AD

AD 680235

USAAVLABS TECHNICAL REPORT 68-70

STATIC AND DYNAMIC VISCOELASTIC BEHAVIOR OF FIBER REINFORCED MATERIALS AND STRUCTURES

By

Z. Hashin

M. A. Brull

T. Y. Chu

Z. Zudans

October 1968

U. S. ARMY AVIATION MATERIEL LABORATORIES
FORT EUSTIS, VIRGINIA

CONTRACT DAAJ02-67-C-0050

THE FRANKLIN RESEARCH INSTITUTE LABORATORIES

PHILADELPHIA, PENNSYLVANIA

136

DEPARTMENT OF THE ARMY
U. S. ARMY AVIATION MATERIEL LABORATORIES
FORT EUSTIS, VIRGINIA 23604

This program was carried out under Contract DAAJ02-67-C-0050 with The Franklin Research Institute Laboratories.

The data contained in this report are the result of research conducted to obtain a better understanding of the viscoelastic behavior of fiber reinforced materials and the time dependence of this behavior in fiber reinforced viscoelastic structures.

The report has been reviewed by the U.S. Army Aviation Materiel Laboratories and is considered to be technically sound. It is published for the exchange of information and the stimulation of future research.

Task 1F162204A17001
Contract DAAJ02-67-C-0050
USAAVLABS Technical Report 68-70
October 1968

STATIC AND DYNAMIC VISCOELASTIC BEHAVIOR OF FIBER REINFORCED MATERIALS AND STRUCTURES

Final Report

F-C2041

by

Z. Hashin
M. A. Brull
T. Y. Chu
Z. Zudans

Prepared by

The Franklin Institute Research Laboratories
Philadelphia, Pennsylvania

for

U. S. Army Aviation Materiel Laboratories
Fort Eustis, Virginia

SUMMARY

Time-dependent properties of uniaxially fiber reinforced materials composed of linear viscoelastic matrix and elastic fibers are investigated. Sample calculations are given for the static and dynamic properties of a viscoelastic fiber reinforced material. In addition, the behavior of fiber reinforced viscoelastic structures is investigated, and a number of practical problems for beams, plates, and shells, subjected to static and dynamic loadings, are analyzed.

BLANK PAGE

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF ILLUSTRATIONS (Cont'd)

LIST OF SYMBOLS

A	Cross-sectional area
$A^*_{\alpha\beta\gamma\lambda}$	Elasticity tensor for plane stress theory referred to the plate axes
$\bar{A}_{\alpha\beta\gamma\lambda}$	$A^*_{\alpha\beta\gamma\lambda}$ referred to the material axes
$B^*_{ijk\ell}$	See Equation (6)
B^*_{ij}	Nonvanishing components of $B^*_{ijk\ell}$
$B^*_{\alpha\beta\gamma\lambda}$	Plate stiffness tensor referred to the plate axes
$\bar{B}_{\alpha\beta\gamma\lambda}$	Plate stiffness tensor referred to the material axes
c	Volume fraction of fiber
C	Shear factor
C'	Integration constants
$C^*_{ijk\ell}$	Effective relaxation moduli
$\hat{C}^*_{ijk\ell}$	Laplace transform of $C^*_{ijk\ell}$
C^*_{ij}	Nonvanishing components of $C^*_{ijk\ell}$
$\hat{C}^*_{ij}$	Nonvanishing components of $\hat{C}^*_{ijk\ell}$
CCA	Composite cylinder assemblage
$D_{ijk\ell}$	Complex moduli
$D^R_{ijk\ell}$	Real part of $D_{ijk\ell}$
$D^I_{ijk\ell}$	Imaginary part of $D_{ijk\ell}$
$D^*_{ijk\ell}$	Effective complex moduli
D^*_{ij}	Nonvanishing components of $D^*_{ijk\ell}$

E	Young's modulus
E_a^*	Effective axial Young's relaxation modulus
$\hat{E}_a^*$	Laplace transform of E_a^*
e_a^*	Effective axial Young's creep compliance
$\hat{e}_a^*$	Laplace transform of e_a^*
$\check{E}_a^*$	Effective complex axial Young's modulus
e_{ij}	Deviatoric components of the strain tensor
f	Known function
FRM	Fiber reinforced material
g	Known function
G	Shear modulus
G_a^*	Effective axial shear relaxation modulus
G_T^*	Effective transverse shear relaxation modulus
$\hat{G}_a^*$	Laplace transform of G_a^*
$\hat{G}_T^*$	Laplace transform of G_T^*
g_a^*	Effective axial shear creep compliance
g_T^*	Effective transverse shear creep compliance
$\hat{g}_a^*$	Laplace transform of g_a^*
$\hat{g}_T^*$	Laplace transform of g_T^*
$\check{G}_a^*$	Effective complex axial shear modulus

LIST OF SYMBOLS (Cont'd)

$\check{G}_T^*$ Effective complex transverse shear modulus

H Heaviside unit step function

h Plate thickness

I Moment of inertia

I_m Matrix bulk creep compliance

Im Imaginary part of a quantity

$\hat{I}_m$ Laplace transform of I_m

$J_{ijk\ell}$ See Equation (7)

J_m Matrix shear creep compliance

$\hat{J}_m$ Laplace transform of J_m

K_T^* Effective plane strain bulk relaxation modulus

k_T^* Effective plane strain bulk creep compliance

$\hat{K}_T^*$ Laplace transform of K_T^*

$\hat{k}_T^*$ Laplace transform of k_T^*

k Plane strain bulk modulus

K_m Matrix bulk relaxation modulus

$\hat{K}_m$ Laplace transform of K_m

$\check{K}_T^*$ Effective complex plane strain bulk modulus

$\check{K}_m$ Complex bulk modulus of matrix

$\check{k}_m$ Complex plane strain bulk modulus of matrix

LIST OF SYMBOLS (Cont'd)

ℓ	Bar length
LT	Laplace transform
LT^{-1}	Inverse Laplace transform
$M_{\alpha\beta}$	Stress couples
n_i	Components of outward normal
n	Integer
p	Laplace transform variable
P_o	Axial load in gap
Q_α	Shear force
q_{mn}	Time dependence of displacement
R_e	Real part of a complex quantity
RVE	Representative volume element
S	Surface
s	Stress deviator
S_u	Surface on which displacement is prescribed
S_T	Surface on which traction is prescribed
$S^*_{ijk\ell}$	Effective creep compliance
$\bar{S}^*_{ijk\ell}$	Laplace transform of $S^*_{ijk\ell}$
s_{ij}	Deviatoric components of the stress tensor
TD	Transform domain
T_i	Traction
t	Time
u_i	Displacements
U	Displacement mode

LIST OF SYMBOLS (Cont'd)

V	Volume
V_α	Shear force
v	Volume fraction
W	Displacement
w	Deflection
x_1,x_2,x_3	Cartesian coordinate system defined by Figure 1
$\underline{x}$	Abbreviation of x_1,x_2,x_3

GREEK SYMBOLS

σ_{ij}	Stress tensor
$\bar{\sigma}_{ij}$	Average stress tensor
σ	Isotropic part of the stress tensor
$\bar{\varepsilon}_{ij}$	Average strain tensor
ε	Isotropic part of the strain tensor
ε_{ij}	Strain tensor
$\hat{\bar{\sigma}}_{ij}$	Laplace transform of $\bar{\sigma}_{ij}$
$\hat{\bar{\varepsilon}}_{ij}$	Laplace transform of $\bar{\varepsilon}_{ij}$
$\bar{\sigma}$	Uniform average stress
$\bar{\varepsilon}$	Uniform average strain
λ	Lamé modulus
ν	Poisson's ratio
κ_m	See Equation (54)
Γ_m	See Equation (55)
τ	Dummy variable

LIST OF SYMBOLS (Cont'd)

ω	Frequency
ω_n	Resonant frequency
δ	Loss angle
ρ	Density
ι	$\sqrt{-1}$
$\phi(t)$	A function
ϑ	Torsion mode
ξ	Axial coordinate
$\tilde{\nu}_a^*$	Transform domain Poisson's ratio

SUBSCRIPTS

f	Fiber
i	Vector component
$ijk\ell$	Tensor component
m	Matrix

SUPERSCRIPTS

E	Elastic
I	Imaginary part of a complex quantity
R	Real part of a complex quantity
VE	Viscoelastic
*	Effective values
$\hat{}$	Laplace transformed quantities
$\check{}$	Complex quantities
$\tilde{}$	Amplitude of periodic functions

BLANK PAGE

1. INTRODUCTION

The purpose of the present investigation is to study the dynamic behavior of uniaxially fiber reinforced material (FRM) for the case of elastic fibers and linear viscoelastic matrix.

A large number of investigations have dealt with the static elastic behavior of uniaxially FRM. Of these, the most fruitful approach, in the writer's opinion, has been the analysis given by Hashin and Rosen [1], which was based on a geometrical model named the composite cylinder assemblage (CCA). The advantages of this approach are here briefly summarized:

(a) The geometrical model is based on a random (though special) geometry.

(b) The mathematical analysis of the model is rigorous.

(c) The analysis yields simple closed expressions for four of the five elastic moduli of the CCA.

(d) Experimental verification of the results has been excellent.

The relative simplicity of the approach makes it possible to extend the approach to more complicated cases. Thus, the CCA model has been used successfully by Hashin et al [2], to analyze the elastic properties of biaxially FRM. Another extension which is of primary importance for present purposes is a recent analysis by Hashin [3] of the static viscoelastic properties of a uniaxially FRM on the basis of the CCA model. Our purpose here is to use the same model for prediction of the dynamic properties of a viscoelastic uniaxially FRM. Such a study is of primary practical importance since:

(a) In many cases, the matrix of a FRM is time dependent and, to a good approximation, linearly viscoelastic.

(b) Frequently, structural elements of FRM are exposed to vibrations.

Vibration analysis on the basis of elastic properties will predict resonant frequencies that should be avoided. However, the time-dependent and dissipative nature of the matrix produces damping of vibrations, which is of beneficial effect. Free vibrations will rapidly die out, while forced vibrations will have a less dangerous character than that predicted on the basis of pure elastic behavior. Thus, vibration analysis which takes into account the viscoelastic nature of the materials should lead to more realistic design criteria.

2. SUMMARY OF STATIC VISCOELASTIC BEHAVIOR OF UNIAXIALLY FRM

The present summary is based on the results obtained in Reference [3]. The static stress-strain relations of any anisotropic heterogeneous viscoelastic material may be written in the form

$$\bar{\sigma}_{ij}(t) = \int_0^t C^*_{ijk\ell}(t - \tau) \frac{d\bar{\varepsilon}_{k\ell}(\tau)}{d\tau} , \qquad (1)$$

$$\bar{\varepsilon}_{ij}(t) = \int_0^t S^*_{ijk\ell}(t - \tau) \frac{d\bar{\sigma}_{k\ell}(\tau)}{d\tau} , \qquad (2)$$

where $\bar{\sigma}_{ij}$ are average stresses, $\bar{\varepsilon}_{ij}$ are average strains, $C^*_{ijk\ell}$ are the effective relaxation moduli, and $S^*_{ijk\ell}$ are the effective creep compliances. The range of subscripts is 1, 2, 3, and repeated subscripts indicated summation over 1, 2, 3. It should be borne in mind that because of the dual nature of (1) and (2), the $C^*_{ijk\ell}$ and $S^*_{ijk\ell}$ are related. This relationship will be given later. It is convenient to take the Laplace transform (LT) of (1) and (2). We define the LT, $\hat{\phi}(p)$ of a function $\phi(t)$ by

$$\hat{\phi}(p) = \int_0^\infty e^{-pt} \phi(t) \, dt. \qquad (3)$$

By use of the convolution theorem and the rules for LT of derivatives, the LT of (1) and (2) assumes the form

$$\hat{\bar{\sigma}}_{ij}(p) = p \, \hat{C}^*_{ijk\ell}(p) \, \hat{\bar{\varepsilon}}_{k\ell}(p) , \qquad (4)$$

$$\hat{\bar{\varepsilon}}_{ij}(p) = p \, \hat{S}^*_{ijk\ell}(p) \, \hat{\bar{\sigma}}_{k\ell}(p) . \qquad (5)$$

It is now seen that the p $\hat{C}^*_{ijk\ell}$ and p $\hat{S}^*_{ijk\ell}$ matrices must be reciprocal. This is the relation between the effective relaxation moduli and the creep compliances. This relationship can be transformed to the time domain. However, it then becomes very complicated and is not very useful in that form.

It is convenient to define the following quantities:

$$B^*_{ijk\ell}(p) = p \, \hat{C}^*_{ijk\ell}(p) , \qquad (6)$$

$$J^*_{ijk\ell}(p) = p \, \hat{S}^*_{ijk\ell}(p) . \qquad (7)$$

Then (4) and (5) assume the form

$$\hat{\bar{\sigma}}_{ij}(p) = B^*_{ijk\ell}(p)\ \hat{\bar{\varepsilon}}_{k\ell}(p) \ , \tag{8}$$

$$\hat{\bar{\varepsilon}}_{ij}(p) = J^*_{ijk\ell}(p)\ \hat{\bar{\sigma}}_{k\ell}(p) \ . \tag{9}$$

The relationships (8) and (9) bear a formal resemblance to elasticity stress-strain laws. Therefore, the quantities $B^*_{ijk\ell}$ are called transform domain (TD) moduli, and the $J^*_{ijk\ell}$ are called TD compliances.

The results given so far apply for any anisotropic viscoelastic material. A uniaxially fiber reinforced material is transversely isotropic, with respect to an axis in fiber direction. As a consequence of this symmetry, it may be shown that, as in the elastic case, there are only five independent material functions (i.e., relaxation moduli or creep compliances). The relation (8) can then be simplified to the form

$$\hat{\bar{\sigma}}_{11}(p) = B^*_{11}(p)\ \hat{\bar{\varepsilon}}_{11}(p) + B^*_{12}(p)\ \hat{\bar{\varepsilon}}_{22}(p) + B^*_{12}(p)\ \hat{\bar{\varepsilon}}_{33}(p) \ , \tag{10}$$

$$\hat{\bar{\sigma}}_{22}(p) = B^*_{12}(p)\ \hat{\bar{\varepsilon}}_{11}(p) + B^*_{22}(p)\ \hat{\bar{\varepsilon}}_{22}(p) + B^*_{23}(p)\ \hat{\bar{\varepsilon}}_{33}(p) \ , \tag{11}$$

$$\hat{\bar{\sigma}}_{33}(p) = B^*_{12}(p)\ \hat{\bar{\varepsilon}}_{11}(p) + B^*_{23}(p)\ \hat{\bar{\varepsilon}}_{22}(p) + B^*_{22}(p)\ \hat{\bar{\varepsilon}}_{33}(p) \ , \tag{12}$$

$$\hat{\bar{\sigma}}_{12}(p) = 2B^*_{44}(p)\ \hat{\bar{\varepsilon}}_{12}(p) \ , \tag{13}$$

$$\hat{\bar{\sigma}}_{13}(p) = 2B^*_{44}(p)\ \hat{\bar{\varepsilon}}_{13}(p) \ , \tag{14}$$

$$\hat{\bar{\sigma}}_{23}(p) = \left[B^*_{22}(p) - B^*_{23}(p)\right]\ \hat{\bar{\varepsilon}}_{23}(p) \ , \tag{15}$$

where x_1 is fiber direction and x_2, x_3 are transverse directions (Figure 1). A similar simplification holds for (9). It is seen that (10) through (15) contain the five independent quantities B^*_{11}, B^*_{12}, B^*_{22}, B^*_{23}, and B^*_{44}. In view of (6), we may define the quantities $\hat{C}^*_{11}$, $\hat{C}^*_{12}$, $\hat{C}^*_{22}$, $\hat{C}^*_{23}$, and $\hat{C}^*_{44}$, which are related to the former quantities by

$$B^*_{mn}(p) = p\hat{C}^*_{mn}(p) \quad ; \quad mn = 11,\ 22,\ 23,\ 44. \tag{16}$$

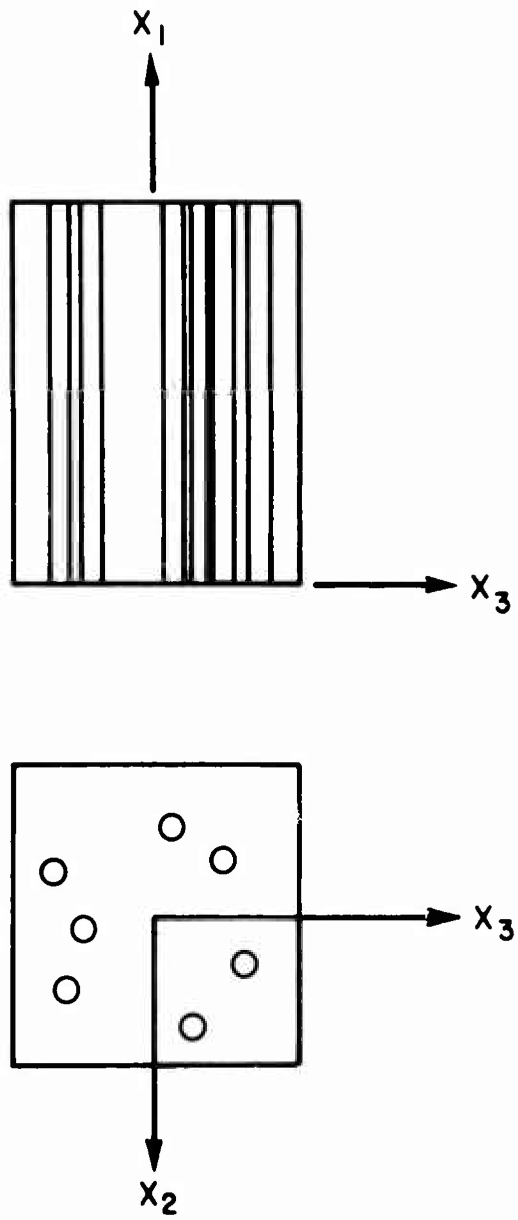

Figure 1. Geometrical Model of the Material.

When (16) is inserted into (10) through (15) and inverted to the time domain, we have

$$\bar{\sigma}_{11}(t) = \int_0^t \left[C_{11}^*(t-\tau) \frac{d\bar{\varepsilon}_{11}(\tau)}{d\tau} + C_{12}^*(t-\tau) \frac{d\bar{\varepsilon}_{22}(\tau)}{d\tau} + C_{12}^*(t-\tau) \frac{d\bar{\varepsilon}_{33}(\tau)}{d\tau} \right] d\tau \,, \tag{17}$$

$$\bar{\sigma}_{22}(t) = \int_0^t \left[C_{12}^*(t-\tau) \frac{d\bar{\varepsilon}_{11}(\tau)}{d\tau} + C_{22}^*(t-\tau) \frac{d\bar{\varepsilon}_{22}(\tau)}{d\tau} + C_{23}^*(t-\tau) \frac{d\bar{\varepsilon}_{33}(\tau)}{d\tau} \right] d\tau \,, \tag{18}$$

$$\bar{\sigma}_{23}(t) = \int_0^t \left[C_{12}^*(t-\tau) \frac{d\bar{\varepsilon}_{11}(\tau)}{d\tau} + C_{23}^*(t-\tau) \frac{d\bar{\varepsilon}_{22}(\tau)}{d\tau} + C_{22}^*(t-\tau) \frac{d\bar{\varepsilon}_{33}(\tau)}{d\tau} \right] d\tau \,, \tag{19}$$

$$\bar{\sigma}_{12}(t) = 2 \int_0^t C_{44}^*(t-\tau) \frac{d\bar{\varepsilon}_{12}(\tau)}{d\tau} \, d\tau \,, \tag{20}$$

$$\bar{\sigma}_{13}(t) = 2 \int_0^t C_{44}^*(t-\tau) \frac{d\bar{\varepsilon}_{13}(\tau)}{d\tau} \, d\tau \,, \tag{21}$$

$$\bar{\sigma}_{23}(t) = \int_0^t \left[C_{22}^*(t-\tau) - C_{23}^*(t-\tau) \right] \frac{d\bar{\varepsilon}_{23}(\tau)}{d\tau} \, d\tau \,. \tag{22}$$

We can write similar relations for expressions of strains in terms of stresses. Thus, for example, the analogue of (17) would be

$$\bar{\varepsilon}_{11}(t) = \int_0^t \left\{ S_{11}^*(t-\tau) \frac{d\bar{\sigma}_{11}(\tau)}{d\tau} + S_{12}^*(t-\tau) \frac{d\bar{\sigma}_{22}(\tau)}{d\tau} + S_{12}^*(t-\tau) \frac{d\bar{\sigma}_{33}(\tau)}{d\tau} \right\} d\tau \,, \tag{23}$$

etc.

The relations of strains in terms of stresses involve the five effective creep compliances S_{11}^*, S_{12}^*, S_{22}^*, S_{23}^*, S_{44}^*, which are related to the five relaxation moduli.

The above choice of effective relaxation moduli and creep compliances is not very advantageous from a physical point of view. To bring out physically significant quantities, it is necessary to use special states of average strain and average stress. The simplest situations are presented by (20), (21), and (22), which all correspond to pure shears. We adopt the notation

$$C^*_{44}(t) = G^*_a(t) \ , \tag{24}$$

$$C^*_{22}(t) - C^*_{23}(t) = 2G^*_T(t) \ . \tag{25}$$

It is then seen that (20) and (21) are the stress histories produced by shearing strain histories, where the shears are in planes parallel to the fibers. Thus, $G^*_a(t)$ is the effective *axial* relaxation shear modulus.

By similar reasoning, $G^*_T(t)$ is the effective *transverse* relaxation shear modulus, for shear in planes normal to the fibers.

The creep compliances $g^*_a(t)$ and $g^*_T(t)$ correspond to (24) and (25). The meaning of these is brought out by the stress-strain relations

$$\bar{\varepsilon}_{12}(t) = \frac{1}{2}\int_o^t g^*_a(t-\tau)\,\frac{d\bar{\sigma}_{12}(\tau)}{d\tau}\,d\tau \ , \tag{26}$$

$$\bar{\varepsilon}_{13}(t) = \frac{1}{2}\int_o^t g^*_a(t-\tau)\,\frac{d\bar{\sigma}_{13}(\tau)}{d\tau}\,d\tau \ , \tag{27}$$

$$\bar{\varepsilon}_{23}(t) = \frac{1}{2}\int_o^t g^*_T(t-\tau)\,\frac{d\bar{\sigma}_{23}(\tau)}{d\tau}\,d\tau \ . \tag{28}$$

Note that the Laplace transforms of (20) through (22), (24), (25), and (26) through (28) yield the relations

$$p\hat{G}^*_a(p) = \frac{1}{p\ \hat{g}^*_a(p)} \ , \tag{29}$$

$$p\hat{G}^*_T(p) = \frac{1}{p\ \hat{g}^*_T(p)} \ . \tag{30}$$

We now turn to the more complicated relations, (17) through (19), between normal average stresses and strains. First, assume the average state of strain to be

$$\bar{\varepsilon}_{22}(t) = \bar{\varepsilon}_{33}(t) = \bar{\varepsilon}(t) \ ,$$

$$\bar{\varepsilon}_{11}(t) = \bar{\varepsilon}_{12}(t) = \bar{\varepsilon}_{13}(t) = \bar{\varepsilon}_{23}(t) = 0 \ . \tag{31}$$

This is an isotropic plane strain system, in planes normal to the fibers. Introducing (31) into (17) through (19), we have

$$\bar{\sigma}_{22}(t) = \bar{\sigma}_{33}(t) = \bar{\sigma}(t) \ , \tag{32}$$

$$\bar{\sigma}(t) = 2 \int_0^t K_T^*(t-\tau) \frac{d\bar{\varepsilon}(\tau)}{d\tau} d\tau \ , \tag{33}$$

$$2K_T^*(t) = C_{22}^*(t) + C_{23}^*(t) \ . \tag{34}$$

Consequently, $K_T^*(t)$ is called the effective plane strain relaxation bulk modulus.

If we assume the average state of stress to be

$$\bar{\sigma}_{22}(t) = \bar{\sigma}_{33}(t) = \bar{\sigma}(t) \ , \tag{35}$$

$$\bar{\sigma}_{13}(t) = \bar{\sigma}_{12}(t) = \bar{\sigma}_{23}(t) = 0 \ , \tag{36}$$

and retain the plane strain condition

$$\bar{\varepsilon}_{11}(t) = 0 \ , \tag{37}$$

then it is easily shown that

$$\bar{\varepsilon}_{22}(t) = \bar{\varepsilon}_{33}(t) = \bar{\varepsilon}(t) \ , \tag{38}$$

$$\bar{\varepsilon}(t) = \frac{1}{2} \int_0^t k_T^*(t-\tau) \frac{d\bar{\sigma}(\tau)}{d\tau} d\tau \ , \tag{39}$$

where $k_T^*(t)$ is the effective plane strain bulk creep compliance. It is related to K_T^* by

$$p\hat{K}_T^*(p) = \frac{1}{p\hat{k}_T^*(p)} \ . \tag{40}$$

Now assume the mixed state of average stress and strain to be

$$\bar{\varepsilon}_{11}(t) = \bar{\varepsilon}(t) \ , \tag{41}$$

$$\bar{\sigma}_{22}(t) = \bar{\sigma}_{33}(t) = 0 \ , \tag{42}$$

and all other average shear stresses and strains vanish. This situation corresponds to uniaxial straining in the fiber direction of a cylinder with free lateral surface. Under these circumstances, the relation between uniaxial average stresses and strains $\bar{\sigma}_{11}(t)$ and $\bar{\varepsilon}_{11}(t)$ is given by

$$\bar{\sigma}_{11}(t) = \int_0^t E_a^*(t-\tau) \frac{d\bar{\varepsilon}_{11}(\tau)}{d\tau} d\tau , \tag{43}$$

where $E_a^*(t)$ is the effective axial Young's relaxation modulus. The inversion of the relation (43) is

$$\bar{\varepsilon}_{11}(t) = \int_0^t e_a^*(t-\tau) \frac{d\bar{\sigma}_{11}(\tau)}{d\tau} d\tau , \tag{44}$$

where $e_a^*(t)$ is the effective axial Young's creep compliance. The functions of E_a^* and e_a^* are related in the Laplace transform domain by

$$p\hat{E}_a^*(p) = \frac{1}{p\hat{e}_a^*(p)} . \tag{45}$$

For discussion concerning the Poisson's effect associated with uniaxial stressing, see Reference [3].

We shall now proceed to discuss some explicit results obtained in Reference [3] for effective relaxation moduli and creep functions of viscoelastic fiber reinforced materials. It is again recalled that all results are based on the composite cylinder assemblage model.

It is assumed that the fibers are isotropic elastic. The fiber elastic moduli are given the subscript f. Various moduli to be used are

λ_f	Lamé modulus,
G_f	Shear modulus,
ν_f	Poisson's ratio,
E_f	Young's modulus,
$k_f = \lambda_f + G_f$	Plane strain bulk modulus.

It is assumed that the matrix is isotropic viscoelastic. In order to write the matrix stress-strain relations, let the stresses and strains first be split into isotropic and deviatoric parts. Thus,

$$\sigma_{ij} = \sigma\delta_{ij} + s_{ij} \quad , \ \sigma = 1/3\ \sigma_{kk} , \tag{46}$$

$$\varepsilon_{ij} = \varepsilon \delta_{ij} + e_{ij} \quad , \quad \varepsilon = 1/3 \; \varepsilon_{kk} \; . \tag{47}$$

The most general viscoelastic stress-strain relations at any point in the matrix have the form

$$\sigma(t) = 3 \int_0^t K_m(t-\tau) \frac{d\varepsilon(t)}{d\tau} \; , \tag{48}$$

$$s_{ij}(t) = 2 \int_0^t G_m(t-\tau) \frac{de_{ij}(\tau)}{d\tau} \, d\tau \; , \tag{49}$$

$$\varepsilon(t) = 1/3 \int_0^t I_m(t-\tau) \frac{d\sigma(\tau)}{d\tau} \, d\tau \; , \tag{50}$$

$$e_{ij}(t) = 1/2 \int_0^t J_m(t-\tau) \frac{ds_{ij}(\tau)}{d\tau} \, d\tau \; . \tag{51}$$

Here, $K_m(t)$ and $G_m(t)$ are the matrix bulk and shear relaxation moduli, respectively, and $I_m(t)$ and $J_m(t)$ are the matrix bulk and shear creep compliances, respectively. The relaxation moduli and creep compliances are connected through their Laplace transforms, as in previous examples, by the relations

$$p\hat{K}_m(p) = \frac{1}{p\hat{I}_m(p)} \; , \tag{52}$$

$$p\hat{G}_m(p) = \frac{1}{p\hat{J}_m(p)} \; . \tag{53}$$

The matrix properties as expressed by the functions $K_m(t)$, $G_m(t)$, $I_m(t)$, and $J_m(t)$ may be regarded as experimentally measured information. It is also customary to approximate viscoelastic stress-strain relations by time differential operator relations which may be described by spring-dashpot models. Although these operator relations are a less satisfactory description than the general relations (48) through (51), they are used for reasons of mathematical convenience. It should be noted that (48) through (51) include spring-dashpot model relations as special cases.

After Reference [3], we introduce for reasons of convenience the following notation:

$$\kappa_m(p) = p\hat{K}_m(p) \; , \tag{54}$$

$$\Gamma_m(p) = p\hat{G}_m(p) \ . \tag{55}$$

The Laplace transform of $k_T^*(t)$ is then given by the following expression:

$$p\hat{K}_T^*(p) = \kappa_m(p) + \frac{1}{3}\Gamma_m(p) + \left\{ \frac{1}{k_f - \kappa_m(p) - \frac{1}{3}\Gamma_m(p)} + \frac{1-c}{\kappa_m(p) + \frac{4}{3}\Gamma_m(p)} \right\}^{-1} c \tag{56}$$

where c is the fiber volume fraction.

Frequently it is permissible to assume that a viscoelastic material behaves elastically for isotropic stress and that the visco-elastic effect is confined to shear. Then $\kappa_m(p)$ in (56) is replaced by an elastic bulk modulus of the matrix, K_m^E. Also, in practice, the fibers are usually very much stiffer than the matrix. In such cases, the fibers may be assumed to be rigid. This last simplification when used in (56) yields the simple result

$$\hat{K}_T^*(p) = \hat{K}_m(p) + \frac{1}{3}\hat{G}_m(p) + [\hat{K}_m(p) + \frac{4}{3}\hat{G}_m(p)]\frac{c}{1-c} \ , \tag{57}$$

which may be directly inverted to the time domain and then becomes

$$K_T^*(t) = K_m(t) + \frac{1}{3}G_m(t) + [K_m(t) + \frac{4}{3}G_m(t)]\frac{c}{1-c} \ . \tag{58}$$

The Laplace transform of $G_a^*(t)$ is, according to Reference [3],

$$\hat{G}_a^*(p) = \frac{G_f(1+c) + p\hat{G}_m(p)(1-c)}{G_f(1-c) + p\hat{G}_m(p)(1+c)}\hat{G}_m(p) \ . \tag{59}$$

If the fibers are assumed to be rigid relative to the matrix, (59) simplifies to

$$\hat{G}_a^*(p) = \frac{1+c}{1-c}\hat{G}_m(p) \ , \tag{60}$$

which can be directly inverted to the time domain to read

$$G_a^*(t) = \frac{1+c}{1-c}G_m(t) \ . \tag{61}$$

The Laplace transform of $E_a^*(t)$ is given approximately, but with a high degree of accuracy, by

$$\hat{E}_a^*(p) = (1 - c)\ \hat{E}_m(p) + \frac{c}{p} E_f \ , \tag{62}$$

which, when inverted to the time domain, reads

$$E_a^*(t) = (1 - c)\ E_m(t) + c\ E_f\ H(t) \ , \tag{63}$$

where H(t) is the Heaviside unit step function.

Results for the viscoelastic Poisson's effect and the shear relaxation modulus $G_T^*(t)$ are more complicated and will not be discussed here. For discussion, see Reference [3] and subsequent developments in this report.

3. VIBRATIONS AND COMPLEX MODULI OF HOMOGENEOUS, ANISOTROPIC VISCOELASTIC BODIES

We shall begin with a discussion of the concept of complex moduli for anisotropic and homogeneous viscoelastic materials. The dynamic equations of such materials in the absence of body forces may be written in the following form:

$$\sigma_{ij,j} = \rho \frac{\partial^2 u_i}{\partial t^2} , \tag{64}$$

$$\sigma_{ij}(\underline{x},t) = \int_o^t C_{ijk\ell}(t-\tau) \frac{\partial \varepsilon_{k\ell}(\underline{x},\tau)}{\partial \tau} d\tau , \tag{65}$$

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) . \tag{66}$$

Here σ_{ij}, u_i, and ε_{ij} are the local stresses, displacements, and small strains, respectively; ρ is the density; and $C_{ijk\ell}$ are the relaxation moduli. It is seen that there are 6 stresses, 3 displacements, and 6 strains (15 quantities in all). Now there are 3 equations (64), 6 equations (65), and 6 equations (66) (15 equations in all). The stresses and strains can be easily eliminated from (64) through (66) by substitution of (66) into (65) and the resulting expressions into (64). Thus, one obtains 3 equations for the three u_i. This will not be done here since the resulting equations are very complicated and will not be needed here.

With Equations (64) through (66) there must be associated boundary and initial conditions. A general kind of such conditions for a body of volume V and surface S is given by

$$u_i(S,t) = u_i^o \quad \text{on } S_u , \tag{67}$$

$$T_i(S,t) = T_i^o \quad \text{on } S_T , \tag{68}$$

$$u_i(\underline{x},o) = f_i(\underline{x}) , \tag{69}$$

$$\frac{\partial u_i}{\partial t}(\underline{x},o) = g_i(\underline{x}) . \tag{70}$$

Here, T_i are tractions expressed by

$$T_i = \sigma_{ij} n_j , \tag{71}$$

and on the surface S, n_i are the components of the outward unit normal. The displacements are prescribed by (67) as functions of time and surface coordinates on the part S_u of the surface. Similarly, the tractions are prescribed on the part S_T of the surface; of course, $S_u + S_T = S$. Conditions (69) and (70) express the fact that the displacements u_i and the velocities $\frac{\partial u_i}{\partial t}$ are given at initial time zero.

We do not wish to look into the dynamic problem in such generality. Instead, we consider the problem of steady-state vibrations. In this case the displacements have the form

$$u_i(\underline{x},t) = \tilde{u}_i(\underline{x})\ e^{i\omega t}\ . \tag{72}$$

Here, $\tilde{u}_i(\underline{x})$ are three space-variable functions, $i = \sqrt{-1}$, and ω is the circular frequency. Insertion of (72) into (66) yields

$$\varepsilon_{ij}(\underline{x}) = \tilde{\varepsilon}_{ij}(\underline{x})\ e^{i\omega t}\ , \tag{73}$$

where

$$\tilde{\varepsilon}_{ij} = \frac{1}{2}\ (\tilde{u}_{i,j} + \tilde{u}_{j,k})\ . \tag{74}$$

If we insert (73) into (65), we find that

$$\sigma_{ij}(\underline{x},t) = \tilde{\varepsilon}_{k\ell}(\underline{x})\ i\omega \int_0^t C_{ijk\ell}(t-\tau)\ e^{i\omega\tau}\ d\tau\ . \tag{75}$$

However, it should be noted that a physical approximation is involved in this substitution, for (65) is based on the Boltzmann superposition principle in time and is therefore limited to gentle time variations. This leads us to the conclusion that (75) is valid for vibrations which are not too rapid. Since

$$\omega = 2\pi n\ , \tag{76}$$

where n is the number of cycles per unit time, the validity of (75) *must be limited to small frequency.*

It is convenient to perform the change of variable

$$\tau' = t - \tau \tag{77}$$

in (75). Then the equation transforms into

$$\sigma_{ij}(\underline{x},t) = \tilde{\varepsilon}_{k\ell}(\underline{x})\ i\omega\ e^{i\omega t} \int_0^t C_{ijk\ell}(\tau')e^{-i\omega\tau'}d\tau'\ . \tag{78}$$

The time dependence on the right side of (78) is not a sinusoidal vibration, for besides the $e^{i\omega t}$ factor the integral is dependent on t because of the upper limit of the integral. The assumptions are now made that t is so large that it can be given infinite value in the upper limit of the integral and that this integral then converges. Note that $e^{i\omega t}$ is not affected by making t large, because it is purely oscillatory for all times. Then we find that

$$\sigma_{ij}(\underline{x},t) = \tilde{\varepsilon}_{k\ell}(\underline{x})\ i\omega\ e^{i\omega t} \int_0^\infty C_{ijk\ell}(\tau')e^{-i\omega\tau'}\,d\tau' \ . \tag{79}$$

We introduce the notation

$$i\omega \int_0^\infty C_{ijk\ell}(\tau')\ e^{-i\omega\tau'}\ d\tau' = D_{ijk\ell}(i\omega) \ . \tag{80}$$

Then (79) can be written in the form

$$\sigma_{ij}(\underline{x},t) = \tilde{\sigma}_{ij}(\underline{x})\ e^{i\omega t} \ , \tag{81}$$

$$\tilde{\sigma}_{ij}(\underline{x}) = D_{ijk\ell}(i\omega)\ \tilde{\varepsilon}_{k\ell}(\underline{x}) \ . \tag{82}$$

It is seen that now the stress time variation is of the nature of a sinusoidal oscillation and that this happens only after a very long time has elapsed. This is physically reasonable since a steady state of vibrations can be achieved only after sufficient time has elapsed for transient effects to die out.

Because of the formal resemblance of (82) to an elasticity stress-strain relation, the quantities $D_{ijk\ell}$ are called the *complex moduli*. They may also be written in the form

$$D_{ijk\ell}(i\omega) = D^R_{ijk\ell}(\omega) + iD^I_{ijk\ell}(\omega) \ , \tag{83}$$

where $D^R_{ijk\ell}$ and $D^I_{ijk\ell}$ and the real and imaginary parts, respectively, of $D_{ijk\ell}$.

If (81) and (72) are substituted into (64), the exponential cancels out and we find that

$$\tilde{\sigma}_{ij,j} + \rho\omega^2\ \tilde{u}_i = 0 \ . \tag{84}$$

It is now seen that the space-dependent parts of the stresses, displacements, and strains enter into Equations (74), (82), and (84). These are precisely the equations governing the space-dependent parts of these quantities in the problem of elastic vibrations, with elastic moduli $C_{ijk\ell}$ replaced by complex moduli $D_{ijk\ell}$.

We still have to discuss boundary and initial conditions for the space-dependent parts. For this purpose, (67) and (68) have to be taken in the oscillatory form

$$u_i(S,t) = \tilde{u}_i^o(S)\, e^{i\omega t} \quad \text{on } S_u \,, \tag{85}$$

$$T_i(S,t) = \tilde{T}_i^o(S)\, e^{i\omega t} \quad \text{on } S_T \,. \tag{86}$$

In view of (71), (72), and (73), the boundary conditions (85) and (86) assume the form

$$\tilde{u}_i(S) = \tilde{u}_i^o \,, \tag{87}$$

$$\tilde{T}_i(S) = \tilde{T}_i^o \,. \tag{88}$$

With these boundary conditions, the analogy with the elastic vibration problem is complete. Equations (74), (82), and (84) together with (87) and (88) define a unique solution to the problem. No initial conditions can be prescribed. This is logical, for the assumed oscillatory nature of all variables prescribes the time variation. Furthermore, this time variation is steady state; it has, so to speak, "neither a beginning nor an end", and therefore initial conditions cannot be imposed.

In practice, boundary conditions of type (85) and (86) will be given either in the form

$$u_i(S,t) = \tilde{u}_i^o(S)\cos\omega t \quad \text{on } S_u \,, \tag{89}$$

$$T_i(S,t) = \tilde{T}_i^o(S)\cos\omega t \quad \text{on } S_T \,, \tag{90}$$

or in the form

$$u_i(S,t) = \tilde{u}_i^o(S)\sin\omega t \quad \text{on } S_u \,, \tag{91}$$

$$T_i(S,t) = \tilde{T}_i^o(S)\sin\omega t \quad \text{on } S_T \,. \tag{92}$$

It is recalled that

$$\cos\omega t = \mathrm{Re}(e^{i\omega t}) \,,$$
$$\sin\omega t = \mathrm{Im}(e^{i\omega t}) \,. \tag{93}$$

Since the coefficients $D_{ijk\ell}$ in (92) are complex, the solution for $\tilde{u}_i$ and $\tilde{\sigma}_{ij}$ will also be complex. If the boundary conditions are of type (89) and (90), the real part of the solution is taken. If the boundary conditions are of type (91) and (92), the imaginary part of the solution is taken.

In view of the similarity to the elastic vibration problem, we can now formulate an analogy: To solve a viscoelastic vibration problem, take the solution, for the corresponding elastic vibration problem. In this solution replace the elastic moduli by the complex moduli. The resulting solution has a real part which corresponds to boundary conditions of type (89) and (90) and an imaginary part which corresponds to boundary conditions of type (91) and (92).

In view of further developments, it is important to point out an interesting relation between the relaxation moduli and the complex moduli. Let the Laplace transforms of the relaxation moduli $C_{ijk\ell}(t)$ be defined, as in Section 2, by

$$\hat{C}_{ijk\ell}(p) = \int_0^\infty C_{ijk\ell}(t)\, e^{-pt}\, dt. \tag{94}$$

Furthermore, the LT of (65) is by the convolution theorem

$$\hat{\sigma}_{ij}(\underline{x},p) = p\, \hat{C}_{ijk\ell}(p)\, \hat{\varepsilon}_{k\ell}(\underline{x},p) \ . \tag{95}$$

In analogy with the definition (6) for effective TD moduli, we define here the local TD moduli

$$B_{ijk\ell}(p) = p\, \hat{C}_{ijk\ell}(p) \ . \tag{96}$$

If we now compare (96) in terms of (94) with (80), we note the very important relationship

$$D_{ijk\ell}(i\omega) = B_{ijk\ell}(i\omega) \ . \tag{97}$$

That is: *If the TD moduli are known as functions of the LT variable p, then the complex moduli are given by replacement of p by $i\omega$.*

4. EFFECTIVE COMPLEX MODULI OF UNIAXIALLY FIBER REINFORCED MATERIALS

The extension of the theory which has been developed in Section 3, to composite bodies such as fiber reinforced materials is not a simple matter. To try to find a detailed solution of the dynamic equations in fibers and matrix is an absolutely hopeless task, and any treatment must be limited to utilization of effective physical properties. The questions which arise now are: is it permissible to use effective physical properties in the equations developed in Section 3, and will the results of the calculation yield the macroscopic dynamic behavior of viscoelastic fiber reinforced specimens? The answer to these questions is: theoretically no but practically yes, if the frequencies of vibration are not high.

To discuss this statement, it should be remembered that calculation of static effective properties such as effective relaxation moduli is based on statistically homogeneous states of stress and strain [3,4]. Such states of stress and strain occur in fiber reinforced materials in static equilibrium for certain boundary conditions, which are called homogeneous boundary conditions [3]. However, in dynamic problems, statistically homogeneous states of stress and strain apparently can not occur, because even for homogeneous bodies there does not exist a single case of a dynamic solution in which the states of stress or strain do not vary in space.

On the other hand, experience teaches us that if continuum theory with effective properties is applied to dynamic problems, for moderate frequencies, the results predicted approximate quite well global quantities such as beam deflections and resonant frequencies. The investigation of the limitations of this approximation is extremely difficult, and to date these limitations have not been explored. Therefore, it will be assumed in the present work that continuum theory with effective properties is meaningful for fiber reinforced composites.

Some justification for this may be provided by the following argument. It may be imagined that a fiber reinforced specimen is divided into many representative volume elements (RVE). Each RVE is a small part of the specimen, yet its cross section contains many fibers. The stresses and strains in such an element may be thought to be locally statistically homogeneous. Then the relation between the average stress and strain in any element is given by (1) and (2), although, in contrast to the static case, the averages vary from one RVE to another. The equations of motion of an element then assume a form similar to (64) through (66) with partial space derivatives replaced by increments of quantities over space increments, i.e., sides of cubic RVE. The resulting difference equations are approximated by the differential equations (64) through (66).

Adoption of the fundamental assumptions discussed above, in conjunction with the results given in Section 2 for effective relaxation moduli, leads directly to calculation of the effective complex moduli. The transverse isotropy of a uniaxially fiber reinforced material, with respect to the x_1 axis in fiber direction, simplifies (82) in the same way that (8) simplifies to the form (10) through (15), as a consequence of the same symmetry. We can therefore write out (82) in the following form

$$\tilde{\bar{\sigma}}_{11} = D^*_{11}\,\tilde{\bar{\varepsilon}}_{11} + D^*_{12}\,\tilde{\bar{\varepsilon}}_{22} + D^*_{12}\,\tilde{\bar{\varepsilon}}_{33}\,, \tag{98}$$

$$\tilde{\bar{\sigma}}_{22} = D^*_{12}\,\tilde{\bar{\varepsilon}}_{11} + D^*_{22}\,\tilde{\bar{\varepsilon}}_{22} + D^*_{23}\,\tilde{\bar{\varepsilon}}_{23}\,, \tag{99}$$

$$\tilde{\bar{\sigma}}_{33} = D^*_{12}\,\tilde{\bar{\varepsilon}}_{11} + D^*_{23}\,\tilde{\bar{\varepsilon}}_{22} + D^*_{22}\,\tilde{\bar{\varepsilon}}_{33}\,, \tag{100}$$

$$\tilde{\bar{\sigma}}_{12} = 2\,D^*_{44}\,\tilde{\bar{\varepsilon}}_{12}\,, \tag{101}$$

$$\tilde{\bar{\sigma}}_{13} = 2\,D^*_{44}\,\tilde{\bar{\varepsilon}}_{13}\,, \tag{102}$$

$$\tilde{\bar{\sigma}}_{23} = (D^*_{22} - D^*_{23})\,\tilde{\bar{\varepsilon}}_{23}\,. \tag{103}$$

Here, $\tilde{\bar{\sigma}}_{ij}$ and $\tilde{\bar{\varepsilon}}_{ij}$ are the space-dependent parts of the local averages $\bar{\sigma}_{ij}$ and $\bar{\varepsilon}_{ij}$, which are of the oscillatory form

$$\bar{\sigma}_{ij} = \tilde{\bar{\sigma}}_{ij}(\underline{x})\,e^{i\omega t}\,, \tag{104}$$

$$\bar{\varepsilon}_{ij} = \tilde{\bar{\varepsilon}}_{ij}(\underline{x})\,e^{i\omega t}\,. \tag{105}$$

The D^* coefficients in (98) through (103) are the effective complex moduli. In the $D_{ijk\ell}$ notation appearing in (82), they have the following meaning:

$$D^*_{11} = D^*_{1111}\,, \tag{106}$$

$$D^*_{12} = D^*_{1122} = D^*_{1133}\,, \tag{107}$$

$$D^*_{22} = D^*_{2222} = D^*_{3333}\,, \tag{108}$$

$$D^*_{23} = D^*_{2233}\,, \tag{109}$$

$$D^*_{44} = D^*_{1212} = D^*_{1313} . \qquad (110)$$

Denoting the left sides of (106) through (110) by D^*_{mn} and using the result (97), we have

$$D^*_{mn}(\iota\omega) = B^*_{mn}(\iota\omega) . \qquad (111)$$

The meaning of (111) is as follows: To find the effective complex moduli D^*_{mn}, it is necessary to replace the variable p in the TD moduli B^*_{mn} by the variable $\iota\omega$. The $B^*_{mn}(p)$ are known on the basis of the composite cylinder assemblage model and have been discussed in Section 2. Thus, we have a straightforward method to give expressions for the effective complex moduli.

It is also quite obvious that we can define physically meaningful complex moduli, such as axial and transverse complex shear moduli, axial and transverse complex Young's moduli, and a plane strain complex bulk modulus, in analogy to similar moduli defined in Section 2. We shall give these moduli the same notation as the static relaxation moduli with an added v above them. Thus, in analogy with (24), (25), and (34),

$$\check{G}^*_a(\iota\omega) = D^*_{44}(\iota\omega) , \qquad (112)$$

$$2\check{G}^*_T(\iota\omega) = D^*_{22}(\iota\omega) - D^*_{23}(\iota\omega) , \qquad (113)$$

$$2\check{K}^*_T(\iota\omega) = D^*_{22}(\iota\omega) + D^*_{23}(\iota\omega) . \qquad (114)$$

The complex axial Young's modulus is defined for states of stress and strain analogous to (41) and (42). We take a cylinder and apply average uniaxial oscillatory stress

$$\bar{\sigma}_{11} = \tilde{\bar{\sigma}}_{11} e^{\iota\omega t} , \qquad (115)$$

producing uniaxial oscillatory average strain

$$\bar{\varepsilon}_{11} = \tilde{\bar{\varepsilon}}_{11} e^{\iota\omega t} , \qquad (116)$$

then

$$\tilde{\bar{\sigma}}_{11} = \check{E}^*_a(\iota\omega) \tilde{\bar{\varepsilon}}_{11} . \qquad (117)$$

The explicit expressions of the effective complex moduli of the fiber reinforced material will involve the complex moduli of the fibers and the matrix. Therefore, the meaning of these will be briefly discussed. The fibers being assumed perfectly elastic and isotropic, their complex moduli are simply their isotropic elastic moduli. The stress-strain relation for fiber material is

$$\sigma_{ij} = \lambda_f \varepsilon_{kk}\delta_{ij} + 2G_f \varepsilon_{ij} \; . \tag{118}$$

If σ_{ij} and ε_{ij} have the oscillatory forms

$$\sigma_{ij} = \tilde{\sigma}_{ij} \, e^{i\omega t} \; , \tag{119}$$

$$\varepsilon_{ij} = \tilde{\varepsilon}_{ij} \, e^{i\omega t} \; , \tag{120}$$

then insertion of (119) and (120) into (118) gives simply

$$\tilde{\sigma}_{ij} = \lambda_f \tilde{\varepsilon}_{kk}\delta_{ij} + 2G_f \tilde{\varepsilon}_{ij} \; . \tag{121}$$

We see that the elastic moduli λ_f and G_f are trivially the complex moduli. The physical meaning of this is that in a perfectly elastic body oscillatory stresses and strains are in phase.

For the viscoelastic matrix the situation is of course different. It has been described in detail for a general viscoelastic anisotropic material in Section 3, and the present isotropic material is a special case. Going through the same arguments which lead to (82), we find for the present isotropic viscoelastic matrix a relation of the form

$$\tilde{\sigma}_{ij} = \check{\lambda}_m(i\omega) \, \tilde{\varepsilon}_{kk} \, \delta_{ij} + 2\check{G}_m(i\omega) \, \tilde{\varepsilon}_{ij} \; . \tag{122}$$

Here, $\tilde{\sigma}_{ij}$ and $\tilde{\varepsilon}_{ij}$ are the time-independent parts of (119) and (120). $\check{\lambda}_m(i\omega)$ and $\check{G}_m(i\omega)$ are the complex moduli of the matrix, and, since they are functions of $i\omega$, the stresses and strains are out of phase. The complex moduli of the matrix must be determined by a vibration experiment, in which a specimen is subjected to oscillatory stresses and strains with various frequencies ω. Note that $\tilde{\sigma}_{ij}$ and $\tilde{\varepsilon}_{ij}$ are the amplitudes of the stress and strain variations. The relation between these yields the complex moduli.

In analogy to elasticity, various other related complex moduli can be defined; e.g.,

$$\check{E}_m(i\omega) = \frac{\check{G}_m \, (3\check{\lambda}_m + 2\check{G}_m)}{\check{\lambda}_m + \check{G}_m} \tag{123}$$

$$\check{K}_m(\iota\omega) = \check{\lambda}_m + \frac{2}{3}\check{G}_m , \quad (124)$$

$$k_m(\iota\omega) = \check{\lambda}_m + \check{G}_m , \quad (125)$$

where (123) is the complex Young's modulus, (124) is the complex three-dimensional bulk modulus, and (125) is the complex plane strain bulk modulus. Generally, the complex Young's modulus is the easiest to measure, and the complex shear modulus can also be determined experimentally. All the other related complex moduli can then be computed.

We now proceed to construct detailed expressions for some of the effective complex moduli of a uniaxially fiber reinforced material by use of the method outlined. We start with the effective plane strain bulk modulus as given by (50), and according to what has been said before, the transform variable p is replaced by $\iota\omega$ everywhere. We then have

$$\check{K}^*_T(\iota\omega) = \kappa_m(\iota\omega) + \frac{1}{3}\Gamma_m(\iota\omega)$$

$$+ \left\{ \frac{1}{k_f - \kappa_m(\iota\omega) - \frac{1}{3}\Gamma_m(\iota\omega)} + \frac{1-c}{\kappa_m(\iota\omega) + \frac{4}{3}\Gamma_m(\iota\omega)} \right\}^{-1} c . \quad (126)$$

The left side of (56) is the effective TD plane strain bulk modulus. Therefore, according to (96) and (97), the left side of (126) is the effective plane strain complex modulus $\check{K}^*_T(\iota\omega)$. Also, $K_m(p)$ and $\Gamma_m(p)$ in (56) are TD bulk and shear moduli, respectively, of the matrix. Therefore, $\kappa_m(\iota\omega)$ and $\Gamma_m(\iota\omega)$ are the complex bulk and shear moduli, respectively, of the matrix; i.e.,

$$\kappa_m(\iota\omega) = \check{K}_m(\iota\omega) , \quad (127)$$

$$\Gamma_m(\iota\omega) = \check{G}_m(\iota\omega) . \quad (128)$$

So (126) assumes the form

$$\check{K}^*_T(\iota\omega) = \check{K}_m(\iota\omega) + \frac{1}{3}\check{G}_m(\iota\omega)$$

$$+ \left\{ \frac{1}{k_f - \check{K}_m(\iota\omega) - \frac{1}{3}\check{G}_m(\iota\omega)} + \frac{1-c}{\check{K}_m(\iota\omega) + \frac{4}{3}\check{G}_m(\iota\omega)} \right\}^{-1} c . \quad (129)$$

When the fibers are assumed to be rigid in comparison to the matrix, which is a very good approximation for a fiber reinforced material, we multiply (57) by p on both sides. Taking into account (54) and (55), we then have

$$\check{K}^*_T(\iota\omega) = \kappa_m(\iota\omega) + \frac{1}{3}\,\Gamma_m(\iota\omega) + [\kappa_m(\iota\omega) + \frac{4}{3}\,\Gamma_m(\iota\omega)]\,\frac{c}{1 - c}\ . \qquad (130)$$

Again the left side is $\check{K}^*_T(\iota\omega)$, and in view of (127) and (128), we can rewrite (130) in the form

$$\check{K}^*_T(\iota\omega) = \check{K}_m(\iota\omega) + \frac{1}{3}\,\check{G}_m(\iota\omega) + [\check{K}_m(\iota\omega) + \frac{4}{3}\,\check{G}_m(\iota\omega)]\,\frac{c}{1 - c}\ . \qquad (131)$$

It is often convenient to split a complex modulus into its real and imaginary parts. Thus, we may write

$$\check{K}^*_T(\iota\omega) = K^{*R}_T(\omega) + \iota\,K^{*I}_T(\omega)\ , \qquad (132)$$

$$\check{K}_m(\iota\omega) = K^R_m(\omega) + \iota\,K^I_m(\omega)\ , \qquad (133)$$

$$\check{G}_m(\iota\omega) = G^R_m(\omega) + \iota\,G^I_m(\omega)\ . \qquad (134)$$

It is seen that (131) separates very simply into real and imaginary parts. Thus,

$$K^{*R}_T = K^R_m + \frac{1}{3}\,G^R_m + [K^R_m + \frac{4}{3}\,G^R_m]\,\frac{c}{1 - c}\ , \qquad (135)$$

$$K^{*I}_T = K^I_m + \frac{1}{3}\,G^I_m + [K^I_m + \frac{4}{3}\,G^I_m]\,\frac{c}{1 - c}\ . \qquad (136)$$

On the other hand, the separation of (129) into real and imaginary parts is quite complicated. It is seen that K^{*R}_T and K^{*I}_T will be functions of all of the real and imaginary parts of $\check{K}_m(\iota\omega)$ and $\check{G}_m(\iota\omega)$.

We now turn to the effective axial shear modulus $\check{G}^*_a(\iota\omega)$. We multiply (59) on both sides by p. Taking into account (55), the following expression is obtained:

$$p\,\check{G}^*_a(p) = \frac{G_f(1 + c) + \Gamma_m(p)(1 - c)}{G_f(1 - c) + \Gamma_m(p)(1 + c)}\,\Gamma_m(p)\ . \qquad (137)$$

Now p is again replaced by $\iota\omega$. Then the left side of (137) becomes $\check{G}^*_a(\iota\omega)$, and in view of (128), (137) is rewritten as

$$\check{G}_a^*(\iota\omega) = \frac{G_f(1 + c) + \check{G}_m(\iota\omega)(1 - c)}{G_f(1 - c) + \check{G}_m(\iota\omega)(1 + c)} \check{G}_m(\iota\omega) \ . \tag{138}$$

If the fibers are assumed to be rigid, then on the basis of (60),

$$\check{G}_a^*(\iota\omega) = \frac{1 + c}{1 - c} \check{G}_m(\iota\omega) \ . \tag{139}$$

This simple expression separates easily into real and imaginary parts. Writing

$$\check{G}_a^*(\iota\omega) = G_a^{*R}(\omega) + \iota \ G_a^{*I}(\omega) \ , \tag{140}$$

and using (134), expression (139) can be rewritten as

$$\check{G}_a^{*R} = \frac{1 + c}{1 - c} G_m^R \ , \tag{141}$$

$$\check{G}_a^{*I} = \frac{1 + c}{1 - c} G_m^I \ . \tag{142}$$

The shear loss angle δ_m of the matrix is defined by

$$\tan \delta_m = \frac{G_m^I}{G_m^R} \ . \tag{143}$$

Similarly, an axial shear loss angle δ_a can be defined for the composite by

$$\tan \delta_a^* = \frac{\check{G}_a^{*I}}{\check{G}_a^{*R}} \ . \tag{144}$$

Inserting (141) and (142) into (144) and comparing with (143), we obtain the significant result

$$\tan \delta_a^* = \tan \delta_m \ . \tag{145}$$

Thus, the presence of rigid fibers does not affect the loss angle.

Discussion of the effective complex axial Young's modulus is in all respects similar. Multiplication of both sides of (62) by p gives

$$p \ \hat{E}_a^*(p) = (1 - c) \ p \ \hat{E}_m(p) + c \ E_f \ , \tag{146}$$

which yields for the complex modulus

$$\check{E}_a^*(\iota\omega) = (1 - c)\ \check{E}_m(\iota\omega) + c\ E_f\ . \tag{147}$$

Separation into real and imaginary parts is straightforward; i.e.,

$$E_a^{*R} = (1 - c)\ E_m^R + c\ E_f\ , \tag{148}$$

$$E_a^{*I} = (1 - c)\ E_m^I\ . \tag{149}$$

5. LONGITUDINAL VIBRATIONS OF VISCOELASTIC FIBER REINFORCED ROD

We consider now a typical vibration problem of a rod which is built in at one end and is subjected to sinusoidal time-variable force at the other end (Figure 2). The fibers are in the direction of the axis of the rod.

The appropriate end conditions are

$$u\ (o,t) = 0\ , \tag{150}$$

$$\sigma\ (\ell,t) = E^{*} \left.\frac{\partial u}{\partial x}\right|_{x=\ell} = \sigma_o e^{i\omega t}\ , \tag{151}$$

where u (o,t) is the longitudinal displacement, ω is the frequency, and

$$\sigma_o = \frac{P_o}{A}\ , \tag{152}$$

where A is the cross-sectional area. In accordance with the general discussion in Sections 3 and 4, we consider first the associated elastic problem. The governing differential equation is the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}\ , \tag{153}$$

where

$$c^2 = \frac{E_a^{*}}{\rho}\ , \tag{154}$$

in which E_a^{*} is the effective axial Young's modulus, and ρ is the average density of the material. We set

$$u\ (x,t) = U(x)\ e^{i\omega t}\ . \tag{155}$$

Substitution of (155) into (153) gives the ordinary differential equation

$$\frac{d^2U}{dx^2} + \frac{\omega^2}{c^2}\ U = 0\ . \tag{156}$$

Substitution of (155) into (150) and (151) gives the end conditions

$$U(o) = 0\ , \tag{157}$$

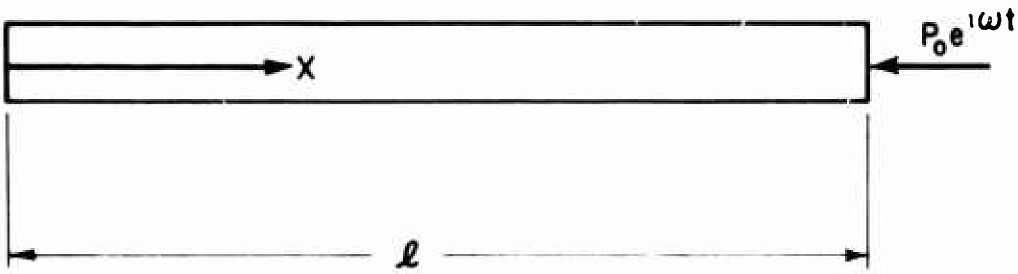

Figure 2. Longitudinal Vibrations of Viscoelastic Fiber Reinforced Rod.

$$\frac{dU(\ell)}{dx} = \frac{\sigma_o}{E_a^*} . \tag{158}$$

The solution of (156) with end conditions (157) and (158) is

$$U(x) = \frac{\sigma_o c}{\omega E_a^*} \frac{\sin\left(\frac{\omega x}{c}\right)}{\cos\left(\frac{\omega \ell}{c}\right)} . \tag{159}$$

The resonant frequencies ω_n are found by letting

$$\cos\left(\frac{\omega \ell}{c}\right) = 0 \tag{160}$$

so that U becomes unbounded. Equation (160) in conjunction with (154) yields

$$\omega_n = \frac{(2n + 1)\pi}{2\ell} \quad \frac{E_a^*}{\rho} , \tag{161}$$

and the first resonant frequency is accordingly

$$\omega_o = \frac{\pi}{2\ell} \sqrt{\frac{E_a^*}{\rho}} . \tag{162}$$

According to the theory in Sections 3 and 4, the modulus E_a^* in (159) must now be replaced by the effective complex modulus $\check{E}_a^*(i\omega)$ as given by (147), (148), and (149). It is convenient to represent $\check{E}_a^*(\imath\delta)$ in the usual form

$$\check{E}_a^*(\imath\omega) = |\check{E}_a^*| \, e^{\imath\delta^*} , \tag{163}$$

where

$$|\check{E}_a^*| = \imath\sqrt{(E_a^{*R})^2 + (E_a^{*I})^2} , \tag{164}$$

$$\tan\delta^* = \frac{E_a^{*I}}{E_a^{*R}} . \tag{165}$$

In view of (148) and (149),

$$|\check{E}_a^*| = \sqrt{(v_m E_m^R + v_f E_f)^2 + (v_m E_m^I)^2} , \tag{166}$$

$$\tan\delta^* = \frac{v_m E_m^I}{v_m E_m^R + v_f E_f} , \tag{167}$$

where v_m and v_f are the volume fractions ot fibers and matrix, respectively.

A typical example of a viscoelastic fiber reinforced material is an epoxy reinforced by glass or boron fibers. For such materials it is generally true that

$$\frac{E_f}{E_m} > 25 \qquad \frac{E_m^I}{E_m^R} < 0.1 . \tag{168}$$

The E_m in the left part of (168) is the *static* Young's modulus, which may be interpreted as $E_m^R(\omega)$ at zero frequency. Experiments described in Reference [5] have shown that for an epoxy, E_m^R is a very slowly increasing function of ω, it being necessary to increase the frequency from 10 to 10^5 in order to produce a 50% increase in E_m^R. Therefore, the relative magnitude of E_f to E_m^R may be taken as about the same as that of E_f to E_m.

In view of (168) and the subsequent remarks, we can approximate (166) with great accuracy by

$$|\overset{\vee}{E}{}^{*}_{a}| = v_m E^R_m + v_f E_f = E^*_a(\omega) \quad . \tag{169}$$

The expression (169) is the effective elastic axial Young's modulus of a uniaxially fiber reinforced material with the matrix Young's modulus E_m replaced by $E^R_m(\omega)$. In the following it will be denoted by $E^*_a(\omega)$. It should be noted that because of (168) and the subsequent remarks, $E^*_a(\omega)$ is very nearly equal to the static effective axial Young's modulus.

It also follows from (167) and (168) that

$$\tan \delta^* << 1 \ ; \tag{170}$$

therefore

$$\tan \delta^* \simeq \delta^* \ . \tag{171}$$

In view of (163) and (169),

$$\sqrt{\overset{\vee}{E}{}^{*}_{a}(\iota\omega)} \simeq (E^*_a)^{1/2} e^{\iota\delta^*/2} \quad . \tag{172}$$

Define $\tilde{c}$ by

$$\tilde{c} = \sqrt{\frac{\overset{\vee}{E}{}^{*}_{a}(\iota\omega)}{\rho}} \quad . \tag{173}$$

From (154) and (172),

$$\tilde{c} \simeq c\, e^{\iota\delta^*/2} \quad . \tag{174}$$

Now replacement of E^*_a by $\overset{\vee}{E}{}^{*}_{a}(\iota\omega)$ in (159) leads to

$$U(x) = \frac{\sigma_o \tilde{c}}{\omega \overset{\vee}{E}{}^{*}_{a}(\iota\omega)} \; \frac{\sin\left(\frac{\omega x}{\tilde{c}}\right)}{\cos\left(\frac{\omega \ell}{\tilde{c}}\right)} \ . \tag{175}$$

We use the approximations (172) and (174) in (175) to obtain

$$\tilde{u}(x) \simeq \frac{\sigma_o c}{E^*_a(\omega)} \, e^{-\iota\delta^*/2} \, \frac{\sin\left(\frac{\omega x}{c} \, e^{-\iota\delta^*/2}\right)}{\cos\left(\frac{\omega \ell}{c} \, e^{-\iota\delta^*/2}\right)} \ , \tag{176}$$

where $\bar{u}(x)$ is the space-dependent part of $u(x,t)$ in the viscoelastic vibration problem.

By Euler's formula,

$$e^{-\iota\delta^*/2} = \cos\frac{\delta^*}{2} - \iota\sin\frac{\delta^*}{2} , \tag{177}$$

and since by (170) and (171) δ^* is a very small angle,

$$\cos\delta^*/2 \simeq 1 , \tag{178}$$

$$\sin\delta^*/2 \simeq \delta^*/2 . \tag{179}$$

Thus,

$$e^{-\iota\delta^*/2} \simeq 1 - \iota\delta^*/2 , \tag{180}$$

$$\sin\left(\frac{\omega x}{c} e^{-\iota\delta^*/2}\right) \simeq \sin\left(\frac{\omega x}{c} - \iota\frac{\omega x\delta^*}{2c}\right) , \tag{181}$$

$$\cos\left(\frac{\omega\ell}{c} e^{-\iota\delta^*/2}\right) \simeq \cos\left(\frac{\omega\ell}{c} - \iota\frac{\omega\ell\delta^*}{2c}\right) . \tag{182}$$

Using the usual expression for the sin and cos of a complex number, we obtain

$$\sin\left(\frac{\omega x}{c} e^{-\iota\delta^*/2}\right) \simeq \sin\left(\frac{\omega x}{c}\right)\cosh\left(\frac{\omega\delta^* x}{2c}\right) - \iota\cos\left(\frac{\omega x}{c}\right)\sinh\left(\frac{\omega\delta^* x}{2c}\right), \tag{183}$$

$$\cos\left(\frac{\omega\ell}{c} e^{-\iota\delta^*/2}\right) \simeq \cos\left(\frac{\omega\ell}{c}\right)\cosh\left(\frac{\omega\delta^*\ell}{2c}\right) + \iota\sin\left(\frac{\omega\ell}{c}\right)\sinh\left(\frac{\omega\delta^*\ell}{2c}\right), \tag{184}$$

where cosh and sinh are the usual hyperbolic functions. It would now be very desirable to show that $\frac{\omega\delta^*\ell}{2c}$ is a very small expression, so that we could use small argument approximations for the hyperbolic functions. Consider a typical glass fiber reinforced epoxy. Typical material properties are

$$E_m^R \simeq 0.5 \times 10^6 \text{ psi}(*)$$

$$E_f = 10 \times 10^6 \text{ psi}.$$

(*)As stated before E_m^R is really a very slowly increasing function of frequency. The present constant value is merely an order of magnitude.

We choose a large matrix loss angle, $\tan \delta_m = 0.1$. Then

$$E_m^I = 0.05 \times 10^6 \text{ psi} .$$

We take as volume fractions

$$v_m = 0.4 \qquad v_f = 0.6 .$$

Then from (148) and (167),

$$E_a^* = 6.2 \times 10^6 \text{ psi}$$

$$\tan \delta^* = 0.005 .$$

Take as typical density

$$\rho = 3.0 \text{ (relative to water)}.$$

Then it follows that

$$c = 12{,}770 \text{ ft/sec}.$$

Let the length of the rod be

$$\ell = 10.0 \text{ ft}.$$

Then

$$\frac{\omega\delta^*\ell}{2c} = 2.5 \times 10^{-6}\ \omega .$$

If

$$\frac{\omega\delta^*\ell}{2c} \leq 0.05 ,$$

we have with great accuracy

$$\sinh\left(\frac{\omega\delta^*\ell}{2c}\right) = \frac{\omega\delta^*\ell}{2c} , \tag{185}$$

$$\cosh\left(\frac{\omega\delta^*\ell}{2c}\right) = 1 . \tag{186}$$

This corresponds to

$$\omega \leq 20{,}000 .$$

Since

$$\omega = 2\pi n \ ,$$

the frequency n satisfies

$$n \leq 3180 \text{ cycles/sec.}$$

This is a high frequency for which the present theory may not be valid anymore. Thus, it is seen that (185) and (186) may be safely assumed.

Since $o \leq x \leq \ell$, we have with better accuracy than (185) and (186)

$$\sinh\left(\frac{\omega\delta^* x}{2c}\right) = \frac{\omega\delta^* x}{2c} \ , \tag{187}$$

$$\cosh\left(\frac{\omega\delta^* x}{2c}\right) = 1 \ . \tag{188}$$

Introducing (185) through (188) into (183) and (184), then introducing the resulting approximations and (186) into (176), we find the first order of magnitude in $\frac{\omega\delta^* x}{2c}$ and $\frac{\omega\delta^* \ell}{2c}$.

$$\tilde{u}(x) = \frac{\sigma_o c}{\omega E_a^*(\omega)} \frac{\sin\left(\frac{\omega x}{c}\right)}{\cos\left(\frac{\omega \ell}{c}\right)} (1 - \imath\ \delta^*/2) \ . \tag{189}$$

Since

$$u(x,t) = \tilde{u}(x)\ e^{\imath\omega t} \ , \tag{190}$$

we have from (189) and (190)

$$u\,(x,t) = \frac{\sigma_o c}{\omega E_a^*} \frac{\sin\left(\frac{\omega x}{c}\right)}{\cos\left(\frac{\omega \ell}{c}\right)} [\cos\ \omega t + \delta^*/2 \sin\ \omega t + \imath(\sin\ \omega t - \delta^*/2 \cos\ \omega t)] \tag{191}$$

Now if the forcing stress (151) has a cosine variation, i.e.,

$$\sigma(\ell,t) = \sigma_o \cos\ \omega t = \text{Re}\ (\sigma_o\ e^{\imath\omega t}) \ , \tag{192}$$

then the displacement is the real part of (191); thus,

$$u_c\ (x,t) = \frac{\sigma_o c}{\omega E_a^*\ (\omega)} \frac{\sin\left(\frac{\omega x}{c}\right)}{\cos\left(\frac{\omega \ell}{c}\right)} (\cos \omega t + \delta^*/2 \sin \omega t)\ . \qquad (193)$$

If

$$\sigma(\ell,t) = \sigma_o \sin \omega t = I_m(\sigma_o e^{i\omega t})\ , \qquad (194)$$

then the displacement is the imaginary part of (191). Thus,

$$u_s\ (x,t) = \frac{\sigma_o c}{\omega E_a^*\ (\omega)} \frac{\sin\left(\frac{\omega x}{c}\right)}{\cos\left(\frac{\omega \ell}{c}\right)} (\sin \omega t - \delta^*/2 \cos \omega t)\ . \qquad (195)$$

From the elastic analysis at the beginning of this section, it is seen that the elastic solutions corresponding to (192), (193), and (194), (195) are, respectively,

$$u_c\ (x,t) = \frac{\sigma_o c}{\omega E_a^*\ (\omega)} \frac{\sin\left(\frac{\omega x}{c}\right)}{\cos\left(\frac{\omega \ell}{c}\right)} \cos \omega t\ , \qquad (196)$$

$$u_s\ (x,t) = \frac{\sigma_o c}{\omega E_a^*\ (\omega)} \frac{\sin\left(\frac{\omega x}{c}\right)}{\cos\left(\frac{\omega \ell}{c}\right)} \sin \omega t\ . \qquad (197)$$

It is thus seen that in the viscoelastic vibration problem, there are out-of-phase components which are proportional to δ^*.

It is instructive to represent (193) and (195) in different forms. It is seen that these expressions may be rewritten as

$$u_c\ (x,t) = \frac{\sigma_o c \sqrt{1 + \delta^{*2}/4}}{\omega E_a^*\ (\omega)} \frac{\sin\left(\frac{\omega x}{c}\right)}{\cos\left(\frac{\omega \ell}{c}\right)} \cos (\omega t - \phi)\ , \qquad (198)$$

$$u_s\ (x,t) = \frac{\sigma_o c \sqrt{1 + \delta^{*2}/4}}{\omega E_a^*\ (\omega)} \frac{\sin\left(\frac{\omega x}{c}\right)}{\cos\left(\frac{\omega \ell}{c}\right)} \sin (\omega t - \phi)\ , \qquad (199)$$

where

$$\tan \phi = \frac{\delta^*}{2} . \tag{200}$$

Comparison of (198) and (199) with the elastic results (196) and (197) shows that the amplitude of the viscoelastic vibrations is increased by the factor $\sqrt{1 + \delta^{*2}/4}$ and that there is a phase lag ϕ whose magnitude is given by (200). Since, as has been shown before, δ^* is very small for a typical fiber reinforced material, the amplitudes are practically the same as in elastic vibrations, and also the phase lag can be safely ignored. It is also seen that because of the occurrence of cos $(\omega\ell/c)$ in the denominators in (198) and (199), the resonant frequencies are the same as for the elastic vibrations and are thus given by (161) and (162).

The very important conclusion is that for all practical purposes, the viscoelastic effect can be ignored in the problem of longitudinal vibrations of fiber reinforced rods made of typical fiber reinforced materials.

It should be noted that this phenomenon is due to the fact that the fibers are very much stiffer than the matrix and that the matrix loss angle is small. In cases where these conditions are not fulfilled, the foregoing conclusions may become invalid, and it may be necessary to carry through the analysis without the approximations which were introduced here. However, such cases do not seem to be of interest for fiber reinforced materials.

6. TORSIONAL VIBRATIONS OF VISCOELASTIC FIBER REINFORCED CYLINDER

As a second typical problem, we consider a fiber reinforced cylinder which is built in at one end and is subjected to a sinusoidal torque, in time, at the other end. Again we consider first the associated elastic problem.

The basic variable is the angle of twist $\theta(x,t)$. The governing differential equation for θ is

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\theta}{\partial t^2} , \qquad (201)$$

where

$$c^2 = \frac{G_a^* C}{\rho I} , \qquad (202)$$

(see, e.g., Reference [6]). G_a^* is the effective axial shear modulus, and C is a number which depends upon the geometry of the cross section. In terms of the Saint Venant torsion function $\phi(y,z)$, C is given by

$$C = \iint_{(A)} (y^2 + z^2 + y\frac{\partial\phi}{\partial z} - z\frac{\partial\phi}{\partial y})\, dydz ,$$

the integral being taken over the cross section, which is referred to a y,z coordinate system with origin at the centroid (see, e.g., [7]). Note that $G_a^* C$ is the torsional rigidity of the cylinder.

Furthermore, ρ is the density of the material and I is the polar moment of inertia of the section.

The torque M(x,t) is given by

$$M(x,t) = G_a^* C \frac{\partial\theta}{\partial x} . \qquad (203)$$

The end conditions are

$$\theta(o,t) = 0 , \qquad (204)$$

$$M(\ell,t) = G_a^* C \frac{\partial\theta}{\partial x}\bigg|_{x=\ell} = M_o e^{i\omega t} . \qquad (205)$$

In (205), M_o is the amplitude of the sinusoidal forcing torque and ω is the frequency.

We seek a solution of the form

$$\theta(x,t) = \vartheta(x)\ e^{i\omega t} \ . \tag{206}$$

Introduction of (206) into (201), (204), and (205) yields the ordinary differential equation

$$\frac{d^2\vartheta}{dx^2} + \frac{\omega^2}{c^2}\vartheta = 0 \tag{207}$$

and the end conditions

$$\vartheta(o) = 0 \ , \tag{208}$$

$$\left.\frac{d\vartheta}{dx}\right|_{x=\ell} = \frac{M_o}{G_a^* C} \ . \tag{209}$$

The solution of (207), subject to (208) and (209), is

$$\vartheta(x) = \frac{c\ M_o}{\omega\ G_a^*\ C}\ \frac{\sin\left(\frac{\omega x}{c}\right)}{\cos\left(\frac{\omega \ell}{c}\right)} \ ; \tag{210}$$

therefore, the complete solution is

$$\theta(x,t) = \frac{c\ M_o}{\omega\ G_a^*}\ \frac{\sin\left(\frac{\omega x}{c}\right)}{\cos\left(\frac{\omega \ell}{c}\right)}\ e^{i\omega t} \ . \tag{211}$$

The condition of resonance is obtained from

$$\cos\left(\frac{\omega \ell}{c}\right) = 0 \ , \tag{212}$$

then

$$\frac{\omega \ell}{c} = \frac{2n+1}{2}\pi \ , \tag{213}$$

and from (202) and (213),

$$\omega_n = \frac{2n + 1}{2} \frac{\pi}{\ell} \sqrt{\frac{G_a^* C}{\rho I}} , \tag{214}$$

with first resonant frequency

$$\omega_o = \frac{\pi}{2\ell} \sqrt{\frac{G_a^* C}{\rho I}} \tag{215}$$

We now consider the case when the matrix is viscoelastic. Then the space-dependent part of the angle $\theta(x,t)$ is denoted by $\tilde{\theta}(x)$ and is found by replacement of G_a^* in (210) by the effective complex axial shear modulus $\check{G}_a^*$. An expression for $\check{G}_a^*$ has been given before for elastic fibers and viscoelastic matrix (138). We shall here use the simplified form (139) for rigid fibers in order to simplify the analysis. Since in a typical fiber reinforced material with epoxy matrix the ratio of fiber to matrix stiffness is about 25, at least, this is in general an excellent approximation. In order to avoid confusion with propagation velocity c, we shall denote the fiber volume fraction by v_f. Then we have

$$\check{G}_a^*(i\omega) = \frac{1 + v_f}{1 - v_f} \check{G}_m(i\omega) , \tag{216}$$

$$\check{G}_m(i\omega) = G_m^R + i G_m^I , \tag{217}$$

$$\tan \delta^* = \tan \delta_m = \tan \delta = \frac{G_m^I}{G_m^R} . \tag{218}$$

Expression (217) can be rewritten with the help of (218) in the form

$$\check{G}_m(i\omega) = G e^{i\delta} , \tag{219}$$

where

$$G = \sqrt{(G_m^R)^2 + (G_m^I)^2} = G_m^R \sqrt{1 + \tan^2 \delta} . \tag{220}$$

Hence, from (216) to (220),

$$\check{G}_a^*(i\omega) = \frac{1 + v_f}{1 - v_f} G_m^R \sqrt{1 + \tan^2\delta}\; e^{i\delta} . \tag{221}$$

We recall that the elastic effective axial shear modulus G_a is given, in the case of the rigid fibers, by

$$G_a^* = \frac{1 + v_f}{1 - v_f} G_m \tag{222}$$

(see References [1 and 3]), where G_m is the elastic shear modulus of the matrix.

We adopt the notation

$$\frac{1 + v_f}{1 - v_f} G_m^R (\omega) = G_a^* (\omega) , \tag{223}$$

in an analogy to the notation used in (169). Then from (221) and (223),

$$\check{G}_a^* (i\omega) = G_a^* (\omega) \sqrt{1 + \tan^2\delta}\; e^{i\delta} . \tag{224}$$

Denoting c as a function of $\check{G}_a^*(i\omega)$ by $\tilde{c}$, we have from (202) and (224),

$$\tilde{c}^2 = \frac{G_a^* (\omega)\, C}{\rho\, I} \sqrt{1 + \tan^2\delta}\; e^{i\delta} . \tag{225}$$

Therefore, from (202) and (225),

$$\tilde{c} = c(1 + \tan^2 \delta)^{1/4} e^{i\delta/2} . \tag{226}$$

Replacing c by (226) and G_a^* by (224) in (210), we obtain

$$\tilde{\theta} (x) = \frac{c\, M_o}{\omega\, G_a^* (\omega)\, C} \frac{e^{-i\delta/2}}{(1 + \tan^2 \delta)^{1/4}} \frac{\sin[\frac{\omega}{c} (1 + \tan^2\delta)^{-1/4} e^{-i\delta/2} x]}{\cos[\frac{\omega}{c} (1 + \tan^2\delta)^{-1/4} e^{-i\delta/2} \ell]} . \tag{227}$$

Since for the usual viscoelastic materials in general

$$\tan \delta \le 0.1 ,$$

it is seen that $\tan^2 \delta$ can be safely ignored in comparison to 1. Thus,

$$\tilde{\theta}(x) = \frac{c\, M_o}{\omega\, G_a^*(\omega)\, C}\, e^{-\imath\delta/2}\, \frac{\sin\left(\frac{\omega}{c}\, e^{-\imath\delta/2} x\right)}{\cos\left(\frac{\omega}{c}\, e^{-\imath\delta/2} \ell\right)} \,. \tag{228}$$

There is no difficulty in finding the real and imaginary parts of (228). The results, however, are very clumsy, and we shall therefore continue the calculation for $x = \ell$.

We have from (228),

$$\tilde{\theta}(\ell) = \frac{c\, M_o}{\omega\, G_a^*(\omega)\, C}\, (\cos \delta/2 - \imath \sin \delta/2)\, \frac{\sin[\frac{\omega\ell}{c}\, (\cos \delta/2 - \imath \sin \delta/2)]}{\cos[\frac{\omega\ell}{c}\, (\cos \delta/2 - \imath \sin \delta/2)]} \,. \tag{229}$$

We use the notation

$$\frac{\omega\ell}{c} \cos \delta/2 = \alpha \,, \qquad \text{(a)}$$

$$\frac{\omega\ell}{c} \sin \delta/2 = \beta \,. \qquad \text{(b)} \tag{230}$$

Using the usual expressions for the sine and cosine of complex quantities (see, e.g., Reference [8]), we find

$$\tilde{\theta}(\ell) = \frac{c\, M_o}{\omega\, G_a^*(\omega)\, C}\, (\cos \delta/2 - \imath \sin \delta/2)\, \frac{\sin(2\alpha) - \imath \sinh(2\beta)}{\cos(2\alpha) + \cosh(2\beta)} \,. \tag{231}$$

Thus, the angle of twist at $x = \ell$ is given by

$$\theta(\ell,t) = \tilde{\theta}(\ell)\, e^{\imath\omega t} = \frac{c\, M_o}{\omega C_a^*(\omega) C}\, \frac{\sin(2\alpha) - \imath \sinh(2\beta)}{\cos(2\alpha) + \cosh(2\beta)} \cdot e^{\imath(\omega t - \delta/2)} \,. \tag{232}$$

If (205) has the form

$$M(\ell,t) = M_o \cos \omega t \,, \tag{233}$$

then

$$\theta_c(\ell,t) = \text{Re}\left[\tilde{\theta}(\ell)\, e^{\imath\omega t}\right] . \tag{234}$$

If (205) has the form

$$M(\ell,t) = M_o \sin \omega t \; , \tag{235}$$

then

$$\theta_s(\ell,t) = \text{Im}\left[\tilde{\theta}(\ell)\, e^{i\omega t}\right] . \tag{236}$$

It follows from (234) and (232) that

$$\theta_c(\ell,t) = \frac{c\, M_o}{\omega G_a^*(\omega) C} \; \frac{\sin(2\alpha)\cos(\omega t - \delta/2) + \sinh(2\beta)\sin(\omega t - \delta/2)}{\cos(2\alpha) + \cosh(2\beta)} \; . \tag{237}$$

From (236) and (232),

$$\theta_s(\ell,t) = \frac{c\, M_o}{\omega G_a^*(\omega) C} \; \frac{\sin(2\alpha)\sin(\omega t - \delta/2) - \sinh(2\beta)\cos(\omega t - \delta/2)}{\cos(2\alpha) + \cosh(2\beta)} \; . \tag{238}$$

It is convenient to represent (237) and (238) in different forms. It is easily seen that (237) can be rearranged as

$$\theta_c(\ell,t) = \frac{c\, M_o}{\omega G_a^*(\omega) C} \; \frac{\sqrt{\sin^2(2\alpha) + \sinh^2(2\beta)}}{\cos(2\alpha) + \cosh(2\beta)} \cos(\omega t - \delta/2 - \psi) \; , \tag{239}$$

where

$$\tan \psi = \frac{\sinh(2\beta)}{\sin(2\alpha)} \; . \tag{240}$$

It is thus seen that (239) is a cosine vibration with a phase lag relative to moment input (233). When (234) is written in the form

$$\theta_c(\ell,t) = \text{Amp}\,[\theta c]\, \cos(\omega t - \varphi) \; , \tag{241}$$

it is seen that the amplitude is given by

$$\text{Amp}\,[\theta_c] = \frac{c\, M_o}{\omega G_a^*(\omega) C} \; \frac{\sqrt{\sin^2(2\alpha) + \sinh^2(2\beta)}}{\cos(2\alpha) + \cosh(2\beta)} \tag{242}$$

and that the phase lag is given by

$$\varphi = \tan^{-1}\left[\frac{\sinh(2\beta)}{\sin(2\alpha)}\right] + \delta/2 \ . \tag{243}$$

Similarly, (238) can be written in the form

$$\theta_s(\ell,t) = \frac{c\ M_o}{\omega G_a^*(\omega) C} \ \frac{\sqrt{\sin^2(2\alpha) + \sinh^2(2\beta)}}{\cos(2\alpha) + \cosh(2\beta)} \ \sin(\omega t - \delta/2 - \psi) \ , \tag{244}$$

where ψ is again given by (240). Thus, we can also here use the representation

$$\theta_s(\ell,t) = \text{Amp}\ [\theta_s]\ \sin(\omega t - \varphi) \ ,$$

where Amp $[\theta_s]$ is given by (242) and φ is given by (243).

Figure 3 shows a plot of Amp $[\theta_c]$ as a function of input frequency for a viscoelastic fiber reinforced circular cylinder. Also shown is the amplitude variation with ω for the elastic vibrations, with $G_m^R(\theta)$ taken as the matrix elastic shear modulus.

The data used are as follows:

$\ell = 5.0$ ft. length of cylinder

$d = 4.0$ in. diameter of section

$\rho = 3.0$ density relative to water

$$G_m^R(\omega) = G_m^R(0)\ [1 + \frac{1}{4}\log_{10}\omega],\ 1 \leq \omega \leq 10^4\ \text{sec}^{-1}$$

$$G_m^R(0) = 0.5 \times 10^6 \text{ psi}$$

$$\tan\delta_m = 0.1$$

$$v_m = 0.4 \qquad v_f = 0.6.$$

It is seen that in the neighborhood of the first elastic resonant frequency, the viscoelastic amplitude has a peak which is about five times that of the initial amplitude. The next peak which comes behind the second elastic resonant frequency is already very damped out. The damping increases further in the neighborhood of higher elastic resonant frequencies.

It is thus seen that the viscoelastic nature of the matrix has a very beneficial effect on torsional vibrations.

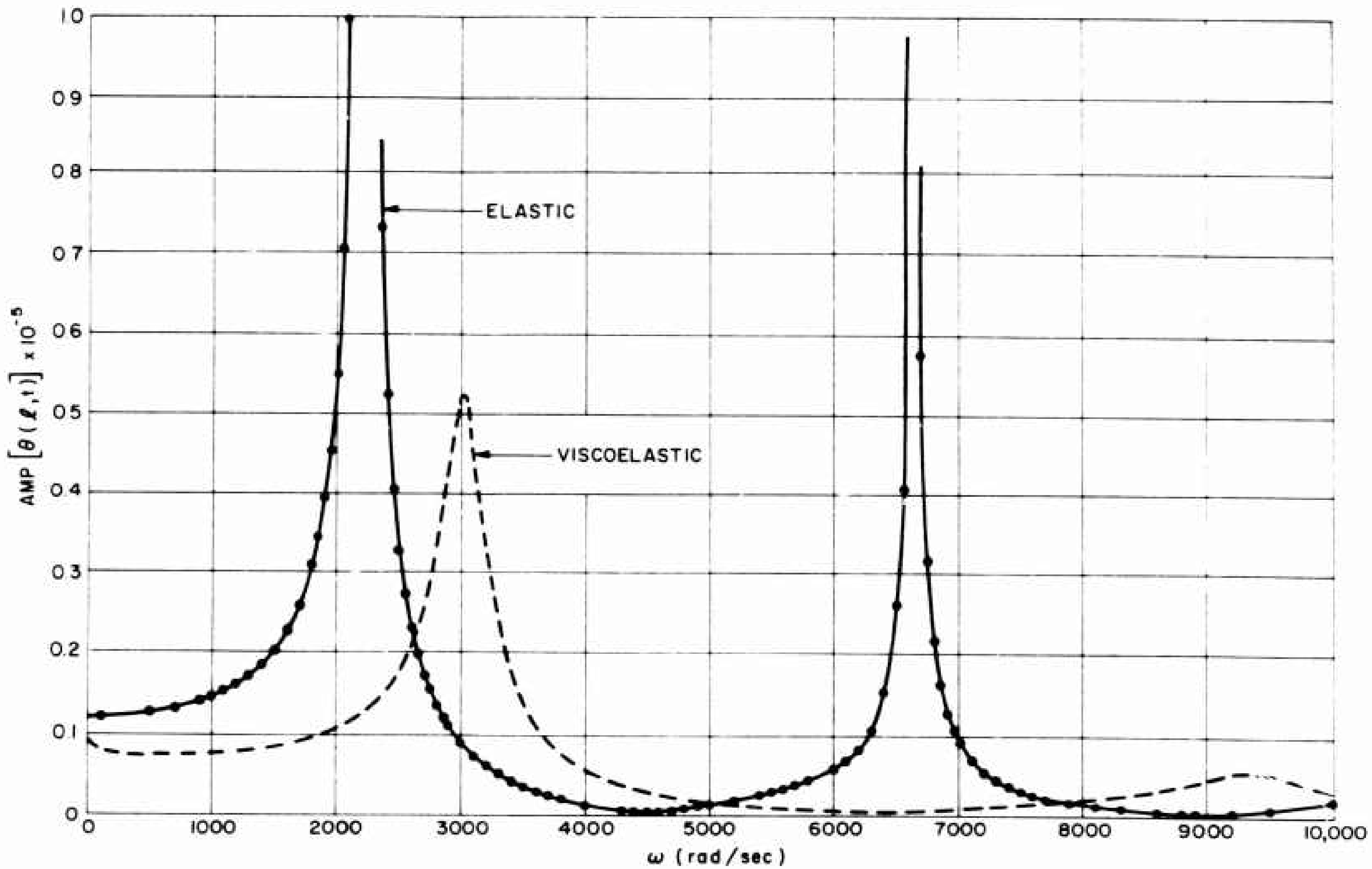

Figure 3. Amplitude of Angle of Twist.

7. TRANSVERSE VIBRATIONS OF UNIAXIALLY FIBER REINFORCED BEAM

We now consider the problem of the forced vibrations of a beam of constant cross section. Let x be measured along the beam axis, which is also the direction of fiber reinforcement. The equation of motion of an elastic fiber reinforced beam is

$$a^2 \frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} = \frac{1}{\rho A} p(x,t) \ , \tag{246}$$

$$a^2 = \frac{E_a^* I}{\rho A} \ , \tag{247}$$

where w is the transverse deflection, ρ is the average density, p is the load per unit length on the beam as a function of position and time, I is the moment of inertia I_y, and A is the cross-sectional area.

We consider the important case of a simply supported beam which is loaded at $x = \xi$ by the concentrated force $P_o e^{i\omega t}$, Figure 4. For this case the boundary conditions are

$$w(o,t) = \frac{\partial^2 w}{\partial x^2}(o,t) = w(\ell,t) = \frac{\partial^2 w}{\partial x^2}(\ell,t) = 0 \ , \tag{248}$$

and the loading function p(x,t) assumes the form

$$p(x,t) = P_o \delta\,(x-\xi)\, e^{i\omega t} \ , \tag{249}$$

where δ is the delta function.

The solution to the problem described by (246) through (249) is well known and is given by

$$w^E(x,t) = \frac{2P_o e^{i\omega t}}{E_a^* I \ell} \sum_{n=1}^{\infty} \frac{\sin(\alpha_n \xi)\,\sin(\alpha_n x)}{\alpha_n^4 \left(1 - \frac{\omega^2}{\omega_n^2}\right)} \ , \tag{250}$$

where

$$\alpha_n = \frac{n\pi}{\ell} \ , \tag{251}$$

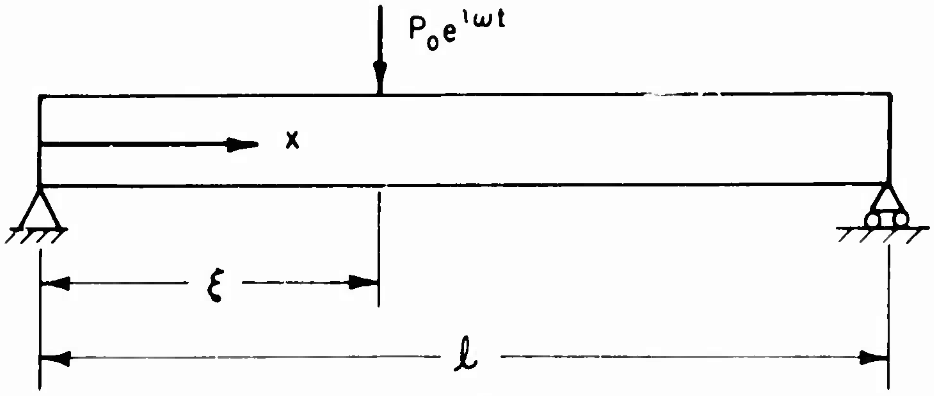

Figure 4. Lateral Vibrations of Viscoelastic Fiber Reinforced Beam.

$$\omega_n^2 = \alpha_n^4 \frac{E_a^* I}{\rho A} \, . \tag{252}$$

We rewrite the solution (250) in the form

$$w^E(x,t) = e^{i\omega t} \sum_{n=1}^{\infty} w_n^E(x) \, , \tag{253}$$

where

$$w_n^E(x) = \frac{2P_o}{E_a^* I \ell} \frac{\sin(\alpha_n \xi) \sin(\alpha_n x)}{\alpha_n^4 \left(1 - \frac{\omega^2}{\omega_n^2}\right)} \, . \tag{254}$$

In view of (252), Equation (254) can be written as

$$w_n^E(x) = \frac{2P_o}{\ell} \frac{\sin(\alpha_n \xi) \sin(\alpha_n x)}{E_a^* I \alpha_n^4 - \omega^2 \rho A} \, . \tag{255}$$

The complex viscoelastic solution then has the form

$$w^{VE}(x) = e^{i\omega t} \sum_{n=1}^{\infty} \tilde{w}_n(x) \, , \tag{256}$$

where

$$\tilde{w}_n(x) = \frac{2P_o}{\ell} \frac{\sin(\alpha_n \xi) \sin(\alpha_n x)}{\check{E}_a^*(\iota\omega) I \alpha_n^4 - \omega^2 \rho} . \qquad (257)$$

In view of (147) through (149),

$$\check{E}_a^*(\iota\omega) = v_m E_m^{*R} + v_f E_f + \iota v_m E_m^I . \qquad (258)$$

We rewrite (258) in the form

$$E_a^*(\iota\omega) = E_a^*(\omega)(1 + \iota \tan \delta^*) , \qquad (259)$$

where E_a^* is given by (169) and also by (148), and $\tan \delta^*$ is given by (167). Substitution of (259) into (257) and rearrangement yields

$$\tilde{w}_n(x) = \frac{2P_o}{\ell} \frac{E_a^*(\omega) I \alpha_n^4 - \omega^2 \rho - \iota E_a^*(\omega) I \alpha_n^4 \tan \delta^*}{(E_a^*(\omega) I \alpha_n^4 - \omega^2 \rho)^2 + (E_a^*(\omega) I \alpha_n^4 \tan \delta^*)^2} \sin(\alpha_n \xi) \sin(\alpha_n x) . \qquad (260)$$

We shall adopt the notation

$$\tilde{\omega}_n^2 = \alpha_n^4 \frac{E_a^*(\omega) I}{\rho A} , \qquad (261)$$

which is analogous to (252). Note that the right side of (261) is a function of ω because of the presence of $E_a^*(\omega)$. However, because of the very slow variation of $E_a^*(\omega)$ with ω (see Section 5), (261) is numerically quite close to the value of (252). With the notation (261), Equation (260) can be rewritten in the form

$$\tilde{w}_n(x) = \frac{2P_o}{E_a^*(\omega) I \ell} \frac{1 - \frac{\omega^2}{\tilde{\omega}_n^2} - \iota \tan \delta^*}{\alpha_n^4 \left[\left(1 - \frac{\omega^2}{\tilde{\omega}_n^2}\right)^2 + \tan^2 \delta^*\right]} \sin(\alpha_n x) \sin(\alpha_n \xi) . \qquad (262)$$

It follows that

$$e^{\iota\omega t}\,\tilde{w}_n(x) = \frac{2P_o}{E_a^*(\omega) I \ell \, \alpha_n^4 \left[\left(1 - \frac{\omega^2}{\tilde{\omega}_n^2}\right)^2 + \tan^2 \delta^*\right]}$$

$$\cdot \left\{\left(1 - \frac{\omega^2}{\tilde{\omega}_n^2}\right) \cos \omega t + \tan \delta^* \sin \omega t + \iota \left[\left(1 - \frac{\omega^2}{\tilde{\omega}_n^2}\right) \sin \omega t - \tan \delta^* \cos \omega t\right]\right\}$$

$$\cdot \sin (\alpha_n x) \sin (\alpha_n \xi) \quad . \qquad (263)$$

This completely defines the dynamic viscoelastic solution (256). Thus, if the applied force is $P_o \cos \omega t$,

$$w_c^{VE}(x,t) = \frac{2P_o}{E_a^*(\omega) I \ell} \sum_{n=1}^{\infty} \frac{\left(1 - \frac{\omega^2}{\tilde{\omega}_n^2}\right) \cos \omega t + \tan \delta^* \sin \omega t}{\alpha_n^4 \left[\left(1 - \frac{\omega^2}{\tilde{\omega}_n^2}\right) + \tan^2 \delta^*\right]} \sin(\alpha_n \xi) \sin(\alpha_n x) . \qquad (264)$$

If the applied force is $P_o \sin \omega t$, we have

$$w_s^{VE}(x,t) = \frac{2P_o}{E_a^*(\omega) I \ell} \sum_{n=1}^{\infty} \frac{\left(1 - \frac{\omega^2}{\tilde{\omega}_n^2}\right) \sin \omega t - \tan \delta^* \sin \omega t}{\alpha_n^4 \left[\left(1 - \frac{\omega^2}{\tilde{\omega}_n^2}\right) + \tan^2 \delta^*\right]} \sin(\alpha_n \xi) \sin(\alpha_n x) . \qquad (265)$$

Defining

$$A = \sum_{n=1}^{\infty} \frac{\left(1 - \frac{\omega^2}{\tilde{\omega}_n^2}\right) \sin(\alpha_n \xi) \sin(\alpha_n x)}{\alpha_n^4 \left[\left(1 - \frac{\omega^2}{\tilde{\omega}_n^2}\right) + \tan^2 \delta^*\right]} \qquad (266)$$

$$B = \sum_{n=1}^{\infty} \frac{\tan \delta^* \sin(\alpha_n \xi) \sin(\alpha_n x)}{\alpha_n^4 \left[\left(1 - \frac{\omega^2}{\tilde{\omega}_n^2}\right) + \tan^2 \delta^*\right]} \qquad (267)$$

Equations (264) and (265) may be written in the form

$$w_c^{VE}(x,t) = \text{Amp}\left[w_c^{VE}\right] \cos(\omega t - \varphi) , \qquad (268)$$

$$w_s^{VE}(x,t) = \text{Amp}\left[w_s^{VE}\right] \sin(\omega t - \varphi) , \qquad (269)$$

where

$$\text{Amp}\left[w_c^{VE}\right] = \text{Amp}\left[w_s^{VE}\right] = \text{Amp}\left[w^{VE}\right] = \frac{2P_o}{E_a^*(\omega) I \ell} \left[A^2 + B^2\right]^{1/2} \qquad (270)$$

and

$$\varphi = \tan^{-1} \frac{B}{A} . \qquad (271)$$

Since δ^* is a very small angle, it is reasonable to neglect $\tan^2 \delta^*$ in the denominator of (271), unless ω is almost equal to $\tilde{\omega}_n$. Barring this eventuality, if one remembers that $E_a^*(\omega)$ is a very slowly increasing function of ω, it appears that the amplitude is very nearly that of elastic vibrations with Young's modulus $E_a^*(0)$. Also, if ω is not close to $\tilde{\omega}_n$, the phase lag defined by (271) is very small and can be neglected.

It is seen that the denominator in (270) cannot vanish, and thus there is no resonance in theory. Yet the denominator can still become very small. Consider the case

$$\omega = \tilde{\omega}_k . \qquad (272)$$

Since $\tilde{\omega}_n$ as defined by (261) involves $E_a^*(\omega)$, it is seen that (272) is an equation for ω which must be solved. Let a solution of (272) be denoted by ω_k^{ρ}. Then (266) and (267) assume the form

$$A = \sum_{\substack{n=1 \\ n\neq k}}^{\infty} \frac{\left(1 - \frac{\omega_k^{c\,2}}{\tilde{\omega}_n^2}\right) \sin(\alpha_n \xi)\, \sin(\alpha_n x)}{\alpha_n^4 \left[\left(1 - \frac{\omega_k^{o}}{\tilde{\omega}_n^2}\right)^2 + \tan^2 \delta^*\right]} \qquad \text{(a)}$$

$$B = \sum_{\substack{n=1 \\ n\neq k}}^{\infty} \frac{\tan \delta^* \sin(\alpha_n \xi)\, \sin(\alpha_n x)}{\alpha_n^4 \left[\left(1 - \frac{\omega_k^{o}}{\tilde{\omega}_n}\right)^2 + \tan^2 \delta^*\right]} + \frac{\sin(\alpha_k \xi)\, \sin(\alpha_k x)}{\alpha_k^4 \tan \delta^*} \; . \qquad \text{(b)} \qquad (273)$$

It is seen that for small k, i.e., the first solutions of (272), the last term in (273b) will become very large; thus, we have a pseudo-resonance condition.

It is also seen that when (272) is fulfilled, Equation (267) gives

$$\tan \varphi \to \infty, \; \varphi = \frac{\pi}{2} \; , \qquad (274)$$

which implies a phase reversal of the vibration, since then

$$\cos(\omega t - \varphi) = \sin \omega t$$

$$\sin(\omega t - \varphi) = - \cos \omega t \; . \qquad (275)$$

In order to obtain an idea of the comparison of elastic and viscoelastic vibrations, the deflection at the center of a beam loaded at its center has been calculated numerically. In that case,

$$\xi = x = \frac{\ell}{2} \; ; \qquad (276)$$

therefore,

$$\sin(\alpha_n \xi)\, \sin(\alpha_n x) = \sin^2\left(\frac{n\pi}{2}\right) = \begin{cases} 1 & n \text{ odd} \\ o & n \text{ even} \; . \end{cases} \qquad (277)$$

Thus for that case,

$$A = \sum_{n=1,3,5}^{\infty} \frac{1 - \frac{\omega^2}{\bar{\omega}_n^2}}{\alpha_n^4\left[\left(1 - \frac{\omega^2}{\bar{\omega}_n^2}\right) + \tan^2 \delta^*\right]} , \tag{278}$$

$$B = \tan \delta^* \times \sum_{n=1,3,5}^{\infty} \frac{1}{\alpha_n^4\left[\left(1 - \frac{\omega^2}{\bar{\omega}_n^2}\right) + \tan^2 \delta^*\right]} . \tag{279}$$

The amplitude (279) has been calculated numerically as a function of ω for a beam of length $\ell = 50$ ft and section 2 in. x 4 in. The material characteristics chosen were as follows:

$\rho = 3.0$ density relative to water

$\tan \delta_m = 0.1$

$E_f = 27 \times 10^6$ psi

$E_m^R(\omega) = E_m^R(o)\ [1 + \frac{1}{4} \log_{10}\omega],\quad 1 \leq \omega \leq 10^4\ (\sec^{-1})$

$E_m^R(o) = 1.35 \times 10^6$ psi

$v_m = 0.4 \qquad v_f = 0.6.$

The variation of the amplitude with ω is shown in Figure 5 for elastic vibrations using $E_a^*(o)$ as elastic modulus and for viscoelastic vibrations. Elastic analysis is shown by a full curve, and viscoelastic analysis is shown by circles. It is seen that for the first elastic resonant frequency resonance is attained for the viscoelastic vibration also. At second elastic resonant frequency, there is already a small deviation between the two, but still the viscoelastic amplitude is very large.

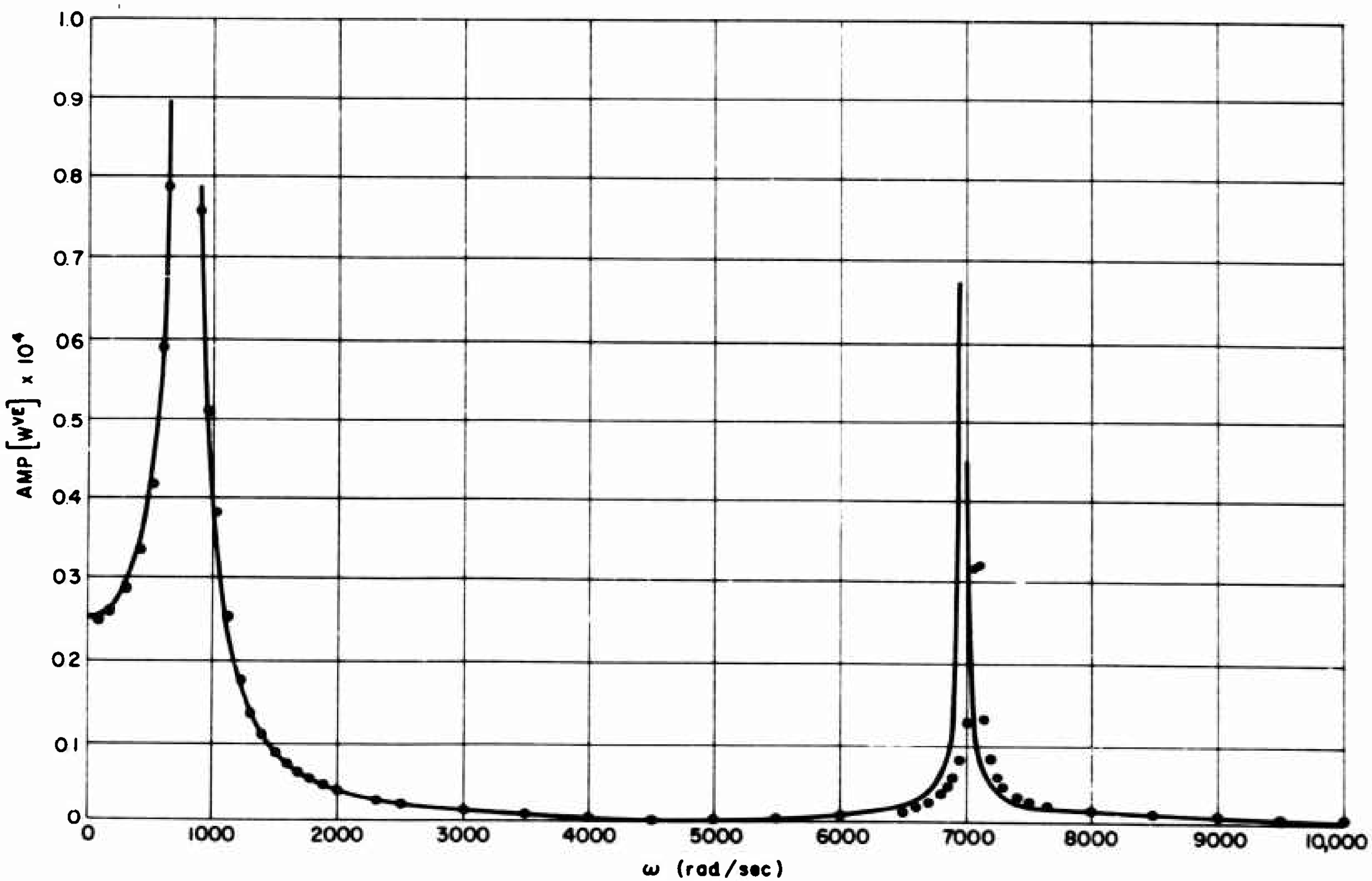

Figure 5. Amplitude of Lateral Deflection of Beam.

8. SAMPLE CALCULATIONS OF VISCOELASTIC PROPERTIES

Results for static viscoelastic properties of uniaxially fiber reinforced materials have been given in Section 2. It has been seen that considerable simplification is achieved when it is assumed that the fibers are perfectly rigid relative to the matrix. However, this assumption is not permissible in the calculation of $E_a^*(t)$; but fortunately, the assumption is not needed here, since (63) is an explicit, simple expression incorporating finite elastic stiffness of fibers.

In order to obtain an idea of the approximation involved in the assumption of rigid fibers, we perform a sample calculation of the axial shear relaxation modulus $G_a^*(t)$ for the case of elastic fibers. The Laplace transform of $G_a^*(t)$ is given by (59). To be specific, it is assumed that the matrix behaves according to the simple Maxwell model. Thus, the deviation matrix stress-strain relation has the form

$$\dot{e}_{ij} = \frac{\dot{s}_{ij}}{2G_m} + \frac{s_{ij}}{2\eta_m} , \tag{280}$$

where G_m is the elastic (initial) shear modulus, η_m is a viscosity coefficient, and a dot indicates time derivative. The theory given in Reference [3] includes the use of time differential operator stress-strain relations of phases, such as (280), for computation of effective viscoelastic properties. It is also possible to tie in (280) directly with a more general stress-strain relation of type (49). To see this, take the LT of (49), which is

$$\hat{s}_{ij}(p) = 2p\, \hat{G}_m(p)\, \hat{e}_{ij}(p) . \tag{281}$$

Now the LT of (280) is

$$p\hat{e}_{ij}(p) - e_{ij}(o) = \frac{p\, \hat{s}_{ij}(p)}{2G_m} - \frac{s_{ij}(o)}{2G_m} + \frac{\hat{s}_{ij}(p)}{2\eta_m} . \tag{282}$$

Evidently

$$e_{ij}(o) = \frac{s_{ij}(o)}{2G_m} , \tag{283}$$

which expresses the initial elastic response. Taking into account (283) and (282) in (281) and rearranging yields

$$\hat{s}_{ij}(p) = \frac{2p\eta_m}{1 + T_m p}\, \hat{e}_{ij}(p) , \tag{284}$$

where

$$T_m = \frac{\eta_m}{G_m} \tag{285}$$

is the relaxation time of the Maxwell matrix. Comparison of (284) with (281) shows that

$$p\hat{G}_m(p) = \frac{\eta_m p}{1 + T_m p} \,. \tag{286}$$

Now (286) is substituted into (59), yielding

$$\hat{G}_a^*(p) = \frac{G_f(1+c)\ (1+T_m p) + \eta_m(1-c)p}{G_f(1-c)\ (1+T_m p) + \eta_m(1-c)p} \cdot \frac{\eta_m}{1 + T_m p} \,. \tag{287}$$

Equation (287) is easily inverted into the time domain by the method of partial fractions. The final result is

$$\frac{G_a^*(t)}{G_m} = \frac{\phi\left[\frac{4c}{1+c}\right]}{\phi(1-c) + 1+c} \exp\left[-\frac{\phi(1-c)}{\phi(1-c) + 1+c} \cdot \frac{t}{T_m}\right] + \frac{1-c}{1+c} \exp\left[-\frac{t}{T_m}\right] , \tag{288}$$

where

$$\phi = \frac{G_f}{G_m} \,. \tag{289}$$

When the fibers are rigid, $\phi \to \infty$ and then (288) reduces to

$$\frac{G_a^*(t)}{G_m} = \frac{1+c}{1-c} \exp\left[-\frac{t}{T_m}\right] . \tag{290}$$

Since $G_m \exp\left[-\frac{t}{T_m}\right]$ is the relaxation modulus of the Maxwell matrix, (290) is in accordance with the general result (60).

In order to assess the numerical importance of ϕ, (288) has been plotted in Figure 6 as a function of t/T_m for several values of ϕ, assuming that $c = 0.5$. It is seen that for $\phi \geq 25$, which is a typical value for fiber reinforced materials, the rigid fiber assumption gives a good approximation. Therefore, this idealization is considered to be justified.

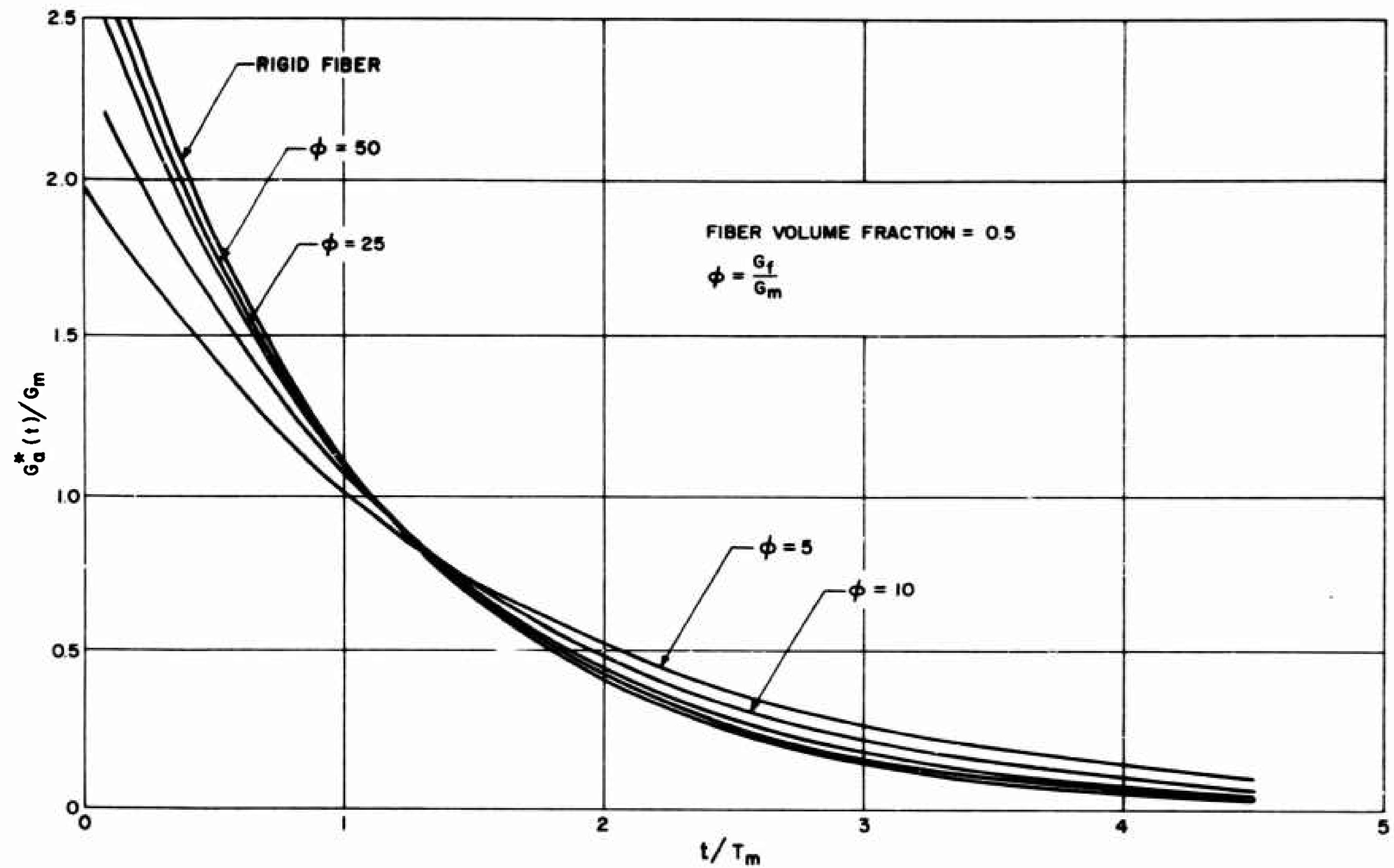

Figure 6. Effective Axial Shear Relaxation Modulus (Maxwell Model Matrix).

Sample calculations of viscoelastic properties of fiber reinforced materials have been performed for a polyisobutylene matrix. The matrix properties have been taken from Tobolsky and Castiff [10]. Initial elastic moduli are

$$K_m(o) = 2.32 \times 10^5 \text{ psi}, \; G_m(o) = 1.85 \times 10^5 \text{ psi}.$$

It is assumed that the matrix is elastic in dilatation. Plots of the shear relaxation modulus $G_m(t)$ and the shear creep compliance $g_m(t)$ are shown in Figures 7 and 8. With the assumption of rigid fibers, the axial shear relaxation modulus $G_a^*(t)$ is given by (61). Plots of $G_a^*(t)$ for various c are shown in Figure 9. The axial shear creep compliance is simply given by

$$g_a^*(t) = \frac{1-c}{1+c} g_m(t) \; . \tag{291}$$

For derivation of (291), see Reference [3]. Equation (291) is plotted for various c in Figure 10.

The plane strain bulk relaxation modulus is given by (58). In the present case, it is assumed that the matrix is elastic in dilatation.

Therefore,

$$K_m(t) = K_m \, H(t) \; , \tag{292}$$

where H(t) is the Heaviside unit step function. Then (58) assumes the form

$$K_T^*(t) = \frac{K_m}{1-c} H(t) + \frac{1 + 3c}{3(1-c)} G_m(t) \; . \tag{293}$$

A plot of (293) for various c is shown in Figure 11.

The situation with respect to the plane strain bulk compliance $k_T^*(t)$ is, however, not so simple. The relationship between K_T^* and k_T^* is expressed in terms of their Laplace transforms by (40). When this expression is written in the form

$$\hat{K}_T^*(p) \; \hat{k}_T^*(p) = \frac{1}{p^2} \; , \tag{294}$$

inversion by use of the convolution theorem for Laplace transforms leads to

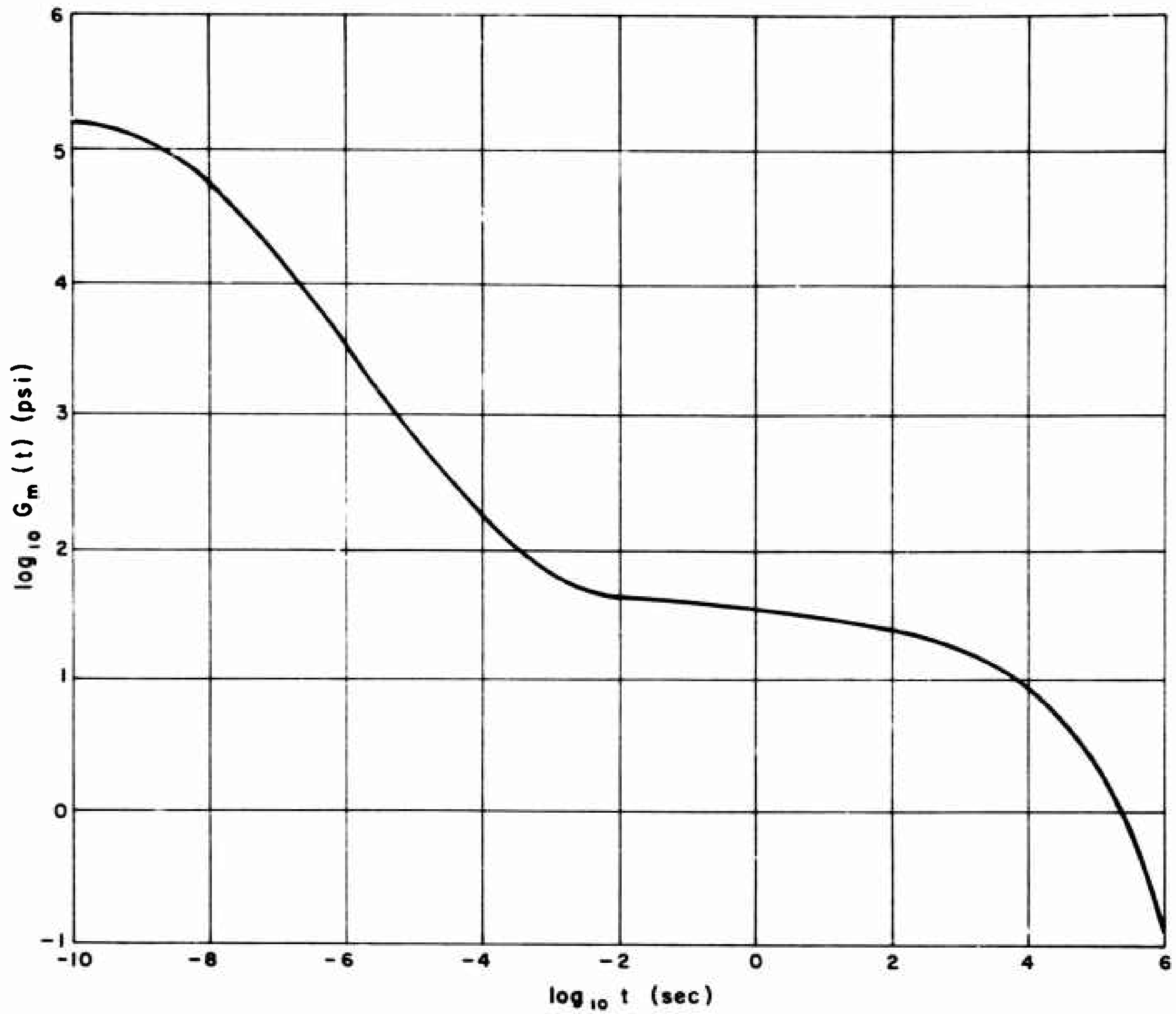

Figure 7. Shear Relaxation Modulus of Polyisobutylene.

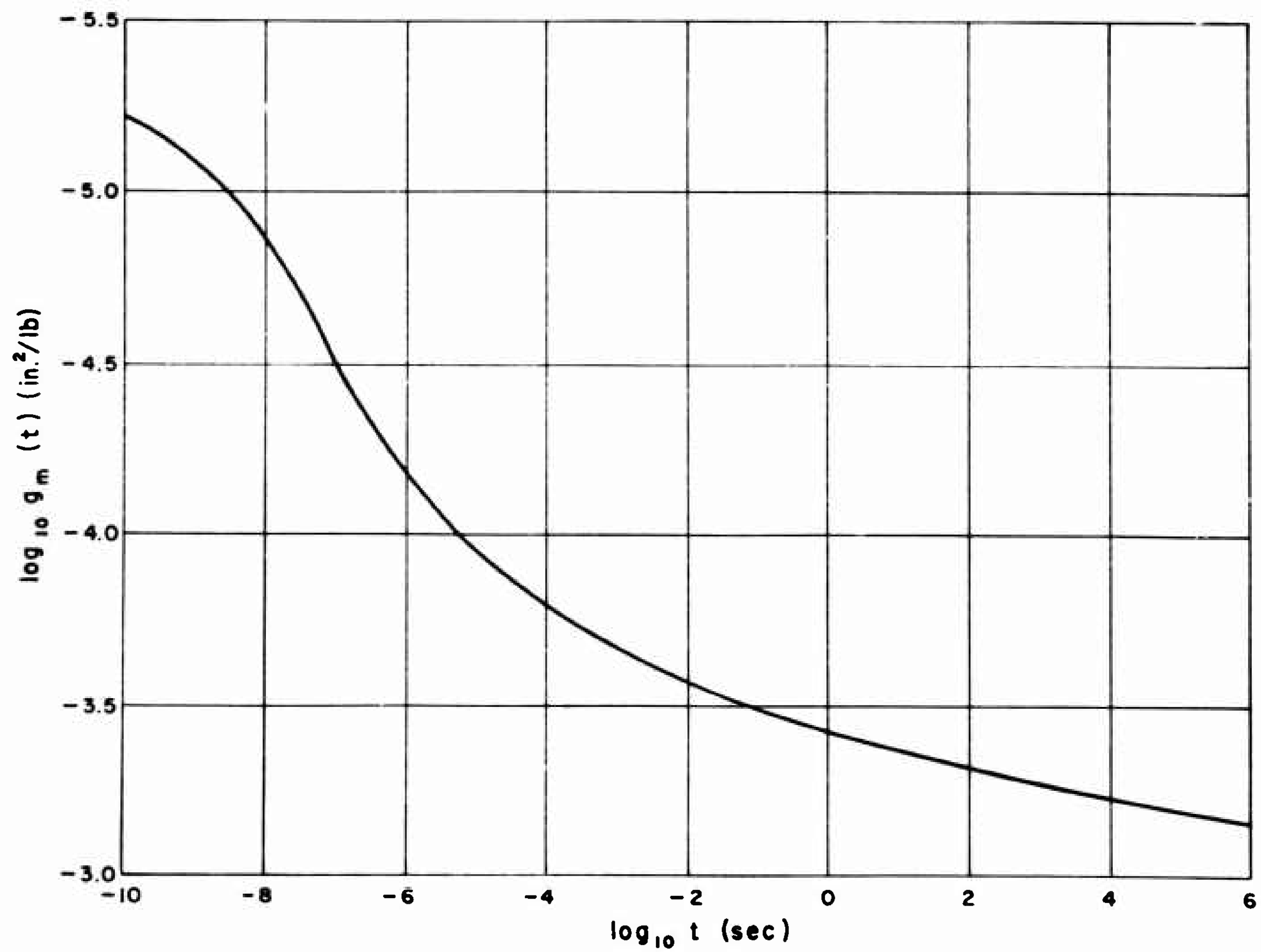

Figure 8. Shear Creep Compliance of Polyisobutylene.

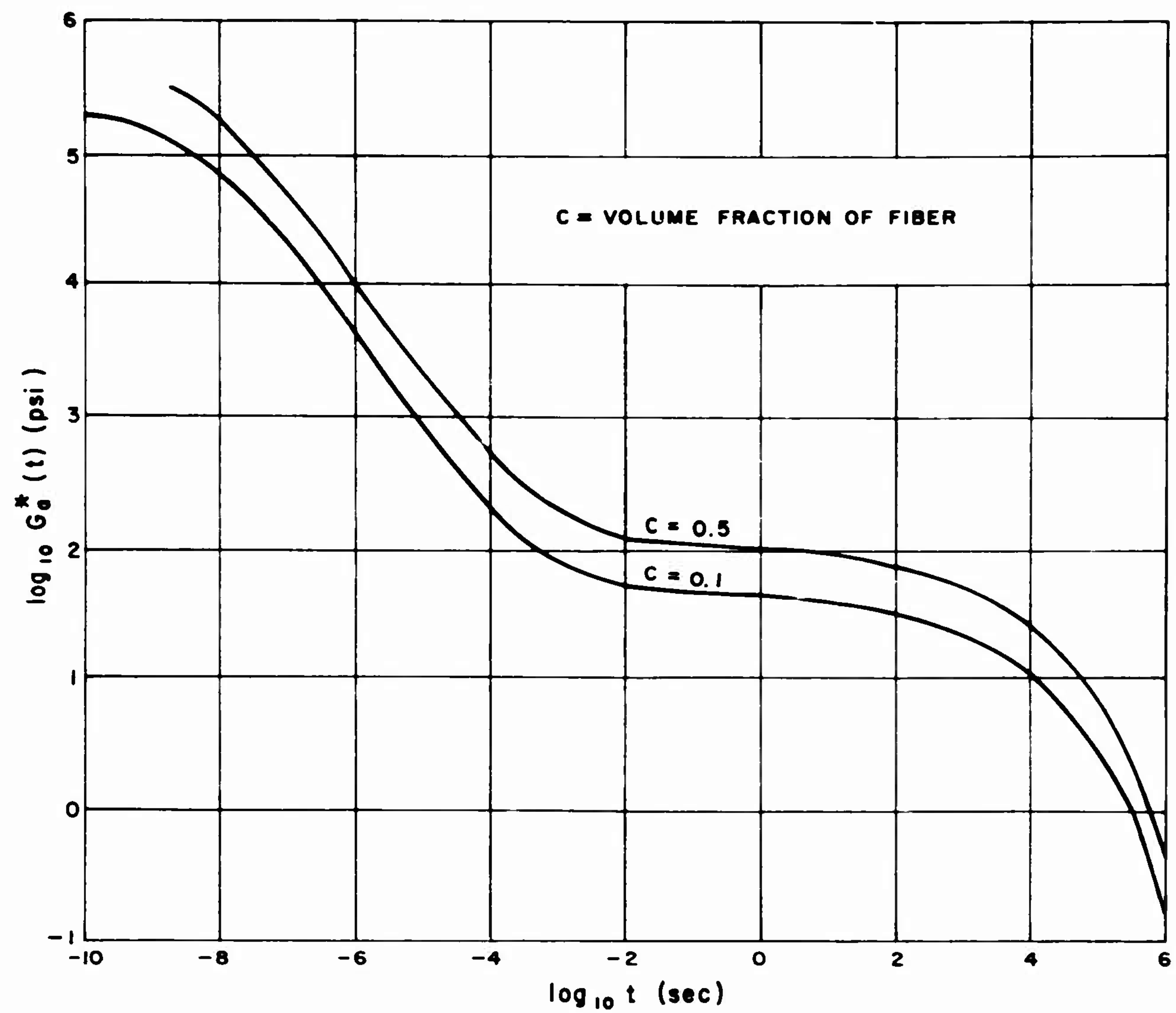

Figure 9. Effective Axial Shear Relaxation Modulus, $G_a^*(t)$ (Polyisobutylene Matrix).

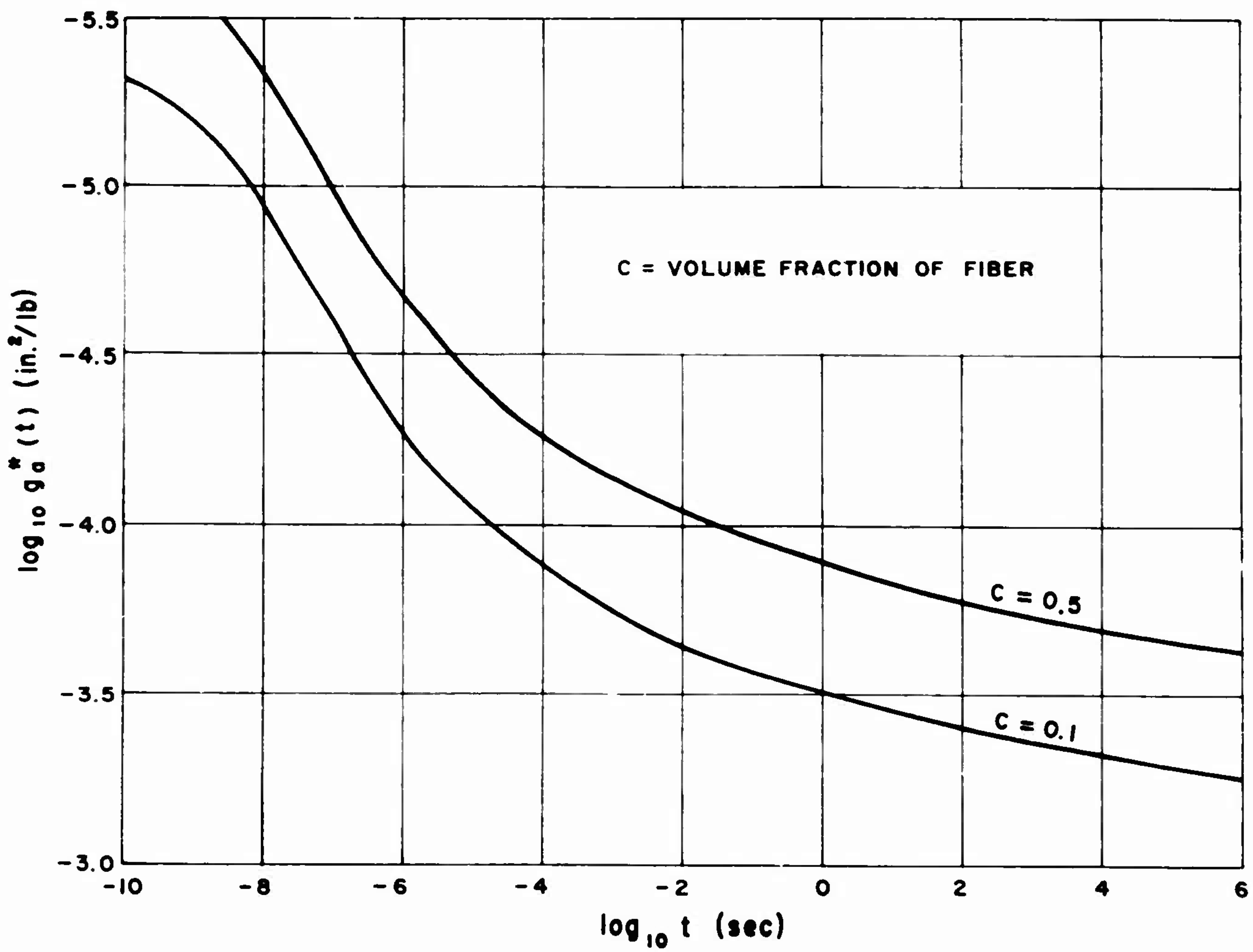

Figure 10. Effective Axial Shear Creep Compliance, $g_a^*(t)$ (Polyisobutylene Matrix).

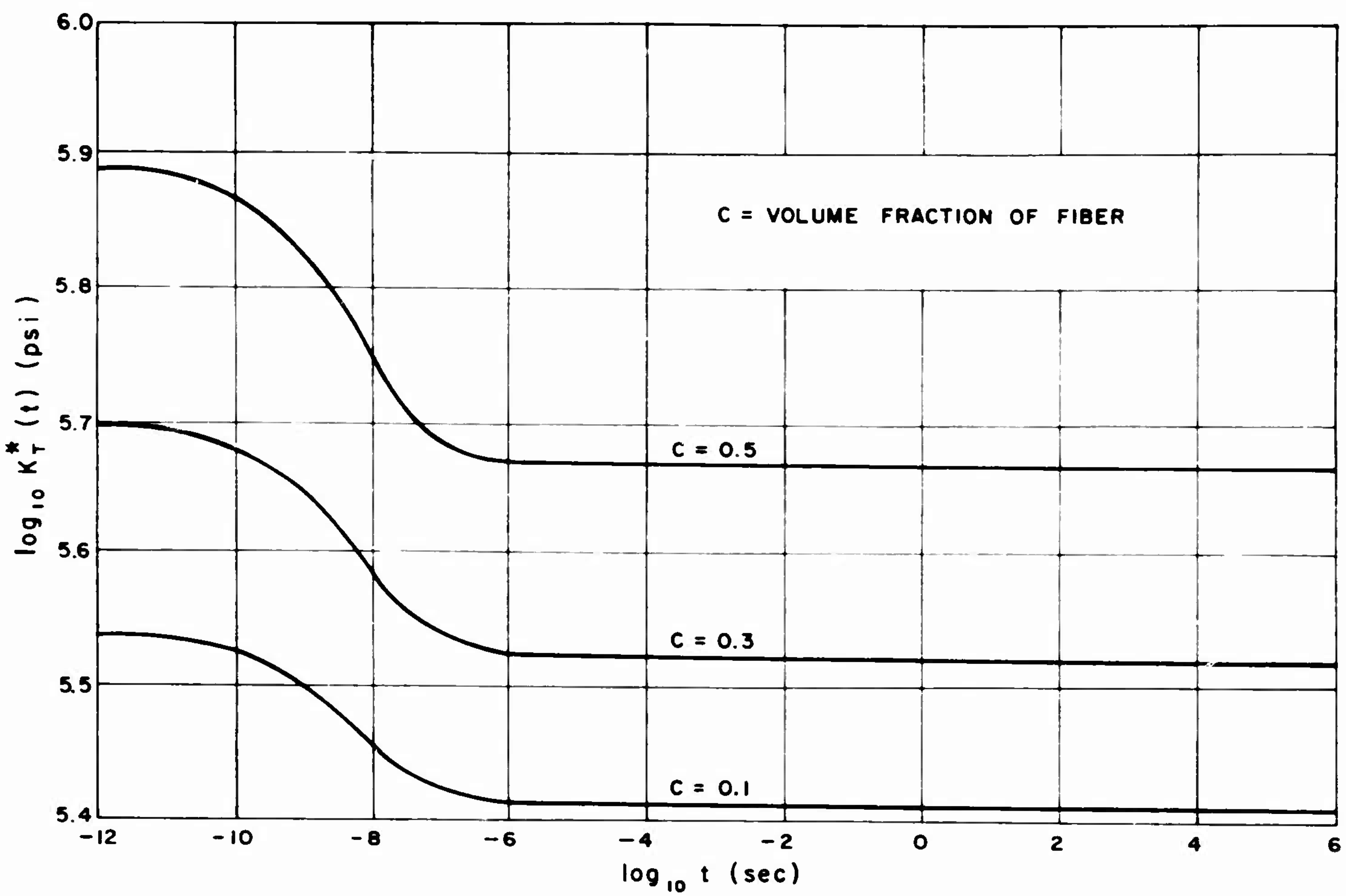

Figure 11. Effective Plane Strain Bulk Relaxation Modulus, $K_T^*(t)$ (Polyisobutylene Matrix).

$$\int_o^t K_T^*(t-\tau)\ k_T^*(\tau)\ d\tau = t\ . \tag{295}$$

Equation (295) is a Volterra integral equation with kernel $K_T^*(t-\tau)$ and unknown function k_T^*. Introduction of (293) into (295) yields

$$\frac{K_m}{1-c}\int_o^t H(t-\tau)\ k_T^*(\tau)\ d\tau + \frac{1+3c}{3(1-c)}\int_o^t G_m(t-\tau)\ k_T^*(\tau)\ d\tau = t\ . \tag{296}$$

Now

$$H(t-\tau) = 1 \qquad t>\tau\ , \qquad (a)$$

$$H(t-\tau) = o \qquad t>\tau\ . \qquad (b) \tag{297}$$

It is seen that (297b) is impossible since $o\leq\tau\leq t$. Therefore, $H(t-\tau)$ in (296) is simply replaced by unity. Then (296) takes the form

$$\int_o^t \left[\frac{K_m}{1-c} + \frac{1+3c}{3(1-c)}G_m(t-\tau)\right] k_T^*(\tau)\ d\tau = t\ . \tag{298}$$

The integral equation (298) has been solved numerically by the method of Lee and Rogers [11] and by the use of the given experimental information K_m and $G_m(t)$. Plots of k_T^* for various values of c are shown in Figure 12.

The Young's relaxation modulus $E_a^*(t)$ is given by the simple expression (63). Here it is not permissible to assume rigid fibers, since the (63) would become infinite. While (63) is easily evaluated, there is little point in doing this; for it has been shown in (63) that for the usual stiff fibers which are encountered in practice, the time dependence of (63) is negligible. A very good approximation for (63) is simply

$$E_a^* \cong cE_f = \text{constant},\ t>o\ . \tag{299}$$

Similarly, the Young's creep compliance $e_a^*(f)$ is very well approximated by the inverse of (299). Thus,

$$e_a^* \cong \frac{1}{cE_f} = \text{constant},\ t>o\ . \tag{300}$$

Another important viscoelastic modulus which has not been discussed in detail, so far, is $G_T^*(t)$. The definition of this modulus is given by (23) and (25). Unfortunately, the composite cylinder assemblage

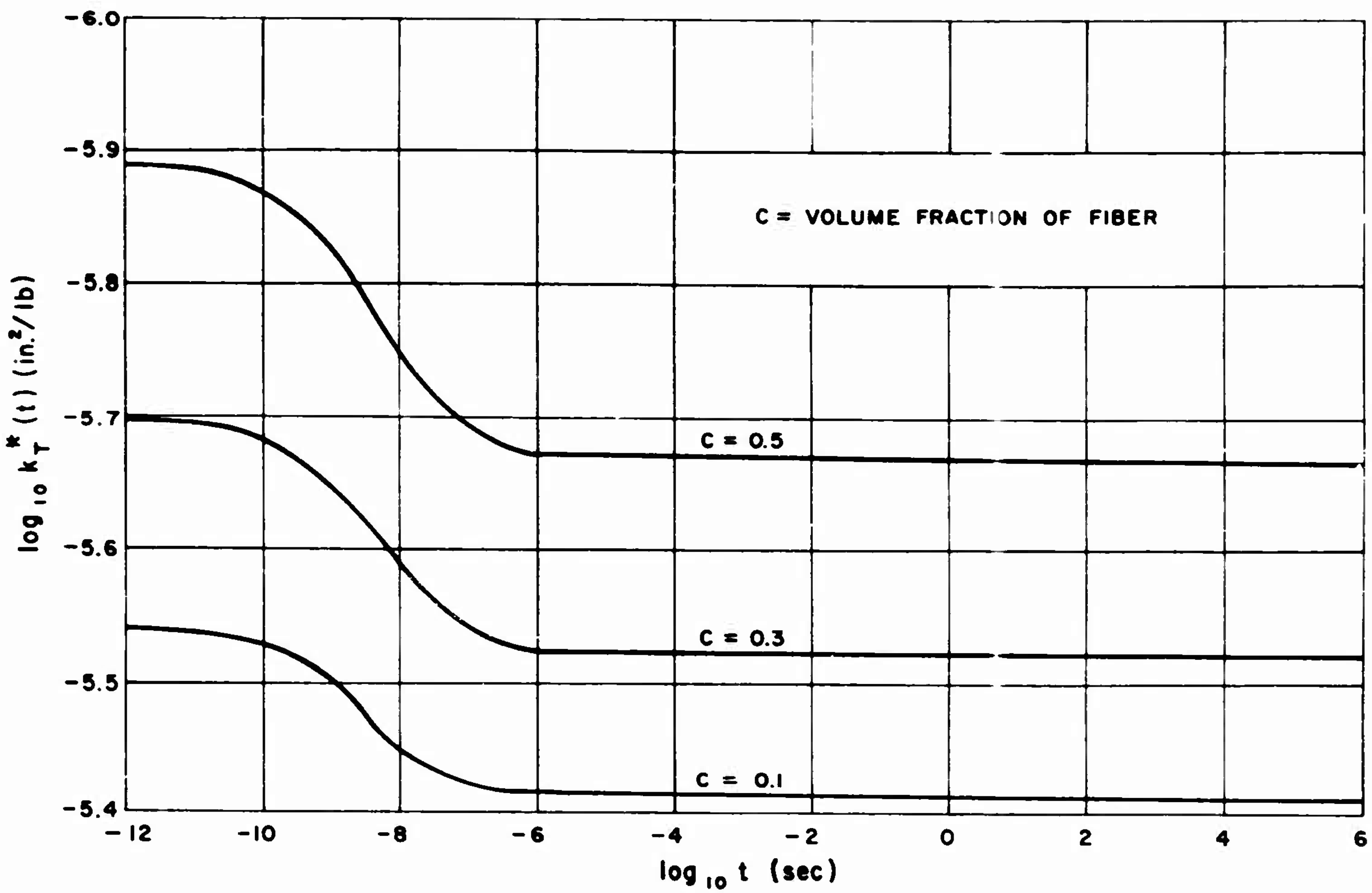

Figure 12. Effective Plane Strain Bulk Creep Compliance, $k_T^*(t)$ (Polyisobutylene Matrix).

analysis performed in Reference [1] has not yielded a closed result for the corresponding G_T^{*E} in the elastic case. Only lower and upper bounds have been obtained. The lower bound for G_T^{*E} is given by

$$G_{T(-)}^{*E} = G_m^E \left\{1 + v_f \Bigg/ \left[\frac{G_m^E}{G_f^E - G_m^E} + \frac{k_m^E + 2G_m^E}{2(k_m^E + G_m^E)} v_m\right]\right\} \tag{301}$$

where superscripts E denote elastic moduli; see Reference [2], page 19. The upper bound for G_T^{*E} is very complicated and will not be reproduced here. In the event that the bounds are close, the elastic-viscoelastic correspondence principle for effective properties, as expounded in Reference [3], may be used to convert (301) into the approximate Laplace transform, $\bar{G}_T^*(p)$ of $G_T^*(t)$ for the viscoelastic fiber reinforced material. (Strictly speaking, this is not legal). Considerable simplification is achieved if the fibers are taken to be rigid and the matrix is taken to be incompressible. Then it is easily shown that (301) in transform domain reduces to (60). We thus obtain the approximation

$$G_T^*(t) \sim \frac{1+c}{1-c} G_m(t) = G_a^*(t) \ . \tag{302}$$

It is to be expected that for rigid fibers and a nearly incompressible matrix such as a polymer, (302) will provide a fair approximation.

We now turn to the computation of effective complex moduli. Again, a polyisobutylene matrix is chosen, and the fibers are assumed to be rigid. The matrix is again assumed to be elastic in dilatation. This implies that the complex bulk modulus (133) of the matrix satisfies the conditions

$$K_m^R(\omega) = K_m = \text{constant}, \quad \text{(a)}$$

$$K_m^I(\omega) = o \ . \quad \text{(b)} \tag{303}$$

The complex shear modulus $G_m(i\omega)$ values have again been taken from Reference [10]. A plot of $G_m^R(\omega)$ is shown in Figure 15 (curve c = o), and a plot of $G_m^I(\omega)$ is given in Figure 16 (curve c = o), both plots being vs. log ω.

K_T^{*R}, K_T^{*I}, G_a^{*R}, G_a^{*I}, E_a^{*R}, and E_a^{*I} have been computed on the basis of Equations (135), (136), (141), (142), (148), and (149), respectively. The results for various c are shown in Figures 13 through 18.

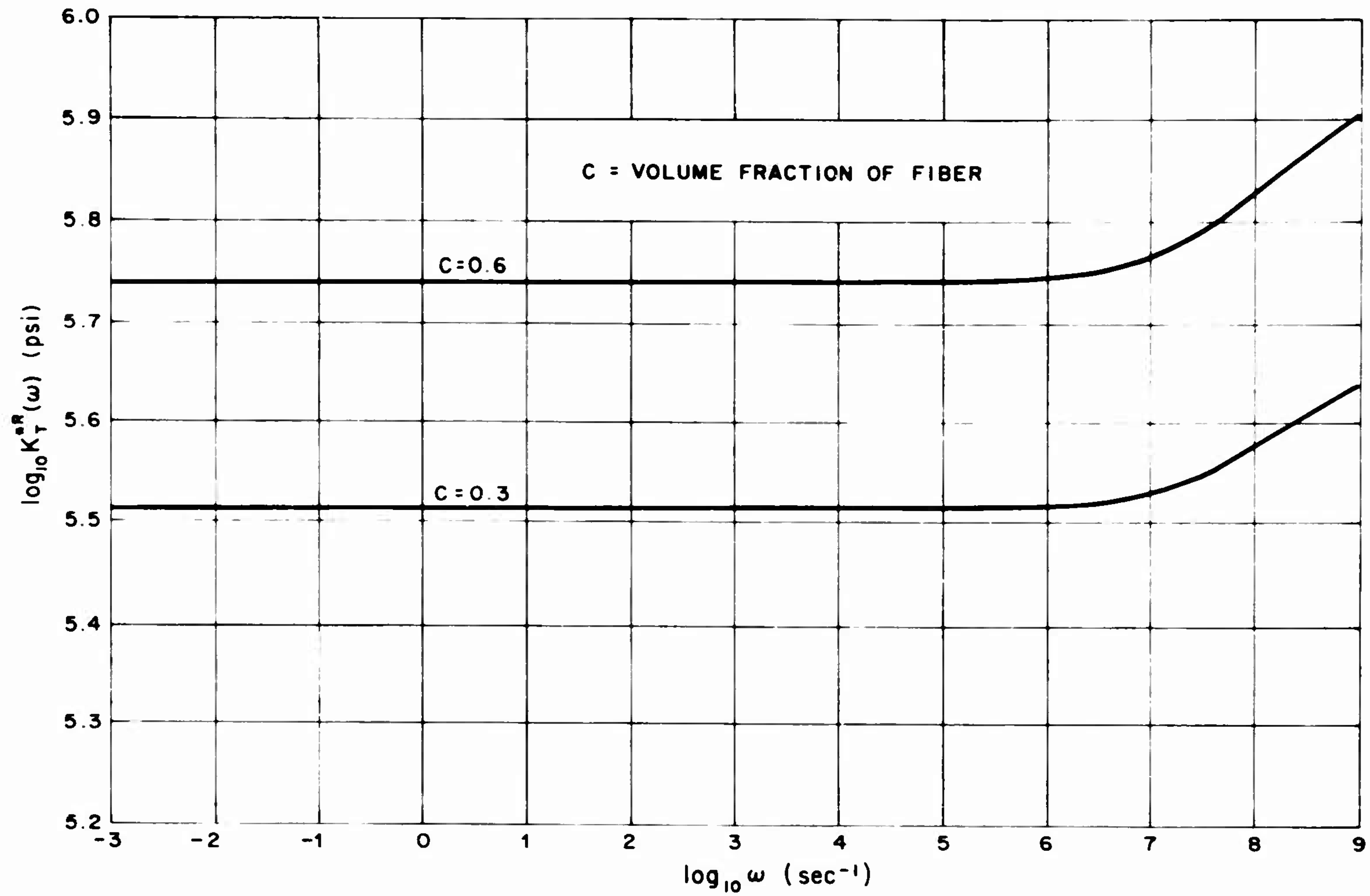

Figure 13. Real Part of Complex Bulk Modulus, $K_T^{*R}(\omega)$ (Polyisobutylene Matrix).

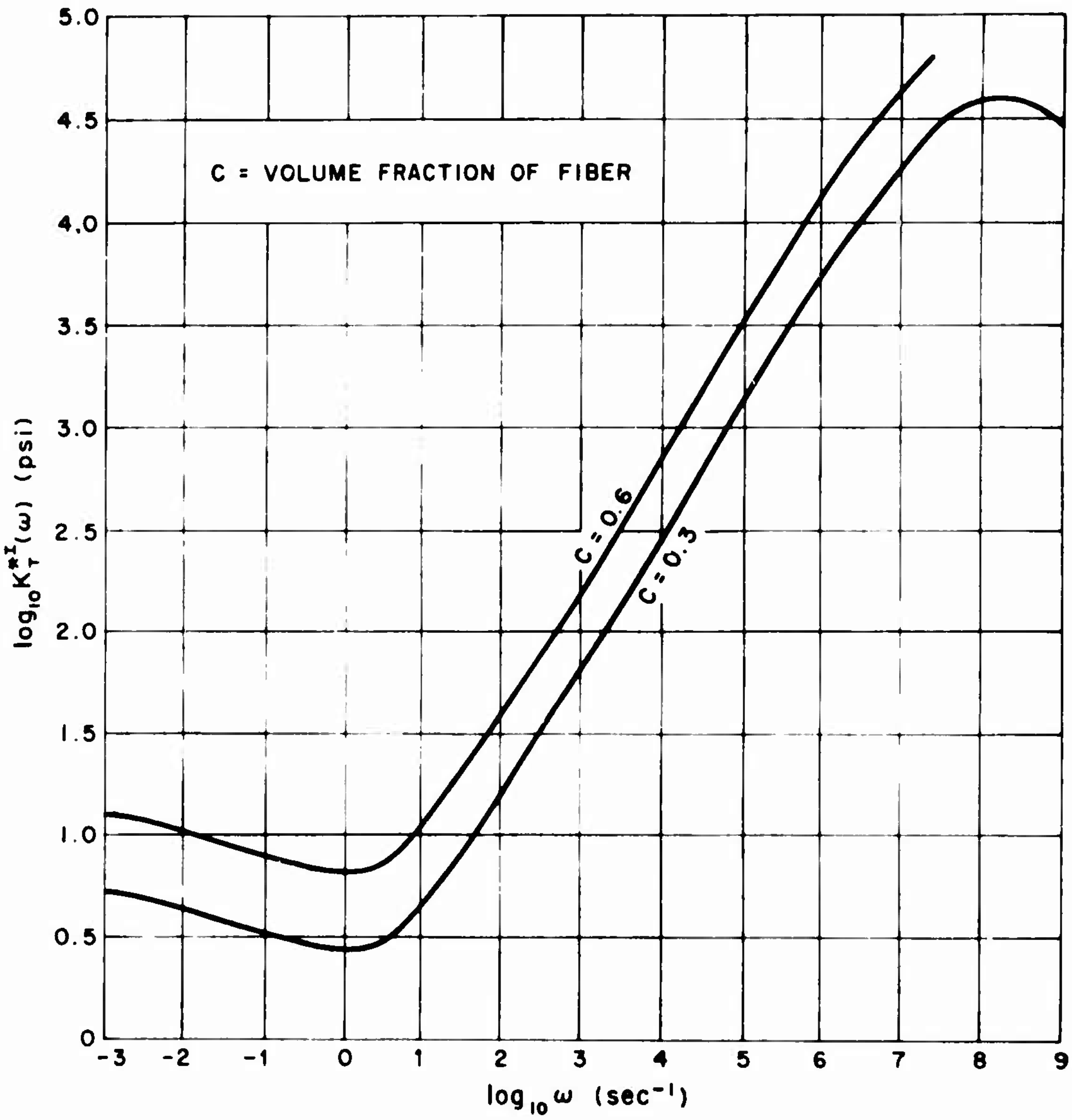

Figure 14. Imaginary Part of Complex Bulk Modulus, $K_T^{*I}(\omega)$ (Polyisobutylene Matrix).

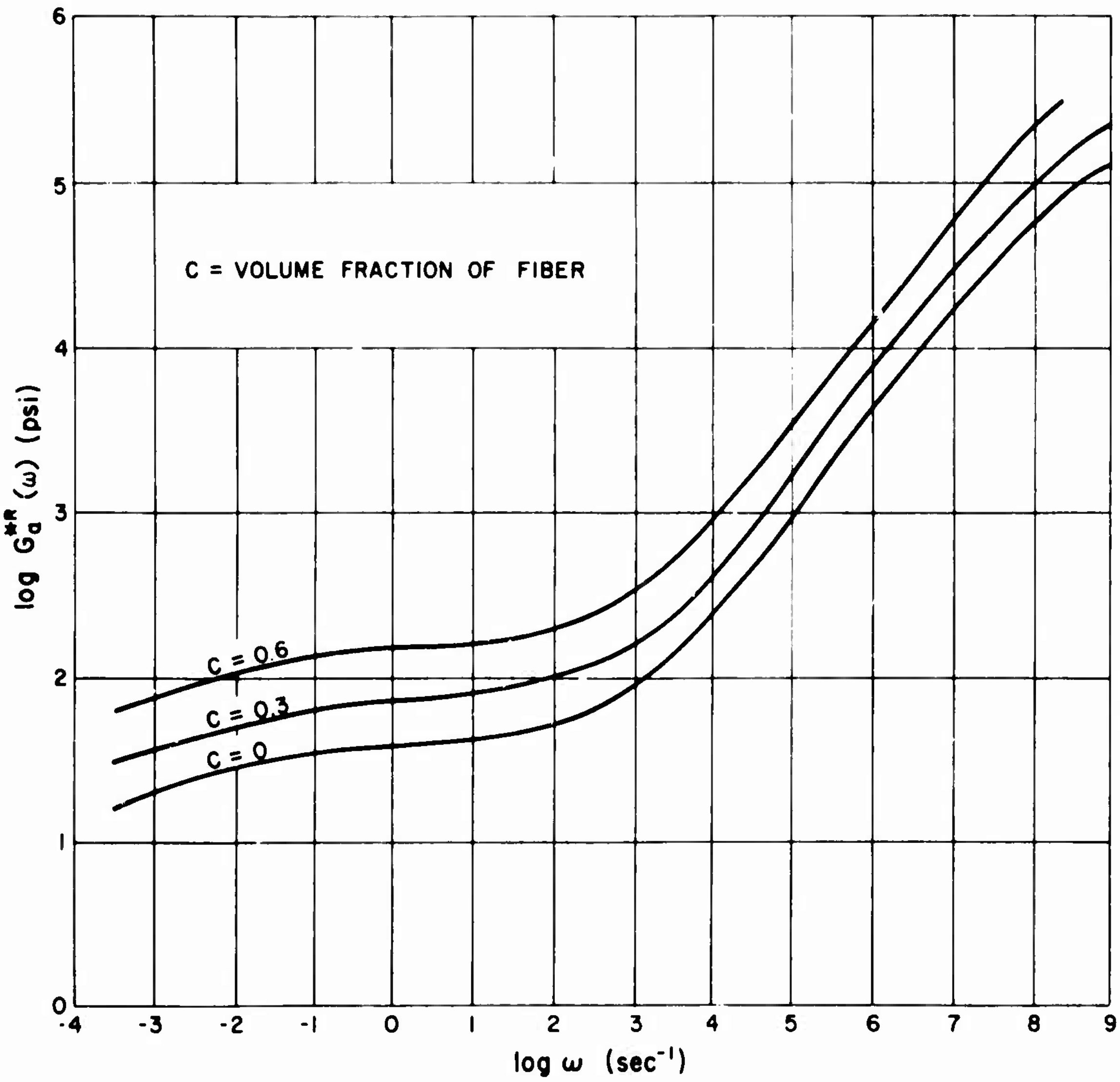

Figure 15. Real Part of Complex Axial Shear Modulus, $G_a^{*R}(\omega)$(Polyisobutylene Matrix).

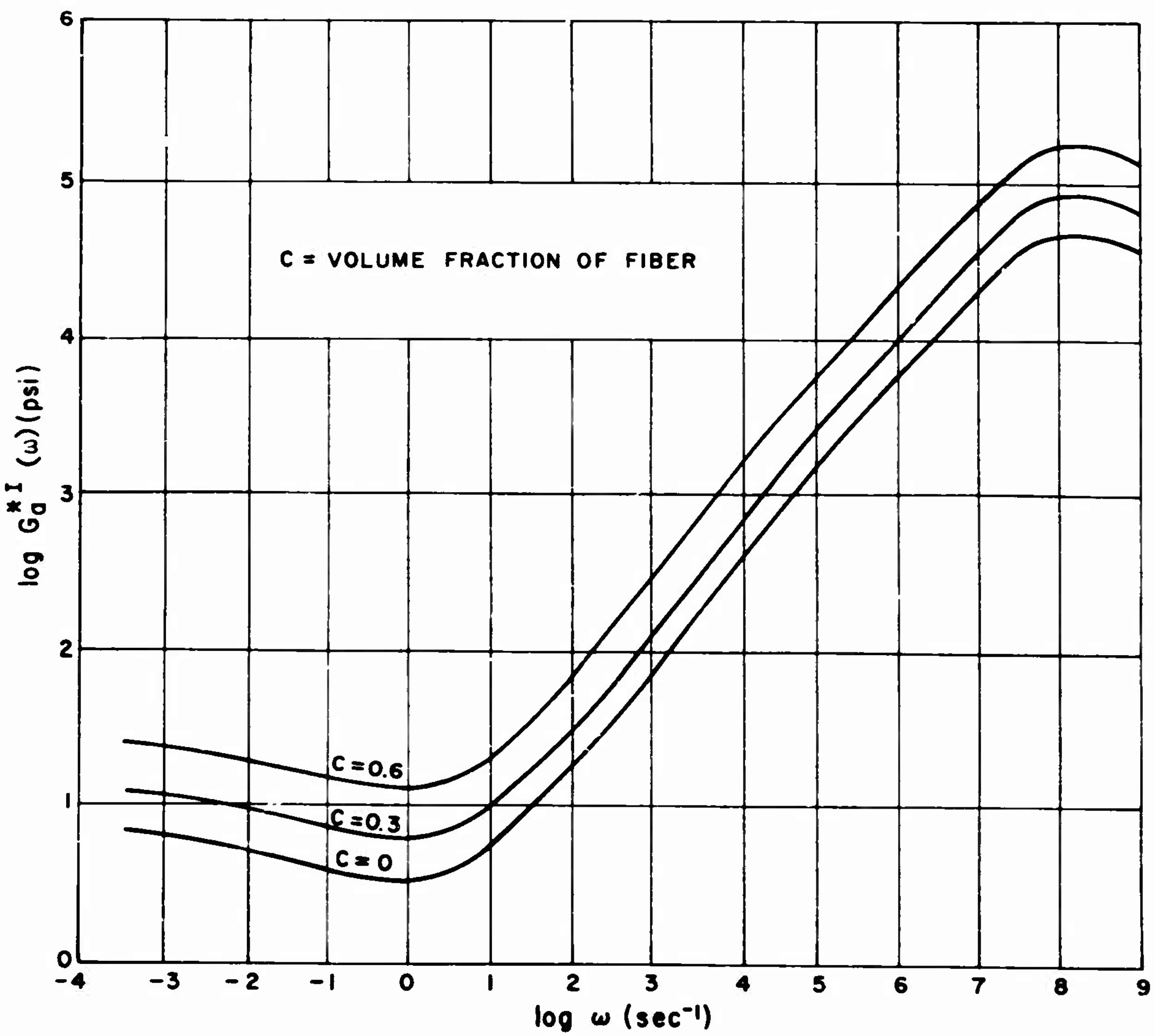

Figure 15. Imaginary Part of Complex Axial Shear Modulus, $G_a^{*I}(\omega)$ (Polyisobutylene Matrix).

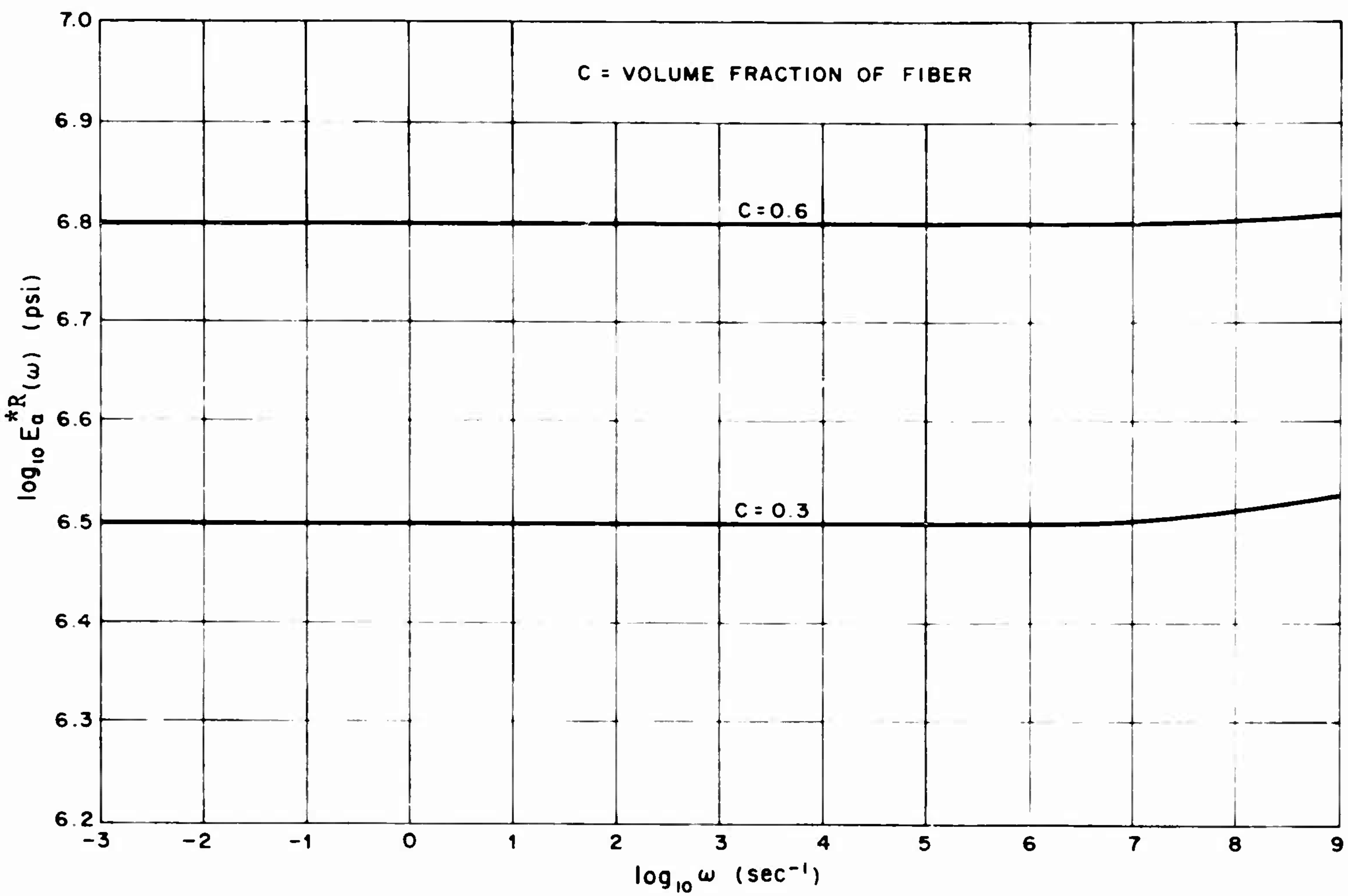

Figure 17. Real Part of Complex Axial Young's Modulus, $E_a^{*R}(\omega)$ (Polyisobutylene Matrix).

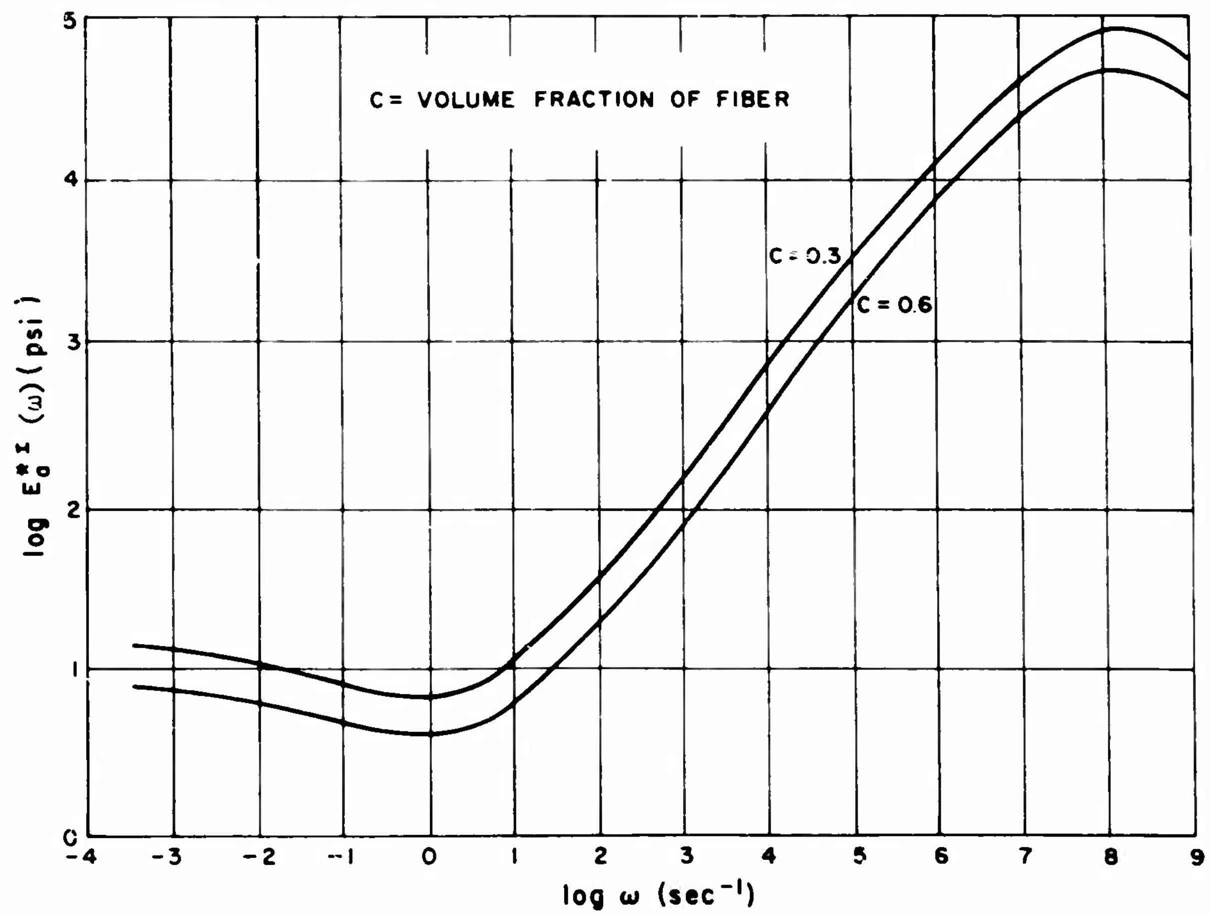

Figure 18. Imaginary Part of Axial Young's Modulus, $E_a^{*I}(\omega)$ (Polyisobutylene Matrix).

9. LATERAL VIBRATIONS OF FIBER REINFORCED VISCOELASTIC PLATES

It has been shown in previous sections that the solution of viscoelastic problems for fiber reinforced materials may be reduced to the solution of associated elastic problems in a manner identical to viscoelastic analysis of homogeneous materials. This is done either by application of the elastic-viscoelastic correspondence principle (References [12] and [13]) or, for steady-state vibrations, by the use of suitably defined complex moduli (Reference [14]). In this section we consider the lateral vibrations of fiber reinforced viscoelastic plates, for which the associated elastic problems are those of lateral vibration of anisotropic plates.

DERIVATION OF GOVERNING EQUATIONS

The governing equations for anisotropic plates, based on the Kirchhoff hypothesis, are easily developed by integration of the equations of linear elasticity across the plate thickness. It will be remembered that the displacements are of the form

$$u_\alpha = x_3\gamma_\alpha\,(x_1,x_2) \quad \alpha = 1,2 \qquad \text{(a)}$$

$$u_3 = w = w\,(x_1,x_2)\ , \qquad \text{(b)} \tag{304}$$

and that the neglect of transverse shear deformation leads to the relations

$$\gamma_\alpha = -\,w,_\alpha\ , \tag{305}$$

where

$$w,_\alpha = \frac{\partial w}{\partial x_\alpha}\ . \tag{306}$$

The equilibrium equations for an element of plate are easily obtained, with the result

$$\frac{\partial M_{\alpha\beta}}{\partial x_\beta} - Q_\alpha = 0\ , \tag{307}$$

$$\frac{\partial Q_\alpha}{\partial x_\alpha} + p = \rho h\ddot{w}\ , \tag{308}$$

where dots denote partial derivatives with respect to time. It should be noted that the rotatory inertia has been neglected in Equation (307). This is consistent with the neglect of transverse shear deformation and will limit the present theory to frequencies which cannot approach the frequency of the first thickness shear mode. This limitation is, however, not severe for thin plates.

The stress-strain relation for anisotropic plates may be written as

$$M_{\alpha\beta} = B^*_{\alpha\beta\gamma\lambda} K_{\gamma\lambda} , \tag{309}$$

where

$$K_{\alpha\beta} = \frac{1}{2} (\gamma_{\alpha,\beta} + \gamma_{\beta,\alpha}) . \tag{310}$$

The plate stiffness tensor $B^*_{\alpha\beta\gamma\lambda}$ is defined as

$$B^*_{\alpha\beta\gamma\lambda} = B^*_{\alpha\beta\gamma\lambda} \frac{h^3}{12} , \tag{311}$$

where $A^*_{\alpha\beta\gamma\lambda}$ are the components of the elasticity tensor for plane stress theory for the material referred to the plate axes. It should be noted that $B^*_{\alpha\beta\gamma\lambda}$ could also be interpreted as a linear integral time operator or as complex moduli for either formulation of the viscoelastic problem.

Substitution of Equation (310) into Equation (309) gives

$$M_{\alpha\beta} = - B^*_{\alpha\beta\gamma\lambda} w,_{\gamma\lambda} . \tag{312}$$

The governing equation for anisotropic plates is then obtained from Equations (307), (308), and (312); the result is

$$B^*_{\alpha\beta\gamma\lambda} w,_{\gamma\lambda\alpha\beta} + \rho h\ddot{w} = p(x_1,x_2,t) , \tag{313}$$

where p is the transverse load per unit area.

The appropriate boundary conditions are that on each edge we must prescribe either w or the effective shear V_α and either the slope or the bending moment. It will be remembered that due to the neglect of transverse shear deformation, the problem is of the fourth order and does not allow us to prescribe the shear resultant and twisting moment independently; we are therefore led to defining the effective shear V_α as

$$V_\alpha = Q_\alpha + \frac{\partial M_{\alpha\beta}}{\partial x_\beta} \quad \text{(no sum on } \beta) . \tag{314}$$

Expanding Equation (313), we obtain the governing equation in the form

$$B^*_{1111} \frac{\partial^4 w}{\partial x_1^4} + (B^*_{1211} + B^*_{2111} + B^*_{1112} + B^*_{1121}) \frac{\partial^4 w}{\partial x_1^3 \partial x_2} + (B^*_{2211} + B^*_{1212}$$

$$+ B^*_{1221} + B^*_{1122} + B^*_{2112} + B^*_{2121}) \frac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + (B^*_{2212} + B^*_{2221} + B^*_{1222}$$

$$+B^*_{2122}) \frac{\partial^4 w}{\partial x_1 \partial x_3^3} + B^*_{2222} \frac{\partial^4 w}{\partial x_2^4} + \rho h \frac{\partial^2 w}{\partial t^2} = p(x_1, x_2, t) \ . \qquad (315)$$

We recall that the stiffness tensor $B^*_{\alpha\beta\gamma\lambda}$ must satisfy the symmetry relations $B^*_{\alpha\beta\gamma\lambda} = B^*_{\beta\alpha\gamma\lambda} = B^*_{\alpha\beta\lambda\gamma} = B^*_{\lambda\gamma\alpha\beta}$, and the above equation therefore reduces to

$$B^*_{1111} \frac{\partial^4 w}{\partial x_1^4} + 4B^*_{1112} \frac{\partial^4 w}{\partial x_1 \partial x_2^3} + (2B^*_{1122} + 4B^*_{1212}) \frac{\partial^4 w}{\partial x_1^2 \partial x_2^2}$$

$$+ 4B^*_{2221} \frac{\partial^4 w}{\partial x_1 \partial x_3^3} + B^*_{2222} \frac{\partial^4 w}{\partial x_2^4} + \rho h \frac{\partial^2 w}{\partial t^2} = p(x_1, x_2, t) \ . \qquad (316)$$

It will be recalled that the stiffness components $B^*_{\alpha\beta\gamma\lambda}$ appearing in all previous equations are referred to plate coordinates. When the plate material is fiber reinforced with unidirectionally oriented fibers, the material is orthotropic with respect to axes, which are respectively along the fiber direction and perpendicular to the fiber direction (Reference [1]). It is clear that plate stiffnesses can be defined with respect to material-oriented axes as

$$\bar{D}_{\alpha\beta\gamma\lambda} = \bar{A}_{\alpha\beta\gamma\lambda} \frac{h^3}{12} \ , \qquad (317)$$

where $\bar{A}_{\alpha\beta\gamma\lambda}$ are the components of the elasticity tensor referred to material axes. Clearly, then, the stiffness tensor $B^*_{\alpha\beta\gamma\lambda}$ (or $\bar{B}_{\alpha\beta\gamma\lambda}$) obeys the transformation law for a surface tensor of rank 4 which is

$$B^*_{\alpha\beta\gamma\lambda} = \alpha_{\alpha\nu} \, \alpha_{\beta\pi} \, \alpha_{\gamma\eta} \, \alpha_{\lambda\xi} \, \bar{B}^*_{\nu\pi\eta\xi} \ , \qquad (318)$$

where $\alpha_{\alpha\beta}$ are the direction cosines of the plate axes with respect to the material axes. If we consider a rectangular plate with fibers oriented at an angle θ to the x_k direction, it is easily shown that the stiffness tensor referred to the plate axes is obtained from that referred to the material axes by the matrix equation

$$\begin{bmatrix} D_{1111} \\ D_{2222} \\ D_{1122} \\ D_{1212} \\ D_{1112} \\ D_{1222} \end{bmatrix} = [\ A\ ] \begin{bmatrix} \bar{B}^*_{1111} \\ \bar{B}^*_{2222} \\ \bar{B}^*_{1212} \\ \bar{B}^*_{1122} \end{bmatrix} . \tag{319}$$

Elements of the matrix [A] are given as follows:

$$\left.\begin{aligned}
A_{11} &= A_{22} = \cos^4\theta \\
A_{12} &= A_{21} = \sin^4\theta \\
A_{13} &= A_{23} = 4 \sin^2\theta \cos^2\theta \\
A_{14} &= A_{24} = 2 \sin^2\theta \cos^2\theta \\
A_{31} &= A_{32} = \sin^2\theta \cos^2\theta \\
A_{33} &= - A_{13} \\
A_{34} &= \sin^4\theta + \cos^4\theta \\
A_{41} &= A_{42} = A_{31} \\
A_{43} &= (\sin^2\theta - \cos^2\theta)^2 \\
A_{44} &= - A_{14} \\
A_{51} &= - A_{62} = \sin\theta \cos^3\theta \\
A_{52} &= - A_{61} = - \sin^3\theta \cos\theta \\
A_{53} &= - A_{63} = 2 (\sin^2\theta - \cos^2\theta) \sin\theta \cos\theta \\
A_{54} &= - A_{64} = (\sin^2\theta - \cos^2\theta) \sin\theta \cos\theta
\end{aligned}\right\} . \tag{320}$$

It is clear from Equation (319) that the fiber reinforced plate will, in general, be anisotropic for $\theta \neq 0$. This special type of anisotropy is often denoted by the term related-orthotropic [15].

VIBRATIONS OF ORTHOTROPIC FIBER REINFORCED PLATES

We first consider the simplified case of fibers oriented along the x_1 axis. The plate is then orthotropic, and Equation (316) then reduces to the form

$$D^*_{1111} \frac{\partial^4 w}{\partial x_1^4} + 2D^*_{1122} \frac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + D^*_{2222} \frac{\partial^4 w}{\partial x_2^4} + \rho h \frac{\partial^4 w}{\partial t^2} = p(x_1, x_2, t) , \qquad (321)$$

where

$$D^*_{1111} = B^*_{1111} \qquad (a)$$

$$D^*_{1122} = B^*_{1122} + 2B^*_{1212} \qquad (b)$$

$$D^*_{2222} = B^*_{2222} . \qquad (c) \qquad (322)$$

The stress-strain relations (309) reduce for this case to

$$M_{11} = -\left[B^*_{1111} \frac{\partial^2 w}{\partial x_1^2} + B^*_{1122} \frac{\partial^2 w}{\partial x_2^2}\right] , \qquad (a)$$

$$M_{22} = -\left[B^*_{2211} \frac{\partial^2 w}{\partial x_1^2} + B^*_{2222} \frac{\partial^2 w}{\partial x_1^2}\right] , \qquad (b)$$

$$M_{12} = M_{21} = -\ 2B^*_{1212} \frac{\partial^2 w}{\partial x_1 \partial x_2} . \qquad (c) \qquad (323)$$

It will be of interest to consider the case of a rectangular plate of length "a" and width "b" simply supported along all edges and subjected to a sinusoidally varying concentrated load at a point (ξ_1, ξ_2).

For a plate simply supported along all edges, the boundary conditions are

$$w = M_{11} = 0 \qquad \text{at } x_1 = 0,\ a\ , \qquad \text{(a)}$$

$$w = M_{22} = 0 \qquad \text{at } x_2 = 0,\ b\ , \qquad \text{(b)} \qquad (324)$$

and we note that, since w diminishes along these edges, the moment boundary condition, by virtue of equations (323), reduces to

$$\frac{\partial^2 w}{\partial x_1^2} = 0\ , \qquad \text{at } x_1 = 0,\ a\ , \qquad \text{(a)}$$

$$\frac{\partial^2 w}{\partial x_2^2} = 0\ , \qquad \text{at } x_2 = 0,\ b\ . \qquad \text{(b)} \qquad (325)$$

If we consider a sinusoidally time-varying concentrated load of magnitude P applied at the point $(\xi_1,\ \xi_2)$ of the plate, the loading function $p(x_1,\ x_2,\ t)$ may be expressed as

$$p(x_1,\ x_2,\ t) = P\delta(x_1-\xi_1)\ \delta\ (x_2-\xi_2)\ e^{i\omega t}\ . \qquad (326)$$

It is clear from the boundary conditions, (324) and (325), that the solution to the present problem may be obtained as a Fourier series in the functions $\sin \frac{mx_1}{a}$ and $\sin \frac{nx_2}{b}$. We therefore expand $p(x_1,\ x_2,\ t)$ in such a Fourier series and obtain

$$p(x_1,\ x_2,\ t) = \frac{4P}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (\sin \frac{m\pi\xi_1}{a} \sin \frac{m\pi x_1}{a} \sin \frac{n\pi\xi_2}{b} \sin \frac{n\pi x_2}{b})\ e^{i\omega t}\ . \qquad (327)$$

We now seek a solution for $w(x_1,\ x_2,\ t)$ in the form

$$w(x_1,\ x_2,\ t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn}(t) \sin \frac{m\pi x_1}{a} \sin \frac{n\pi x_2}{b}\ . \qquad (328)$$

Substituting Equations (327) and (328) into Equation (321) and equaling coefficients of like terms in the Fourier series yields one ordinary differential equation for the functions $q_{mn}(t)$; the result is

$$\ddot{q}_{mn} + \frac{1}{\rho h}\left[\left(\frac{m\pi}{a}\right)^4 D^*_{1111} + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 D^*_{1122} + \left(\frac{n\pi}{b}\right)^4 D^*_{2222}\right] q_{mn}$$

$$= \frac{4P}{\rho hab} e^{\iota\omega t} \sin\frac{m\pi\xi_1}{a} \sin\frac{n\pi\xi_2}{b} . \tag{329}$$

This equation may be written in the form

$$\ddot{q}_{mn} + \omega^2_{mn} q_{mn} = p_{mn} e^{\iota\omega t} , \tag{330}$$

where

$$\omega^2_{mn} = \frac{1}{\rho h}\left[\left(\frac{m\pi}{a}\right)^4 D^*_{1111} + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 D^*_{1122} + \left(\frac{n\pi}{b}\right)^4 D^*_{2222}\right] \tag{331}$$

and

$$p_{mn} = \frac{4P}{\rho hab} \sin\frac{m\pi\xi_1}{a} \sin\frac{n\pi\xi_2}{b} . \tag{332}$$

It will be of particular interest to consider only the steady-state solution of Equation (330), since in the viscoelastic case transients will be damped out. Thus, we consider a solution of the form

$$q_{mn}(t) = \rho_{mn} e^{\iota\omega t}$$

and substitute this expression into Equation (330). We then obtain the steady-state solution as

$$q_{mn}(t) = \frac{p_{mn}}{\omega^2_{mn} - \omega^2} e^{\iota\omega t} . \tag{333}$$

Thus, the elastic solution to the problem may be written in the form

$$w(x_1, x_2, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{p_{mn} e^{\iota\omega t}}{\omega^2_{mn} - \omega^2} \sin\frac{m\pi x_1}{a} \sin\frac{n\pi x_2}{b} . \tag{334}$$

As was clear in the case of the beam solution of Section 7, this solution may be written in the form

$$w(x_1, x_2, t) = e^{\iota\omega t} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W^E_{mn}(x_1, x_2) , \tag{335}$$

where

$$W^{E}_{mn}(x_1, x_2) = \frac{4P}{\rho hab} \frac{\sin \frac{m\pi\xi_1}{a} \sin \frac{n\pi\xi_2}{b} \sin \frac{m\pi x_1}{a} \sin \frac{n\pi x_2}{b}}{\omega^2_{mn} - \omega^2} . \qquad (336)$$

The passage to the viscoelastic solution may now be accomplished in a manner entirely analogous to that of Section 7. Thus, for the viscoelastic case, we introduce into Equation (331) appropriate complex stiffnesses for D^*_{1111}, D^*_{2222}, and D^*_{1122}. These complex stiffnesses are completely defined by Equation (331) if the components of the elasticity tensor are replaced by the corresponding complex moduli. They can be written explicitly in terms of complex moduli as

$$\tilde{D}^*_{1111} = \frac{h^3}{12} \frac{\check{E}^*_a}{1 - \frac{\check{E}^*_T}{\check{E}^*_a} \tilde{\nu}^{*2}_a} , \qquad (a)$$

$$\tilde{D}^*_{1122} = \frac{h^3}{12} \left[\frac{\tilde{\nu}^*_a \check{E}^*_T}{1 - \frac{\check{E}^*_T}{\check{E}^*_a} \tilde{\nu}^{*2}_a} + 2\check{G}^*_a \right] , \qquad (b)$$

$$\tilde{D}^*_{2222} = \frac{h^3}{12} \frac{\check{E}^*_T}{1 - \frac{\check{E}^*_T}{\check{E}^*_a} \tilde{\nu}^{*2}_a} . \qquad (c) \qquad (337)$$

Under these conditions, Equation (331) is then replaced by the expression

$$\bar{\omega}^2_{mn} = \frac{1}{\rho h} \left[\left(\frac{m\pi}{a}\right)^4 \tilde{D}^*_{1111} + 2 \left(\frac{m\pi}{a}\right)^2 \left(\frac{n\pi}{b}\right)^2 \tilde{D}^*_{1122} + \left(\frac{n\pi}{b}\right)^4 \tilde{D}^*_{2222} \right]$$

$$= \omega^2_{mn} \left[1 + \iota \tan \delta_{mn}\right] . \qquad (338)$$

It should be noted that $\bar{\omega}_{mn}$ and δ_{mn} are both functions of frequency ω, since the complex moduli and stiffnesses are frequency dependent. Thus, the viscoelastic solutions to the problem can now be written in the form

$$\bar{w}(x_1, x_2, t) = e^{i\omega t} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn}^{VE} (x_1, x_2) , \tag{339}$$

where

$$W_{mn}^{VE} (x_1, x_2) = \frac{4P}{\rho hab} \frac{\sin \frac{m\pi\xi_1}{a} \sin \frac{n\pi\xi_2}{b} \sin \frac{m\pi x_1}{a} \sin \frac{n\pi x_2}{b}}{\bar{\omega}_{mn}^2 - \omega^2 + i\bar{\omega}_{mn}^2 \tan \delta_{mn}} . \tag{340}$$

Equation (340) may be written as

$$W_{mn}^{VE} (x_1, x_2) =$$

$$\frac{4P}{\rho hab} \frac{\bar{\omega}_{mn}^2 - \omega^2 - i\bar{\omega}_{mn}^2 \tan \delta_{mn}}{(\bar{\omega}_{mn}^2 - \omega^2)^2 + \bar{\omega}_{mn}^4 \tan^2 \delta_{mn}} \sin \frac{m\pi\xi_1}{a} \sin \frac{n\pi\xi_2}{b} \sin \frac{m\pi x_1}{a} \sin \frac{n\pi x_2}{b} . \tag{341}$$

Following the development previously presented in Equations (260) through (263), we can obtain the dynamic viscoelastic solution to the problem.

To illustrate the dynamic viscoelastic solution (339), we consider a rectangular plate of length "a" and width "b" simply supported along all edges and subjected to a sinusoidally varying concentrated load at the center $\left(\frac{a}{2}, \frac{b}{2}\right)$. See Figure 19. The dimensions of the plate are

$$a = 60 \text{ in.}, \; b = 30 \text{ in.}, \; h = 0.375 \text{ in.}$$

The matrix material of the plate is taken to be polyisobutylene, whose properties are given in Section 8. It is assumed that the matrix is elastic in dilatation. The material properties of the fiber and the density of the plate are as follows:

$$K_f = 5.83 \times 10^6 \text{ psi}$$

$$G_f = 4.375 \times 10^6 \text{ psi}$$

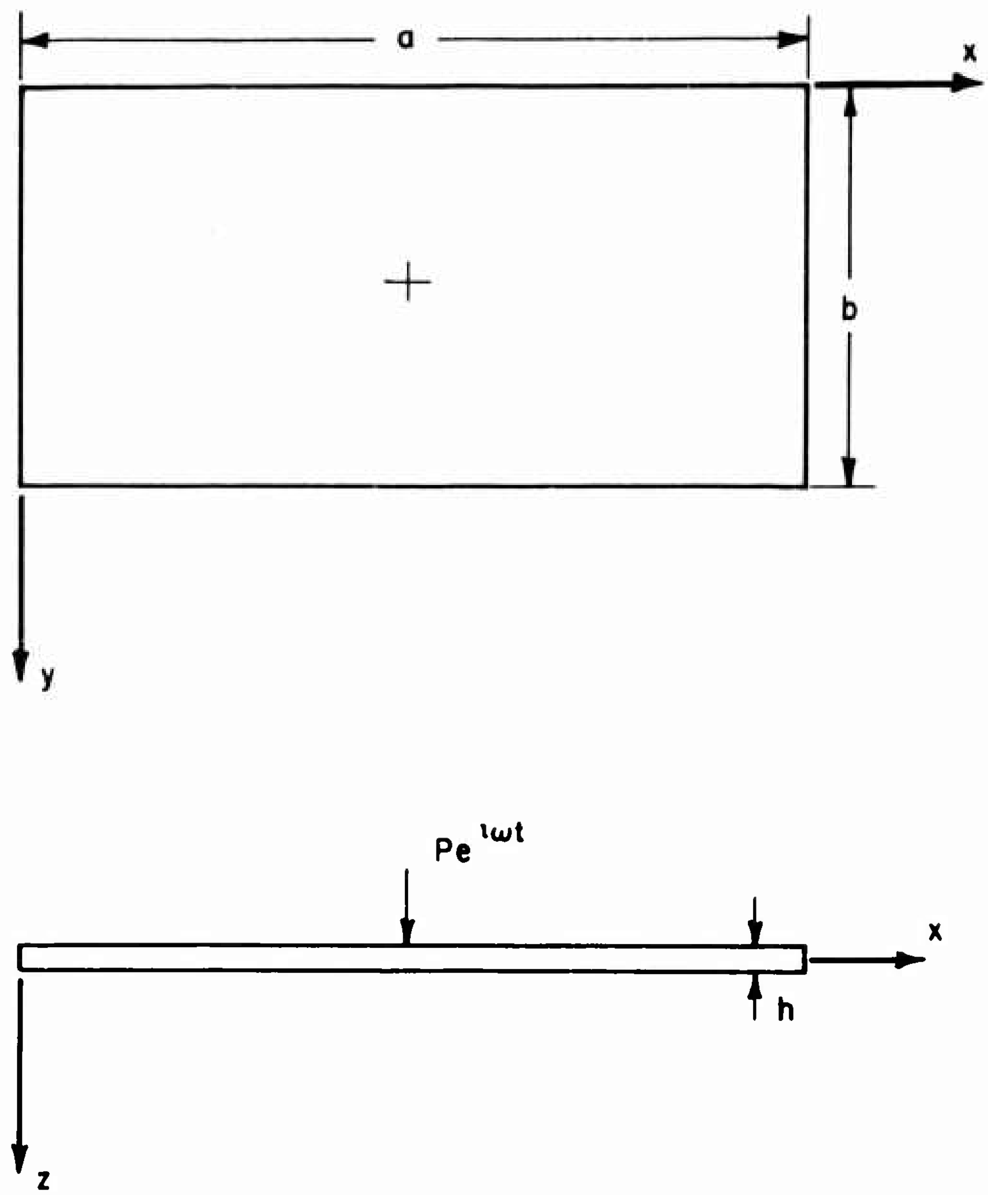

Figure 19. Lateral Vibrations of Rectangular Plate.

$\rho = 3.0$ (density relative to water)

$c = 0.5$ (volume fraction of fiber).

From (339) and (340), the deflection of the plate at $\left(\frac{a}{2}, \frac{b}{2}\right)$ is found to be

$$w\left(\frac{a}{2}, \frac{b}{2}, t\right) = e^{\imath\omega t} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{4P}{\rho hab} \frac{1}{\bar{\omega}_{mn}^{2}(1+\imath \tan \delta_{mn}) - \omega^{2}}, \tag{342}$$

where

$$\bar{\omega}_{mn}^{2}(1+\imath \tan \delta_{mn}) = \frac{\pi^4}{\rho hab}\left[\frac{m^4}{4}\tilde{D}_{1111}^{*} + 2m^2 n^2 \tilde{D}_{1122}^{*} + 4n^4 \tilde{D}_{2222}^{*}\right] . \tag{343}$$

Using (337), the deflection at the center of the plate can be computed from (342). The variations of the amplitude with ω are plotted in Figure 20. As in the case of beams, it is seen that there is no true resonance. However, for forcing frequencies close to the first natural frequency of the plate, large response amplitude is encountered and resonance is attained.

Finally, we note that the algebraic and numerical calculations for this case are at least one order of magnitude more complex than those for the case of beams, which was treated previously. This is due to the appearance of four distinct moduli in Equation (331), each of which has different frequency dependent real and imaginary parts; this causes the "loss tangent", $\tan \delta_{mn}$, to depend on the mode under consideration. Some simplifications may, however, be possible for particular materials.

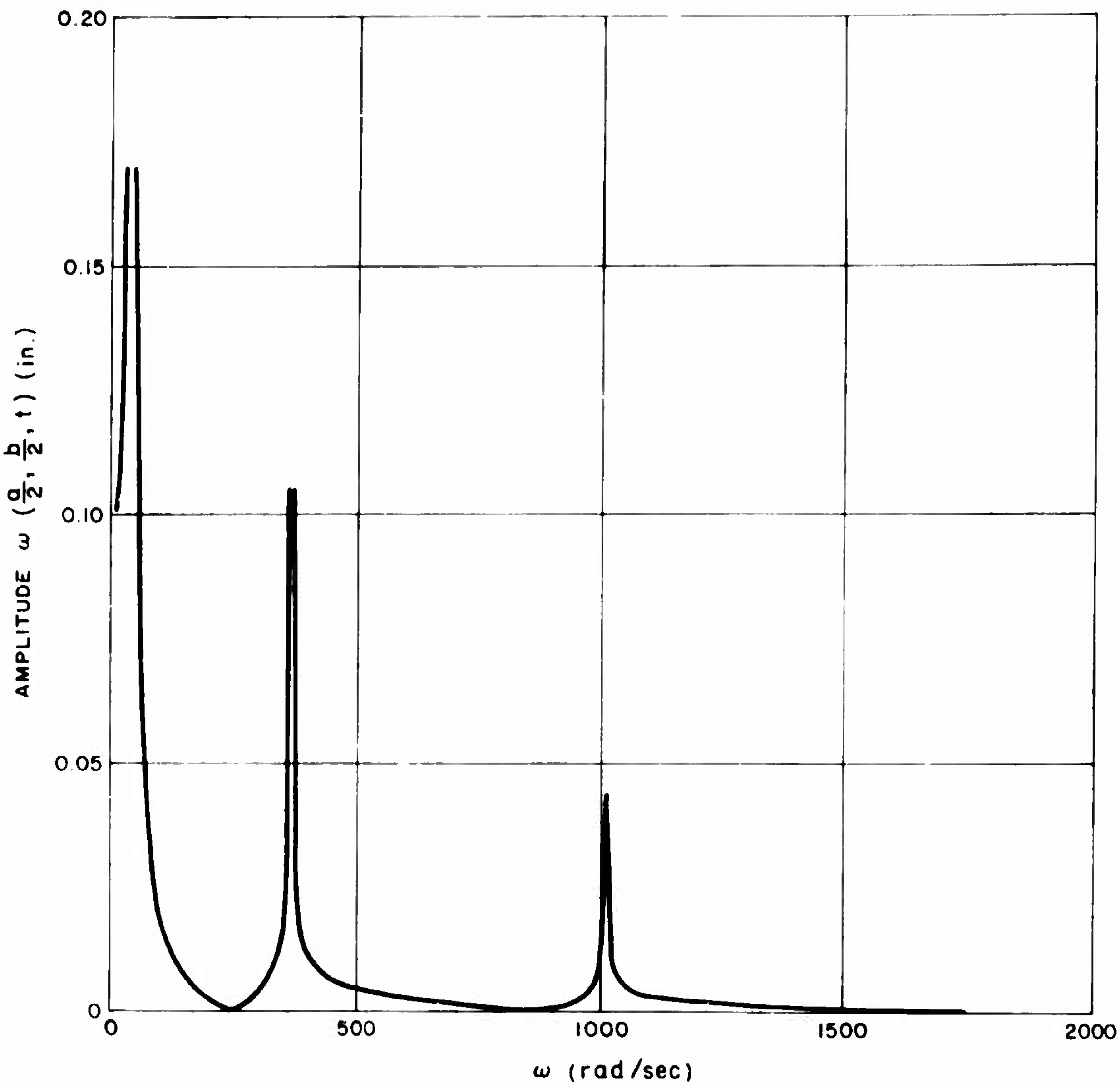

Figure 20. Amplitude of Lateral Deflection of Plate.

10. ELEMENTARY THEORY OF BENDING AND SHEARING OF VISCOELASTIC FIBER REINFORCED BEAMS

We consider a beam of constant cross section. The system of axes to which the beam is referred is shown in Figure 21. It is assumed for the sake of simplicity that the y-axis is an axis of symmetry of the cross section. The beam is uniaxially reinforced by fibers in the x-direction.

It will be our purpose to develop a theory of such beams which is analogous to the Euler-Bernoulli theory of homogeneous beams. We note in this respect that for homogeneous elastic and viscoelastic beams in pure bending, the Euler-Bernoulli theory is rigorous, provided that the bending moments on the terminal beam sections are applied through a normal stress distribution which varies linearly over the section. Suppose that such a stress distribution is applied on the terminal sections of a uniaxially fiber reinforced beam. The beam may be thought of as being represented by the composite cylinder assemblage model. Since there are a very large number of composite cylinders passing through the section, the normal stress σ_{xx} on any composite cylinder end section is uniform. Then it is known that in such a composite cylinder, the axial strain ε_{xx} will be uniform throughout fiber and matrix (see Reference [1]), except for some local perturbation at the end regions. Since the strain ε_{xx} is proportional to the stress σ_{xx}, it follows that ε_{xx} is very nearly linearly distributed over the section of the cylinder. We therefore conclude that in pure bending, the Euler-Bernoulli theory should be applicable with high accuracy for a fiber reinforced beam.

Because of the assumption that plane sections remain plane and since y is an axis of symmetry of the section, we write, as in the usual strength of materials development,

$$\varepsilon_{xx}(x,y,t) = \frac{y}{R(x,t)} , \qquad (344)$$

where R(x,t) is the radius of curvature of the deformed beam, which here depends also on time because of the time dependence of the material. The average stress $\bar{\sigma}_{xx}$ on the section of a composite cylinder is then given by

$$\bar{\sigma}_{xx}(x,y,t) = \int_0^t E_a^*(t-\tau) \frac{\partial \varepsilon_{xx}(x,y,\tau)}{\partial \tau} d\tau . \qquad (345)$$

Note that the y-coordinate represents positions of central axes of composite cylinders only. Since there are very many of these, y may be approximated by the continuous variable y.

Introducing (344) into (345), we have

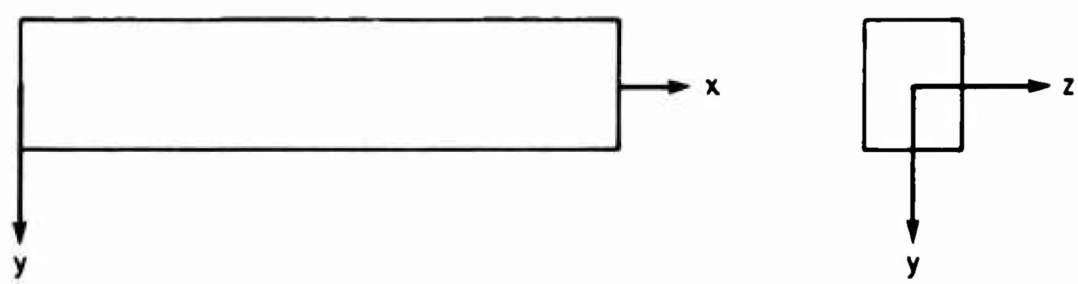

Figure 21. Beam Coordinate System.

$$\bar{\sigma}_{xx}(x,y,t) = y\, B(x,t) \,, \tag{346}$$

where

$$B(x,t) = \int_0^t E_a^*(t-\tau) \frac{\partial}{\partial \tau} \left[\frac{1}{R(x,\tau)} \right] d\tau \,. \tag{347}$$

In the absence of axial forces on the beam, equilibrium demands that

$$\int_{(A)} \bar{\sigma}_{xx}(x,y,t)\, dA = 0 \,. \tag{348}$$

Introducing (346) into (348), it follows that

$$\int_{(A)} y dA = 0 \,. \tag{349}$$

This implies that the linear distribution (346) must pass through the centroid of the section and that the stress and strain vanish at the centroid.

From moment equilibrium at any section, we have

$$M_{xx}(x,t) = M(x,t) = \int_{(A)} \bar{\sigma}_{xx}(x,y,t)\, ydA \,. \tag{350}$$

Introducing (346) into (350), we find that

$$M(x,t) = B(x,t)\, I \,, \tag{351}$$

where

$$I = I_{zz} = \int y^2 dA \tag{352}$$

is the moment of inertia with respect to the z-axis. Combining (351) with (346), we have

$$\bar{\sigma}_{xx}(x,y,t) = \frac{M(x,t)}{I}\, y \,, \tag{353}$$

which is the exact analogue of the strength of materials stress for an *elastic* beam. It is thus seen that the stress $\bar{\sigma}_{xx}$ is independent of the viscoelastic nature of the material. It should be noted that in accordance with the present development, the moment M is reckoned to be positive if it produces tension below the neutral axis and compression above the neutral axis of the section.

In order to analyze the deflection of the center line w(x,t), we assume this deflection to be small in comparison to section dimension. Then

$$R(x,t) \approx - \frac{\partial^2 w(x,t)}{\partial x^2} \,. \tag{354}$$

The negative sign in (354) stems from the choice of positive y downward and bending moment sign convention. It follows from (354) and (347) that

$$B(x,t) = - \frac{\partial^2}{\partial x^2} \int_0^t E_a^*\,(t-\tau)\, \frac{\partial w(x,\tau)}{\partial \tau}\, d\tau \,. \tag{355}$$

Introducing (355) into (351), we have

$$M(x,t) = - I \frac{\partial^2}{\partial x^2} \int_0^t E_a^*\,(t-\tau)\, \frac{\partial w(x,\tau)}{\partial \tau}\, d\tau \,. \tag{356}$$

Equation (356) is the governing equation for the beam deflection. End conditions for w have, of course, to be specified.

The present form of (356) is somewhat inconvenient. We shall now proceed to transform it into a different form. For this purpose let us take the time Laplace transform of both sides. Then

$$\hat{M}(x,p) = - I \frac{\partial^2}{\partial x^2} \left[p \hat{E}_a^* (p) \hat{w} (x,p) \right] , \qquad (357)$$

where we have used the notation (3) for Laplace transforms. Let us consider the very important case where the load is applied to the beam at time $t = 0$ and is then held constant.

In that case,

$$M(x,t) = M(x) H(t) , \qquad (358)$$

where $H(t)$ is the Heaviside unit step function defined as

$$H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 . \end{cases}$$

It follows that

$$\hat{M}(x,p) = \frac{1}{p} M(x) . \qquad (359)$$

From (45),

$$p \hat{E}_a^*(p) = \frac{1}{p\hat{e}_a^*(p)} , \qquad (45)$$

where $\hat{e}_a^*(p)$ is the transform of the Young's creep compliance $e_a^*(t)$ which enters into the relation

$$\varepsilon_{xx}(t) = \int_0^t e_a^*(t-\tau) \frac{\partial \bar{\sigma}_{xx}(\tau)}{\partial \tau} d\tau . \qquad (360)$$

Thus, $e_a^*(t)$ is the longitudinal strain produced in a fiber reinforced material by constant unit average stress $\bar{\sigma}_{xx}$.

Introducing (359) and (45) into (357), we find that

$$\frac{\partial^2 \hat{w}(x,p)}{\partial x^2} = - \frac{M(x)}{I} \hat{e}_a^* (p) , \qquad (361)$$

which can be directly inverted into the time domain to give

$$\frac{\partial^2 w(x,t)}{\partial x^2} = - \frac{M(x)}{I} e_a^* (t) . \qquad (362)$$

Equation (362) is the analogue of the well-known differential equation for an elastic beam. As is usually done for elastic beams we shall now make the assumption that, although derived for the case of pure bending, Equation (362) is also applicable when shear forces are present. Then (362) represents the differential equation of viscoelastic fiber reinforced beams in bending.

We now consider end conditions for w. Typical end conditions are

$$\text{simple support at } x_o \qquad w(x_o,t) = 0 \qquad \text{(a)}$$

$$\text{built-in support at } x_o \qquad \frac{\partial w}{\partial x}(x_o,t) = 0 \ . \qquad \text{(b)} \qquad (363)$$

Set

$$w(x,t) = W(x)\, e_a^*(t) \ . \qquad (364)$$

Then Equation (362) reduces to

$$\frac{d^2W(x)}{dx^2} = -\frac{M(x)}{I} \ . \qquad (365)$$

Also, introduction of (364) into (363) gives

$$W(x_o) = 0 \qquad \text{(a)}$$

$$\frac{dW(x_o)}{dx} = 0 \ . \qquad \text{(b)} \qquad (366)$$

It is seen that (365) together with (366) represents an elastic deflection problem with unit Young's modulus. The viscoelastic solution is then obtained from (364). Since in any elastic beam deflection problem EI occurs in the deflection denominator, we can now formulate the following simple rule: *In order to find the deflection of a viscoelastic beam, simply take the elastic deflection and replace the reciprocal of the Young's modulus by the viscoelastic Young's compliance.*

As an application, consider a simply supported beam, of length ℓ, which is loaded uniformly by p_o per unit length. The elastic deflection is

$$w^E(x) = \frac{p_o x}{24E_a^* I}(\ell^3 - 2\ell x^2 + x^3) \ . \qquad (367)$$

The viscoelastic deflection is

$$w^{VE}(x,t) = \frac{p_o x e_a^*(t)}{24I} (\ell^3 - 2\ell x^2 + x^3) \ . \tag{368}$$

The maximum deflection is at the center $x = \ell/2$

$$w^{VE}(\ell/2,t) = \frac{5p_o\ell^4}{384I} e_a^*(t) \ . \tag{369}$$

It has been shown in Reference [3] that the viscoelastic effect in uniaxial stress is negligible for conventional fiber reinforced materials in which the fibers are very stiff in comparison to the matrix. In other words, if the creep compliance $e_a^*(t)$ is replaced by the initial compliance $e_a^*(o)$, the error produced is insignificant; but according to the development given in Reference [3],

$$e_a^*(o) = \frac{1}{E_a^*(o)} \ , \tag{370}$$

where from (63)

$$E_a^*(o) = (1-c)\ E_m(o) + cE_f \ , \ t \geq o \ . \tag{371}$$

(It should be recalled that c is the volume fraction of fibers and that $E_m(o)$ is the initial (elastic) Young's modulus of the matrix.) Thus, (368) assumes the form

$$w^{VE}(x,t) \cong \frac{p_o x}{24E_a^*(o)I} (\ell^3 - 2\ell x^3 + x^3) \ . \tag{372}$$

Equation (372) is no longer time dependent and is precisely the elastic solution (367) with $E_a^*(o)$ used as an elastic Young's modulus. It is also quite clear that similar conclusions would hold for bending deflection analysis for any loading, and we may thus state the following important conclusion: *Under time-constant load, the bending deflections of uniaxially fiber reinforced viscoelastic beams, in which the fibers are very much stiffer than the matrix, may be approximated with high accuracy by the time-independent deflection of elastic beams with Young's modulus taken as the initial* $E_a^*(o)$.

We now turn to the effect of shearing forces on the deflection of fiber reinforced beams, and we shall adopt the usual strength of materials approach. It will first be recalled that bending stresses are given by (353). In the event of constant load for $t \geq o$, the moment is given by (358); consequently, the bending stresses are given by

$$\bar{\sigma}_{xx}(x,y) = \frac{M(x)}{I} y, \quad t \geq o . \tag{373}$$

It will be recalled that the shear stress $\bar{\sigma}_{xy}$ follows according to strength of materials simply from equilibrium considerations with the use of the bending stresses only. Since (373) is the exact analogue of the strength of materials bending stress, it follows immediately that the shear stress σ_{xy} is also given by the strength of materials expression. Thus,

$$\bar{\sigma}_{xy}(x,y) = \frac{S(x)Q(y)}{b(y)I} , \tag{374}$$

where

S(x) = Shear force at x-section,

Q(y) = First area moment with respect to the neutral axis of the part of the section above y,

b(y) = Width of the section at height y,

I = Moment of inertia I_{zz}.

The overbar in (374) denotes local averages. The maximum shear stress always occurs at y = 0. It may be conveniently written in the form

$$\bar{\sigma}_{xy\ max} = \bar{\sigma}_{xy}(x,o) = k \frac{S(x)}{A} , \quad t \geq o , \tag{375}$$

where A is the area of the section and k is a geometrical section shape factor given by

$$k = \frac{AQ(o)}{Ib(o)} . \tag{376}$$

For a rectangular section, $k = \frac{3}{2}$; for a circular section, $k = \frac{4}{3}$.

The shear strain produced on the neutral axis by the stress (375) is given by

$$\bar{\varepsilon}_{xy}(x,o) = \frac{1}{2} g_a^*(t) \bar{\sigma}_{xy}(x,o) . \tag{377}$$

This expression needs some explanation. Since the stress (375) is constant in time for $t \geq o$, the strain produced is directly proportional to the appropriate creep compliance. This creep compliance is $g_a^*(t)$ since the shearing action proceeds in places parallel and transverse to the fibers.

On the basis of the treatment given in Section 9, the creep compliance for stiff fibers can be very well approximated by the creep compliance for rigid fibers. Accordingly, we use

$$g_a^*(t) = \frac{1-c}{1+c} g_m(t) . \tag{378}$$

According to strength of materials theory, there is an additional shear deflection w_s which is superposed on the usual Euler-Bernoulli bending deflection w_b. Since w_b produces no shear, we have

$$\bar{\varepsilon}_{xy}(x,o) = \frac{1}{2} \frac{\partial w_s}{\partial x} . \tag{379}$$

Substituting (375) into (377) and substituting the resulting expression into (379), we find that

$$\frac{\partial w_s}{\partial x} = \frac{k}{A} S(x) g_a^*(t) . \tag{380}$$

Now from equilibrium

$$\frac{dS(x)}{dx} = - p(x) , \tag{381}$$

where p(x) is the load per unit length of the beam, taken as positive in the y (downward) direction. Differentiating both sides of (380) and using (381), we obtain

$$\frac{\partial^2 w_s}{\partial x^2} = - \frac{k}{A} p(x) g_a^*(t) . \tag{382}$$

This is the governing equation for the shear deflection of the beam.

Equation (382) is similar to (362) and can be solved by analogous methods. We set

$$w_s(x,t) = W_s(x) g_a^*(t) . \tag{383}$$

Introducing (383) into (382), we obtain for W_s the differential equation

$$\frac{d^2 W_s}{dx^2} = - \frac{k}{A} p(x) . \tag{384}$$

As an example, we consider again the case of a simply supported uniformly loaded beam with p_o per unit length for $t \geq o$. The bending moment is then given by

$$M(x) = \frac{p_o x}{2} (\ell - x) \quad . \tag{385}$$

Introducing (385) into (365) and integrating, we have

$$W_b(x) = -\frac{p_o}{2I}\left(\frac{\ell x^3}{6} - \frac{x^4}{12}\right) + C_1^b x + C_2^b \quad . \tag{386}$$

From (364) and (386),

$$w_b(x,t) = -\frac{p_o e_a^*(t)}{2I}\left(\frac{\ell x^3}{6} - \frac{x^4}{12}\right) + C_1^b e_a^*(t)\, x + C_2^b e_a^*(t) \quad . \tag{387}$$

Integration of (384) for constant p_o yields

$$W_s(x) = -\frac{kp_o}{2A} x^2 + C_1^s x + C_2^s \quad . \tag{388}$$

Consequently, from (383),

$$w_s(x,t) = -\frac{kp_o g_a^*(t)}{2A} x^2 + C_1^s\, g_a^*(t) x + C_2^s\, g_a^*(t) \quad . \tag{389}$$

The total deflection is then given by

$$w(x,t) = w_b(x,t) + w_s(x,t)$$

$$= -\frac{p_o e_a^*(t)}{2I}\left(\frac{\ell x^3}{6} - \frac{x^4}{12}\right) - \frac{kp_o g_a^*(t)}{2A} x^2 + C_1(t)\, x + C_2(t) \quad , \tag{390}$$

where

$$C_1(t) = C_1^b\, e_a^*(t) + C_1^s\, g_a^*(t) \quad , \qquad \text{(a)}$$

$$C_2(t) = C_2^b\, e_a^*(t) + C_2^s\, g_a^*(t) \quad . \qquad \text{(b)} \tag{391}$$

The end conditions of simple support are

$$w(o,t) = w(\ell,t) = o \; . \tag{392}$$

Insertion of (390) into (392) yields

$$C_2(t) = o \; , \tag{393}$$

$$C_1(t) = \frac{p_o \ell^3}{24I} e_a^*(t) + \frac{p_o k \ell}{2A} g_a^*(t) \; . \tag{394}$$

Hence, finally,

$$w(x,t) = \frac{p_o x}{24I} (\ell^3 - 2\ell x^2 + x^3) \; e_a^*(t) + \frac{p_o kx}{2A} (\ell - x) \; g_a^*(t) \; . \tag{395}$$

The first part of (395) is recognized as the bending deflection w_b (368), while the second part is the shear deflection w_s.

Inspection of the expressions obtained for w_b and w_s shows a fundamental difference. According to previous discussion the time dependence of w_b is negligible, and for all practical purposes this is an elastic deflection which does not vary in time. However, $g_a^*(t)$ in w_s has a considerable time variation. Indeed, the rigid fiber approximation (378) shows that $g_a^*(t)$ is directly proportional to the shear compliance of the matrix $g_m(t)$, and this compliance can assume very large values, and, theoretically speaking, can even become unbounded. We thus arrive at the conclusion that a *viscoelastic fiber reinforced beam can become unserviceable due to excessive shear deflection.*

As an example, consider the deflection (395) at the center of the beam, where it attains its maximum. We have

$$w(\ell/2,t) = \frac{5p_o \ell^4}{384E_a^* I} + \frac{kp_o \ell^2}{8A} \frac{1-c}{1+c} g_m(t) \; , \tag{396}$$

where we have neglected the time dependence of $e_a^*(t)$ in accordance with previous discussion and have replaced it by $1/E_a^*$. Also, the rigid fiber approximation (378) has been used for $g_a^*(t)$. Denoting the first term on the right side of (396) by δ_e, we rearrange (396) in the form

$$w(\ell/2,t) = \delta_e \left[1 + \frac{48k\rho^2 E_a^*}{5\ell^2} \frac{1-c}{1+c} g_m(t) \right] \; , \tag{397}$$

where ρ is the radius of gyration of the section; i.e.,

$$\rho = \sqrt{\frac{I}{A}}\,. \qquad (398)$$

A plot of (397) is shown in Figure 22 for a beam of rectangular section. The dimensions of the beam are

$$b = 2.0 \text{ in.}$$

$$h = 4.0 \text{ in.}$$

$$\ell = 60.0 \text{ in.}$$

The material characteristics chosen are as follows:

$$E_f^E = 27.0 \times 10^6 \text{ psi}$$

$$E_m(o) = 3.62 \times 10^5 \text{ psi}$$

$$v_f = .5.$$

The matrix is taken to be polymethyl methacrylate (Reference [9]), whose shear creep compliance is shown in Figure 23.

The deflection is plotted against logarithmic time starting out at 1 sec, when the shear deflection is negligible in comparison to the bending deflection. The initial deflection doubles (assuming continued linearity in log t) after 10^{107} hrs. The enormous amount of time required to double the deflection shows that there is no danger of failure within a reasonable lifetime of the beam.

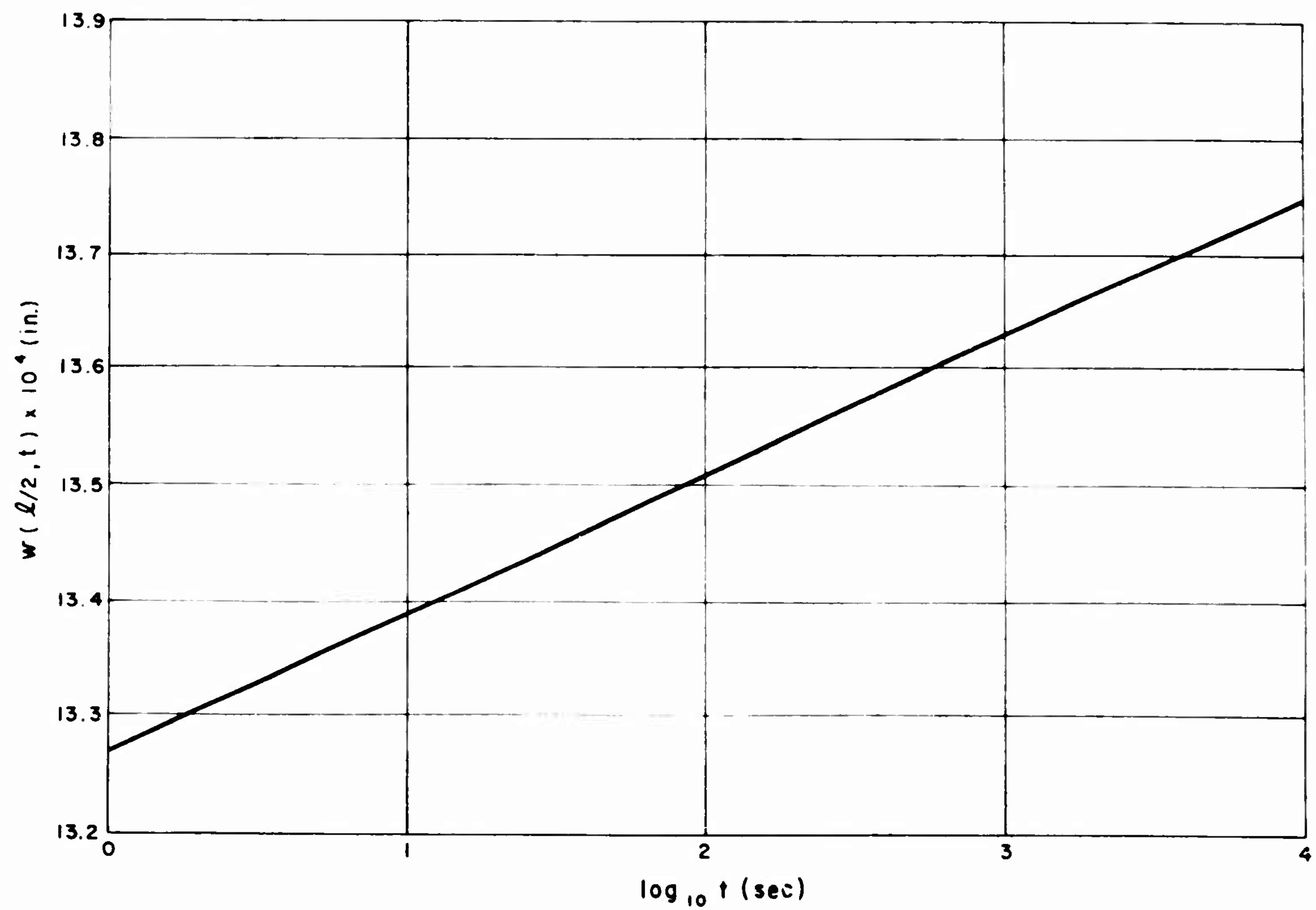

Figure 22. Lateral Deflection of Beam.

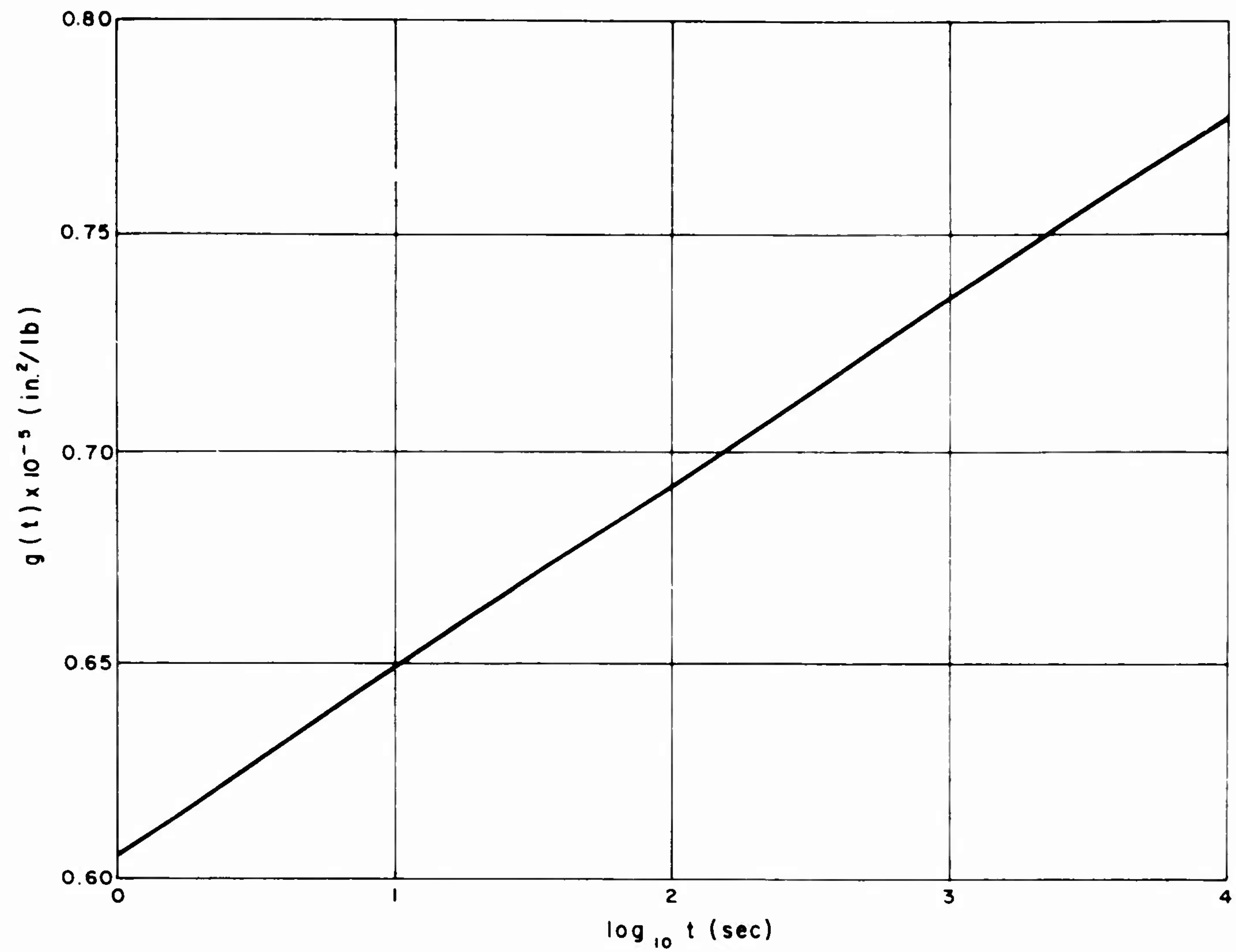

Figure 23. Shear Creep Compliance of Polymethyl Methacrylate.

11. TORSION OF FIBER REINFORCED CYLINDER

The theory of torsion of uniaxially reinforced cylinders when the fibers are in the direction of the generators fortunately proves to be very simple. Consider a cylinder whose x_1-axis is in the generator and fiber direction, while x_2, x_3 are the transverse axes in the plane of the transverse section A. We make an isotropic elastic torsion-type assumption

$$u_1 = \alpha(t)\ \phi\ (x_2, x_3)\ , \qquad \text{(a)}$$

$$u_2 = -\ \alpha(t)\ x_3,\ x_1\ , \qquad \text{(b)}$$

$$u_3 = \alpha(t)\ x_2,\ x_1\ , \qquad \text{(c)} \qquad (399)$$

where $\alpha(t)$ is the time-variable angle of twist per unit length of cylinder and ϕ is the torsion function. The only nonvanishing strains are then

$$\varepsilon_{12} = \frac{1}{2}\,\alpha(t)\left(\frac{\partial\phi}{\partial x_2} - x_3\right), \qquad \text{(a)}$$

$$\varepsilon_{13} = \frac{1}{2}\,\alpha(t)\left(\frac{\partial\phi}{\partial x_3} + x_2\right). \qquad \text{(b)} \qquad (400)$$

Because of (20), (21), and (24), the only nonvanishing stresses are then

$$\sigma_{12} = 2\int_0^t G_a^*(t-\tau)\,\frac{\partial\varepsilon_{12}(\tau)}{\partial\tau}\,d\tau\ , \qquad \text{(a)}$$

$$\sigma_{13} = 2\int_0^t G_a^*(t-\tau)\,\frac{\partial\varepsilon_{13}(\tau)}{\partial\tau}\,d\tau\ . \qquad \text{(b)} \qquad (401)$$

Inserting (400) into (401) we find

$$\sigma_{12} = \left(\frac{\partial\phi}{\partial x_2} - x_3\right)\int_0^t G_a^*(t-\tau)\,\frac{d\alpha(\tau)}{d\tau}\,d\tau\ , \qquad \text{(a)}$$

$$\sigma_{13} = \left(\frac{\partial\phi}{\partial x_3} + x_2\right)\int_0^t G_a^*(t-\tau)\,\frac{d\alpha(\tau)}{d\tau}\,d\tau\ . \qquad \text{(b)} \qquad (402)$$

Noting that

$$L(t) = \int_{o}^{t} G_a^{*}(t-\tau) \frac{d\alpha(\tau)}{d\tau} d\tau \tag{403}$$

where L(t) is a known function of time, the only surviving equilibrium equation is

$$\frac{\partial\sigma_{12}}{\partial x_2} + \frac{\partial\sigma_{13}}{\partial x_3} = o. \tag{404}$$

Substitution of (402) into (404) yields the equation

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2} = o. \tag{405}$$

Satisfaction of the zero traction boundary conditions on the curved surface of the cylinder, for the whole time range, leads precisely as in the elastic torsion problem (see, e.g., Sokolnikoff [7]) to the ϕ boundary condition

$$\frac{d\phi}{dn} = x_3 \cos(x_2, \vec{n}) - x_2 \cos(x_3, \vec{n}) , \tag{406}$$

on the contour C of the section A, where $\vec{n}$ is the normal to the contour. Equations (405) and (406) define ϕ uniquely except for an immaterial arbitrary constant. We conclude that *the present function ϕ is the same as that in the isotropic elastic torsion problem.*

The torque on the cylinder is given by

$$M_T = \iint_{(A)} (x_2\sigma_{13} - x_3\sigma_{12}) \, dx_2 \, dx_3 . \tag{407}$$

Substituting (402) with (403) into (407), we find that

$$M_T = L(t) \iint_{(A)} \left(x_2^2 + x_3^2 + x_2 \frac{\partial\phi}{\partial x_2} - x_3 \frac{\partial\phi}{\partial x_2}\right) dx_2 \, dx_3 . \tag{408}$$

We note that the initial elastic torsional rigidity of the cylinder is given by

$$D_{e\ell.} = G_a^{*}(o) \iint_A \left(x_2^2 + x_3^2 + x_2 \frac{\partial\phi}{\partial x_3} - x_3 \frac{\partial\phi}{\partial x_2}\right) dx_2 \, dx_3 . \tag{409}$$

Therefore (408) may be rewritten as

$$M_T(t) = \frac{L(t)}{G_a^*(o)} D_{e\ell.} \quad . \tag{410}$$

Furthermore, the stresses (402) in the cylinder can be written as

$$\sigma_{12} = \frac{L(t)}{\alpha(o)G_a^*(o)} \left(\frac{\partial\phi}{\partial x_2} - x_3 \right) \alpha(o)G_a^*(o) \quad , \qquad \text{(a)}$$

$$\sigma_{13} = \frac{L(t)}{\alpha(o)G_a^*(o)} \left(\frac{\partial\phi}{\partial x_3} + x_2 \right) \alpha(o)G_a^*(o) \quad . \qquad \text{(b)} \tag{411}$$

It is seen that if "elastic" stresses are defined by

$$\sigma_{12}^{e\ell.} = \left(\frac{\partial\phi}{\partial x_2} - x_3 \right) \alpha(o)G_a^*(o) \quad , \qquad \text{(a)}$$

$$\sigma_{13}^{e\ell.} = \left(\frac{\partial\phi}{\partial x_3} + x_2 \right) \alpha(o)G_a^*(o) \quad . \qquad \text{(b)} \tag{412}$$

then (411) assumes the form

$$\sigma_{12} = \frac{L(t)}{\alpha(o)G_a^*(o)} \sigma_{12}^{e\ell.} \quad , \qquad \text{(a)}$$

$$\sigma_{13} = \frac{L(t)}{\alpha(o)G_a^*(o)} \sigma_{13}^{e\ell.} \quad . \qquad \text{(b)} \tag{413}$$

As an application let us consider the case when a viscoelastic fiber reinforced cylinder is subjected to a constant torque M_T° for $t > o$. In that event

$$M_T(t) = M_T^\circ \, H(t) \tag{414}$$

where H(t) is the Heaviside unit step function. How take the Laplace transforms of (403), (410), and (414), and combine the results. This gives

$$\hat{\alpha}(p) = \frac{M_T{}^\circ \; G_a^*(o)}{p^2 \hat{G}_a^*(p) D_{e\ell.}} \quad . \tag{415}$$

If (29) is recalled, it is seen that (415) assumes the form

$$\hat{\alpha}(p) = \frac{M_T{}^\circ \; G_a^*(o)}{D_{e\ell.}} \hat{g}_a^*(p) \; . \tag{416}$$

This result can be directly inverted into the time domain, obtaining

$$\hat{\alpha}(t) = \frac{M_T{}^\circ \; G_a^*(o)}{D_{e\ell.}} g_a^*(t) \; . \tag{417}$$

To find the stresses, take the Laplace transform of (413), using (403). The result is

$$\hat{\sigma}_{12} = \frac{p \; \hat{G}_a^*(p)\hat{\alpha}(p)}{\alpha(o)G_a^*(o)} \sigma_{12}{}^{e\ell.} \; , \qquad (a)$$

$$\hat{\sigma}_{13} = \frac{p \; \hat{G}_a^*(p)\hat{\alpha}(p)}{\alpha(o)G_a^*(o)} \sigma_{13}{}^{e\ell.} \; . \qquad (b) \tag{418}$$

Introduce (415) into (418) to obtain

$$\hat{\sigma}_{12} = \sigma_{12}{}^{e\ell.} \frac{M_T{}^\circ}{p \; \alpha(o) \; D_{e\ell.}} \; , \qquad (a)$$

$$\hat{\sigma}_{13} = \sigma_{13}{}^{e\ell.} \frac{M_T{}^\circ}{p \; \alpha(o) \; D_{e\ell.}} \; . \qquad (b) \tag{419}$$

Because of (417),

$$\alpha(o) = \frac{M_T{}^\circ \; G_a^*(o)}{D_{e\ell.}} g_a^*(o) \; , \tag{420}$$

but

$$g_a^*(o) = \frac{1}{G_a^*(o)} \ . \tag{421}$$

Therefore,

$$\alpha(o) = \frac{M_T{}^\circ}{D_{e\ell.}} \ . \tag{422}$$

Introducing into (419) and inverting, it follows that

$$\sigma_{12} = \sigma_{12}{}^{e\ell.} H(t) \ , \quad (a)$$

$$\sigma_{13} = \sigma_{13}{}^{e\ell.} H(t) \ . \quad (b) \tag{423}$$

Using (412), (423) can be rewritten in the form

$$\sigma_{12} = \frac{M_T{}^\circ G_a^*(o)}{D_{e\ell.}} \left(\frac{\partial\phi}{\partial x_2} - x_3 \right) H(t) \quad (a)$$

$$\sigma_{13} = \frac{M_T{}^\circ G_a^*(o)}{D_{e\ell.}} \left(\frac{\partial\phi}{\partial x_3} + x_2 \right) H(t). \quad (b) \tag{424}$$

Equations (424) imply that if the torque is kept constant in time, the stresses also are constant in time and are those obtained for the elastic problem.

To give an example of the preceding theory, we consider the torsion (414) of a circular cylinder of radius a, in which the fibers are rigid. In this case, the torsional shear stress $\sigma_{r\theta}$ is simply given by

$$\sigma_{r\theta} = \frac{M_T{}^\circ}{\pi\frac{a^4}{2}} r, \qquad \sigma_{r\theta_{max}} = \frac{2M_T{}^\circ}{\pi a^3} \ . \tag{425}$$

In order to compute the angle of twist from (417), we note that for a circular cylinder

$$D_{e\ell.} = G_a^*(o) \frac{\pi a^4}{2} \ . \tag{426}$$

Furthermore, it has been shown in Reference [3] that

$$g_a^{\star}(t) = \frac{1-c}{1+c}\, g_m(t) \ , \tag{427}$$

where $g_m(t)$ is the shear creep compliance of the matrix. Inserting (426) and (427) into (417), we find that

$$\alpha(t) = \frac{M_T^{\circ}}{\pi \frac{a^4}{2}} \frac{1-c}{1+c}\, g_m(t) \ . \tag{428}$$

It is seen that if $g_m(t)$ becomes unbounded, which is quite often the case, then $\alpha(t)$ becomes unbounded.

12. QUASI-STATIC DEFORMATION OF VISCOELASTIC FIBER REINFORCED PLATES

We consider here the case of quasi-static bending by transverse loads of a thin, rectangular fiber reinforced plate. The reinforcement direction is taken as x_1. The middle surface of the plate of thickness h is referred to the plane Cartesian system of axes x_1, x_2. The x_3-axis is chosen normal to the middle plane, pointing downward (see Figure 19).

The deflection of the middle surface is

$$u_3(x_1, x_2, o, t) = w(x_1, x_2, t) , \tag{429}$$

where t is the time. We use the usual Kirchhoff hypothesis of thin plate bending, according to which the deformations are given by

$$u_1(x_1, x_2, x_3, t) = - x_3 \frac{\partial w}{\partial x_1} , \tag{430}$$

$$u_2(x_1, x_2, x_3, t) = - x_3 \frac{\partial w}{\partial x_2} . \tag{431}$$

The strains in $x_1 - x_2$ planes associated with (429), (430), and (431) are then given by

$$\varepsilon_{11} = - x_3 \frac{\partial^2 w}{\partial x_1^2} , \tag{432}$$

$$\varepsilon_{22} = - x_3 \frac{\partial^2 w}{\partial x_2^2} , \tag{433}$$

$$\varepsilon_{12} = - x_3 \frac{\partial^2 w}{\partial x_1 \partial x_2} . \tag{434}$$

It will be convenient to continue the analysis in Laplace transform domain. The Laplace transforms of (432) through (434) are written compactly as

$$\hat{\varepsilon}_{ij}(x_1, x_2, x_3, p) = - x_3 \cdot \frac{\partial^2 \hat{w}(x_1, x_2, p)}{\partial x_i \, \partial x_j} . \tag{435}$$

$$i,j = 1,2$$

The Laplace transformed viscoelastic stress-strain law of the uniaxially reinforced plate material may be written in the form

$$\hat{\varepsilon}_{11} = \frac{\hat{\sigma}_{11}}{p\hat{E}^*_a(p)} - \frac{\tilde{\nu}^*_a(p)}{p\hat{E}^*_a(p)} \hat{\sigma}_{22} , \tag{436}$$

$$\hat{\varepsilon}_{22} = - \frac{\tilde{\nu}^*_a(p)}{p\hat{E}^*_a(p)} \hat{\sigma}_{11} + \frac{\hat{\sigma}_{22}}{p\hat{E}^*_T(p)} , \tag{437}$$

$$\hat{\varepsilon}_{12} = \frac{\hat{\sigma}_{12}}{2p\hat{G}^*_a(p)} . \tag{438}$$

Here $\hat{E}^*_a(p)$ is the LT of the axial Young's relaxation modulus, $\hat{E}^*_T(p)$ is the LT of the transverse (normal to fibers) Young's relaxation modulus, $\hat{G}^*_a(p)$ is the LT of the axial shear relaxation modulus, and $\tilde{\nu}^*_a(p)$ is a transform domain "Poisson's ratio" which will be discussed later. It has been assumed in (436) through (438) that σ_{33} can be neglected, as is usually done in theory of plates. The form of (436) through (438) follows directly from the discussion given in Section 2, according to which a Laplace transformed viscoelastic stress-strain law is completely analogous to an elastic stress-strain law with p-multiplied transforms of relaxation moduli replacing elastic moduli; hence, the form of (436) through (438).

$$\hat{\sigma}_{11} = p \frac{\hat{E}^*_a \hat{\varepsilon}_{11} + \tilde{\nu}^*_a \hat{E}^*_T \hat{\varepsilon}_{22}}{1 - \frac{\hat{E}^*_T}{\hat{E}^*_A} \tilde{\nu}^{*2}_a} , \tag{439}$$

$$\hat{\sigma}_{22} = p \frac{\hat{E}^*_t (\hat{\varepsilon}_{22} + \tilde{\nu}^*_a \hat{\varepsilon}_{11})}{1 - \frac{\hat{E}^*_T}{\hat{E}^*_a} \tilde{\nu}^{*2}_a} , \tag{440}$$

$$\hat{\sigma}_{12} = 2p\hat{G}^*_a \hat{\varepsilon}_{12} . \tag{441}$$

The LT of the bending and twisting moments in the plate are given by

$$\hat{M}_{ij} = \int_{-h/2}^{h/2} \hat{\sigma}_{ij}\, x_3 dx_3 \; . \qquad i,j = 1,2 \tag{442}$$

Substituting (435) into (439) through (441) and substituting the resulting expressions into (442), we find that

$$\hat{M}_{11} = - p \frac{h^3}{12} \frac{\hat{E}_a^* \hat{w}_{,11} + \tilde{\nu}_a^* \hat{E}_T^* \hat{w}_{,22}}{1 - \frac{\hat{E}_T^*}{\hat{E}_a^*} \tilde{\nu}_a^{*2}} \tag{443}$$

$$\hat{M}_{22} = - p \frac{h^3}{12} \frac{\hat{E}_T^* (\tilde{\nu}_a^* \hat{w}_{,11} + \hat{w}_{,22})}{1 - \frac{\hat{E}_T^*}{\hat{E}_a^*} \tilde{\nu}_a^{*2}} \; , \tag{444}$$

$$\hat{M}_{12} = - \frac{h^3}{12} 2 \hat{G}_a^* \hat{w}_{,12} \; . \tag{445}$$

Here, $\hat{w}_{,11}$, $\hat{w}_{,22}$ and $\hat{w}_{,12}$ denote the second partial derivatives of $\hat{w}$ with respect to x_1 and x_2.

The quasi-static equilibrium equation of a plate in terms of bending moments is given by

$$\frac{\partial^2 M_{11}}{\partial x_1^2} + 2 \frac{\partial^2 M_{12}}{\partial x_1 \partial x_2} + \frac{\partial^2 M_{22}}{\partial x_2^2} + q(x_1, x_2, t) = 0 \; , \tag{446}$$

where q is the transverse load per unit area of plate, positive downward. See, e.g., Timoshenko and Woinowsky-Kreiger [16]. Taking the LT of (446) and substituting $\hat{M}_{ij}$ from (443) through (445), we find that

$$\hat{D}_{1111}^* \frac{\partial^4 \hat{w}}{\partial x_1^4} + 2 \hat{D}_{1122}^* \frac{\partial^4 \hat{w}}{\partial x_1^2 \partial x_2^2} + \hat{D}_{2222}^* \frac{\partial^4 \hat{w}}{\partial x_2^4} = \hat{q}(x_1, x_2, p) \; , \tag{447}$$

where

$$\hat{D}^{\star}_{1111} = \frac{h^3}{12} \frac{p\,\hat{E}^{\star}_{a}}{1 - \frac{\hat{E}^{\star}_{a}}{\hat{E}^{\star}_{a}} \tilde{\nu}^{\star 2}_{a}} , \tag{448}$$

$$\hat{D}^{\star}_{1122} = \frac{h^3 p}{12} \left[\frac{\tilde{\nu}^{\star}_{a} \hat{E}^{\star}_{T}}{1 - \frac{\hat{E}^{\star}_{T}}{\hat{E}^{\star}_{a}} \tilde{\nu}^{\star 2}_{a}} + 2\hat{G}^{\star}_{a} \right] , \tag{449}$$

$$\hat{D}^{\star}_{2222} = \frac{h^3}{12} \frac{p\,\hat{E}^{\star}_{T}}{1 - \frac{\hat{E}^{\star}_{T}}{\hat{E}^{\star}_{a}} \tilde{\nu}^{\star 2}_{a}} . \tag{450}$$

Equation (449) is the governing differential equation of the LT of the deflection of the plate.

We now consider a typical boundary condition. For a freely supported rectangular plate, $0 \le x_1 \le a$, $0 \le x_2 \le b$.

$$w(0, x_2, t) = w(a, x_2, t) = w(x_1, 0, t) = w(x_1, b, t) = 0 , \tag{451}$$

$$w,_{11}(0, x_2, t) = w,_{11}(a, x_2, t) = w,_{22}(x_1, 0, t) = w,_{22}(x_1, b, t) = 0 . \tag{452}$$

For a built-in plate, (452) is replaced by

$$w,_{1}(0, x_2, t) = w,_{1}(a, x_2, t) = w,_{2}(x_1, 0, t) = w,_{2}(w_1, 0, t) = 0 . \tag{453}$$

Since our governing equation is for the LT $\hat{w}$ of w, the boundary conditions (451) through (453) have to be Laplace transformed. They obviously remain in the same form after Laplace transformation. For a simply supported plate, we thus have

$$\hat{w}(0, x_2, p) = \hat{w}(a, x_2, p) = \hat{w}(x_1, 0, p) = \hat{w}(x_1, b, p) = 0 , \tag{454}$$

$$\hat{w},_{11}(0, x_2, p) = \hat{w},_{11}(a, x_2, p) = \hat{w},_{22}(x_1, 0, p) = \hat{w},_{22}(x_1, b, p) = 0 . \tag{455}$$

Suppose now that the load on the plate has the form

$$q(x_1, x_2, t) = q_o(x_1, x_2)\, \phi(t) \ , \tag{456}$$

where $\phi(t)$ is any function of time. Then the Laplace transform of (456) is given by

$$\hat{q}(x_1, x_2, p) = q_o\, \hat{\phi}(p) \ , \tag{457}$$

and this expression now has to be used on the right side of (447).

Let q_o in (456) be expanded into the double Fourier series

$$q_o(x_1, x_2) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn} \sin \alpha_m x_1 \sin \beta_n x_2 \ , \tag{458}$$

where

$$\alpha_m = \frac{m\pi}{a} \ , \tag{459}$$

$$\beta_n = \frac{n\pi}{b} \ , \tag{460}$$

$$q_{mn} = \frac{4}{ab} \int_o^a \int_a^b q_o(x_1, x_2) \sin \alpha_m x_1 \sin \beta_n x_2 \, dx_1 dx_2 \ . \tag{461}$$

[illegible] a solution of (447) in the form

$$\hat{w}(x_1, x_2, p) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \hat{w}_{mn} \sin \alpha_m x_1 \sin \beta_n x_2 \ . \tag{462}$$

It is seen that (462) satisfies all the boundary conditions (454) and (455) of the simply supported plate. Now substitute (458) into (457), and then substitute the resulting expression and (462) into (447). Equate coefficient of $\sin \alpha_m x_1 \sin \beta_n x_2$ on both sides. This procedure yields

$$\hat{w}_{mn}(p) = \frac{q_{mn}\, \hat{\phi}(p)}{\hat{D}^*_{1111}\, \alpha_m^4 + 2\hat{D}^*_{1122}\, \alpha_m^2 \beta_n^2 + \hat{D}^*_{2222}\, \beta_n^4} \ . \tag{463}$$

Equation (463) must be inverted into the time domain to yield

$$w_{mn}(t) = LT^{-1}\,[\hat{w}_{mn}(p)] \ . \tag{464}$$

It follows from (462) and (464) that

$$w(x_1, x_2, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn}(t) \sin \alpha_m x_1 \sin \beta_n x_2 \ . \tag{465}$$

This completes the *formal* solution of the problem of the simply supported rectangular viscoelastic plate.

The numerical exploitation of the solution which has been given here unfortunately involves immense difficulties. The quantities $\hat{D}^*_{1111}$, $\hat{D}^*_{1122}$, and $\hat{D}^*_{2222}$ in the denominator of (463) are defined in terms of transforms of viscoelastic relaxation moduli, (448) through (450), which are themselves complicated functions of transforms of viscoelastic relaxation moduli of matrix and of elastic moduli of fibers. Analytical inversion is, in general, out of the question, and the inversion should be carried our numerically by the use of computers. Here no attempt will be made to carry out numerical inversion by computer. Instead, we shall approximate the viscoelastic effective properties of the fiber reinforced material with the aim of making an analytical inversion possible. The purpose is thus not to solve one case precisely from a numerical point of view but, rather, to obtain an estimate of the viscoelastic behavior of [illegible]r reinforced plates in bending. We shall assume, whenever convenient, that the matrix is incompressible. This is generally a fair approximation for polymeric viscoelastic materials.

The first quantity to be discussed is $\tilde{\nu}^*_a(p)$. Its interpretation is as follows: suppose that the fiber reinforced material is subjected to a uniaxial stress constant in time for t>o, $\sigma_{11}(t) = \sigma_{11} H(t)$, in fiber direction. This produces an axial average strain $\bar{\varepsilon}_{11}(t)$ with transform $\hat{\bar{\varepsilon}}_{11}(p)$ and a transverse average strain $\bar{\varepsilon}_{22}(t) = \bar{\varepsilon}_{33}(t)$ with transform $\hat{\bar{\varepsilon}}_{22}(p) = \hat{\bar{\varepsilon}}_{33}(p)$. The transform domain Poisson's ratio $\tilde{\nu}^*_a(p)$ is defined by

$$\tilde{\nu}^*_a(p) = - \frac{\hat{\bar{\varepsilon}}_{22}(p)}{\hat{\bar{\varepsilon}}_{11}(p)} \ . \tag{466}$$

A physical time-dependent Poisson's ratio may be defined by

$$\nu_a^*(t) = -\frac{\bar{\varepsilon}_{22}(t)}{\bar{\varepsilon}_{11}(t)} . \qquad (467)$$

Note that $\tilde{\nu}_a^*(p)$ is *not* the transform of $\nu_a^*(t)$.

It has been shown in Reference [3] that for incompressible viscoelastic matrix and stiff elastic fibers, $\nu_a^*(t)$ is time independent and is given by

$$\nu_a^*(t) \tilde{=} \frac{1}{2} v_m + \nu_f v_f = \nu_a^* \qquad (\text{^} \leq t \leq \infty). \qquad (468)$$

Here, v_m and v_f are the volume fractions of matrix and fibers, respectively, and ν_f is the elastic fibers' Poisson's ratio. The middle quantity in (468) is henceforth denoted ν_a^*.

It follows from (467) and (468) that

$$\hat{\bar{\varepsilon}}_{22}(p) \tilde{=} - \nu_a^* \hat{\bar{\varepsilon}}_{11}(p) . \qquad (469)$$

Comparison of (469) with (466) then shows that

$$\tilde{\nu}_a^*(p) \tilde{=} \nu_a^* . \qquad (470)$$

Next, we consider $\hat{E}_a^*(p)$, which is the transform of the axial Young's relaxation modulus. It has been shown in Reference [3] that $E_a^*(t)$ is practically time independent and is given by

$$E_a^*(t) \tilde{=} (E_m v_m + E_f v_f) H(t) = E_a^* H(t) , \qquad (471)$$

where E_m and E_f are respectively matrix and .iber elastic Young's moduli, and E_a^* without time argument henceforth denotes the middle quantity in (471). Hence, we have from (471),

$$\hat{E}_a^*(p) = \frac{E_a^*}{p} . \qquad (472)$$

Now the transverse Young's modulus E_T^* of a uniaxially fiber reinforced material is much smaller than the axial modulus E_a^*; see Hashin and Rosen [1]. The same is true to a larger extent for a viscoelastic fiber reinforced material; i.e.,

$$E_a^*(t) >> E_T^*(t) \ . \tag{473}$$

Equation (473) is physically obvious, since for axial loads the stiff elastic fibers take the major part of the load, while for transverse loads the weak viscoelastic matrix takes the major part of the load. Since

$$\hat{E}_a^*(p) = \int_o^\infty E_a^*(t)\ e^{-pt}\ dt \ , \tag{474}$$

$$\hat{E}_T^*(p) = \int_o^\infty E_T^*(t)\ e^{-pt}\ dt \ , \tag{475}$$

it follows from (473) through (475) that

$$\hat{E}_a^*(p) >> \hat{E}_T^*(p) \ . \tag{476}$$

A typical value of (468) is given by

$$v_f = v_m = 0.5, \ \nu_f = 0.2 \ .$$

Then

$$\nu_a^* = 0.35 \ . \tag{477}$$

Using (476) and (477) in (448) through (450), it is seen that the quantity $\tilde{\nu}_a^{*2}\ \hat{E}_T^*/\hat{E}_a^*$ can be safely neglected in comparison to unity. Furthermore, we introduce (470) and (472) into (448) through (450). We then obtain the approximations

$$\hat{D}_{1111}^* \tilde{=} \frac{h^3}{12} E_a^* \ , \tag{478}$$

$$\hat{D}_{1122}^* \tilde{=} \frac{h^3}{12} p\ (\nu_a^*\ \hat{E}_T^* + 2\hat{G}_a^*) \ , \tag{479}$$

$$\hat{D}_{2222}^* \tilde{=} \frac{h^3}{12} p\ \hat{E}_T^* \ . \tag{480}$$

We now have to consider the quantity $\hat{E}_T^*$, which is the transform of the transverse Young's relaxation modulus (normal to fibers). The *elastic* E_T^* can be expressed in terms of other moduli in the following way (see Reference [2], p. 16):

$$E_T^* = \frac{4G_T^*}{1 + G_T^*/K_T^* + 4G_T^* \nu_a^{*2}/E_a^*} . \tag{481}$$

If the matrix is incompressible or even nearly incompressible, then K_T^* becomes very much larger then G_T^*, and, consequently, the ratio G_T^*/K_T^* can be neglected in the denominator of (481). In addition, for stiff fibers E_a^* is generally very much larger than G_T^* and ν_a^{*2} is a small quantity. So the third term in the denominator of (481) can also be neglected. We thus obtain the approximation

$$E_T^* \cong 4G_T^* , \tag{482}$$

valid for incompressible matrix and stiff fibers.

In transform domain, the relation (482) becomes

$$\hat{E}_T^* \cong 4\hat{G}_T^* . \tag{483}$$

According to the discussion given in Section 8 for incompressible matrix and stiff fibers, the transverse relaxation modulus $G_T^*(t)$ is approximately given by (302). Taking the transform of this equation and introducing it into (483), we obtain

$$\hat{E}_T^* \cong 4\hat{G}_a^* = 4 \frac{1 + c}{1 - c} \hat{G}_m(p) , \tag{484}$$

where c is the volume fraction of fibers. Consequently, we can now further approximate (478) through (480), obtaining

$$\hat{D}_{1111}^* \cong \frac{h^3}{12} E_a^* , \tag{485}$$

$$\hat{D}_{1122}^* \cong \frac{h^3}{12} 2p\hat{G}_a^* (1 + 2\nu_a^*) , \tag{486}$$

$$\hat{D}_{2222}^* \cong \frac{h^3}{12} 4p\hat{G}_a^* . \tag{487}$$

We now assume that the load on the plate is constant in time for $t>o$. This means that $\phi(t)$ in (456) is given by

$$\phi(t) = H(t) \; . \tag{488}$$

Consequently,

$$\hat{\phi}(p) = \frac{1}{p} \; . \tag{489}$$

Introducing (484) through (488) into (463), we obtain

$$\frac{h^3}{12} \hat{w}_{mn}(p) = \frac{1}{E^*_a \alpha^4_m + 4 \frac{1 + c}{1 - c} \hat{G}_m(p) \; [(1 + 2\nu^*_a) \; \alpha^2_m \; \beta^2_n + \beta^4_n]} \left(\frac{1}{p}\right) \; . \tag{490}$$

Equation (490) cannot be inverted without specification of the functional form of $\hat{G}_m(p)$. We make the simple assumption that the visco-elastic matrix is characterized by a simple Maxwell model. In that event, $p \; G_m(p)$ is given by (286). Introducing this result into (490), we have, after rearrangement,

$$\frac{h^3}{12} \hat{w}_{mn}(p) = \frac{q_{mn}}{TA_{mn}} \frac{1 + T_p}{p \; (E^*_a \; \alpha^4_{mn}/TA_{mn} + p)} \; , \tag{491}$$

(no sum on m,n)

where

$$A_{mn} = E^*_a \; \alpha^4_m + 4G_m \frac{1 + c}{1 - c} \; [1 + 2\nu^*_a) \; \alpha^2_m \; \beta^2_n + \beta^4_n] \; . \tag{492}$$

Equation (491) is easily inverted after separation of the right side into partial fractions. The result is

$$w_{mn}(t) = \frac{q_{mn}}{h^3/12} \left\{ \left(\frac{1}{A_{mn}} - \frac{1}{E^*_a \; \alpha^4_m}\right) \exp\left(- \frac{E^*_a \; \alpha^4_m}{A_{mn}} \cdot \frac{t}{T}\right) + \frac{1}{E^*_a \; \alpha^4_m} \; H(t) \right\} \; . \tag{493}$$

Equations (465) and (493) specify the deflection of the plate. It is of interest to compute the initial deflection of the plate $w(x_1, x_2, o)$ and the final deflection $w(x_1, x_2, \infty)$. From (493), we have

$$w_{mn}(o) = \frac{q_{mn}}{h^3 A_{mn}/12}, \tag{494}$$

$$w_{mn}(\infty) = \frac{q_{mn}}{h^3 E_a^* \alpha_m^4/12}. \tag{495}$$

It is seen that in spite of the fact that the Maxwell matrix by itself has unbounded creep deformations, the plate deformation has a finite limit. This phenomenon is due to the effect of the fibers.

Figure 24 shows the variation with nondimensional time t/T of the deflection at the center of a square, simply supported, visco-elastic fiber reinforced plate. The plate and material data are as follows:

$$E_m = 5.0 \times 10^5 \text{ psi}$$

$$E_f = 10.5 \times 10^6 \text{ psi}$$

$$\nu_f = 0.2$$

$$G_m = 1.7 \times 10^5 \text{ psi}$$

$$c = 0.5$$

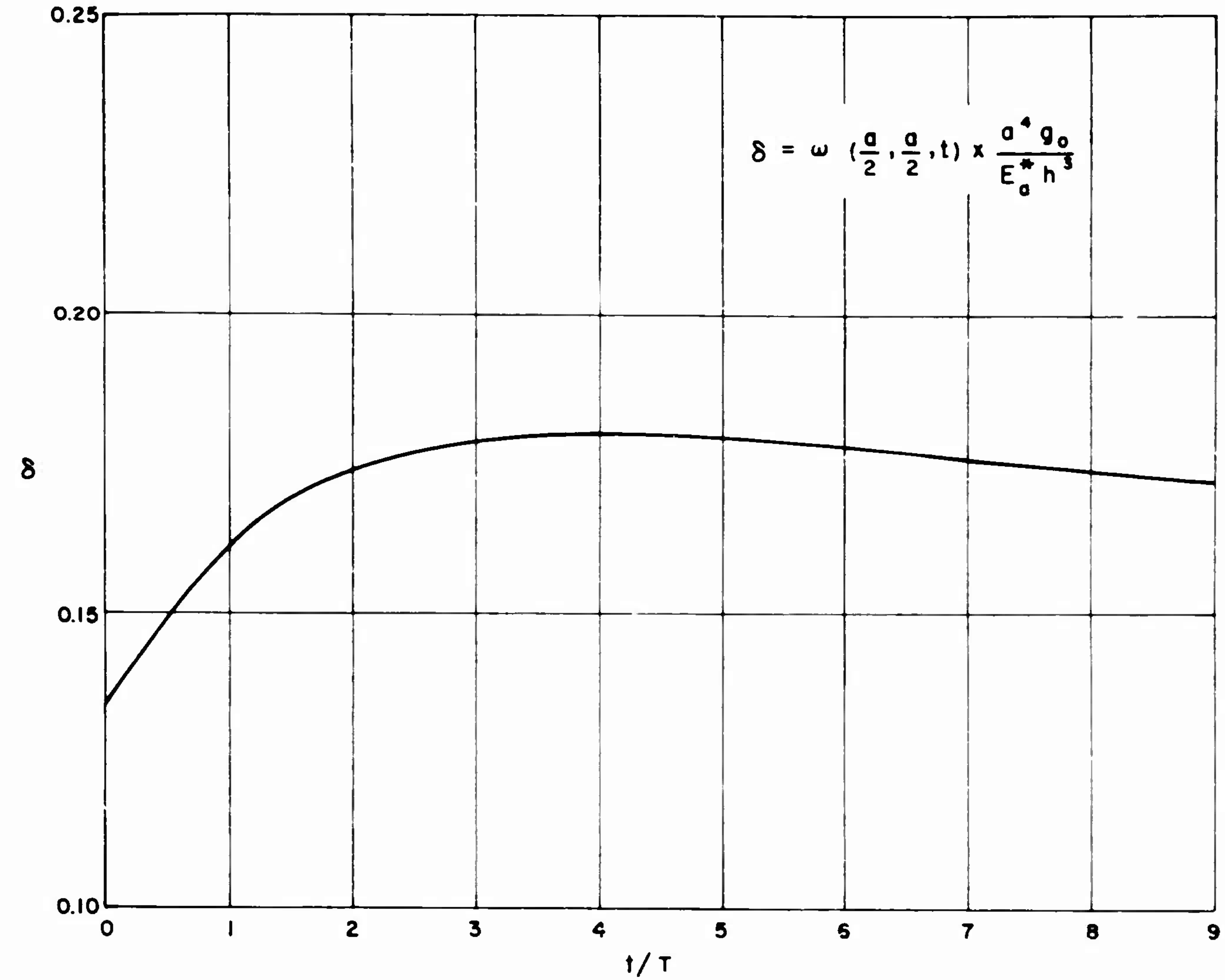

Figure 24. Lateral Deflection of Plate.

13. AXISYMMETRICALLY LOADED VISCOELASTIC FIBER REINFORCED CIRCULAR CYLINDRICAL SHELL

The shell is shown in Figure 25 with its coordinate system. Axisymmetry under axisymmetric loading requires that material properties also be axisymmetric. For uniaxial reinforcement, this is possible only for two cases: when the fibers are in generator direction or when the fibers are in circumferential direction, normal to generators. In view of previous notation for material properties where x_1 always was in fiber direction, we can here make the following identification:

Longitudinal reinforcement	$x \to x_1$	$\theta \to x_2$	(496)
Circumferential reinforcement	$x \to x_2$	$\theta \to x_1$.	(497)

We consider a static shell whose only load per unit shell area is in z-direction, normal to the shell surface,

$$q_2 = q = q(x) \ . \qquad (498)$$

In the axisymmetric case the shear membrane force $N_{x\theta}$, the twisting moment $M_{x\theta}$, and the shear force Q_θ all vanish. Furthermore there is no θ dependence in any quantity. Simplifying the shell equilibrium equations accordingly (see, e.g., Reference [17]), we obtain

$$\frac{dN_x}{dx} = o \qquad (499)$$

$$\frac{dQ_x}{dx} - \frac{N_\theta}{R} + q(x) = o \ , \qquad (500)$$

$$\frac{dM_x}{dx} - Q_x = o \ , \qquad (501)$$

where N_x, N_θ are normal membrane forces, M_x is normal bending moment, and Q_x is the shear force. Elimination of Q_x from (500) and (501) yields

$$\frac{d^2M_x}{dx^2} - \frac{N_\theta}{R} + q(x) = o \ , \qquad (502)$$

while (499) implies that

$$N_x = N_o = \text{const.} \qquad (503)$$

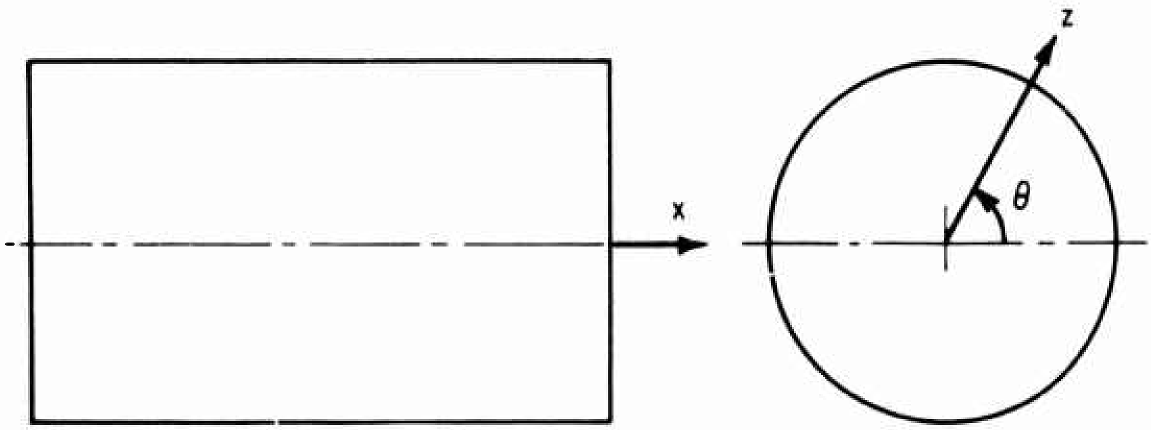

Figure 25. Cylindrical Shell Coordinate System.

In order to simplify the discussion, we shall derive the equations of an elastic fiber reinforced shell, and we shall then proceed to the viscoelastic case via the correspondence principle. On the basis of the Kirchhoff assumption, axisymmetry, and the thin shell approximation $h/R << 1$, the only surviving strains in the shell are

$$\varepsilon_{xx} = \frac{du}{dx} - z\frac{d^2w}{dx^2}, \tag{504}$$

$$\varepsilon_{\theta\theta} = \frac{w}{R}. \tag{505}$$

Here, z is measured from the shell middle surface, normal to it and positive outward. u and w are displacements in x- and z-directions, respectively.

Let it now be assumed that the reinforcement is longitudinal. Then (496) applies, and the stress-strain law is given by

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E_a^*} - \frac{\nu_a^*}{E_a^*}\sigma_{\theta\theta}, \tag{506}$$

$$\varepsilon_{\theta\theta} = -\frac{\nu_a^*}{E_a^*}\sigma_{xx} + \frac{\sigma_{\theta\theta}}{E_T^*}. \tag{507}$$

where E_a^* is the axial Young's modulus, ν_a^* is the axial Poisson's ratio, and E_T^* is the transverse Young's modulus. Solving (506) and (507) for the stresses yields

$$\sigma_{xx} = A_{xx} \varepsilon_{xx} + A_{x\theta} \varepsilon_{x\theta} , \tag{508}$$

$$\sigma_{\theta\theta} = A_{x\theta} \varepsilon_{xx} + A_{\theta\theta} \varepsilon_{\theta\theta} , \tag{509}$$

where

$$A_{xx} = \frac{E_a^*}{1 - \frac{E_T^*}{E_a^*} \nu_a^{*2}} , \tag{510}$$

$$A_{x\theta} = \frac{\nu_a^* E_T^*}{1 - \frac{E_T^*}{E_a^*} \nu_a^{*2}} \tag{511}$$

$$A_{\theta\theta} = \frac{E_T^*}{1 - \frac{E_T^*}{E_a^*} \nu_a^{*2}} . \tag{512}$$

The shell membrane forces and bending moments are given by

$$N_x = \int_{-h/2}^{h/2} \sigma_{xx} \, dz \quad ; \quad N_\theta = \int_{-h/2}^{h/2} \sigma_{\theta\theta} \, dz , \tag{513}$$

$$M_x = \int_{-h/2}^{h/2} \sigma_{xx} \, z \, dz \quad ; \quad M_\theta = \int_{-h/2}^{h/2} \sigma_{\theta\theta} \, z \, dz . \tag{514}$$

We now introduce (504) and (505) into (508) and (509), and we introduce the resulting expressions into (513) and (514). Thus, shell membrane forces and bending moments are obtained in terms of u and w. These are in turn introduced into the equilibrium Equations (502) and (503). This procedure leads to

$$\frac{h^3}{12} A_{xx} \frac{d^4 u}{dx^4} + \frac{h}{R} (A_{x\theta} \frac{du}{dx} + A_{\theta\theta} \frac{w}{R}) = q \ , \qquad (515)$$

$$h(A_{xx} \frac{du}{dx} + A_{x\theta} \frac{w}{R}) = N_o \ . \qquad (516)$$

Elimination of $\frac{du}{dx}$ yields the equation

$$\frac{d^4 w}{dx^4} + \gamma w = \frac{12q}{h^3 A_{xx}} - \frac{12 A_{x\theta}}{Rh^3 A_{xx}^2} N_o \ , \qquad (517)$$

where

$$\gamma = \frac{12}{R^2 h^2} \frac{A_{xx} A_{\theta\theta} - A_{x\theta}^2}{A_{xx}^2} \ . \qquad (518)$$

The membrane force N_o in (517) is equal to the constant prescribed membrane force in x-directions at the end sections x = 0, L of the cylinder. If there is no membrane force prescribed, N_o vanishes.

Usual boundary conditions at x = 0, L prescribe any two of the quantities w, $\frac{dw}{dx}$, M_x, and Q_x.

For a viscoelastic fiber reinforced shell cylinder, E_a^*, E_T^*, and ν_a^* have to be replaced by $p\hat{E}_a^*(p)$, $p\hat{E}_T^*(p)$, and $\tilde{\nu}_a^*(p)$ in an analogy to the treatment given in Section 12. The load q is now q(x,t), a function of time also, and has to be replaced by its transform $\hat{q}(x,p)$.

Let it be assumed for simplicity that the cylinder is not subjected to longitudinal membrane load at the ends. Then (517), written for the Laplace transform of w(x,t) which is denoted by $\hat{w}$, becomes

$$\frac{d^4 \hat{w}}{dx^4} + \tilde{\gamma}(p)\hat{w} = \frac{12\hat{q}}{h^3 \tilde{A}_{xx}(p)} \ , \qquad (519)$$

where

$$\tilde{\gamma}(p) = \frac{12}{R^2 h^2} \frac{\tilde{A}_{xx}(p)\, \tilde{A}_{\theta\theta}(p) - \tilde{A}_{x\theta}^2(p)}{\tilde{A}_{xx}^2(p)} \ , \qquad (520)$$

and

$$\tilde{A}_{xx}(p) = \frac{p\hat{E}^*_a}{1 - \frac{\hat{E}^*_T}{\hat{E}^*_a}\tilde{\nu}^{*2}_a} , \tag{521}$$

$$\tilde{A}_{x\theta}(p) = \frac{p\tilde{\nu}^*_a \hat{E}^*_T}{1 - \frac{\hat{E}^*_T}{\hat{E}^*_a}\tilde{\nu}^{*2}_a} , \tag{522}$$

$$\tilde{A}_{\theta\theta}(p) = \frac{p\hat{E}^*_T}{1 - \frac{\hat{E}^*_T}{\hat{E}^*_a}\tilde{\nu}^{*2}_a} . \tag{523}$$

For very stiff fibers and nearly incompressible viscoelastic matrix, the approximations used in Section 12 apply in a completely analogous way. These are contained in Equations (485) through (487) for the approximate TD plate stiffnesses. In the present case, Equations (521) through (523) become

$$\tilde{A}_{xx}(p) \cong E^*_a , \tag{524}$$

$$\tilde{A}_{x\theta}(p) \cong 4\nu^*_a p \hat{G}^*_a , \tag{525}$$

$$\tilde{A}_{\theta\theta}(p) \cong 4p\hat{G}^*_a . \tag{526}$$

Introduction of (524) and (526) into (520) yields

$$\tilde{\gamma}(p) = \frac{48p\,\hat{G}^*_a}{R^2h^2E^*_a} . \tag{527}$$

In the elastic case, Equation (517) is easily solved in general, and the solution is then fitted to the boundary conditions. It should, however, be borne in mind that the homogeneous solution of (517) contains the terms $\exp[(-\gamma)^{1/4}]$. In the viscoelastic case, γ has to be replaced by $\tilde{\gamma}(p)$ as given by (527). Thus the Laplace transform $\hat{w}$ will be obtained in a very complicated form, and the inversion will prove to be extremely difficult. Indeed, analytical inversion is out of the question. It is therefore more convenient to use a Fourier method for the solution of (519), since this results in a much simpler form of $\hat{w}$.

Let

$$q(x,t) = \sum_{n=1}^{\infty} q_n(t) \sin \alpha_n x \ , \tag{528}$$

$$\alpha_n = \frac{n\pi}{L} \ . \tag{529}$$

Then

$$\hat{q}(x,p) = \sum_{n=1}^{\infty} \hat{q}_n(p) \sin \alpha_n x \ . \tag{530}$$

Assume that a solution of (519) is the form

$$\hat{w} = \sum_{n=1}^{\infty} \hat{w}_n(p) \sin \alpha_n x \ . \tag{531}$$

Introduce (530) and (531) into (519), also taking into account (524). Equating terms of identical sine functions, we have

$$\hat{W}_n(p) = \frac{12\ \hat{q}_n}{h^3 E_a^*[\tilde{\gamma}(p) + \alpha_n^4]} \ , \tag{532}$$

where $\tilde{\gamma}(p)$ is given by (527). The solution in form (531) satisfies boundary conditions of simple support at $x = 0, L$; i.e.,

$$\left.\begin{aligned} w(x,t) = o \quad &\rightarrow \quad \hat{w}(x,p) = o \\ \frac{d^2 w(x,t)}{dx^2} = o \quad &\rightarrow \quad \frac{d^2 \hat{w}(x,p)}{dx^2} = o \end{aligned}\right\} x = 0,\ L. \tag{533}$$

Let

$$w_n(t) = LT^{-1}\,\hat{w}_n(p). \tag{534}$$

Then the solution to the problem of the simply supported shell is given by

$$w(x,t) = \sum_{n=1}^{\infty} w_n(t)\sin\alpha_n x\ . \tag{535}$$

Suppose now that the loading is uniform and does not vary in time. Then

$$q(x,t) = q_o\,H(t)\ ,$$

$$\hat{q}(x,p) = q_o\,\frac{1}{p}\ . \tag{536}$$

The Fourier expansion of 1 in $o \le x \le L$ is given by

$$1 = \frac{4}{\pi}\sum_{n=1,3,5}^{\infty}\frac{1}{n}\sin\alpha_n x\ . \tag{537}$$

Hence, in the present case,

$$\hat{q}_n = \frac{4q_o}{\pi np} \qquad n = 1,\ 3,\ 5\ \text{--}\ . \tag{538}$$

Suppose again that the matrix is represented by a simple Maxwell model. Then as in Section 12,

$$p\hat{G}_a^* = \frac{1+c}{1-c}\,\frac{\eta_m p}{1+Tp}\ . \tag{539}$$

Introducing (539) into (527) and introducing the resulting expression together with (538) into (532), we obtain

$$\hat{w}_n(p) = \frac{12}{E_a^* h^3}\,\frac{(1+Tp)^2}{p\{\phi Tp + \phi\beta T^2 p^2 + \alpha_n^4\,(1+Tp)^2\}}\ , \tag{540}$$

where

$$\phi = \frac{12}{R^2 h^2}\,\frac{4\,(1+c)}{1-c}\,\frac{G_m}{E_a^*}\ , \tag{541}$$

$$\beta = 1 - 4\nu_a^{*2} \frac{1 + c}{1 - c} \frac{G_m}{E_a^*} . \qquad (542)$$

Here, G_m is the initial matrix shear modulus. Equation (540) can be written in the form

$$\hat{w}_n(p) = \frac{12}{E_a^* h^3} \frac{4q_o}{\pi n \alpha_n^4} \frac{1 + 2Tp + T^2p^2}{p\left(1 + \frac{2\zeta p}{\omega} + \frac{p^2}{\omega^2}\right)} , \qquad (543)$$

where

$$\frac{1}{\omega} = T \sqrt{1 + \frac{\phi\beta}{\alpha_n^4}} \qquad \text{(a)} \qquad (544)$$

$$\zeta = \frac{2 + \frac{\phi}{\alpha_n^4}}{2\sqrt{1 + \frac{\phi\beta}{\alpha_n^4}}} \qquad \text{(b)}$$

The Laplace inverse transform of (543) is

$$w_n(t) = \frac{12}{E_a^* h^3} \frac{4q_o}{\pi n \alpha_n^4} \left\{ 1 + \frac{1}{\sqrt{1 - \zeta^2}} [(1 + 2T\zeta\omega_1 - T^2\omega^2 + 2T^2\zeta^2\omega^2)^2 + \omega^2 (1 - \zeta^2) (2T\zeta\omega)^2]^{1/2} e^{-\zeta\omega t} \sin (\omega_1 \sqrt{1 - \zeta^2}\, t + \psi) \right\} \qquad (545)$$

where

$$\psi = \tan^{-1} \frac{2T\omega\sqrt{1 - \zeta^2}\,(1 - T\zeta\omega)}{T^2\omega^2 (2\zeta^2 - 1) + 1 - 2T\zeta\omega} - \tan^{-1} \frac{\sqrt{1 - \zeta^2}}{- \zeta} . \qquad (546)$$

REFERENCES

1. Hashin, Z., and Rosen, B. W., *The Elastic Moduli of Fiber Reinforced Materials* , J. Appl. Mech., Vol. 31, No. 223, 1964.

2. Hashin, Z., Brull, M. A., Chu, T. Y., and Zudans, Z., Determination of the Effective Elastic Properties for Biaxially Fiber Reinforced Materials, FIRL Final Report F-B2455, USAAVLABS Technical Report 67-41, U. S. Army Aviation Materiel Laboratories, Fort Eustis, Virginia, August 1967, AD 662 774.

3. Hashin, Z., *Viscoelastic Fiber Reinforced Materials* , AIAA Journal, Vol. 4, No. 1411, 1966.

4. Hashin, Z., *Viscoelastic Behavior of Heterogeneous Media* , J. Appl. Mech., Vol. 32, No. 630, 1965.

5. Bodner, S. R., and Lifshitz, J. M., *Experimental Investigations on the Dynamic Strength of Composites* , EOAR, USAF, Contract AF(052)-951, Report No. 5, MML, Technion - Israel Institute of Technology, Haifa, Israel.

6. Nowacki, W., Dynamics of Elastic Systems, J. Wiley, New York, 1963.

7. Sokolnikoff, I. S., Mathematical Theory of Elasticity, McGraw-Hill, New York, 1956.

8. Churchill, R. V., Complex Variables and Applications, McGraw-Hill, New York, 1948.

9. Sato, K., Nakane, H., Hideshima, T., and Iwayanagi, S., *Creep of Polymethyl Methacrylate at Low Temperature* , J. Phys. Soc., Japan, Vol. 9, No. 413, 1954.

10. Tobolsky, A. V., and Castiff, E., *Elastoviscous Properties of Polyisobutylene (and Other Amorphous Polymers) from Stress Relaxation Studies IX* , J. Poly. Sci., Vol. 19, No. 111, 1956.

11. Lee, E. H., and Rogers, T. G., *Solution of Viscoelastic Stress Analysis Problems Using Measured Creep or Relaxation Junctions* , J. Appl. Mech., Vol. 85, No. 127, 1963.

12. Brull, M. A., *A Structural Theory Incorporating the Effect of Time-Dependent Elasticity* , Proceedings of First Midwestern Conference on Solid Mechanics, Urbana, Ill., 1963.

13. Lee, E. H., *Viscoelastic Stress Analysis* , Structural Mechanics, edited by Goodier and Hill, Pergamon Press, New York, 1960.

14. Bland, D. R., *The Theory of Linear Viscoelasticity* , Pergamon Press, New York, 1960.

15. Kingsbury, H. B., Stresses and Deformations of Amisotropic Plates and Shells, Ph. D. Dissertation, University of Pennsylvania, 1965.

16. Timoshenko, S., and Woinowsky-Kreiger, S., Theory of Plates and Shells, 2nd ed., McGraw-Hill, New York, 1959.

17. Flugge, W., Stresses in Shells, Springer-Verlag, New York, 1960.

Unclassified
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) The Franklin Institute Research Laboratories Philadelphia, Pennsylvania	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

STATIC AND DYNAMIC VISCOELASTIC BEHAVIOR OF FIBER REINFORCED MATERIALS AND STRUCTURES

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Final Report

5. AUTHOR(S) (First name, middle initial, last name)

Z. Hashin T. Y. Chu
M. A. Brull Z. Zudans

6. REPORT DATE October 1968	7a. TOTAL NO. OF PAGES 136	7b. NO. OF REFS 17

8a. CONTRACT OR GRANT NO. DAAJ02-67-C-0050 b. PROJECT NO. Task 1F162204A17001 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) USAAVLABS Technical Report 68-70 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) F-C2041

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale; its distribution is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY US Army Aviation Materiel Laboratories Fort Eustis, Virginia

13. ABSTRACT

Time-dependent properties of uniaxially fiber reinforced materials composed of linear viscoelastic matrix and elastic fibers are investigated. Sample calculations are given for the static and dynamic properties of a viscoelastic fiber reinforced material. In addition, the behavior of fiber reinforced viscoelastic structures is investigated, and a number of practical problems for beams, plates, and shells, subjected to static and dynamic loadings, are analyzed.

DD FORM 1 NOV 65 1473 REPLACES DD FORM 1473, 1 JAN 64, WHICH IS OBSOLETE FOR ARMY USE.

Unclassified
Security Classification

Unclassified

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Viscoelastic properties of materials						
Fiber reinforced material						
Uniaxial fibers						
Complex moduli						
Relaxation moduli						
Creep compliance						
Shell stiffness tensor						

Unclassified

Security Classification