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RM-5927-PR
MARCH 1969



# WAITING TIME TILL SERVICE FOR A SIMPLE INVENTORY MODEL

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# PREFACE

This Memorandum uses a single-facility, single-consumable item inventory model to derive an exact expression for the steady-state expected time that a customer waits until a facility satisfies his demand for a unit of stock. This effort continues RAND's work in inventory theory, especially that concerned with multi-echelon techniques, as described in C. C. Sherbrooke, METRIC: A Multi-Echelon Technique for Recoverable Item Control, RM-5078-PR, November 1966; B. L. Miller, A Real-Time METRIC for the Distribution of Serviceable Assets, RM-5687-PR, October 1968; and B. L. Fox and D. M. Landi, Optimization Problems with One Constraint, RM-5791-PR, October 1968.

R. M. Simon is a consultant to the RAND Corporation.

## SUMMARY

This Memorandum describes a single-facility, single-consumable item inventory model in which demands are Poisson, procurement lead-times are either deterministic or exponential, and a continuous review (s, S) replenishment policy is employed. We derive an exact expression for the steady-state expected time that a customer waits to be satisfied. Although this performance measure is not usually studied for inventory models, it is useful for imbedding the facility in a multi-echelon structure.

## ACKNOWLEDGMENT

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## CONTENTS

# I. INTRODUCTION

We wish to consider a facility stocking a single consumable item for which the customer demands form a Poisson process and are backlogged when stock is not available. It is assumed that the facility uses a continuous review (s, S) replenishment policy, and that the procurement lead-time is either deterministic or follows an exponential probability distribution. That is, when the facility inventory position (stock-on-hand plus on-order minus backlogged demands) falls to the level s, the facility places an order for S - s units from its supplier. A procurement lead-time later, this order arrives at the facility.

We shall be interested in the steady-state expected time that a customer waits until the facility satisfies his demand for a unit of stock. This performance measure is not usually studied, because it is not directly related to the cost structure of most models. The measure is of interest, however, particularly if the facility is to be imbedded in a multi-echelon model [6], [7].

The methods we use to obtain the steady-state expected waiting time can also be used to obtain the entire stationary distribution of this random variable.

## II. THE DERIVATION

We use the following notation:

$(s, S) \equiv$ replenishment policy

$Q \equiv S - s$

$L \equiv$ procurement lead-time

$\{D(t), t \geq 0\} \equiv$ demand process; a Poisson process with parameter $\lambda$

$\{B(t), t \geq 0\} \equiv$ stochastic process of facility backorders (negative backorders denote stock-on-hand)

$W(n) \equiv$ random time of arrival of the nth customer to enter the system

$\tau(n) \equiv$ random waiting time for the nth customer to enter the system till his demand is satisfied.

To obtain the stationary expected value of $\tau(n)$, we proceed as follows. Select any positive integer n. Then

$$E[\tau(n)] = \sum_{b=-s}^{\infty} E[\tau(n) | B(W(n)^-) = b] \ . \ Pr[B(W(n)^-) = b].$$

If $B(W(n)^-) < 0$, then the customer will be served immediately. Thus,

$$E[\tau(n)] = \sum_{b=0}^{\infty} E[\tau(n) | B(W(n)^-) = b] \ . \ Pr[B(W(n)^-) = b]. \qquad (1)$$

It follows easily from the properties of the Poisson process [8], that for $b \geq 0$,

$$\lim_{m \to \infty} \frac{1}{m} \sum_{n=1}^{m} Pr[B(W(n)^-) = b] = Pr[B(t^*) = b],$$

where $\Pr[B(t^*) = b]$ denotes the stationary probability of b facility backorders.† Passing to the limit in (1), we obtain

$$E[\tau] = \lim_{m\to\infty} \frac{1}{m} \sum_{n=1}^{m} E[\tau(n)]$$

$$= \sum_{b=1}^{\infty} \alpha(b) \cdot \Pr[B(t^*) = b - 1], \qquad (2)$$

where

$$\alpha(b) \equiv \lim_{m\to\infty} \frac{1}{m} \sum_{n=1}^{m} E[\tau(n) \mid B(W(n)^-) = b - 1].$$

DETERMINISTIC PROCUREMENT LEAD-TIMES

Assuming that L is a constant, let us first consider how to calculate $\alpha(b)$. We assume that $s \geq 0$, and hence the order that will satisfy the customer that just entered the facility has already been placed. Thus, the interval until this customer is satisfied is smaller than L. In fact, the interval is L minus the time elapsed since the placement of the order that will satisfy the customer. To obtain the steady-state expected value of this elapsed time, we use the following theorem.

THEOREM:

Consider a Poisson process $\{N(t), t \geq 0\}$ with parameter $\lambda$. Suppose that we are given positive integers j, k, m such that

$$j \leq k < m.$$

†Here, all stationary probabilities are considered to exist in Cesàro mean.

Suppose also that we are given real numbers L, t such that

$$0 < L < t.$$

Let

$$\theta(j,k,m,L,t) \equiv E[t - W(N(t) - j) \,|\, W(N(t)) = t,\ k < N(t) - N(t - L) \le m].$$

Then

$$\theta(j,k,m,L,t) = jL \sum_{h=k+1}^{m} [(\lambda L)^{h-1}/h!] \Big/ \sum_{h=k+1}^{m} [(\lambda L)^{h-1}/(h - 1)!]. \qquad (3)$$

The proof of the theorem appears in the Appendix. Let us now use it to obtain $\alpha(b)$. Figure 1 depicts the sequence of events of interest.

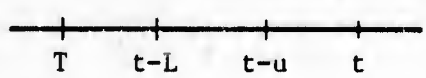


Figure 1

At time t (for large t) a customer demand enters the facility, causing the total number of backorders to become $b \ge 1$. The order that will satisfy this demand has already been placed, say at time $t - u$. The order is still outstanding and hence $u < L$. We are interested in the expected value of u, since $\alpha(b) = L - E(u)$.[†] We observe the following about u:

1. If $Q \ge b \ge 1$, then time $(t - u)$ corresponds to the arrival time of the $(s + b)$th previous customer to enter the facility before t; that is, the demand triggering the order that will satisfy the demand entered at t. If $2Q \ge b \ge Q + 1$, then time $(t - u)$ corresponds to the arrival time of the $(b - (Q - s))$th previous customer to enter the facility before t. In general,

[†]We see from the independent increments of the Poisson process that for large $n, E[\tau(n) \,|\, B(W(n)^-) = b - 1]$ is independent of n.

if $(n + 1)Q \geq b \geq nQ + 1$, then time $(t - u)$ corresponds to the arrival time of the $(b - (nQ - s))$th previous demand to enter the facility before t. These statements hold true for all cases $Q > s$, $Q = s$, and $Q < s$.

2. b depot backorders at time t provide us with some information concerning where in the interval $(t - L, t)$ the time $(t - u)$ occurs. Let T denote the most recent time before $(t - L)$ at which the facility places an order. Since the inventory position at the facility after placing the order at T is $s + Q$, and since this order will arrive by time t, we have

$$D(t) - D(T^{+}) = s + Q + b.$$

Thus,

$$D(t) - D(t - L) + D(t - L) - D(T^{+}) = s + Q + b. \tag{4}$$

But by definition of T, the facility places no orders in the interval $(T^{+}, t - L)$, hence

$$0 \leq D(t - L) - D(T^{+}) < Q.$$

Using this inequality in (4) we obtain,

$$s + b < D(t) - D(t - L) \leq s + Q + b.$$

Using the above two observations, it follows that calculating the expected value of u given that $B(t^{-}) = b - 1$, and that a customer enters the facility at t is equivalent to calculating

$$E[t - W(N(t) - j) | W(N(t)) = t,\ s + b < D(t) - D(t - L) \leq s + Q + b],$$

where $j = b(\text{mod } Q) + s$. Thus the theorem applies, and since $\alpha(b) = L - E(u)$, we obtain

$$\alpha(b) = L - \left\{(x + s)L \sum_{h=s+b+1}^{s+b+Q} [(\lambda L)^{h-1}/h!] \Big/ \sum_{h=s+b+1}^{s+b+Q} [(\lambda L)^{h-1}/(h - 1)!]\right\}, \quad (5)$$

where $x \equiv b (\mathrm{mod}\ Q)$.

Hadley and Whitin [2, p. 184] have shown that for $b \geq 0$,

$$\Pr[B(t^*) = b] = (1/Q) \sum_{h=s+b+1}^{s+b+Q} \exp(-\lambda L)(\lambda L)^h/h!. \qquad (6)$$

Using expressions (5) and (6) in (2), we simplify and obtain

$$E[\tau] = L \cdot P_{out} - (1/\lambda) \sum_{n=0}^{\infty} \sum_{x=1}^{Q} (x + s)\Pr[B(t^*) = nQ + x], \qquad (7)$$

where $P_{out} = \Pr[B(t^*) \geq 0]$.

EXPONENTIAL SERVICE TIMES

We assume here that procurement lead-times are independent, identically distributed random variables each following the exponential probability law with mean L.

Given that $B(W(n)^-) = b - 1$, let $c(b)$ denote the number of orders outstanding (including the one possibly triggered by the arrival of the nth customer), and let $a(b)$ denote the number of orders that must arrive before the nth customer is served. It is easily seen that $a(b) = ([b/Q]) + 1$ and that $c(b) = ([(s + Q + b)/Q])^{\dagger}$, where $([z])$ denotes the greatest integer k such that $k \leq z$.

Since the excess random variable of an exponentially distributed random variable is exponentially distributed with the same mean, the times from $W(n)$ until each of the $c(b)$ outstanding orders arrives are

$^{\dagger}$This expression assumes that $B(0) = -(s + Q)$. When we pass to the Cesàro limit, the initial condition becomes immaterial.

exponentially distributed with mean L. Thus, $E[\tau(n)|B(W(n)^-) = b - 1]$ equals the expected value of the a(b)th order statistic of c(b) random variables independently and identically distributed as an exponential random variable with mean L. Using the technique found in [3, p. 160], it is easily shown that the expected value of this order statistic is given by

$$\alpha(b) = L \sum_{h=0}^{a(b)-1} 1/c(b - h) . \tag{8}$$

That $\alpha(b)$ equals this expression follows from the independence of the right-hand side with respect to n.

For exponentially distributed procurement lead-times, the expression for $\Pr[B(t^*) = b]$ is quite complex. If $Q = 1$, then it follows from Palm's theorem [4] that $(s + b + 1)$ follows the Poisson probability law with mean $\lambda L$. Using this result with expression (8) in (2), we obtain

$$E[\tau] = L \exp(-\lambda L) \sum_{b=1}^{\infty} [(\lambda L)^{s+b}/(s + b)!] \sum_{h=0}^{b-1} 1/(s + b + 1 - h). \tag{9}$$

For general $Q \geq 1$, the distribution of $B(t^*)$ can be obtained from Markov chain theory [1]. For $n = 0, 1, \ldots$ and $k = 0, 1, \ldots, Q - 1$,

$$\Pr[B(t^*) = k + (n - 1)Q - s] = \sum_{\ell=n}^{\infty} (-1)^{\ell-n} \binom{\ell}{n} (r_\ell)^{k+1} x_\ell, \tag{10}$$

where $r_\ell = \lambda L/(\lambda L + \ell)$, and

$$x_\ell = (r_1 \cdot r_2 \cdots r_{\ell-1})^Q/[Q(1 - r_1^Q)(1 - r_2^Q)\ldots(1 - r_\ell^Q)].$$

Using expressions (8) and (10) in (2), $E[\tau]$ can be calculated.

For $Q = 1$, the stationary distribution of $B(t^*)$ depends only upon the mean of the procurement lead-time distribution [4]. From expressions (5) and (8) it can easily be shown that $E[\tau]$ is always shorter for a deterministic procurement lead-time L than with an exponentially distributed procurement lead-time with mean L.

## III. AN APPROXIMATION

The following approximation to $E[\tau]$ has been used in other studies [6]:

$$E[\tau] = \lim_{t\to\infty} E[\text{total waiting time in } (0, t]/D(t)]$$

$$= \lim_{t\to\infty} E[\int_0^t \hat{B}(z)dz/D(t)],$$

where $\hat{B}(z) = \max\{0, B(z)\}$. Thus,

$$E[\tau] = \lim_{t\to\infty} E[\frac{1}{t}\int_0^t \hat{B}(z)dz/(D(t)/t)]$$

$$\approx \lim_{t\to\infty} \left[E[\frac{1}{t}\int_0^t \hat{B}(z)dz]/E[D(t)/t]\right]$$

$$= E[\hat{B}(t^*)]/\lambda, \tag{11}$$

where

$$E[\hat{B}(t^*)] = \sum_{b=1}^{\infty} b \cdot \Pr[B(t^*) = b].$$

The approximation is used because the random variables $D(t)/t$ and $\frac{1}{t}\int_0^t \hat{B}(z)dz$ are not independent. The approximation (11) may be calculated by using expressions (6) or (10) to evaluate $E[\hat{B}(t^*)]$.

Numerical calculations performed for the case of deterministic procurement lead-times indicate that the approximate expression (11) is accurate to within 1 percent of the exact expression. Numerical calculations were also performed for the case of exponentially distributed procurement lead-times with $Q = 1$. In this case, the approximation underestimated the true value by 17 to 56 percent.

# APPENDIX

## PROOF OF THEOREM

Since the Poisson process has independent increments, we may assume that $t = L$. Thus by the definition of $\theta(j,k,m,L,t)$,

$$\theta(j,k,m,L,t) = L - E[W(N(L) - j) | W(N(L)) = L, k < N(L) \le m]. \quad (12)$$

Let us denote the second term of (12) by $\theta'(j,k,m,L)$. Now

$$\theta'(j,k,m,L) = \sum_{\ell=k+1}^{m} E[W(N(L) - j) | W(N(L)) = L, N(L) = \ell] \cdot$$

$$\Pr[N(L) = \ell | W(N(L)) = L, k < N(L) \le m]. \quad (13)$$

As mentioned in [5, p. 142],

$$E[W(N(L) - j) | W(N(L)) = L, N(L) = \ell]$$

is equal to the expected value of the $(\ell - j)$th order statistic of $(\ell - 1)$ random variables independently and identically distributed as a uniform random variable on the interval $(0, L]$. This expected value is easily seen to be $(\ell - j)L/\ell$. Using this result in (13) yields

$$\theta'(j,k,m,L) = \sum_{\ell=k+1}^{m} [(\ell - j)L/\ell] \cdot \Pr[N(L) = \ell | W(N(L)) = L, k < N(L) \le m]. \quad (14)$$

For $\ell \in \{k + 1, \ldots, m\}$ we have,

$$\Pr[N(L) = \ell | W(N(L)) = L, k < N(L) \le m] = \gamma(L;\lambda,\ell) / \sum_{h=k+1}^{m} \gamma(L;\lambda,h),$$

where $\gamma(\cdot;\mu,r)$ denotes the Gamma probability density function with parameters $\mu$ and $r$. Thus,

$$\Pr[N(L) = \ell \,|\, W(N(L)) = L,\ k < N(L) \le m]$$

$$= \left(\frac{\lambda}{(\ell-1)!}(\lambda L)^{\ell-1}e^{-\lambda L}\right) \Big/ \left(\sum_{h=k+1}^{m} \frac{\lambda}{(h-1)!}(\lambda L)^{h-1}e^{-\lambda L}\right)$$

$$= [(\lambda L)^{\ell-1}/(\ell-1)!]/[\sum_{h=k+1}^{m} (\lambda L)^{h-1}/(h-1)!].$$

Using this in (14) we obtain,

$$\theta'(j,k,m,L) = \sum_{\ell=k+1}^{m} \{[(\ell-j)L/\ell][(\lambda L)^{\ell-1}/(\ell-1)!]\}/(\sum_{h=k+1}^{m} (\lambda L)^{h-1}/(h-1)!)$$

$$= L\{\sum_{\ell=k+1}^{m} (\lambda L)^{\ell-1}/(\ell-1)!\}/\{\sum_{h=k+1}^{m} (\lambda L)^{h-1}/(h-1)!\}$$

$$- jL\{\sum_{\ell=k+1}^{m} (\lambda L)^{\ell-1}/\ell!\}/\{\sum_{h=k+1}^{m} (\lambda L)^{h-1}/(h-1)!\}$$

$$= L - jL\{\sum_{h=k+1}^{m} (\lambda L)^{h-1}/h!\}/\{\sum_{h=k+1}^{m} (\lambda L)^{h-1}/(h-1)!\}.$$

Using this result in (12) we obtain expression (3), which proves the theorem.

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10. ABSTRACT

The description of a single-facility, single-consumable item inventory model in which demands are Poisson, procurement lead-times are either deterministic or exponential, and a continuous review (s, S) replenishment policy is employed. An exact expression is derived for the steady-state expected time that a customer waits to be satisfied. Although this performance measure is not usually studied for inventory models, it is useful for imbedding the facility in a multi-echelon structure.

11. KEY WORDS

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