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NONLINEAR ANALYSIS AND SYNTHESIS OF GENERALIZED TRACKING SYSTEMS

by

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NONLINEAR ANALYSIS AND SYNTHESIS OF GENERALIZED TRACKING SYSTEMS* (PART I)

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ABSTRACT

Using the theory of Markov processes, the response probability density function $p(\underline{y}, t)$ of a generalized tracking system is shown to be the solution to a $(N + 1)$ - dimensional Fokker-Planck equation. The vector $\underline{y} = (\phi, y_1 \ldots, y_N)$ is Markov and ϕ represents the system phase error reduced modulo 2π.

It is further shown that the response distributions $[p(\phi, t), p(y_k, t); k = 1, 2, \ldots, N]$ of the state variables satisfy a set of second-order partial differential equations. In the steady state, this set of equations become ordinary, first-order differential equations for which the exact solutions which specify the marginal probability densities $[p(\phi), p(y_k); k = 1, 2, \ldots, N]$, are determined by two sets of conditional expectations. In particular, the marginal density $p(\phi)$ of the phase error is embedded in a knowledge of the conditional expectations $\{E(y_k \mid \phi), k = 1, 2, \ldots, N\}$ while the marginal density $p(y_k)$ of the state variable y_k is embedded in a knowledge of the conditional expectation $E(g(\phi) \mid y_k); k = 1, 2, \ldots, N$. Here $g(\phi)$ belongs to that class of nonlinearities for which $g(\phi)$ is an odd function. The conditional expectations are approximated by two methods and, for the case of greatest interest, i.e., $g(\phi) = \sin\phi$, the conditional expectation $E(y_1 \mid \phi)$ is measured via computer simulation methods. Agreement of the simulation results with those obtained from theoretical considerations are within less than one percent of each other.

Synthesis procedures for effecting stochastic optimization of a generalized tracker are presented. For zero detuning it is shown that the tracker which minimizes the mean-squared value of the phase-error is obtained when $N = 0$ and $g(\phi)$ is proportional to sgn (ϕ). In the case of nonzero detuning it is shown

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that the loop which minimizes the mean-squared phase-error is second-order, i.e., $N = 1$, and the optimum nonlinearity for this loop is also proportional to sgn ϕ. For large loop signal-to-noise ratios and equal mean-squared loop phase-errors, the second-order PLL system requires a loop signal-to-noise ratio, ρ, of approximately $\rho/2$ times larger than that required in the optimum second-order tracker. At low loop signal-to-noise ratios it is shown that all trackers perform approximately the same. For second-order trackers some results are presented to the case where $g(\phi)$ must be physically realizable.

The field of probability currents is derived and it is shown that this field is rotational when there is zero detuning in the loop. From these currents the average number of cycles slipped per unit of time is derived. With zero detuning this average is zero. Finally, the diffusion coefficient, representing the rate with which the phase-error is undergoing diffusion, is given as well as the expected value of the time interval during which the loop remains locked.

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A. INTRODUCTION

In modern communication, radar tracking, missile guidance and navigation systems, synchronization and tracking are generally accomplished by cross-correlating a locally generated reference signal with the received signal to produce a measurement of the error. In practice this reference signal is developed by means of a nonlinear device. A wide variety of non-linear functions are available and the choice considerably influences system performance. For example, system performance could be mean-squared tracking error, moments of the mean time to first loss of synchronization or the minimum acquisition time. The basic concept associated with a generalized tracking loop is best illustrated by considering first the familiar phase-locked loop (PLL) system of Fig. 1 and its well-known equivalent model (Ref. 1) illustrated in Fig. 2. Assuming that the phase detector is a multiplier and that the additive noise $n_1(t)$ which is present at the input is white and Gaussian with single-sided spectral density N_o watts/hertz, we may describe system operation by the stochastic differential equation (Ref. 1 - 10):

$$\dot{\phi}(t) = \dot{\theta}(t) - AKF(p)\left[R_{rs}(\phi) + n(t)/A\right]$$

$$p = d/dt \tag{1}$$

where $R_{rs}(\phi) = \sin\phi$. Here $\phi(t) = \theta(t) - \hat{\theta}(t)$ is the instantaneous phase-error of the voltage control oscillator (VCO) with respect to the input signal $s[t, \theta(t)]$, $F(p)$ is the transfer function of the loop filter in operator form where p is the Heaviside operator, A^2 is the power in the input signal component, K is the open-loop gain and n(t) may be shown (Ref. 1 and 7) to be white Gaussian noise if the input noise process $n_1(t)$ is white and Gaussian, $\theta(t)$ is the process to be tracked and $\hat{\theta}(t)$ is the tracker estimate of $\theta(t)$.

Frequently a PLL system must operate in conditions where external fluctuations due to additive noise are so intense that classical linear PLL theory does not adequately characterize loop performance nor explain loop behavior (Ref. 1 and 2). The problem of investigating the effects of external noise on system accuracy has been carried through to completion for the linear model (Ref. 1 and 2). As a consequence certain approximate analysis have evolved for explaining and characterizing loop performance in the region of operation where direct linearization cannot be used, e. g., the quasi-linear analysis, (Ref. 3), the linear spectral method (Ref. 2 and 4) and the Volterra series methods (Ref. 5).

The statistical dynamics of a first-order PLL were given in Ref. 2, 7, and 8. For a PLL system with integrating filter certain exact results have been reported (Ref. 9). An approximate solution for the joint probability distribution of the phase-error and phase-error rate was given in Ref. 10 for a system with a proportional-plus-integral control type loop filter. The mean time to first slip was evaluated by Viterbi (Ref. 1 and 8) for a first-order loop. Later, Tausworthe (Ref. 11) derived a Fokker-Planck equation whose solution for the mean time to first slip was shown to agree with Viterbi's for the first-order loop. In addition, Tausworthe (Ref. 11) obtained an approximate solution for the mean time to first slip in a second order PLL.

It can be shown that the stochastic differential equation of operation for the delay-locked loop (DLL) (Ref. 12) with noise present, or the N^{th} order tan-locked loop (TLL) when noise is absent (Ref. 13), is identical with (1) except for the nonlinearity in the loop. In practice, a wide variety of nonlinearities can be synthesized from appropriate modifications of a DLL or a TLL, and hence, there formally exists the possibility of synthesizing the optimum nonlinearity having first determined the best choice of F(p).

Since this is true, and among other things, a generalized tracking loop like that of Fig. 3 is of interest. The equivalent loop model is shown in Fig. 4 while the stochastic differential equation of operation is given by (1) with $R_{rs}(\phi)$ replaced by $g(\phi)$. In what follows we shall assume (without loss in generality) that $R_{rs}(\phi)$ is periodic. Physically speaking, $R_{rs}(\phi) = g(\phi)$ is the normalized cross-correlation function (double-frequency terms neglected) between the input signal component $s[t, \theta(t)]$ and the control signal r(t). It will be of interest to do signal design on $s[t, \theta(t)]$ and $r(t)$, under appropriate constraints, and select that $F(p)$ which will optimize loop performance, i.e., we first seek the loop filter $F(p)$ which will optimize loop performance subject to some performance criterion and then optimize further by a choice of the nonlinearity.

An exact solution to the problem of analyzing and synthesizing an optimum tracker is formally possible on the basis of the theory of Markov processes (Ref. 6). The possibility of obtaining exact results enables (1), the nonlinear effects to be understood, (2) the limits of application of the various approximate methods and loop theories to be accessed, (3) one to perform stochastic optimization of the loop in the nonlinear region of operation, and (4) an optimum tracking theory to be developed. Thus a comparison of performance between the optimum tracker and any implemented suboptimum tracker, such as the PLL, can be made to see if the additional complexity required to mechanize the former is warranted. Also, it is just as satisfying to know what the optimum tracker is as it is to know what the channel capacity of a communication system is even though each are unattainable in practice.

This paper presents rather general results for the response probability density function of the phase-error ϕ of the generalized tracker in Fig. 3. In particular, a $(N + 1)$ - dimensional Fokker-Planck equation is derived whose solution is denoted by the multidimensional probability density $p(\underline{y}, t)$ where $\underline{y}$ is a vector Markov process in the state variables $(\phi, y_1, \ldots, y_N)$. This $(N + 1)$ - dimensional Fokker-Planck equation is then reduced to a second-order partial differential equation whose solution is $p(\phi, t)$, i.e., the probability density of the phase-error process reduced modulo 2π. In the steady state, it is shown that this one-dimensional equation becomes an ordinary first-order differential equation for which the form of an exact solution is known. From this equation and its solution, it is shown that the steady-state probability density $p(\phi)$ for an $(N + 1)^{th}$ order tracker is completely determined by the set of conditional expectations $E[y_k | \phi]$, $k = 1, 2, \ldots, N$.

In addition, general formulas for determining the probability densities $p(y_k, t)$ of the state variables y_k; $k = 1, 2, \ldots, N$, are given. It is shown that the probability densities, $p(y_k, t)$; $k = 1, 2, \ldots, N$, are solutions to second-order, partial differential equations. In the steady state, these partial differential equations become first-order ordinary differential equations for which the solution is known. From these equations we show that the probability densities $p(y_k)$ of the state variables y_k, $k = 1, 2, \ldots, N$, are completely determined by the set of conditional expectations $\{E[g(\phi)|y_k], k = 1, 2, \cdots, N\}$, respectively.

For each projection of $\underline{y}$, we evaluate the probability current and show that the probability field is rotational when zero detuning exists in the loop. From these results we are then able to deduce the average number of phase-jumps (cycles-slipped) per unit of time.

The expectations $\{E(y_k | \phi)\}$ are appoximated by two methods. The first approximation is based upon the linear tracking theory while the second is based upon a modification and generalization of techniques due to Viterbi (Ref. 1) and Holmes (Ref. 14). To check on the validity of these approximations $E(y_1|\phi)$ is measured via computer simulation methods for the case of greatest interest, i.e., $N = 1$, $g(\phi) = \sin \phi$. The variance of the phase-error is also computed for both approximations and the results are compared with measurements obtained earlier (Ref. 10) via hardware simulation.

Stochastic optimization for the $(N + 1)$ - order tracker is carried out, its performance evaluated and compared with that of a second-order PLL. It is further shown that a first-order tracker with nonlinearity $g(\phi) = L \operatorname{sgn} \phi$ minimizes the mean square error over the class of linear filters and periodic nonlinearities when zero detuning exists within the loop.

We close by concluding that specification of loop response, hence, performance of a generalized tracking system in the nonlinear region of operation, is tantamount to possessing a knowledge of the set of expectations $\{E(y_k | \phi), E(g(\phi) | y_k); k = 1, 2, \ldots, N\}$. This knowledge may be obtained via computer simulation techniques (Ref. 15 and 16) or as is done here, one may approximate them; hence, stochastic optimization of the loop in the non-linear region of operation is possible. The utility of the results become obvious when one considers the complexity of the equipment (Ref. 10) and the amount of time required to measure only a few of the many statistical parameters at work in the loop, e.g., mean square phase-error or mean time to first loss of synchronization. On the other hand, the amount of time required to measure the marginal densities and the moments thereof via computer simulation methods (Ref. 15 and 16) is also prohibitive of time. Although we do not demonstrate the results here, it can also be shown (Ref. 17) that the mean time to first slip and all moments of this random event are also embedded in a knowledge of $E(y_k \mid \phi)$, $k = 1, 2, \ldots, N$. Thus, measurement or approximation of these conditional expectations when used in conjunction with the results presented here is all that is required to produce loop behavior and to carry out system synthesis in the non-linear region of operation.

B. SYSTEM MODEL

If we write the loop filter transfer function in the partial fraction expansion

$$F(p) = F_1 + \sum_{k=1}^{N} \frac{1 - F_k}{1 + \tau_k p} \tag{2}$$

we have that $F(p) = 1$ if $F_k = 1$ for all $k = 1, 2, \ldots, N$. For example, if $N = 1$, we have the proportional-plus-integral control filter of considerable practical interest, i.e.,

$$F(p) = \frac{1 + \tau_2 p}{1 + \tau_1 p} \tag{3}$$

where $F_1 = \tau_2/\tau_1$.

If we substitute (2) into (1) and assume an input of the form $\theta(t) = (\omega - \omega_0)t + \theta$ where θ is a constant, we can rewrite (1) as

$$\dot{\phi}(t) = \Omega_0 - F_1[AKg(\phi) + Kn(t)] - \sum_{k=1}^{N} \frac{1 - F_k}{1 + \tau_k p}[AKg(\phi) + Kn(t)] \tag{4}$$

where $\Omega_0 = \omega - \omega_0$ is the loop detuning. Introducing the state variable y_k

$$y_k = -\left[\frac{1 - F_k}{1 + \tau_k p}\right][AKg(\phi) + Kn(t)] \tag{5}$$

for $k = 1, 2, \ldots, N$, we can replace (4) by the equivalent system of $N + 1$, first-order, stochastic differential equations,* i. e.,

$$\dot{y}_0 = \dot{\phi} = \Omega_0 - F_1 AKg(\phi) + \sum_{k=1}^{N} y_k - F_1 Kn(t)$$

$$\dot{y}_1 = -\frac{y_1}{\tau_1} - \frac{(1 - F_1)\left[AKg(\phi) + Kn(t)\right]}{\tau_1}$$

$$\vdots$$

$$\dot{y}_k = -\frac{y_k}{\tau_k} - \frac{(1 - F_k)\left[AKg(\phi) + Kn(t)\right]}{\tau_k} \qquad (6)$$

$$\vdots$$

$$\dot{y}_N = -\frac{y_N}{\tau_N} - \frac{(1 - F_N)\left[AKg(\phi) + Kn(t)\right]}{\tau_N}$$

where $y_0 = \phi$. Written this way, it is clear from (6) that the coordinates y_0, $y_1, \ldots, y_N$ form components of a $(N + 1)$ - dimensional Markov vector $\underline{y} = (\phi, y_1, \ldots, y_N)$ since each component, $\dot{y}_k$, depends only upon the present values of $\underline{y}$ and a white Gaussian noise process. For convenience, we define the vector $\underline{y}_0' = (y_1, \ldots, y_N)$. Also in what follows we shall refer to a specific component of $\underline{y}$ as the projection of $\underline{y}$.

The most complete characterization of the state $\underline{y}$ of the tracking loop is its statistical description by means of the probability (response) density function of $\underline{y}$, viz., $P(\underline{y}, t)$. The response distribution $P(\underline{y}, t)$, of course, can be formally determined by using the theory of Markov processes. In the next section we expose the procedure.

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*By defining the Markov extension such that ϕ is one component of the vector process allows one to proceed with the analysis. Heretofore, the vector process has been defined such that ϕ is a weighted sum of the projections thus leading to formidable mathematical difficulties if $N > 1$. Also such an extension clearly depicts that the components of $\underline{y}'$ are Markov for a fixed ϕ. Further, if the τ_k's are much greater than one (narrowband loop) then it is clear that the y_k variables are slowly varying random processes.

C. THE (N + 1) - DIMENSIONAL FOKKER-PLANCK EQUATION

Given the fact that the components of $\underline{y}$ form a vector Markov process, $P(\underline{y}, t)$ satisfies the (N + 1) - dimensional Fokker-Planck (F - P) equation (Ref. 6), viz.,

$$\frac{\partial P(\underline{y}, t)}{\partial t} = -\sum_{k=0}^{N} \frac{\partial}{\partial y_k} [K_k(\underline{y}, t)\ P(\underline{y}, t)] + \frac{1}{2} \sum_{k=0}^{N} \sum_{\ell=0}^{N} \frac{\partial^2}{\partial y_k\, \partial y_\ell} [K_{\ell k}(\underline{y}, t)\ P(\underline{y}, t)] \tag{7}$$

where the intensity coefficients $K_k(\underline{y}, t)$ and $K_{\ell k}(\underline{y}, t)$ are defined by the formulas

$$\begin{aligned} K_k(\underline{y}, t) &= \lim_{\Delta t \to 0} \frac{E[\Delta y_k \mid \underline{y}]}{\Delta t} \\ K_{\ell k}(\underline{y}, t) &= \lim_{\Delta t \to 0} \frac{E[\Delta y_\ell\, \Delta y_k \mid \underline{y}]}{\Delta t} \end{aligned} \tag{8}$$

and the $E[\cdot \mid \underline{y}]$ denotes mathematical expectation of the enclosed quantity given $\underline{y}$. In the case of a stationary random process, the coefficients $K_k(\underline{y}, t)$ and $K_{\ell k}(\underline{y}, t)$ do not depend upon t.

In what follows the concept of probability current density

$$I_k(\underline{y}, t) \triangleq \left\{ \left[K_k(\underline{y}, t) - \frac{1}{2} \sum_{\ell=0}^{N} \frac{\partial}{\partial y_\ell} K_{\ell k}(\underline{y}, t) \right] P(\underline{y}, t) \right\} ; k = 0, 1, \ldots, N \tag{9}$$

will be of use when evaluating the average number of phase-jumps or cycles-slipped per unit of time. Thus we can write the F-P equation in the form

$$\frac{\partial P(\underline{y}, t)}{\partial t} = -\sum_{k=0}^{N} \frac{\partial}{\partial y_k} I_k(\underline{y}, t) \tag{10}$$

The vector $\underline{I}(\underline{y}, t) \triangleq \left[I_0(\underline{y}, t), \ldots, I_N(\underline{y}, t)\right]$ may be interpreted as a probability current density vector. We will also need to consider the probability surface current of the k^{th} projection of $\underline{I}(\underline{y}, t)$, i.e.,

$$I_k(y_k, t) \triangleq \underbrace{\int \cdots \int}_{\text{N-Fold}} I_k(\underline{y}, t)\, d\underline{y}_k' \tag{11}$$

where $d\underline{y}_k' \triangleq dy_0, dy_1, \ldots dy_{k-1}\, dy_{k+1}, \ldots dy_N$. The probability current of the k^{th} projection describes the amount of probability crossing the hyperplane $y_k = y_k'$ in the positive direction per unit time.* Just as the equation of heat conduction involves a flow of heat, (7) involves a flow of probability.

Before we proceed we will give a graphic physical interpretation of the F-P equation which will prove useful and somewhat picturesque. For every sample function of a vector Markov process, the vector trajectory $\underline{y}(t)$ can be thought of as being swept out from $\underline{y}(t_0)$ by a point in an (N + 1)-dimensional space $\underline{y} = (\phi, y_1, \ldots, y_N)$. The position of this point at time t, i.e., $[y_0(t), \ldots, y_N(t)]$, may be envisioned as a Brownian particle undergoing diffusion in (N + 1) - space as a function of time. The set of sample functions of the process $\underline{y}(t)$, is the ensemble of trajectories which move about in a random manner. The fraction of time that the particle spends in any region of the probability space R' is proportional to the total probability in that region.

The coefficients required for (7) can be straightforwardly evaluated by using (6) and (8). The differential equation whose solution describes the probability density $P(\underline{y}, t)$ can then be written by mere substitution of these K_k and $K_{\ell k}$ into (7). Such a computation yields

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*A geometric interpretation of (11) is possible. Define $\underline{n}_k$ to be a unit vector pointed along the positive direction of the y_k-axis. Then $\underline{I}(y, t) \cdot \underline{n}_k = I_k(y, t)$ represents the probability current density flowing in the positive y_k direction, and $I_k(y, t)\, dy_k'$ represents the amount of probability surface current flowing through the differential surface area dy_k'. Integrating over the surface gives the total probability surface current flowing through the hyperplane $y_k = y_k'$.

$$\frac{\partial P(\underline{y}, t)}{\partial t} = -\frac{\partial}{\partial \phi}\left\{\left[\Omega_0 - F_1 AKg(\phi) + \sum_{k=1}^{N} y_k\right] P(\underline{y}, t)\right\}$$

$$+ \frac{F_1^2 N_0 K^2 \partial^2 P(\underline{y}, t)}{4\partial\phi^2} + \sum_{k=1}^{N}\left(\frac{\partial}{\partial y_k}\left[\frac{y_k}{\tau_k} + \left(\frac{1 - F_k}{\tau_k}\right) AKg(\phi)\right] P(\underline{y}, t)\right.$$

$$\left. + \left\{\frac{(1 - F_k)^2 N_0 K^2}{4\tau_k^2} \frac{\partial^2}{\partial y_k^2} + \frac{F_1(1 - F_k) N_0 K^2}{4\tau_k} \frac{\partial^2}{\partial\phi\partial y_k}\right\} P(\underline{y}, t)\right) \tag{12}$$

$$+ \sum_{k \neq \ell \neq 0}^{N} \sum^{N} \frac{(1 - F_k)(1 - F_\ell) K^2 N_0}{4\tau_k \tau_\ell} \frac{\partial^2}{\partial y_k \partial y_\ell} P(\underline{y}, t)$$

Certain special cases of (12) are of great practical importance. For example, with $N = 1$ and $F_k = 0$ for all $k \neq 1$, (12) becomes the F-P equation for the PLL system with the proportional-plus integral-control loop filter given in (3).

D. INITIAL CONDITIONS AND BOUNDARY CONDITIONS

In order to obtain solutions to the F-P equation we have to supplement it with initial conditions and boundary conditions. For our purposes we specify the initial distribution $P(\underline{y}, t_0)$ at time $t = t_0$ to be

$$P(\underline{y}, t_0) = \prod_{n=0}^{N} \delta[y_n - y_n(t_0)] \tag{13}$$

and the subsequent evolution of the distribution is found from (12).

The boundary conditions themselves are determined by the physics of the problem. If $P[\phi, \underline{y}_0', t]$ is to be a probability density and since $K_k(\underline{y}, t)$ and $K_{\ell k}(\underline{y}, t)$ are periodic in ϕ, we have

$$\lim_{t\to\infty} P\left[\phi, \underline{y}_0', t\right] = \lim_{t\to\infty} P\left[\phi \pm 2n\pi, \underline{y}_0', t\right] = 0 \tag{14}$$

and in the steady-state $P\left[\phi, \underline{y}_0', t\right]$ has an unbounded variance. This condition is directly traceable to the cycle-slipping (phase-jumps) phenomenon associated with generalized tracking systems. Thus to obtain a probability distribution with a finite variance in the steady-state we shall consider the solution of (7) reduced modulo 2π. Let

$$\tilde{p}(\phi, \underline{y}_0', t) \triangleq \sum_{n=-\infty}^{\infty} P\left[\phi + 2n\pi, \underline{y}_0', t\right] \tag{15}$$

for all ϕ. Note that $\tilde{p}$ is periodic in ϕ but, as such, it is not a probability density function since it is an infinite sum of density functions each with unit area. Thus, to obtain a solution which has the properties of a density function we define

$$p(\phi, \underline{y}_0', t) = \begin{cases} \tilde{p}(\phi, \underline{y}_0', t) & \text{in any } \phi \text{ interval of } 2\pi \text{ width around any lock point } 2n\pi. \\ 0 & \text{elsewhere} \end{cases} \tag{15a}$$

To justify that $\tilde{p}(\phi, \underline{y}_0', t)$ is a solution, we note that $P(\underline{y}, t)$ is a solution to (7) in the region R' for which $y_j = \pm\infty, j = 0, \ldots, N$. The function $p(\underline{y}, t)$ is defined in a region R which is the hyperslab formed by two hyperplanes located 2π radians apart. Hence, since each term of (15) is a solution to (12) in R', the sum is also a solution in R.

Along any edge of the surface Γ of the hyperslab R, i.e., the edge $y_j = \pm\infty$ for any and all $j = 1, \cdots, N$, we have the boundary conditions*

$$p(\phi, \underline{y}_0', t)\Big|_{y_j=\pm\infty} = y_j p(\phi, \underline{y}_0', t)\Big|_{y_j=\pm\infty} = 0 \tag{16}$$

We also note that since $p(\underline{y}, t)$ is a probability density as defined above, then

$$\int_R p(\underline{y}, t)\, dR = 1 \tag{17}$$

*See footnote on page 13.

As a consequence of (17) we have*, along any edge of Γ

$$\frac{\partial}{\partial y_j} p(\phi, \underline{y}_0', t)\Big|_{y_j=\pm\infty} = 0 \quad ; \; j = 1, 2, \ldots, N \tag{18}$$

From (6), and assuming that $g(\pi) = g(-\pi) = 0$, we have

$$p(-\pi, \underline{y}_0', t) = p(\pi, \underline{y}_0', t)$$

It follows from this that

$$\frac{\partial p(-\pi, \underline{y}_0', t)}{\partial y_j} = \frac{\partial p(\pi, \underline{y}_0', t)}{\partial y_j} \qquad ; \qquad j = 1, 2, \cdots, N \tag{19}$$

If we set $\Omega_0 = 0$, we note from (6) that**

$$p(\underline{y}, t) = p(-\underline{y}, t)$$

Finally, if

$$\frac{\partial p(\pi, \underline{y}_0', t)}{\partial \phi} = \frac{\partial p(-\pi, \underline{y}_0', t)}{\partial \phi}$$

we can write in vector notation

$$\nabla p\Big|_{\phi=\pi} = \nabla p\Big|_{\phi=-\pi} \tag{20}$$

where ∇ is the differential operator for space, the del operator,

$$\nabla = \left[\frac{\partial}{\partial y_0}, \frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_N}\right]$$

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*The fact that $p(\underline{y}, t) = 0$ at $y_j = \pm\infty$ is a consequence of the definition of a random variable. Condition (18), and the fact that $y_j p(\underline{y}, t)$ in (16) approaches zero as y_j approaches infinity, requires that $p(\underline{y}, t)$ approaches infinity faster than $[y_j]^{1+\varepsilon}$, $\varepsilon > 0$.

**Note, these boundary conditions are necessary for finding the marginal density functions $p(y_k, t)$, $k = 1, 2, \cdots, N$ but are not required for finding $p(\phi, t)$.

Hence, since from (18) ∇p is zero at the boundaries $y_j = \pm\infty$, $j = 1, 2, \cdots, N$,

$$\int_R \nabla p \cdot d\Gamma = 0$$

where Γ is the surface of the region R.

In operator form, (12) may now be written as

$$\nabla \cdot \underline{I}(\underline{y}, t) + \frac{\partial p(\underline{y}, t)}{\partial t} = L[p(\underline{y}, t)] + \frac{\partial p(\underline{y}, t)}{\partial t} = 0 \tag{21}$$

where L, identified from (12), is an elliptic-parabolic operator (Ref. 18) if $F_k \leq 1$ for all $k = 1, 2, \ldots, N$. Using Gauss's Theorem it is interesting to note that one can write

$$\int_R \nabla \cdot \underline{I}\, dR = \int_\Gamma \underline{n} \times \underline{I}\, d\Gamma = 0$$

where R is the volume (probability space) bounded by surface Γ and $\underline{n}$ is unit vector normal to the surface of Γ and directed positively outwards. From Maxwell's field equations we know that the divergence of the current density $\underline{J}$ is just the time rate of change of the charge density ρ. Also the divergence of the flux density $\underline{D}$ is equal to ρ. Here if we interpret ρ as the probability density $p(\underline{y}, t)$ and $\underline{\mathcal{D}}$ as a probability flux density then we may write $\nabla \cdot \underline{\mathcal{D}} = p(\underline{y}, t)$, i.e., the net probability flux flowing out of a volume dR at time t is just equal to the probability of being in that volume at time t.

Interesting enough if we integrate both sides of (12) with respect to y_j for all $j \neq k$ and make use of the boundary conditions (16) and (18) we arrive at*

*Oscillator instabilities may be included here by replacing K_{00} by $K_{00} + \int_{-\infty}^{\infty} R_O(\tau) d\tau$ where $R_O(\tau)$ is the correlation function of the random process $O(t) = [\dot{\theta}_1(t) - \dot{\theta}_2(t)]$; $\theta_1(t)$ represents the phase instabilities in $s[t, \theta(t)]$ and $\theta_2(t)$ represents the instabilities in the CRG output $r(t)$. To make this replacement valid, and remain within the framework of Markov process theory, the correlation time of $O(t)$ must be small when compared to the response time of the loop.

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial \phi}\left\{\left[\Omega_0 + \sum_{j\neq k\neq 0}^{N} E(y_j, t|\phi, y_k) + y_k - F_1 AKg(\phi)\right]p\right\}$$

$$+ \frac{F_1^2 K^2 N_0}{4} \frac{\partial^2 p}{\partial \phi^2} + \frac{(1 - F_k)^2 N_0 K^2}{4\tau_k^2} \frac{\partial^2 p}{\partial y_k^2} \tag{22}$$

$$+ \left\{\frac{\partial}{\tau_k \, \partial y_k}\left[y_k + (1 - F_k)\, AKg(\phi)\right]p\right\} + \frac{F_1(1 - F_k)N_0 K^2}{4\tau_k} \frac{\partial^2 p}{\partial \phi \partial y_k}$$

where $p = p(\phi, y_k, t)$ is the modulo 2π density in ϕ. In (22), $E(y_j, t|\phi, y_k)$ is the conditional expectation of y_j given ϕ and y_k for all $j \neq k \neq 0$. This expectation is taken with respect to $p(y_j, t|\phi, y_k)$.

Unfortunately, the general solution to (21) or (22) cannot be found analytically. In the next section we confine ourselves to the problem of determining the marginal probability density of the phase process $\phi(t)$ reduced modulo 2π, i.e., $p(\phi, t)$ for an $(N + 1)^{th}$ order loop.

E. DIFFERENTIAL EQUATION FOR THE MARGINAL PROBABILITY DENSITY $p(\phi, t)$

To find $p(\phi, t)$, we first need a differential equation whose solution is indeed $p(\phi, t)$. This is easily found by integrating both sides of (22) with respect to y_k and using the boundary conditions (16) and (18). Without belaboring the details we have

$$\frac{\partial p(\phi, t)}{\partial t} = -\frac{\partial}{\partial \phi}\left\{\left[\Omega_0 - F_1\, AKg(\phi) + \sum_{k=1}^{N} E(y_k, t|\phi)\right] p(\phi, t)\right\}$$

$$+ \frac{F_1^2 N_0 K^2}{4} \frac{\partial^2}{\partial \phi^2} p(\phi, t) \tag{23}$$

where $E(y_k, t \mid \phi)$ denotes the conditional expectation of y_k given ϕ. In (23), and throughout this paper, any sum is considered empty if the lower index exceeds the upper index.

In the steady state, i.e., limit as t approaches infinity, $p(\phi, t)$ approaches $p(\phi)$ such that the partial differential equation (23) becomes the ordinary differential equation

$$\frac{F_1^2 N_0 K^2}{4} \frac{d^2 p(\phi)}{d\phi^2} - \frac{d}{d\phi}\left\{\left[\Omega_0 - F_1 AKg(\phi) + \sum_{k=1}^{N} E(y_k \mid \phi)\right] p(\phi)\right\} = 0 \qquad (24)$$

For $F_1 = 1$, $N = 0$, we have $E(y_1 \mid \phi) = 0$, and (24) reduces to well known results (Refs. 1, 7, 8, 9) for the first-order Markov process.

If we define the nonlinear restoring force $h(\phi)$ as*

$$h(\phi) \triangleq \frac{4\left[\Omega_0 - F_1 AKg(\phi) + \sum_{k=1}^{N} E(y_k \mid \phi)\right]}{F_1^2 N_0 K^2} \qquad (25)$$

then (24) has the steady-state solution (Ref. 6)

$$p(\phi) = C_0 \exp\left[\int^{\phi} h(x)dx\right]\left\{1 + D_0 \int_{-\pi}^{\phi} \exp\left[-\int^{y} h(x)dx\right] dy\right\} \qquad (26)$$

To evaluate the constants C_0 and D_0, we must utilize the boundary conditions

$$\int_{-\pi}^{\pi} p(\phi)\, d\phi = 1 \qquad (27)$$

*The function $h(\phi)$ plays the same role here as the electric field strength in electromagnetic theory.

$$p(\pi) = p(-\pi) \tag{28}$$

Thus using (28) we obtain

$$D_0 = \frac{A(-\pi) - A(\pi)}{A(\pi)\int_{-\pi}^{\pi} A^{-1}(y)\, dy} \tag{29}$$

where

$$A(z) = \exp\left[\int^{z} h(x)dx\right] \tag{30}$$

By means of (27) the normalization constant C_0 can be evaluated.

If $D_0 = 0$, we then have*

$$p(\phi) = C_0 \exp\left[\int^{\phi} h(y)dy\right] \tag{31}$$

For $g(\phi) = \sin \phi$, $N = 0$, (31) reduces to a well known result, Refs. (1, 7, 8, and 9). Since $g(\phi)$ is periodic and since $p(\phi)$ is continuous we may also write (Ref. 19)

$$p(\phi) = C_0' \exp\left[\int^{\phi} h(x)dx\right] \cdot \int_{\phi}^{\phi+2\pi} \exp\left[-\int^{x} h(z)dz\right] dx \tag{32}$$

for any ϕ belonging to an interval of width 2π centered about any stable lock point $2n\pi$.

Equations (26) and (32) are remarkable in that they hold <u>for all order loops</u> and a broad class of nonlinearities. In fact, it is clear from (25) and (32) that the probability density function of the phase-error of an $(N + 1)$ - order loop is <u>completely determined by the set of conditional expectations</u> $E[y_k \mid \phi]$, $k = 1, \ldots, N$. Interestingly enough $E(y_k \mid \phi)$ is the minimum mean-square error estimate of y_k given ϕ.

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*Using linear tracking theory it is easy to show that in the linear region (Ref. 11) that (cont'd. on p. 18).

$$\sum_{k=1}^{N} E(y_k \mid \phi) = (AKF_1 - \ell^2(0)/2W_L)\phi$$

when $\Omega_0 = 0$ and $\ell(\tau)$ is the unit impulse response of the loop. Thus

$$p(\phi) = C_0 \exp\left[- \frac{4A}{N_0 KF_1} \int^{\phi} g(x)dx + \frac{2\phi^2}{N_0 K^2 F_1^2} \{AKF_1 - \ell^2(o)/2W_L\}\right]$$

for all $N \geq 1$ and ϕ belongs to an interval of width 2π centered about any stable lock point $2n\pi$.

F. DIFFERENTIAL EQUATIONS FOR THE MARGINAL PROBABILITY DENSITIES, $p(y_k, t)$; $k = 1, 2, \ldots, N$.

If we integrate (12) from $-\pi$ to π, use the appropriate boundary conditions, and then integrate over the domain $|y_j| \leq \infty$ for all $j \neq k \neq 0$, we arrive at (see Appendix A) the set of partial differential equations:

$$\frac{(1 - F_k)^2 N_0 K^2}{4\tau_k^2} \frac{\partial^2 p(y_k, t)}{\partial y_k^2} + \frac{\partial}{\tau_k \partial y_k} \{[y_k + (1 - F_k) AKE\{g(\phi), t \mid y_k\}] p(y_k, t)\}$$

$$+ K_{00} \left\{ \frac{\partial}{\partial \phi} p(\phi, t \mid y_k) \Big|_{-\pi}^{\pi} \right\} p(y_k, t) = \frac{\partial p(y_k, t)}{\partial t} \qquad (33)$$

for all $k = 1, 2, \ldots, N$. In (33), the quantity $E[g(\phi), t \mid y_k]$ is the conditional expectation of $g(\phi)$ given y_k at time t. In the steady state, (33) becomes a set of ordinary differential equations,

$$\frac{(1 - F_k)^2 N_0 K^2}{4\tau_k} \frac{d^2 p(y_k)}{dy_k^2} + \frac{d}{dy_k} \{[y_k + (1 - F_k) AKE(g(\phi) \mid y_k)] p(y_k)\}$$

$$+ K_{00} \left\{ \frac{d}{d\phi} [p(\phi) \mid y_k] \Big|_{-\pi}^{\pi} \right\} p(y_k) = 0 \qquad (34)$$

for all $k = 1, 2, \cdots, N$. Assuming that the derivatives in (34) at $\pm\pi$ are zero (which is probably true when $\Omega_0 = 0$) we can write

$$p(y_k) = C_k \exp\left[-\int^{y_k} g_k(x)dx\right]\left\{1 + D_k \int_{-\infty}^{y_k} \exp\left[\int^{u} g_k(z)dz\right] du\right\} \tag{35}$$

where the nonlinear restoring force $g_k(y_k)$ is defined by

$$g_k(y_k) = \frac{4\tau_k}{(1 - F_k)^2 N_0 K^2}\left[y_k + (1 - F_k)\, AKE(g(\phi) \mid y_k)\right] \tag{36}$$

for $k = 1, 2, \ldots, N$. In order to obtain explicit solutions for the $p(y_k)$ it appears that the conditional expectations must either be approximated or measured by the method of computer simulation. It is remarkable that the marginal distributions $p(y_k)$ are determined by the conditional expectation $\{E(g(\phi) \mid y_k);\ k = 1, 2, \ldots, N\}$; hence, specifying loop performance is also dependent upon this knowledge.* The constant C_k is a normalization constant determined from

$$\int_{-\infty}^{\infty} p(y_k)dy_k = 1$$

while the constant D_k is determined from the boundary condition $p(y_k) = 0$ at $y_k = \pm\infty$. Thus

$$D_k = \frac{B_k(-\infty) - B_k(\infty)}{B_k(\infty)\displaystyle\int_{-\infty}^{\infty} B_k^{-1}(z)\, dz} \tag{37}$$

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*These expectations are the minimum mean-square error estimates of $g(\phi)$ given y_k.

where

$$B_k(u) = \exp\left[-\int^{u} g_k(x)dx\right] \tag{38}$$

for all k = 1, 2, . . . , N. We therefore see that when the D_k's are zero

$$p(y_k) = C_k \exp\left[\int^{y_k} -[g_k(u)]\, du\right] \tag{39}$$

for all k.

From (6) it is clear that, in the steady state, the means of the coordinates y_k are

$$\overline{y_k} = -(1 - F_k)(AK)\,\overline{g(\phi)};\ k = 1, 2, \ldots, N \tag{40}$$

where the over bar "———" denotes the statistical average. Further, for small signal strengths, i.e., small AK, it is seen that y_k are uncorrelated, zero mean Gaussian random variables having variance

$$\sigma_k^2 = (1 - F_k)^2 \frac{K^2 N_0}{4\tau_k};\ k = 1, 2, \ldots, N \tag{41}$$

Also, from (2), (6) and (40) it is clear that the mean of the phase-error rate $\dot{\phi}$ is given by

$$\overline{\dot{\phi}} = \Omega_0 - AKF_1\overline{g(\phi)} + \sum_{k=1}^{N} \overline{y_k} = \Omega_0 - AK\overline{g(\phi)}F(0) \tag{42}$$

whereupon the mean of the phase-error rate in the steady-state becomes, using (40) and (42)

$$\overline{\dot{\phi}} = \Omega_0 - AK\overline{g(\phi)}\left[1 + \sum_{k=2}^{N} (1 - F_k)\right] \tag{43}$$

for $N < 2$. For $N = 1$,

$$\overline{\dot{\phi}} = \Omega_0 - AK\overline{g(\phi)} \tag{44}$$

in the steady state.

G. LOOPS WITH $F_k = 0$; $k = 1, 2, \ldots, N$

Going back to (23), we see that when $F_k = 0$, for all $k = 0, 1, \cdots, N$, corresponding to a loop filter $F(p) = \sum_{k=1}^{N} (1 + \tau_k p)^{-1}$, the reduced F-P equation degenerates, and the technique previously used fails. There is then no other alternative than to solve (12) with $F_k = 0$ for all $k = 1, 2, \cdots, N$, although it is possible to reduce it somewhat by integrating with respect to $y_1, \cdots, y_N$ and using the boundary conditions (18) and (19).

This procedure yields the reduced F-P equation, viz.,

$$\frac{\partial}{\partial y_j}\left\{\left[\frac{y_j}{\tau_j} + \frac{AKg(\phi)}{\tau_j}\right]p(\phi, y_j, t)\right\} + \frac{K^2N_0}{4\tau_j^2}\frac{\partial}{\partial y_j}p(\phi, y_j, t)$$

$$- \frac{\partial}{\partial \phi}\left\{\left[\Omega_0 + y_j + \sum_{k \neq j \neq 1}^{N} E(y_k, t|\phi, y_j)\right]p(\phi, y_j, t)\right\} = \frac{\partial p(\phi, y_j, t)}{\partial t} \tag{45}$$

For the case $N = 1, \Omega_0 = 0$, we obtain in the steady state,

$$p(\phi, \dot{\phi}) = C \exp\left[-\frac{\tau\rho}{2G_\ell}\dot{\phi}^2 - \rho\int^{\phi} g(x)\, dx\right]$$

$$|\dot{\phi}| < \infty \quad ; \quad \phi \epsilon \left[(2k - 1)\pi, (2k + 1)\pi\right] \text{ for } k \text{ any integer.} \tag{46}$$

where $\dot{\phi} = y_1$, $G_\ell = AK$ is the open loop gain, and ρ is the signal-to-noise ratio existing in the loop bandwidth $W_L = 2b_L = AK/2$, i.e., $\rho = A^2/N_0 b_L = 2A^2/N_0 W_L$. The parameter C is a normalization constant.

Note that the density $p(\phi, \dot{\phi})$ can be written as $p(\phi)p(\dot{\phi})$; ϕ and $\dot{\phi}$ are therefore statistically independent random variables. Further, we note that for $\Omega_0 = 0$, $p(\phi)$ is identical to the expression for a first-order loop and that $\dot{\phi}$ is a

Gaussian random variable with variance $G_\ell/\tau\rho$. Finally, we point out that if we integrate (45) with respect to ϕ from $-\pi$ to π our method degenerates yielding "0 = 0."

H. EVALUATING THE EXPECTATIONS $E(y_k|\phi)$ AND THE STEADY STATE DENSITY $p(\phi)$ FOR AN (N + 1) -ORDER TRACKER

The conditional expectation may be approximated with a great deal of accuracy by modifying and generalizing a method due to Viterbi (Ref. 8) and Holmes (Ref. 14). If we take the expectation of both sides of the differential equation (6) for $y_k(v)$ conditioned upon the phase variable $\phi(t)$ and multiply both sides by $\exp(v/\tau_k)$ we may write

$$\int_{-\infty}^{t} \frac{d}{dv}\left\{E\left[y_k(v)|\phi(t)\right] \exp(v/\tau_k)\right\} dv = -\frac{AK(1-F_k)}{\tau_k}\int_{-\infty}^{t} E\left[g\{\phi(v)\}|\phi(t)\right]\exp(v/\tau_k)\, dv \tag{47}$$

Integrating both sides of (47) from minus infinity to t with $t_0 = -\infty$ and introducing the change of variables $v = t - \tau$ yields* (see Ref. 19)

$$E\left[y_k(t)|\phi(t)\right] = -\frac{AK(1-F_k)}{\tau_k}\int_{0}^{\infty} \exp(-\tau/\tau_k) E\left[\left(g\{\phi(t-\tau)\} - \overline{g\{\phi(t-\tau)\}}\right)|\phi(t)\right] d\tau$$
$$- AK(1-F_k)\,\overline{g\{\phi(t-\tau)\}} \tag{48}$$

for all $k = 1, 2, \cdots, N$. The expectation under the integral sign may be estimated using the Orthogonality Principle* (OP) to find the best $\rho_G(\tau)$ such that $E\left[\{g(\phi_2) - \overline{g(\phi_2)}\}|\phi_1\right]$ is estimated by $\rho_G(\tau)\left[g(\phi_1) - \overline{g(\phi_1)}\right]$ in the best linear mean square sense. Here $\phi_2 = \phi(t-\tau)$, $\phi_1 = \phi(t)$ and $\rho_G(\tau)$ approaches zero as τ approaches minus infinity. First, we define the error ϵ,

$$\epsilon \triangleq E_{\phi_1}\left[\left\{E_{\phi_2}\left(\{g(\phi_2) - \overline{g(\phi_2)}\}|\phi_1\right) - \rho_G(\tau)\left[g(\phi_1) - \overline{g(\phi_1)}\right]\right\}^2\right]$$

*Note that $g[\phi(t-\tau)]$ is strictly stationary since the process $\phi(t)$ reduced modulo 2π is strictly stationary. See Section 9-5 in, Probability, Random Variables, and Stochastic Processes, by A. Papoulsis, McGraw Hill, N.Y., 1965.

Let

$$y(\phi_1) = E_{\phi_2}[g(\phi_2) - \overline{g(\phi_2)}|\phi_1] \; ; \quad x(\phi_1) = [g(\phi_1) - \overline{g(\phi_1)}]$$

and write

$$\epsilon = E_{\phi_1}[(y(\phi_1) - \rho_G(\tau)x(\phi_1))^2]$$

Thus the function $\rho_G(\tau)$ which produces the best linear mean square estimate $E[g\{\phi(t-\tau)\} - \overline{g\{\phi(t-\tau)\}}|\phi_1]$ is easily shown, using the OP, to be given by*

$$\rho_G(\tau) = \frac{R_g(\tau) - (\bar{g})^2}{\sigma_G^2} = \frac{R_G(\tau)}{\sigma_G^2} \tag{49}$$

where for a stationary ϕ process, $\overline{g(\phi_1)} = \overline{g(\phi_2)} = \bar{g}$ and $G = g(\phi) - \bar{g}$. The minimum mean square error $\epsilon_m(\tau)$ is given by $\epsilon_m(\tau) = \sigma_G^2[1 - \rho_G^2(\tau)]$. Replacing the expectation $E[(g(\phi_2) - \bar{g})|\phi_1]$ by the best linear mean square estimate $\rho_G(\tau)[g(\phi) - \bar{g}]$ in (48) we have

$$\hat{E}[y_k(t)|\phi(t)] = -\frac{AK(1 - F_k)g(\phi)}{\tau_k\sigma_G^2}\int_0^\infty \rho_G(\tau) \exp(-\tau/\tau_k)\, d\tau - AK(1 - F_k)\bar{g} \tag{50}$$

and the hat "^" is used to denote the fact that we have used the linear estimate in (49). If the loop is designed such that the correlation time of $\rho_G(\tau)$ is much less than τ_k^{-1} (narrow band loop) then, to a good approximation,**

$$\hat{E}[y_k(t)|\phi(t)] \approx \frac{AK(1 - F_k)\, g(\phi)}{2\tau_k\sigma_G^2} S_G(0) - AK(1 - F_k)\,\bar{g} \tag{51}$$

for all $k = 1, 2, \ldots, N$. In (51) $S_G(0)$ is the spectral density of $G(\phi) = g(\phi) - \bar{g}$ at the origin. From (1) it is clear that $S_G(0) = N_0/2A^2$ so that (also see Ref. 2)

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*A random process $x(t)$ is said to be of the separable class if it is second order stationary and satisfies $E[x(t+\tau)|x(t)] = \rho_x(\tau)\, x$ for all τ. The correlation function $\rho_x(\tau) = E[x(t+\tau)\, x(t)]/E(x^2)$.

**Throughout this paper the symbol "≈" is used to denote approximately equal to and the symbol "≅" implies asymptotically equal. Strictly speaking, the integral in (50) is less than or equal to that given in (51).

$$\hat{E}[y_k(t)|\phi(t)] \approx \frac{N_0 K(1 - F_k)}{4A\tau_k \sigma_G^2} g(\phi) + \bar{y}_k + \tau_k \bar{\dot{y}}_k \tag{52}$$

for all $k = 1, 2, \cdots, N$ and from (6), $\bar{y}_k + \tau_k \bar{\dot{y}}_k = -AK(1 - F_k)\bar{g}$. Note $\hat{E}[y_k(t)|\phi(t)]$ is independent of t since $\phi(t)$ reduced mod 2π is stationary.

At this point, in the development of a working theory, it appears that the goodness of the assumption which lead one from (48) to (51) must be justified by direct measurement of $E[y_k(t)|\phi(t)]$. The measurement of $E[y_k(t)|\phi(t)]$ can readily be adapted for simulation on a digital computer. The fact that $E[(y_1|\phi)]$ is sinusoidal in the steady state for a PLL when $\Omega_0 = 0$, $N = 1$, has been verified by computer simulation techniques. Typical results from the simulation for a low signal-to-noise ratio case are shown in Fig. 5 along with a plot of $\hat{E}[y_1(t)|\phi(t)]$ to accentuate the interference.

From (25), (40) and (52) the restoring force, for an N + 1-order tracker, becomes*

$$h(\phi) = [\beta(N) - \alpha(N) g(\phi)] \tag{53}$$

where

$$\beta(N) = \frac{4}{N_0 F_1^2 K^2}\left[\Omega_0 - AK\bar{g}\left\{F(0) - F_1\right\}\right]$$

(54)

$$\alpha(N) = \left[\frac{4A}{N_0 F_1 K} - \frac{1}{AKF_1^2 \sigma_G^2} \sum_{k=1}^{N} \frac{1 - F_k}{\tau_k}\right]$$

Thus $p(\phi)$ may be obtained from (32) and (54) for an (N + 1)-order tracker. The factor $\beta(N) > 0$ is responsible for the asymmetry in $p(\phi)$; hence, $p(\phi)$ will be symmetric if the loop is designed such that $\beta(N) = 0$, i.e.,

*The function $U(\phi) = -\int h(\phi)\, d\phi$ is analogous to the electrostatic scalar potential. On the other hand, $h(\phi) = -\nabla U(\phi)$ plays the role of the electric field strength in electromagnetic field theory.

$$F(0) - F(\infty) = \frac{\Omega_o}{AK\bar{g}} = \sum_{k=1}^{N} (1 - F_k) \tag{55}$$

where $F(\infty) = F_1$. It is clear that for $\Omega_o \neq o$ this equation can never be satisfied by the design of a first-order system. Further, if the factor $1/\alpha(N)$ is interpreted as a "variance", this variance is minimized with a choice of $N = 1$, i.e., a second-order loop.

For the N + 1-order PLL the solution to (24) can be written from (32) as* (see Ref. 19)

$$p(\phi) = \frac{\exp\left[\beta\phi + \alpha \cos \phi\right]}{4\pi^2 \exp(-\pi\beta) \left| I_{j\beta}(\alpha) \right|^2} \cdot \int_{\phi}^{\phi+2\pi} \exp\left[-\beta x - \alpha \cos x\right] dx \tag{56}$$

where we have written β for $\beta(N)$ and α for $\alpha(N)$. The functions $I_k(x)$ are modified Bessel functions of imaginary order and of argument x. The density function in (56) has been experimentally observed in the laboratory (Ref. 10) for the case where $\Omega_0 = 0$.

The mean value of the phase-error may be found from (56) and the well-known Bessel function expansions of $\exp(\pm x \cos \phi)$. Without belaboring the details (see Ref. 19) we have

$$\bar{\phi} = \int_{-\pi}^{\pi} \phi \, p(\phi) \, d\phi$$

$$\bar{\phi} = \frac{2 \sinh \pi\beta}{\pi \left| I_{j\beta}(\alpha) \right|^2} \sum_{m=1}^{\infty} \frac{m I_m(\alpha)}{m^2 + \beta^2} \left[\frac{I_o(\alpha)}{m} + \frac{I_m(\alpha)}{4m} + \sum_{\substack{k=1 \\ k \neq m}}^{\infty} \frac{2m(-1)^k I_k(\alpha)}{m^2 - k^2} \right] \tag{57}$$

It is clear from (57) that with $\beta(N) = 0, \bar{\phi} = 0$. Furthermore, $\bar{\phi}^2$ is given by (Ref. 19)

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*As previously pointed out ϕ belongs to any interval of width 2π centered about any stable lock point $2n\pi$; n any integer. For convenience we set n = 0 in what follows, [See (15) and 15a)].

$$\overline{\phi^2} = \int_{-\pi}^{\pi} \phi^2 p(\phi)\, d\phi$$

$$\overline{\phi^2} = \frac{\sinh \pi\beta}{\pi |I_{j\beta}(\alpha)|^2} \left\{ \frac{I_0(\alpha)}{\beta} \left[\frac{\pi^2 I_0(\alpha)}{3} + 4 \sum_{k=1}^{\infty} \frac{(-1)^k I_k(\alpha)}{k^2} \right] \right.$$

$$+ 2\beta I_o(\alpha) \sum_{k=1}^{\infty} \frac{I_k(\alpha)}{k^2(\beta^2 + k^2)} \tag{58}$$

$$\left. + 2\beta \sum_{k=1}^{\infty} \frac{(-1)^k I_k(\alpha)}{\beta^2 + k^2} \left[\left(\frac{\pi^2}{3} + \frac{1}{2k^2} \right) I_k(\alpha) + 4 \sum_{\substack{m=1 \\ m \neq k}}^{\infty} \frac{(-1)^m (k^2 + m^2) I_m(\alpha)}{\left(k^2 - m^2\right)^2} \right] \right\}$$

The variance $\sigma_\phi^2 = \overline{\phi^2} - (\bar{\phi})^2$ is minimized when the loop is designed such that $\beta = 0$ and α is maximized (Ref. 19). Thus, σ_ϕ^2 is minimized by the choice of a second-order loop designed such that $\beta = 0$. For this case we have, from (58), that

$$\sigma_\phi^2 \Big|_{min} = \frac{\pi^2}{3} + \frac{4}{I_0(\alpha)} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} I_k(\alpha) \tag{59}$$

Finally, the moments of sin ϕ and the phase-error rate $\dot{\phi}$, are given respectively by (Ref. 19)

$$\overline{\sin^2 \phi} = \frac{1}{2\alpha} \mathrm{RE} \left[\frac{I_{1+j\beta}(\alpha) + I_{j\beta-1}(\alpha)}{I_{j\beta}(\alpha)} \right] - \frac{\beta}{\alpha} \left[\overline{2 \sin \phi} - \frac{\beta}{\alpha} \right] \tag{60}$$

$\{\mathrm{RE}[\cdot]$ - Denotes real part of $[\cdot]\}$

$$\overline{\dot{\phi}^2} = K_{00}^2 \left[\alpha^2 \overline{(\sin \phi)^2} + \beta^2 - 2\alpha\beta\, \overline{\sin \phi} \right]$$

$$\overline{\sin\phi} = \frac{\beta}{\alpha} - \frac{\sinh \pi\beta}{\pi\alpha}\left|I_{j\beta}(\alpha)\right|^{-2}; \quad \overline{\cos\phi} = RE\left[\frac{I_{j\beta+1}(\alpha) + I_{j\beta-1}(\alpha)}{2I_{j\beta}(\alpha)}\right]$$

$$\overline{\dot\phi} = \Omega_o - AKF(o)\,\overline{\sin\phi} \tag{61}$$

$$\sigma_{\phi}^2 = K_{00}^2\alpha^2\sigma_{\sin\phi}^2 + K_{00}\left(2 - 2K_{00}\beta^2\right)$$

$$\sigma_{\sin\phi}^2 = \overline{\sin^2\phi} - [\overline{\sin\phi}]^2$$

From (60) we see that, when $\beta = 0$, the mean phase-error rate $\overline{\dot\phi} = \bar\omega - \omega_o = \Omega_o$. Thus the mean frequency of the CRG $\bar\omega$ agrees with the frequency of $s[t, \theta(t)]$ on the average, i.e., $\bar\omega = \omega$.

It is convenient to give a physical interpretation of our solution in terms of the concept of "potential wells" containing a Brownian particle. As previously mentioned the position of the trajectory $\underline{y}(t)$ at time t can be thought of as representative of the position of a particle undergoing Brownian motion. In fact, the motion of the ϕ-projection along the ϕ-axis can be interpreted as the motion of a particle in an external force field $h(\phi)$ whose dependence on the position is nonlinear. The function $U(\phi) = -\int^{\phi} h(x)dx$, represents the potential at ϕ and $\{U(\phi) - U(\phi + 2\pi)\}$ represents the potential difference a distance of 2π radians apart, (See footnote p. 24).

Figure 6 illustrates the normalized function $\beta^{-1}U(\phi)$ versus ϕ for various values of β/α and for the case of a PLL. In fact all positions of possible phase-lock are found when the restoring force $h(\phi)$ is zero. If $\beta > \alpha$ phase-lock is not possible since there exists no "well-bottoms" B_n. The particle (phase-error) will slip from well-to-well without coming to rest. If, however, $\beta = \alpha$ then the solutions to $\sin\phi = 1$ specify inflection points of $U(\phi)$. This corresponds to the situation where the frequency difference between the CGR and the incoming signal is a constant greater than zero, i.e., the so-called <u>false-lock</u> frequency. Hence, $\beta(N) = \alpha(N)$ defines the condition where a <u>limit cycle</u> occurs which, in practice, is usually referred to as a <u>false-lock</u> condition. If $\beta < \alpha$ well-bottoms (B_n - stable points of <u>potential</u> minima) occur at $\phi = 2n\pi + \sin^{-1}(\beta/\alpha)$ and well-tops (T_n - unstable points of potential maxima) occur at

$\phi = (2n + 1)\pi + \sin^{-1}(\beta/\alpha)$ where n is any integer. The slipping occurs more rapidly the greater is β and the shallower the well depth. The depth of the wells are proportional to α. In otherwords, one wishes to design the loop such that for any external conditions, β is minimized and α is maximized; a fact which we have just observed analytically. The shape of the potential wells is obviously dependent upon the nonlinearity $g(\phi)$. Hence, we would intutively expect that the "best" shape occurs when the well-sides are straight provided the loop is designed such that $\beta = 0$. In fact, it is conjectured that an optimum signal acquisition device is one in which $g(\phi)$ is synthesized such that $\partial U(\phi, t)/\partial\phi$ is a symmetric square-wave centered about zero for all t.

I. NET FLOW OF PROBABILITY PER UNIT OF TIME THROUGH THE HYPERPLANE $y_k = y_k'$ AND THE AVERAGE NUMBER OF CYCLES SLIPPED PER UNIT OF TIME FOR AN (N+1)-ORDER TRACKER

Due to the additive noise, discontinuities of oscillator (CGR) synchronization arises. The local oscillator may slip or gain a cycle of oscillations relative to the oscillations of the external signal $s[t, \theta(t)]$. In tracking applications the average number of cycles slipped per unit time is an important parameter as it is indicative of the error introduced into any doppler measurement made to obtain velocity and changes in range.

To calculate the average number of cycles slipped per unit of time in the steady state, we make use of the concept of probability current (introduced earlier) in the ϕ direction. The average flow of probability through the hyperlane $\phi = \phi'$ in the positive ϕ direction per unit of time is, from (9), (11) and the boundary conditions, equal to

$$I_0(\phi) = K_0\tilde{p}(\phi) - \frac{K_{00}}{2}\frac{d}{d\phi}\tilde{p}(\phi) \tag{62}$$

where $\tilde{p}(\phi)$ is the periodic extension of $p(\phi)$ and K_0 and K_{00} are defined in (8). Using (26) in (62) we find that $I(\phi)$ is given by*

*Here we drop the subscript "zero" on $I_0(\phi)$ and write $I(\phi)$ for $I_0(\phi)$.

$$I(\phi) = \begin{cases} -\dfrac{N_0 K^2 F_1^2 C_o D_o}{4}; \text{ Generalized Tracker} \\ \dfrac{N_0 F_1^2 K^2 \sinh \pi\beta}{8\pi^2 \left| I_{j\beta}(\alpha) \right|^2}; \text{ PLL} \end{cases} \tag{63}$$

for all ϕ and $k = 0, \cdots, N$. This says that the flow of probability per unit time through the hyperplane $\phi = \phi'$ is constant.* If $\Omega_o = 0$, then $D_o = 0$, $\beta = 0$ and $I(\phi) = 0$ and there is no flow of probability through the hyperplane $\phi = \phi'$.

In practice it is sometimes convenient to know the average number of cycles slipped per unit of time independent of direction. Denote the average number of cycles slipped to-the-right (positive ϕ-direction) per unit of time by $I_+(\phi)$ and the average number of cycles slipped to-the-left (negative ϕ-direction) per unit of time by $I_-(\phi)$. The parameter I_+ represents the average number of trajectories of the ϕ projection traveling in the positive ϕ-direction per unit time which pass through the hyperplane $\phi' = \phi$. Similarily I_- represents the average number of trajectories of the ϕ projection traveling in the negative ϕ-direction per unit time which pass through the same hyperplane. Thus $I = I_+ - I_-$ represents the (net) average number of cycles slipped per unit time. Extending the arguments due to Tikhonov (Ref. 7) for the case when $N = 0$, we find that the ratio of I_+ to I_- is given by

$$\frac{I_+}{I_-} = \exp\left[2\pi\beta\right] = \exp\left[U(\phi) - U(\phi + 2\pi)\right] \tag{64}$$

for the $(N + 1)$-order tracker. For $\beta = 0$ we see from (64) that the "to-the-left" cycles slipped equal the "to-the-right" cycles slipped. Thus the total number of cycles slipped per unit of time independent of direction is $\bar{S} = I_+ + I_-$, viz,

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* From Maxwells field equations we know that the curl of the magnetic field intensity $\underline{H}$ is equal to the current density $\underline{J}$ in the steady state. Here the probability surface current $I(\phi)$ is analogous to $\underline{J}$ so that the curl of the ϕ-projection of the electromagnetic field generated by electronic charges flowing in the PLL circuit is equal to $I(\phi)$. This curl is zero when the detuning is zero.

$$\bar{S} = \begin{cases} -\dfrac{N_0 F_1^2 K^2 C_o D_o \cosh \pi\beta}{4 \sinh \pi\beta} \; ; \text{ Generalized Tracker} \\ \\ \dfrac{N_0 F_1^2 K^2 \cosh \pi\beta}{8\pi^2 \left| I_{j\beta}(\alpha) \right|^2} \; ; \text{ PLL} \end{cases} \qquad (65)$$

The quantity $D = (2\pi)^2 \bar{S}$ is the diffusion coefficient representing the rate with which $\phi(t)$ is undergoing diffusion while Dt accounts for the fact that (14) is true.**

The probability currents $I_k(y_k)$; $k = 1, 2, \cdots, N$, may be easily computed from (11) and (35) in a manner similar to that described in this section. Thus the probability current flowing through the hyperplane $y_k = y_k'$ per unit of time in the positive direction of the y_k-axis is $I_k(y_k) = -K_{kk} C_k D_k / 2$ for all $k = 1, 2, \cdots, N$. Again we find that the net flow of probability is constant and the field is rotational. When $D_k = 0$, $I_k(y_k) = 0$, for all $k = 1, 2, \cdots, N$. This says that there is no net flow of probability in the y_k-direction.

J. THE PROBABILITY DISTRIBUTION p(Φ) FOR THE SECOND-ORDER TRACKER

If we now consider the case where $N = 1$, $F_1 = \tau_2/\tau_1$ and define*

$$r \triangleq AKF_1\tau_2 \triangleq \frac{AK\tau_2^2}{\tau_1}; \quad W_L = \frac{r+1}{2\tau_2} \text{ for } r\tau_1 \ll \tau_2 \qquad (66)$$

*The loop bandwidth W_L is defined as

$$W_L \triangleq 2b_L \triangleq \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \left| H(s) \right|^2 ds$$

where H(s) the closed transfer function when the loop is linearized (Refs. 1 and 2). Using the linear theory one may compute the system damping behavior. For $r < 4$ the system is under-damped, for $r = 4$ the system is critically damped and for $r > 4$ the system is over-damped. Most tracking systems are designed such that $r = 2$. This corresponds to $1/\sqrt{2}$ damping.

**The expected value of the time intervals between cycle slipping events is given by $\Delta T \approx [\bar{S}]^{-1}$. This assumes that the instants in time when phase synchronization is lost are statistically independent.

where r is related to the loop damping ($\zeta = \sqrt{r/4}$) and W_L is the loop bandwidth as defined from linear tracking theory. Thus $h(\phi)$ becomes

$$h(\phi) = -\left(\frac{r+1}{r}\right) \rho\, g(\phi) + \left(\frac{r+1}{r}\right)^2 \left[\frac{\rho\Omega_0}{2W_L} + \frac{\rho E(y_1 \mid \phi)}{2W_L}\right] \tag{67}$$

and $\rho = 2A^2/N_0 W_L$, i.e., the signal-to-noise ratio in the loop bandwidth W_L. As an alternative, it is instructive to approximate the conditional expectation using linear tracking theory. Thus

$$E(y_1 \mid \phi) \approx (1 - F_1)\left\{\frac{2rW_L}{(1+r)^2}\phi - \Omega_0\left(1 + \frac{F_1}{1+r}\right)\right\} \tag{68}$$

Therefore, within this approximation,

$$h(\phi) \approx -\left(\frac{r+1}{r}\right) \rho g(\phi) + \left(\frac{r+1}{r}\right)^2 \left\{(1 - F_1)\frac{r\rho\phi}{(1+r)^2} + \frac{\rho\Omega_0}{2W_L} - \frac{\Omega_0\, \rho(1-F_1)}{2W_L}\left(1 + \frac{F_1}{1+r}\right)\right\} \tag{69}$$

If $F_1 \ll 1$, then

$$p(\phi) = C_0 \exp\left[-\left(\frac{r+1}{r}\right) \rho \int^{\phi} g(x)\, dx + \frac{\rho}{2r}\phi^2\right] \tag{70}$$

which is independent of Ω_0. Thus, within the linear approximation, it is seen that a second-order loop with $F_1 \ll 1$ will always pull in lock so that $p(\phi)$ is symmetric. This, of course, is why second-order loops are used in practice. Contrast this with the solution, $N = 0$, for a first-order loop with $\Omega_0 \neq 0$ where it is noted from (26) that $p(\phi)$ is never symmetric.

On the other hand, from the linear least-squares estimate (52) and from (54) we write

$$h(\phi) \approx -\left[\left(\frac{r+1}{r}\right) \rho - \frac{1-F_1}{r\sigma_G^2}\right] g(\phi) + \left(\frac{r+1}{r}\right)^2 \frac{\rho}{2W_L} [\Omega_0 - AK(1-F_1)\bar{g}] \tag{71}$$

If the loop is designed such that $F_1 \ll 1$ and $\Omega_0 = AK\bar{g}$ for the range of interest in ρ we have, from (31),

$$p(\phi) = C_0 \exp\left[-\alpha(1) \int^{\phi} g(x)\, dx\right] \tag{72}$$

The variance of ϕ for a PLL mechanization can be obtained by first solving for $\overline{\sin^2\phi}$ from

$$\overline{\sin^2 \phi} = \int_{-\pi}^{\pi} \sin^2 \phi\, p(\phi) d\phi = \frac{1}{a}\left[\frac{I_1(a)}{I_0(a)}\right] \tag{73}$$

and the corresponding variance, given in (59), with

$$a = a(1) = \left(\frac{r+1}{r}\right)\rho - \frac{1}{r\sigma_G^2} \tag{74}$$

can be obtained from (72), (73) and (74) by numerical integration on a digital computer. Figure 7 illustrates the results for various values of r. The circles in Fig. 7 correspond to points obtained in the laboratory (Ref. 10) by means of a hardware mechanization with r = 4 and $g(\phi) = \sin\phi$. Thus, the assumptions which lead us from (48) to (51) appear to be particularly good for narrow band loops and the range of interest of ρ in practice.

It is interesting to obtain asymptotic solutions to (22) for the case $\tau_1 \gg 1$, i.e., $r \ll 1$. From (22) we write

$$- \frac{d}{d\phi} \left\{ \left[\Omega_0 + \sum_{k=1}^{N} E(y_k \; \phi) - AKF_1 g(\phi) \right] p(\phi | y_1) \right\} + \frac{F_1^2 K^2 N_0}{4} \frac{d^2 p(\phi | y_1)}{d\phi^2} \cong 0 \qquad (75)$$

for $F_1 \ll 1$. The solution to (75) is, therefore, identical to the solution to (24) so that, within the approximation, we have that $p(\phi) \cong p(\phi \mid y_k)$ for $r \ll 1$. Using this fact and Bayes rule we find that

$$\lim_{r \to 0} p(y_k \mid \phi) \cong p(y_k) \qquad (76)$$

and since $p(\phi \mid y_k) \simeq p(\phi)$ we have

$$\lim_{r \to 0} E(g(\phi) \mid y_k) \cong \int_{-\pi}^{\pi} g(\phi) \; p(\phi) d\phi \qquad (77)$$

and since $p(y_k \mid \phi) \simeq p(y_k)$ we have

$$\lim_{r \to 0} E(y_k \mid \phi) \cong \int_{-\infty}^{\infty} y_k p(y_k) \; dy_k \qquad (78)$$

If the loop is designed such that $\beta(N) \simeq 0$ for the range of interest in Ω_0, we have, from the symmetry of $p(\phi)$, that

$$\lim_{r \to 0} E(g(\phi) \mid y_k) \cong 0 \qquad (79)$$

since $g(\phi)p(\phi)$ is odd for all $k=1, 2, \ldots, N$. Consequently, from (76), (77) and (78) we find

$$\lim_{r \to 0} E(y_k \mid \phi) \cong 0 \tag{80}$$

so

$$\lim_{r \to 0} p(\phi) \cong C_0 \exp \left[\left(-\frac{r+1}{r} \right) \rho \int^{\phi} g(x)\, dx \right] \tag{81}$$

for all $N > 0$. From (71) we also note, at the other extreme for r, that

$$\lim_{r \to \infty} p(\phi) \cong C_0 \exp \left[-\rho \int^{\phi} g(x)\, dx \right] \tag{82}$$

for all $N \leq 1$ when $\beta(N) = 0$.

K. SYNTHESIS OF OPTIMUM TRACKING LOOPS

The problem of choosing the "best" loop filter F(p) as well as the "best" nonlinearity so as to provide for an optimum tracking loop depends upon what is meant by optimum. For example, during the signal acquisition mode the performance index is acquisition time. After the signal has been acquired, the problem becomes one of either tracking or data demodulation, or both, and the performance index changes. Hence, after acquisition a design based upon minimum acquisition time becomes suboptimum. For the case of phase-coherent communications one would want to minimize the mean-squared phase error, i.e., minimize the functional

$$\min_{F(p),\, g(\phi)} G\,[F(p),\, g(\phi)] = \sigma_{\phi}^2 \tag{83}$$

subject to the linearity constraint on F(p), and a gain constraint on the class of nonlinearities $\{g(\phi)\}$. In the case of tracking, one desires to maximize the expected time to loss of phase synchronization (Ref. 17). In general, however, it is conceivable that such an optimization technique may be formidable or a solution to (82) may not even exist if F(p) is restricted to being linear. Be that as it may, however, a few results are presently available for the case where one constrains the loop filter to be of the form $F(p) = 1$ or $F(p) = 1/(1 + \tau_2 p)$ and then selects that nonlinearity $g(\phi)$ such that the mean square error is minimum. This turns out to be equivalent (Refs. 20 and 21) to minimizing the area under the tail of the density $p(\phi)$. For these cases it can be shown (Refs. 20 and 21) that the optimum nonlinearity is

$$g(\phi) = \operatorname{sgn}[\sin\phi] \tag{84}$$

where $F(p) = 1$, or $F_1 = 1$ and it is assumed that $\Omega_0 = 0$. The restoring force $h(\phi)$ becomes rectangular for this $g(\phi)$.

In practice, it is desirable to design the loop such that $p(\phi)$ is symmetric so that no bias is introduced in the phase measurement. From (54) it can be seen that any asymmetry in $p(\phi)$ of (32) is due to the $\beta(N)$. By proper design of the loop filter $\beta(N)$ can be made arbitrarily small for reasonable frequency offsets. This requires $N \geq 1$ so that

$$p(\phi) = C_0 \exp\left[-\alpha(N)\int^{\phi} g(\phi)d\phi\right] \tag{85}$$

if (55) holds. Paralleling the arguments due to Stiffler and Shaft (Refs. 20 and 21) we have, for the N + 1- order tracker with $p(\phi)$ defined by (85), that the nonlinearity which minimizes the mean-squared phase error is also given by

$$g(\phi) = \operatorname{sgn}[\sin\phi] \tag{86}$$

for all N. Since $[\alpha(N)]^{-1}$ can be interpreted as a variance then σ_ϕ^2 is minimized over the choice of F(p) when $\alpha(N)$ is maximized, i.e., the potential wells are deepest. This is accomplished with $N = 1$, i.e., a second-order tracker. Hence we are lead to believe, on the basis of the preceding argument, that a second-order loop designed such that $\beta(1) = 0$ with $g(\phi)$ given by (86) is, for all practical purposes, that tracker which minimizes the variance of the phase error.

For the second-order tracker the minimum mean-square error obtainable is, from (74), (85) and (86)

$$(\sigma_\phi^2)_{min} = \frac{2\left[1 - \left\{1 + \pi\alpha + (\pi\alpha)^2/2\right\} \exp(-\pi\alpha)\right]}{\alpha^2 \left[1 - \exp(-\pi\alpha)\right]} \tag{87}$$

where $\alpha = \alpha(1)$ is given in (54). For high signal-to-noise ratios, $\alpha = \rho$ and

$$(\sigma_\phi^2)_{min} \cong \frac{2}{\rho^2}; \ \rho \gg 1 \tag{88}$$

At low values of ρ,

$$(\sigma_\phi^2)_{min} \cong \frac{\alpha\pi^3}{3} \cdot \frac{1}{\exp(\pi\alpha) - 1} \tag{89}$$

Thus, for large signal-to-noise ratios, the improvement in σ_ϕ^2 offered by the second-order tracker with optimum nonlinearity is $I = 10 \log_{10}(\rho/2)$ db better than a second-order phase-locked loop with $g(\phi) = \sin\phi$. For equal phase variances and large signal-to-noise ratios, the second-order tracker with optimum nonlinearity, requires that

$$\rho_{PLL} = \rho^2/2 \tag{90}$$

where ρ is the signal-to-noise ratio in the optimum second-order tracker. For low signal-to-noise ratios, see Figs. 7 and 8, the improvement is not as drastic and the performance of both trackers approach each other.

If for an (N + 1)-order tracker, with $\Omega_0 = 0$, we desire to minimize the variance of ϕ then it is clear from (32) and (54) that the factor β is zero and $\alpha(N)$ is maximized when $N = 0$. The minimum mean square error is still given by (87) with $\alpha = \rho = 4A/N_0K$. Thus, we are lead to the conclusion that the optimum tracker is one for which the nonlinearity is given by (86) and $F(p) = 1$ when

$\Omega_0 = 0$. Hence a first-order system is optimum from the point of view of producing the minimum phase-error variance when $\Omega_0 = 0$. There are other approaches which may be used, aside from signal design to vary $R_{rs}(\phi)$. We shall not discuss all of them in detail here; however, astute methods of instrumentation, as in n^{th} order tanlock (Ref. 13) and delay locked loops (Ref. 12), are possible.

L. THE OPTIMUM REFERENCE SIGNAL FOR A SQUARE-WAVE INPUT

Since a square wave cross-correlation function is difficult to mechanize, there are various other approaches which can be used to vary $R_{rs}(\phi)$ which are interesting. For example, for a first-order loop (Ref. 22), Layland constrained the signals $[t, \theta(t)]$ to be a square-wave (the case of digital communications) and maximized p(0) with respect to r(t) under a unit power constraint. If the same methodology is applied here for the optimum second-order tracker then it is easy to show that (Ref. 22) the optimum control signal is given by

$$r(\tau) = D'\left\{\int_{\tau}^{\pi/2} (\phi - \tau)\, w(\phi)\, d\phi + \left(\frac{\pi}{2} - \tau\right) w(\pi/2) \cdot \right.$$

$$\int_0^{\pi/2} \exp\left[-\frac{2\alpha}{\pi}\int_0^{\pi/2} (\pi/2 - \max\{\zeta, \eta\})\, r(\eta)\right] d\eta\, d\zeta$$

$$\left. + \, w(\pi/2)\int_0^{\pi/2} (\pi/2 - \max\{\zeta, \eta\}) \exp\left[-\frac{2\alpha}{\pi}\int_0^{\pi/2} [\pi/2 - \max(\zeta, \eta)] r(\eta) d\eta\right] d\zeta\right\} \tag{91}$$

where D' is a normalizing constant for $\tau \epsilon [0, \pi/2]$. The parameter α is defined in (54) and

$$w(\phi) = \exp\left[-\frac{2\alpha}{\pi}\int_0^{\phi} (\phi - \eta)\, r(\eta) d\eta\right] \tag{92}$$

Note: $r(\tau)$ is implicitly a function of itself through $w(\phi)$.

which is independent of Ω_0 for the second-order tracker designed such that $\beta = 0$. Even though (91) is recursive and does not admit to a solution in closed form, it can be evaluated by employing an iterative numerical procedure on a digital computer. It is readily shown that the variance of the phase-error becomes inversely proportional to $\sqrt{\alpha^{-3}}$ for large signal-to-noise ratios. Fig. 10 illustrates $r(\tau)$ for various values of α. In the limit as α approaches infinity $r(\tau)$ becomes a delta function train $R_{rs}(\phi)$ is given by (86).

M. CONCLUSIONS AND PROSPECTS

The long-standing problem of evaluating the response distributions, or marginal probability densities $p(y_k)$; $k = 0, 1, \ldots, N$; hence, performance of a generalized tracking system operating in the nonlinear region is reduced to one of finding the conditional expectations, $E(y_k \mid \phi)$, $E[g(\phi) \mid y_k]$; $k = 1, 2, \ldots, N$. In the steady-state we have shown that the response distributions (marginal probability densities) are solutions to ordinary, first-order, differential equations for which the form of the exact solution is known. The case of greatest practical interest, i.e., $N = 1$ or the second-order loop, has been studied in some detail. Based upon these results, it is felt that the response distributions, hence, performance, of nonlinear filters in general to random or deterministic signals perturbed by random noise, is always embedded in a knowledge of certain sets of conditional expectations. Stated another way the conditional expectations are necessary but not sufficient to specify the system response distributions whereas system performance (e.g., moments of the response distribution) is embedded in a knowledge of the conditional expectations. In addition, the probability currents $I_k(y_k)$ have been evaluated for all components of $\underline{y}$ and related to the physics of the loop. It is possible to study this problem for the class of loop filters for which

$$F(p) = F_1 + \sum_{k=1}^{N} \frac{1 - F_k}{\tau_k p} \tag{93}$$

by the same methodology. The only difference in our solution is that the non-linear restoring force $g_k(y_k)$ in all solutions is replaced by

$$g_k(y_k) = -\frac{1 - F_k}{\tau_k} [AK\, g(\phi)] \tag{94}$$

for all k = 1, 2, ..., N.

In summary, the marginal probability densities for a generalized tracking loop are then given by

$$p(\phi) = C_0 \exp\left[\frac{2}{K_{00}} \int E(\dot{\phi} \mid \phi) d\phi\right] \left\{1 + D_0 \int_{-\pi}^{\phi} \exp\left[-\frac{2}{K_{00}} \int^{y} E(\dot{\phi} \mid \phi) d\phi\right] dy\right\} \tag{95}$$

where C_0 and D_0 are found from (27) and (29) respectively and

$$E(\dot{\phi} \mid \phi) = \Omega_0 - F_1 AK\, g(\phi) + \sum_{k=1}^{N} E(y_k \mid \phi) \tag{96}$$

with $K_{00} = N_0 K^2 F_1^2 / 2$. The probability distribution for the k^{th} projection of $\underline{y}'$ is

$$p(y_k) = C_k \exp\left[\frac{2}{K_{kk}} \int E(\dot{y}_k \mid y_k) dy_k\right] \left\{1 + D_k \int_{-\infty}^{y_k} \exp\left[\frac{-2}{K_{kk}} \int^{y} E[\dot{x}_k \mid x_k] dx_k\right] dy\right\} \tag{97}$$

where C_k and D_k are found from the normalization condition and (37) respectively.

For $N \le 1$ and the case where $\beta(N) = 0$ we have the limiting cases

$$\lim_{r\to\infty} p(\phi) = C_0 \exp\left[-\rho \int g(\phi)d\phi\right] \tag{98}$$

On the other hand

$$\lim_{r\to 0} p(\phi) = C_0 \exp\left[-\rho \left(\frac{r+1}{r}\right) \int g(\phi)d\phi\right] \tag{99}$$

for all $N \ge 1$.

In addition, we have shown that

$$\hat{E}\left[y_k(t)|\phi(t)\right] = \frac{N_0 K (1 - F_k)\, g(\phi)}{4A\tau_k \sigma_G^2} + \bar{y}_k + \tau_k \overline{\dot{y}_k} \tag{100}$$

which results in

$$p(\phi) = C_0 \exp\left[-\alpha(N) \int^{\phi} g(x)\, dx\right] \tag{101}$$

when $\beta(N)$ is zero. The parameter $\alpha(N)$ is defined in (54). For $(N + 1)$-order trackers designed such that $\beta(N) = 0$ the optimum nonlinearity is given by $g(\phi) = \text{sgn}\,[\sin\phi]$.

For a second-order tracker, results are presented for the case where $R_{rs}(\phi)$ must be realizable. At large signal-to-noise ratios, ρ, the variance of the phase-error is proportional to $\sqrt{\alpha^{-3}}$ as opposed to α^{-2} for a second-order loop with $g(\phi) = \text{sgn}\,(\sin\phi)$. It can be shown that if the length of the pseudo-random sequence is optimized, the mean squared phase error in a delay-locked loop exhibits the α^{-2} dependence on ρ and r. It is not possible, for all practical purposes, to do better than the optimum second-order tracker discussed here.

Finally, for $\Omega_0 = 0$ we have been able to perform stochastic optimization over F(p) and g(ϕ) and found that the variance of the phase error is minimized when F(p) = 1 and g(ϕ) is given in (86).

The loop filter defined in (2) admits only those filters which have simple poles along the real-axis in the left half plane provided the τ_k's are positive real and the F_k's ≤ 1. If one allows for complex τ_k's and F_k's in (2) then a broader class of filters is admitted; however, the differential equations for the y_k's become complex and occur in conjugate pairs. On the basis of (6), the solution (26) for p(ϕ) is real since the sum of the conditional expectations $E(y_k \mid \phi)$ is real; however, the solutions for $p(y_k)$ are no longer valid since they become complex and occur in conjugate pairs. One may return to (6) and produce a new vector Markov process, which is real, by adding and subtracting the conjugate pairs of the y_k's. The solution to the revised Fokker-Planck equation will then render solutions for the marginal densities which are real.

It is more interesting, although more difficult, to consider the general loop filter $F(s) = F_1 + F'(s)$ where

$$F'(s) = \frac{N(s)}{D(s)} = \frac{\sum_{k=1}^{N} p_k\, s^{k-1}}{\sum_{k=1}^{N+1} q_k\, s^{k-1}} \tag{102}$$

where the q_n's and p_n's are real. If one defines the vector Markov process

$$\begin{aligned} y_1 &= -\frac{K\,[Ag(\phi) + n(t)]}{D(s)} \\ y_2 &= \dot{y}_1 \\ y_3 &= \dot{y}_2 \\ &\vdots \\ y_N &= \dot{y}_{N-1} \end{aligned} \tag{103}$$

then, from (102) and (103)

$$y_0 = \phi$$

$$\dot{y}_0 = \dot{\phi}(t) = \dot{\theta}(t) - KF_1 [Ag(\phi) + n(t)] + \sum_{k=1}^{N} p_k y_k$$

$$\dot{y}_1 = y_2$$

$$\dot{y}_2 = y_3 \qquad (104)$$

$$\vdots$$

$$\dot{y}_N = -\frac{1}{q_{N+1}} \sum_{k=1}^{N} q_k y_k - \frac{K}{q_{N+1}} [Ag(\phi) + n(t)]$$

and we again have the vector Markov process $\underline{y} = (\phi, \underline{y}_0^{\prime})$. From (7) and (8) the F-P equation is easily set up and may be integrated, as before, over the projections of $\underline{y}_0^{\prime}$ giving

$$\frac{\partial p(\phi, t)}{\partial t} = -\frac{\partial}{\partial \phi}\left\{\left[\Omega_0 - KF_1 Ag(\phi) + \sum_{k=1}^{N} p_k E(y_k \mid \phi, t)\right] p(\phi, t)\right\}$$

$$+ \frac{F_1^2 K^2 N_0}{4} \frac{\partial^2 p(\phi, t)}{\partial \phi^2} \qquad (105)$$

when the boundary conditions are applied. For N = 1, (105) agrees in form with (23); however, for N > 1 the expectations $E(y_k | \phi, t)$ are, in general, different from those defined in (23). In the steady state the solution to (105) is given by (26) with a restoring force

$$h(\phi) = \frac{1}{c_0}\left[\Omega_0 - KF_1 Ag(\phi) + \sum_{k=1}^{N} p_k E(y_k \mid \phi)\right] \qquad (106)$$

and it is understood that the expectations $E(y_k \mid \phi)$ are to be determined from the vector process defined in (104). Corresponding formulas for the densities of the y_k's defined in (104) may easily be obtained. The details will be presented in a future paper.

Our method has been extended to the case of angle demodulation (phase or frequency) by generalized modulation tracking loops. The signal $\theta(t)$ is assumed to be a Gaussian process with a rational spectral density. It is also possible to produce results for the more general problem of tracking and demodulation where

$$\theta(t) = \sum_{j=1}^{J} a_j t^j + \sum_{q=1}^{Q} \theta_q(t) \tag{107}$$

is composed of a deterministic process which angle modulates the transmitter due to motion and Q statistically independent Gaussian processes with a rational density. Corresponding results for a Gaussian process with irrational spectra have not as yet been forthcoming.

Further, we have been able to produce the loop response (marginal distributions) for two generalized tracking loops in cascade. In either tracker, the angle modulation to be tracked may be composed of deterministic and/or Gaussian angle modulation. The domain of interest in this problem turns out to be a truncated hyperslab which appears as a skewed rectangle through the phase-error section (ϕ_1, ϕ_2). We shall report on these methods in a forthcoming paper.

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Appendix A

There are two approaches which one may take to produce (33). The simplest approach is to integrate (22) from $-\pi$ to $+\pi$ and make use of the boundary condition $p[\pi, y_k] = p[-\pi, y_k]$ and the fact that the probability current

$$\left[\frac{F_1^2 N_0 K^2}{4} \frac{\partial}{\partial \phi} + \frac{F_1(1 - F_k) N_0 K^2}{4\tau_k} \frac{\partial}{\partial y_k} \right] p(\phi, y_k, t) = \frac{\partial}{\partial \phi} [p(\phi, y_k, t)] \Big|_{-\pi}^{\pi} \tag{A-1}$$

This is a result of (20a).

An alternate approach which gives more insight into what happens at the boundary $\phi = \pm\pi$ is to begin with (12) and integrate out the variable ϕ. This produces a partial differential for which the solution is $p(\underline{y}', t)$. The remaining (N-1) of the y_k variables may be integrated out producing (33). We presently exhibit this method. For convenience we drop the zero subscript on $\underline{y}'$.

To produce (33) from (12) we reduce (12) term by term. Consider first the terms

$$I_1 = \int_{-\pi}^{\pi} \sum_{\ell \neq k \neq 0}^{N} \sum^{N} \frac{(1 - F_k)(1 - F_\ell) K^2 N_0}{4\tau_k \tau_\ell} \frac{\partial^2 p(\underline{y}, t)}{\partial y_k \partial y_\ell} \, d\phi \tag{A-2}$$

Carrying out the integration yields the condition

$$\int_{-\pi}^{\pi} p(\underline{y}, t) d\phi = \int_{-\pi}^{\pi} p(\underline{y}', t) p(\phi | \underline{y}', t) d\phi = p(\underline{y}', t) \tag{A-3}$$

so that

$$I_1 = \sum_{k \neq \ell \neq 0}^{N} \sum^{N} \frac{(1 - F_k)(1 - F_\ell) K^2 N_0}{4\tau_k \tau_\ell} \frac{\partial^2}{\partial y_k \partial y_\ell} p(\underline{y}', t) \tag{A-4}$$

Next consider the terms

$$I_2 = \sum_{k=1}^{N} \frac{F_1(1 - F_k)N_0K^2}{4\tau_k} \frac{\partial}{\partial y_k} \int_{-\pi}^{\pi} \frac{\partial}{\partial\phi} p(\phi, \underline{y}', t)d\phi \tag{A-5}$$

$$I_2 = \sum_{k=1}^{N} \frac{F_1(1 - F_k)N_0K^2}{4\tau_k} \frac{\partial}{\partial y_k} p(\phi, \underline{y}', t)\Big|_{-\pi}^{\pi}$$

The terms

$$I_3 = \sum_{k=1}^{N} \int_{-\pi}^{\pi} \frac{(1 - F_k)^2N_0K^2}{4\tau_k^2} \frac{\partial^2}{\partial y_k^2} p(\phi|\underline{y}', t)p(\underline{y}', t)d\phi =$$

$$\sum_{k=1}^{N} \frac{(1 - F_k)^2N_0K^2}{4\tau_k^2} \frac{\partial^2}{\partial y_k^2} p(\underline{y}', t) \tag{A-6}$$

and

$$I_4 = \int_{-\pi}^{\pi} \sum_{k=1}^{N} \left\{\frac{\partial}{\partial y_k} \left[\frac{y_k}{\tau_k} + \left(\frac{1 - F_k}{\tau_k}\right) AK\ g(\phi) \right]p(\phi|\underline{y}', t)p(\underline{y}', t)\right\} d\phi$$

$$= \sum_{k=1}^{N} \frac{\partial}{\partial y_k} \left[\frac{y_k}{\tau_k} + \left(\frac{1 - F_k}{\tau_k}\right) AK\ E(g(\phi)\ |\underline{y}')\right]p(\underline{y}', t) \tag{A-7}$$

Next consider

$$I_5 = \int_{-\pi}^{\pi} \frac{F_1^2N_0K^2}{4} \frac{\partial^2}{\partial\phi^2} p(\phi|\underline{y}', t)p(\underline{y}', t)d\phi \tag{A-8}$$

$$I_5 = \frac{F_1^2N_0K^2}{4} \frac{\partial}{\partial\phi} p(\underline{y}, t)\Big|_{-\pi}^{\pi}$$

Finally consider the terms

$$
\begin{aligned}
I_6 &= \int_{-\pi}^{\pi} \frac{\partial}{\partial \phi}\left[\Omega_0 - F_1 AK\, g(\phi) + \sum_{k=1}^{N} y_k\right] p(\underline{y}, t)\, d\phi \\
&= \left[\Omega_0 - F_1 AK\, g(\phi) + \sum_{k=1}^{N} y_k\right] p(\underline{y}, t)\Big|_{-\pi}^{\pi}
\end{aligned}
\qquad \text{(A-9)}
$$

From (16) these terms are all zero. If we now sum the terms

$$
I_1 + I_2 + I_3 + I_4 + I_5 \qquad \text{(A-10)}
$$

and integrate out all variables in $\underline{y}'$ except y_k we have (33) when we recognize the normal derivative $(I_2 + I_5)$

$$
\left[\frac{F_1^2 N_0 K^2}{4} \frac{\partial}{\partial \phi} + \sum_{k=1}^{N} \frac{F_1(1 - F_k) N_0 K^2}{4\tau_k} \frac{\partial}{\partial y_k}\right] p(\phi, y_k, t) \qquad \text{(A-11)}
$$

$$
= K_{00} \frac{\partial}{\partial \phi} p(\phi, y_k, t)\Big|_{-\pi}^{\pi}
$$

from (20a).

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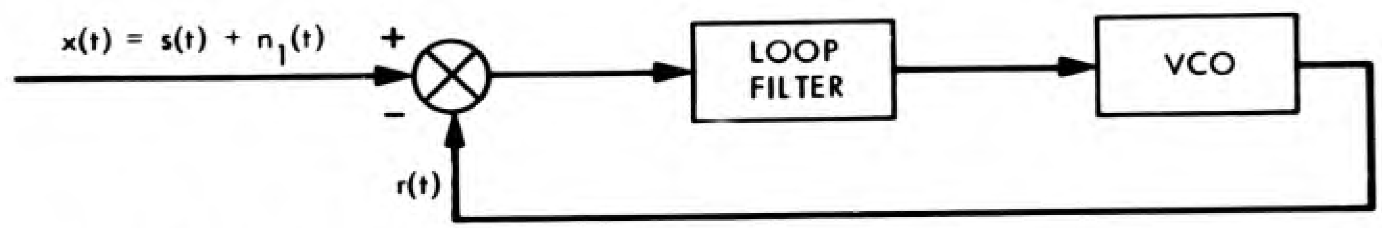


$$s(t) = \sqrt{2} A \sin \left[\omega_0 t + \theta(t)\right]$$

$$r(t) = \sqrt{2} \cos \left[\omega_0 t + \hat{\theta}(t)\right]$$

Figure 1. Phase-Locked Loop Mechanization

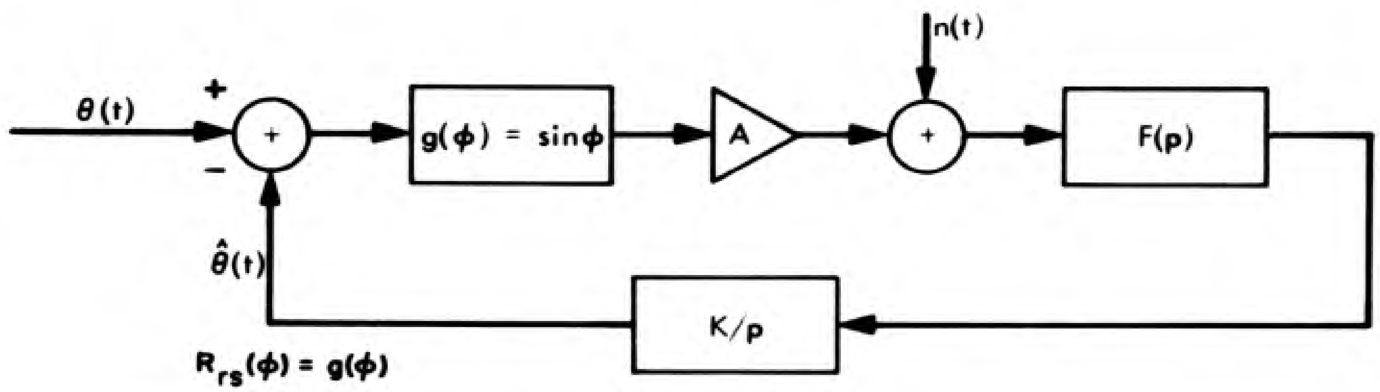


Figure 2. Equivalent Model for a PLL

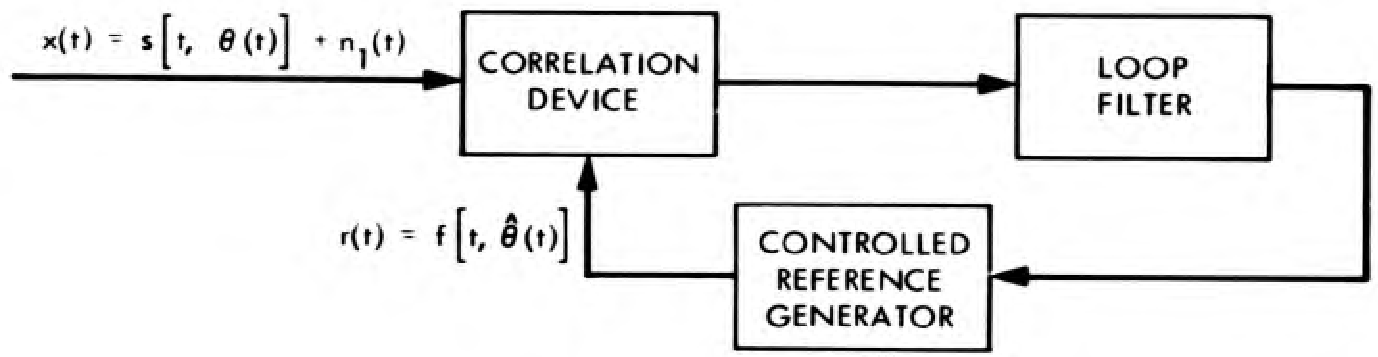


Figure 3. Generalized Tracking Loop

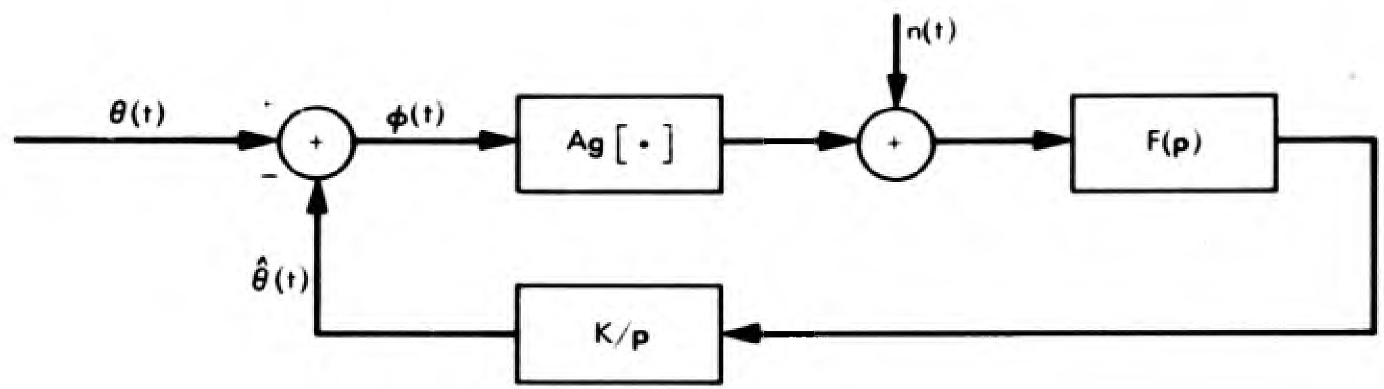


Figure 4. Equivalent Model for a Generalized Tracking Loop

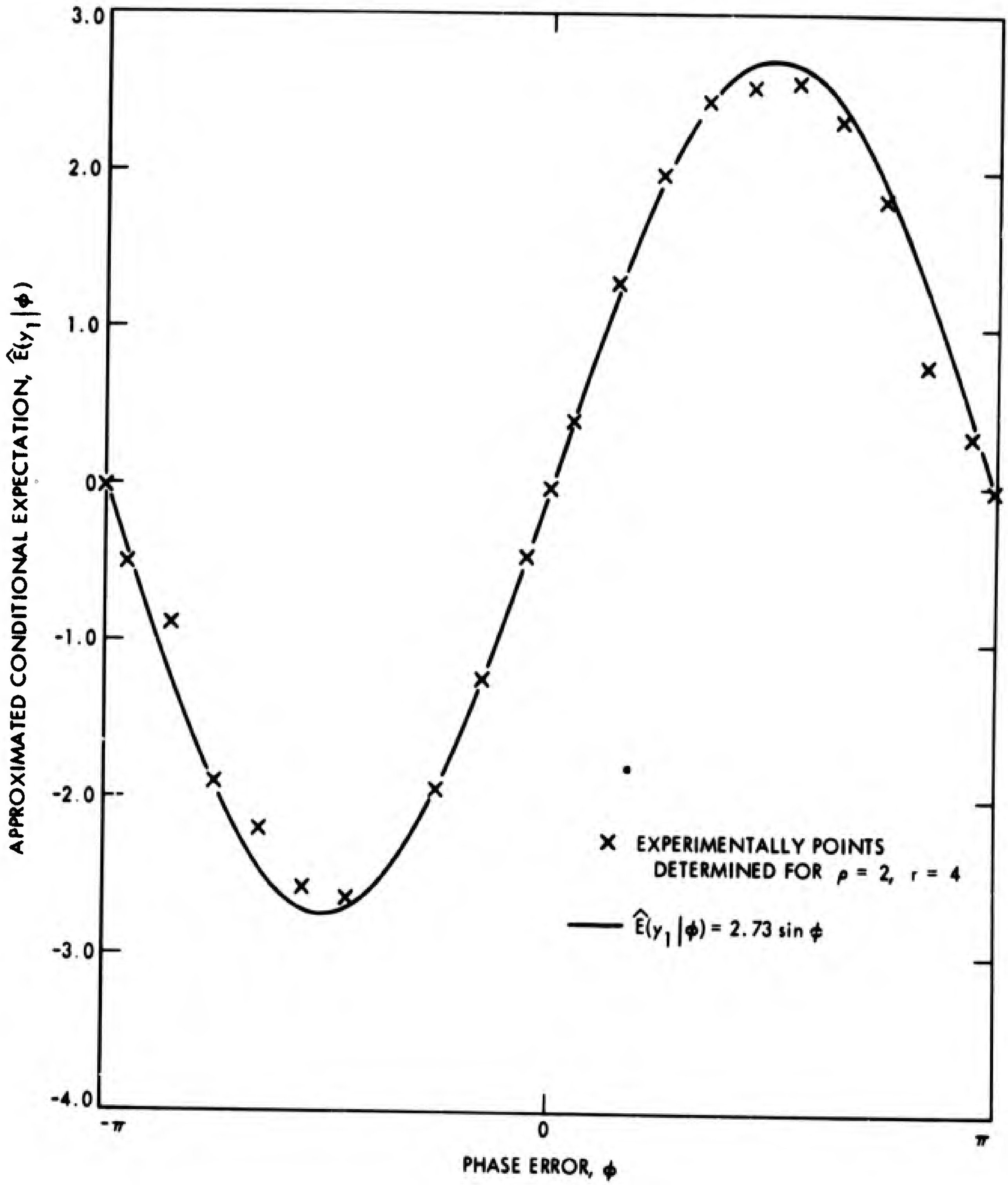
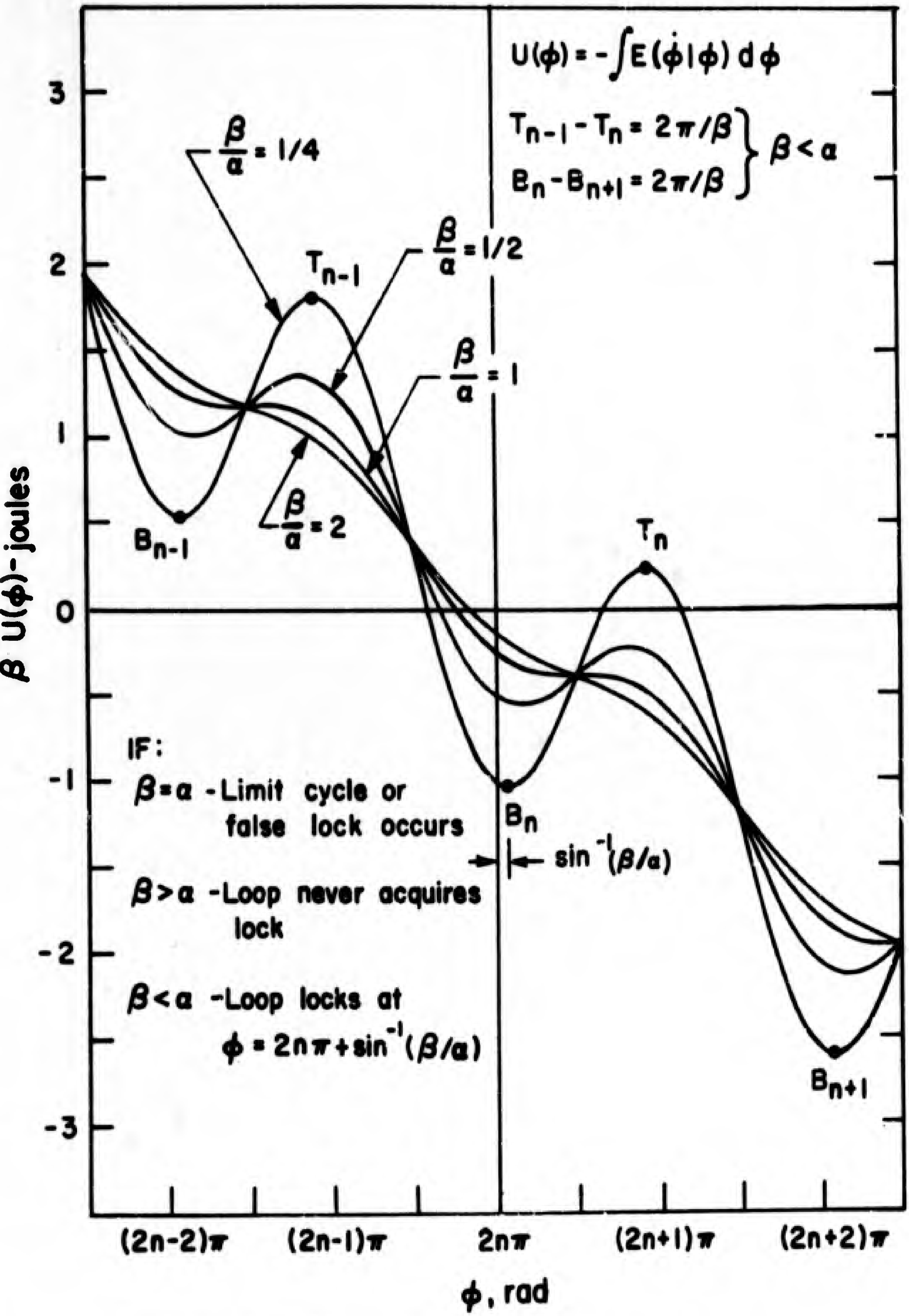
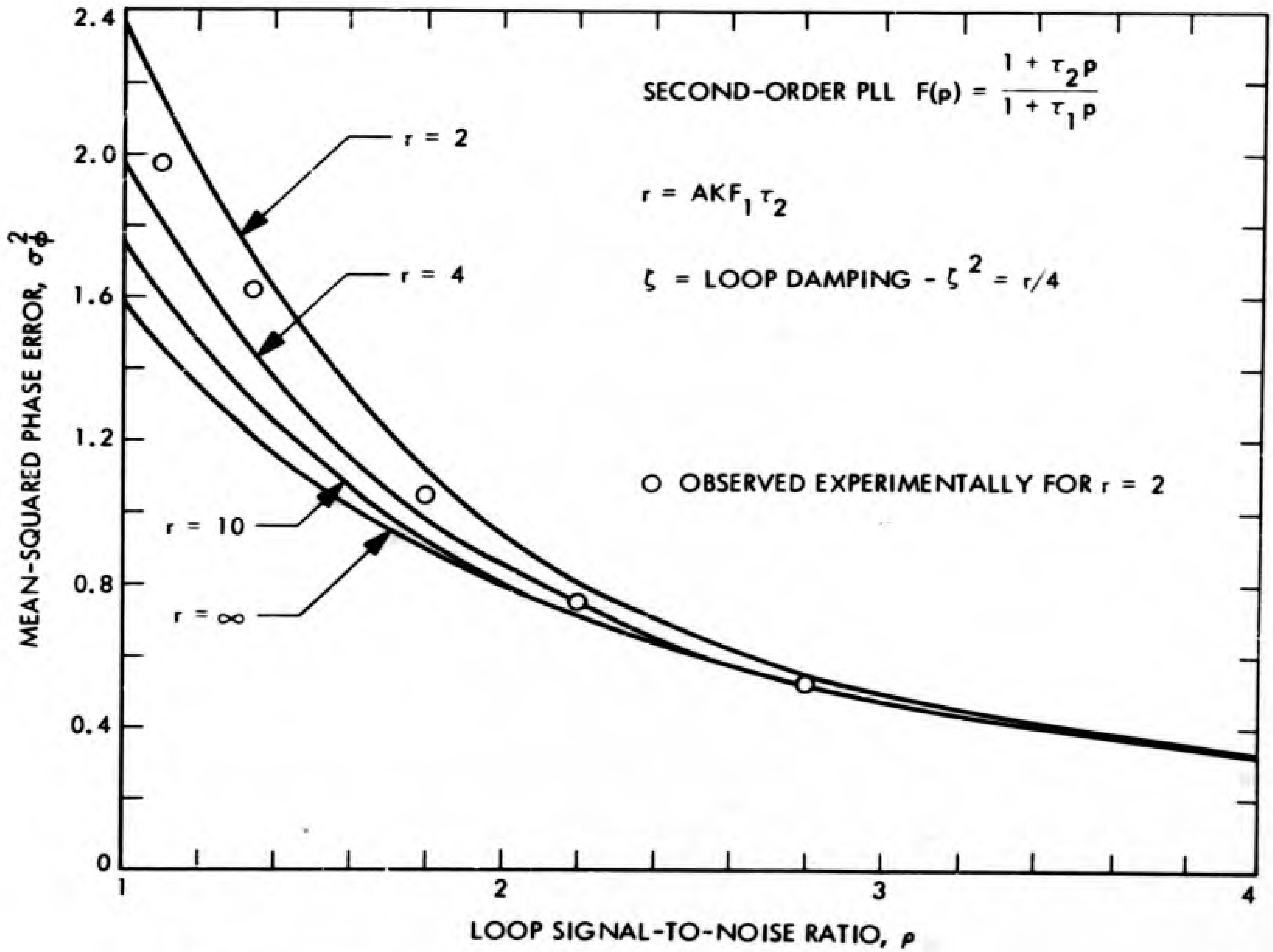


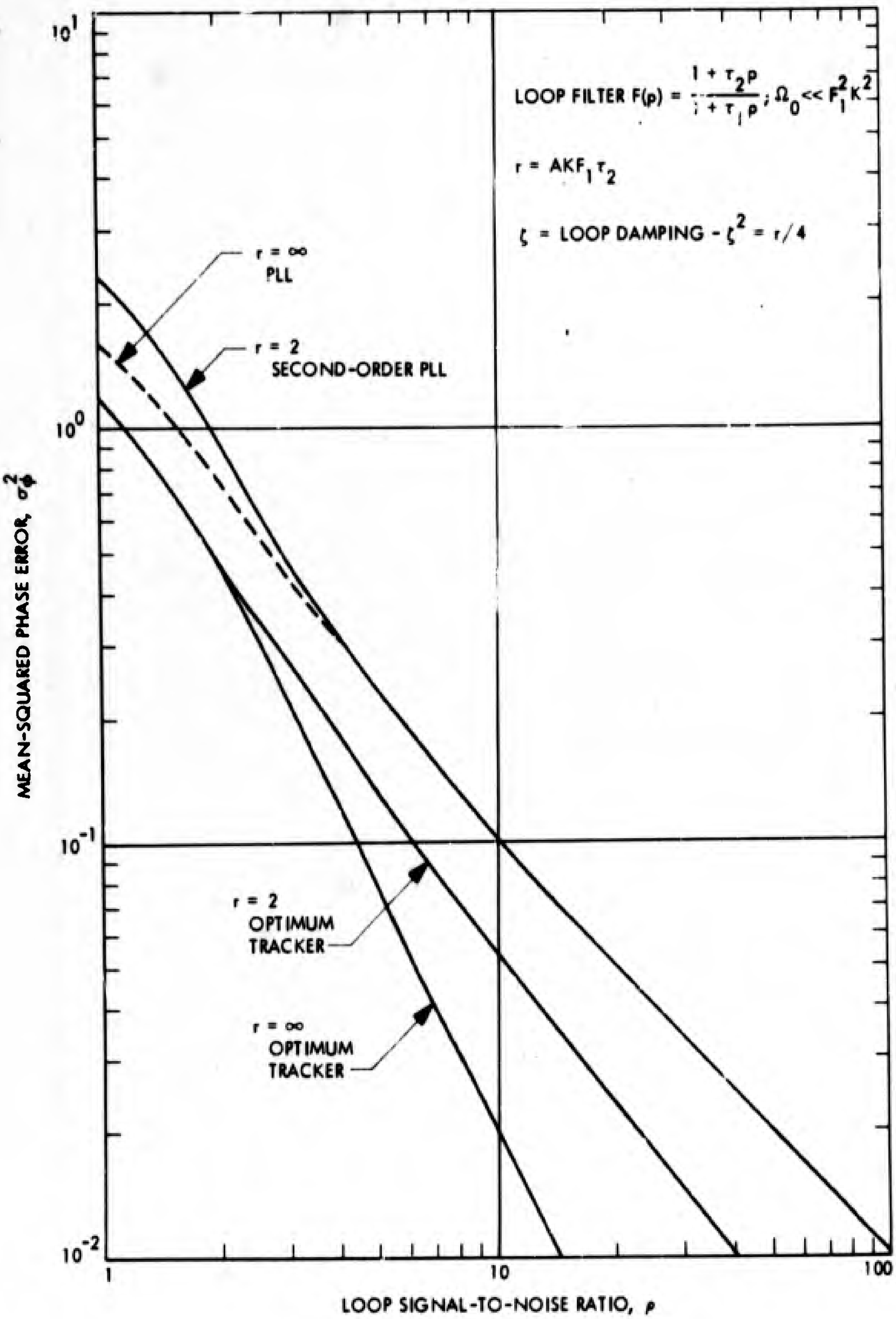
Figure 5. Comparison of $\hat{E}(y_1|\phi)$ With Results Obtained via Computer Simulation



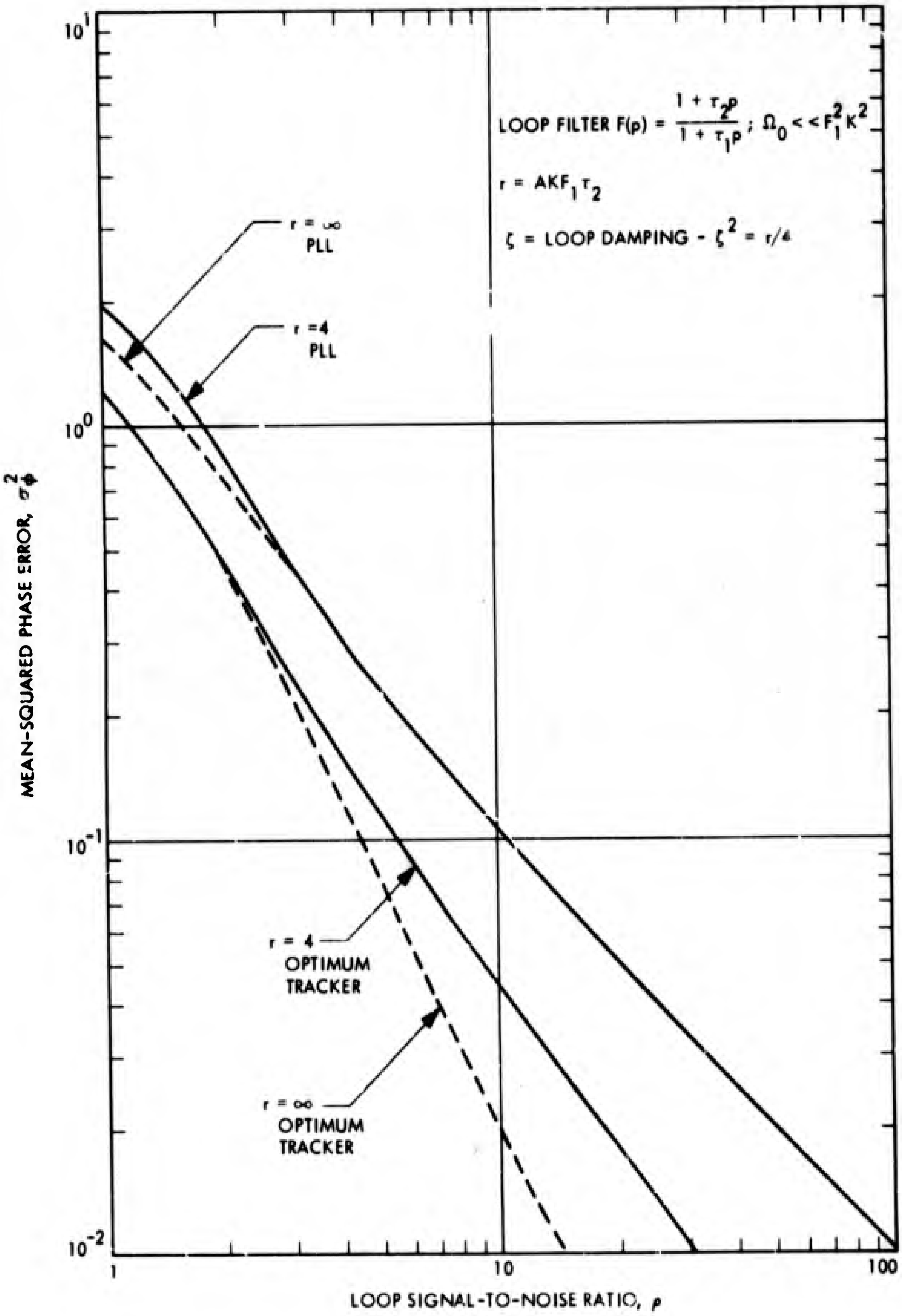
6. Motion of the Phase-Error from Well-to-Well as a Function of the Potential $U(\phi)$



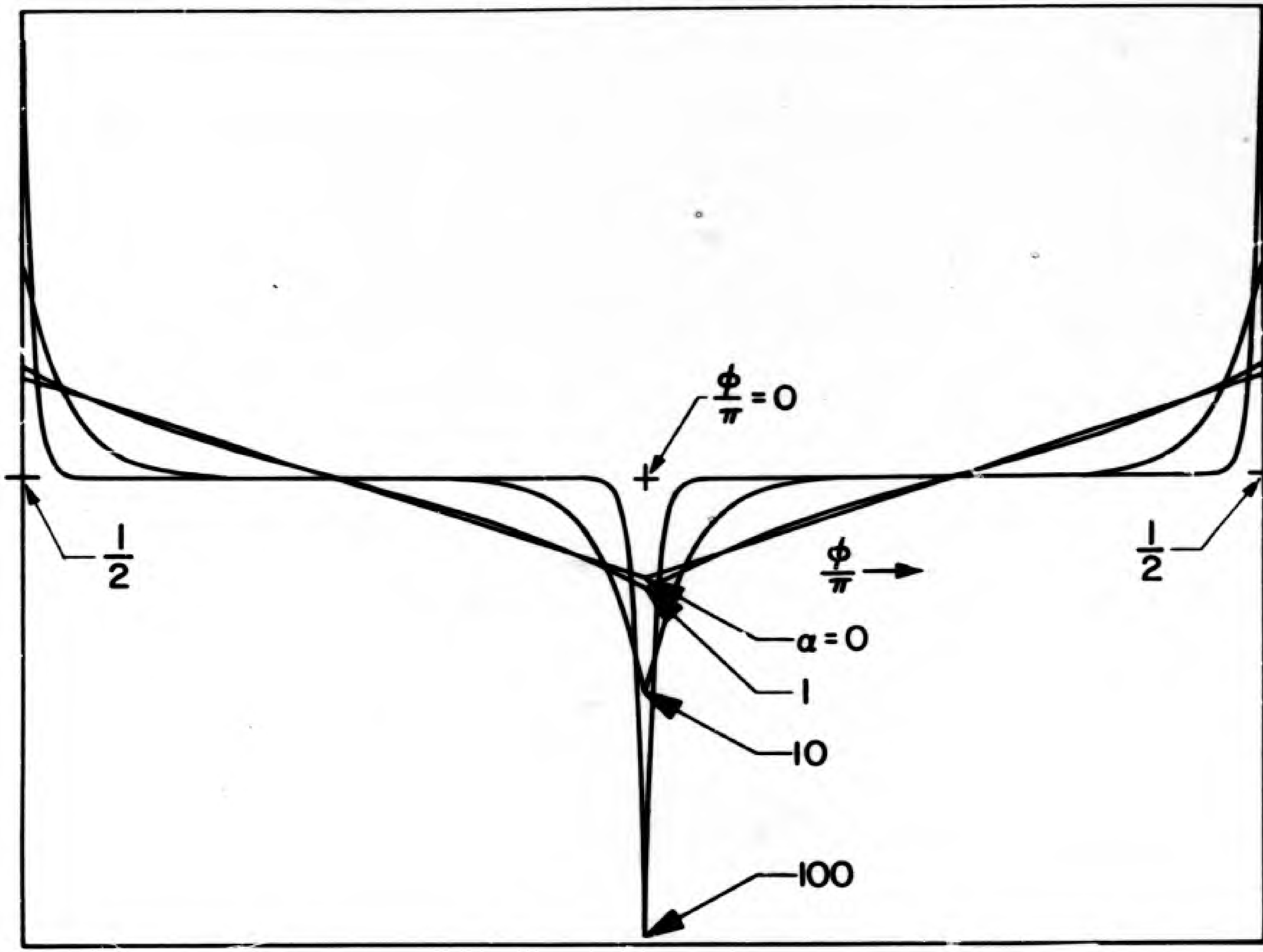
7. Variance of the Phase-Error Vs Loop Signal-to-Noise Ratio ρ



8. Variance of the Phase-Error Vs Loop Signal-to-Noise Ratio ρ



9. Comparison of the Variance of the Phase-Error Vs Signal-to-Noise Ratio ρ



10. Optimum Reference Waveform for Various Values of α

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13. ABSTRACT

This report develops a rather general theory for use in the analysis and synthesis of generalized tracking systems. Synthesis procedures for effecting stochastic optimization of a generalized tracker are presented. The results are useful in various applications, e.g., synchronization techniques, radar tracking, missile guidance and navigation, and phase coherent communication systems.

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