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AIR FORCE CAMBRIDGE RESEARCH LABORATORIES

L. G. HANSCOM FIELD, BEDFORD, MASSACHUSETTS

On the Problem of Equal-Ripple Power Pattern Synthesis

HANS STEYSKAL

OFFICE OF AEROSPACE RESEARCH

United States Air Force

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On the Problem of Equal-Ripple Power Pattern Synthesis

HANS STEYSKAL

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Abstract

This report on antenna array pattern synthesis deals with the approximation of power patterns rather than the usual problem of approximating field patterns. After a qualitative comparison of the different characteristics, an optimum approximation is formulated in general terms. The possibility of finding an exact 'equal ripple' solution is discussed, and an approximate 'equal ripple' solution is offered as an approach to this as yet unsolved mathematical problem.

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On the Problem of Equal-Ripple Power Pattern Synthesis

1. INTRODUCTION

An antenna array consists of a number of identical radiating elements. Individual control of the current amplitude and phase of each element ensures a resultant radiation pattern with greater flexibility than can be achieved by means of continuous-aperture antennas whose radiation properties are set by the geometric configuration. The problem is to determine the element excitation that will produce the desired resultant radiation pattern. Although field pattern synthesis has been treated extensively power pattern synthesis has received comparatively little notice. An up-to-date bibliography on this topic is cited by Schell (1969).

The farfield E from a linear array of N+1 isotropic radiators is given (Silver, 1949) by

$$E(r, \theta) = \frac{e^{-ikr}}{r} \sum_{n=0}^{N} I_n e^{ikd_n \sin\theta} , \qquad (1)$$

where k is the wave number and d_n and θ are as shown in Figure 1.

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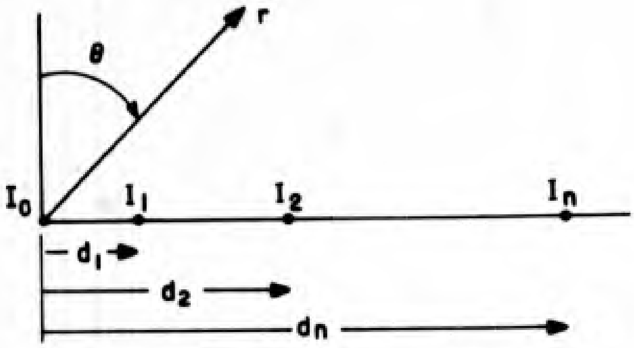

Figure 1. Coordinate System

Since only the angular behavior of the field is required, we separate out

$$E(\theta) = \sum_{n=0}^{N} I_n e^{ikd_n \sin\theta} \quad . \tag{2}$$

The problem of antenna pattern synthesis is to find a suitable set $\{I_n\}_0^N$ so that Eq. (2) approximates a desired function $E_D(\theta)$. If both the amplitude and phase of E_D, which is a complex function, are prescribed, then it is comparatively easy to find the proper set $\{I_n\}_0^N$.

In most applications, only $|E_D|$, the antenna power pattern (commonly called the array factor), is of interest. We write

$$q(\theta) = E\bar{E} = \sum_{n=0}^{N} I_n e^{ikd_n \sin\theta} \sum_{n=0}^{N} \bar{I}_n e^{-ikd_n \sin\theta} \quad , \tag{3}$$

where the overbar marks the complex conjugate and the letter q (for quadratic) represents the power pattern. Power pattern synthesis is a nonlinear problem and hence much more complicated than E-field pattern synthesis, but its closer resemblance to actual conditions is worth the increase in mathematical difficulty.

2. A QUALITATIVE COMPARISON OF FIELD AND POWER PATTERN SYNTHESIS

In this comparison of field and power pattern synthesis, only equally spaced arrays with $d_n = n\lambda/2$ are considered. With a new variable $u = \sin\theta$, Eq. (3) becomes:

$$q(I) = E\bar{E} = \sum_{0}^{N} I_n e^{in\pi u} \sum_{0}^{N} \bar{I}_n e^{-in\pi u} = \sum_{-N}^{N} q_n e^{in\pi u} . \quad (4)$$

2.1 Field Pattern Synthesis

For example, consider a square-top pattern, as in Figure 2(a). Since the phase of E_D is not specified it is usually set equal to a constant, with the resulting E-field pattern as in Figure 2(b) and power pattern (q) as in Figure 2(c). A comparison of Figures 2(b) and (c) shows what happens to the new harmonics created by the multiplication of E by $\bar{E}$:

i) The slope (risetime) is practically unchanged; there are no new harmonics.

ii) In the mainlobe region the approximation error changes from ϵ to 2ϵ, with the same ripple rate; there are no new harmonics.

iii) In the sidelobe region the approximation error changes from ϵ to ϵ^2, with twice the ripple rate, new harmonics are created, seen mainly in the low sidelobe level and in the sudden transition between the sidelobe region and the flank of the mainlobe region.

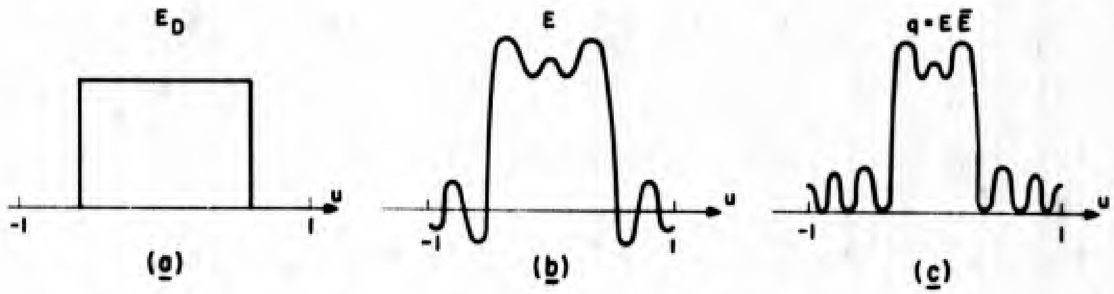

Figure 2. Synthesis of a Field Pattern [(a) desired pattern; (b) synthesized field pattern; (c) power pattern, obtained by squaring the magnitude of the field pattern shown in Figure 2(b)]

When pattern shape is important over an extended region, field pattern synthesis is not a suitable solution because half of the harmonics contained in q go to make up the sidelobe region. The shorter the pattern region, the less significant this selective property of field pattern synthesis. If the desired pattern is a δ function, then the whole interval $-1 \leq u \leq 1$ is a sidelobe region; field pattern synthesis would not in this case imply an unjustified stressing of one or the other part of the pattern since E and q synthesis would both yield the same optimum pattern. This can also be seen by studying E in the complex plane, as Taylor and Whinnery (1951) have done.

2.2 Power Pattern Synthesis

The skewness inherent in the particular use of the harmonics for E-pattern synthesis can be avoided by approximating the desired power pattern p_D directly with a real trigonometric sum:

$$q = \sum_{-N}^{N} q_n e^{in\pi u} , \qquad (5\underline{a})$$

with the additional condition

$$q(u) \geq 0 , \quad -1 \leq u \leq 1 , \qquad (5\underline{b})$$

which is a necessary and sufficient condition that q be a physically realizable power pattern of an N+1 element array, that is, that q can be written $q = E\bar{E}$ (Silver, 1949). Using q in the power pattern synthesis problem has the following consequences:

i) The ripple rate will be the same in the mainlobe region as in the sidelobe region since the harmonics are not used selectively. This implies a doubling of the ripple rate in the mainlobe region (compared with the rate obtainable in field pattern synthesis with constant phase).

ii) It will be critical that q really take on small values in the sidelobe region. Since the final approximation will not be squared as in field pattern synthesis, the power pattern synthesis method must be able to handle a greater 'dynamic' range to closer 'tolerances.'

iii) The form of Eq. (5a) may give the impression that q synthesis allows twice as many degrees of freedom as E synthesis, but the condition expressed in Eq. (5b) limits q to only N+1 free variables. Nevertheless, q synthesis utilizes twice the number of freedoms as E synthesis because the latter presets the phase of the field.* As previously seen, however, the improvement in approximation is usually not twice as good but depends very much on the particular pattern to be synthesized.

---

* After E has been written

$$E = e^{in\pi u/2} \sum_{-N/2}^{N/2} I_{n+N/2} e^{in\pi u} = e^{iN\pi u/2} \sum_{-N/2}^{N/2} I_m e^{im\pi u} ,$$

the second factor is made real by setting $I_{-m} = \bar{I}_m$, leaving only half the number of I's to be determined. Here an odd number of elements was assumed, but the same result holds for an even number—only the formulation is slightly different.

iv) Since $q = E\bar{E}$ is quadratic in $\{I_n\}_0^N$, q synthesis is no longer a linear problem. There is therefore no 1-to-1 correspondence between q and $\{I_n\}$ but normally there exist several such sets, all producing the same power pattern (Silver, 1949; Taylor and Whinnery, 1951).

3. THE APPROXIMATION PROBLEM IN GENERAL

Desired power patterns $p_D(u)$ are usually discontinuous and nonphysical. The process of approximating p_D by a power pattern q can be very nicely formulated in a Hilbert space, which allows for a clear and attractive geometric interpretation in handling the discontinuous functions. Since the desired approximation q will be a sum of harmonics it seems reasonable to let the basis vectors be combinations of $\{e^{in\pi u}\}_{-\infty}^{\infty}$, forming a complete basis for the class $L^2[-1, 1]$. We are thereby limited to dealing with absolutely square integrable p_D's in the following, but this is no great restriction. As usual, we choose the length or norm of a vector x to be the root of the inner product, or $\|x\|^2 = (x, x)$.

Without implying loss of generality, we now number our orthonormal basis vectors e_n from $-\infty$ to $+\infty$, and we choose them so that a vector e_m contains only the harmonics $\{e^{in\pi u}\}_{-m}^{m}$. The set Q, which is composed of all possible power patterns (nonnegative trigonometric sums that are composed of $\{e^{in\pi u}\}_{-N}^{N}$), obviously forms a subset in the 2N+1 dimensional space spanned by $\{e_n\}_{-N}^{N}$.

The whole synthesis problem now resolves itself into finding the particular $q \in Q$ that leads to $\|p_D - q\| =$ minimum, and can be solved in two steps (see Figure 3):

i) Find the projection p_{2N+1} of p_D on the subspace spanned by $\{e_n\}_{-N}^{N}$. This approximates p_D by a finite trigonometric sum

$$p_{2N+1} = \sum_{-N}^{N} p_n e^{in\pi u} .$$

ii) Find the $q \in Q$ that gives $\|p_{2N+1} - q\| = \min$. This approximates p_{2N+1} by the optimum q, satisfying the condition $q \geq 0$.

Our approximation to p_D must satisfy two conditions: (a) it must be a trigonometric sum with 2N+1 terms, and (b) it must be nonnegative. As we have outlined them, these two conditions have been essentially separated (or uncoupled) so that they can be fulfilled independently to facilitate analysis.

To understand how $\|p_D - q\|$, $q \in Q$ is minimized, consider

$$\|p_D - q\|^2 = (p_D - p_{2N+1} + p_{2N+1} - q,\ p_D - p_{2N+1} + p_{2N+1} - q) = \|p_D - p_{2N+1}\|^2 +$$

$$+ \|p_{2N+1} - q\|^2 + (p_D - p_{2N+1},\ p_{2N+1} - q) + (p_{2N+1} - q,\ p_D - p_{2N+1})$$

$$= \underbrace{\|p_D - p_{2N+1}\|^2}_{\text{error due to finite number of terms}} + \underbrace{\|p_{2N+1} - q\|^2}_{\text{error due to condition } q \geq 0} .$$

Since $p_D - p_{2N+1}$ and $p_{2N+1} - q$ are orthogonal, their inner product is zero; and minimization of one is independent of the other.

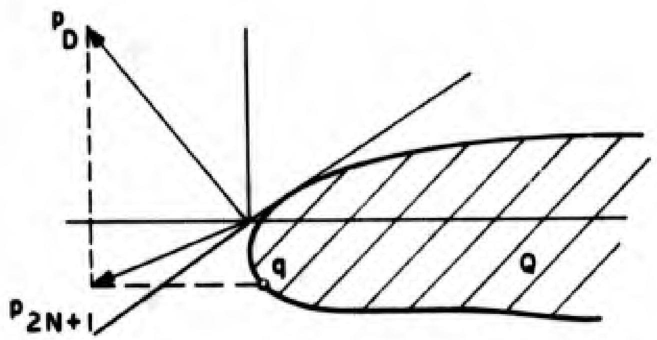

Figure 3. Geometric Interpretation of Pattern Synthesis

We now want to show that Q is a convex set, from which it will follow that there exists one and only one minimum $\|p_{2N} - q\|$. A set is called convex if the line segment between any two points of the set is also contained within the set. The line segment between q_1 and $q_2 \in Q$ is

$$q_1 + \lambda(q_2 - q_1), \quad 0 \leq \lambda \leq 1 ,$$

and since

$$q_1 + \lambda(q_2 - q_1) = (1 - \lambda)q_1 + \lambda q_2,$$

which is clearly included in Q for all λ, it follows that Q is a convex set.

Consider now the minimization of $\|p - q\|$, with $p \notin Q$, $q \in Q$, and suppose there exists a local minimum q_1 and a global minimum q_2 (Figure 4). Join q_1 and q_2 by a line, which because of the convexity lies within Q. Then

$$\|p - q_2\| = \text{min.} \longrightarrow \|q_2 - p + \lambda(q_1 - q_2)\| \geq q_2 - p\| \quad \text{for some } \lambda,\ 0 \leq \lambda \leq 1.$$

$$\|q_2 - p + \lambda(q_1 - q_2)\| = \|(1 - \lambda)q_2 - p + \lambda q_1 + \lambda p - \lambda p\| = \|(1 - \lambda)(q_2 - p) +$$
$$+ \lambda(q_1 - p)\| \leq (1 - \lambda)\|q_2 - p\| + \lambda\|q_1 - p\| .$$

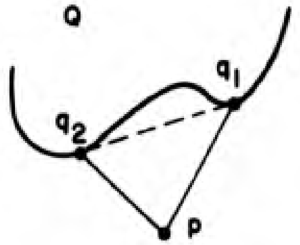

Figure 4. Geometry for the Possibility of a Global and a Local Minimum

Therefore,

$$(1 - \lambda)\|q_2 - p\| + \lambda\|q_1 - p\| \geq \|q_2 - \lambda(q_1 - q_2)\| \geq \|q_2 - p\| ,$$

which gives

$$\|q_1 - p\| \geq \|q_2 - p\| .$$

Similarly, starting with $\|p - q_1\| = \text{min.}$ gives

$$\|q_2 - p\| \geq \|q_1 - p\| .$$

The only possibility left is thus

$$\|q_1 - p\| = \|q_2 - p\|.$$

Note that the convexity of Q and its implication hold in any normed linear space (Banach space).

The choice of the inner product has decisive significance since it determines the norm (distance) in our space. Usually, the inner product is taken as

$$(p_1, p_2) = \int \rho p_1 \bar{p}_2 du ,$$

where ρ is a weighting function, $\rho > 0$. Since our approximation q to p_D is determined by $\|p_D - q\| = \min.$, we must choose the inner product so that the norm not only expresses the properties desired in p_D but the way we want p_D approximated.

With the right inner product our problem is in principle solved. The Gram-Schmidt orthogonalization procedure gives us an orthonormal basis system $\{e_n\}_{-\infty}^{\infty}$ where every e_n is a linear combination of harmonics from the set $\{e^{in\pi u}\}_{-\infty}^{\infty}$. We express p_D in this space,

$$p_{2N+1} = \sum_{-N}^{N} (p_D, e_n)e_n ,$$

and then find the q that minimizes $\|p_{2N+1} - q\|$.

The case of least mathematical complexity is that for which p_D is approximated in the least-mean-square sense. Here $\rho \equiv 1$, and p_{2N+1} becomes the normal Fourier series truncated after 2N+1 terms. This case has been treated by Schell (1969).

4. THE PROBLEM OF SPECIFYING APPROXIMATION CRITERIA

We now discuss how we want to approximate a given nonphysical power pattern p_D, and our requirements must be formulated mathematically as a minimization criterion. This approximation problem has received scant notice in the literature. The only case for which the approximation criterion is explicitly stated and the corresponding optimum q found seems to be the one treated by Schell. Two consequences of this minimization criterion are (1) the sidelobe level cannot be controlled, and (2) large sidelobes, independent of the number of terms used in the approximation, occur in the vicinity of discontinuities (Gibbs phenomenon). The following approximation requirements, considered to come close to many realizable specifications, are more flexible.

Given a prescribed pattern p_D, which has one continuous pattern region with step discontinuities at one or both ends, find a realizable power pattern approximation q that gives

i) An equal relative error in the pattern region.

ii) Equal sidelobes whose height can be controlled but at the cost of an increased error in the pattern region and steepness of the pattern flank.

This q will be the optimum power pattern. It will show equal ripple on the logarithmic scale normally used in studying antenna patterns but will have two different ripple amplitudes - one in the mainlobe region and one in the sidelobe region - except at the discontinuities where the error is of course always at least half the height of the discontinuity.

It is inadvisable to aim for an equal-ripple amplitude in the mainlobe and sidelobe regions since an amplitude of 0.1 (say) represents a small error when the main pattern has a value of approximately 1, but a 0.1 (-10dB) sidelobe level may be prohibitively high. In the majority of such cases the antenna system designer prefers to trade higher pattern ripple for lower sidelobes.

5. THE QUESTION OF THE INNER PRODUCT

We now consider the problem of finding an inner product that norms our space to approximate the desired p_D. Studying the approximation of a square-top pattern [Figure 2(a)] in the least-mean-square sense will reveal the properties our inner product must possess. The general behavior of the error $p_D - p_{2N+1}$ will be characterized (Sommerfeld, 1949) by:

i) Mainlobe and sidelobe regions that have errors of equal magnitude; the errors in the two regions therefore contribute equally to the integral:

$$\int |p_D - p_{2N+1}|^2 du .$$

ii) The error (overshoot) in the vicinity of the step discontinuity will always be $\simeq 9$ percent of the height of the step (Gibbs phenomenon). The further the distance from the discontinuity, the smaller the error amplitude.

These two observations make it clear how to choose the ρ in a new inner product

$$(p_1, p_2) = \int \rho p_1 \bar{p}_2 du .$$

Set $\rho = \rho_1 \rho_2$, where

i) ρ_1 is a measure of the relative accuracy with which we want to approximate p_D in the different regions (high value of ρ_1 gives high accuracy)

ii) ρ_2 is chosen so as to counteract the Gibbs phenomenon, and will thus make the ripple amplitude equal and constant on both sides of a step discontinuity.

Since we can specify ρ_1 for any desired sidelobe level, the only remaining question is, Can we find ρ_2? The solution of this purely mathematical problem is the optimum power pattern q. A survey of the mathematical literature (Achieser, 1956; Ch[illegible]y, 1966; Walsh, 1960; Soaty and Bram, 1964) reveals that this is unfortunately a problem that has not yet been solved.* In fact, surprisingly little is known about the pointwise convergence or local approximation properties of trigonometric sums. Existence theorems for solving and methods for estimating the maximum error in harmonic approximation of continuous functions are all that can be found. There exists no simple analytic method for finding the Chebyshev approximation f_T, that is, the function f_T that satisfies $S_u p | f(u) - f_T(u) | = \min.$, to even a continuous function f. The only way to find f_T is through a system of first-order differential equations. The case we are interested in is much, much harder mathematically since we consider functions with discontinuities and thus want piecewise equal-ripple approximations.

6. AN APPROXIMATE SOLUTION TO THE POSED OPTIMIZATION PROBLEM

Section 5 indicates that there exists no analytic method for finding our optimum power pattern q and so an approximate solution of the problem seems justified. The problem can still be nicely viewed in a linear space, and our approximation method also involves two steps. First we find a trigonometric sum p_{2N+1} that is 'almost' the power pattern of the required properties and then we approximate p_{2N+1} with a power pattern q in the least-mean-square sense.

Let us consider the second approximation in more detail, pictured in a 2N+1 dimensional space X with least-square norm $\| \quad \|_X$. We would really like to minimize the distance between p_{2N} and Q with an equal-ripple norm $\| \quad \|_Y$ in another space Y. The reason we work in X is that the norm in Y is not known explicitly but it is clear that X can be mapped into Y by a continuous linear transformation A.

For p_{2N+1} there exist two possibilities – either $p_{2N+1} \in Q$ or $p_{2N+1} \notin Q$ – and this is true whether we are in X or in Y. For $p_{2N+1} \in Q$ we get the same solution $p_{2N+1} = q$ in X and in Y. For $p_{2N+1} \notin Q$ we get different solutions q_X and q_Y – but since X and Y are related by A, if the point q_X is close to p_{2N+1} in X, the point q_Y will be close to p_{2N+1} in Y since A cannot drastically change the local properties. Figure 5 illustrates the situation.

If we start out with a p_{2N+1} close to Q and minimize $\| p_{2N+1} - q \|_X$ instead of $\| p_{2N+1} - q \|_Y$, the error is a second-order effect. Another major advantage of the least-mean-square norm is that it requires the least complicated mathematics.

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*The Chebyshev weighting function $1/\sqrt{1 - x^2}$ results in an expansion of p_D in Chebyshev polynomials but by no means guarantees an equal-ripple approximation.

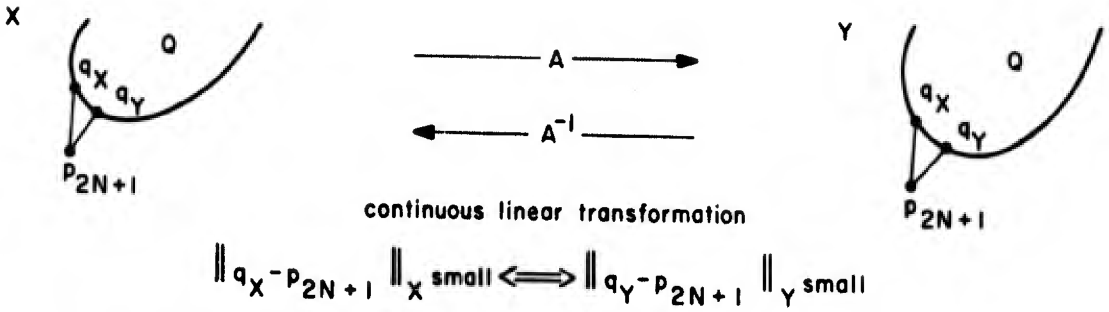


Figure 5. Relationships Between the Approximate Solution and the Exact Location

6.1 The Construction of p_{2N+1}

To find a first approximation p_{2N+1} we will use the Chebyshev polynomials T_{2N}. Dolph uses the Chebyshev polynomials $T_n(x)$ to construct an equal-ripple approximation to a δfunction. Using such functions to sample the desired pattern p_D, we can express $T_n(x)$ in a closed form:

$$T_n(x) = \cos(n \arccos x) .$$

Substituting $x = x_o \cos\frac{\pi}{2}u$, where $x_o > 1$, yields the function

$$T_n(x_o, u) = \cos\left[n \arccos\left(x \cos\frac{\pi}{2}u\right)\right] ,$$

whose general appearance is as shown in Figure 6.

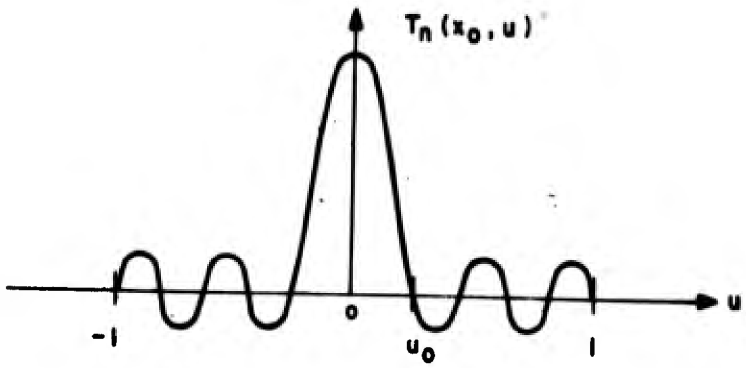


Figure 6. A Chebyshev Polynomial Pattern

Since $T_n(x_o, u)$ is a polynomial in $\cos\frac{\pi}{2}u$ of order n, $T_{2N}(x_o, u)$ is a trigonometric sum of 2N+1 terms, and

$$T_{2N}(x_o, u) = \sum_{m=-N}^{N} K_m e^{im\pi u} .$$

We can now approximate a desired p_D by sampling with $T_{2N}(x_o, u)$ functions and if the sampling distance Δu coincides with the first zero u_o of $T_{2N}(x_o, u)$ then two adjacent samples will not interfere with each other (Figure 7). Furthermore,

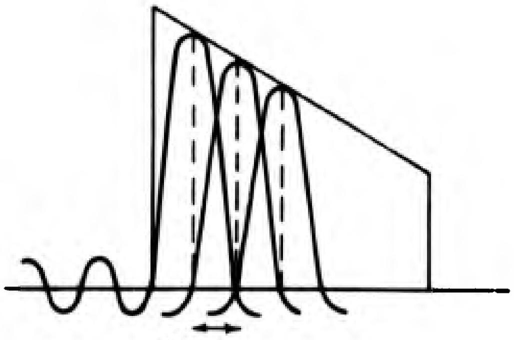


Figure 7. Result of Using Several Chebyshev Sampling Functions to Approximate a Desired Pattern

if we neglect the sidelobes of the individual T function, our approximate pattern gives an error that is proportional to the amplitude of the samples and thus a relative error that is constant. In some cases it may be advantageous to sample with a $\Delta u < u_o$, giving less ripple in the pattern region. (See Appendix A.)

A few expansions will explain the behavior of the sidelobes of the T_{2N} function.

$$T_{2N}(x_o, u) = \cos(2N \operatorname{arc}\cos x \cos\tfrac{\pi}{2}u), \qquad x_o \cos\tfrac{\pi}{2}u \le 1 ;$$

$$T_{2N}(x_o, u) = \cosh(2N \operatorname{arc}\cosh x \cos\tfrac{\pi}{2}u), \qquad x_o \cos\tfrac{\pi}{2}u \ge 1 .$$

We call $T_{2N}(x_o, 0) = R$ the mainlobe/sidelobe ratio. In other words,

$$R = \cosh(2N \operatorname{arc}\cosh x_o),$$

$$x_o = \cosh\left(\frac{\text{arc cosh } R}{2N}\right),$$

and

$$\text{arc cosh } R \simeq \log 2R - \frac{1}{4R^2} - \dots .$$

In all practical cases, $R >> 1$. Therefore,

$$x_o \simeq \cosh\left(\frac{\log 2R}{2N}\right).$$

If, in addition, N is sufficiently large,

$$x_o \simeq 1 + \frac{1}{2}\left(\frac{\log 2R}{2N}\right)^2 ;$$

and if we set $x_o = 1 + \delta$,

$$\delta \simeq \frac{1}{2}\left(\frac{\log 2R}{2N}\right)^2 << 1 .$$

The first zero u_o of $T_{2N}(x_o, u)$ is given by

$$T_{2N}(x_o, u_o) = 0 ,$$

for which

$$2N \text{ arc cos } x_o \cos\frac{\pi}{2}u = \frac{\pi}{2} .$$

Hence,

$$(1 + \delta) \cos\frac{\pi}{2}u_o = \cos\frac{\pi}{4N} .$$

Expanding the cosine terms in a series and neglecting a δu_o^2 term, we obtain

$$u_o \simeq \frac{1}{\pi N}\sqrt{\log^2 2R + (\frac{\pi}{2})^2} \simeq \frac{\log 2R}{\pi N} . \qquad (6)$$

In the sidelobe region,

$$T_{2N}(x_o, u) = \cos 2N \text{ arc cos } x_o \cos\frac{\pi}{2}u = \cos 2N\phi(u),$$

where

$$\phi(u) = \arccos x_o \cos\frac{\pi}{2}u = \arccos[\cos\frac{\pi}{2}u + \delta\cos\frac{\pi}{2}u] \simeq$$

$$\simeq \arccos\cos\frac{\pi}{2}u + \delta\cos\frac{\pi}{2}u \frac{-1}{\sqrt{1-(\cos\frac{\pi}{2}u)^2}} + \frac{\delta(\cos\frac{\pi}{2}u)^2}{2} \frac{\cos\frac{\pi}{2}u}{\left[1-(\cos\frac{\pi}{2}u)^2\right]^{3/2}} + \ldots$$

$$\simeq \frac{\pi}{2}u - \delta\cot\frac{\pi}{2}u + \frac{\delta^2}{2}(\cot\frac{\pi}{2}u)^3 + \ldots .$$

For ϕ to be accurately represented by the first two terms only,

$$\delta\cot\frac{\pi}{2}u >> \frac{\delta^2}{2}(\cot\frac{\pi}{2}u)^3 ,$$

or

$$\frac{\delta}{2}(\cot\frac{\pi}{2}u)^2 << 1,$$

$$\frac{1}{4}\left(\frac{\log 2R}{2N}\right)^2 (\cot\frac{\pi}{2}u)^2 << 1,$$

$$\frac{\log 2R}{4N} << \log\frac{\pi}{2}u \simeq \frac{\pi}{2}u ,$$

$$u >> \frac{\log 2R}{2\pi N} \simeq \frac{u_o}{2} .$$

In the whole sidelobe region, $\phi(u)$ is thus excellently approximated by the first two terms only, and

$$T_{2N}(x_o, u) \simeq \cos(\pi N u - 2N\delta\cot\frac{\pi}{2}u) .$$

Usually also, $N\delta << 1$, which makes $T_{2N}(x_o, u)$ almost perfectly periodic over the greater part of the sidelobe region.

Consider now the synthesis of a square-top pattern as in Figure 8.

The sidelobes from different samples can give a constant interference pattern only in a region where they are almost periodic, that is, in the vicinity of $|u| \simeq 1$. Say they all add up in a certain way at $u = -1$. As u approaches zero they gradually get out of phase, but the picture does not change much because even close to the point $u = -0.5$, most of the samples and their mainlobes are still far away and will thus have maintained their phase pattern.

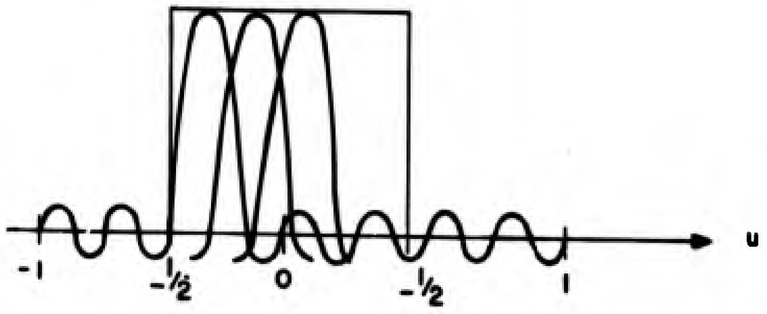
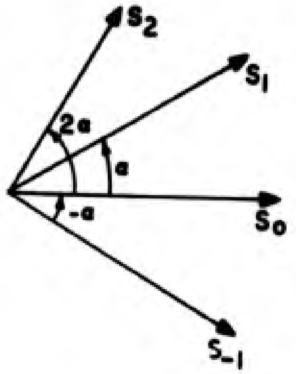


Figure 8. Approximation of a Square-Top Pattern by Chebyshev Sampling Functions

With the help of phasors it is easy to construct the resultant sidelobe level at $|u| = 1$. We call the samples S_n, with $n = 0, \pm 1, \pm 2, \ldots$, and normalize them to $S_n \leq 1$. Using the phase angle given by Eq. (6) and $\cot\frac{\pi}{2}u \simeq \frac{\pi}{2}(1-u)$, we can represent the picture as in Figure 9.



$$a = \pi N x_0 \Delta u$$

$$S_n = |S_n| e^{i\pi N x_0 n \Delta u}$$

Figure 9. Phasors of the Sampling Functions at $|u| = 1$

The final sidelobe level is the vector sum of all the phasors. A relatively small change in Δu can drastically change this level if we have many samples. This is a very attractive feature because if we find that for one particular Δu all the sidelobes (phasors) add up together, we can change that Δu to make the sidelobes partially cancel without appreciably affecting the approximation in the pattern region.

The question now is how to choose the angle between the phasors. The two characteristic cases feature:

i) One dominant sample – this sample dominates even the sidelobe behavior.

ii) Several samples of equal unit height. We do not of course want all the samples to add up, but because of the aperiodicity it is also useless to try to make them cancel out completely. A good intermediate value seems to be 1, and this is further explained in the following.

So far we have constructed a harmonic sum

$$p'_{2N+1} = \sum_n S_n T_{2N}(x_o, u-n\Delta u) , \tag{7}$$

which in the sidelobe region oscillates about zero with almost constant amplitude. We now add a final touch. We add 1 to p'_{2N+1} to make it $p_{2N+1} = p'_{2N+1} + 1$. The significance of the added constant is very great. It essentially eliminates the Gibbs phenomenon in the subsequent approximation by a power pattern q.

Qualitatively this can be explained as follows. (See Figure 10.) If we try a least-mean-square fit to a step, the better fit that results in the a region causes an overshoot (Gibbs phenomenon) in the b region. This trade-off between fit in a and fit in b regions will always exist since the slope of the step can never be matched.

In fitting a q to p'_{2N+1} we have a similar situation. At c, p'_{2N+1} goes negative and its slope at c will be greater than any that q can match since q cannot cross the zero line. Adding a constant equal to the sidelobe level makes it possible for p_{2N+1} to be matched by a q.

Let us sum up our synthesis method for p_{2N+1} for a desired p_D and sidelobe level R.

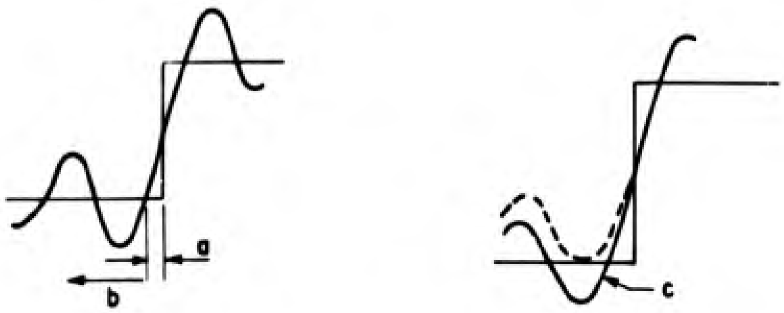


Figure 10. Reduction of Gibbs Phenomenon at a Discontinuity of the Desired Pattern [(left) a least-mean-square fit; (right) addition of a constant to the approximation function (allows a better match to an array power pattern)]

Step 1. Since we sill eventually add 1, we start out with Dolph-Chebyshev patterns whose sidelobe level is 2R. The sidelobe level 2R leads to the sampling distance $\Delta u \simeq u_o$ and δ.

Step 2. Knowing Δu, we find number of samples and their amplitudes $\{S_n\}$.

Step 3. With Δu, δ, and $\{S_n\}$, we construct the phasor diagram and adjust Δu so the phasors add up to 1. (Should this not be possible, as in one of the examples in Sec. 7, we start over again with a larger R.)

Step 4. Add the final sidelobe level (1) to the sum of Dolph-Chebyshev patterns.

The final expression for p_{2N+1} becomes

$$
\begin{aligned}
p_{2N+1} &= 1 + \sum_n S_n T_{2N}(x_o, u - n\Delta u) \\
&= 1 + \sum_n S_n \sum_{m=-N}^{N} K_m e^{im\pi(u-n\Delta u)} \\
&= 1 + \sum_{-N}^{N} K_m \left(\sum_n S_n e^{-i\pi mn\Delta u} \right) e^{im\pi u} ,
\end{aligned}
\tag{8}
$$

where $\{K_m\}$ are the excitation coefficients (currents) belonging to $T_{2N}(x_o, u)$.

Although Steps 1 to 4 can be carried out easily with a slide rule in about 15 minutes, we still need a computer to tell us how good (or bad) the approximation really is.

6.2 The Approximation of p_{2N+1} by q

As mentioned in Sec. 3, we now minimize $\| p_{2N+1} - q \|$ in the least-mean-square sense by the method developed by Schell, who uses a variational technique to obtain an algebraic equation system for those q that make $\| p_{2N+1} - q \|$ stationary. The equation system is solved by iteration on a computer, and since there exists only one minimum it is easy to check that we iterate toward the right q. An additional feature of this technique is that not only q but even one set of corresponding currents $\{I_n\}$ is found.

7. EXAMPLES

This method for power pattern synthesis has been checked by running a few examples on a computer. We assumed a 21-element array, which let us sample p_D with $T_{40}(x_o, u)$ functions. This is a comparatively high-order Chebyshev polynomial, and we used Drane's asymptotic expression for the $\{K_m\}$ in

$$T_{40}(x_o, u) = \sum_{-N}^{N} K_m e^{im\pi u} .$$

The error in these K_m is of the order $1/N = 1/20$, and how this affects the sidelobe level can be seen in Figures 11 to 16, which show the approximate Dolph-Chebyshev patterns for the range R = 40 to 10000. The accuracy is apparently quite good up to $R \simeq 1000$. To see how the iterative mean-square match of q to p_{2N+1} works under near ideal conditions we added the sidelobe level of 1 to these Chebyshev functions and let the computer find a q. The results can be seen in Figures 17 to 22. With increasing R, the more pronounced even a small error becomes in the sidelobe region. Patterns with low sidelobes will thus require many iterations.

The next examples, from nine samples each, are three square-top patterns:

$$p_D(u) = \begin{cases} \text{const.}, & |u| < \frac{1}{2} \\ 0, & |u| > \frac{1}{2} \end{cases}$$

for which Figures 23 to 25 show $|p'_{2N+1}|$ computed by Eq. (7) from the following data:

Sidelobe Level (dB)	R	Δu
20	300	0.1106
26	800	0.1108
32	3200	0.1101

Adding 1 to P'_{2N+1} gave the power patterns shown in Figures 26 to 28.

The last pattern tried was a cosec^2 pattern, of large dynamic range, given by

$$P_D = \begin{cases} 0, & u \leq 0, \quad u > 0.7 \\ \text{cosec}^2 u, & 0 < u < 0.7 \end{cases}$$

with required sidelobes at -20, -26, and -32dB. Figures 29 to 31 show P'_{2N+1} computed from the following data:

Sidelobe Level(dB)	R	Δ_u	S_0	S_1	S_2	S_3	S_4	S_5
20	250	0.1018	1	0.25	0.111	0.063	0.040	0.028
26	800	0.1192	0	1	0.25	0.111	0.063	0.040
32	3200	0.1418	0	1	0.25	0.111	0.063	0

The corresponding power patterns are shown in Figures 32 to 34, and the results are compiled in Table 1.

Table 1. Sidelobe Levels(dB) of Power Patterns

Pattern	R	P'_{2N+1}		q		Final Error
		Design	Achieved	Design	Achieved	
T	40	16	15.9	13	13.1	-0.1
T	100	20	19.9	17	17.0	0
T	400	26	25.7	23	22.7	0.3
T	1000	30	29.3	27	26.1	0.9
T	4000	36	34.8	33	30.2	2.8
T	10000	40	37.3	37	31.9	5.1
square top	300	23	23.1	20	19.8	0.2
square top	800	29	27.0	26	24.2	1.8
square top	3200	35	33.3	32	28.2	3.8
$cosec^2$	250	24	22.1	20	20.4	-0.4
$cosec^2$	800	29	28.7	26	25.4	0.6
$cosec^2$	3200	35	34.8	32	29.1	2.9

The convergence rate of the iterative scheme expressed in terms of sidelobe level versus number of iterations can be seen from Table 2.

Table 2. Sidelobe Levels(dB) After Iteration

Pattern	R	400	800	1200	2000	2400
13-dB T_{2N}	40	13.1				
17-dB T_{2N}	100	17.0				
23-dB T_{2N}	400	22.7				
27-dB T_{2N}	1000	25.5	26.1			
33-dB T_{2N}	4000	28.6	29.3	29.7	30.2	
37-dB T_{2N}	10000	30.0	30.8	31.5	31.9	
20-dB square top	300	18.9	19.6	19.8		
26-dB square top	800	22.2		23.6		24.2
32-dB square top	3200	25.7		27.5		28.2
20-dB $cosec^2$	250	20.4				
26-dB $cosec^2$	800	24.7	25.4			
32-dB $cosec^2$	3200	26.8	27.8	28.5	29.1	

(The computer performed 1000 iterations in five minutes.)

8. COMMENT

In the assumed case of a 21-element array, the proposed synthesis method obviously works quite well for sidelobe levels down to -30 dB. Below that level the method is rather inefficient. The approximations made in $\{K_m\}$, the accuracy of computed values, and the convergence rate of the iterative method for finding q must be reexamined to determine their roles in the unsatisfactory results at large sidelobe ratios. A slow convergence rate seems to be a main offender. The ultimate error $\|p_{2N+1} - q\|$ is not so large that it can be considered a cause. It would be interesting to further study the error committed by minimizing $\|p_{2N+1} - q\|$ in the least-mean-square norm instead of in an equal-ripple norm.

Another related problem is to find a method that would give all sets of currents belonging to the same q or, alternatively, give one particular set that would be optimum with respect to some criterion. An attractive criterion would be "minimum amplitude variation in the currents" or something similar.

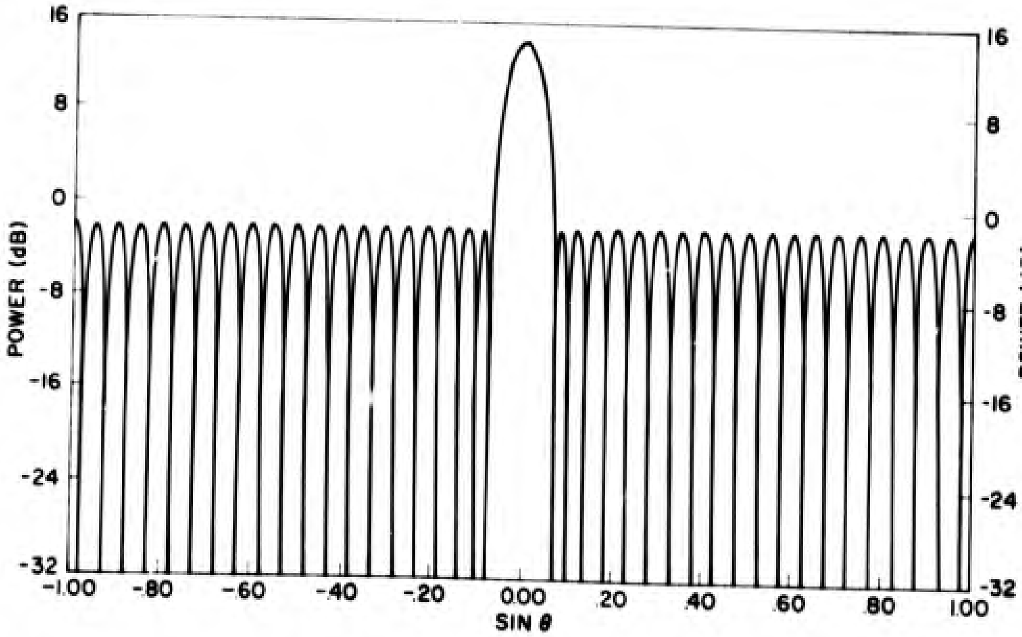


Figure 11. Dolph Chebyshev Patterns of a 21-Element Array, Obtained by Using Drane's Asymptotic Expression: Ratio of Main Beam to Sidelobe Level R = 40

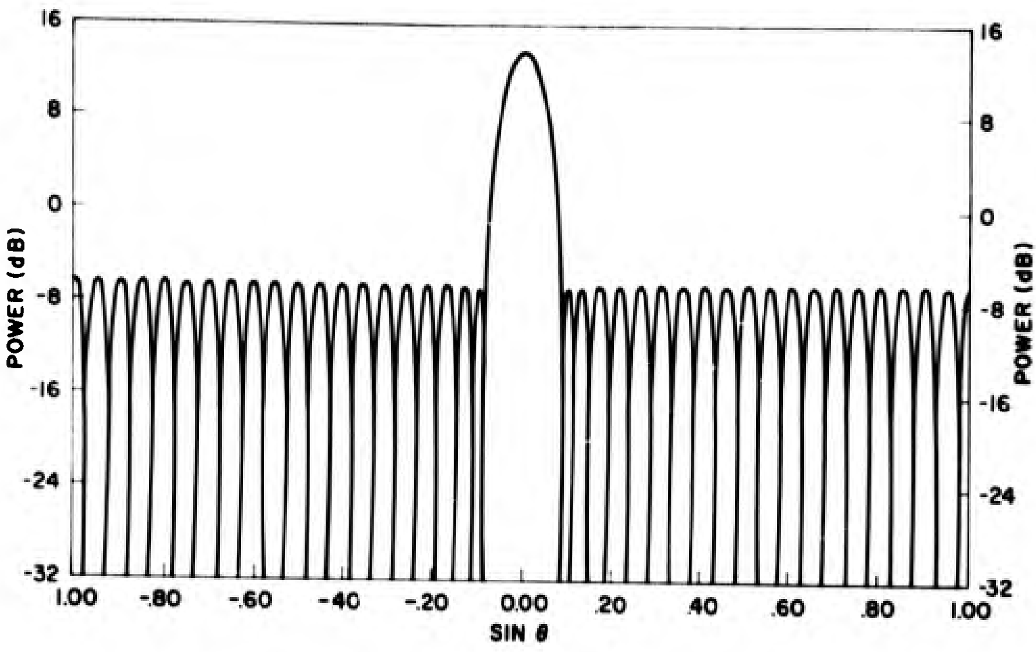


Figure 12. Dolph Chebyshev Patterns of a 21-Element Array, Obtained by Using Drane's Asymptotic Expression: Ratio of Main Beam to Sidelobe Level R = 100

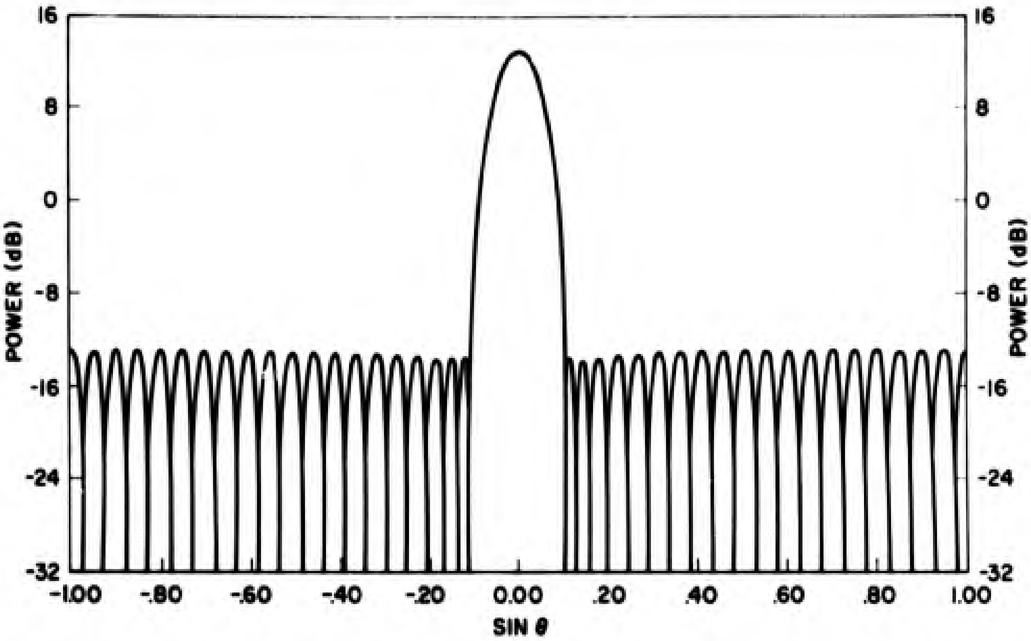


Figure 13. Dolph Chebyshev Patterns of a 21-Element Array, Obtained by Using Drane's Asymptotic Expression: Ratio of Main Beam to Sidelobe Level R = 400

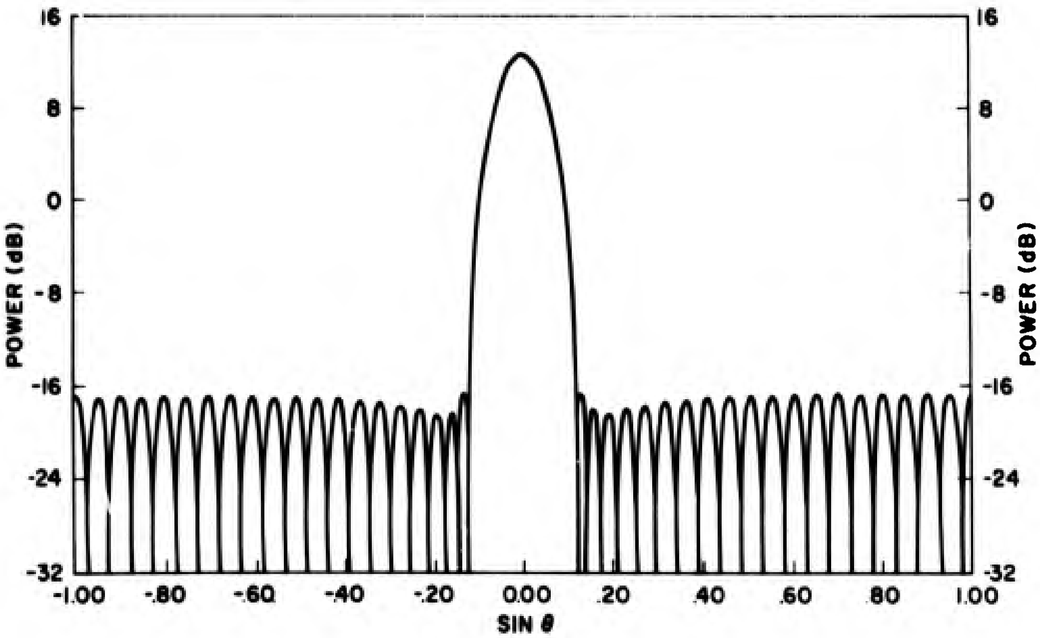


Figure 14. Dolph Chebyshev Patterns of a 21-Element Array, Obtained by Using Drane's Asymptotic Expression: Ratio of Main Beam to Sidelobe Level R = 1000

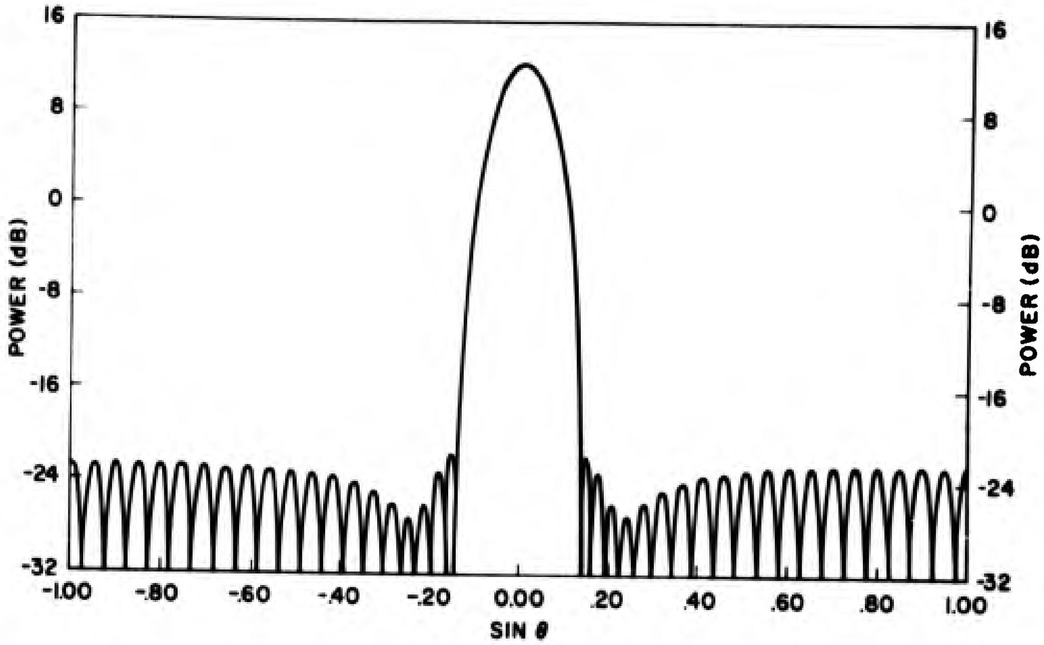


Figure 15. Dolph Chebyshev Patterns of a 21-Element Array, Obtained by Using Drane's Asymptotic Expression: Ratio of Main Beam to Sidelobe Level R = 4000

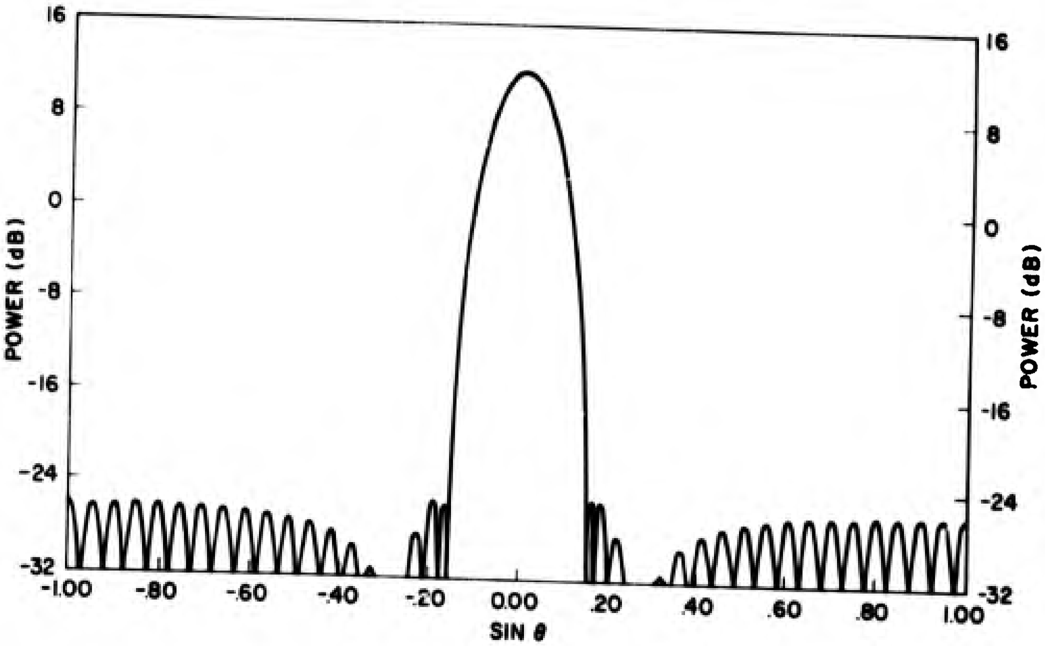


Figure 16. Dolph Chebyshev Patterns of a 21-Element Array, Obtained by Using Drane's Asymptotic Expression: Ratio of Main Beam to Sidelobe Level R = 10000

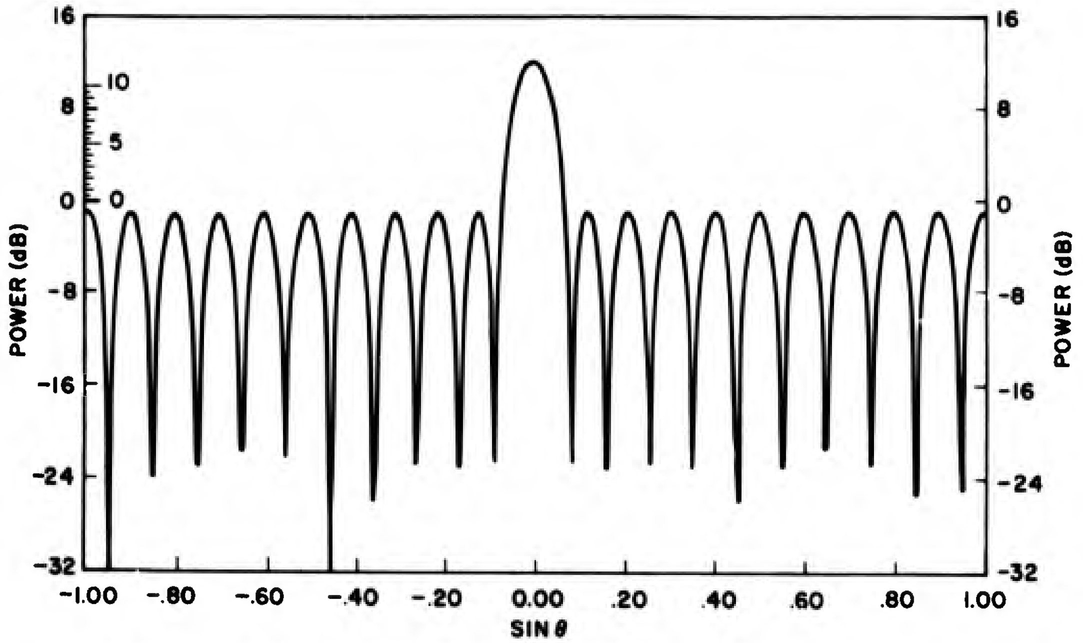


Figure 17. Iterative Least-Mean-Square Match to the Patterns in Figures 11 to 16: R = 40

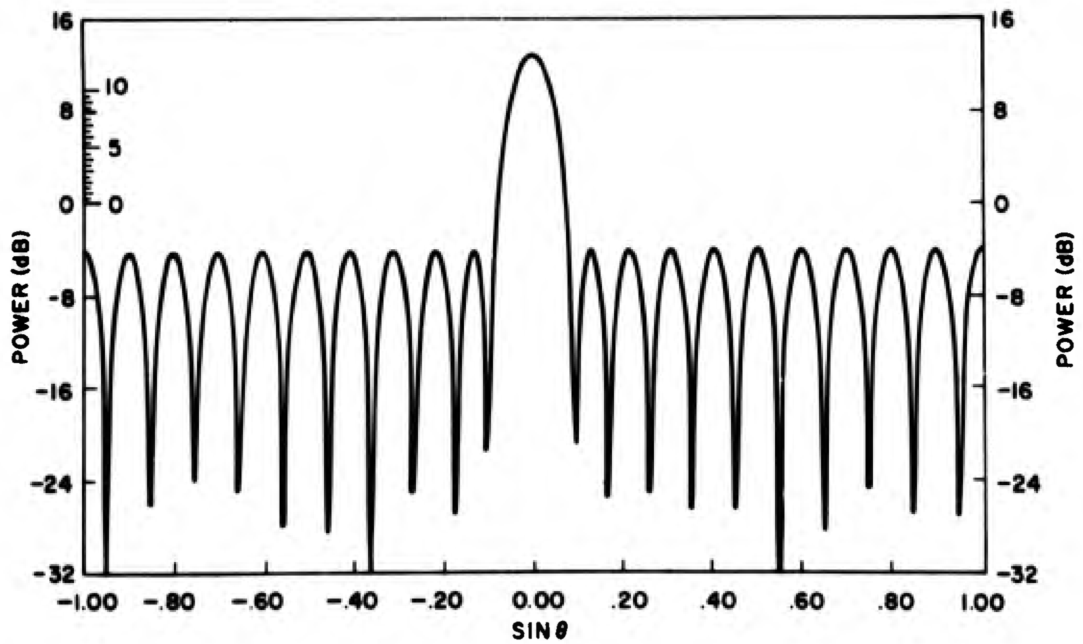


Figure 18. Iterative Least-Mean-Square Match to the Patterns in Figures 11 to 16: R = 100

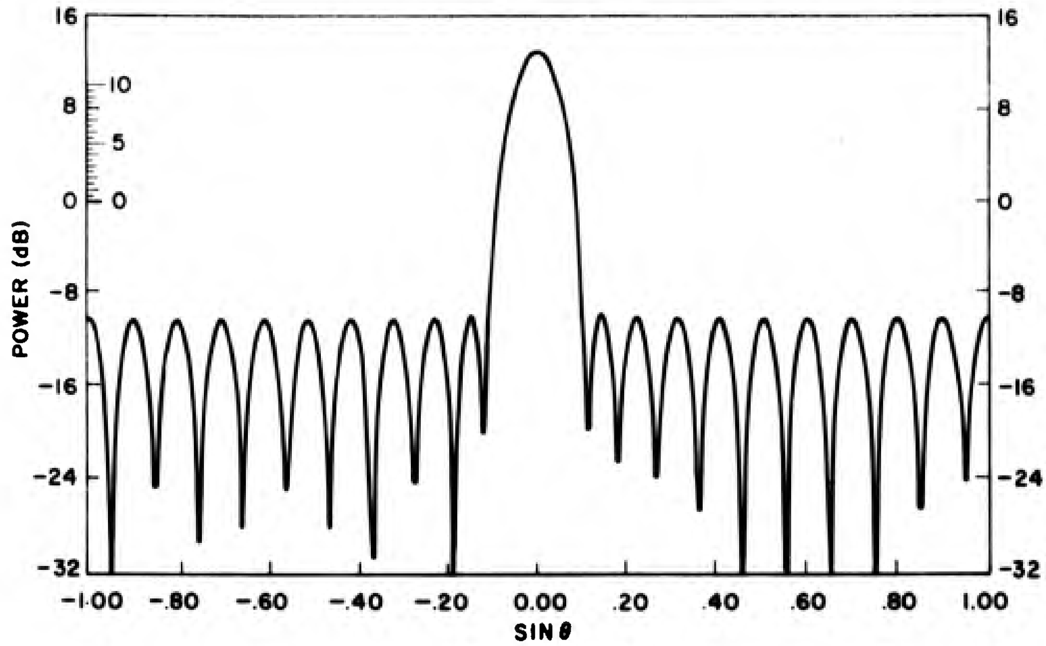


Figure 19. Iterative Least-Mean-Square Match to the Patterns in Figures 11 to 16: R = 400

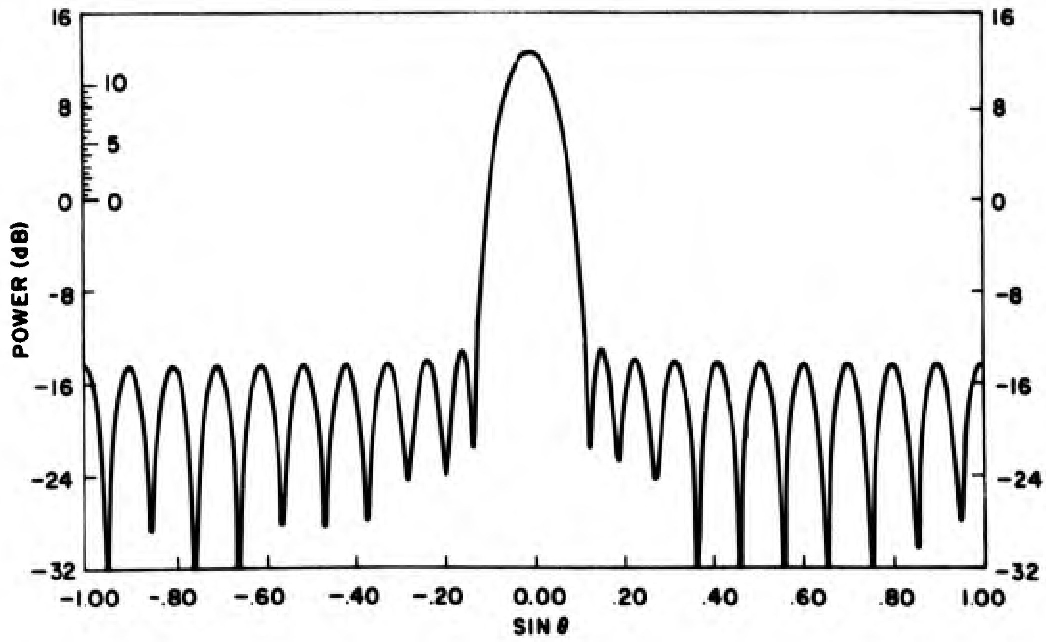


Figure 20. Iterative Least-Mean-Square Match to the Patterns in Figures 11 to 16: R = 1000

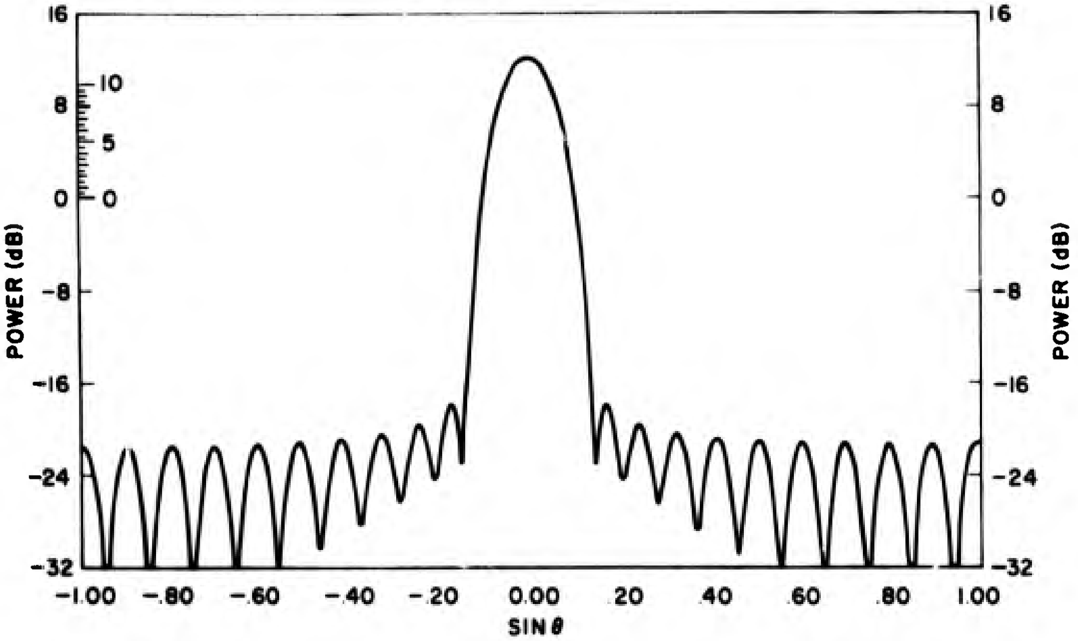


Figure 21. Iterative Least-Mean-Square Match to the Patterns in Figures 11 to 16: R = 4000

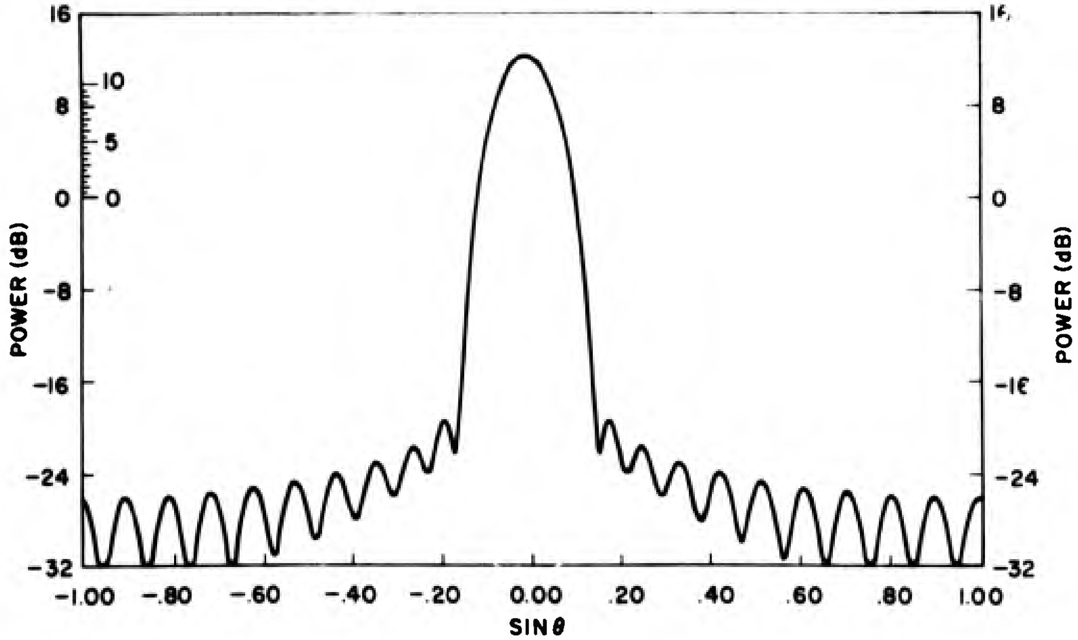


Figure 22. Iterative Least-Mean-Square Match to the Patterns in Figures 11 to 16: R = 10000

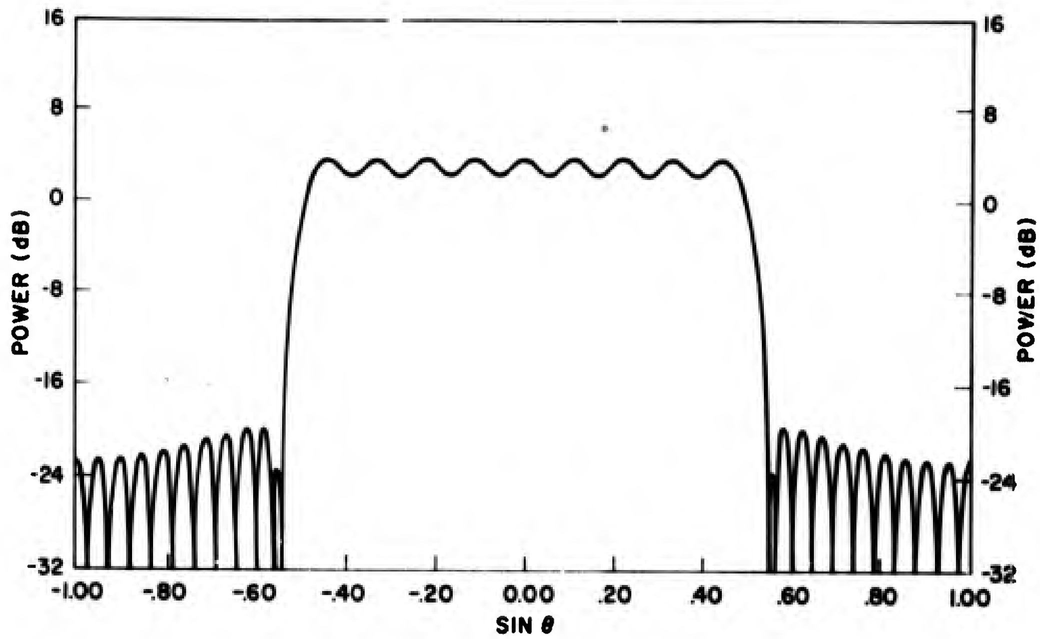


Figure 23. Approximation of a Square-Top Pattern by Using Chebyshev Patterns as Sampling Functions: Desired Sidelobe Level R = 300; Sample Spacing Δu = 0. 1106 [9 samples]

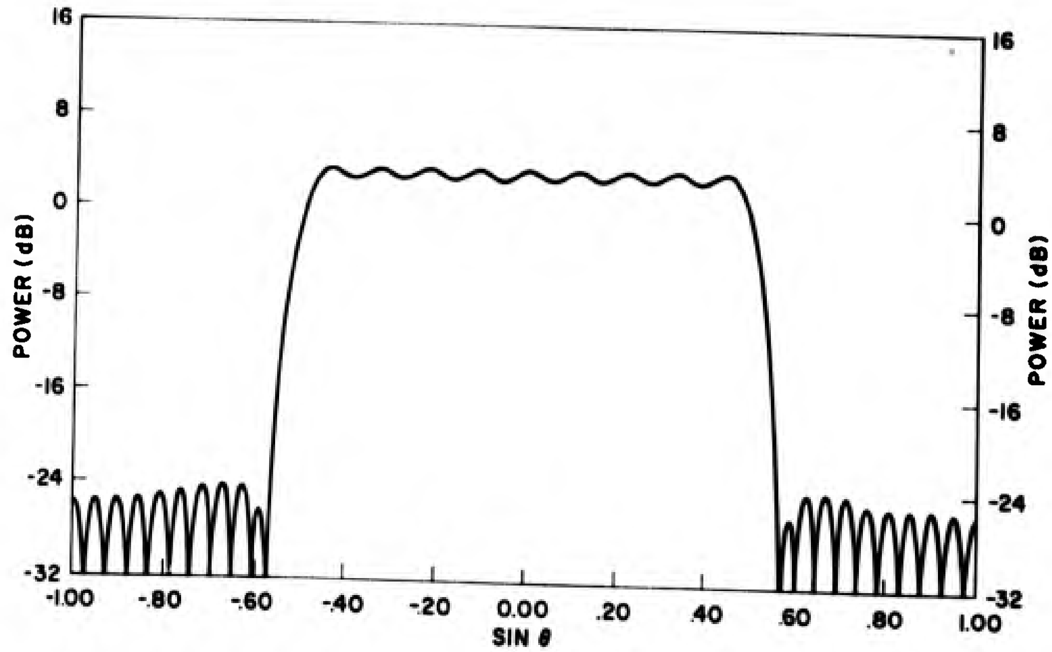


Figure 24. Approximation of a Square-Top Pattern by Using Chebyshev Patterns as Sampling Functions: Desired Sidelobe Level R = 800; Sample Spacing Δu = 0. 1108 [9 samples]

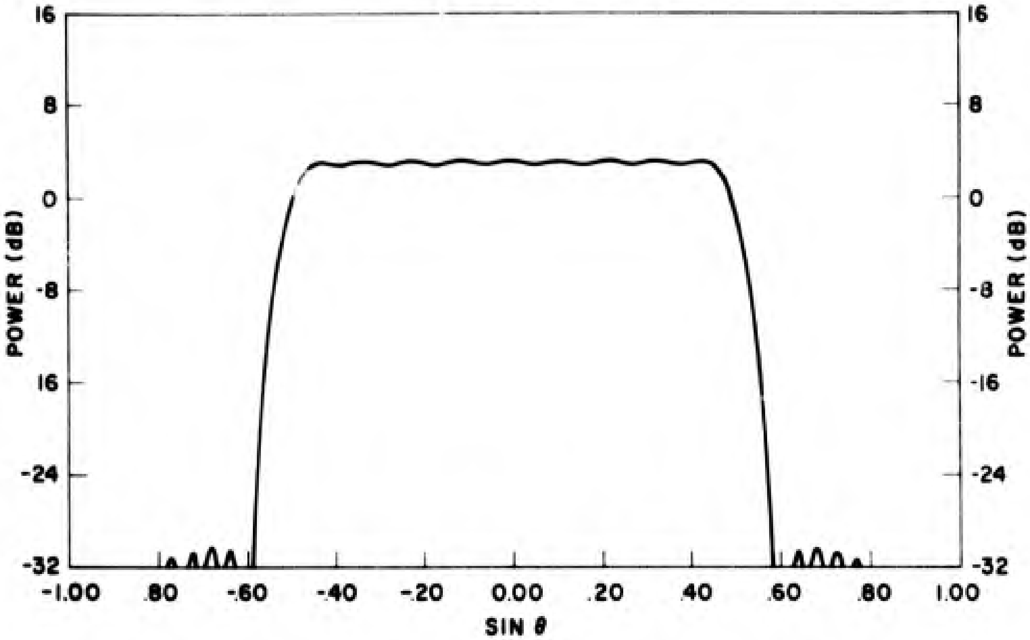


Figure 25. Approximation of a Square-Top Pattern by Using Chebyshev Patterns as Sampling Functions: Desired Sidelobe Level R = 3200; Sample Spacing Δu = 0. 1101 [9 samples]

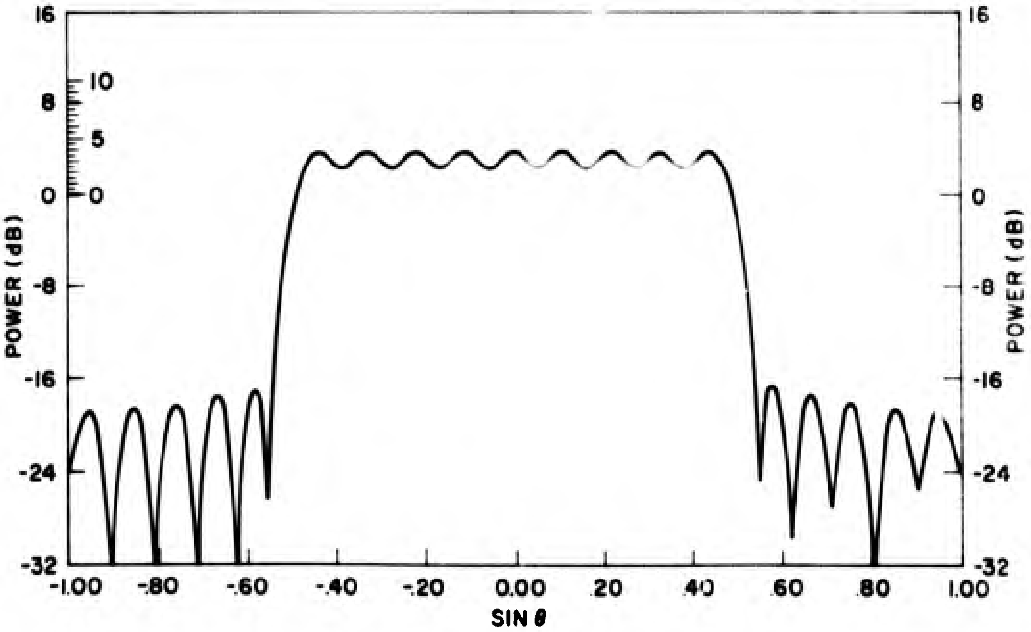


Figure 26. Power Pattern Synthesis Results, Obtained by an Iterative Least-Mean-Square Match to the Patterns in Figures 23 to 25: Desired Sidelobe Level R = 300

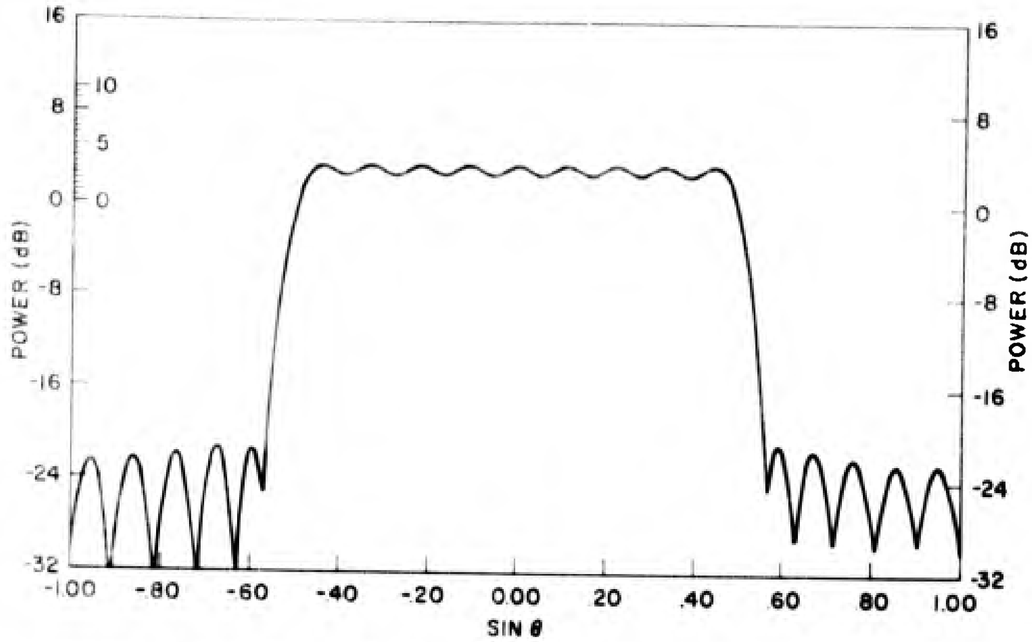


Figure 27. Power Pattern Synthesis Results, Obtained by an Iterative Least-Mean-Square Match to the Patterns in Figures 23 to 25: Desired Sidelobe Level R = 800

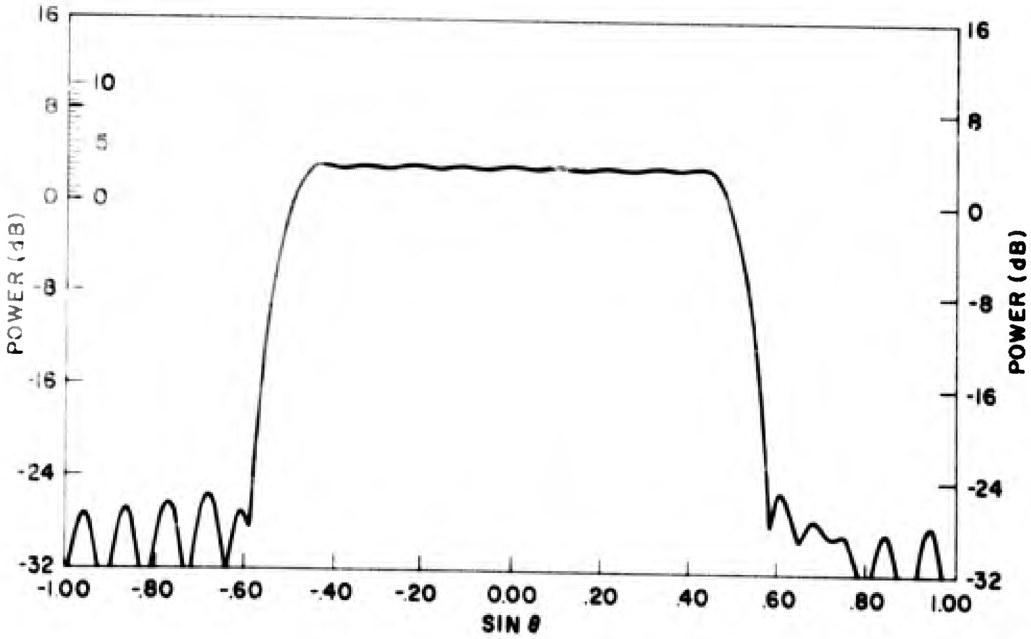


Figure 28. Power Pattern Synthesis Results, Obtained by an Iterative Least-Mean-Square Match to the Patterns in Figures 23 to 25: Desired Sidelobe Level R = 3200

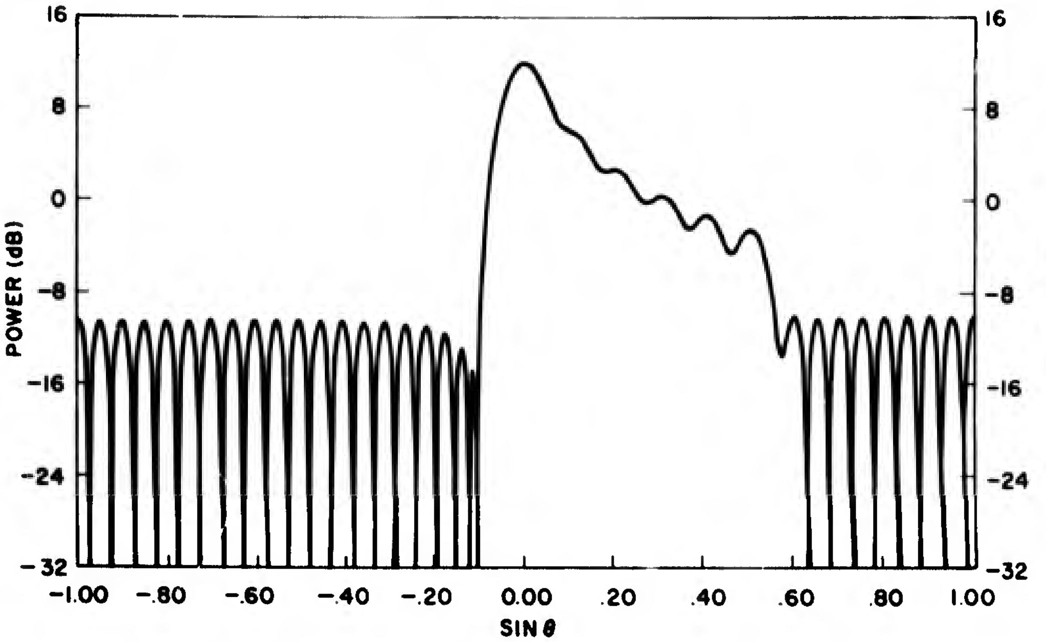


Figure 29. Approximation of a $Cosec^2$ Pattern by Chebyshev Sampling Patterns: R = 250; Sample Spacing Δu = 0. 1018

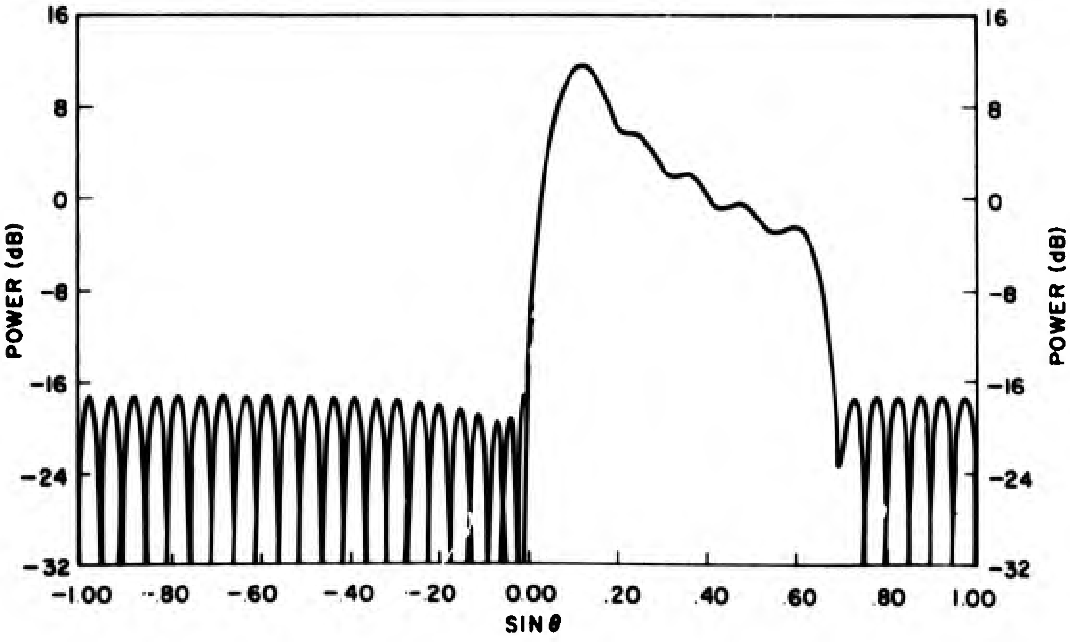


Figure 30. Approximation of a $Cosec^2$ Pattern by Chebyshev Sampling Patterns: R = 800; Sample Spacing Δu = 0. 1192

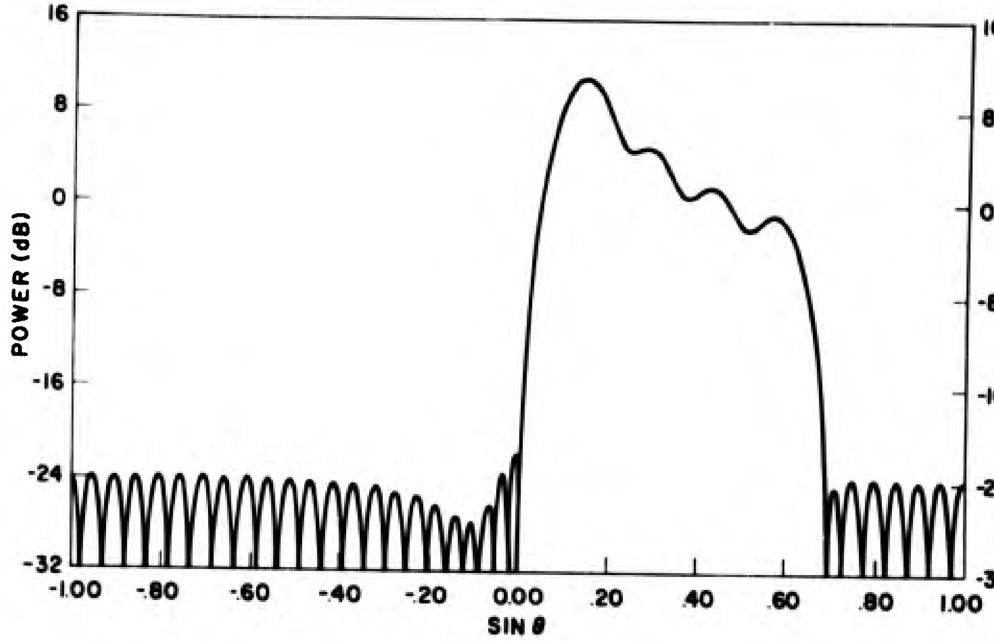


Figure 31. Approximation of a $Cosec^2$ Pattern by Chebyshev Sampling Patterns: R = 3200; Sample Spacing Δu = 0. 1418

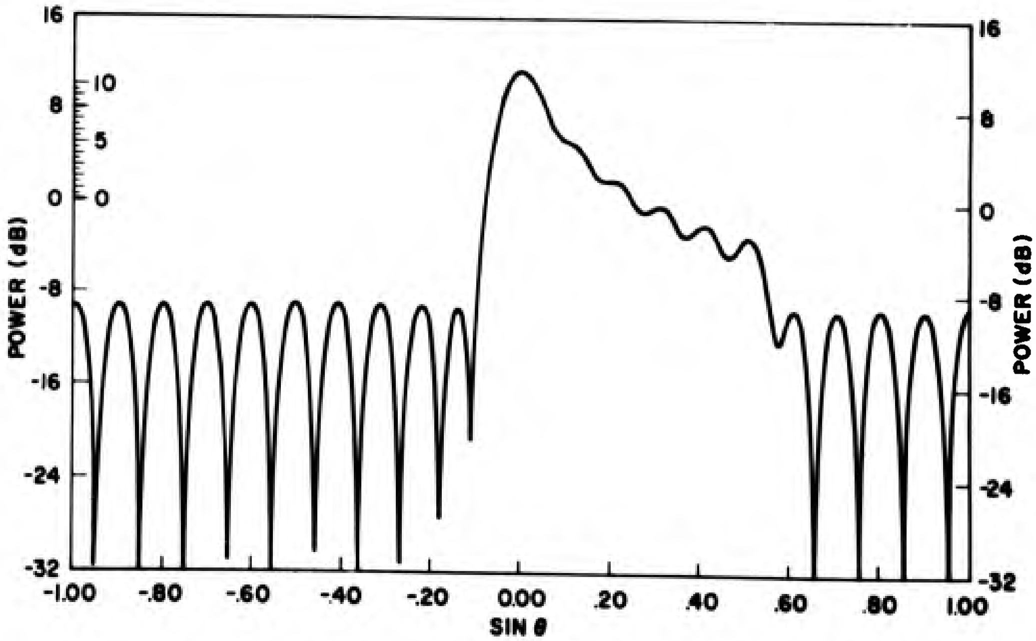


Figure 32. Power Pattern Synthesis Results for the $Cosec^2$ Patterns: R = 250

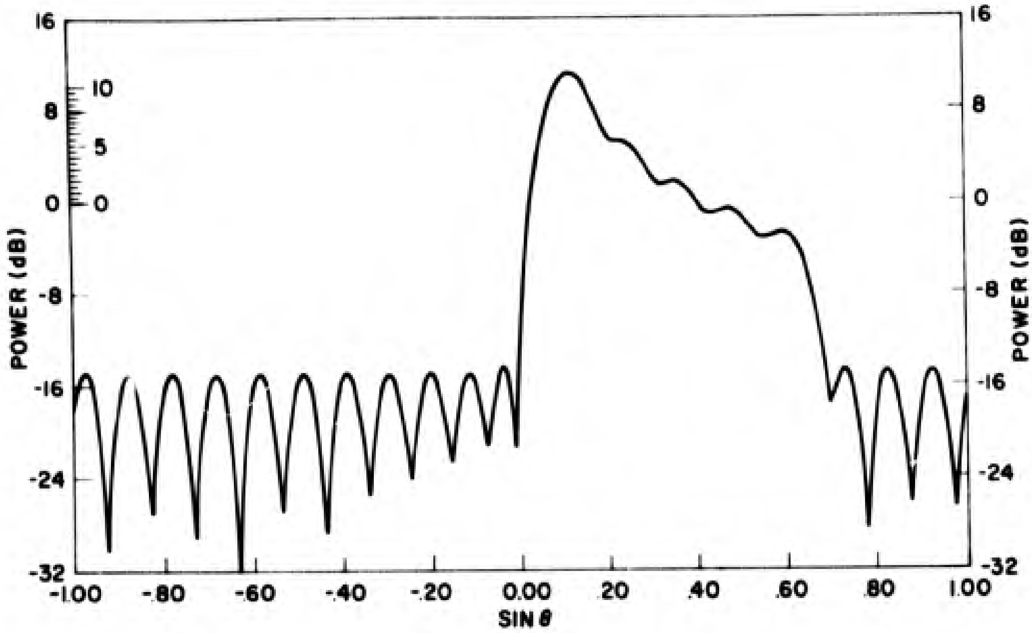


Figure 33. Power Pattern Synthesis Results for the Cosec^2 Patterns: R = 800

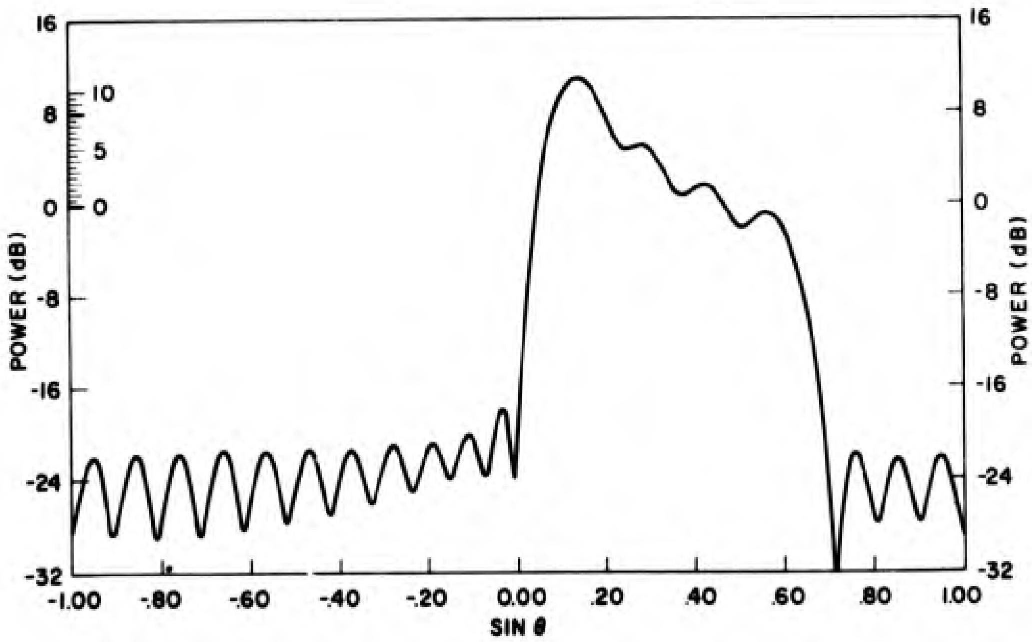


Figure 34. Power Pattern Synthesis Results for the Cosec^2 Patterns: R = 3200

Acknowledgments

I am indebted to Dr. Allan C. Schell for suggesting the problem of power pattern synthesis and for helpful discussions during the course of the work. Frederick Cleveland, Major, USAF, helped greatly by doing the computer programing.

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Appendix A

A Comment on the Sampling Distance

In the case of patterns of large dynamic range, such as cosec^2 patterns, two adjacent samples will differ greatly in amplitude. To prevent the larger sample from dominating its smaller neighbor, the obvious choice for the distance Δu is $\Delta u = u_o$.

What error can this sampling distance cause if we synthesize a square-top pattern? The maximum relative error ϵ_o will appear half way between the two samples, and will be:

$$\epsilon_o = 1 - 2\,\frac{T_{2N}\left(x_o, \frac{u_o}{2}\right)}{T_{2N}(x_o, 0)} \simeq 1 - 2\,\frac{e^{2N\,\text{arc}\cosh x_o \cos\frac{\pi}{2}\frac{u_o}{2}}}{e^{2N\,\text{arc}\cosh x_o}}$$

$$\simeq 1 - 2\,e^{2N\left\{\text{arc}\cosh x_o\left[1 - \frac{1}{2}\left(\frac{\pi u_o}{4}\right)^2\right] - \text{arc}\cosh x_o\right\}}$$

$$\simeq 1 - 2\,e^{-\left(\frac{N\pi^2}{16}u_o^2\frac{x_o}{\sqrt{x_o^2-1}}\right)} \simeq 1 - 2\,e^{-\left(\frac{N\pi^2}{16}u_o^2\frac{1}{\sqrt{2\delta}}\right)}$$

$$\simeq 1 - 2\,e^{-\left(\frac{\log^2 2R + \left(\frac{\pi}{2}\right)^2}{8\log 2R}\right)}$$

The relative error ϵ_o is thus a function of R only, and this function varies rather slowly as the following numerical values indicate:

R	Relative Error ϵ_o (%)
100	3
400	17
1000	26
10000	34

In the case of the square-top pattern the error in the pattern region can be decreased by moving the samples closer together. An alternative choice of Δu is thus $\Delta u = u_1$, where u_1 is the 3-dB beamwidth defined by

$$T_{2N}(x_o, \frac{u_1}{2}) = \frac{1}{2} T_{2N}(x_o, 0) .$$

By an expansion involving the same approximations as in the calculation of ϵ_o we find

$$u_1 \simeq \frac{2}{\pi N}\sqrt{\log 4 \log 2R} .$$

The maximum relative error will now appear at the sampling points. The contributors at one such point from the left and right neighbor will be

$$\epsilon_1 = 2\frac{T_{2N}(x_o, u_1)}{T_{2N}(x_o, 0)} ,$$

and expanding this expression just as previously, using u_1 instead of u_o gives

$$\epsilon_1 \simeq 2\, e^{-x_o \log 16} \simeq \frac{1}{8} .$$

The error with this sampling distance is thus constant, equal to 12.5 percent for all values of N and R.

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13. ABSTRACT

This report on antenna array pattern synthesis deals with the approximation of power patterns rather than the usual problem of approximating field patterns. After a qualitative comparison of the different characteristics, an optimum approximation is formulated in general terms. The possibility of finding an exact 'equal ripple' solution is discussed, and an approximate 'equal ripple' solution is offered as an approach to this as yet unsolved mathematical problem.

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