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OBLIQUE LOADING OF UNIDIRECTIONAL FIBER COMPOSITES; SHEAR LOADING

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January 1969

U. S. ARMY AVIATION MATERIEL LABORATORIES
FORT EUSTIS, VIRGINIA

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SAN DIEGO, CALIFORNIA

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The data contained in this report are the result of research conducted to determine the mechanism of load transfer through a composite material consisting of fibers embedded in a matrix with shear loads parallel to the fiber direction.

The report has been reviewed by the U.S. Army Aviation Materiel Laboratories and is considered to be technically sound. It is published for the exchange of information and the stimulation of future research.

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Final Report

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for

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FORT EUSTIS, VIRGINIA

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ABSTRACT

Stress fields in the components of a multifiber, unidirectional composite that are caused by shear loading in axial directions are studied and solutions are presented. Computations for various volumetric contents and material constants are described, as well as experiments which showed good correlation with the theoretical results from this work. Further, it is shown that certain mathematical solutions normally used for such problems are not compatible with present boundary conditions, and that the problem becomes two-dimensional.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF ILLUSTRATIONS (Continued)

LIST OF SYMBOLS

$A = B\sqrt{3}$	length of the characteristic element
A_1, A_2	integration constants
a	radius of the fiber
a_j, b_j, c_k, d_k, a_m	integration constants
b	width of the characteristic element
B_1, B_2	integration constants
C_1, C_2	integration constants
G	composite shear modulus
G_f	fiber shear modulus
G_m	matrix shear modulus
Im	imaginary part of the preceding function
K_1, K_2, K_3	constants defined in equations (19) to (22) and (19) to (29)
ℓ	length of the fiber
P_i	Papkovich-Neuber potential connected with direction i
Re	real part of the preceding function
T	externally applied shear load
u_x, u_y, u_z	displacements in directions of the coordinate axis x,y,z
V_f	volume percentage of fibers in a composite
x, y, z	coordinate axis
α, β, γ	independent constants in the arguments of the Laplace solutions
γ	shearing angle
∂	partial derivative

LIST OF SYMBOLS (Continued)

δ	longitudinal displace due to shear
δ_{ij}	Kronecker delta
∇	Laplace operator
ϵ_{ij}	strain in plane perpendicular to the coordinate i , in direction of the coordinate j
ρ	radius of the fiber (used in the computer program only)
ν	Poisson's ratio
σ_{ij}	stress component on surface perpendicular to the coordinate i in direction of the coordinate j
τ	engineering notation for shear stress
φ	cylindrical coordinate angle defined in Figure 3
I	superscript refers to fiber
II	superscript refers to matrix

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INTRODUCTION

The problem of shear loading, together with axial and transverse loading, constitutes the general case of oblique loading of a composite. A previous contract, DA 44-177-AMC-441(T), resulted in the solution of the axial loading case of a finite-length, multifiber composite. The model selected was a multifiber composite of hexagonal arrangement, loaded in shear in such a manner as to represent an external attack on surfaces parallel to the fibers.

The computer program obtained takes into account the fiber and matrix modulus, the fiber radius, and the fiber volumetric content. To obtain the stresses, analytical solutions have been used and adjusted to the boundary conditions at the interfaces between fibers and matrix, to certain repeating symmetry conditions, and to other geometrical conditions characterizing the problem. The boundary conditions were matched in 368 points for each fiber and the hexagonally surrounding resin.

The composite longitudinal shear modulus obtained for a fibrous composite was compared with the results from previous work done in this field. A design formula has been derived.

TECHNICAL DISCUSSION

The geometrical arrangement of the reinforcements in the matrix is assumed to be such that a central fiber is surrounded by six symmetrically spread fibers which, in turn, are surrounded by six others. Figure 1 shows the plane geometry of the composite.

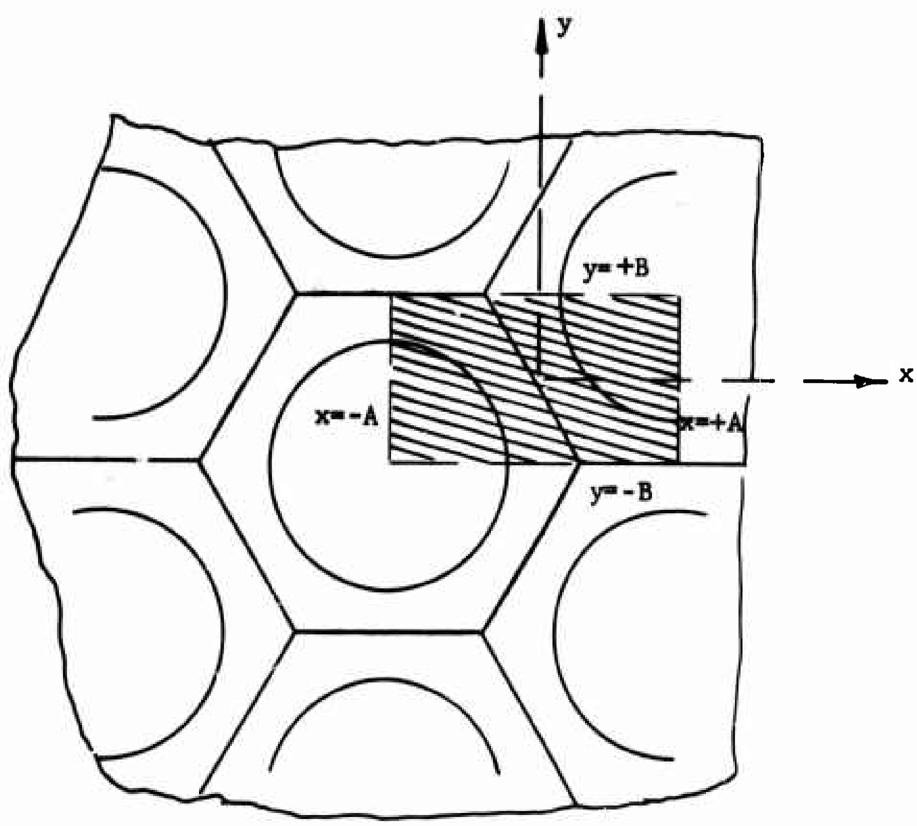


Figure 1. Plane Geometry of the Composite With the Characteristic Element Shaded.

Figure 2 shows a composite before and after shear deformation.

It is assumed that the material constants, such as modulus of elasticity and Poisson's ratio of both the reinforcement and the matrix, shall differ from one another. The transmission of loads from one material into the other is expressed by the boundary condition equations at the interfaces. The kinematic boundary conditions will be assumed as complete contact between the two materials so that no slippage will be possible.

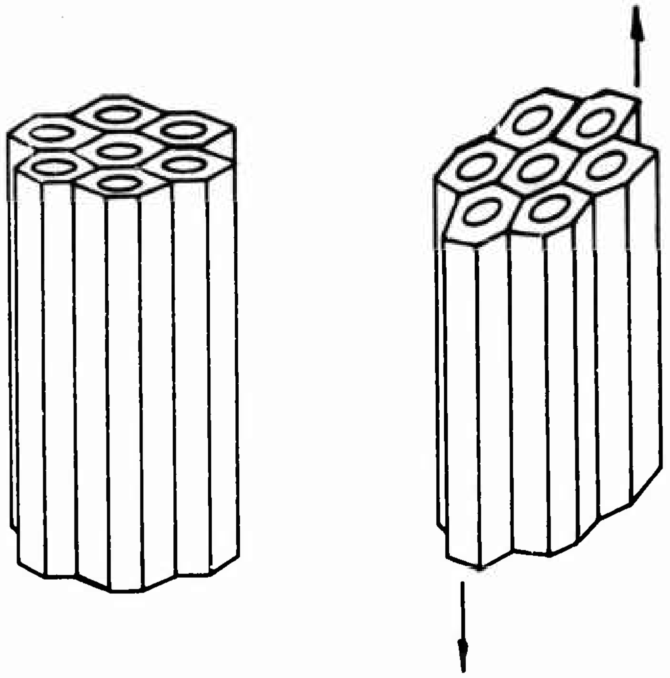


Figure 2. Composite Before and After Deforming.

As in the axial loading case,* an attempt was made to express the solutions of the Navier equations in terms of Neuber-Papkovitch potentials P_i . These potentials are defined as being solutions of the Laplace equation

$$\nabla^2 P_i = 0 \qquad (1)$$

where $i = 0,1,2,3$.

In equation (1), P_0 is scalar while the other terms are vector components. In rectangular coordinates, as shown in Figure 3, the displacement solutions of the Navier equation are

$$u_i = P_i - \frac{1}{4(1-\nu)} \frac{\partial}{\partial x_i} (x_j P_j + P_0) \qquad (2)$$

where $i = x,y,z$ and
$j = x,y,z$.

* Contract DA 44-177-AMC-441(T).

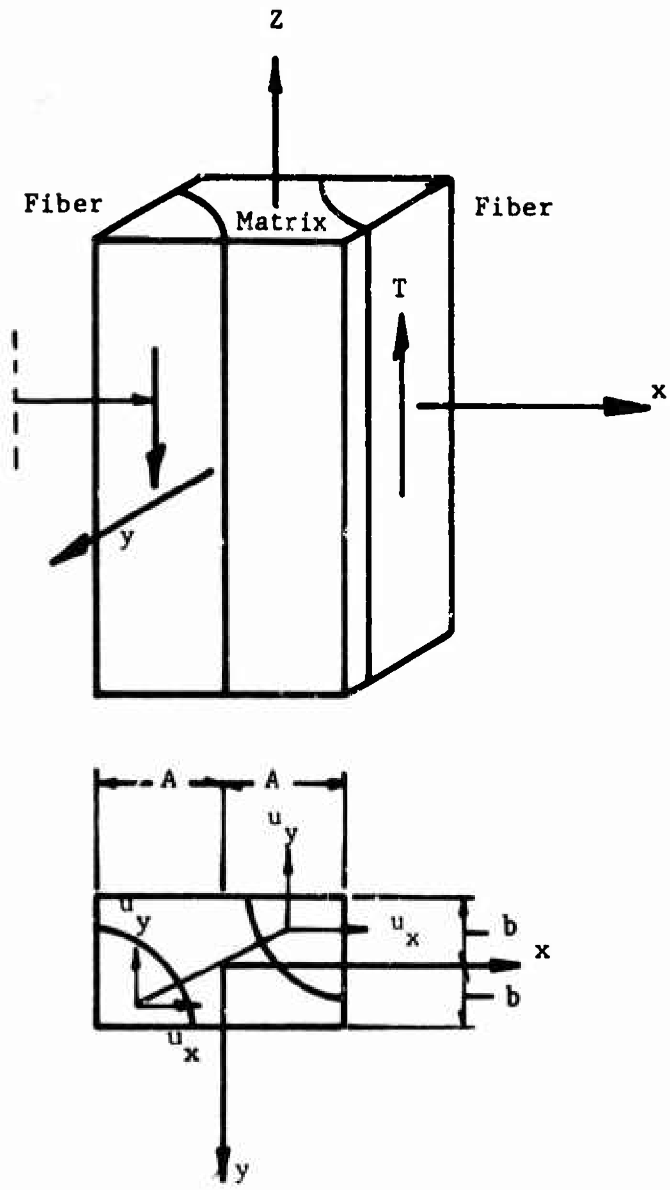


Figure 3. Constant From the Hexagonal System for Definition of the Coordinates Used in the Analysis and in the Computation.

Applying a shear T on the surface $x = \pm A$, one would assume that the following symmetry conditions would be adequate:

$$u_x\ (x,y,z) = -u_x\ (-x,-y,z) \tag{3}$$

$$u_x\ (x,y,z) = u_x\ (x,y,-z) \tag{4}$$

$$u_y(x,y,z) = -u_y(-x,-y,z) \quad (5)$$

$$u_y(x,y,z) = u_y(x,y,-z) \quad (6)$$

$$u_z(x,y,z) = -u_z(-x,-y,z) \quad (7)$$

$$u_z(x,y,z) = u_z(x,y,-z) \quad (8)$$

But because the solutions of equation (1) are of the form

$$P_i = A_i \cdot \frac{\sin}{\cos}(\alpha x) \cdot \frac{\sin}{\cos}(\beta y) \cdot \frac{\sinh}{\cosh}(\gamma z) \quad (9)$$

it is impossible to find in equation (9) a combination of solutions which satisfies the symmetry conditions represented by equations (3) through (8). On the basis of physical considerations alone, there can be no objection to these symmetry conditions. The problem, however, stems from the fact that, for large displacements, the surfaces $x = \pm A$ would try to come closer to each other during the application of the shear T. Since, in the theory of elasticity, only infinitesimally small displacements are taken into account, the surfaces $x = \pm A$ do not come closer while shear is applied. The interpretation of this apparent contradiction is that it is impossible to find an equilibrated and compatible solution which satisfies the symmetry conditions. In reference to the present problem, it can be stated that the solutions of equation (9) will never satisfy the symmetry conditions in equations (3) through (8).

In the following discussion, symmetry conditions will be presented which comply with the solutions of equation (9). Nevertheless, a prerequisite to further use of these symmetry conditions (which will now be compatible with the solutions of elasticity) is the determination of whether or not the proposed symmetry conditions are adequate for the problem. A set of symmetry conditions compatible with the solutions is given below.

$$u_x(x,y,z) = u_x(-x,-y,z) \quad (10)$$

$$u_x(x,y,z) = -u_x(x,y,-z) \quad (11)$$

$$u_y(x,y,z) = u_y(-x,-y,z) \quad (12)$$

$$u_y\ (x,y,z) = -u_y\ (x,y,-z) \quad (13)$$

$$u_z\ (x,y,z) = -u_z\ (-x,-y,z) \quad (14)$$

$$u_z\ (x,y,z) = u_z\ (x,y,-z) \quad (15)$$

Proceeding then with conventional procedures, we must select such combinations of equation (9) which, when introduced into equation (2), satisfy the conditions in equations (10) through (15). These combinations are presented below.

$$P_x = (A_1 \sin \alpha x \sin \beta y + A_2 \cos \alpha x \cos \beta y) \sinh \gamma z \quad (16)$$

$$P_y = (B_1 \sin \alpha x \sin \beta y + B_2 \cos \alpha x \cos \beta y) \sinh \gamma z \quad (17)$$

$$P_z = (C_1 \sin \alpha x \cos \beta y + C_2 \cos \alpha x \sin \beta y) \cosh \gamma z \quad (18)$$

$$P_0 = \text{constant}$$

In equations (16) through (18), A, B, C, α, and β are independent constants, while $\gamma^2 = \alpha^2 + \beta^2$. Additionally, from the geometry shown in Figures 1 and 3, it could be assumed that, at the surface $x = \pm A$, the displacements u_z in axial direction z are constant.

Also constant at this surface is the displacement in the x direction. The shear in the y direction in the plane $x = \pm A$ is zero. Consequently, we have the following boundary conditions at the planes $x = \pm A$:

$$u_z\ (A,y,z) = K_1 \quad (19)$$

$$u_z\ (-A,y,z) = -K_1 \quad (20)$$

$$u_x\ (A,y,z) = K_2 \quad (21)$$

$$u_x\ (-A,y,z) = -K_2 \quad (22)$$

$$\sigma_{xy}\ (A) = \sigma_{xy}\ (-A) = 0 \quad (23)$$

Using Figure 3, we can visualize that, at the surface $y = \pm B$,

$$\sigma_{yx}(x,B,z) = 0 \tag{24}$$

$$\sigma_{yx}(x,-B,z) = 0 \tag{25}$$

$$\sigma_{yz}(x,B,z) = 0 \tag{26}$$

$$\sigma_{yz}(x,-B,z) = 0 \tag{27}$$

$$u_y(x,B,z) = K_3 \tag{28}$$

$$u_y(x,-B,z) = -K_3 \tag{29}$$

Since u_x and u_y are antisymmetric functions in z (equations 11 and 13), then $K_2=0$, on the basis of equations (21) and (22), and $K_3=0$, from equations (28) and (29). Consequently, the only nonhomogeneous boundary condition is $u_z=K_1$. If only integral boundary conditions are imposed at both free ends, then these boundaries do not contribute to the solution of the problem; they are automatically satisfied because of the equilibrium conditions and because there are no applied forces of known distribution at that surface. Since the fiber's length is large compared with its diameter, it is permissible to assume that the displacements u_x , u_y , and u_z do not depend on z . This assumption is valid for the entire fiber and becomes invalid only on one or two fiber diameters from the fiber ends. (For the ends, a perturbation analysis will be performed later.) With this assumption, the displacements in equation (2) will be reduced to the following form:

$$u_x = P_x - \frac{1}{4(1-\nu)} \frac{\partial}{\partial x} (xP_x + yP_y) \tag{30}$$

$$u_y = P_y - \frac{1}{4(1-\nu)} \frac{\partial}{\partial y} (xP_x + yP_y) \tag{31}$$

$$u_z = P_z \tag{32}$$

From the displacements in equation (2) or, more specifically, in equations (30) through (32), the strains are obtained by the relation

$$\epsilon_{ij} = \frac{1}{2}\left[\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right] \tag{33}$$

Hook's law[1] furnishes the relation between stress and strain:

$$\sigma_{ij} = \frac{E}{1+\nu}\left\{\epsilon_{ij} + \delta_{ij}\frac{\nu}{1-2\nu}\epsilon_{kk}\right\} \tag{34}$$

From equations (33) and (34), we obtain

$$\sigma_{ij} = \frac{E}{2(1+\nu)}\left[\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} + \delta_{ij}\frac{2\nu}{1-2\nu}\frac{\partial u_k}{\partial x_k}\right] \tag{35}$$

In order to calculate the stresses in terms of Neuber-Papkovitch potentials, the expressions for the displacements given in equation (2) will be introduced into equation (35), as follows:

$$\sigma_{ij} = \frac{E}{2(1+\nu)}\left[\frac{\partial}{\partial x_i}\left\{P_j - \frac{1}{4(1-\nu)}\frac{\partial}{\partial x_j}(x_k P_k + P_0)\right\} + \frac{\partial}{\partial x_j}\left\{P_i - \frac{1}{4(1-\nu)}\frac{\partial}{\partial x_i}(x_\ell P_\ell + P_0)\right\} + \delta_{ij}\frac{2\nu}{1-2\nu}\frac{\partial}{\partial x_m}\left\{P_m - \frac{1}{4(1-\nu)}\frac{\partial}{\partial x_m}(x_n P_n + P_0)\right\}\right] \tag{36}$$

or

$$\sigma_{ij} = \frac{E}{2(1+\nu)}\left[\frac{\partial P_j}{\partial x_i} + \frac{\partial P_i}{\partial x_j} + \delta_{ij}\frac{2\nu}{1-2\nu}\frac{\partial P_m}{\partial x_m} - \frac{1}{2(1-\nu)}\frac{\partial^2}{\partial x_i \partial x_j}(x_k P_k + P_0) - \delta_{ij}\frac{2\nu}{4(1-2\nu)(1-\nu)}\frac{\partial^2}{\partial x_m \partial x_m}(x_n P_n + P_0)\right] \tag{37}$$

By using engineering notations σ_x, σ_y, σ_{xy}, and so forth, we can write equation (37) in another form:

$$\sigma_{xx} = \sigma_x = \frac{E}{2(1+\nu)}\left[\frac{\partial P_x}{\partial x} - \frac{1}{2(1-\nu)}\left(x\frac{\partial^2 P_x}{\partial x^2} + y\frac{\partial^2 P_y}{\partial x^2}\right) + \frac{\nu}{1-\nu}\frac{\partial P_y}{\partial y}\right] \tag{38}$$

$$\sigma_{yy} = \sigma_y = \frac{E}{2(1+\nu)}\left[\frac{\partial P_y}{\partial y} - \frac{1}{2(1-\nu)}\left(x\frac{\partial^2 P_x}{\partial y^2} + y\frac{\partial^2 P_y}{\partial y^2}\right) + \frac{\nu}{1-\nu}\frac{\partial P_x}{\partial x}\right] \tag{39}$$

$$\sigma_{zz} = \sigma_z = \frac{E}{2(1+\nu)}\frac{\nu}{(1-\nu)}\left[\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y}\right] \tag{40}$$

$$\sigma_{xy} = \sigma_{yx} = \frac{E}{2(1+\nu)}\left[\frac{1-2\nu}{2(1-\nu)}\left(\frac{\partial P_x}{\partial y} + \frac{\partial P_y}{\partial x}\right) - \frac{1}{2(1-\nu)}\left(x\frac{\partial^2 P_x}{\partial x \partial y} + y\frac{\partial^2 P_y}{\partial x \partial y}\right)\right] \tag{41}$$

$$\sigma_{xz} = \sigma_{zx} = \frac{E}{2(1+\nu)}\frac{\partial P_z}{\partial x} \tag{42}$$

$$\sigma_{yz} = \sigma_{zy} = \frac{E}{2(1+\nu)}\frac{\partial P_z}{\partial y} \tag{43}$$

Some important conclusions characterizing the shear problem can be made from the stress equations, (38) through (43) In these equations, (38) through (41) depend only on P_x and P_y, and (42) and (43) depend only on P_z. According to the theorem that a harmonic function is zero if the boundary conditions are completely homogeneous, the potentials P_x and P_y introduced into the homogeneous boundary conditions expressed in equations (21) through (27), keeping in mind that K_2 is also zero, must be zero. Therefore,

$$P_x(x,y,z) = P_y(x,y,z) = 0 \tag{44}$$

Consequently, during shear application the following displacements and stresses are zero, not only at the boundaries but also in the whole composite:

$$u_x = u_y = \sigma_x = \sigma_y = \sigma_z = \sigma_{xy} = 0 \qquad (45)$$

In conclusion, this problem must be recognized as two-dimensional, having only the following displacement and stresses:

$$u_z \, , \, \sigma_{xz} \quad \text{and} \quad \sigma_{yz}$$

Remembering the definition of the potentials P (equation 1) and their relationship with the displacements (equation 2) and the stresses (equations 42 and 43), the longitudinal shear problem can then be reduced to the solutions of the Laplace equation and their adjustment to the boundary conditions.

$$\nabla^2 P_z = 0$$

Furthermore, since $u_z = P_z$,

$$\nabla^2 u_z = 0 \qquad (46)$$

The expressions for the shear stresses are then expressed by the well-known relation

$$\sigma_{xz} = \frac{E}{2(1+\nu)} \frac{\partial u_z}{\partial x} \qquad (47)$$

$$\sigma_{yz} = \frac{E}{2(1+\nu)} \frac{\partial u_z}{\partial y} \qquad (48)$$

The displacements and stresses must satisfy certain symmetry and boundary conditions, which are restated in the following paragraphs.

SYMMETRY CONDITION

$$u_z(x,y) = -u_z(-x,-y) \tag{49}$$

BOUNDARY CONDITIONS

$$\sigma_{yz}(x,B,z) = 0 \tag{50}$$

$$\sigma_{yz}(x,-B,z) = 0 \tag{51}$$

$$u_z(A,y,z) = K_1 \tag{52}$$

$$u_z(-A,y,z) = -K_1 \tag{53}$$

Along the interfaces, the contact shear stresses must be continuous in both materials I and II (fiber and resin).

$$\sigma^{I}_{xz} \cos \varphi + \sigma^{I}_{yz} \sin \varphi = \sigma^{II}_{xz} \cos \varphi + \sigma^{II}_{yz} \sin \varphi \tag{54}$$

Expressed by equations (47) and (48), the boundary condition in equation (54) is

$$G^{I} \frac{\partial u_z}{\partial x} \cos \varphi + G^{I} \frac{\partial u_z}{\partial y} \sin \varphi = G^{II} \frac{\partial u_z}{\partial x} \cos \varphi + G^{II} \frac{\partial u_z}{\partial y} \sin \varphi \tag{55}$$

where

$$G^{I,II} = \frac{E^{I,II}}{2(1+\nu^{I,II})}$$

and where the superscripts I and II designate reinforcement and matrix respectively.

The first approach tried was the finite difference method, using a square net of 112 nodes, a Laplace operator of second-order approximation $[O(h^2)]$, and a linear interpolation along the interface curved boundary. The following problem appeared in the solution obtained for a test case with 65% reinforcement.

In the fiber, near the interface, the displacement function in the z direction u_z had a convex curvature with respect to the displacement in the resin. This is illogical, for the following reason: Close to the surface, where the transition takes place from one material into the other, the stress must increase rather than decrease, as a convex curvature would indicate. This is shown, in exaggerated form, in Figure 4.

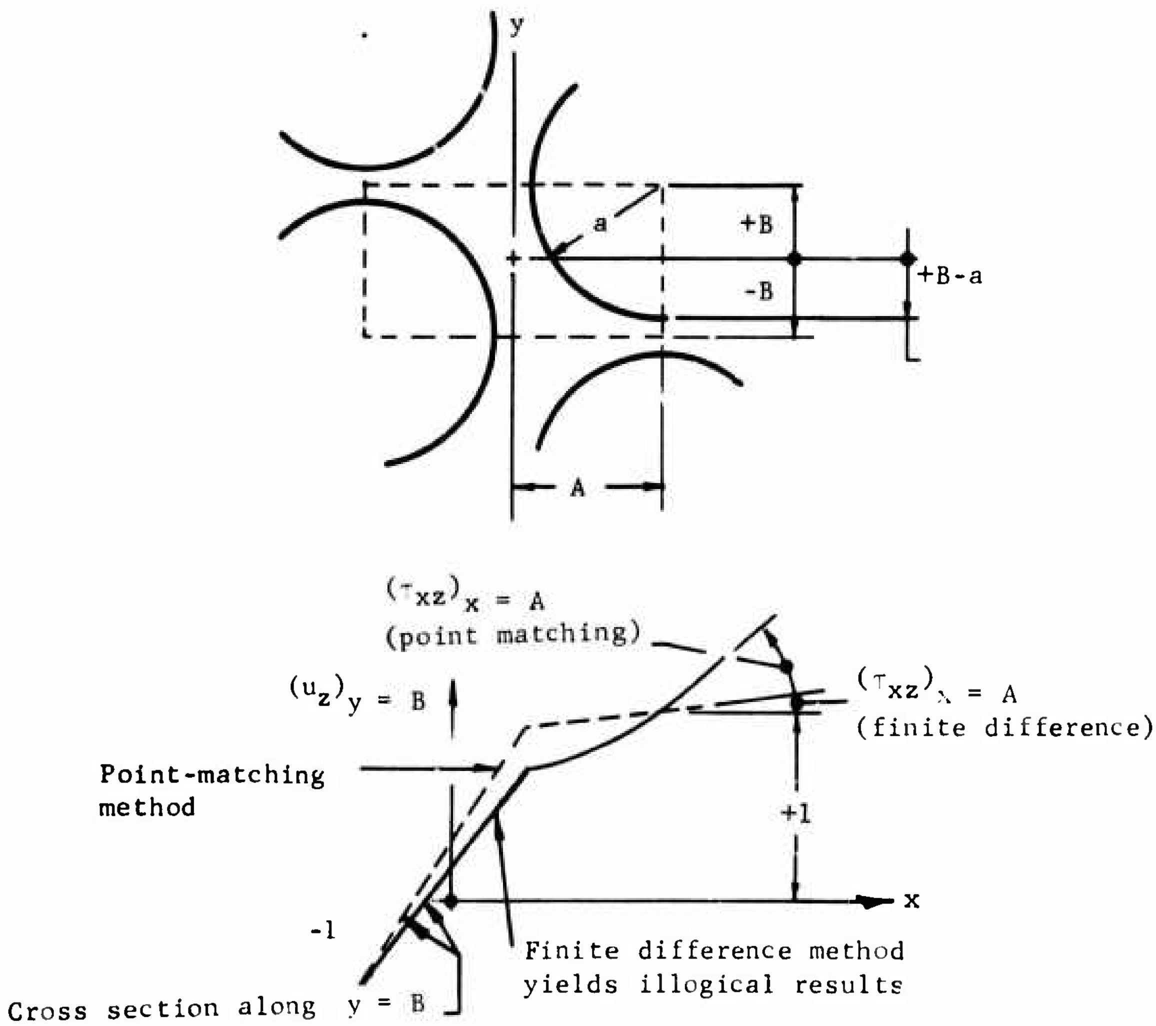


Figure 4. Shear Displacement in the Axial Direction, Obtained by the Finite Difference Method and Compared With the Point-Matching Method.

Since the modulus of elasticity is calculated from the relation

$$\sigma_{xz} = G \frac{\partial u_z}{\partial x} \tag{56}$$

that will say from the slope of the displacement u_z as shown later in equation 61, the two methods will give two different moduli of elasticity, corresponding to the slopes of the two curves in Figure 4, in $x = A$. The relation between the slope of the displacement in the reinforcement and that in the resin is obtained from the fact that, at the interface, the shear stresses are equal in both materials:

$$G^{I} \left(\frac{\partial u_z}{\partial x}\right)^{I} = G^{II} \left(\frac{\partial u_z}{\partial x}\right)^{II} \tag{57}$$

or

$$\frac{G^{I}}{G^{II}} = \frac{\left(\frac{\partial u_z}{\partial x}\right)^{II}}{\left(\frac{\partial u_z}{\partial x}\right)^{I}} \tag{58}$$

This relation can be very large in number. In composite materials, it ranges from 10 to 180; it is therefore difficult to represent, let alone satisfy, the boundary conditions at the interface with the finite difference method. In the point-matching method, different functions for both materials will be used.

METHOD OF SOLUTION

A boundary collocation (point matching) method is used. This method essentially consists of choosing a set of linearly independent functions which satisfy the differential equation exactly. The total solution comprises the solution of individual functions, each one multiplied by a free constant. These constants are determined by postulating that the boundary conditions at a certain number of points must be satisfied. If the number of boundary conditions is equal to the number of undetermined coefficients in the functions, a system of linear equations with exactly as many equations as unknowns will result. If, however, more boundary points are used in the equations than there are free constants, an "overdetermined" system of linear equations will arise, which can be solved by the classical least-squares method.

With boundary collocation methods, it is usually safer to specify more boundary conditions (about twice as many seems to be a good rule of thumb). This results in a smoother overall solution.

The success of any collocation method depends, to a large extent, on a suitable choice of the set of functions by which the solution is to be approximated. The following functions would satisfy both the Laplace equation and the symmetry condition:

$$w^{I}(x,y) = \sum_{m=1}^{\infty}\left[A_m^I \sin\left(\frac{m\pi}{A}x\right)\cosh\left(\frac{m\pi}{A}y\right)+ B_m^I \cos\left(\frac{m\pi}{A}x\right)\sinh\left(\frac{m\pi}{A}y\right)+\right.$$

$$\left. C_m^I \sinh\left(\frac{m\pi}{B}x\right)\cos\left(\frac{m\pi}{B}y\right)+ D_m^I \cosh\left(\frac{m\pi}{B}x\right)\sin\left(\frac{m\pi}{B}y\right)\right]+ E_0^I x + F_0^I y$$

and a similar function for w^{II} (all superscripts in the function are replaced by II).

These functions were programmed first, but their oscillatory nature did not allow a reasonable fit of the boundary conditions along the line $x = A$.

A second set of functions, which also satisfies the differential equations, the symmetry conditions, and, in addition, the boundary conditions along $y = \pm B$, is

$$u_z^I(x,y) = \sum_{m=1,2,3}^{\infty}\left[A_m^I \sinh\left(\frac{m\pi}{B}x\right)\cos\frac{m\pi}{B}y + B_m^I \cosh\left(\frac{(2m-1)\pi}{2B}x\right)\times\right.$$

$$\left.\sin\left(\frac{(2m-1)\pi}{2B}y\right)\right]+ E_0^I x$$

and a similar function for w^{II} .

These functions have the same basic oscillatory character and are therefore not suitable as a solution to the shear problem.

The functions which were finally chosen are polynomials, and $u_z^I(x,y)$ and $u_z^{II}(s,y)$ were assumed to be of the following form. These functions are mentioned because they seem to be, but are not, usable for the collocation method. The present case is one of the examples which has shown that not all the functions that satisfy the Laplace equation and the symmetry conditions can be used successfully in a collocation method.

$$u_z^I(x,y) = \sum_{j=1}^{j_{max}} a_j^I \, Re\,(x+iy)^{(2j-1)} + b_j^I \, Im(x+iy)^{(2j-1)} +$$

$$\sum_{k=1}^{k_{max}} c_k^I \, Re(x+iy)^{2k} + d_k^I \, Im(x+iy)^{(2k)} + E_o^I \qquad (59)$$

$$u_z^{II}(x,y) = \sum_{m=1}^{m_{max}} a_m^{II} \, Re(x+iy)^{(2m-1)} + b_m^{II} \, I_m(x+iy)^{2m-1} \qquad (60)*$$

The above expression for u_z^I does not satisfy the symmetry condition. Since, in the basic element, the two areas representing fibers are not connected, it is not necessary that the same function be used in the two fiber areas. The function u_z^{II}, which extends over a connected area both to the right and to the left of the y-axis and above and below the x-axis, must satisfy the symmetry condition $u_z^{II}(x,y) = -u_z^{II}(-x,-y)$. The above function has this property.

The appendix gives details on the boundary points where the boundary conditions were established.

* This function was proposed by Dr. George Burgin of Decision Sciences.

NUMERICAL RESULTS

Figure 5 represents the level lines of displacements due to shear for a 65% glass fiber composite.

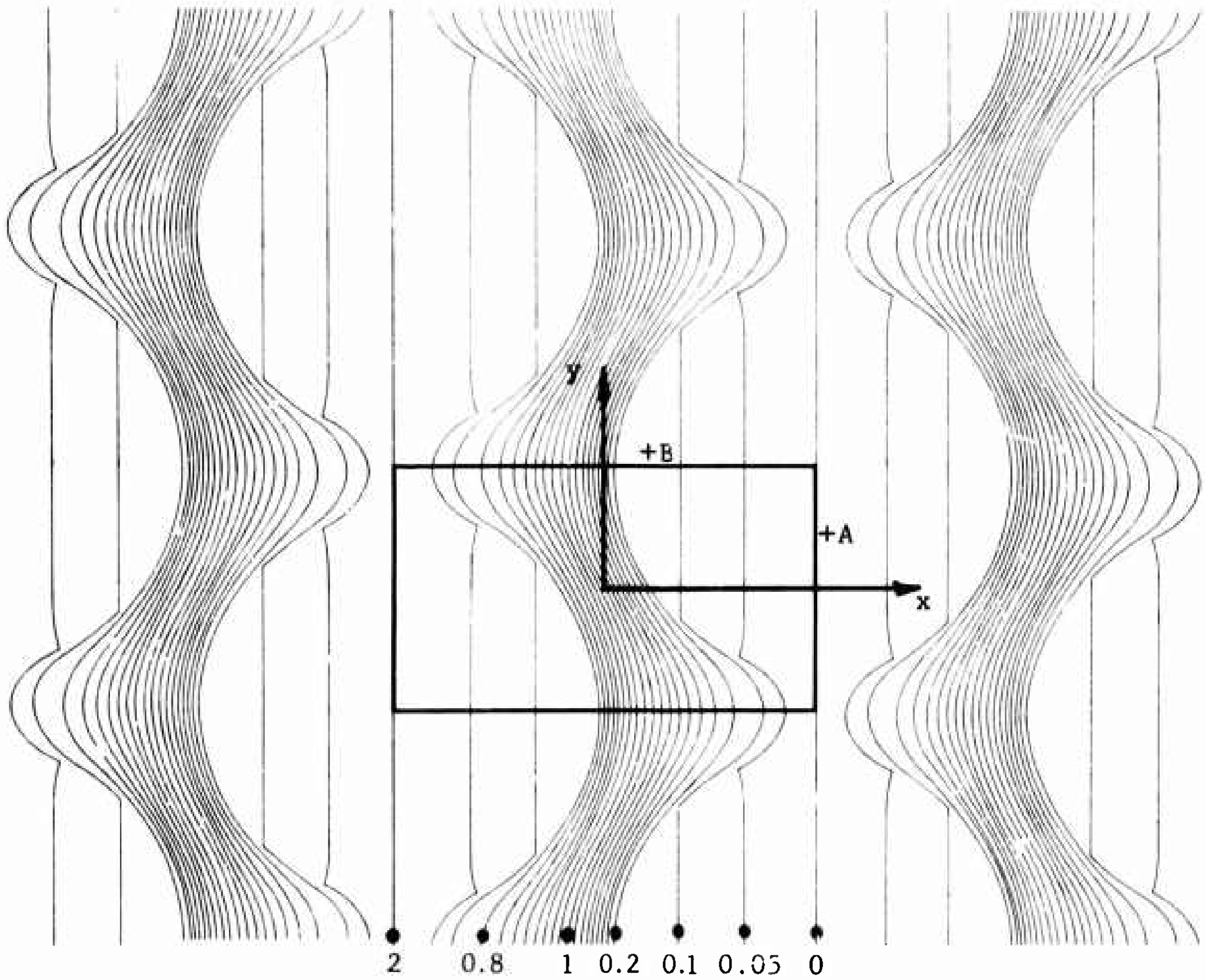


Figure 5. Level Lines of Displacements When Axial Shear is Applied at a Surface Parallel to the Surface ±A.

DISPLACEMENTS

Figures 6 through 8 show the displacements in a composite due to longitudinal shear, selected to produce ±1 displacement at the surfaces x = ±A .

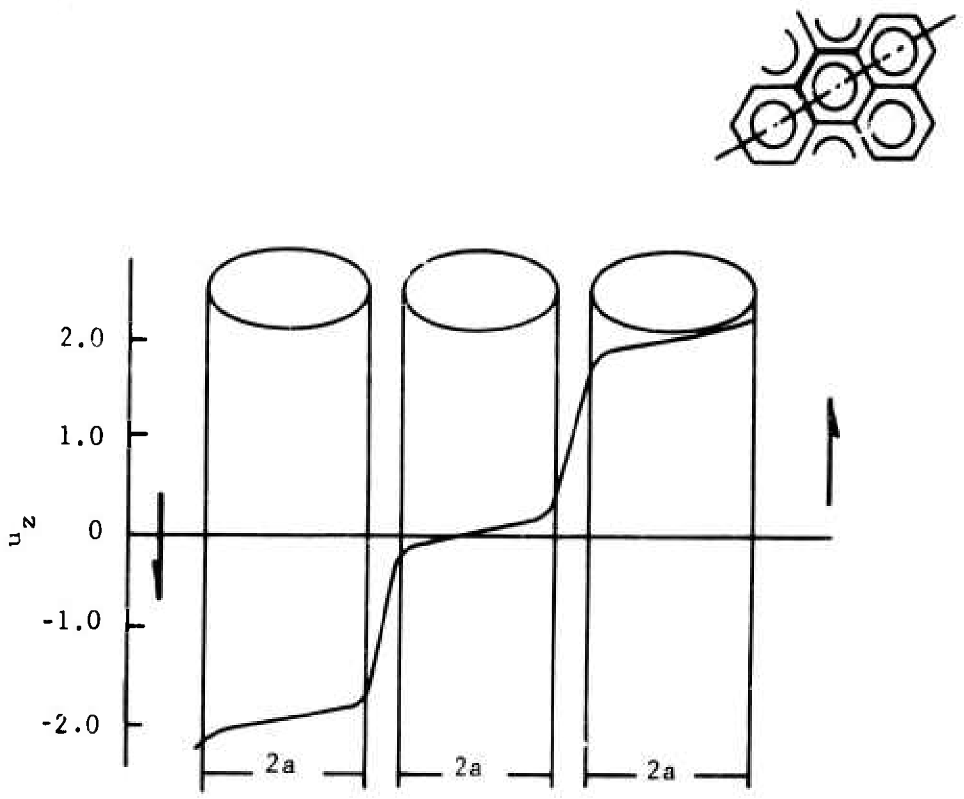


Figure 6. Displacement Functions due to Displacement ±1 at the Boundary $x = \pm A$.

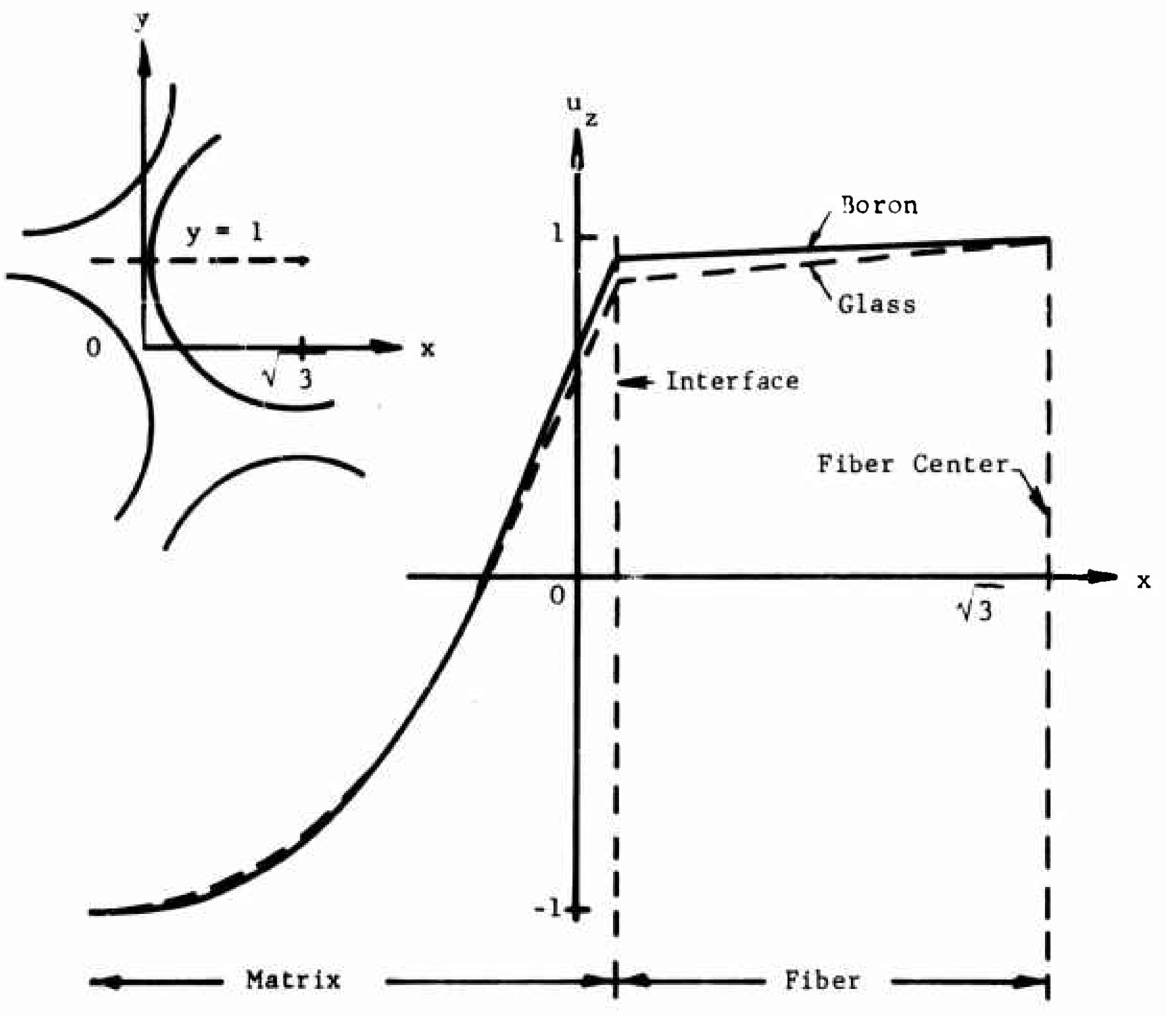


Figure 7. Axial Displacement Versus x-Axis at y = +1.

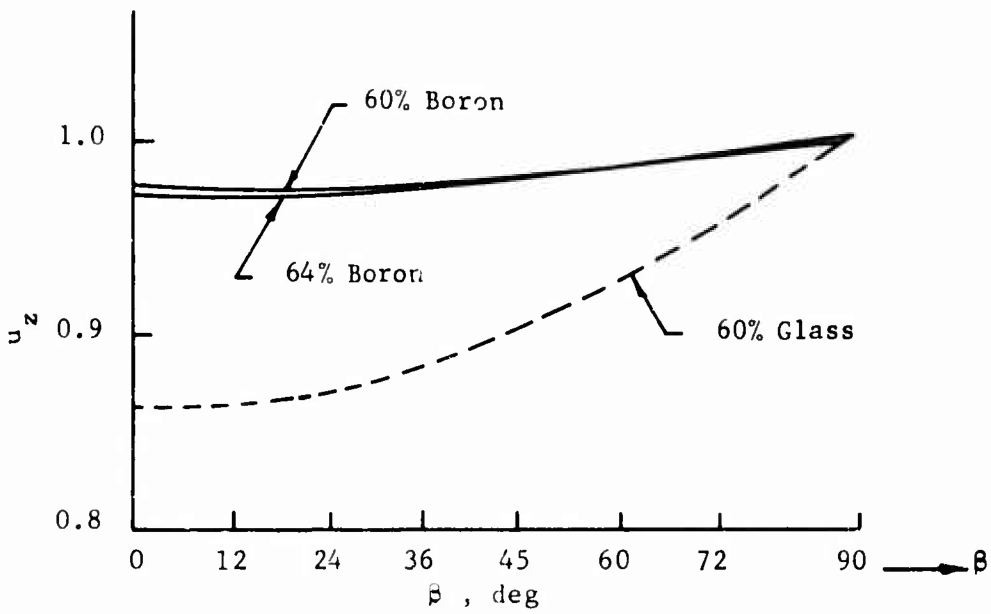


Figure 8. Axial Displacement at Interface due to Displacement ±1 at x = ±A.

STRESS DISTRIBUTIONS

Figures 9 through 12 show the stress distribution in the composite component that is caused by longitudinal shear at the surface $x = \pm A$. In these representations, the stresses generated are divided by the applied stress, so that the numbers shown are the shear stress concentrations. The applied stress is expressed by the average applied load at the boundary surface

$$\frac{T}{2B\ell} = \bar{\sigma}_{xz}(A) = \frac{1}{2B}\int_{-B}^{+B} (\sigma_{xz})_{x=A}\, dy \qquad (61)$$

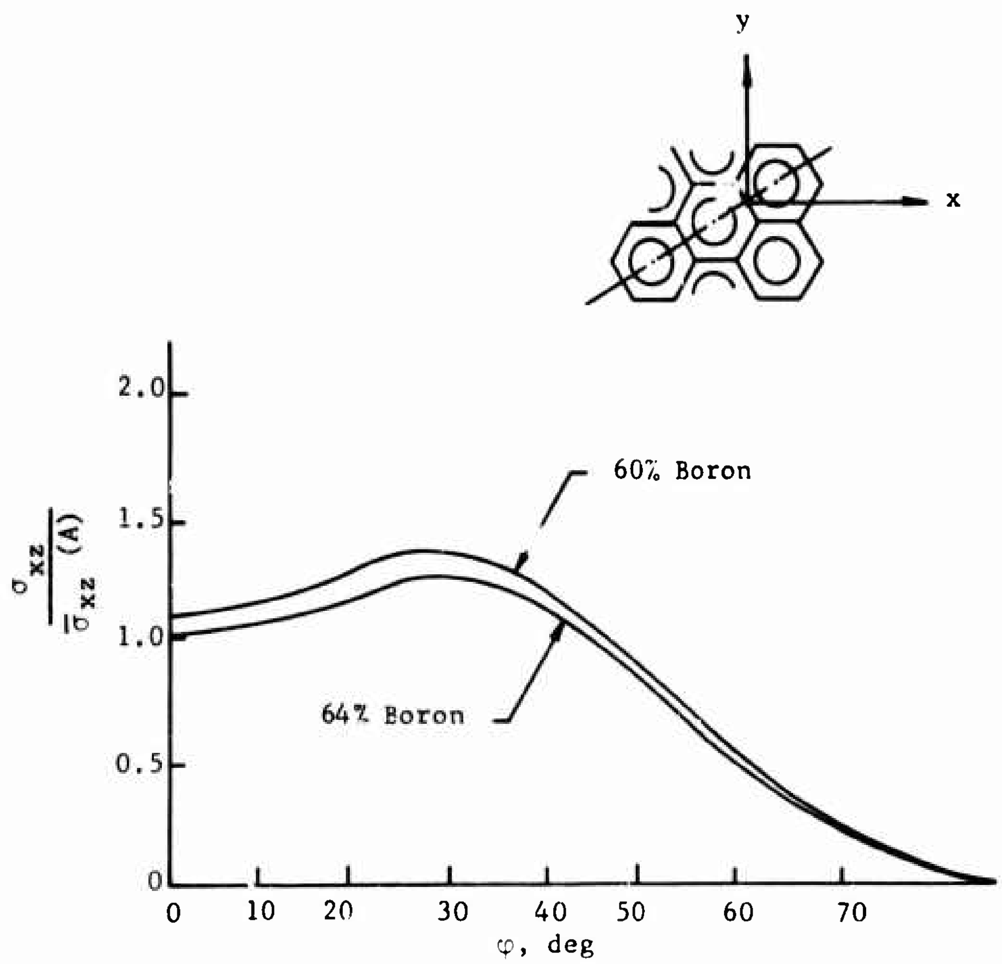


Figure 9. Shear Stress at the Interface due to Longitudinal Applied Shear Load $\bar{\sigma}_{xz}(A)$.

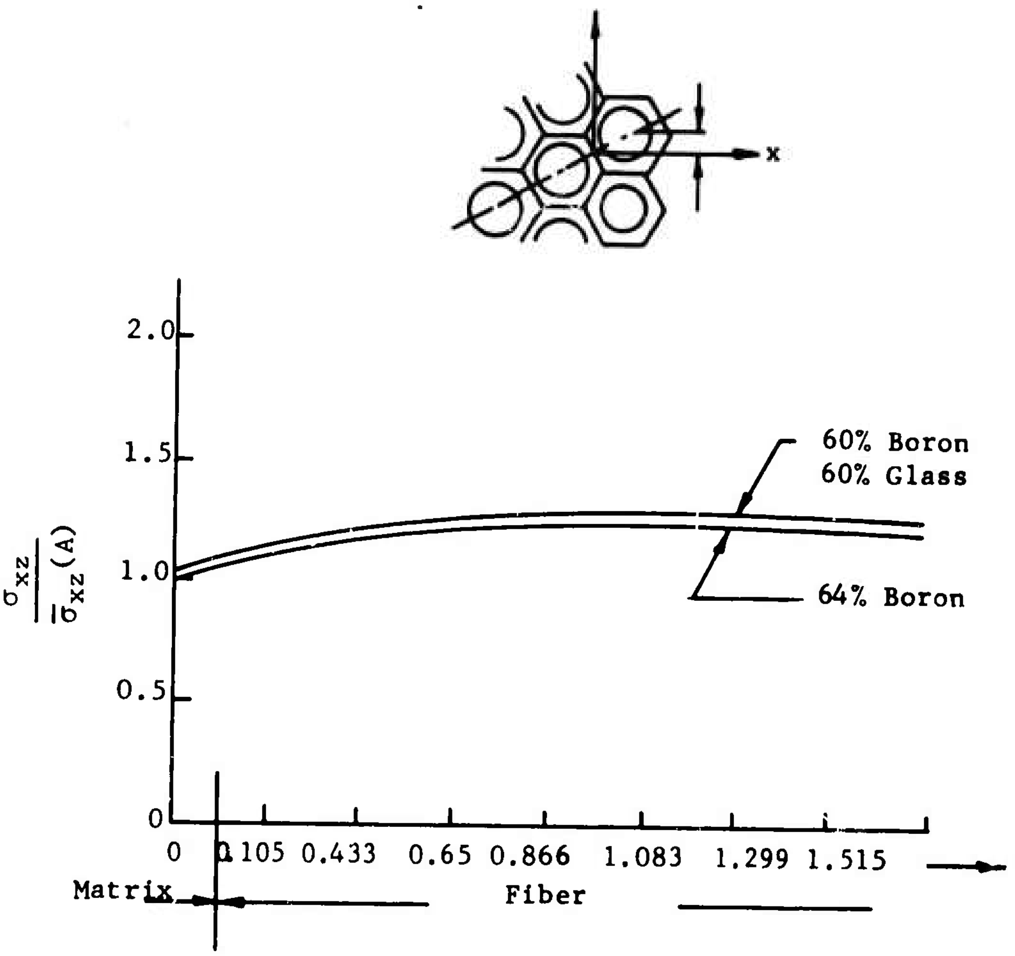


Figure 10. Axial Shear σ_{xz} as Function of x at y = 1.

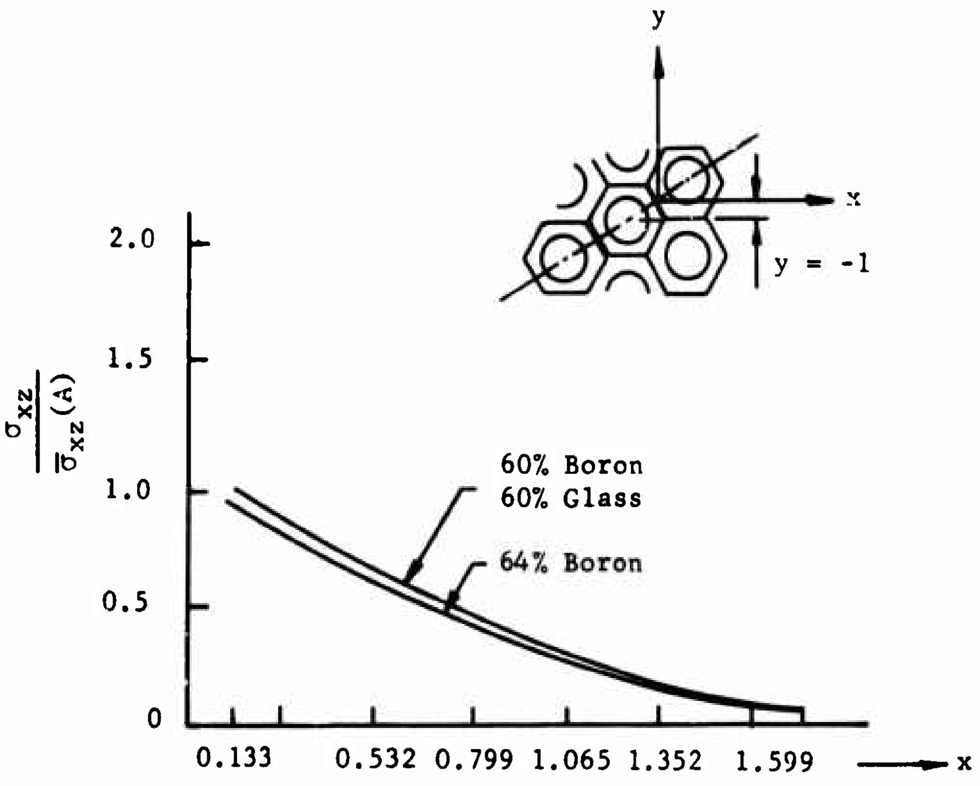


Figure 11. Shear Stress σ_{xz}^{II} as Function of x for y = -1.

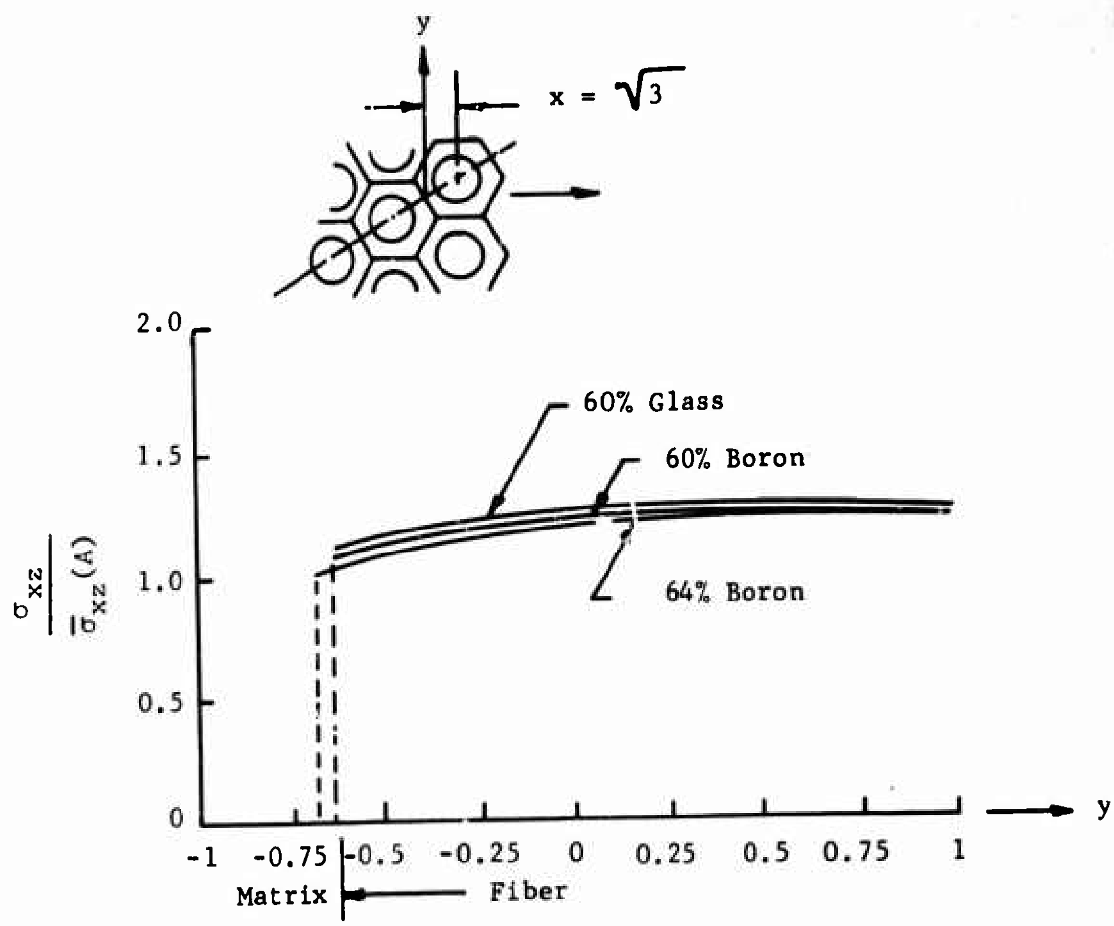


Figure 12. Axial Shear σ_{xz} as a Function of y at $x = \sqrt{3}$.

LONGITUDINAL SHEAR MODULUS OF A COMPOSITE

The total shear modulus of the composite can be calculated from the applied shear load necessary to produce the shearing angle γ . Expressing the total shear load T at the surfaces $x = \pm A$ by the shear stress produced at the surfaces in both materials, we obtain

$$T = \int_{-B}^{+B} (\sigma_{xz})_{x=A} \, \ell dy$$

$$= \int_{-B}^{B-a} (\sigma_{xz})^{II}_{x=A} \, \ell dy + \int_{B-a}^{B} (\sigma_{xz})^{I}_{x=A} \, \ell dy$$

$$= G^{II} \int_{-B}^{B-a} \left(\frac{\partial u_z}{\partial x}\right)^{II}_{x=A} \ell dy + G^{I} \int_{B-a}^{B} \left(\frac{\partial u_z}{\partial x}\right)^{I}_{x=A} \ell dy \qquad (62)$$

In addition to the expression in equation (62), the total shearing force of the composite can be expressed by the composite shear modulus G:

$$\frac{T}{2\ell B} = G\gamma = G \frac{\delta}{A} \qquad (63)$$

In a hexagonal arrangement,

$$A = B\sqrt{3}$$

or (64)

$$G = \frac{1}{2}\sqrt{3} \frac{T}{\delta \ell}$$

Consequently, the composite shear modulus is

$$G = \frac{1}{2\delta}\sqrt{3} \left\{ G^{II} \int_{-B}^{B-a} \left(\frac{\partial u_z}{\partial x}\right)^{II}_{x=A} dy + G^{I} \int_{B-a}^{B} \left(\frac{\partial u_z}{\partial x}\right)^{I}_{x=A} dy \right\} \qquad (65)$$

From the computer program, as described in the Appendix, the composite shear modulus G has been presented as the function of the fiber modulus G_f, the matrix modulus G_m, and the volume percentage of the fiber V_f (see Figure 13).

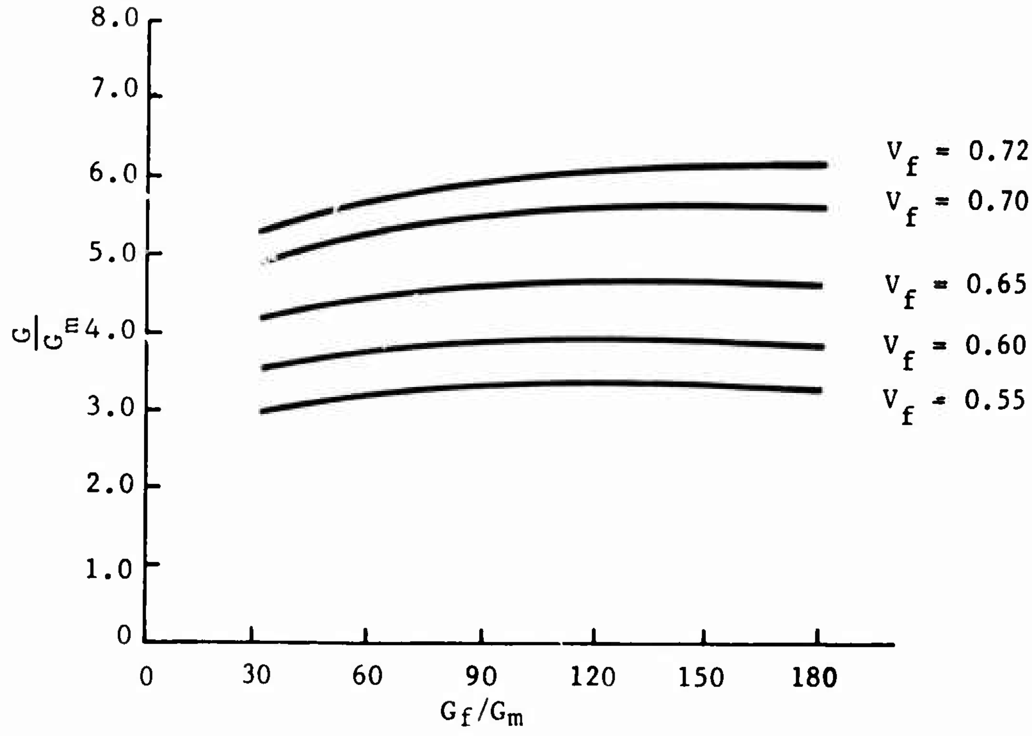


Figure 13. The Longitudinal Shear Modulus of a Unidirectional Composite as Computed From the Computer Program.

In the following test, the computer results are compared with the results of the approximation formula by Rosen, Foye, Ekvall and Berg,[2-5] the computer results of Tsai,[6] and the test results of Adams and Doner[7]. The results calculated from Rosen's formula[2]

$$\frac{G}{G_m} = \frac{\frac{G_f}{G_m}(1 + V_f) + (1 - V_f)}{\frac{G_f}{G_m}(1 - V_f) + (1 + V_f)}$$

agree quite well with our computer results except when $V_f = 0.8$. Among all existing formulas, this one is the most exact compared with our computer results (see Figure 14). The results of Foye's formula[3]

$$\frac{G}{Gm} = \frac{\frac{G_f}{G_m}}{V + (1 - V)\frac{G_f}{G_m}}$$

are 28%-34% lower than our computer results.

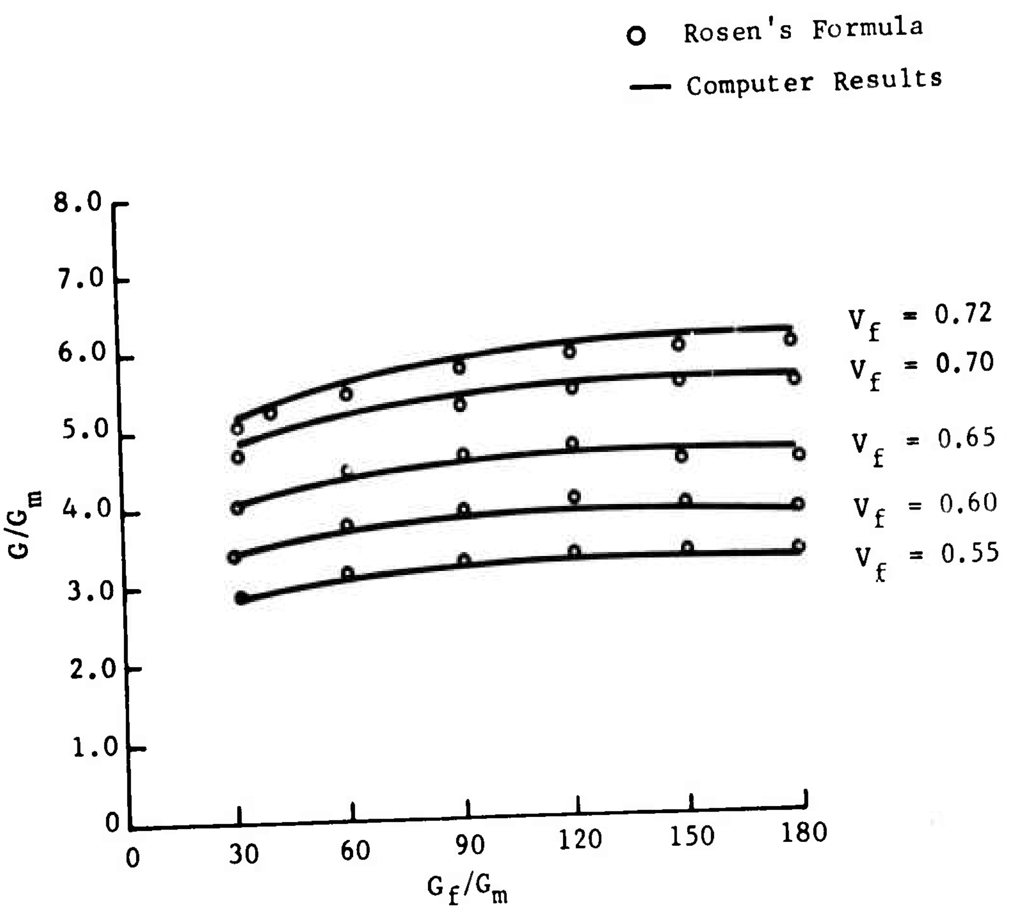


Figure 14. Computer Results Compared With Formula by Rosen.

The results from the equation given by Ekvall[4]

$$\frac{G}{G_m} = \frac{\dfrac{G_f}{G_m}}{\dfrac{R}{L}\,\Phi + \left(1 - \dfrac{R}{\ell}\right)\dfrac{G_f}{G_m}}$$

where

$$\frac{R}{\ell} = 1.05\sqrt{V_f}$$

and

$$\Phi = \int_0^{\pi/2} \frac{\sin\theta\, d\theta}{\dfrac{G_m}{G_f} + \sin\theta\left(1 - \dfrac{G_m}{G_f}\right)}$$

$$= \frac{\pi}{2\left(1 - \dfrac{G_m}{G_f}\right)} - \frac{\dfrac{G_m}{G_f}}{\left(1 - \dfrac{G_m}{G_f}\right)\sqrt{\left(1 - \dfrac{G_m}{G_f}\right)^2 - \left(\dfrac{G_m}{G_f}\right)^2}} \times$$

$$\ell n \frac{1 - \sqrt{\left(1 - \dfrac{G_m}{G_f}\right)^2 - \left(\dfrac{G_m}{G_f}\right)^2}}{1 + \sqrt{\left(1 - \dfrac{G_m}{G_f}\right)^2 - \left(\dfrac{G_m}{G_f}\right)^2}} - \ell n \frac{1 - \dfrac{G_m}{G_f} - \sqrt{\left(1 - \dfrac{G_m}{G_f}\right)^2 - \left(\dfrac{G_m}{G_f}\right)^2}}{1 - \dfrac{G_m}{G_f} + \sqrt{\left(1 - \dfrac{G_m}{G_f}\right)^2 - \left(\dfrac{G_m}{G_f}\right)^2}}$$

are as much as twice the results of our program.

The results from Berg's formula[5]

$$\frac{G}{Gm} = \frac{G_f}{G_m}\left[\frac{\pi}{2B} - \frac{2A}{B\sqrt{A^2 - B^2}}\tan^{-1}\frac{(A-B)}{\sqrt{A^2 - B^2}}\right] + 1 - 1.2125\sqrt{V_f}$$

where

$$A = 0.8247\sqrt{\frac{1}{V_f}\left(\frac{G}{G_m}\right)}$$

and

$$B = 0.866\left(1 - \frac{G_f}{G_m}\right)$$

are about 10%-25% below our computer results. Tsai's computer results[6] are 27%-67% higher (Figure 15).

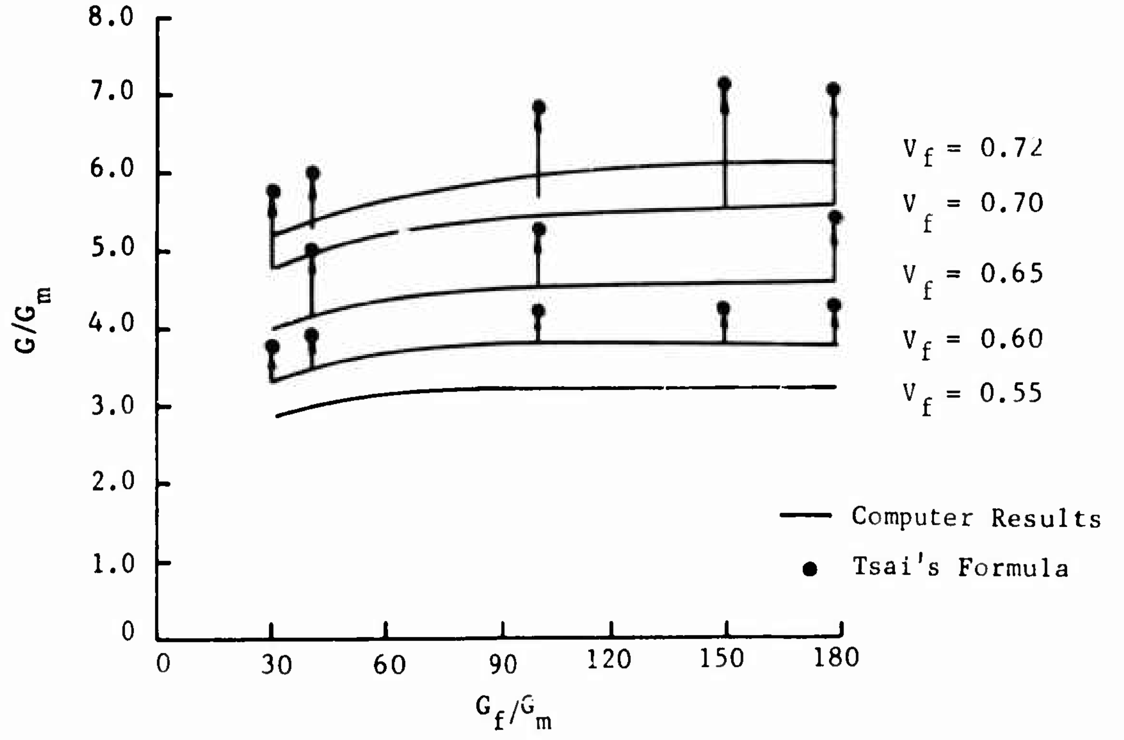


Figure 15. Computer Results Compared With Formula by S. Tsai.

Figure 16 is a comparison of our computer results with the results of Foye's formula.[3]

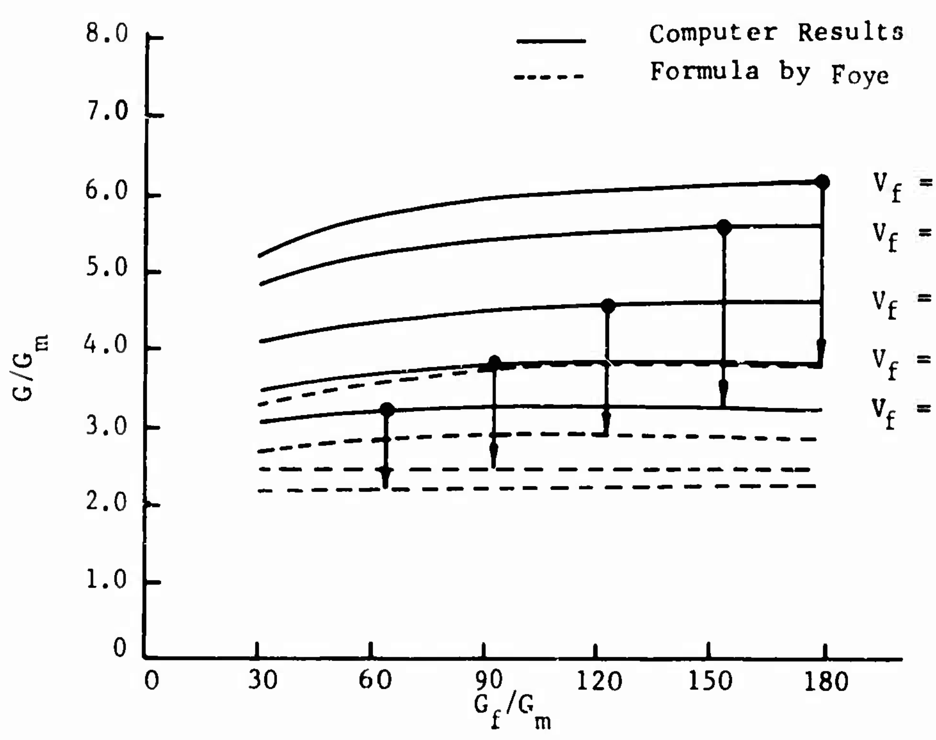


Figure 16. Computer Results Compared With Formula by Foye.

Test results obtained by Adams and Doner[7] are slightly higner but very close to the computer results obtained under this program (Figure 17).

An approximate equation developed from the present computer program is

$$\frac{G}{Gm} = (11.78\, V_f^{\,2} - 13\, V_f + 3.78)\, \ln\left(\frac{G_f}{G_m}\right) + 6.66\, V_f - 1.27 \qquad (66)$$

Figure 18 shows equation (66) in comparison with the computer results obtained in this work.

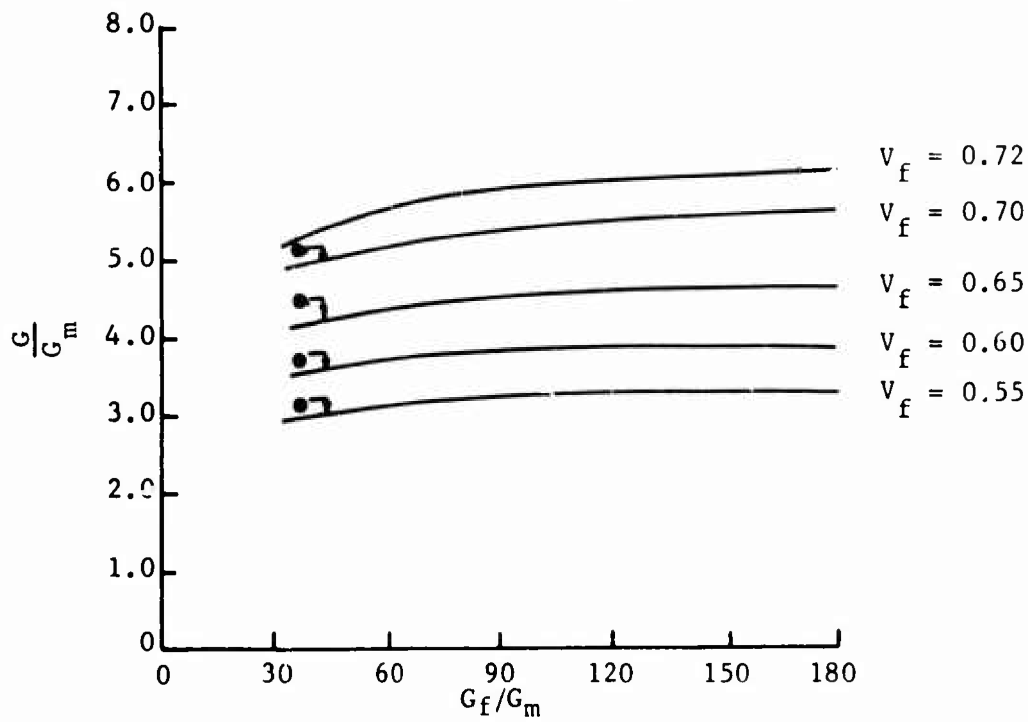


Figure 17. Computer Results Compared With Test Results From Adams and Doner.

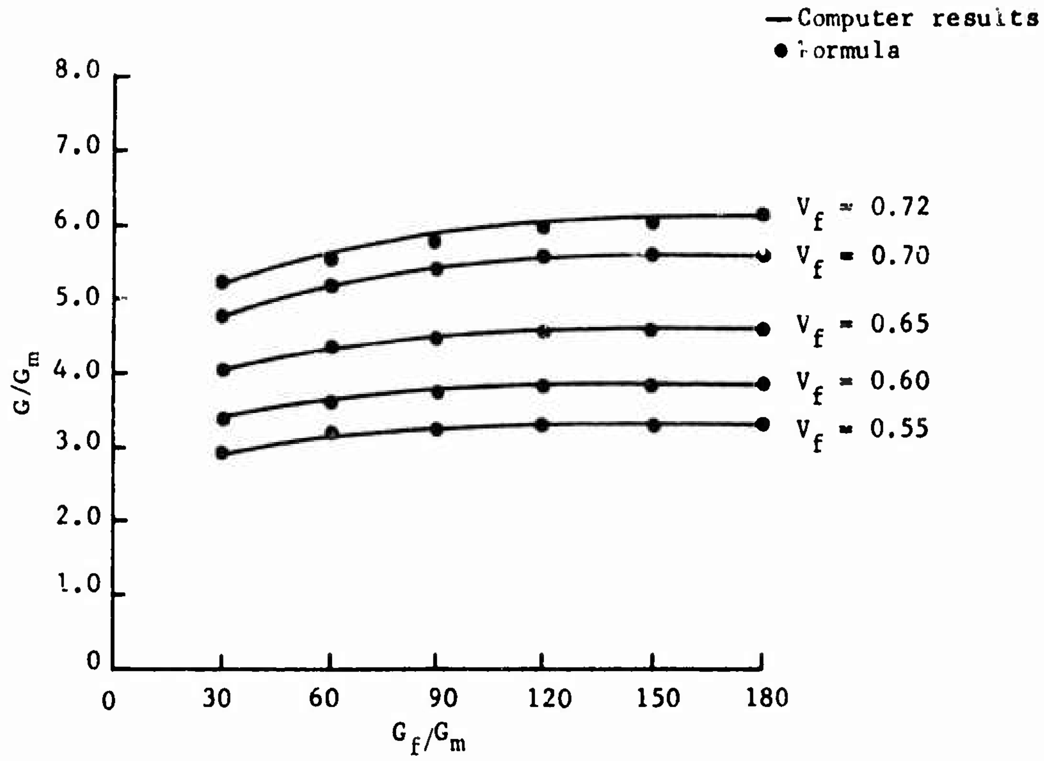


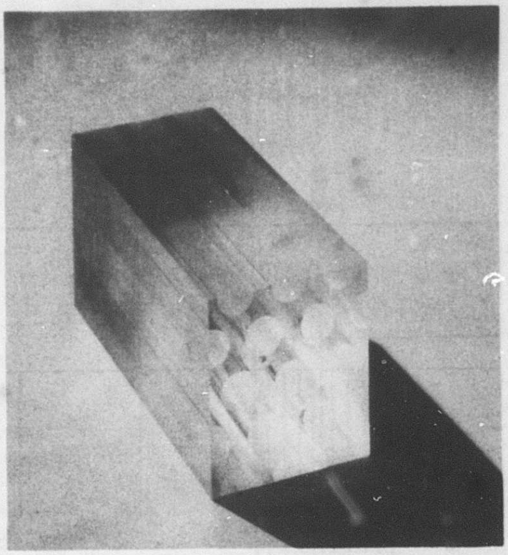
Figure 18. Approximation Formula (66) Developed During This Contract Compared With the Computer Results.

EXPERIMENTS

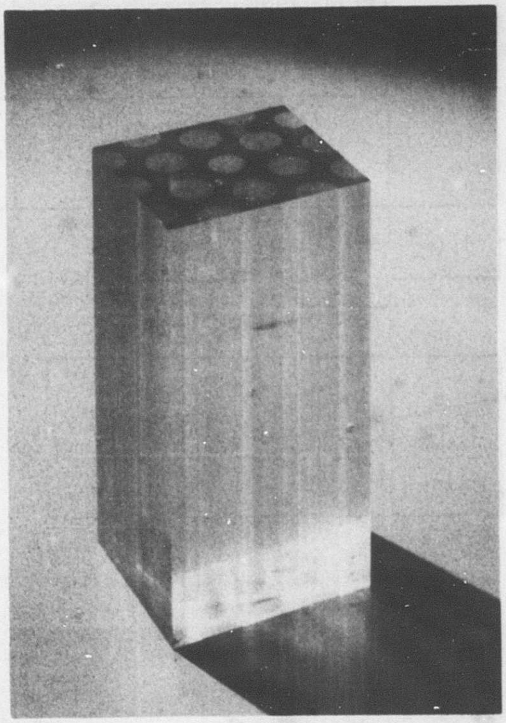
The experimental portion of this contract was conducted primarily to provide clarification of the stress concentrations at the end of the fibers.

For this purpose, two test specimens such as those shown in Figure 19 were fabricated. The resin used was epoxy type. Glass fibers measuring 0.28 inch in diameter were embedded, in a hexagonal array, into the resin. Using an epoxy resin as the adhesive, the specimen was attached to a steel test frame, as illustrated in Figure 20. With this frame, shear loads could be applied to the specimen. The specimen was analyzed by photoelastic methods, first without load and then with incrementally increasing loads. Figures 21 and 22 show the specimen illuminated by polarized light, in the unloaded state and loaded with 800 pounds of pressure, respectively. It is possible to see only small stress concentrations at the end of the fibers. The depth of this change in the stress is, more or less, two fiber diameters along the axis.

In conclusion, the stress analysis with its assumptions made for long fibers, as developed during this contract, is justified. The results are exact along the fiber, with the exception of a very small part close to the end. In practical use, the length of this region (no more than four fiber diameters) is about 1/2000 for the shortest fiber. Therefore, the influence of stress concentrations at the fiber end is negligible for the computation of composite shear modulus.



(a)



(b)

Figure 19. Shear Test Specimen.

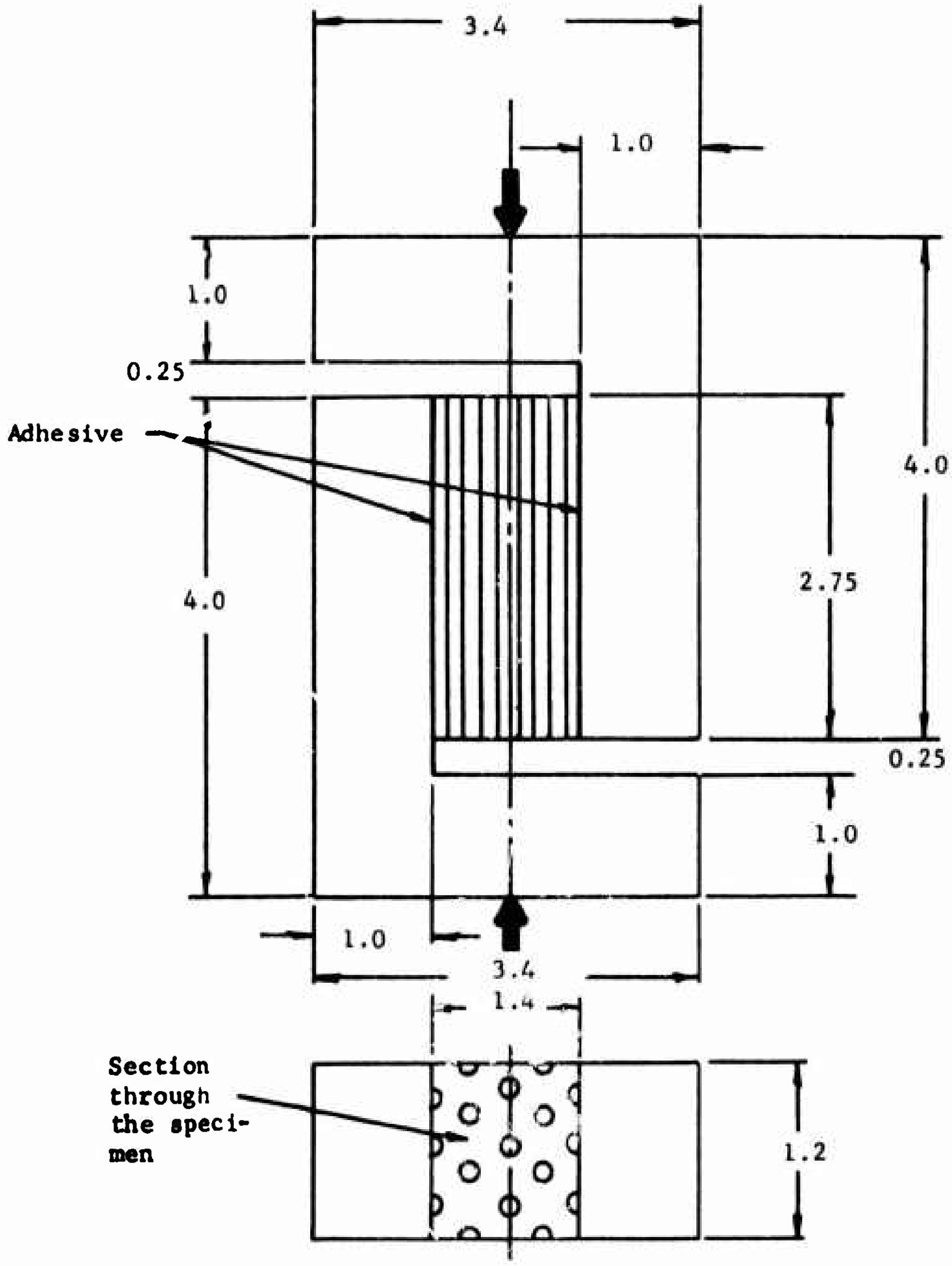


Figure 20. Test Picture Frame for the Shear Test.

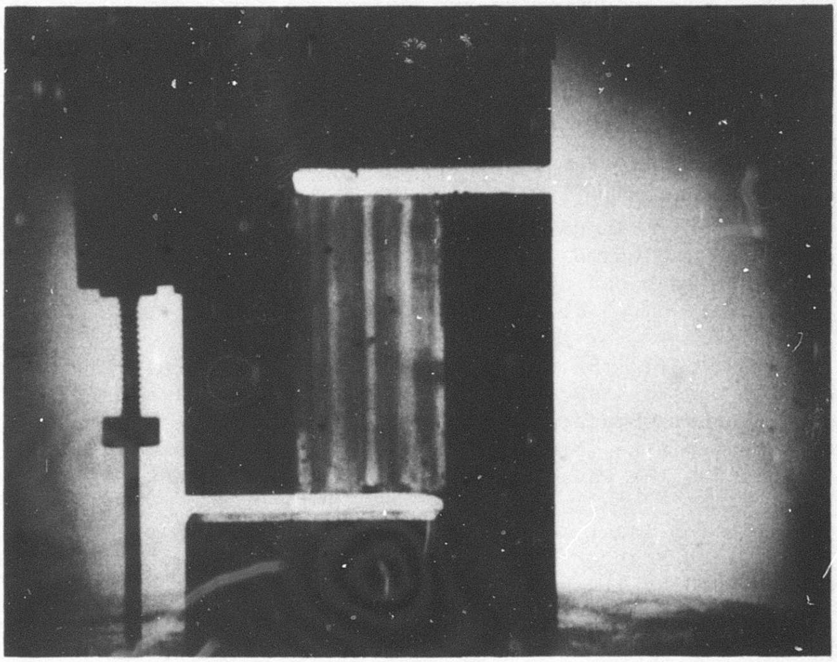


Figure 21. Unloaded Test Specimen.

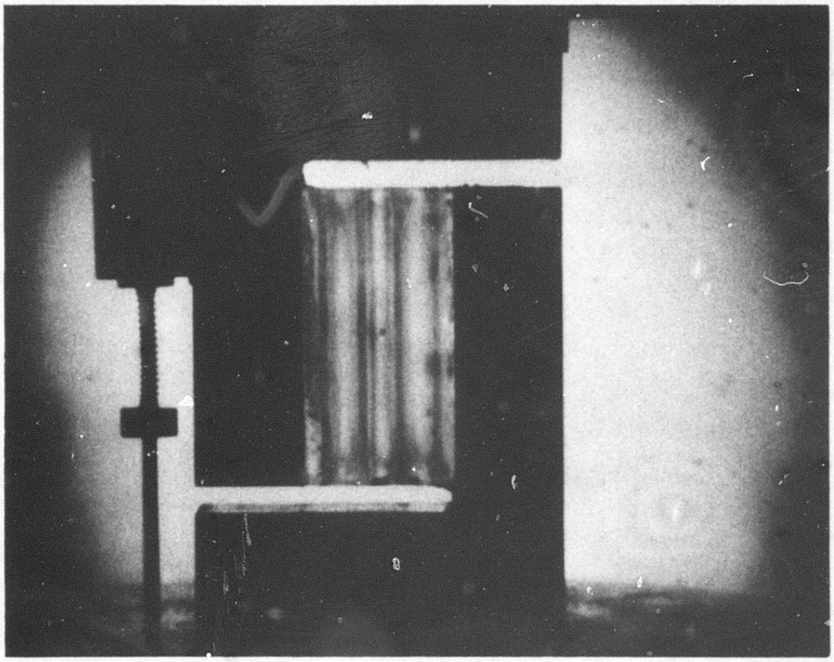


Figure 22. Loaded Test Specimen.

CONCLUSIONS AND RECOMMENDATIONS

It appears, from this analysis of a composite material subjected to longitudinal shear loading, that micromechanical stresses occur which are larger than the stresses applied to the composite as a whole. This report illustrates, for example, that the stress σ_{nz} along the interface is 35% larger than the applied average stress $\bar{\sigma}_{xz}(A)$ for an angle $\phi = 28^{\circ}$ (reference Figure 9). In other words, a stress concentration exists which is induced by the inclusion of fibers in the matrix. This form of stress concentration can be known only by using the micromechanical analysis based on the theory of elasticity, as used in this report.

For the most part, the shear modulus of elasticity obtained by this micromechanical analysis correlated well with test results. It must be noted, however, that obtaining the shear modulus by experimental methods (torsioned tubes, plates, etc.) is difficult, from the standpoint of accuracy. This was the cause of the variation in the results reported.

The formula for shear modulus which emerged from this work yields a high degree of accuracy for most composites used in structures. This formula can be used easily by the designer.

Once the internal behavior of composite materials under load is understood, then it will be possible to attain material optimization by judiciously selecting the most adequate combination of geometry and mechanical properties of the fibers and the matrix. This contract effort was directed to this end. Work is being continued on the micromechanics program in the area of other forms of loading, such as transverse loading.

REFERENCES CITED

1. W. Flugge, Handbook of Engineering Mechanics, McGraw-Hill, New York, 1962, p. 33-35

2. B. W. Rosen, N. F. Dow, and Z. Haskin, Mechanical Properties of Fibrous Composite, NASA CR-31, Contract NAS-470, General Electric Company, April 1964

3. R. L. Foye, "Compression Strength of Unidirectional Composite," Paper No. 66-143, presented at the 1966 AIAA Conference, 24-26 January 1966

4. J. C. Ekvall, Elastic Properties of Orthotropic Monofilament Laminates, ASME 61-AY-56, March 1961

5. K. R. Berg, "The Effect of Fiber Spacing in Composites on Modulus of Laminate Configurations Used in Aerospace Structures," Paper presented at the 8th Annual Structures, Structural Dynamics, and Materials Conference of the AIAA/ASME, March 1967

6. S. W. Tsai, D. F. Adams, and D. R. Doner, Analysis of Composite Structures, NASA CR-620, November 1966

7. D. F. Adams, D. R. Doner, R. L. Thomas, Mechanical Behavior of Fiber-Reinforced Composite Materials, AFML-TR-67-96, Air Force Materials Laboratory, Wright-Patterson Air Force Base, Ohio, May 1967

APPENDIX

NUMERICAL CALCULATIONS AND COMPUTER PROGRAM

The program presently uses the following upper summation limits:

$$JMAX = KMAX = MMAX = 10$$

This gives a total number of 61 unknown coefficients. Ninety-one conditions at the boundary points are specified, so that a linear system of 91 equations with 61 unknowns must be solved.

The above expression for u_z^I equation (59) does not satisfy the symmetry condition. Since in the basic element the two areas representing fibers are not connected, it is not necessary that the same function be used in the two fiber areas. The function w^{II}, which extends over a connected area both to the right and to the left of the y-axis and above and below the x-axis, has to satisfy the symmetry condition $w^{II}(x,y) = -w^{II}(-x,-y)$. The function given in equation (60) has this property.

Figures 23 and 24 show the points u: ' r the boundary collocation for the two different cases $a<A$ and $a>A$. Tables I and İ show the 91 equations generated for both cases.

DESCRIPTION OF THE PROGRAM

The program is written in FORTRAN 63 (similar to FORTAN IV) for a Control Data Computer Model 3600. It consists of approximately 1200 FORTRAN statements. The following general flow chart gives an idea about the options available in the program.

THE SUBROUTINES AND THEIR PURPOSES

ALBERTO — This is the main program. It reads and prints the input data. It calls the necessary subroutines to calculate the equations for the coefficients, to solve these equations, to calculate displacements and stresses, and to calculate the ratio of the two shear moduli

$$G_{Matrix}/G_{Total}$$

GETMTRX — This routine generates equations at 91 boundary points. Uses subroutine power.

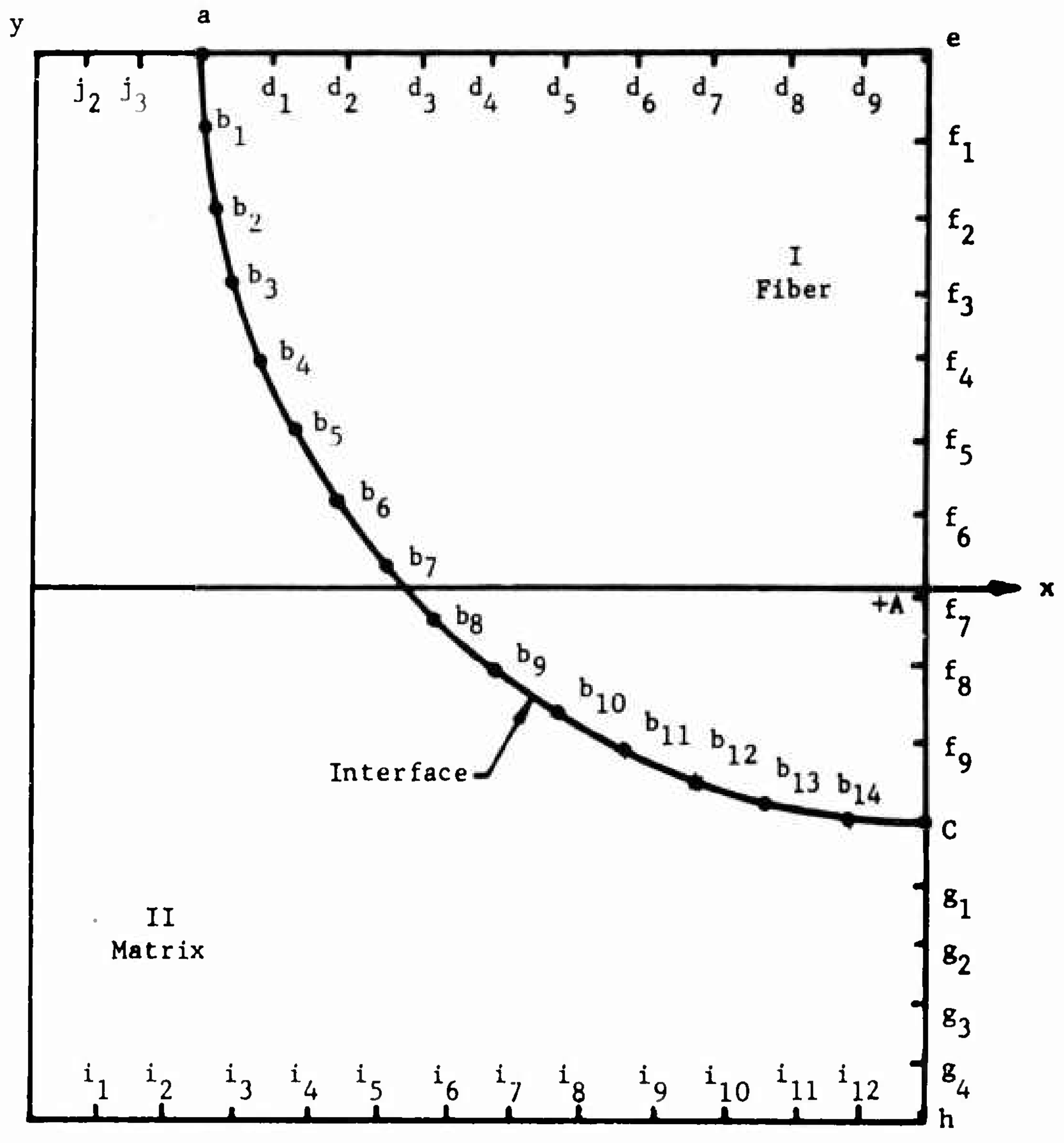


Figure 23. Arrangement of Boundary Points for $a < A$.

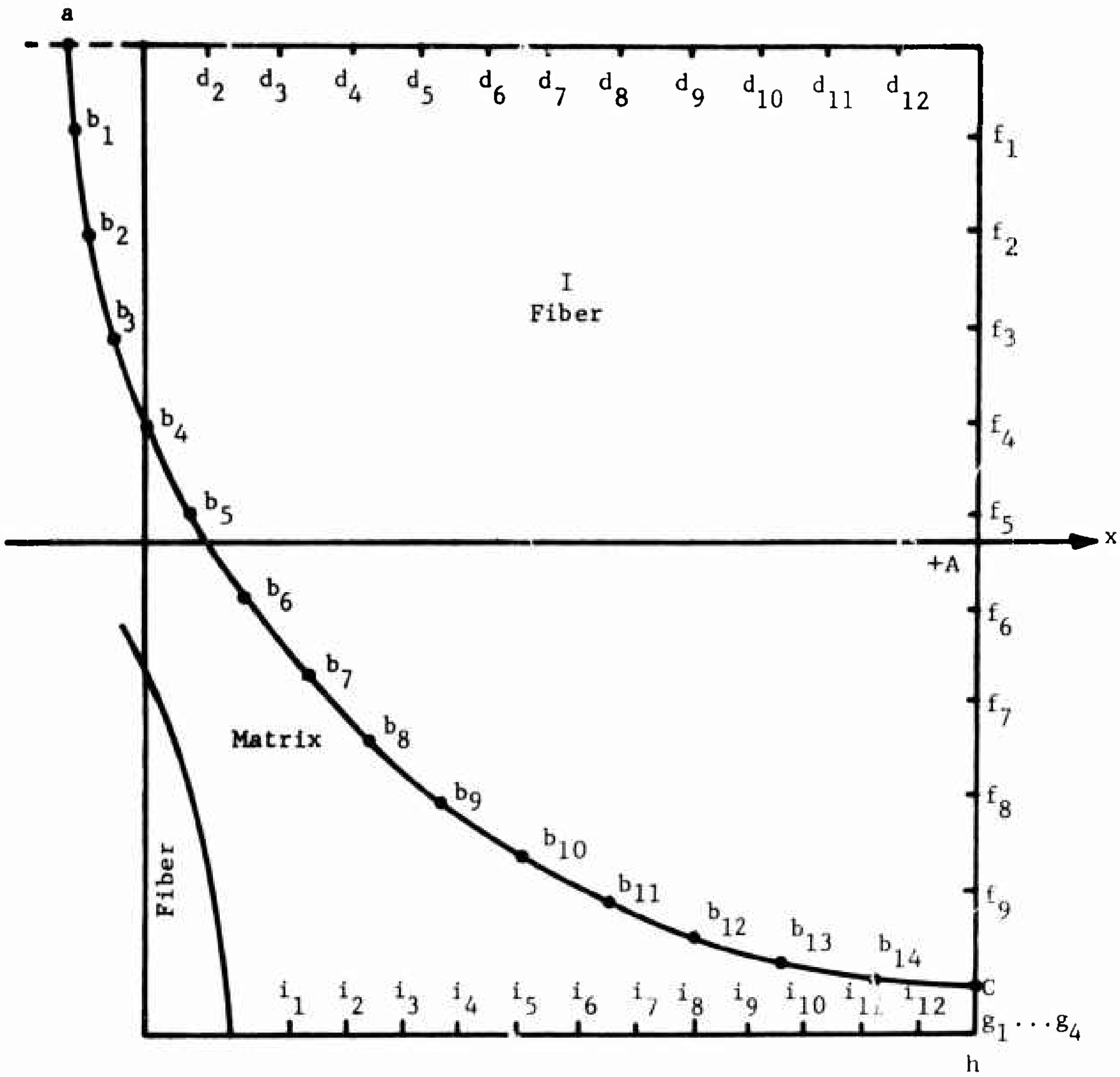


Figure 24. Arrangement of 61 Location Points for $a < A$.

TABLE I EQUATIONS GENERATED IF $\rho<a$			
Point	Condition	No. of Points	No. of Equation
a	$u_z^I = u_z^{II}$	1	1
a	$\tau_{xz}^I = \tau_{xz}^{II}$	1	2
a	$\tau_{yz}^I = 0$	1	3
a	$\tau_{yz}^{II} = 0$	1	4
b	$u_z^I = u_z^{II}$	14	5-18
b	$\tau_{xz}^I \cos\varphi + \tau_{yz}^I \sin\varphi = \tau_{xz}^{II} \cos\varphi + \tau_{yz}^{II} \sin\varphi$	14	19-32
c	$u_z^I = k$	1	33
c	$u_z^{II} = k$	1	34
c	$\tau_{yz}^I = \tau_{yz}^{II}$	1	35
d	$\tau_{yz}^I = 0$	9	36-44
e	$\tau_{yz}^I = 0$	1	45
e	$u_z^I = k$	1	46

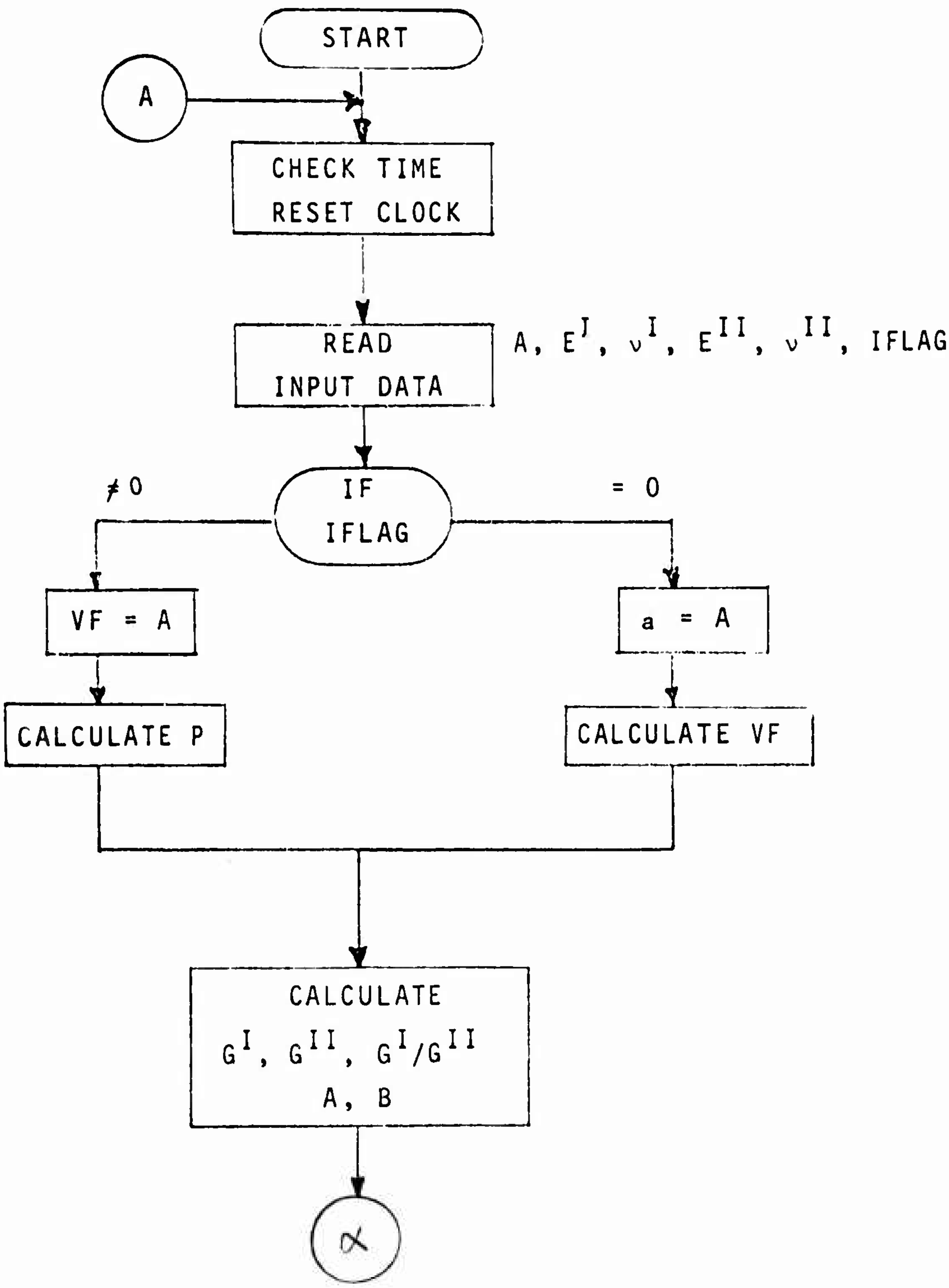
TABLE I Continued			
Point	Condition	No. of Points	No. of Equation
f	$u_z^{I} = k$	9	47-55
g	$u_z^{II} = k$	4	56-59
h	$^{II} = k$	1	60
h	$\tau_{yz}^{II} = 0$	1	61
i	$\tau_{yz}^{II} = 0$	12	62-73
j	$\tau_{yz}^{II} = 0$	3	74-76
f	$\partial u_z^{I}/\partial y = 0$	9	77-85
c	$\partial u_z^{I}/\partial y = 0$	1	86
c	$\partial u_z^{II}/\partial y = 0$	1	87
g	$\partial u_z^{II}/\partial y = 0$	4	88-91

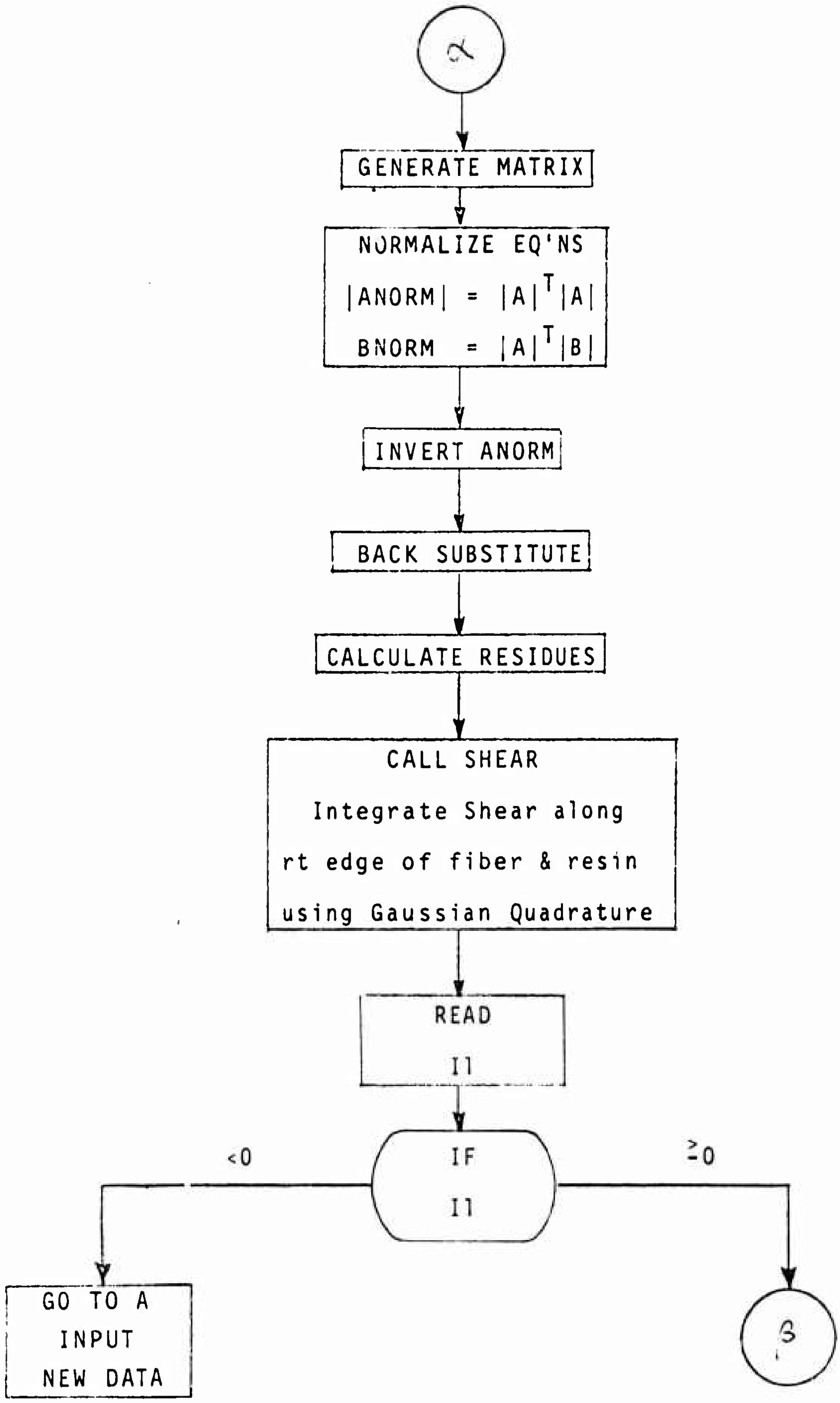
TABLE II. EQUATIONS GENERATED IF $\rho > a$

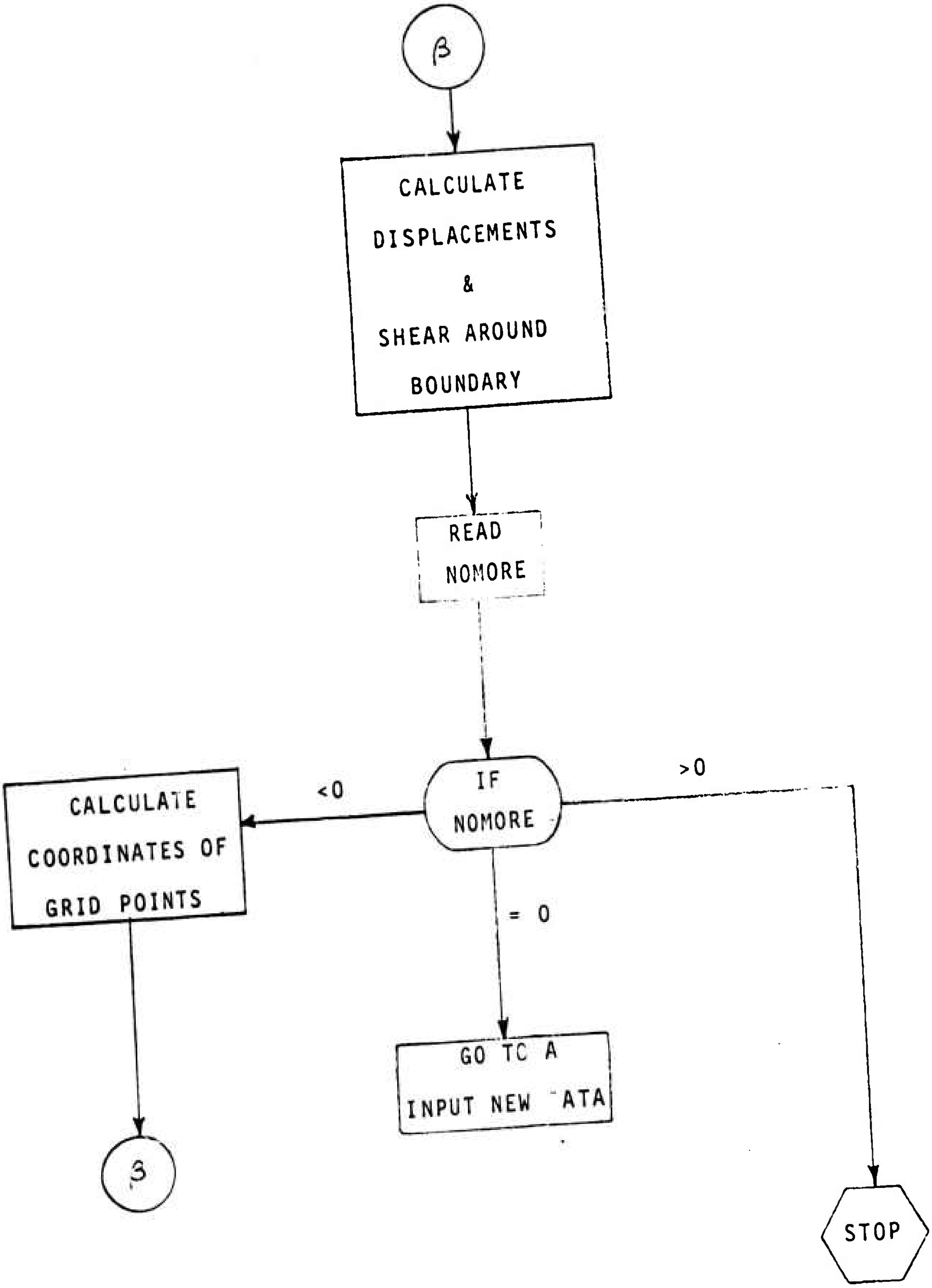
Point	Condition	No. of Points	No. of Equation
d	$\tau_{yz}^{I} = 0$	12	36-47
e	$\tau_{yz}^{I} = 0$	1	48
e	$u_{z}^{I} = k$	1	49
f	$u_{z}^{I} = k$	9	50-58
g	$u_{z}^{II} = k$	4	59-62
h	$u_{z}^{II} = k$	1	63
h	$\tau_{yz}^{II} = 0$	1	64
i	$\tau_{yz}^{II} = 0$	12	65-76

NOTES:
1. Equations 1 through 35 are the same as for the case $a < A$.
2. There are no points j.
3. Equations 77 through 91 are the same as for the case $a < A$.

FLOW DIAGRAM OF PROGRAM ALBERTO




α
GENERATE MATRIX
NORMALIZE EQ'NS
|ANORM| = |A|T|A|
BNORM = |A|T|B|
INVERT ANORM
BACK SUBSTITUTE
CALCULATE RESIDUES
CALL SHEAR
Integrate Shear along
rt edge of fiber & resin
using Gaussian Quadrature
READ
I1
IF
I1
<0
≥0
GO TO A
INPUT
NEW DATA
β


β
CALCULATE
DISPLACEMENTS
&
SHEAR AROUND
BOUNDARY
READ
NOMORE
IF
NOMORE
<0
>0
= 0
CALCULATE
COORDINATES OF
GRID POINTS
GO TC A
INPUT NEW ATA
STOP

POWER This subroutine computes both the real and the imaginary parts of the function

$$(x + iy)^n , \quad \text{where} \quad n \leq 20$$

The routine is first entered through the entry point, initial, in order to calculate binomial coefficients. The entry points POWER, DPOWERX, and DPOWERY are used to calculate

$$f(x,y) = (x + iy)^n , \quad \frac{\partial f}{\partial x} , \quad \frac{\partial f}{\partial y}$$

respectively, and where n is an odd positive integer. The entry points EPOWER, DEPOWERX, and DEPOWERY are used when n is an even number.

AMINUS Function subroutine to compute $(-1)^n$ where n is an integer.

PWRF Function subroutine to compute x^n where n is an integer. If $x = o$, PWRF will return with $x^n = o$, even when $n < o$.

NORMEQ Generates the normal equations by

1. Transposing matrix A
2. Multiplying and forming the matrix product A^TA
3. Multiplying and forming the matrix product A^Tb for the right-hand side of equations

Additional Capability: List the matrix A using the subroutine LISTARAY.

LISTARAY A subroutine to list matrices, labelling rows and columns.

NWMATINV UCSD Library Subroutine. Inverts matrices, solves linear systems of equations, and computes determinant. It uses a Gaussian method with two-dimensional pivot search.

ZEROLIST — UCSD Library routine will zero out variables specified by the list defined in the NAMELIST statement.

PRTCOEF — Output subroutine

W1 — Additional entry points: W2 , TAUXZ1 , TAUXZ2 , TAUYZ1 , and TAUYZ2 . This routine computes w^{I} , w^{II} , $\tau_{xz}{}^{I}$, $\tau_{xz}{}^{II}$, etc.

GRID — This routine computes coordinates of the 127 points used in the finite difference method.

SHEAR — Integrates (Gaussian integration) τ_{xz} along right-hand boundary.

USAGE OF THE PROGRAM

Card 1

Field 1: VF (Volumetric Factor, e.g., 0.6) if IFLAG $\neq$ 0

or

RHO (Radius of Fiber) if IFLAG = 0

If V_F is specified, RHO is computed as

$$\rho = 2\sqrt{\frac{2\sqrt{3}\ V_F}{\pi}}$$

If RHO is specified, V_F is computed as

$$V_F = \frac{\pi \rho^2}{\infty\sqrt{3}}$$

Field 2: E^{I} , Young's modulus, fiber, psi

Field 3: ν^{I} , Poisson's ratio fiber

Field 3: E^{II} , Young's modulus, psi, matrix

Field 4: ν^{II} , Poisson's ratio, matrix

Field 5: IFLAG , See description for Field 1

<u>Card 2</u> Format (I4)

IPRT

If IPRT ≥0, do <u>not</u> print the matrix of the coefficients for the linear equations

If IPRT <0, <u>print</u> matrix of the coefficients

<u>Card 3</u> Format (I4)

NOMORE

If	NOMORE	< 0	Call GRID
	NOMORE	= 0	Input new data
	NOMORE	> 0	STOP

NOMORE <0 calls GRID, which causes the program to generate the coordinates as used in the finite difference method and to calculate displacements and stresses at these points. If card 3 has a NOMORE <0, a second card, 3a, has to follow which has a NOMORE ≥0.

```
      PROGRAM ALBERTO
C
C     SHEAR PROBLEM, INTEGRATION CONSTANTS ARE FOUND BY POSTULATING
C     BOUNDARY CONDITIONS AT A NUMBER OF POINTS, SO THAT TOTAL NUMBER OF
C     EQUATIONS IS MUCH LARGER THAN NUMBER OF UNKNOWN
C     COEFFICIENTS. EQUATIONS ARE SOLVED BY LEAST SQUARES METHOD
C     NORMEQ GENERATES THE NORMAL EQUATIONS
C
C *** IT = TOTAL NUMBER OF EQUATIONS GIVEN BY BOUNDARY CONDITIONS
C *** NBREQ = NUMBER OF NORMAL - EQUATIONS (=NUMBER OF UNKNOWN COEFFICIENTS)
C
C *** PROGRAMMED SEPTEMBER 13,1967
C         REVISED SEPTEMBER 22, 1967
C         MODIFIED OCTOBER 9,1967 TO USE FUNCTION, (X + IY)**(2*N+1)
C         MODIFIED OCTOBER 24, 1967 TO USE FUNCTION, (X + IY)**N IN FIBER
C
C *** NODD  = MAXIMUM ODD POWER USED IN BOTH FIBER AND RESIN
C     NEVEN = MAXIMUM EVEN POWER USED IN FIBER
C
      COMMON/INPT/A,B,RHO,E1,FNU1,E2,FNU2
      COMMON/MTRX/AA(100,61), RHS(100)
C ***   100 IS THE MAXIMUM NUMBER OF BOUNDARY EQUATIONS PERMITTED
      COMMON/LABL/INDEX2(100), INDEX(100)
      NAMELIST/LABEL/INDEX2, INDEX
      DIMENSION BACK(100)
      COMMON/NORMTRX/ANDRM(61,61),BNORM(61)
      COMMON/COEFF/A1(20), A2(20), B1(20), E01
      COMMON/NUMBER/NODD, NEVEN, NBREQ, NVAR, IT
      COMMON/POINTS/XX(150),YY(150),JFLAG(150)
      DIMENSION IFLAGG(5), JFORM(2)
      TYPE INTEGER STAR
      DATA (IFLAGG =  8HFIBER    ,   8HRESIN    ,   8HINTRFACE,
     *                8H         ,   8H              )
      DATA(JFORM(1) = 8H(1H+,95X)
      DATA(NVAR = 20), (NBREQ = 61), (NCOL = 40), (NUM = 20),
     *     (NODD = 19), (NEVEN = 20), ( IT   = 91)
      DATA(PI = 3.1415926)
    6 L = LAPSTIME(LL)
      CALL STARTIME
      READ 1, A,E1,FNU1,E2,FNU2,IFLAG
    1 FORMAT(5E10.4,I10)
      IF (EOF, 50) 3161, 4
    4 JFORM(2) = 8H,A8,//)
C
      IF (IFLAG) 11, 12
   11 VF = A
      RHO = 2.*SQRTF(2.*SQRTF(3.)*VF/PI)
      GO TO 13
   12 RHO = A
      VF = RHO*RHO*PI/(8.*SQRTF(3.))
   13 G1 = E1/(2.*(1.+FNU1))
      G2 = E2/(2.*(1.+FNU2))
      G1OVERG2 = G1/G2
      B = 1.0
      A=B*SQRTF(3.)
      PRINT 2,B,VF,RHO,E1,FNU1,E2,FNU2,G1,G2,G1OVERG2
    2 FORMAT(1H1,20X,16HINPUT PARAMETERS,///20X,16(1H-)///
     18H B       ,E20.5/
     18H VF          ,E20.5/
     28H RHO     ,E20.5/
     38H E1      ,E20.5/
```

```
      48H NU1       ,E20.5/
      58H E2        ,E20.5/
      68H NU2        ,E20.5/
      18H G1            ,E20.5/
      18H G2            ,E20.5/
      18H G1/G2         ,E20.5//)
C
      L1 = L / 1000            $           L2 = L  -  L1 * 1000
      PRINT 603, L1, L2
  603 FORMAT(5HOTIME,I13,1H.I3)
C
      PRINT 602, IT, NODD, NEVEN, NBREQ
  602 FORMAT(41H4NUMBER OF BOUNDARY EQUATIONS SPECIFIED =,I5,/26HOMAXIMU
     *M ODD POWER USED  =,I5,/26HOMAXIMUM EVEN POWER USED =,I5,/20HONUMB
     *ER OF COLUMNS =  ,I5)
C
C     GENERATE MATRIX AND RIGHT HAND SIDES OF EQUATIONS
C
      CALL GETMTRX
C
C
C     FORM NORMAL EQUATIONS
C
      CALL NORMEQ
C
C
C     SOLVE EQUATIONS
C
  400 CALL NWMATINV(ANORM, BNORM, BACK, 61, NBREQ, 1, DET, IDET)
C
      IDET20 = 20*IDET
      PRINT 971,DET,IDET20
  971 FORMAT (///,1X, 4HDET=,E20.5,5X,     10HTIMES 10**,I4)
      IF(DET .NE. 0.) GO TO 318
C
  317 READ 972, N1, STAR
      IF (STAR .EQ. 1H*) 6, 317
  972 FORMAT(A8,A1)
C
C
C *** BACK SUBSTITUE SOLUTIONS INTO ORIGINAL MATRIX
C
  318 DO 319 I=1,IT
      BACK(I) = 0.
  319 CONTINUE
C
      DO 320 JJ=1,IT
      DO 320 KK=1,NBREQ
      BACK(JJ) = BACK(JJ) + BNORM(KK) * AA(JJ,KK)
  320 CONTINUE
C
C ***    CALCULATE SUM OF SQUARES OF RESIDUES
C
      SUMSQRES = 0.
      DO 335 J=1,IT
  335 SUMSQRES = SUMSQRES + (RHS(J) - BACK(J))**2
C
  340 PRINT 904,SUMSQRES
  904 FORMAT(1H1,8X,1HJ,11X,7HBACK(J),14X,6HRHS(J),25X,17HSUM RESIDUES S
     *Q =,E20.5,/)
```

```
      PRINT 903, (J, BACK(J), RHS(J), INDEX(J), J=1,IT)
  903 FORMAT (I10, 2E20.5, 10X, A8)
C
:     STORE COEFFICIENTS
C
      DO 27 J=1,60
   27 A1(J) = 0.
      DO 28 J=1,NVAR
      A1(J) = BNORM(J)
      A2(J) = BNORM(J+NVAR)
   28 CONTINUE
         DO 30 J=1,NUM
         B1(J) = BNORM(J+NCOL)
   30    CONTINUE
      E01 = BNORM(NBREQ)
C
C     PRINT THE COEFFICIENTS
C
      CALL PRTCOEF
C
      CALL SHEAR
C
      READ 5, I1
      IF(I1) 6, 40, 40
    5 FORMAT(9I4)
C
   40 I = 0
C
   99 I = I+1
      X = XX(I)              $              Y = YY(I)
      IFLAG = JFLAG(I)              $             LABELS = INDEX2(I)
C
      GO TO (100, 200, 100, 250) IFLAG
C
C     IFLAG=1...FIBER
C     IFLAG=2...RESIN
C     IFLAG=3...INTERFACE
C     IFLAG=4...STOP
C
  100 CONTINUE
      PRINT 3, IFLAGG(IFLAG), X, Y
    3 FORMAT(/,1H0,A8,20X,3HX =,E15.4,5X,3HY =,E15.4)
      PRINT JFORM, LABELS
      CALL W1(X,Y,WW1)
      CALL TAUXZ1(X,Y,TAUXZZ1)
      CALL TAUYZ1(X,Y,TAUXYY1)
      PRINT 902,WW1,TAUXZZ1,TAUXYY1
  902 FORMAT(4H W1=,E15.7,5X,10HTAU XZ 1 =,E15.7,5X,10HTAU YZ 1= ,
     1 E15.7//)
      GO TO (99, 99, 201) IFLAG
C
  200 CONTINUE
      PRINT 3, IFLAGG(IFLAG), X, Y
      PRINT JFORM, LABELS
  201 CALL W2(X,Y,WW2)
      CALL TAUXZ2(X,Y,TAUXZZ2)
      CALL TAUYZ2(X,Y,TAUXYY2)
      PRINT 901,WW2,TAUXZZ2,TAUXYY2
  901 FORMAT(4H W2=,E15.7,5X,10HTAU XZ 2 =,E15.7,5X,10HTAU YZ 2= ,     E
     115.7//)
```

```
      IF(IFLAG .NE. 3) GO TO 99
C
      COSTHETA = (A-X)/RHO          $         SINTHETA = (B-Y)/RHO
      FTAU1 = TAUXZZ1 * COSTHETA    +    TAUXYY1 * SINTHETA
      FTAU2 = TAUXZZ2 * COSTHETA    +    TAUXYY2 * SINTHETA
      PRINT 905, FTAU1, FTAU2
  905 FORMAT(24X,10HF(TAU 1) =,E15.7,5X,10HF(TAU 2) =,E15.7,//)
      GO TO 99
C
  250 READ 5, IFLAG
C
      IF (IFLAG) 300, 305, 315
C                                              THIS VARIABLE IS CALLED  NOMORE   IN
C     IFLAG = -1 . . . CALL GRID                       PROGRAM DESCRIPTION
C     IFLAG =  0 . . . INPUT NEW DATA
C     IFLAG = +1 . . . STOP
C
  300 CALL ZEROLIST(LABEL)       $      CALL GRID
      JFORM(2) = 8H,I8,//)       $      GO TO 40
C
  305 PRINT 906
  310 PRINT 500              $              GOTO 6
  906 FORMAT(///,1H0, 35X,20H E N D    P O I N T S)
  500 FORMAT(1H1)
C
  315 PRINT 906
  316 L = LAPSTIME(LL)
 3161 L1 = L / 1000            $             L2 = L  -  L1 * 1000
      PRINT 603, L1, L2
      END
```

```
      SUBROUTINE GETMTRX
C
C ***     IN THE FIBER, USE THE FUNCTION
C                (X + IY)**M    ,  M = 0,1,2,3, . . .
          IN THE RESIN, USE THE FUNCTION
C                (X + IY)**N    ,  N = 1,3,5, . . .
C
C
      COMMON/INPT/AA,B,RHO,E1,FNU1,E2,FNU2
      COMMON/MTRX/ A(100,61), RHS(100)
      NAMELIST/MATRIX/A, RHS
      COMMON/NUMBER/NODD, NEVEN, NBREQ, NVAR, IT
      COMMON/COMPLX/Z(20)
      EQUIVALENCE(Z,ZEVEN)
      DIMENSION ZEVEN(20)
      DIMENSION XX(20),YY(20),CT(20),ST(20)
      COMMON/POINTS/XPT(150), YPT(150), JFLAG(150)
      COMMON/LABL/LABEL(100), LABEL2(100)
      DIMENSION SCR(100)
      DIMENSION NUMBER(14)
      DATA(IB = 62000000B), (ID = 64000000B), (IF = 66000000B),
     *    (IG = 67000000B), (II = 71000000B), (IJ = 41000000B)
      DATA(NUMBER = 1R1, 1R2, 1R3, 1R4, 1R5, 1R6, 1R7, 1R8, 1R9,
     *               2R10,2R11,2R12,2R13,2R14)
      DATA(PI=3.141592653)
C
      CALL ZEROLIST(MATRIX)          $          CALL INITIAL
      G1=E1/(2.*(1.+FNU1)) $ G2=E2/(2.*(1.+FNU2)) $ CG=G2/G1
      NCOL = 2*NVAR
      NUM = NEVEN
C
C     EQUATION 1 (POINT A) W1-W2=0
C
      X=AA-RHO $ Y=B
C
      Z(1) = XPT(1) = X          $          Z(2) = YPT(1) = Y
      LABEL(1) = 4RA
      LABEL2(1) = 8HA1,2 - W
      JFLAG( 1) = 3
      I1 = 2
      CALL POWER
      DO 10 N=1,NVAR
      A(1,N) = Z(N)          $          A(1,N+NVAR) = -Z(N)
   10 CONTINUE
C
      ZEVEN(1) = X          $          ZEVEN(2) = Y
      CALL EPOWER
      DO 11 N=1,NUM
      A(1,NCOL+N) = ZEVEN(N)
   11 CONTINUE
      A(1,NBREQ) = 1.
C
C     EQUATION 2 (POINT A) TAUXZ1/G1-TAUXZ2/G1=0
C
      LABEL2(2) = 8HA12 TAUX
      Z(1) = X     $     Z(2) = Y
      CALL DPOWERX
      DO 20 N=1,NVAR
      A(2,N) = Z(N)          $          A(2,N+NVAR) = -Z(N)*CG
   20 CONTINUE
C
```

```
      ZEVEN(1) = X             $           ZEVEN(2) = Y
      CALL DEPOWERX
      DO 21 N=1,NUM
      A(2,N+NCOL) = ZEVEN(N)
   21 CONTINUE
C
C     EQUATION 3 (POINT A) TAUYZ1/G1=0
C
      LABEL2(3) = 8HA1   TAUY
      Z(1) = X      $    Z(2) = Y
      CALL DPOWERY
      DO 30 N=1,NVAR
      A(3,N) = Z(N)
   30 CONTINUE
C
      ZEVEN(1) = X             $           ZEVEN(2) = Y
      CALL DEPOWERY
      DO 31 N=1,NUM
      A(3,N+NCOL) = ZEVEN(N)
   31 CONTINUE
C
C     EQUATION 4 TAUYZ2/G2=0 IN POINT A
C
      LABEL2(4) = 8HA2   TAUY
      Z(1) = X            $            Z(2) = Y
      CALL DPOWERY
      DO 40 N=1,NVAR
      A(4,N+NVAR) = Z(N)
   40 CONTINUE
C
C                                                                  B
C     EQUATIONS 5 THRU 18 POINTS B W1-W2=0
C
C     GENERATE THE 14 POINTS XX AND YY
C
      DELTA=6.*PI/180.
C
      LABEL2(5) = 8HB1,2 - W
      DO 50 I=1,14 $ FI=I
      THETA=FI*DELTA
      XX(I)=AA-RHO*COSF(THETA) $ CT(I)=COSF(THETA)
      YY(I)=B-RHO*SINF(THETA) $ ST(I)=SINF(THETA)
   50 CONTINUE
C
      DO 70 I=1,14
      Z(1) = XPT(I1) = XX(I)       $            Z(2) = YPT(I1) = YY(I)
      LABEL(I1) = 18 + NUMBER(I)
      JFLAG(I1) = 3
      I1 = I1 + 1
      CALL POWER
C
      DO 60 N=1,NVAR
      A(I+4,N) = Z(N)           $            A(I+4,N+NVAR) = -Z(N)
   60 CONTINUE
C
      ZEVEN(1) = XX(I)       $           ZEVEN(2) = YY(I)
      CALL EPOWER
      DO 61 N=1,NUM
      A(I+4,N+NCOL) = ZEVEN(N)
   61 CONTINUE
```

```
      A(I+4,NBREQ) = 1.
   70 CONTINUE
C
C     EQUATIONS 19 THRU 32 F(TAU1)-F(TAU2)=0 POINTS B
C
      LABEL2(19) = 8HB F(TAU)
      DO 90 I=1,14
      Z(1) = XX(I)       $       Z(2) = YY(I)
      CALL DPOWERX
      DO 75 N=1,NVAR
   75 SCR(N) = Z(N) * CT(I)
C
      ZEVEN(1) = XX(I)       $          ZEVEN(2) = YY(I)
      CALL DEPOWERX
      DO 76 N=1,NUM
      SCR(N+NVAR) = ZEVEN(N) * CT(I)
   76 CONTINUE
      Z(1) = XX(I)       $       Z(2) = YY(I)
      CALL DPOWERY
      DO 80 N=1,NVAR
      A(I+18,N) = SCR(N) + Z(N)*ST(I)
      A(I+18,N+NVAR) = -A(I+18,N) * CG
   80 CONTINUE
C
      ZEVEN(1) = XX(I)       $          ZEVEN(2) = YY(I)
      CALL DEPOWERY
      DO 81 N=1,NUM
      A(I+18,N+NCOL) = SCR(N+NVAR) + ZEVEN(N) * ST(I)
   81 CONTINUE
   90 CONTINUE
C
C     EQUATION 33 POINT C W1=K
C
      X=AA $ Y=B-RHO
C
      LABEL2(33) = 8HC1  -  W
      Z(1) = XPT(I1) = X          $          Z(2) = YPT(I1) = Y
      LABEL(I1) = 4RC
      JFLAG(I1) = 3
      I1 = I1 + 1
      CALL  OWER
      DO 1  N=1,NVAR
      A(33,N) = Z(N)
  100 CONTINUE
C
      ZEVEN(1) = X           $          ZEVEN(2) = Y
      CALL EPOWER
      DO 101 N=1,NUM
      A(33,N+NCOL) = ZEVEN(N)
  101 CONTINUE
      A(33,NBREQ) = RHS(33) = 1.
C
C
C     EQUATION 34 POINT C W2=K
C
      LABEL2(34) = 8HC2  -  W
      Z(1) = X          $          Z(2) = Y
      CALL POWER
      DO 120 N=1,NVAR
      A(34,N+NVAR) = Z(N)
```

```
  120 CONTINUE
      RHS(34) = 1.
C
C     EQUATION 35 POINT C (TAUYZ1-TAUYZ2)/G1=0
C
      LABEL2(35) = 8HC12 TAUY
      Z(1) = X          $     Z(2) = Y
      CALL DPOWERY
      DO 140 N=1,NVAR
      A(35,N) = Z(N)            $          A(35,N+NVAR) = -Z(N) * CG
  140 CONTINUE
C
      ZEVEN(1) = X            $          ZEVEN(2) = Y
      CALL DEPOWERY
      DO 141 N=1,NUM
      A(35,N+NCOL) = ZEVEN(N)
  141 CONTINUE
C                                                                      D
C     POINT D EQUATIONS 36 THRU 44  TAUZY1/G1=0
C
C     GENERATE THE 9 COORDINATES XX AND YY
C
      XO = AA - RHO           $          Y = B
      IF(XO) 150, 150, 151
  150 IDPT = 12          $     DPT = 13.        $       GO TO 152
  151 IDPT = 9           $     DPT = 10.
  152 DELTAX = RHO/DPT
C
      LABEL2(36) = 8HD - TAUY
A     DO 153 I=1,IDPT             $          FI = I
      XX(I)=XO+FI*DELTAX
  153 CONTINUE
C
      DO 160 I=1,IDPT
      Z(1) = XPT(I1) = XX(I)        $          Z(2) = YPT(I1) = Y
      LABEL(I1) = ID + NUMBER(I)
      JFLAG(I1) = 1
      I1 = I1 + 1
      CALL DPOWERY
      DO 155 N=1,NVAR
      A(I+35,N) = Z(N)
  155 CONTINUE
C
      ZEVEN(1) = XX(I)         $          ZEVEN(2) = Y
      CALL DEPOWERY
      DO 156 N=1,NUM
      A(I+35,N+NCOL) = ZEVEN(N)
  156 CONTINUE
  160 CONTINUE
      I2 = 36 + IDPT
C                                                                      E
C     EQUATION 45 POINT E TAUZY1/G1=0
C
      X=AA $ Y=B
C
      Z(1) = XPT(I1) = X            $          Z(2) = YPT(I1) = Y
      LABEL(I1) = 4RE
      LABEL2(I2) = 8HE - TAUY
      JFLAG(I1) = 1
      I1 = I1 + 1
```

```
      CALL DPOWERY
      DO 170 N=1,NVAR
      A(I2,N) = Z(N)
  170 CONTINUE
C
      ZEVEN(1) = X          $          ZEVEN(2) = Y
      CALL DEPOWERY
      DO 171 N=1,NUM
      A(I2,N+NCOL) = ZEVEN(N)
  171 CONTINUE
      I2 = I2 + 1
C
C     EQUATION 46 POINT E W1=K
C
      LABEL2(I2) = 8HE    -   W1
      Z(1) = X      $      Z(2) = Y
      CALL POWER
      DO 190 N=1,NVAR
      A(I2,N) = Z(N)
  190 CONTINUE
C
      ZEVEN(1) = X          $          ZEVEN(2) = Y
      CALL EPOWER
      DO 191 N=1,NUM
      A(I2,N+NCOL) = ZEVEN(N)
  191 CONTINUE
      A(I2,NBREQ) = RHS(I2) = 1.
C                                                        F
      EQUATIONS 47 THRU 55 POINTS F W1=K
C
      LABEL2(I2+1) = 8HF    -   W
      X=AA $ YO=B $ DELTAY=-RHO/10.
      IFPT = I1 - 1
      DO 200 I=1,9 $ FI=I
  200 YY(I)=YO+FI*DELTAY
C
      DO 220 I=1,9
      Z(1) = XPT(I1) = X          $          Z(2) = YPT(I1) = YY(I)
      LABEL(I1) = IF + NUMBER(I)
      JFLAG(I1) = 1
      I1 = I1 + 1
      CALL POWER
      DO 210 N=1,NVAR
      A(I+I2,N) = Z(N)
  210 CONTINUE
C
      ZEVEN(1) = X          $          ZEVEN(2) = YY(I)
      CALL EPOWER
      DO 211 N=1,NUM
      A(I+I2,N+NCOL) = ZEVEN(N)
  211 CONTINUE
      A(I+I2,NBREQ) = RHS(I+I2) = 1.
  220 CONTINUE
      I2 = I2 + 9

      XPT(I1) = AA          $          YPT(I1) = B-RHO
      LABEL(I1) = 4RC       $          ICPT = I1 - 1
      JFLAG(I1) = 3
      I1 = I1 + 1
C                                                        G
```

```
C      EQUATIONS 56 THRU 59 POINTS G W2=Y
C
       LABEL2(I2+1) = 8HG    -   W2
       YO=B-RHO $ DELTAY=-(2.*B-RHO)/5.
C
       DO 250 I=1,4 $ FI=I
       Y=YO+FI*DELTAY
       JFLAG(I1) = 2
       LABEL(I1) = IG + NUMBER(I)
C
       Z(1) = XPT(I1) = X          $          Z(2) = YPT(I1) = Y
       I1 = I1 + 1
       CALL POWER
       DO 240 N=1,NVAR
       A(I+I2,N+NVAR) = Z(N)
  240  CONTINUE
       RHS(I+I2) = 1.
  250  CONTINUE
       I2 = I2 + 5
C
C      EQUATION 60 POINT H W2=K
C
       X=AA $ Y=-B
       LABEL(I1) = 4RH
       LABEL2(I2) = 8HH    -   W2
       JFLAG(I1) = 2
C
       Z(1) = XPT(I1) = X          $          Z(2) = YPT(I1) = Y
       I1 = I1 + 1
       CALL POWER
       DO 260 N=1,NVAR
       A(I2,N+NVAR) = Z(N)
  260  CONTINUE
       RHS(I2) = 1.
       I2 = I2 + 1
C
C      EQUATION 61 POINT H TAUYZ2/G2=0
C
       LABEL2(I2) = 8HH - TAUY
       Z(1) = X     $     Z(2) = Y
       CALL DPOWERY
       DO 280 N=1,NVAR
       A(I2,N+NVAR) = Z(N)
  280  CONTINUE
C
C      EQUATIONS 62 THRU 73 POINTS I TAUYZ2/G2=0
C
       XO = AA - RHO
       IF(XO) 290, 290, 291
  290  AO = XO + AA   $   XO = -XO    $   IPT = 0    $   GO TO 292
  291  AO =      AA   $   XO = 0.     $   IPT = 1
  292  DELTAX = AO / 13.      $       Y = -B
       LABEL2(I2+1) = 8HI - TAUY
C
       DO 300 I=1,12 $ FI=I
       X = XO + FI*DELTAX
       JFLAG(I1) = 2
       LABEL(I1) = II + NUMBER(I)
       Z(1) = XPT(I1) = X          $          Z(2) = YPT(I1) = Y
       I1 = I1 + 1
```

H

I

```
      CALL DPOWERY
C
      DO 295 N=1,NVAR
      A(I+12,N+NVAR) = Z(N)
  295 CONTINUE
  300 CONTINUE
      IF(IPT) 301, 321
C
C     EQUATIONS 74 THRU 76 POINTS J TAUYZ2/G2=0
C
  301 DELTAX = (AA-RHO)/3.          $           Y = B
      LABEL2(74) = 8HJ - TAUY
C
      DO 320 I=1,3 $ FI=I
      X=(FI-1)*DELTAX
      JFLAG(I1) = 2
      LABEL(I1) = IJ + NUMBER(I)
      Z(1) = XPT(I1) = X          $           Z(2) = YPT(I1) = Y
      I1 = I1 + 1
      CALL DPOWERY
C
      DO 310 N=1,NVAR
      A(I+73,N+NVAR) = Z(N)
  310 CONTINUE
  320 CONTINUE
C
C *** EQUATION 77-86 DW1/DY=0 ALONG POINTS F AND IN POINT C
C
C
  321 LABEL2(77) = 8HF  DW/DY
      DO 340 I=1,10
      Z(1) = XPT(IFPT+I)      $      Z(2) = YPT(IFPT+I)
      CALL DPOWERY
      DO 330 N=1,NVAR
      A(I+76,N) = Z(N)
  330 CONTINUE
C
      ZEVEN(1) = XPT(IFPT+I)          $          ZEVEN(2) = YPT(IFPT+I)
      CALL DEPOWERY
      DO 331 N=1,NUM
      A(I+76,N+NCOL) = ZEVEN(N)
  331 CONTINUE
  340 CONTINUE
      LABEL2(86) = 8HC1 DW/DY
C
C *** EQUATIONS 87 - 91 DW2/DY=0 IN POINT C AND ALONG POINTS G
C
      LABEL2(87) = 8HC2 DW/DY
      X= AA $ YO=B-RHO $ DELTAY=-(2.*B-RHO)/5.
      DO 450 I = 1,5  $ FI = I
      Z(1) = XPT(ICPT+I)       $       Z(2) = YPT(ICPT+I)
      CALL DPOWERY
      DO 440 N=1,NVAR
      A(I+86,N+NVAR) = Z(N)
  440 CONTINUE
  450 CONTINUE
      LABEL2(88) = 8HG  DW/DY
      JFLAG(I1) = 4
      RETURN $ END
```

```
      SUBROUTINE POWER
C
C  ****       THIS SUBROUTINE COMPUTES VALUES OF THE FUNCTION
         (X + IY)**N    ,   N = 1, 2, 3, . . .
C        AND ITS FIRST DERIVATIVES WITH RESPECT TO X AND Y
C
      COMMON/COMPLX/Z(10)
      COMMON/NUMBER/NODD, NEVEN, NBREQ, NVAR, IT
      TYPE COMPLEX Z,Z1
      EQUIVALENCE (Z, DZ)
      DIMENSION DZ(20), C(230)
      COMMON/LABL/A(200)
      NAMELIST/LABEL/A,C
C
      IF(Z(1) .EQ. (0., 0.)) GO TO 110
      Z1 = Z(1) **2            $          NUM = (NODD+1)/2
      GO TO 10
C
C ***
      ENTRY EPOWER
C
      IF(Z(1) .EQ. (0., 0.)) GO TO 110
      Z1 = Z(1) **2            $          NUM = NEVEN/2
      Z(1) = Z1
C
   10 DO 20 I=2,NUM
         Z(I) = Z(I-1) * Z1
   20 CONTINUE           $              RETURN
C
C ***
      ENTRY DPOWERX
C
      X = DZ(1)              $              Y = DZ(2)
      DZ(1) = 1. $      DZ(2) = 0.
      IF(X .EQ. 0.    .AND.    Y .EQ. 0.)  GO TO 110
      NO = 0      $      N1 = 3      $      N2 = NODD
      GO TO 50
C ***
      ENTRY DEPOWERX
C
      X = DZ(1)              $              Y = DZ(2)
      IF(X .EQ. 0.    .AND.    Y .EQ. 0.)  GO TO 110
      NO = 1      $      N1 = 2      $      N2 = NEVEN
C
   50 DO 70 N=N1,N2,2
      NCOMB = N * (N+1)/2  -  2          $              NN = N-1
      DO 65 I=1,2          $      II=I-1       $      SUM = 0.
C
   55 DO 60 K=IT,NN,2            $              FK = N-K
      SUM=SUM+ FK * C(NCOMB+K) * PWRF(X,N-K-1) * PWRF(Y,K) * AMINUS(K/2)
   60 CONTINUE
C
   65 DZ(N+II-NO) = SUM
   70 CONTINUE
      RETURN
C
C ***
      ENTRY DPOWERY
C
      X = DZ(1)              $              Y = DZ(2)
```

```
      DZ(1) = 0. $    DZ(2) = 1.
      IF(X .EQ. 0.    .AND.    Y .EQ. 0.)  GO TO 110
      NO = 0      $      N1 = 3      $      N2 = NODD
      GO TO 80
C ***
      ENTRY DEPOWERY
C
      X = DZ(1)             $             Y = DZ(2)
      IF(X .EQ. 0.    .AND.    Y .EQ. 0.)  GO TO 110
      NO = 1      $      N1 = 2      $      N2 = NEVEN
C
   80 DO 100 N=N1,N2,2
      NCOMB = N * (N+1)/2  -  2           $           NN = N
      DO 95  I=1,2      $        I1 = I-1        $       SUM = 0.
C
   85 DO 90 K=I1,NN,2              $              FK = K
      SUM=SUM+ FK * C(NCOMB+K) * PWRF(X,N-K) * PWRF(Y,K-1) * AMINUS(K/2)
   90 CONTINUE
C
   95 DZ(N+I1-NO) = SUM
  100 CONTINUE
      RETURN
C
  110 DO 120 N=3,20
      DZ(N) = 0.
  120 CONTINUE              $              RETURN
C
C
C ***
      ENTRY INITIAL
C
      CALL ZEROLIST(LABEL)
      A(2) = A(3) = 1.
      NMAX = XMAXOF(NODD, NEVEN) + 3
      IPOS = 1
C
      DO 150 N=5,NMAX              $              NN = N - 2
      DO 145 J=1,NN                $              K = N - J
      A(K) = A(K)+ A(K-1)
      IF(A(K)) 145, 145, 144
C
C        THE ARRAY  C, CONTAINS THE BINOMIAL COEFFICIENTS
C
  144 C(IPOS) = A(K)
      IPOS = IPOS + 1
C
  145 CONTINUE
  150 CONTINUE
      RETURN          $          END
```

```
      FUNCTION AMINUS(N)
      IF(N) 5, 25, 10
    5 M = -N              $         GO TO 15
   10 M = N
   15 M = M .AND. 1       $        IF(M) 20, 25
   20 AMINUS = -1         $        RETURN
   25 AMINUS =  1         $        RETURN         $         END
```

```
      FUNCTION PWRF(X,N)
      PWRF = 1.
      IF (N) 10, 90
   10 IF(X) 30, 20
   20 PWRF = 0.                 $              RETURN
   30 IF(X-1.) 40, 90
   40 IF(N) 50, 90, 70
   50 M = -N                    $              DO 60 I=1,M
   60 PWRF = PWRF / X           $              RETURN
   70 DO 80 I=1,N
   80 PWRF = PWRF * X
   90 RETURN                    $              END
```

```
      SUBROUTINE NORMEQ
C
C     FORMS THE NORMAL EQUATIONS FOR THE SOLUTION OF THE
C     OVERDETERMINED SYSTEM
C
      COMMON/NORMTRX/ANORM(61,61),BNORM(61)
      COMMON/MTRX/AA(100,61), RHS(100)
      COMMON/NUMBER/NODD, NEVEN, NBREQ, NVAR, IT
      DIMENSION ATR(61,100)
C
      DO 20 J = 1,NBREQ
      DO 10 K = 1,NBREQ
   10 ANORM(J,K) = 0.0
   20 BNORM(J) = 0.0
C
C     TRANSPOSE THE REDUCED MATRIX
C
  150 DO 200 J=1,IT
      DO 200 K = 1,NBREQ
      ATR(K,J) = AA(J,K)
  200 CONTINUE
C
C     MULTIPLY ATR TIMES AA
C
      DO 350 J = 1,NBREQ
      DO 340 K = 1,NBREQ
      DO 330 KK = 1,IT
  330 ANORM(J,K) = ANORM(J,K) + ATR(J,KK)*AA(KK,K)
  340 CONTINUE
  350 CONTINUE
C
C     FORM RIGHT HAND SIDES OF NORMAL EQUATIONS
C
      DO 380 J = 1,NBREQ
      DO 380 K = 1,IT
  380 BNORM(J) = BNORM(J) + ATR(J,K) * RHS(K)
C
  110 READ 700,I1, I2
  700 FORMAT(2I4)
      IF(I1) 390, 400, 400
  390 CALL LISTARAY(AA,100,IT,NBREQ,1)
  400 RETURN             $            END
```

```
      SUBROUTINE NWMATINV(A,B,INDEX,NMAX,N,M,DETERM,IDET)
      DIMENSION A(NMAX,1),B(NMAX,1),INDEX(1)
      EQUIVALENCE(IROW,JROW,IR),(ICOLUM,JCOLUM,IC)
      EQUIVALENCE(AMAX,T,SWAP,IAMAX),(PIVOT,TEMP,ITEMP)
      DATA (MINUS=6000000000000000B)
C**   INITIALIZATION.
      DETERM=1.0
      IDET = 0
      DO 20 J=1,N
   20 INDEX(J)=MINUS
      DO 550 I=1,N
C**   SEARCH FOR ELEMENT OF LARGEST MAGNITUDE.
      AMAX=0.0
      DO 105 J1=1,N
      IF(-INDEX(J1)) 105,105,60
   60 DO 100 K1=1,N
      IF(-INDEX(K1)) 100,100,80
   80 TEMP=A(K1,J1)
      IF(TEMP)83,100,82
   82 TEMP=-TEMP
   83 IF(-ITEMP-IAMAX) 100,100,84
   84 AMAX=-TEMP
      IR=K1
      IC=J1
  100 CONTINUE
  105 CONTINUE
      IF(AMAX)120,115
  115 DETERM=0
      IDET = 0
      RETURN
  120 IROW=IR
      ICOLUM=IC
      INDEX(ICOLUM)=INDEX(ICOLUM).AND..NOT.MINUS
      IF(.NOT.(IROW-ICOLUM))260,140
  140 DETERM=-DETERM
C**   EXCHANGE ROWS.
      DO 200 L=1,N
      SWAP=A(IROW,L)
      A(IROW,L)=A(ICOLUM,L)
  200 A(ICOLUM,L)=SWAP
      IF(.NOT.M)260,210
  210 DO 250 L=1, M
      SWAP=B(IROW,L)
      B(IROW,L)=B(ICOLUM,L)
  250 B(ICOLUM,L)=SWAP
C**   SAVE PIVOT INFORMATION. DO DETERMINANT.
  260 INDEX(I)=IROW*100000000B+ICOLUM+INDEX(I)
      PIVOT   =A(ICOLUM,ICOLUM)
      DETERM=DETERM*PIVOT
C
C  **    KEEP DETERMINANT BETWEEN 1.OE-20 AND 1.OE-20
  270 DETERMI = ABSF(DETERM)
      IF(DETERMI .GT. 1.E-20) GO TO 275
      DETERM = DETERM * 1.E20
      IDET = IDET - 1
      GO TO 270
  275 IF(DETERMI .LT. 1.E20) GO TO 300
      DETERM = DETERM / 1.E20
      IDET = IDET + 1
      GO TO 270
```

```
  300 CONTINUE
C**   REDUCE LEADING COEF TO 1.
      A(ICOLUM,ICOLUM)=1.0
      DO 350 L=1,N
  350 A(ICOLUM,L)=A(ICOLUM,L)/PIVOT
      IF(.NOT.M)380,360
  360 DO 370 L=1,M
  370 B(ICOLUM,L)=B(ICOLUM,L)/PIVOT
C**   SUBSTITUTE FOR NTH VARIABLE.
  380 DO 550 L1=1,N
      IF(.NOT.(L1-ICOLUM)) 550,400
  400 T=A(L1,ICOLUM)
      A(L1,ICOLUM)=0.0
      DO 450 L=1,N
  450 A(L1,L)=A(L1,L)-A(ICOLUM,L)*T
      IF(.NOT.M)550,460
  460 DO 500 L=1,M
  500 B(L1,L)=B(L1,L)-B(ICOLUM,L)*T
  550 CONTINUE
C**   UNDO ROW EXCHANGES.
      L=N
      DO 710 L2=1,N
      JROW=INDEX(L)/100000000B
      JROW=JROW.AND.77777B
      JCOLUM=INDEX(L).AND.77777B
      IF(.NOT.(JROW-JCOLUM))710,630
  630 DO 705 K=1,N
      SWAP=A(K,JROW)
      A(K,JROW)=A(K,JCOLUM)
  705 A(K,JCOLUM)=SWAP
  710 L=L-1
  740 RETURN
      END
```

```
      SUBROUTINE LISTARAY (A, NMAX, M, N, ISTART)
C
C ***      SET  WIDEOU'  WHEN USING THIS SUBROUTINE
           FOR USE WITH PROGRAM ALBERTO
C
      DIMENSION A(NMAX, 1)
      DIMENSION IC(10), IFORM(10)
      DATA (  IFORM  =  8H(1H1)    ,
     *                  8H(/,6H6RO,
     *                  8H         ,
     *                  8HH  COL,I,
     *                  8H4,3X),6H,
     *                  8H   ROW) ,
     *                  8H(I4,1X, ,
     *                  8H         ,
     *                  8HI6)      ,
     *                  8H(1H5)      )
    2 IFORM(3) =        8HW   ,10(5
      IFORM(8) =        8H10E12.3,
      DO 10 I=ISTART, N, 10
      IM1 = I-1      $     IP9 = I+9      $     LAST = 10
      IF(IP9 .GT. N) 6,7
    6 IP9 = N      $     J = N-ISTART+1      $     LAST = J-(10*(J/10))
      ENCODE(8, 900, IFORM(3)) LAST
      ENCODE(8, 901, IFORM(8)) LAST
  900 FORMAT (4HW  ,,I2,2H(5)
  901 FORMAT (I2,6HE12.3,)
    7 DO 8 ICOUNT = 1, LAST
    8 IC(ICOUNT) = IM1 + ICOUNT
      PRINT IFORM(1)
      PRINT IFORM(2), (IC(J), J=1,LAST)
      PRINT IFORM(7), (J, (A(J,II),II=I, IP9), J, J=1,M)
      PRINT IFORM(2), (IC(J), J=1,LAST)
   10 CONTINUE
      PRINT IFORM(10)
      RETURN     ;     END
```

```
      SUBROUTINE PRTCOEF
      COMMON/COEFF/A1(20), A2(20), B1(20), E01
      COMMON/NUMBER/NODD, NEVEN, NBREQ, NVAR, IT
C
      PRINT 901
  901 FORMAT(1H1,21X,2HA1,20X,2HB1,30X,2HA2,////)
C
      NMAX = XMAXOF(NODD+1, NEVEN)
C
      PRINT 902,(N, A1(N), B1(N), A2(N), N=1,NMAX)
  902 FORMAT(I10,2E20.4,10X,E20.4)
C
      PRINT 903, E01
  903 FORMAT(///,5H E01=,E20.4,/,1H4)
C
      RETURN     $     END
```

```
      SUBROUTINE W1(X, Y, ANS)
      COMMON/INPT/A,B,RHO,E1,FNU1,E2,FNU2
      COMMON/COFFF/A1(20), A2(20), B1(20), E01
      COMMON/COMPLX/Z(20)
      DIMENSION ZEVEN(20)
      EQUIVALENCE(Z,ZEVEN)
      COMMON/NUMBER/NODD, NEVEN, NBREQ, NVAR, IT
C
      NZERO = NODD + 1           $           NUM = NEVEN
      SUM = 0.     $      Z(1) = X      $      Z(2) = Y
      CALL POWER
      DO 10 N=1,NZERO
      SUM = SUM  +  A1(N) * Z(N)
   10 CONTINUE
C
      ZEVEN(1) = X      $      ZEVEN(2) = Y
      CALL EPOWER
      DO 11 N=1,NUM
      SUM = SUM + B1(N) * ZEVEN(N)
   11 CONTINUE
C
      ANS = SUM + E01
      RETURN
C
C ***
      ENTRY W2
C
      NZERO = NODD + 1
      SUM = 0.     $      Z(1) = X      $      Z(2) = Y
      CALL POWER
      DO 20 N=1,NZERO
      SUM = SUM  +  A2(N) * Z(N)
   20 CONTINUE
C
      ANS = SUM
      RETURN
C
C  ***
      ENTRY TAUXZ1
C
      NZERO = NODD + 1           $           NUM = NEVEN
      G1=E1/(2.*(1.+FNU1))
      SUM = 0.     $      Z(1) = X      $      Z(2) = Y
      CALL DPOWERX
      DO 30 N=1,NZERO
      SUM = SUM  +  A1(N) * Z(N)
   30 CONTINUE
C
      ZEVEN(1) = X      $      ZEVEN(2) = Y
      CALL DEPOWERX
      DO 31 N=1,NUM
      SUM = SUM   +   B1(N) * ZEVEN(N)
   31 CONTINUE
C
      ANS = G1 * SUM
      RETURN
C
C  ***
      ENTRY TAUXZ2
C
```

```
      G2=E2/(2.*(1.+FNU2))
      NZERO = NODD + 1
      SUM = 0.      $      Z(1) = X      $      Z(2) = Y
      CALL DPOWERX
      DO 40 N=1,NZERO
      SUM = SUM  +  A2(N) * Z(N)
   40 CONTINUE
C
      ANS = G2 * SUM
      RETURN
C
C  ***
      ENTRY TAUYZ1
C
      G1 = E1/(2.*(1.+FNU1))
      NZERO = NODD + 1             $             NUM = NEVEN
      SUM = 0.      $      Z(1) = X      $      Z(2) = Y
      CALL DPOWERY
      DO 50 N=1,NZERO
      SUM = SUM  +  A1(N) * Z(N)
   50 CONTINUE
C
      ZEVEN(1) = X      $      ZEVEN(2) = Y
      CALL DEPOWERY
      DO 51 N=1,NUM
      SUM = SUM   +   B1(N) * ZEVEN(N)
   51 CONTINUE
C
      ANS = G1 * SUM
      RETURN
C
C  ***
      ENTRY TAUYZ2
C
      G2=E2/(2.*(1.+FNU2))
      NZERO = NODD + 1
      SUM = 0.      $      Z(1) = X      $      Z(2) = Y
      CALL DPOWERY
      DO 60 N=1,NZERO
      SUM = SUM  +  A2(N) * Z(N)
   60 CONTINUE
C
      ANS = G2 * SUM
C
      RETURN     $     END
```

```
      SUBROUTINE SHEAR
      COMMON/INPT/ A, B, RHO, E1, FNU1, E2, FNU2
      DIMENSION A1(5), X1(5), A4(20), X4(20)
      DATA (X1=                                                          GLQ   8
     D .9739065285, .8650633667, .6794095683, .4333953941, .1488743390)  GLQ   9
      DATA (A1=                                                          GLQ  10
     D .0666713443, .1494513491, .2190863625, .2692667193, .2955242247)  GLQ  12
      DATA (X4=                                                          GLQ  19
     D .9982377097, .9907262387, .9772599500, .9579168192, .9328128083,  GLQ  20
     D .9020988070, .8659595032, .8246122308, .7783056514, .7273182552,  GLQ  21
     D .6719566846, .6125538897, .5494671251, .4830758017, .4137792044,  GLQ  22
     D .3419940908, .2681521850, .1926975807, .1160840707, .0387724175)  GLQ  23
      DATA (A4=                                                          GLQ  24
     D .0045212771, .0104982845, .0164210584, .0222458492, .0279370070,  GLQ  25
     D .0334601953, .0387821680, .0438709082, .0486958076, .0532278470,  GLQ  26
     D .0574397691, .0613062425, .0648040135, .0679120458, .0706116474,  GLQ  27
     D .0728865824, .0747231691, .0761103619, .0770398182, .0775059480)  GLQ  28
C
      X = A          $          G2 = E2/(2.*(1. + FNU2))
C
C             40 NODE GAUSSIAN QUADRATURE FOR INTEGRAL ALONG FIBER
C
      TO = (2*B - RHO) / 2.          $         T1 = RHO / 2.      $      F1 =0.
C
      DO 10 N=1,20
      TEMP1 = TO - T1*X4(N)       $       TEMP2 = TO + T1*X4(N)
      CALL TAUXZ1(X, TEMP1, ANS1)
      CALL TAUXZ1(X, TEMP2, ANS2)
      F1 = F1   +  A4(N) * (ANS1 + ANS2)
   10 CONTINUE
      F1 = F1 * T1
C
C             10 NODE GAUSSIAN QUADRATURE FOR INTEGRAL ALONG RESIN
C
      TO = -RHO / 2.       $       T1 = (-2. * B   +  RHO) / 2.    $     F2 = 0.
C
      DO 20 N=1,5
      TEMP1 = TO   -   T1 * X1(N)
      TEMP2 = TO   +   T1 * X1(N)
      CALL TAUXZ2(X, TEMP1, ANS1)
      CALL TAUXZ2(X, TEMP2, ANS2)
      F2 = F2   +   A1(N) * (ANS1 + ANS2)
   20 CONTINUE
      F2 = F2 * T1
C
      SHR = (F1 + F2) / 2.
      GCOMB = SHR * SQRTF(3.)
      GG2 = GCOMB/G2
      PRINT 600, F1, F2, SHR, GCOMB, GG2
C
  600 FORMAT(3H T1,12X,E15.4,/,3HOT2,12X,E15.4,/5HOTBAR,10X,E15.4,/,
     *6HOGCOMB,9X,E15.4,/,5HOG/G2,9X,E15.4,/,1H1)
      RETURN       $       END
```

```
      SUBROUTINE GRID
      COMMON/INPT/A,B,RHO,E1,FNU1,E2,FNU2
      COMMON/POINTS/XX(150),YY(150),JFLAG(150)
      COMMON/LABL/ INDEX(200)
C
      DIMENSION IY1(7), IX1(6),  IY2(7), IX2(6)
      DATA(IY1 =  0,  6, 10, 14, 16, 18, 20)
      DATA(IX1 =  1,  3,  5,  7, 11, 15)
      DATA(IY2 =  0,  4,  6,  8, 10, 14, 18)
      DATA(IX2 = 17, 13,  9,  5,  3,  1)
C
      FX(X)  = A - SQRTF(RHOSQ - (X-B)**2)
      FY(X)  = B - SQRTF(RHOSQ - (X-A)**2)
C
      RHOSQ = RHO * RHO
      J = 1
      DX = A/19.           $           DY = B/11.
C
C ***     POINTS IN FIBER
C
      DO 10 K=1,6
      I1Y = IY1(K)           $           I2Y = IY1(K+1) - 2
      I1X = IX1(K)           $           I2X = 17
      DO 5 IY = I1Y,I2Y,2            $            FFY = IY
      Y = B    -    FFY * DY
C
      DO 5 IX = I1X,I2X,2            $            FFX = IX
      X =    DX * FFX
      XX(J) = X      $      YY(J) = Y      $      JFLAG(J) = 1
      J =    J + 1
    5 CONTINUE
   10 CONTINUE
C
C ***     POINTS IN RESIN
C
      DO 20 K=1,6
      I1Y = IY2(K)           $           I2Y = IY2(K+1) - 2
      I1X = 1                $           I2X = IX2(K)
      DO 15 IY = I1Y, I2Y, 2            $            FFY = IY
      Y =  FFY * DX   -   B
C
      DO 15 IX = I1X, I2X, 2            $            FFX = IX
      X =   DX * FFX
      XX(J) = X      $      YY(J) = Y      $      JFLAG(J) = 2
      J =   J + 1
   15 CONTINUE
   20 CONTINUE
C
C ***     POINTS ALONG INTERFACE
C
      Y = B               $          X = A-RHO
      XX(109) = X         $       YY(109) = Y       $       JFLAG(109) = 3
      Y = B- 2.*DY        $          X = FX(Y)
      XX(110) = X         $       YY(110) = Y       $       JFLAG(110) = 3
      X = DX              $          Y = FY(X)
      XX(111) = X         $       YY(111) = Y       $       JFLAG(111) = 3
      Y = B- 4.*DY        $          X = FX(Y)
      XX(112) = X         $       YY(112) = Y       $       JFLAG(112) = 3
      Y = B- 6.*DY        $          X = FX(Y)
      XX(113) = X         $       YY(113) = Y       $       JFLAG(113) = 3
```

```
      Y = B- 8.*DY       $         X = FX(Y)
      XX(114) = X        $     YY(114) = Y        $    JFLAG(114) = 3
      X =  3.*DX         $         Y = FY(X)
      XX(115) = X        $     YY(115) = Y        $    JFLAG(115) = 3
      Y = B-10.*DY       $         X = FX(Y)
      XX(116) = X        $     YY(116) = Y        $    JFLAG(116) = 3
      Y = B-12.*DY       $         X = FX(Y)
      XX(117) = X        $     YY(117) = Y        $    JFLAG(117) = 3
      X =  5.*DX         $         Y = FY(X)
      XX(118) = X        $     YY(118) = Y        $    JFLAG(118) = 3
      Y = B-14.*DY       $         X = FX(Y)
      XX(119) = X        $     YY(119) = Y        $    JFLAG(119) = 3
      X =  7.*DX         $         Y = FY(X)
      XX(120) = X        $     YY(120) = Y        $    JFLAG(120) = 3
      X =  9.*DX         $         Y = FY(X)
      XX(121) = X        $     YY(121) = Y        $    JFLAG(121) = 3
      Y = B-16.*DY       $         X = FX(Y)
      XX(122) = X        $     YY(122) = Y        $    JFLAG(122) = 3
      X = 11.*DX         $         Y = FY(X)
      XX(123) = X        $     YY(123) = Y        $    JFLAG(123) = 3
      X = 13.*DX         $         Y = FY(X)
      XX(124) = X        $     YY(124) = Y        $    JFLAG(124) = 3
      Y = B-18.*DY       $         X = FX(Y)
      XX(125) = X        $     YY(125) = Y        $    JFLAG(125) = 3
      X = 15.*DX         $         Y = FY(X)
      XX(126) = X        $     YY(126) = Y        $    JFLAG(126) = 3
      X = 17.*DX         $         Y = FY(X)
      XX(127) = X        $     YY(127) = Y        $    JFLAG(127) = 3
      X = A              $         Y = B - RHO
      XX(128) = X        $     YY(128) = Y        $    JFLAG(128) = 3
      INDEX(128) = 500
C
      DO 30 J=1,127
      INDEX(J) = J
   30 CONTINUE
C
      J = 129
      X = A              $          IIY = 0
      DO 40 IY=IIY,11               $          FFY = 2*IY
      Y = B    -   FFY * DY
      XX(J+IY) = X       $      YY(J+IY) = Y      $
      JFLAG(J+IY) = 1       $      INDEX(J+IY) = 1000 + IY
   40 CONTINUE
C
      JFLAG(139) = JFLAG(140) = 2
      JFLAG(141) = 4
C
      RETURN       $       END

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13. ABSTRACT

Stress fields in the components of a multifiber, unidirectional composite that are caused by shear loading in axial directions are studied, and solutions are presented. Computations for various volumetric contents and material constants are described, as well as experiments which showed good correlation with the theoretical results from this work. Further, it is shown that certain mathematical solutions normally used for such problems are not compatible with present boundary conditions, and that the problem becomes two-dimensional.

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