi_e(i)] - A_e^T(i)C(i)D_e(i); \quad S_{p2}(N) = 0 \qquad \text{(B10)}$$

$$P_{p1}(i) = M_{p1}(i) - M_{p1}(i)\Theta_p^T(i)[\Theta_p(i)M_{p1}(i)\Theta_p^T(i)+R_p(i)]^{-1}\Theta_p(i)M_{p1}(i)$$
(B11)

$$P_{p2}(i) = M_{p2}(i) - M_{p1}(i)\Theta_p^T(i)[\Theta_p(i)M_{p1}(i)\Theta_p^T(i)+R_p(i)]^{-1}\Theta_p(i)M_{p2}(i)$$
(B12)

$$P_{p3}(i) = M_{p3}(i) - M_{p2}^T(i)\Theta_p^T(i)]\Theta_p(i)M_{p1}(i)\Theta_p^T(i)+R_p(i)]^{-1}\Theta_p(i)M_{p2}(i)$$
(B13)

$$M_{p1}(i+1) = [I-\Phi_e(i)][P_{p1}[I-\Phi_e^T(i)] - P_{p2}(i)\Psi_e^T(i)] -$$

$$\Psi_e(i)[P_{p2}(i)[I-\Phi_e^T(i)] - P_{p3}(i)\Psi_o^T(i)] + Y(i)Q(i)Y^T(i)\,;$$

$$\langle M_{p1}(0)\rangle_{j,k} = P_o\,; \quad j,k = 0,\ldots,N \tag{B14}$$

$$M_{p2}(i+1) = [I-\Phi_e(i)]P_{p2}(i) - \Psi_e(i)P_{p3}(i)\,; \quad M_{p2}(0) = 0 \tag{B15}$$

$$M_{p3}(i+1) = P_{p3}(i) + \Omega_e(i)R_e(i+1)\Omega_e^T(i)\,; \quad \langle M_{pe}(0)\rangle_{j,k} = \begin{cases} R_e(0)\,; & j=k=0 \\ 0_{r,r} & \text{otherwise} \end{cases} \tag{B16}$$

$$K_{p1}(i+1,j) = [I-P_{p1}(i+1)\Theta_p^T(i+1)R_p^{-1}(i+1)\Theta_p(i+1)][[I-\Phi_e(i)]K_{p1}(i,j) -$$

$$\Psi_e(i)K_{p2}(i,j) - \Gamma_p(i)\Lambda_p(i,j)]\,; \quad K_{p1}(i,i) = P_{p1}(i)\Theta_p^T(i)R_p^{-1}(i) \tag{B17}$$

$$K_{p2}(i+1,j) = K_{p2}(i,j) - P_{p2}^T(i+1)\Theta_p^T(i+1)R_p^{-1}(i+1)\Theta_p(i+1)\,\cdot$$

$$[[I-\Phi_e(i)]K_{p1}(i,j) - \Psi_e(i)K_{p2}(i,j) - \Gamma_p(i)\Lambda_p(i,j)]\,;$$

$$K_{p2}(i,i) = P_{p2}^T(i)\Theta_p^T(i)R_p^{-1}(i) \tag{B18}$$

$$\Lambda_p(i,j) = \begin{cases} [B(i)+\Gamma_p^T(i)S_{p1}(i+1)\Gamma_p(i)]^{-1}\,\Gamma_p^T(i)\,[S_{p1}(i+1)[I-\Phi_e(i)]K_{p1}(i,j) + \\ [S_{p2}(i+1)-S_{p1}(i+1)\Psi_e(i)]K_{p2}(i,j)] \\ 0_{n,q} \quad \text{if} \quad j = i+1,\ldots,N \end{cases} \tag{B19}$$

$$S_{e1}(i) = [I-\Phi_p^T(i)][I-S_{e1}(i+1)\Gamma_e(i)[C(i)+\Gamma_e^T(i)S_{e1}(i+1)\Gamma_e(i)]^{-1}\,\Gamma_e^T(i)]\,\cdot$$

$$S_{e1}(i+1)[I-\Phi_p(i)] - A_p^T(i)B(i)A_p(i)\,; \quad S_{e1}(N) = -A_N \tag{B20}$$

$$S_{e2}(i) = [I-\Phi_p^T(i)][I-S_{e1}(i+1)\Gamma_e(i)[C(i)+\Gamma_e^T(i)S_{e1}(i+1)\Gamma_e(i)]^{-1}\Gamma_e^T(i)] \cdot$$

$$[S_{e2}(i+1)-S_{e1}(i+1)\Psi_p(i)] - A_p^T(i)B(i)D_p(i); \quad S_{e2}(N) = 0 \qquad \text{(B21)}$$

$$P_{e1}(i) = M_{e1}(i) - M_{e1}(i)\Theta_e^T(i)[\Theta_e(i)M_{e1}(i)\Theta_e^T(i)+R_e(i)]^{-1}\Theta_e(i)M_{e1}(i) \qquad \text{(B22)}$$

$$P_{e2}(i) = M_{e2}(i) - M_{e1}(i)\Theta_e^T(i)[\Theta_e(i)M_{e1}(i)\Theta_e^T(i)+R_e(i)]^{-1}\Theta_e(i)M_{e2}(i) \qquad \text{(B23)}$$

$$P_{e3}(i) = M_{e3}(i) - M_{e2}^T(i)\Theta_e^T(i)[\Theta_e(i)M_{e1}(i)\Theta_e^T(i)+R_e(i)]^{-1}\Theta_e(i)M_{e2}(i) \qquad \text{(B24)}$$

$$M_{e1}(i+1) = [I-\Phi_p(i)][P_{e1}(i)[I-\Phi_p^T(i)] - P_{e2}(i)\Psi_p^T(i)] - \Psi_p(i)[P_{e2}(i)[I-\Phi_p^T(i)] -$$

$$P_{e3}(i)\Psi_p^T(i)] + Y(i)Q(i)Y^T(i);$$

$$\langle M_{e1}(0)\rangle_{j,k} = P_o; \quad j,k = 0, \ldots, N \qquad \text{(B25)}$$

$$M_{e2}(i+1) = [I-\Phi_p(i)]P_{e2}(i) - \Psi_p(i)P_{e3}(i); \quad M_{e2}(0) = 0 \qquad \text{(B26)}$$

$$M_{e3}(i+1) = P_{e3}(i) + \Omega_p(i)R_p(i+1)\Omega_p^T(i); \quad \langle M_{e3}(0)\rangle_{j,k} = \begin{cases} R_p(0); & j=k=0 \\ 0_{q,q} & \text{otherwise} \end{cases} \qquad \text{(B27)}$$

$$K_{e1}(i+1,j) = [I-P_{e1}(i+1)\Theta_e^T(i+1)R_e^{-1}(i+1)\Theta_e(i+1)][[I-\Phi_p(i)]K_{e1}(k,j) -$$

$$\Psi_p(i)K_{e2}(i,j) - \Gamma_e(i)\Lambda_e(i,j)]; \quad K_{e1}(i,i) = P_{e1}(i)\Theta_e^T(i)R_e^{-1}(i) \qquad \text{(B28)}$$

$$K_{e2}(i+1,j) = K_{e2}(i,j) - P_{e2}^T(i+1)\Theta_e^T(i+1)R_e^{-1}(i+1)\Theta_e(i+1)[[I-\Phi_p(i)]K_{e1}(i,j) -$$

$$\Psi_p(i)K_{e2}(i,j) - \Gamma_e(i)\Lambda_e(i,j)]; \quad K_{e2}(i,i) = P_{e2}^T(i)\Theta_e^T(i)R_e^{-1}(i) \qquad \text{(B29)}$$

$$\Lambda_e(i,j) = \begin{cases} [C(i)+\Gamma_e^T(i)S_{e1}(i+1)\Gamma_e(i)]^{-1}\,\Gamma_e^T(i)\,[S_{e1}(i+1)[I-\Phi_p(i)]K_{e1}(i,j)\; + \\ [S_{e2}(i+1)-S_{e1}(i+1)\Psi_e(i)]K_{e2}(i,j)] \\ 0_{n,r} \quad \text{if} \quad j = i+1,\ \ldots,\ N \end{cases} \tag{B30}$$

The S_{p3} and S_{e3} equations turn out to be superfluous. The subsidiary system of implicit equations can be expressed in terms of these partitions as:

$$\gamma_{p1}(i) = [I-\Phi_e^T(i)]\,[I-S_{p1}(i+1)\Gamma_p(i)[B(i)+\Gamma_p^T(i)S_{p1}(i+1)\Gamma_p(i)]^{-1}\,\Gamma_p^T(i)]\,\cdot$$
$$[\gamma_{p1}(i+1)-S_{p1}(i+1)\Gamma_e(i)a_e(i)] - A_e^T(i)C(i)a_e(i)\,; \qquad \gamma_{p1}(N) = 0 \tag{B31}$$

$$\delta_{p1}(i+1) = [I-P_{p1}(i+1)\Theta_p^T(i+1)R_p^{-1}(i+1)\Theta_p(i+1)]\,[[I-\Phi_e(i)]\delta_{p1}(i)\; -$$
$$\Psi_e(i)\delta_{p2}(i)-\Gamma_p(i)a_p(i)-\Gamma_e(i)a_e(i)]\,;$$
$$\delta_{p1}(0) = [I-P_{p1}(0)\Theta_p^T(0)R_p^{-1}(0)\Theta_p(0)]\,\overline{\pi}_p(0) \tag{B32}$$

$$\delta_{p2}(i+1) = \delta_{p2}(i) - P_{p2}^T(i+1)\Theta_p^T(i+1)R_p^{-1}(i+1)\Theta_p(i+1)\Big[[I-\Phi_e(i)]\delta_{p1}(i)\; -$$
$$\Psi_e(i)\delta_{p2}(i)-\Gamma_e(i)a_e(i)-\Gamma_p(i)[a_p(i)-[B(i)+\Gamma_p^T(i)S_{p1}(i+1)\Gamma_p(i)]^{-1}\,\cdot$$
$$\Gamma_p^T(i)\gamma_{p1}(i+1)]\Big]\,; \qquad \delta_{p2}(0) = P_{p2}^T(0)\Theta_p^T(0)R_p^{-1}(0)\Theta_p(0)\overline{\pi}_p(0) = 0 \tag{B33}$$

$$a_p(i) = [B(i)+\Gamma_p^T(i)S_{p1}(i+1)\Gamma_p(i)]^{-1}\,\Gamma_p^T(i)\Big[\gamma_{p1}(i+1)+S_{p1}(i+1)[[I-\Phi_e(i)]\delta_{p1}(i)\; -$$
$$\Psi_e(i)\delta_{p2}(i)-\Gamma_e(i)a_e(i)] + S_{p2}(i+1)\delta_{p2}(i)\Big] \tag{B34}$$

$$\gamma_{e1}(i) = [I-\Phi_p^T(i)][I-S_{e1}(i+1)\Gamma_e(i)[C(i)+\Gamma_e^T(i)S_e(i+1)\Gamma_e(i)]^{-1}\Gamma_e^T(\;)][\gamma_e(i+1) - S_{e1}(i+1)\Gamma_p(i)a_p(i)] - A_p^T(i)B(i)a_p(i); \quad \gamma_{e1}(N) = 0 \tag{B35}$$

$$\delta_{e1}(i+1) = [I-P_{e1}(i+1)\Theta_e^T(i+1)R_e^{-1}(i+1)\Theta_e(i+1)][[I-\Phi_p(i)]\delta_{e1}(i) - \Psi_p(i)\delta_{e2}(i) - \Gamma_e(i)a_e(i) - \Gamma_p(i)a_p(i)];$$
$$\delta_{e1}(0) = [I-P_{e1}(0)\Theta_e^T(0)R_e^{-1}(0)\Theta_e(0)]\overline{\pi}_e(0) \tag{B36}$$

$$\delta_{e2}(i+1) = \delta_{e2}(i) - P_{e2}^T(i+1)\Theta_e^T(i+1)R_e^{-1}(i+1)\Theta_e(i+1)\Big[[I-\Phi_p(i)]\delta_{e1}(i) - \Psi_e(i)\delta_{e2}(i)-\Gamma_p(i)a_p(i)-\Gamma_e(i)[a_e(i)-[C(i)+\Gamma_e^T(i)S_{e1}(i+1)\Gamma_e(i)]^{-1}\cdot \Gamma_e^T(i)\gamma_{e1}(i+1)]\Big];$$
$$\delta_{e2}(0) = P_{e2}^T(0)\Theta_e^T(0)R_e^{-1}(0)\Theta_e(0)\overline{\pi}_p(0) = 0 \tag{B37}$$

$$a_e(i) = [C(i)+\Gamma_e^T(i)S_{e1}(i+1)\Gamma_e(i)]^{-1}\Gamma_e^T(i)\Big[\gamma_{e1}(i+1)+S_{e1}(i+1)[[I-\Phi_p(i)]\delta_{e1}(i) - \Psi_p(i)\delta_{e2}(i)-\Gamma_p(i)a_p(i)]+S_{e2}(i+1)\delta_{e2}(i)\Big] \tag{B38}$$

The δ_{p2} and δ_{e2} equations also turn out to be superfluous. The convexity conditions can also be written in terms of these partitions as

$$\left.\begin{matrix}[B(i)+\Gamma_p^T(i)S_{p1}(i+1)\Gamma_p(i)] \\ [C(i)+\Gamma_e^T(i)S_{e1}(i+1)\Gamma_e(i)]\end{matrix}\right\} \text{ positive definite for } i=0,\dots,N-1. \tag{B39}$$

Finally, it is convenient for future use to obtain the Kalman filter equations for the pursuer's and evader's associated optimal control problems in partitioned form. Using $\begin{bmatrix}\hat{\pi}_p(i) \\ \hline \hat{\eta}_{ep}(i)\end{bmatrix}$ for $\hat{\sigma}_{ep}(i)$ and $\begin{bmatrix}\overline{\pi}_p(i) \\ \hline \overline{\eta}_{ep}(i)\end{bmatrix}$ for $\overline{\sigma}_{ep}(i)$, the equations for the pursuer are:

$$\hat{\pi}_p(i) = \bar{\pi}_p(i) + P_{p1}(i)\Theta_p^T(i)R_p^{-1}(i)[z_p(i)-\Theta_p(i)\bar{\pi}_p(i)] \quad \text{(B40)}$$

$$\begin{cases} \bar{\pi}_p(i+1) = [I-\Phi_e(i)]\hat{\pi}_p(i) - \Psi_e(i)\hat{\eta}_{ep}(i) + \Gamma_p(i)u(i) - \Gamma_e(i)a_e(i); \\ \qquad \langle\bar{\pi}_p(0)\rangle_j = \bar{x}_o \\ \underline{\text{or}} \\ \bar{\pi}_p(i+1) = [I-\Gamma_p(i)[B(i)+\Gamma_p^T(i)S_{p1}(i+1)\Gamma_p(i)]^{-1}\Gamma_p^T(i)S_{p1}(i+1)] \cdot \\ \qquad [[I-\Phi_e(i)]\hat{\pi}_p(i)-\Psi_e(i)\hat{\eta}_{ep}(i)] - \Gamma_p(i)[B(i)+\Gamma_p^T(i)S_{p1}(i+1)\Gamma_p(i)]^{-1} \cdot \\ \qquad \Gamma_p^T(i)[\gamma_{p1}(i+1)+S_{p2}(i+1)\hat{\eta}_{ep}(i)] - \Gamma_e(i)a_e(i) \end{cases}$$

(B41)

$$\hat{\eta}_{ep}(i) = \bar{\eta}_{ep}(i) + P_{p2}^T(i)\Theta_p^T(i)R_p^{-1}(i)[z_p(i)-\Theta_p(i)\bar{\pi}_p(i)] \quad \text{(B42)}$$

$$\bar{\eta}_{ep}(i+1) = \hat{\eta}_{ep}(i); \qquad \langle\bar{\eta}_{ep}(0)\rangle_j = 0 \qquad (j = 0, \ldots, N) \quad \text{(B43)}$$

and the equations for the evader's Kalman filter can be expressed as:

$$\hat{\pi}_e(i) = \bar{\pi}_e(i) + P_{e1}(i)\Theta_e^T(i)R_e^{-1}(i)[z_e(i)-\Theta_e(i)\bar{\pi}_e(i)] \quad \text{(B44)}$$

$$\begin{cases} \bar{\pi}_e(i+1) = [I-\Phi_p(i)]\hat{\pi}_e(i) - \Psi_p(i)\hat{\eta}_{pe}(i) - \Gamma_e(i)v(i) - \Gamma_p(i)a_p(i); \quad \langle\bar{\pi}_e(0)\rangle_j = \bar{x}_o \\ \underline{\text{or}} \\ \pi_e(i+1) = [I-\Gamma_e(i)[C(i)+\Gamma_e^T(i)S_{e1}(i+1)\Gamma_e(i)]^{-1}\Gamma_e^T(i)S_{e1}(i+1)] \cdot \\ \qquad [[I-\Phi_p(i)]\hat{\pi}_e(i)-\Psi_p(i)\hat{\eta}_{pe}(i)] - \Gamma_e(i)[C(i)+\Gamma_e^T(i)S_{e1}(i+1)\Gamma_e(i)]^{-1} \cdot \\ \qquad \Gamma_e^T(i)[\gamma_{e1}(i+1)+S_{e2}(i+1)\hat{\eta}_{pe}(i)] - \Gamma_p(i)a_p(i) \end{cases}$$

(B45)

$$\hat{\eta}_{pe}(i) = \bar{\eta}_{pe}(i) + P_{e2}^T(i)\Theta_e^T(i)R_e^{-1}(i)[z_e(i)-\Theta_e(i)\bar{\pi}_e(i)] \quad \text{(B46)}$$

$$\bar{\eta}_{pe}(i+1) = \hat{\eta}_{pe}(i); \qquad \langle\bar{\eta}_{pe}(0)\rangle_j = \bar{x}_o . \quad \text{(B47)}$$

Now consider the discretized game developed in the "Derivation Outline" section:

$$x((i+1)\Delta) = x(i\Delta) + [G_p(i\Delta)u(i\Delta) - G_e(i\Delta)v(i\Delta)]\Delta + [\xi(i\Delta)\Delta] \tag{B48}$$

$$J = \frac{1}{2}\,\varepsilon\left\{x^T(N\Delta)S_f x(N\Delta) + \sum_{i=0}^{N-1}[u^T(i\Delta)B(i\Delta)u(i\Delta) - v^T(i\Delta)C(i\Delta)v(i\Delta)]\Delta\right\} \tag{B49}$$

$$z_p(i\Delta) = H_p(i\Delta)x(i\Delta) + w_p(i\Delta) \tag{B50}$$

$$z_e(i\Delta) = H_e(i\Delta)x(i\Delta) + w_e(i\Delta) \tag{B51}$$

where

$$\begin{bmatrix} \xi(i\Delta) \\ \hline w_p(i\Delta) \\ \hline w_e(i\Delta) \end{bmatrix} \text{ are independent Normal} \left(\begin{bmatrix} 0 \\ \hline 0 \\ \hline 0 \end{bmatrix}, \begin{bmatrix} Q(i\Delta) & 0 & 0 \\ \hline 0 & R_p(i\Delta) & 0 \\ \hline 0 & 0 & R_e(i\Delta) \end{bmatrix} \cdot \frac{1}{\Delta} \right)$$

random variables, and the common prior is Normal$(\bar{x}_o, P_o)$. Since this is a special case of the multistage game examined in Appendix A, its solution can be obtained immediately from that of the more general multistage game by making the following substitutions in the solution presented in Appendix A:

$$i \rightarrow i\Delta$$

$$G_p(i) \rightarrow \Delta G_p(i\Delta)$$

$$G_e(i) \rightarrow \Delta G_e(i\Delta)$$

$$B(i) \rightarrow \Delta B(i\Delta)$$

$$C(i) \rightarrow \Delta C(i\Delta)$$

$$R_p(i) \rightarrow \frac{1}{\Delta} R_p(i\Delta)$$

$$R_e(i) \rightarrow \frac{1}{\Delta} R_e(i\Delta)$$

$$Q(i) \rightarrow \frac{I}{\Delta} Q(i\Delta)\Delta^2$$

Once this is done, it is possible to use the definitions of the Φ, Ψ, Ω, Γ, Θ, A, D, and Y matrices (see Appendix A) to express the solution in terms of the subpartitions of the enlarged variables. That is, the S_{p1}, S_{e1}, P_{p1}, P_{e1}, M_{p1}, and M_{e1} matrices are divided into $(N+1)\times(N+1)$ subpartitions of dimension $n \times n$, corresponding to the N+1 partitions in the π vector; and the other enlarged variables are partitioned into $(N+1) \times (N+1)$ subpartitions (or N+1 subpartitions in the case of vector variables, or variables defined in Appendix A as vertically or horizontally partitioned matrices) of the appropriate dimensions.

Since the real objective here is to obtain first-order equations for small Δ (and hence large N), it is convenient at this point to determine which terms in the subpartition equations are of first-order significance, and which are of higher order, so that the latter need not be computed. This procedure is not altogether straightforward, however, the difficulties being caused by the fact that the number of indices being summed in the subpartition equations is of order $N\left(=\frac{1}{\Delta}\right)$, which becomes infinite as Δ approaches zero, and by the fact that the various subpartitions are of different orders of magnitude as $\Delta \rightarrow 0$. It has been possible, though, to find the orders of magnitude of these subpartitions that are consistent with the equations they satisfy, mainly through a procedure of trial and error. The result is that the "S" subpartitions are of order Δ^2, the "K" and "γ"

subpartitions and the "Λ" matrices are of order Λ, and the "P" and "δ" subpartitions and the "a" vectors are of order unity, with the following exceptions:

$\langle \gamma_{p1}(i) \rangle_N$ is of order unity,

$\langle S_{p1}(i) \rangle_{N,N}$ is of order unity,

$\langle S_{p1}(i\Delta) \rangle_{j,N} = \langle S_{p1}(i\Delta) \rangle_{N,j}^T$ is of order Δ; $j = 0, \ldots, N$,

$\langle S_{p2}(i\Delta) \rangle_{N,j}$ is of order Δ; $j = 0, \ldots, N$,

$\langle P_{p3}(i\Delta) \rangle_{j,j}$ is of order $\frac{1}{\Delta}$, $j = 0, \ldots, i$ (because it contains the covariance of the evader's measurement noise),

and similarly for the evader's parameters. The "M" variables are incorporated into a system of difference equations in the "P" variables, and are therefore not used explicitly. The orders of magnitude of the pursuer's "S" and "P" matrix subpartitions are shown in Figures B1 and B2.

With this result in hand, and defining $\delta(i, j)$ as the Kronecker delta function:

$$\delta(i,j) = \begin{cases} 1; & i = j \\ 0; & i \neq j \end{cases}$$

and $U(i)$ as the unit step function:

$$U(i) = \begin{cases} 1; & i \geq 0 \\ 0; & 1 < 0 \end{cases},$$

it is straightforward but tedious to calculate the first-order subpartition equations as the following:

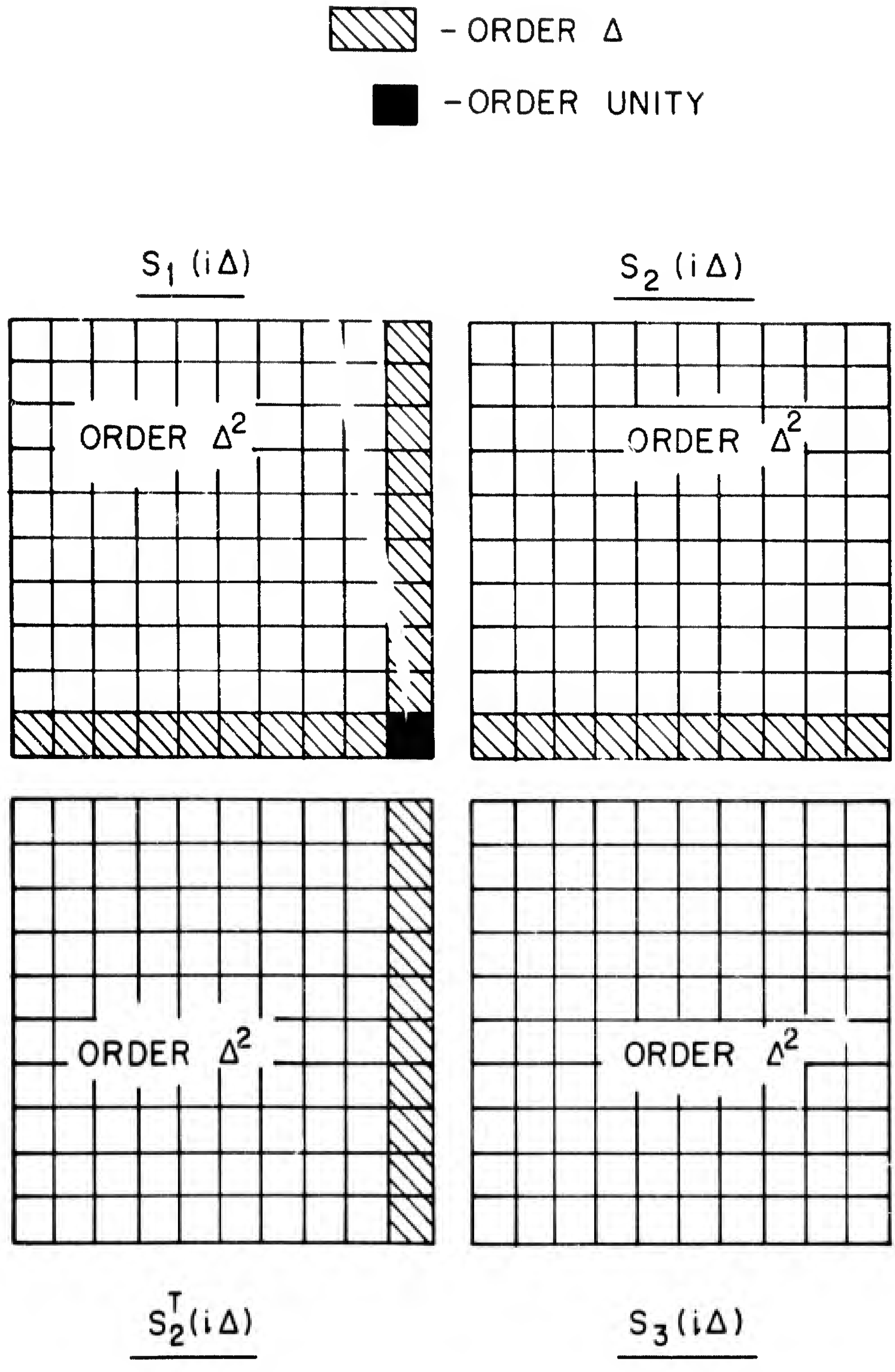

FIG. B1 PARTITION MAGNITUDES – S_p AND S_e MATRICES (DISCRETIZED GAME)

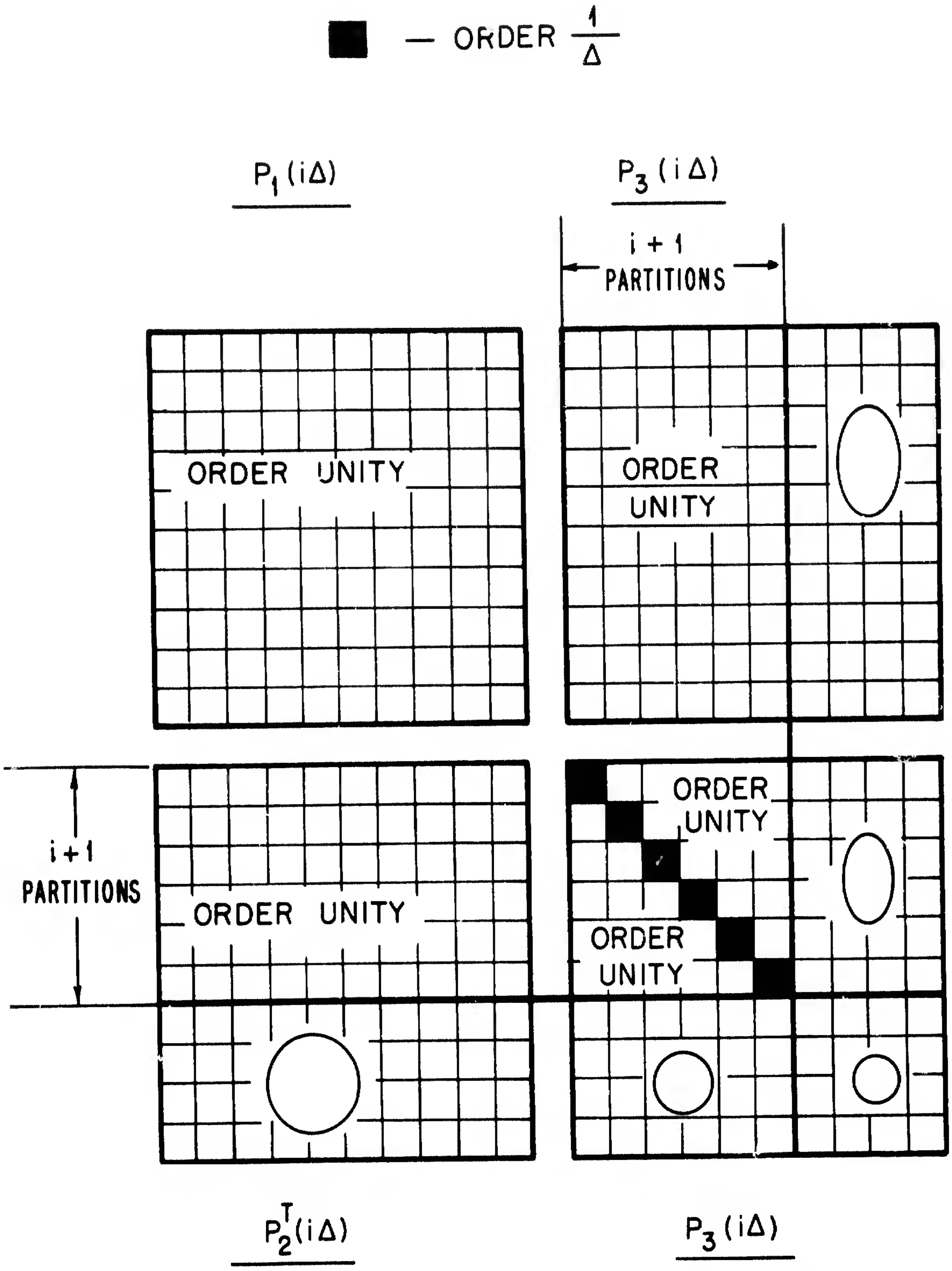

FIG. B2 PARTITION MAGNITUDES – P_p AND P_e MATRICES (DISCRETIZED GAME)

Primary Equation System

$$\langle S_{p1}(i\Delta)\rangle_{j,k} = \langle S_{p1}((i+1)\Delta)\rangle_{j,k} - \Big\{ H_e^T(j\Delta)\Lambda_e^T(i\Delta, j\Delta)C(i\Delta)\Lambda_e(i\Delta,k\Delta)H_e(k\Delta) + \left(\sum_{\alpha=i+1}^{N} \langle S_{p1}((i+1)\Delta)\rangle_{j,\alpha}\right) G_e(i\Delta)\Lambda_e(i\Delta,k\Delta)H_e(k\Delta) + H_e^T(j\Delta)\Lambda_e^T(i\Delta,j\Delta) \cdot G_e^T(i\Delta) \sum_{\alpha=i+1}^{N} \langle S_{p1}((i+1)\Delta)\rangle_{\alpha,k} + \left(\sum_{\alpha=i+1}^{N} \langle S_{p1}((i+1)\Delta)\rangle_{j,\alpha}\right) G_p(i\Delta) \cdot B^{-1}(i\Delta)G_p^T(i\Delta)\left(\sum_{\beta=i+1}^{N} \langle S_{p1}((i+1)\Delta)\rangle_{\beta,k}\right)\Big\} \Delta ;$$

$$\langle S_{p1}(N\Delta)\rangle_{j,k} = \begin{cases} S_f ; & j = k = N \\ 0_{n,n} & \text{otherwise} \end{cases} \qquad \text{(B52)}$$

$$\langle S_{p2}(i\Delta)\rangle_{j,k} = \langle S_{p2}((i+1)\Delta)\rangle_{j,k} - \Big\{ H_e^T(j\Delta)\Lambda_e^T(i\Delta,j\Delta)C(i\Delta)\Lambda_e(i\Delta,k\Delta) + \left(\sum_{\alpha=i+1}^{N} \langle S_{p1}((i+1)\Delta)\rangle_{j,\alpha}\right) G_e(i\Delta)\Lambda_e(i\Delta,k\Delta) + H_e^T(j\Delta)\Lambda_e^T(i\Delta,j\Delta) \cdot G_e^T(i\Delta) \sum_{\alpha=i+1}^{N} \langle S_{p2}((i+1)\Delta)\rangle_{\alpha,k} + \left(\sum_{\alpha=i+1}^{N} \langle S_{p1}((i+1)\Delta)\rangle_{j,\alpha}\right) G_p(i\Delta) \cdot B^{-1}(i\Delta)G_p^T(i\Delta)\left(\sum_{\beta=i+1}^{N} \langle S_{p2}((i+1)\Delta)\rangle_{\alpha,k}\right)\Big\} \Delta ; \quad S_{p2}(N\Delta) = 0_{n,r} \qquad \text{(B53)}$$

$$\langle P_{p1}((i+1)\Delta)\rangle_{j,k} = \langle P_{p1}(i\Delta)\rangle_{j,k} - \Big\{ \langle P_{p1}(i\Delta)\rangle_{j,i} H_p^T(i\Delta)R_p^{-1}(i\Delta)H_p(i\Delta) \cdot \langle P_{p1}(i\Delta)\rangle_{i,k} + U(j-i-1)G_e(i\Delta) \sum_{\alpha=0}^{i} \Lambda_e(i\Delta, \alpha\Delta)[H_e(\alpha\Delta)\langle P_{p1}(i\Delta)\rangle_{\alpha,k} + \langle P_{p2}(i\Delta)\rangle_{k,\alpha}^T] + U(k-i-1) \sum_{\alpha=0}^{i} [\langle P_{p2}(i\Delta)\rangle_{j,\alpha} + \langle P_{p1}(i\Delta)\rangle_{j,\alpha} H_e^T(\alpha\Delta)] \cdot \Lambda_e^T(i\Delta, \alpha\Delta)G_e^T(i\Delta)\Big\} \Delta + U(j-i-1)U(k-i-1)Q(i\Delta)\Delta; \quad \langle P_{p1}(0)\rangle_{j,k} = P_o \qquad \text{(B54)}$$

$$\langle P_{p2}((i+1)\Delta)\rangle_{j,k} = \langle P_{p2}(i\Delta)\rangle_{j,k} - \langle P_{p1}(i\Delta)\rangle_{j,i} H_p^T(i\Delta)R_p^{-1}(i\Delta)H_p(i\Delta) \cdot \langle P_{p2}(i\Delta)\rangle_{i,k} + U(j-i-1)G_e(i\Delta) \sum_{\alpha=0}^{i} \Lambda_e(i\Delta,\alpha\Delta)[H_e(\alpha\Delta)\langle P_{p2}(i\Delta)\rangle_{\alpha,k} + \langle P_{p3}(i\Delta)\rangle_{\alpha,k}] \Delta ; \quad \langle P_{p2}(0)\rangle_{j,k} = 0_{n,r} \tag{B55}$$

$$\langle P_{p3}((i+1)\Delta)\rangle_{j,k} = \langle P_{p3}(i\Delta)\rangle_{j,k} - \langle P_{p2}(i\Delta)\rangle_{i,j}^T H_p^T(i\Delta)R_p^{-1}(i\Delta)H_p(i\Delta) \cdot \langle P_{p2}(i\Delta)\rangle_{i,k}\Delta + \frac{1}{\Delta} R_e((i+1)\Delta)\delta(i+1,j)\delta(i+1,k);$$

$$\langle P_{p3}(0)\rangle_{j,k} = \begin{cases} \frac{1}{\Delta} R_e(0); & j = k = 0 \\ 0_{r,r} & \text{otherwise} \end{cases} \tag{B56}$$

$$\langle K_{p1}((i+1)\Delta,j\Delta)\rangle_k = \langle K_{p1}(i\Delta,j\Delta)\rangle_k - \Big\{ \langle P_{p1}((i+1)\Delta)\rangle_{k,i+1} H_p^T((i+1)\Delta) \cdot R_p^{-1}((i+1)\Delta)H_p((i+1)\Delta)\langle K_{p1}(i\Delta,j\Delta)\rangle_{i+1} + U(k-i-1)\Big[G_p(i\Delta)\Lambda_p(i\Delta,j\Delta) + G_e(i\Delta) \sum_{\alpha=0}^{i} \Lambda_e(i\Delta,j\Delta)[H_e(\alpha\Delta)\langle K_{p1}(i\Delta,j\Delta)\rangle_\alpha + \langle K_{p2}(i\Delta,j\Delta)\rangle_\alpha]\Big]\Big\} \Delta ;$$

$$\langle K_{p1}(i\Delta,i\Delta)\rangle_k = \langle P_{p1}(i\Delta)\rangle_{k,i} H_p^T(i\Delta)R_p^{-1}(i\Delta) \tag{B57}$$

$$\langle K_{p2}((i+1)\Delta,j\Delta)\rangle_k = \langle K_{p2}(i\Delta,j\Delta)\rangle_k - \langle P_{p2}((i+1)\Delta)\rangle_{i+1,k}^T H_p^T((i+1)\Delta) \cdot R_p^{-1}((i+1)\Delta)H_p((i+1)\Delta)\langle K_{p1}(i\Delta,j\Delta)\rangle_{i+1}\Delta ;$$

$$\langle K_{p2}(i\Delta,i\Delta)\rangle_k = \langle P_{p2}(i\Delta)\rangle_{i,k}^T H_p^T(i\Delta)R_p^{-1}(i\Delta) \tag{B58}$$

$$\Lambda_p(i\Delta,j\Delta) = \begin{cases} B^{-1}(i\Delta)G_p^T(i\Delta) \displaystyle\sum_{\alpha=i+1}^{N} \sum_{\beta=0}^{N} [\langle S_{p1}((i+1)\Delta)\rangle_{\alpha,\beta}\langle K_{p1}(i\Delta,j\Delta)\rangle_\beta + \langle S_{p2}((i+1)\Delta)\rangle_{\alpha,\beta}\langle K_{p2}(i\Delta,j\Delta)\rangle_\beta] \\ 0_{m,q} \qquad \text{if } N \geq j > i \end{cases} \tag{B59}$$

$$\langle S_{e1}(i\Delta)\rangle_{j,k} = \langle S_{e1}((i+1)\Delta)\rangle_{j,k} - \Big\{ H_p^T(j\Delta)\Lambda_p^T(i\Delta,j\Delta)B(i\Delta)\Lambda_p(i\Delta,k\Delta)H_p(k\Delta) + \left(\sum_{\alpha=i+1}^{N} \langle S_{e1}((i+1)\Delta)\rangle_{j,\alpha}\right) G_p(i\Delta)\Lambda_p(i\Delta,k\Delta)H_p(k\Delta) + H_p^T(j\Delta)\Lambda_p^T(i\Delta,j\Delta) \cdot G_p^T(i\Delta) \sum_{\alpha=i+1}^{N} \langle S_{e1}((i+1)\Delta)\rangle_{\alpha,k} + \left(\sum_{\alpha=i+1}^{N} \langle S_{e1}((i+1)\Delta)\rangle_{j,\alpha}\right) G_e(i\Delta) \cdot C^{-1}(i\Delta)G_e^T(i\Delta)\left(\sum_{\beta=i+1}^{N} \langle S_{e1}((i+1)\Delta)\rangle_{\alpha,k}\right)\Big\}\Delta ;$$

$$\langle S_{e1}(N\Delta)\rangle_{j,k} = \begin{cases} -S_f \; ; \; j = k = N \\ 0_{n,n} \text{ otherwise} \end{cases} \tag{B60}$$

$$\langle S_{e2}(i\Delta)\rangle_{j,k} = \langle S_{e2}((i+1)\Delta)\rangle_{j,k} - \Big\{ H_p^T(j\Delta)\Lambda_p^T(i\Delta,j\Delta)B(i\Delta)\Lambda_p(i\Delta,k\Delta) + \left(\sum_{\alpha=i+1}^{N} \langle S_{e1}((i+1)\Delta)\rangle_{j,\alpha}\right) G_p(i\Delta)\Lambda_p(i\Delta,k\Delta) + H_p^T(j\Delta)\Lambda_p^T(i\Delta,j\Delta) \cdot G_p^T(i\Delta) \sum_{\alpha=i+1}^{N} \langle S_{e2}((i+1)\Delta)\rangle_{\alpha,k} + \left(\sum_{\alpha=i+1}^{N} \langle S_{e1}((i+1)\Delta)\rangle_{j,\alpha}\right) G_e(i\Delta) \cdot C^{-1}(i\Delta)G_e^T(i\Delta)\left(\sum_{\beta=i+1}^{N} \langle S_{e2}((i+1)\Delta)\rangle_{\beta,k}\right)\Big\}\Delta ; \; \langle S_{e2}(N\Delta)\rangle = 0_{n,q} \tag{B61}$$

$$\langle P_{e1}((i+1)\Delta)\rangle_{j,k} = \langle P_{e1}(i\Delta)\rangle_{j,k} - \Big\{ \langle P_{e1}(i\Delta)\rangle_{j,i} H_e^T(i\Delta)R_e^{-1}(i\Delta)H_e(i\Delta) \cdot \langle P_{e1}(i\Delta)\rangle_{i,k} + U(j-i-1)G_p(i\Delta) \sum_{\alpha=0}^{i} \Lambda_p(i\Delta,\alpha\Delta)[H_p(\alpha\Delta)\langle P_{e1}(i\Delta)\rangle_{\alpha,k} + \langle P_{e2}(i\Delta)\rangle_{k,\alpha}^T] + U(k-i-1) \sum_{\alpha=0}^{i} [\langle P_{e2}(i\Delta)\rangle_{j,\alpha} + \langle P_{e1}(i\Delta)\rangle_{j,\alpha} H_p^T(\alpha\Delta)] \cdot \Lambda_p^T(i\Delta,\alpha\Delta)G_p^T(i\Delta)\Big\} \Delta + U(j-i-1)U(k-i-1)Q(i\Delta)\Delta ; \; \langle P_{e1}(0)\rangle_{j,k} = P_o \tag{B62}$$

$$\langle P_{e2}((i+1)\Delta)\rangle_{j,k} = \langle P_{e2}(i\Delta)\rangle_{j,k} - \Big\{ \langle P_{e1}(i\Delta)\rangle_{j,i} H_e^T(i\Delta) R_e^{-1}(i\Delta) H_e(i\Delta) \cdot \langle P_{e2}(i\Delta)\rangle_{i,k} + U(j-i-1) G_p(i\Delta) \sum_{\alpha=0}^{i} \Lambda_p(i\Delta,\alpha\Delta)[H_p(\alpha\Delta)\langle P_{e2}(i\Delta)\rangle_{\alpha,k} + \langle P_{e3}(i\Delta)\rangle_{\alpha,k}] \Big\} \Delta \; ; \quad \langle P_{p2}(0)\rangle_{j,k} = 0_{n,q} \tag{B63}$$

$$\langle P_{e3}((i+1)\Delta)\rangle = \langle P_{e3}(i\Delta)\rangle_{j,k} - \langle P_{e2}(i\Delta)\rangle_{i,j}^T H_e^T(i\Delta) R_e^{-1}(i\Delta) H_e(i\Delta) \cdot \langle P_{e2}(i\Delta)\rangle_{i,k}\Delta + \frac{1}{\Delta} R_p((i+1)\Delta)\delta(i+1,j)\delta(i+1,k) \; ;$$

$$\langle P_{e3}(0)\rangle_{j,k} = \begin{cases} \frac{1}{\Delta} R_p(0); \; j=k=0 \\ 0_{q,q} \quad \text{otherwise} \end{cases} \tag{B64}$$

$$\langle K_{e1}((i+1)\Delta, j\Delta)\rangle_k = \langle K_{e1}(i\Delta, j\Delta)\rangle_k - \Big\{ \langle P_{e1}((i+1)\Delta)\rangle_{k,i+1} H_e^T((i+1)\Delta) \cdot R_e^{-1}((i+1)\Delta) H_e((i+1)\Delta) \langle K_{e1}(i\Delta, j\Delta)\rangle_{i+1} + U(k-i-1)\Big[G_e(i\Delta)\Lambda_e(i\Delta, j\Delta) + G_p(i\Delta) \sum_{\alpha=0}^{i} \Lambda_p(i\Delta, j\Delta)[H_p(\alpha\Delta)\langle K_{e1}(i\Delta, j\Delta)\rangle_\alpha + \langle K_{e2}(i\Delta, j\Delta)\rangle_\alpha]\Big]\Big\} \Delta \; ;$$

$$\langle K_{e1}(i\Delta, i\Delta)\rangle_k = \langle P_{e1}(i\Delta)\rangle_{k,i} H_e^T(i\Delta) R_e^{-1}(i\Delta) \tag{B65}$$

$$\langle K_{e2}((i+1)\Delta, j\Delta)\rangle_k = \langle K_{e2}(i\Delta, j\Delta)\rangle_k - \langle P_{e2}((i+1)\Delta)\rangle_{i+1,k}^T H_e^T((i+1)\Delta) R_e^{-1}((i+1)\Delta) H_e((i+1)\Delta) \langle K_{e1}(i\Delta, j\Delta)\rangle_{i+1} \Delta \; ;$$

$$\langle K_{e2}(i\Delta, i\Delta)\rangle_k = \langle P_{e2}(i\Delta)\rangle_{i,k}^T H_e^T(i\Delta) R_e^{-1}(i\Delta) \tag{B66}$$

$$\Lambda_e(i\Delta\; j\Delta) = \begin{cases} C^{-1}(i\Delta) G_e^T(i\Delta) \sum_{\alpha=i+1}^{N} \sum_{\beta=0}^{N} \Big(\langle S_{e1}((i+1)\Delta)\rangle_{\alpha,\beta} \langle K_{e1}(i\Delta, j\Delta)\rangle_\beta + \langle S_{e2}((i+1)\Delta)\rangle_{\alpha,\beta} \langle K_{e2}(i\Delta, j\Delta)\rangle_\beta \Big) \\ 0_{k,r} \quad \text{if} \quad N \geq j > i \end{cases} \tag{B67}$$

Subsidiary Equation System

$$\langle \gamma_{p1}(i\Delta)\rangle_j = \langle \gamma_{p1}((i+1)\Delta)\rangle_j - \Bigg\{ H_e^T(j\Delta)\Lambda_e^T(i\Delta,j\Delta)\Bigg[C(i\Delta)a_e(i\Delta) + G_e^T(i\Delta)\sum_{\alpha=i+1}^{N}\langle \gamma_{p1}((i+1)\Delta)\rangle_\alpha\Bigg] + \Bigg[\sum_{\alpha=i+1}^{N}\langle S_{p1}((i+1)\Delta)\rangle_{j,\alpha}\Bigg]\Bigg[G_e(i\Delta)a_e(i\Delta) + G_p(i\Delta)B^{-1}(i\Delta)G_p^T(i\Delta)\sum_{\beta=i+1}^{N}\langle \gamma_{p1}((i+1)\Delta)\rangle_\beta\Bigg]\Bigg\}\Delta; \quad \langle \gamma_{p1}(N\Delta)\rangle_j = 0_{n,1} \tag{B68}$$

$$\langle \delta_{p1}((i+1)\Delta)\rangle_j = \langle \delta_{p1}(i\Delta)\rangle_j - \Big\{\langle P_{p1}((i+1)\Delta)\rangle_{j,i+1} H_p^T((i+1)\Delta)R_p^{-1}((i+1)\Delta)\cdot H_p((i+1)\Delta)\langle \delta_{p1}(i\Delta)\rangle_{i+1} + U(j-i-1)[G_p(i\Delta)a_p(i\Delta)+G_e(i\Delta)a_e(i\Delta)]\Big\}\Delta; \quad \langle \delta_{p1}(0)\rangle_j = \bar{x}_o \tag{B69}$$

$$\langle \delta_{p2}((i+1)\Delta)\rangle_j = \langle \delta_{p2}(i\Delta)\rangle_j - \langle P_{p2}((i+1)\Delta)\rangle_{i+1,j}^T H_p^T((i+1)\Delta)R_p^{-1}((i+1)\Delta)\cdot H_p((i+1)\Delta)\langle \delta_{p1}(i\Delta)\rangle_{i+1}\Delta; \quad \langle \delta_{p2}(0)\rangle_j = 0_{r,1} \tag{B70}$$

$$a_p(i\Delta) = B^{-1}(i\Delta)G_p^T(i\Delta)\sum_{\alpha=i+1}^{N}\Bigg[\langle \gamma_{p1}((i+1)\Delta)\rangle_\alpha + \sum_{\beta=0}^{N}\Big(\langle S_{p1}((i+1)\Delta)\rangle_{\alpha,\beta}\cdot\langle \delta_{p1}(i\Delta)\rangle_\beta + \langle S_{p2}((i+1)\Delta)\rangle_{\alpha,\beta}\langle \delta_{p2}(i\Delta)\rangle_\beta\Big)\Bigg] \tag{B71}$$

$$\langle \gamma_{e1}(i\Delta)\rangle_j = \langle \gamma_{e1}((i+1)\Delta)\rangle_j - \Bigg\{ H_p^T(j\Delta)\Lambda_p^T(i\Delta,j\Delta)\Bigg[B(i\Delta)a_p(i\Delta) + G_p^T(i\Delta)\sum_{\alpha=i+1}^{N}\langle \gamma_{e1}((i+1)\Delta)\rangle_\alpha\Bigg] + \Bigg[\sum_{\alpha=i+1}^{N}\langle S_{e1}((i+1)\Delta)\rangle_{j,\alpha}\Bigg]\Bigg[G_p(i\Delta)a_p(i\Delta) + G_e(i\Delta)C^{-1}(i\Delta)G_e^T(i\Delta)\sum_{\beta=i+1}^{N}\langle \gamma_{e1}((i+1)\Delta)\rangle_\beta\Bigg]\Bigg\}\Delta; \quad \langle \gamma_{e1}(N\Delta)\rangle_j = 0_{n,1} \tag{B72}$$

$$\langle\delta_{e1}((i+1)\Delta)\rangle_j = \langle\delta_{e1}(i\Delta)\rangle_j - \Big\{\langle P_{e1}((i+1)\Delta)\rangle_{j,i+1} H_e^T((i+1)\Delta)R_e^{-1}((i+1)\Delta) \cdot$$
$$H_e((i+1)\Delta)\langle\delta_e(i\Delta)\rangle_{i+1} + U(j-i-1)[G_e(i\Delta)a_e(i\Delta)+G_p(i\Delta)a_p(i\Delta)]\Big\}\Delta ;$$
$$\langle\delta_{e1}(0)\rangle_j = \bar{x}_o \tag{B73}$$

$$\langle\delta_{e2}((i+1)\Delta)\rangle_j = \langle\delta_{e2}(i\Delta)\rangle_j - \langle P_{e2}((i+1)\Delta)\rangle^T_{i+1,j} H_e^T((i+1)\Delta)R_e^{-1}((i+1)\Delta) \cdot$$
$$H_e((i+1)\Delta)\langle\delta_{e1}(i\Delta)\rangle_{i+1}\Delta ; \quad \langle\delta_{e2}(0)\rangle_j = 0_{q,1} \tag{B74}$$

$$a_e(i\Delta) = C^{-1}(i\Delta)G_e^T(i\Delta)\sum_{\alpha=i+1}^{N}\Bigg[\langle\gamma_{e1}((i+1)\Delta)\rangle_\alpha + \sum_{\beta=0}^{N}\Big(\langle S_{e1}((i+1)\Delta)\rangle_{\alpha,\beta} \cdot$$
$$\langle\delta_{e1}(i\Delta)\rangle_\beta + \langle S_{e2}((i+1)\Delta)\rangle_{\alpha,\beta}\langle\delta_{e2}(i\Delta)\rangle_\beta\Big)\Bigg] \tag{B75}$$

To first order, the convexity conditions that constitute part of the sufficient conditions for this discretized game are that the quantities

$$B(i\Delta) + G_p^T(i\Delta)\left(\sum_{\alpha=i+1}^{N}\sum_{\beta=i+1}^{N}\langle S_{p1}((i+1)\Delta)\rangle_{\alpha,\beta}\right)G_p(i\Delta)\Delta \tag{B76}$$

and

$$C(i\Delta) + G_e^T(i\Delta)\left(\sum_{\alpha=i+1}^{N}\sum_{\beta=i+1}^{N}\langle S_{e1}((i+1)\Delta)\rangle\right)G_e(i\Delta)\Delta \tag{B77}$$

be positive definite for $i = 0, \ldots, N-1$. As Δ approaches zero, these conditions are automatically satisfied by the fact that B(i) and C(i) are positive definite, providing that the solution to the primary equation system is asymptotically well behaved. It should be noted that it is reasonable to expect that this proviso not hold in situations where the corresponding deterministic game has a closed-loop conjugate point (the evader escapes), since a degradation in information seems to be in the evader's favor (see the "Problem Statement" section).

It is also straightforward to verify that the minimax strategies corresponding to these parameters can be expressed as:

$$u(i\Delta) = -a_p(i\Delta) - \sum_{j=0}^{i} \Lambda_p(i\Delta, j\Delta) z_p(j\Delta) \quad \text{(pursuer)} \qquad \text{(B78)}$$

and

$$v(i\Delta) = a_e(i\Delta) + \sum_{j=0}^{i} \Lambda_e(i\Delta, j\Delta) z_e(j\Delta) \quad \text{(evader)}, \qquad \text{(B79)}$$

and that the Kalman filter equations can be written in terms of the subpartitions as:

$$\langle \hat{\pi}_p((i+1)\Delta)\rangle_j = \langle \hat{\pi}_p(i\Delta)\rangle_j - U(j-i-1)\Big[G_e(i\Delta)a_e(i\Delta) - G_p(i\Delta)u(i\Delta) + \sum_{\alpha=0}^{i} G_e(i\Delta)\Lambda_e(i\Delta,\alpha\Delta)\Big(H_e(\alpha\Delta)\langle \hat{\pi}_p(i\Delta)\rangle_\alpha + \langle \hat{\eta}_{ep}(i\Delta)\rangle_\alpha\Big)\Big]\Delta + \langle P_{p1}((i+1)\Delta)\rangle_{j,i+1} H_p^T((i+1)\Delta)R_p^{-1}((i+1)\Delta)[z_p((i+1)\Delta) - H_p((i+1)\Delta) \cdot \langle \hat{\pi}_p((i+1)\Delta)\rangle_{i+1}]\Delta ; \quad \langle \hat{\pi}_p(0)\rangle_j = \bar{x}_o \qquad \text{(B80)}$$

$$\langle \hat{\eta}_{ep}((i+1)\Delta)\rangle_j = \langle \hat{\eta}_{ep}(i\Delta)\rangle_j + \langle P_{p2}((i+1)\Delta)\rangle_{i+1,j}^T H_p^T((i+1)\Delta)R_p^{-1}((i+1)\Delta) \cdot [z_p((i+1)\Delta) - H_p((i+1)\Delta)\langle \hat{\pi}_p((i+1)\Delta)\rangle_{i+1}]\Delta ; \quad \langle \hat{\eta}_{ep}(0)\rangle_j = 0_{r,1} \qquad \text{(B81)}$$

$$\langle \hat{\pi}_e((i+1)\Delta)\rangle_j = \langle \hat{\pi}_e(i\Delta)\rangle_j - U(j-i-1)\Big[G_p(i\Delta)a_p(i\Delta) + G_e(i\Delta)v(i\Delta) + \sum_{\alpha=0}^{i} G_p(i\Delta)\Lambda_p(i\Delta,\alpha\Delta)\Big(H_p(\alpha\Delta)\langle \hat{\pi}_e(i\Delta)\rangle_\alpha + \langle \hat{\eta}_{pe}(i\Delta)\rangle_\alpha\Big)\Big]\Delta + \langle P_{e1}((i+1)\Delta)\rangle_{j,i+1} H_e^T((i+1)\Delta)R_e^{-1}((i+1)\Delta)[z_e((i+1)\Delta) - H_e((i+1)\Delta) \cdot \langle \hat{\pi}_e((i+1)\Delta)\rangle_{i+1}]\Delta ; \quad \langle \hat{\pi}_e(0)\rangle_j = \bar{x}_o \qquad \text{(B82)}$$

$$\langle \hat{\eta}_{pe}((i+1)\Delta)\rangle_j = \langle \hat{\eta}_{pe}(i\Delta)\rangle_j + \langle P_{e2}((i+1)\Delta)\rangle^T_{i+1,j}\, H_e^T((i+1)\Delta)R_e^{-1}((i+1)\Delta)\cdot$$
$$[z_e((i+1)\Delta)-H_e((i+1)\Delta)\langle\hat{\pi}_e((i+1)\Delta)\rangle_{i+1}]\Delta\,;\quad \langle\hat{\eta}_{pe}(0)\rangle_j = 0_{q,1} \qquad \text{(B83)}$$

The deterministic game corresponding to this discretized stochastic game is described by the equations

$$x((i+1)\Delta) = x(i\Delta) + [G_p(i\Delta)u(i\Delta) - G_e(i\Delta)v(i\Delta)]\Delta;\; x(0) = \bar{x}_o \qquad \text{(B84)}$$

and

$$J = \frac{1}{2}\, x^T(N\Delta)S_f x(N\Delta) + \frac{1}{2}\sum_{i=0}^{N-1}\Big(u^T(i\Delta)B(i\Delta)u(i\Delta) - v^T(i\Delta)C(i\Delta)v(i\Delta)\Big)\Delta, \qquad \text{(B85)}$$

where the pursuer and evader can measure the state exactly. The calculus of variations can be employed in the usual way (as in Ho, Bryson, and Baron [6]) to derive necessary conditions for the trajectory followed by this deterministic game (i.e., the state and control variable histories) when the players use minimax strategies. Although these necessary conditions weren't stated for the more general multistage game examined in Appendix A because of their algebraic complexity, they can be more succinctly expressed to first order in this case, for small Δ, as

$$u(i\Delta) = -B^{-1}(i\Delta)G_p^T(i\Delta)S(i\Delta)x(i\Delta) \qquad \text{(B86)}$$

$$v(i\Delta) = -C^{-1}(i\Delta)G_e^T(i\Delta)S(i\Delta)x(i\Delta) \qquad \text{(B87)}$$

$$S((i+1)\Delta) = S(i\Delta) + S(i\Delta)[G_p(i\Delta)B^{-1}(i\Delta)G_p^T(i\Delta) - G_e(i\Delta)C^{-1}(i\Delta)G_e^T(i\Delta)]S(i\Delta)\Delta\,;$$
$$S(N\Delta) = S_f \qquad \text{(B88)}$$

From the results obtained in Appendix A on the certainty-coincidence property, these necessary conditions must be satisfied by the noiseless sample path followed by the discretized stochastic game when the pair of minimax strategies determined to first order by equations (B52-75) are used by the pursuer and evader. Since this is true for all sufficiently small Δ, the limiting form of these necessary conditions, as Δ goes to zero, must be satisfied by the noiseless sample path corresponding to the limiting form of the minimax strategies, providing these limits exist. Since the solution to the differential game is treated as such a limiting form of strategy pair, this result will be applicable to the minimax noiseless sample path in the differential game.

Appendix C

THE DIFFERENTIAL GAME

The solution to the stochastic differential game originally posed is obtained by taking the limiting form of the solution to the discretized game of Appendix B as the discretization interval Δ becomes infinitesimal. With an appropriate redefinition of variables, this approach leads quite naturally to a characterization of the (linear) minimax strategies for the differential game in terms of a system of implicit integro-differential equations with split boundary conditions.

As is suggested by the remarks made in section 3E of this report, the next step in solving the problem at hand is to obtain weighting functions L_p, L_e, a_p, and a_e so that the linear strategies determined by them,

$$u(t) = -a_p(t) - \int_0^t \overline{L}_p(t, s) z_p(s)\, ds$$

and

$$v(t) = a_e(t) + \int_0^t \overline{L}_e(t, s) z_e(s)\, ds ,$$

are the limiting forms, in the sense mentioned earlier, of the minimax strategies for the discretized game. This means that $\Lambda_p(i\Delta, j\Delta)$, $\Lambda_e(i\Delta, j\Delta)$, $a_p(i\Delta)$, and $a_e(i\Delta)$, when considered as step functions with continuous arguments, should converge uniformly to these weighting functions as the discretization interval approaches zero. Such weighting functions are characterized by the system of integro-differential equations that results in the limit as $\Delta \to 0$ from the system of first-order difference equations characterizing the solution to the discretized game.

Before passing to the continuous limit, however, it is convenient to make some modifications in the notation for the discretized game solution. First, instead of referring to matrix partitions such as $\langle A(i\Delta)\rangle_{j,k}$, a new matrix function A of three arguments will be defined as

$$A(i\Delta, j\Delta, k\Delta) = \langle A(i\Delta)\rangle_{j,k} .$$

In this way, all three arguments range over the interval $[0, t_f]$ as i, j, and k take on values from 0 to N, no matter how fine the discretization is. These arguments will be considered continuous variables in the continuous limit. Second, the equations will be written in terms of normalized variables in order to avoid the impulse functions that would otherwise result in the asymptotic formulas from the inclusion of terms of varying orders of magnitudes.

These two modifications are incorporated in the following definitions, where j and k are indices taking on integer values from 0 to N:

$$T_p(i\Delta) \triangleq G_p(i\Delta)B^{-1}(i\Delta)G_p^T(i\Delta)$$

$$S_{p1}(i\Delta, j\Delta, k\Delta) \triangleq \frac{1}{\Delta^2}\langle S_{p1}(i\Delta)\rangle_{j,k} ; \quad j \neq N, \quad k \neq N$$

$$S_{p2}(i\Delta, j\Delta, k\Delta) \triangleq \frac{1}{\Delta^2}\langle S_{p2}(i\Delta)\rangle_{j,k} ; \quad j \neq N$$

$$E_{p1}(i\Delta, j\Delta) \triangleq \frac{1}{\Delta}\langle S_{p1}(i\Delta)\rangle_{j,N} ; \qquad j \neq N$$

$$E_{p2}(i\Delta, j\Delta) \triangleq \frac{1}{\Delta} \langle S_{p2}(i\Delta) \rangle_{N,j}$$

$$S_{pN}(i\Delta) \triangleq \langle S_{p1}(i\Delta) \rangle_{N,N}$$

$$P_{p1}(i\Delta, j\Delta, k\Delta) \triangleq \langle P_{p1}(i\Delta) \rangle_{j,k}$$

$$P_{p2}(i\Delta, j\Delta, k\Delta) \triangleq \langle P_{p2}(i\Delta) \rangle_{j,k}$$

$$P_{p3}(i\Delta, j\Delta, k\Delta) \triangleq \langle P_{p3}(i\Delta) \rangle_{j,k} - \frac{1}{\Delta}\, \delta(j,k) U(i-j) R_e(j)$$

$$K_{p1}(i\Delta, j\Delta, k\Delta) \triangleq \frac{1}{\Delta} \langle K_{p1}(i\Delta, j\Delta) \rangle_k$$

$$K_{p2}(i\Delta, j\Delta, k\Delta) \triangleq \frac{1}{\Delta} \langle K_{p2}(i\Delta, j\Delta) \rangle_k$$

$$\bar{L}_p(i\Delta, j\Delta) \triangleq \frac{1}{\Delta} \Lambda_p(i\Delta, j\Delta)$$

$$\gamma_{p1}(i\Delta, j\Delta) \triangleq \frac{1}{\Delta} \langle \gamma_{p1}(i\Delta) \rangle_j \; ; \quad j \neq N$$

$$\gamma_{pN}(i\Delta) \triangleq \langle \gamma_{p1}(i\Delta) \rangle_N$$

$$\delta_{p1}(i\Delta, j\Delta) \triangleq \langle \delta_{p1}(i\Delta) \rangle_j$$

$$\delta_{p2}(i\Delta, j\Delta) \triangleq \langle \delta_{p2}(i\Delta) \rangle_j$$

$$Q_p(i\Delta) \triangleq H^T(i\Delta) R_p^{-1}(i\Delta) H(i\Delta)$$

and an analogous set of definitions for the corresponding variables in the evader's associated optimal control problem.

Using this modified notation, the equations defining the linear minimax strategies in the discretized game (equations B52-79) become in the continuous limit (suppressing the "t" argument for functions of "t" only):

Primary Equation System

$$\dot{S}_{pN} = [\, S_{pN} + \int_t^{t_f} E_{p1}^T(t,\zeta) d\zeta]\; T_p [\, S_{pN} + \int_t^{t_f} E_{p1}(t,\zeta) d\zeta] \; ; \quad S_{pN}(t_f) = S_f \qquad \text{(C1)}$$

$$\frac{\partial E_{p1}(t,\tau)}{\partial t} = \left[H_e^T(\tau)\bar{L}_e^T(t,\tau)G_e^T + \left(E_{p1}(t,\tau) + \int_t^{t_f} S_{p1}(t,\tau,\zeta)d\zeta\right) T_p \right] \cdot \left[S_{pN} + \int_t^{t_f} E_{p1}(t,\zeta)d\zeta \right] ; \quad E_{p1}(t_f,\tau) = 0_{n,n} \tag{C2}$$

$$\frac{\partial E_{p2}(t,\tau)}{\partial t} = \left[S_{pN} + \int_t^{t_f} E_{p1}^T(t,\zeta)d\zeta \right] \left[T_p\left(E_{p2}(t,\tau) + \int_t^{t_f} S_{p2}(t,\zeta,\tau)d\zeta \right) + G_e\bar{L}_e(t,\tau) \right] ; \quad E_{p2}(t_f,\tau) = 0_{n,r} \tag{C3}$$

$$\begin{aligned} \frac{\partial S_{p1}(t,\tau,\sigma)}{\partial t} = {} & H_e^T(\tau)\bar{L}_e^T(t,\tau)\, C\, \bar{L}_e(t,\sigma)H_e(\sigma) + H_e^T(\tau)\bar{L}_e^T(t,\tau)G_e^T \cdot \\ & \left[E_{p1}^T(t,\sigma) + \int_t^{t_f} S_{p1}(t,\zeta,\sigma)d\zeta \right] + \left[E_{p1}(t,\tau) + \int_t^{t_f} S_{p1}(t,\tau,\zeta)d\zeta \right] \cdot \\ & G_e\bar{L}_e(t,\sigma)H_e(\sigma) + \left[E_{p1}(t,\tau) + \int_t^{t_f} S_{p1}(t,\tau,\zeta)d\zeta \right] T_p \cdot \\ & \left[E_{p1}^T(t,\sigma) + \int_t^{t_f} S_{p1}(t,\zeta,\sigma)d\zeta \right] ; \quad S_{p1}(t_f,\tau,\sigma) = 0_{n,n} \end{aligned} \tag{C4}$$

$$\begin{aligned} \frac{\partial S_{p2}(t,\tau,\sigma)}{\partial t} = {} & \left[E_{p1}(t,\tau) + \int_t^{t_f} S_{p1}(t,\tau,\zeta)d\zeta \right] G_e\bar{L}_e(t,\sigma) + H_e^T(\tau)\bar{L}_e^T(t,\tau) \cdot \\ & \left[C\bar{L}_e(t,\sigma) + G_e^T\left(E_{p2}(t,\sigma) + \int_t^{t_f} S_{p2}(t,\zeta,\sigma)d\zeta \right) \right] + \\ & \left[E_{p1}(t,\tau) + \int_t^{t_f} S_{p1}(t,\tau,\zeta)d\zeta \right] T_p \left[E_{p2}(t,\sigma) + \int_t^{t_f} S_{p2}(t,\zeta,\sigma)d\zeta \right] ; \\ & S_{p2}(t_f,\tau,\sigma) = 0_{n,r} \end{aligned} \tag{C5}$$

$$\frac{\partial P_{p1}(t,\tau,\sigma)}{\partial t} = Q - P_{p1}(t,\tau,t)Q_p P_{p1}(t,t,\sigma) - U(\tau-t)G_e \int_0^t \overline{L}_e(t,\zeta)[H_e(\zeta)P_{p1}(t,\zeta,\sigma) + P_{p2}^T(t,\sigma,\zeta)]d\zeta - U(\sigma-t)\int_0^t [P_{p2}(t,\tau,\zeta) + P_{p1}(t,\tau,\zeta)H_e^T(\zeta)] \cdot \overline{L}_e(t,\zeta)G_e^T d\zeta ; \quad P_{p1}(0,t,\sigma) = P_o \tag{C6}$$

$$\frac{\partial P_{p2}(t,\tau,\sigma)}{\partial t} = -P_{p1}(t,\tau,t)Q_p P_{p2}(t,t\ \sigma) - U(\tau-t)G_e^T\left[\overline{L}_e(t,\sigma)R_e(\sigma) + \int_0^t \overline{L}_e(t,\zeta) \cdot [H_e(\zeta)P_{p2}(t,\zeta,\sigma) + P_{p3}(t,\zeta,\sigma)]d\zeta\right]; \quad P_{p2}(0,\tau,\sigma) = 0_{n,r} \tag{C7}$$

$$\frac{\partial P_{p3}(t,\tau,\sigma)}{\partial t} = -P_{p2}^T(t,t,\tau)Q_p P_{p2}(t,t,\sigma) ; \quad P_{p3}(0,\tau,\sigma) = 0_{r,r} \tag{C8}$$

$$\frac{\partial K_{p1}(t,\tau,\sigma)}{\partial t} = -P_{p1}(t,\sigma,t)Q_p K_{p1}(t,\tau,t) - U(\sigma-t) \cdot \left[G_p \overline{L}_p(t,\tau) + G_e \int_0^t \overline{L}_e(t,\zeta)[H_e(\zeta)K_{p1}(t,\tau,\zeta) + K_{p2}(t,\tau,\zeta)]d\zeta\right]; \quad K_{p1}(t,t,\sigma) = P_{p1}(t,\sigma,t)H_p^T(t)R_p^{-1}(t) \tag{C9}$$

$$\frac{\partial K_{p2}(t,\tau,\sigma)}{\partial t} = -P_{p2}^T(t,t,\sigma)Q_p K_{p1}(t,\tau,t) ; \quad K_{p2}(t,t,\sigma) = P_{p2}^T(t,t,\sigma)H_p^T R_p^{-1} \tag{C10}$$

$$\overline{L}_p(t,\tau) = B^{-1}G_p^T \left[S_{pN} + \int_t^{t_f} E_{p1}(t,\zeta)d\zeta\right]K_{p1}(t,\tau,t_f) + \int_0^{t_f}\left(\left[E_{p1}^T(t,\nu) + \int_t^{t_f} S_{p1}(t,\zeta,\nu)d\zeta\right]K_{p1}(t,\tau,\nu) + \left[E_{p2}(t,\nu) + \int_t^{t_f} S_{p2}(t,\zeta,\nu)d\zeta\right]K_{p2}(t,\tau,\nu)\right)d\nu\Bigg\}; \quad t \geq \tau$$

$$\overline{L}_p(t,\tau) = 0_{m,q} ; \quad t < \tau \leq t_f \tag{C11}$$

An analogous set of equations for the evader's corresponding variables is also part of this equation system, the only differences being the interchanging of "B" and "C", "p" and "e", "m" and "k", "q" and "r", and the boundary condition $S_{eN}(t_f) = -S_f$.

Subsidiary Equation System

$$\dot{\gamma}_{pN} = \left[S_{pN} + \int_t^{t_f} E_{p1}^T(t,\zeta)d\zeta\right]\left[G_e a_e + T_e\left(\gamma_{pN} + \int_t^{t_f} \gamma_{p1}(t,\zeta)d\zeta\right)\right] ;$$

$$\gamma_{pN}(t_f) = 0_{n,1} \qquad \text{(C12)}$$

$$\frac{\partial\gamma_{p1}(t,\tau)}{\partial t} = H_e^T(\tau)\bar{L}_e^T(t,\tau)\left[Ca_e + G_e^T\left(\gamma_{pN} + \int_t^{t_f} \gamma_{p1}(t,\zeta)d\zeta\right)\right] +$$

$$\left[E_{p1}(t,\tau) + \int_t^{t_f} S_{p1}(t,\tau,\zeta)d\zeta\right]\left[G_e a_e + T_p\left(\gamma_{pN} + \int_t^{t_f} \gamma_{p1}(t,\zeta)d\zeta\right)\right] ;$$

$$\gamma_{p1}(t_f, \tau) = 0_{n,1} \qquad \text{(C13)}$$

$$\frac{\partial\delta_{p1}(t,\tau)}{\partial t} = -U(\tau-t)[G_e a_e + G_p a_p] - P_{p1}(t,\tau,t)Q_p\delta_{p1}(t,t) ;$$

$$\delta_{p1}(0,\tau) = \bar{x}_o \qquad \text{(C14)}$$

$$\frac{\partial\delta_{p2}(t,\tau)}{\partial t} = -P_{p2}^T(t,t,\tau)Q_p\delta_{p1}(t,t) ; \quad \delta_{p2}(0,\tau) = 0_{n,1} \qquad \text{(C15)}$$

$$a_p = B^{-1}G_p^T\left\{\gamma_{pN} + \int_t^{t_f} \gamma_{p1}(t,\zeta)d\zeta + \left[S_{pN} + \int_t^{t_f} E_{p1}(t,\zeta)d\zeta\right]\delta_{p1}(t,t_f) + \right.$$

$$\int_0^{t_f}\left(\left[E_{p1}^T(t,\nu) + \int_t^{t_f} S_{p1}(t,\zeta,\nu)d\zeta\right]\delta_{p1}(t,\nu) + \right.$$

$$\left.\left.\left[E_{p2}(t,\nu) + \int_t^{t_f} S_{p2}(t,\zeta,\nu)d\zeta\right]\delta_{p2}(t,\nu)\right)d\nu\right\} \qquad \text{(C16)}$$

and an analogous set of equations for the evader's corresponding variables.

The linear minimax control laws are given in terms of these new parameters as:

$$u = -a_p - \int_0^t \bar{L}_p(t,\tau) z_p(\tau)\, d\tau \tag{C17}$$

and

$$v = a_e + \int_0^t \bar{L}_e(t,\tau) z_e(\tau)\, d\tau \tag{C18}$$

Defining $\hat{\pi}_p(i\Delta, j\Delta) \triangleq \langle \hat{\pi}_p(i\Delta) \rangle_j$ and $\hat{\eta}_{ep}(i\Delta, j\Delta) \triangleq \langle \eta_{ep}(i\Delta) \rangle_j$ for the discretized game, and similarly for the evader, the Kalman filter equations become in the continuous limit:

$$\frac{\partial \hat{\pi}_p(t,\tau)}{\partial t} = -U(\tau - t)\left[G_e a_e - G_p u + \int_0^t G_e \bar{L}_e(t,\xi)[H_e(\xi)\hat{\pi}(t,\xi) + \hat{\eta}_{ep}(t,\xi)]d\xi\right] +$$

$$P_{p1}(t,\tau,t) H_p^T R_p^{-1} [z_p(t) - H_p \hat{\pi}_p(t,t)];$$

$$\hat{\pi}(0,\tau) = \bar{x}_o; \quad 0 \leq \tau \leq t_f \tag{C19}$$

$$\frac{\partial \hat{\eta}_{ep}(t,\tau)}{\partial t} = P_{p2}^T(t,t,\tau) H_p^T R_p^{-1} [z_p - H_p \hat{\pi}_p(t,t)]; \quad \eta_{ep}(t,t) = 0 \tag{C20}$$

$$\frac{\partial \hat{\pi}_e(t,\tau)}{\partial t} = -U(\tau - t)\left[G_p a_p + G_e v + G_p \int_0^t \bar{L}_p(t,\xi)[H_p(\xi)\hat{\pi}_e(t,\xi) + \hat{\eta}_{pe}(t,\xi)]d\xi\right] +$$

$$P_{e1}(t,\tau,t) H_e^T R_e^{-1} [z_e - H_e \hat{\pi}_e(t,t)]; \quad \hat{\pi}_e(0,\tau) = \bar{x}_o; \quad 0 \leq \tau \leq t_f \tag{C21}$$

and

$$\frac{\partial \hat{\eta}_{pe}(t,\tau)}{\partial t} = P_{e2}^T(t,t,\tau) H_e^T R_e^{-1} [z_e - H_e \hat{\pi}_e(t,t)]; \quad \hat{\eta}_{pe}(t,t) = 0\,. \tag{C22}$$

It is straightforward to verify by substitution in equations (C2) - (C5) that, if a solution to the primary equation system exists at all, then a solution is available of the form:

$$E^T_{p1}(t,\tau) = E_{p2}(t,\tau)H_e(\tau)\,; \quad 0 \le t,\tau \le t_f \tag{C23}$$

and

$$S_{p1}(t,\tau,\sigma) = S_{p2}(t,\tau,\sigma)H_e(\sigma)\,; \quad 0 \le t,\tau,\sigma \le t_f\,. \tag{C24}$$

Likewise, a solution is also available under these circumstances with the additional property:

$$E^T_{e1}(t,\tau) = E^T_{e2}(t,\tau)H_p(\tau)\,; \quad 0 \le t,\tau \le t_f \tag{C25}$$

and

$$S_{e1}(t,\tau,\sigma) = S_{e2}(t,\tau,\sigma)H_p(\sigma)\,; \quad 0 \le t,\tau,\sigma \le t_f\,. \tag{C26}$$

Henceforth, only solutions of this form will be considered.

From the symmetry of the "S" and "P" matrices in the discretized game, it follows by definition that

$$P_{p1}(t,\tau,\sigma) = P^T_{p1}(t,\sigma,\tau)\,, \qquad P_{e1}(t,\tau,\sigma) = P^T_{e1}(t,\sigma,\tau)\,,$$

$$P_{p3}(t,\tau,\sigma) = P^T_{p3}(t,\sigma,\tau)\,, \qquad P_{e3}(t,\tau,\sigma) = P^T_{e3}(t,\sigma,\tau)\,,$$

$$S_{p1}(t,\tau,\sigma) = S^T_{p1}(t,\sigma,\tau)\,, \quad \text{and} \quad S_{e1}(t,\tau,\sigma) = S^T_{e1}(t,\sigma,\tau)\,.$$

It is also of interest to note that, since by the definition of the enlarged state variables,

$$\hat{\pi}_p(t,\tau) = \hat{\pi}_p(t,t_f) = \varepsilon[x(t)/Z_p(t)] \triangleq \hat{x}_p(t) \tag{C27}$$

and

$$\hat{\eta}_{ep}(t,\tau) = \varepsilon[w_e(\tau)/Z_p(t)] = 0 \tag{C28}$$

for all $t \le \tau \le t_f$, where $Z_p(t)$ denotes the pursuer's measurement history up to time t, and since for all possible measurement histories (for any $0 \le \tau \le t_f$):

$$\hat{\pi}_p(t,\tau) = \delta_{p1}(t,\tau) + \int_0^t K_{p1}(t,\zeta,\tau)\, z_p(\zeta)\, d\zeta \tag{C29}$$

and

$$\hat{\eta}_{ep}(t,\tau) = \delta_{p2}(t,\tau) + \int_0^t K_{p2}(t,\zeta,\tau)\, z_p(\zeta)\, d\zeta\ , \tag{C30}$$

it is necessary that

$$\delta_{p1}(t,\tau) = \delta_{p1}(t,t_f)\ ; \quad 0 \leq t \leq \tau \leq t_f\ , \tag{C31}$$

$$\delta_{p2}(t,\tau) = 0\ ; \quad 0 \leq t \leq \tau \leq t_f\ , \tag{C32}$$

$$K_{p1}(t,\sigma,\tau) = K_{p1}(t,\sigma,t_f)\ ; \quad 0 \leq \sigma \leq t \leq \tau \leq t_f\ , \tag{C33}$$

and

$$K_{p2}(t,\sigma,\tau) = 0\ ; \quad 0 \leq \sigma \leq t \leq \tau \leq t_f\ . \tag{C34}$$

These observations can be used to simplify the equations characterizing the minimax strategies. Defining the new variables (where t, τ, and σ are in the interval $[0, t_f]$):

$$\Omega_p(t) \triangleq S_{pN}(t) + \int_t^{t_f} E_{p1}(t,\tau)\, d\tau\ ,$$

$$\Omega_e(t) \triangleq -S_{eN}(t) - \int_t^{t_f} E_{e1}(t,\tau)\, d\tau\ ,$$

$$\Gamma_p(t,\tau) \triangleq E_{p2}(t,\tau) + \int_t^{t_f} S_{p2}(t,\zeta,\tau)\, d\zeta\ ; \quad \tau \leq t\ ,$$

$$\Gamma_e(t,\tau) \triangleq -E_{e2}(t,\tau) - \int_t^{t_f} S_{e2}(t,\zeta,\tau)\, d\zeta\ ; \quad \tau \leq t\ ,$$

$$K_p(t,\tau) \triangleq K_{p1}(t,\tau,t) = K_{p1}(t,\tau,t_f)\ ; \quad \tau \leq t\ ,$$

$$K_e(t,\tau) \triangleq K_{e1}(t,\tau,t) = K_{e1}(t,\tau,t_f)\ ; \quad \tau \leq t\ ,$$

$$L_p(t,\tau,\sigma) \triangleq K_{p2}(t,\tau,\sigma) + H_e(\sigma) K_{p1}(t,\tau,\sigma)\ ; \quad \tau,\sigma \leq t\ ,$$

$$L_e(t,\tau,\sigma) \triangleq K_{e2}(t,\tau,\sigma) + H_p(\sigma)K_{e1}(t,\tau,\sigma); \quad \tau,\sigma \leq t,$$

$$P_p(t) \triangleq P_{p1}(t,t,t),$$

$$P_e(t) \triangleq P_{e1}(t,t,t),$$

$$M_p(t,\tau) \triangleq P_{p1}(t,t,\tau)H_e^T(\tau) + P_{p2}(t,t,\tau); \quad \tau \leq t,$$

$$M_e(t,\tau) \triangleq P_{e1}(t,t,\tau)H_p^T(\tau) + P_{e2}(t,t,\tau); \quad \tau \leq t,$$

$$N_p(t,\tau,\sigma) \triangleq H_e(\tau)P_{p1}(t,\tau,\sigma)H_e^T(\sigma) + H_e(\tau)P_{p2}(t,\tau,\sigma) + P_{p2}^T(t,\sigma,\tau)H_e^T(\sigma) + P_{p3}(t,\tau,\sigma); \quad \tau,\sigma \leq t,$$

$$N_e(t,\tau,\sigma) \triangleq H_p(\tau)P_{e1}(t,\tau,\sigma)H_p^T(\sigma) + H_p(\tau)P_{e2}(t,\tau,\sigma) + P_{e2}^T(t,\sigma,\tau)H_e^T(\sigma) + P_{e3}(t,\tau,\sigma); \quad \tau,\sigma \leq t,$$

$$\Lambda_p(t,\tau) \triangleq \left[S_{pN}(t) + \int_t^{t_f} E_{p1}(t,\zeta)d\zeta\right]K_{p1}(t,\tau,t_f) + \int_0^{t_f}\left[E_{p1}^T(t,\nu) + \int_t^{t_f} S_{p1}(t,\zeta,\nu)d\zeta\right]K_{p1}(t,\tau,\nu)\,d\nu + \int_0^{t_f}\left[E_{p2}(t,\nu) + \int_t^{t_f} S_{p2}(t,\zeta,\nu)d\zeta\right]K_{p2}(t,\tau,\nu)d\nu; \quad \tau \leq t,$$

and

$$\Lambda_e(t,\tau) \triangleq -\left[S_{eN}(t) + \int_t^{t_f} E_{e1}(t,\zeta)d\zeta\right]K_{e1}(t,\tau,t_f) - \int_0^{t_f}\left[E_{e1}^T(t,\nu) + \int_t^{t_f} S_{e1}(t,\zeta,\nu)d\zeta\right]K_{e1}(t,\tau,\nu)d\nu - \int_0^{t}\left[E_{e2}(t,\nu) + \int_t^{t_f} S_{e2}(t,\zeta,\nu)d\zeta\right]K_{e2}(t,\tau,\nu)d\nu; \quad \tau \leq t,$$

the primary equation system can be expressed in terms of these quantities. This is accomplished by straightforward substitution in the original primary equation system, the use of Leibnitz's rule for differentiating integrals, and the expression of $E_{p1}(t,t)$ (for example) as

$$E_{p1}(t,t) = E_{p1}(t_f, t) - \int_t^{t_f} \frac{\partial E_{p1}(\xi,t)}{\partial \xi} d\xi = - \int_t^{t_f} \frac{\partial E_{p1}(\xi,t)}{\partial \xi} d\xi .$$

The resulting equation system is:

$$\dot{\Omega}_p = H_e^T \int_t^{t_f} [T_p(\xi)\Gamma_p(\xi,t) - T_e(\xi)\Lambda_e(\xi,t)]^T \Omega_p(\xi) d\xi + \left[\Omega_p + \int_t^{t_f} \Gamma_p(t,\xi)H_e(\xi)d\xi\right] T_p \Omega_p ; \quad \Omega_p(t_f) = S_f \tag{C35}$$

$$\frac{\partial \Gamma_p(t,\tau)}{\partial t} = \left[\Omega_p + \int_t^{t_f} \Gamma_p(t,\xi)H_e(\xi)d\xi\right][T_p\Gamma_p(t,\tau) - T_e\Lambda_e(t,\tau)] + H_e^T \int_t^{t_f} \{ [\Gamma_p(\xi,t) - \Lambda_e(\xi,t)]^T T_e(\xi)[\Gamma_p(\xi,\tau) - \Lambda_e(\xi,\tau)] + \Gamma_p^T(\xi,t)[T_p(\xi) - T_e(\xi)]\Gamma_p(\xi,\tau) \} d\xi ; \quad \Gamma_p(t_f,\tau) = 0 ; \quad 0 \leq \tau \leq t_f \tag{C36}$$

$$\dot{P}_p = T_e\left[\int_0^t \Lambda_e(t,\xi)M_p^T(t,\xi)d\xi\right] + \left[\int_0^t \Lambda_e(t,\xi)M_p^T(t,\xi)d\xi\right]^T T_e + Q - P_p Q_p P_p ; \quad P_p(0) = P_o \tag{C37}$$

$$\frac{\partial M_p(t,\tau)}{\partial t} = T_e\left[\Lambda_e(t,\tau)R_e(\tau) + \int_0^t \Lambda_e(t,\xi)N_p^T(t,\tau,\xi)d\xi\right] - P_p Q_p M_p(t,\tau) , \quad 0 \leq \tau \leq t ; \quad M_p(t,t) = P_p H_e^T \tag{C38}$$

$$\frac{\partial N_p(t,\tau,\sigma)}{\partial t} = -M_p^T(t,\tau)Q_pM_p(t,\sigma), \quad 0 \le \tau, \sigma \le t;$$

$$N_p(t,\tau,t) = N_p^T(t,t,\tau) = M_p(t,\tau)H_e^T(\tau) \tag{C39}$$

$$\frac{\partial K_p(t,\tau)}{\partial t} = T_e \int_0^t \Lambda_e(t,\zeta)L_p(t,\tau,\zeta)d\zeta - T_p\Lambda_p(t,\tau) - P_pQ_pK_p(t,\tau),$$

$$0 \le \tau \le t; \quad K_p(t,t) = P_pH_pR_p^{-1} \tag{C40}$$

$$\frac{\partial L_p(t,\tau,\sigma)}{\partial t} = -M_p^T(t,\sigma)Q_pK_p(t,\tau), \quad 0 \le \tau, \sigma \le t;$$

$$L_p(t,t,\sigma) = M_p^T(t,\sigma)H_pR_p^{-1} \tag{C41}$$

$$\dot{\Omega}_e = H_p^T \int_t^{t_f} [T_p(\zeta)\Lambda_p(\zeta,t) - T_e(\zeta)\Gamma_e(\zeta,t)]\Omega_e(\zeta)d\zeta -$$

$$\left[\Omega_e + \int_t^{t_f} \Gamma_e(t,\zeta)H_p(\zeta)d\zeta\right] T_e\Omega_e; \quad \Omega_e(t_f) = S_f \tag{C42}$$

$$\frac{\partial \Gamma_e(t,\tau)}{\partial t} = \left[\Omega_e + \int_t^{t_f} \Gamma_e(t,\zeta)H_p(\zeta)d\zeta\right][T_p\Lambda_p(t,\tau) - T_e\Gamma_e(t,\tau)] +$$

$$H_p^T \int_t^{t_f} \Big\{\Gamma_e^T(\zeta,t)[T_p(\zeta) - T_e(\zeta)]\Gamma_e(\zeta,\tau) - [\Lambda_p(\zeta,t) - \Gamma_e(\zeta,t)]^T\, T_p(\zeta)\,\cdot$$

$$[\Lambda_p(\zeta,\tau) - \Gamma_e(\zeta,\tau)]\Big\}\, d\zeta; \quad \Gamma_e(t_f,\tau) = 0, \quad 0 \le \tau \le t_f \tag{C43}$$

$$\dot{P}_e = Q - T_p\left[\int_0^t \Lambda_p(t,\zeta)M_e^T(t,\zeta)d\zeta\right] - \left[\int_0^t \Lambda_p(t,\zeta)M_e^T(t,\zeta)d\zeta\right]^T T_p - P_eQ_eP_e;$$

$$P_e(0) = P_o \tag{C44}$$

$$\frac{\partial M_e(t,\tau)}{\partial t} = -T_p\left[\Lambda_p(t,\tau)R_p(\tau) + \int_0^t \Lambda_p(t,\zeta)N_e^T(t,\tau,\zeta)d\zeta\right] - P_eQ_eM_e(t,\tau),$$

$$0 \le \tau \le t; \quad M_e(t,t) = P_eH_p^T \tag{C45}$$

$$\frac{\partial N_e(t,\tau,\sigma)}{\partial t} = -M_e^T(t,\tau)Q_eM_e(t,\sigma), \quad 0 \le \tau, \sigma \le t;$$

$$N_e(t,\tau,t) = N_e^T(t,t,\tau) = M_e(t,\tau)H_p^T(\tau) \tag{C46}$$

$$\frac{\partial K_e(t,\tau)}{\partial t} = T_e\Lambda_e(t,\tau) - T_p\int_0^t \Lambda_p(t,\zeta)L_e(t,\tau,\zeta)d\zeta - P_eQ_eK_e(t,\tau),$$

$$0 \le \tau \le t; \quad K_e(t,t) = P_eH_e^TR_e^{-1} \tag{C47}$$

$$\frac{\partial L_e(t,\tau,\sigma)}{\partial t} = -M_e^T(t,\sigma)Q_eK_e(t,\tau), \quad 0 \le \tau, \sigma \le t;$$

$$L_e(t,t,\sigma) = M_e^T(t,\sigma)H_eR_e^{-1} \tag{C48}$$

As a consequence of these definitions, it also follows that

$$\Lambda_p(t,\tau) = \left[\Omega_p + \int_t^{t_f} \Gamma_p(t,\zeta)H_e(\zeta)d\zeta\right]K_p(t,\tau) + \int_0^t \Gamma_p(t,\zeta)L_p(t,\tau,\zeta)d\zeta; \quad \tau \le t \tag{C49}$$

and

$$\Lambda_e(t,\tau) = \left[\Omega_e + \int_t^{t_f} \Gamma_e(t,\zeta)H_p(\zeta)d\zeta\right]K_e(t,\tau) + \int_0^t \Gamma_e(t,\zeta)L_e(t,\tau,\zeta)d\zeta; \quad \tau \le t. \tag{C50}$$

The subsidiary equation system can also be simplified by making the definitions:

$$\gamma_p \triangleq \gamma_{pN} + \int_t^{t_f} \gamma_{p1}(t,\zeta)\,d\zeta$$

$$\gamma_e(t) \triangleq -\gamma_{eN} - \int_t^{t_f} \gamma_{e1}(t,\zeta)\,d\zeta$$

$$\delta_p(t) \triangleq \delta_{p1}(t,t) = \delta_{p1}(t,t_f)$$

$$\delta_e(t) \triangleq \delta_{e1}(t,t) = \delta_{e1}(t,t_f)$$

$$\theta_p(t,\tau) \triangleq \delta_{p2}(t,\tau) + H_e(\tau)\delta_{p1}(t,\tau); \quad 0 \leqslant \tau \leqslant t$$

$$\theta_e(t,\tau) \triangleq \delta_{e2}(t,\tau) + H_p(\tau)\delta_{e1}(t,\tau); \quad 0 \leqslant \tau \leqslant t$$

$$a_p(t) \triangleq \gamma_{pN}(t) + \int_t^{t_f} \gamma_{p1}(t,\zeta)d\zeta + \left[S_{pN}(t) + \int_t^{t_f} E_{p1}(t,\zeta)d\zeta\right]\delta_{p1}(t,t_f) + \int_0^t \left(\left[E_{p1}^T(t,\nu) + \int_t^{t_f} S_{p1}(t,\zeta,\nu)d\zeta\right]\delta_{p1}(t,\nu) + \left[E_{p2}(t,\nu) + \int_t^{t_f} S_{p2}(t,\zeta,\nu)d\zeta\right]\delta_{p2}(t,\nu)\right) d\nu$$

$$a_e(t) \triangleq -\nu_{eN}(t) - \int_t^{t_f} \nu_{e1}(t,\zeta)d\zeta - \left[S_{eN}(t) + \int_t^{t_f} E_{e1}(t,\zeta)d\zeta\right]\delta_{e1}(t,t_f) - \int_0^t \left[E_{e1}^T(t,\nu) + \int_t^{t_f} S_{e1}(t,\zeta,\nu)d\zeta\right]\delta_{e1}(t,\nu)d\nu - \int_0^t \left[E_{e2}(t,\nu) + \int_t^{t_f} S_{e2}(t,\zeta,\nu)d\zeta\right]\delta_{e2}(t,\nu)d\nu$$

Similar manipulations can be performed on the subsidiary equation system to produce a simplified version thereof in terms of the variables just defined. Noting, however, that the resulting equations are linear in these variables, given the solution to the primary equation system, and that the boundary conditions are proportional to $\bar{x}_0$, a solution is sought of the form:

$$\gamma_p(t) = \Pi_p(t)\bar{x}_o$$

$$\gamma_e(t) = \Pi_e(t)\bar{x}_o$$

$$\delta_p(t) = D_p(t)\bar{x}_o$$

$$\delta_e(t) = D_e(t)\bar{x}_o$$

$$\theta_p(t,\tau) = \Theta_p(t,\tau)\bar{x}_o$$

$$\theta_e(t,\tau) = \Theta_e(t,\tau)\bar{x}_o$$

$$a_p(t) = A_p(t)\bar{x}_o$$

$$a_e(t) = A_e(t)\bar{x}_o$$

Further substitution shows that a solution of this form is obtained if the following equation system is satisfied:

$$\dot{\Pi}_p = \left[\Omega_p + \int_t^{t_f} \Gamma_p(t,\zeta)H_e(\zeta)d\zeta\right][T_p\Pi_p - T_eA_e] + H_e^T \int_t^{t_f} [\Gamma_p(\zeta,t) - \Lambda_e(\zeta,t)]^T \cdot T_e(\zeta)[\Pi_p(\zeta) - A_e(\zeta)]d\zeta; \quad \Pi_p(t_f) = 0_{n,n} \tag{C51}$$

$$\dot{D}_p = T_eA_e - T_pA_p - P_pQ_pD_p; \quad D_p(0) = I_n \tag{C52}$$

$$\frac{\partial\Theta_p(t,\tau)}{\partial t} = -M_p^T(t,\tau)Q_pD_p, \quad 0 \leq \tau \leq t; \quad \Theta_p(t,t) = H_eD_p \tag{C53}$$

$$\dot{\Pi}_e = \left[\Omega_e + \int_t^{t_f} \Gamma_e(t,\zeta)H_p(\zeta)d\zeta\right][T_pA_p - T_e\Pi_e] - H_p^T \int_t^{t_f} [\Gamma_e(\zeta,t) - \Lambda_p(\zeta,t)]^T\, T_p(\zeta)[\Pi_e(\zeta) - A_p(\zeta)]d\zeta; \quad \Pi_e(t_f) = 0_{n,n} \tag{C54}$$

$$\dot{D}_e = T_e A_e - T_p A_p - P_e Q_e D_e ; \quad D_e(0) = I_n \tag{C55}$$

$$\frac{\partial \Theta_e(t,\tau)}{\partial t} = -M_e^T(t,\tau) Q_e D_e , \quad 0 \leq \tau \leq t ; \quad \Theta_e(t,t) = H_p D_e \tag{C56}$$

where

$$A_p = \Pi_p + \left[\Omega_p + \int_t^{t_f} \Gamma_p(t,\zeta) H_e(\zeta) d\zeta \right] D_p + \int_0^t \Gamma_p(t,\zeta) \Theta_p(t,\zeta) d\zeta \tag{C57}$$

and

$$A_e = \Pi_e + \left[\Omega_e + \int_t^{t_f} \Gamma_e(t,\zeta) H_p(\zeta) d\zeta \right] D_e + \int_0^t \Gamma_e(t,\zeta) \Theta_e(t,\zeta) d\zeta . \tag{C58}$$

The minimax control laws, when expressed in terms of these new variables, are:

$$u = -B^{-1} G_p^T \left[A_p \bar{x}_o + \int_0^t \Lambda_p(t,\tau) z_p(\tau) d\tau \right] \tag{C59}$$

and

$$v = -C^{-1} G_e^T \left[A_e \bar{x}_o + \int_0^t \Lambda_e(t,\tau) z_e(\tau) d\tau \right] . \tag{C60}$$

As a further consequence of the definitions of these new variables, it should be noted that the following interpretations can be made, assuming that these minimax strategies are employed:

$$P_p(t) = \text{cov}[x(t), x(t)/Z_p(t)]$$

$$M_p(t,\tau) = \text{cov}[x(t), z_e(\tau)/Z_p(t)] ; \quad 0 \leq \tau \leq t$$

$$N_p(t,\tau,\sigma) = \text{cov}[z_e(\tau), z_e(\sigma)/Z_p(t)] ; \quad 0 \leq \tau, \sigma \leq t$$

$$P_e(t) = \text{cov}[x(t), x(t)/Z_e(t)]$$

$$M_e(t,\tau) = \text{cov}[x(t), z_p(\tau)/Z_e(t)] ; \quad 0 \leq \tau \leq t$$

$$N_e(t,\tau,\sigma) = \text{cov}[z_p(\tau), z_p(\sigma)/Z_e(t)] ; \quad 0 \leq \tau, \sigma \leq t .$$

Also, the Kalman filter estimates can be expressed in the following way:

$$\hat{x}_p(t) \triangleq \hat{\pi}_p(t,t) = D_p(t)\bar{x}_o + \int_0^t K_p(t,\tau)z_p(\tau)\,d\tau \,, \tag{C61}$$

$$\hat{x}_e(t) \triangleq \hat{\pi}_e(t,t) = D_e(t)\bar{x}_o + \int_0^t K_e(t,\tau)z_e(\tau)\,d\tau \,, \tag{C62}$$

$$\hat{z}_{ep}(\tau/t) \triangleq H_e(\tau)\hat{\pi}_p(t,\tau) + \hat{\eta}_{ep}(t,\tau) = \Theta_p(t,\tau)\bar{x}_o + \int_0^t L_p(t,\zeta,\tau)z_p(\zeta)d\zeta \,, \tag{C63}$$

$$\hat{z}_{pe}(\tau/t) \triangleq H_p(\tau)\hat{\pi}_e(t,\tau) + \hat{\eta}_{pe}(t,\tau) = \Theta_e(t,\tau)\bar{x}_o + \int_0^t L_e(t,\zeta,\tau)z_e(\zeta)d\zeta \,. \tag{C64}$$

The differential equations for the Kalman filter estimates just defined can be obtained from equations (C19)-(C22) as:

$$\dot{\hat{x}}_p = \frac{\partial\hat{\pi}_p(t,t_f)}{\partial t} = G_p u + T_e\left[A_e\bar{x}_o + \int_0^t \Lambda_e(t,\tau)\hat{z}_{ep}(\tau/t)d\tau\right] + P_pH_p^TR_p^{-1}[z_p - H_p\hat{x}_p]; \quad \hat{x}_p(0) = \bar{x}_o \,, \tag{C65}$$

$$\dot{\hat{x}}_e = \frac{\partial\hat{\pi}_e(t,t_f)}{\partial t} = -G_e v - T_p\left[A_p\bar{x}_o + \int_0^t \Lambda_p(t,\tau)\hat{z}_{pe}(\tau/t)d\tau\right] + P_eH_e^TR_e^{-1}[z_e - H_e\hat{x}_e]; \quad \hat{x}_e(0) = \bar{x}_o \,, \tag{C66}$$

$$\frac{\partial\hat{z}_{ep}(\tau/t)}{\partial t} = H_e(\tau)\frac{\partial\hat{\pi}_p(t,\tau)}{\partial t} + \frac{\partial\hat{\eta}_{ep}(t,\tau)}{\partial t} = M_p^T(t,\tau)H_p^TR_p^{-1}[z_p - H_p\hat{x}_p];$$
$$\hat{z}_{ep}(t/t) = H_e(t)\hat{x}_p(t); \quad 0 \le \tau \le t \,, \tag{C67}$$

and

$$\frac{\partial\hat{z}_{pe}(\tau/t)}{\partial t} = H_p(\tau)\frac{\partial\hat{\pi}_e(t,\tau)}{\partial t} + \frac{\partial\hat{\eta}_{pe}(t,\tau)}{\partial t} = M_e^T(t,\tau)H_e^TR_e^{-1}[z_e - H_e\hat{x}_e];$$
$$\hat{z}_{pe}(\tau/t) = H_p(t)\hat{x}_e(t); \quad 0 \le \tau \le t \,. \tag{C68}$$

Finally, the necessary conditions for the minimax trajectory in the corresponding deterministic game, given in the discretized case by equations (B84) and (B86)-(B88), become in the continuous limit:

$$u = -B^{-1}G_p^T Sx \ , \tag{C69}$$

$$v = -C^{-1}G_e^T Sx \ , \tag{C70}$$

$$\dot{x} = G_p u - G_e v \ ; \quad x(0) = \bar{x}_o \ , \tag{C71}$$

and

$$\dot{S} = S[T_p - T_e]S \ ; \quad S(t_f) = S_f \ . \tag{C72}$$

The necessary conditions given by equations (C69)-(C72) must also be satisfied by the noiseless sample path of the differential game when the linear minimax strategies given by equations (C59) and (C60) are in effect, since the differential game is treated as the limiting case of the discretized game. It is reassuring that the necessary conditions for the noiseless sample path obtained by this analysis agree with the established solution for the corresponding deterministic game as given by equations (1) and (5)-(7).

Appendix D

A LIMITING CASE

In Rhodes and Luenberger [8] it is shown that the solution to the game in which there is no process noise ($Q = 0$) and the evader has no measurements ($R_e = \infty$) is given by:

$$u = -B^{-1}G_p^T[D\hat{x}_p + (S-D)\hat{x}_e] \tag{D1}$$

and

$$v = -C^{-1}G_e^T S\hat{x}_e \tag{D2}$$

where

$$\dot{\hat{x}}_p = G_p u - G_e v + MH_p^T R_p^{-1}[z_p - H_p\hat{x}_p]; \quad \hat{x}_p(0) = \bar{x}_o \tag{D3}$$

$$\dot{\hat{x}}_e = [T_e - T_p]S\hat{x}_e; \quad \hat{x}_e(0) = \bar{x}_o \tag{D4}$$

$$\dot{S} = S[T_p - T_e]S, \quad S(t_f) = S_f \tag{D5}$$

$$\dot{D} = DT_pD; \quad D(t_f) = S_f \tag{D6}$$

$$\dot{M} = -MQ_pM; \quad M(0) = P_o \tag{D7}$$

and where T_p, T_e, Q_p, and Q_e are defined as before. This solution is the same as that obtained from the solution derived in this report by formally letting $Q = 0$ and $R_e = \infty$, as is shown below.

1. The Evader

Looking at Realization III, it is clear from equations (84) and (85) that, since $R_e^{-1} = 0$, the driving terms are zero and hence $L_e \equiv 0$ and

$e_e \equiv 0$. From equation (83), therefore,

$$\dot{\hat{x}}_e = -G_e v - T_p S \hat{x}_e \; ; \quad \hat{x}_e(0) = \bar{x}_o \; , \qquad \text{(D8)}$$

where S is given by equation (D5). From equation (76),

$$v = -C^{-1} G_e^T S \hat{x}_e \; , \qquad \text{(D9)}$$

so that equation (D8) can be rewritten as equation (D4). With equation (D9), this implies that the two evasion strategies agree.

2. The Pursuer

Since $R_e = \infty$, it follows from equations (30), (32), and (34) that $K_e = 0$, $L_e = 0$, and $\Lambda_e = 0$ is a solution to these equations. Therefore, from equation (21), $\Gamma_p = 0$. From Realization I, and the fact that the evasion strategies are the same,

$$v = -C^{-1} G_e^T A_e \bar{x}_o \qquad \text{(equation (18))}$$

$$= -C^{-1} G_e^T S \hat{x}_e \qquad \text{(equation (D2))}$$

$$= -C^{-1} G_e^T S_o \bar{x}_o \qquad \text{(equation (63))}$$

for any $\bar{x}_o$. Therefore,

$$C^{-1} G_e^T A_e = C^{-1} G_e^T S_o \; . \qquad \text{(D10)}$$

Since $\Gamma_p = 0$, it follows from equations (19), (35), and (D6) that

$$Y_p = \Omega_p = D \; . \qquad \text{(D11)}$$

Since it follows from equations (D4) and (64) that $\hat{x}_e = S^{-1} S_o \bar{x}_o$, equation (D1) can be rewritten as

$$u = -B^{-1} G_p^T [D \hat{x}_p + (S_o - D S^{-1} S_o) \bar{x}_o] \; . \qquad \text{(D12)}$$

Also,

$$\frac{d}{dt}[S_o - DS^{-1}S_o] = -\dot{D}S^{-1}S_o + DS^{-1}\dot{S}S^{-1}S_o$$

$$= D[T_p(S_o - DS^{-1}S_o) - T_eS_o]. \quad \text{(D13)}$$

Since equation (D10) holds and $D(t_f) = S(t_f)$, a comparison of equations (D13) and (37) shows that

$$\Pi_p = (S_o - DS^{-1}S_o). \quad \text{(D14)}$$

Using equations (D11) and (D14) in equation (61) and the fact that $\Gamma_p = 0$, Realization II of the pursuer's minimax strategy can be expressed as

$$u = -B^{-1}G_p^T[D\hat{x}_p + (S_o - DS^{-1}S_o)\bar{x}_o],$$

which agrees with the strategy obtained by Rhodes and Luenberger as expressed by equation (D12) <u>if the $\hat{x}_p$'s are the same in both cases</u>.

In the solution obtained in this report, the estimate $\hat{x}_p$ is generated by equation (55), which in this case reduces to

$$\dot{\hat{x}}_p = G_pu + T_eA_e\bar{x}_o + P_pH_p^TR_p^{-1}[z_p - H_p\hat{x}_p]; \quad \hat{x}_p(0) = \bar{x}_o. \quad \text{(D15)}$$

From equations (D2), (D10), and (63), the second term in equation (D15) is just $-G_ev$, so that equation (D15) can be written as

$$\dot{\hat{x}}_p = G_pu - G_ev + P_pH_p^TR_p^{-1}[z_p - H_p\hat{x}_p]; \quad \hat{x}_p(0) = \bar{x}_o, \quad \text{(D16)}$$

which agrees with equation (D3) if $P_p = M$. But, from equation (23) and the fact that $\Lambda_e = 0$,

$$\dot{P}_p = -P_pQ_pP_p; \quad P_p(0) = P_o, \quad \text{(D17)}$$

which agrees with equation (D7) for M. Therefore, the pursuit strategies agree.

The demonstration is similar for the case in which the pursuer has no measurements.