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APPLICATIONS OF A SIMULATION ANALYZER PROGRAM FOR DERIVING AND EVALUATING NUMERICAL INTEGRATION TECHNIQUES

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FOREWORD

This report was prepared under Project 6114, "Simulation Techniques for Aerospace Crew Training," Task 611407, "Mathematical Models and Programming Techniques," in the Simulation Techniques Branch, Training Research Division, Air Force Human Resources Laboratory. The research was begun in January 1968 and was completed August 1968. This report was submitted by the author 14 October 1968.

This technical report has been reviewed and is approved.

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ABSTRACT

This report describes applications of a Simulation Analyzer Program for deriving or evaluating numerical integration methods for use in flight simulation for training. The FORTRAN program was developed in an earlier study, and its theoretical bases and capabilities are briefly presented in this report. The program is used to derive and evaluate optimal integration methods for application to a selected first-order and a second-order differential system. The derived methods are used to solve these systems, and actual solution characteristics are compared with those predicted beforehand by the program. Characteristics of accuracy, actual percent error, stability, and error propagation are shown to be accurately predicted, as is the largest integration interval usable for each problem with each integration method. A thorough description is provided of the five popular simulation-integration techniques in use today and of a recommended procedure for using the Simulation Analyzer Program to derive new integration methods which allow maximization of the integration interval for specific simulation problems. As an example, the program is used to evaluate known methods and derive new methods for the F-100A problem using integration intervals of 0.05 and 0.10. A list of over 70 new integration methods derived by the program, including their stability and truncation-error characteristics, is provided.

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EQUATION

EXPLANATION OF TERMS

Classical: An integration method is said to be classical if its coefficients are uniquely determined by the Taylor series matching method.

Classical to the nth Degree: An integration method is said to be classical to the nth degree if the coefficients of $h^0, h^1, h^2, \ldots h^{n-1}$ in the Taylor series expansion match.

Closed (C_{ij}): An integration method is said to be closed and is denoted C_{ij} if it uses i past values of the dependent variable, the present value of its derivative, and j-1 past values of its derivative to compute the present value of the dependent variable.

Coefficients of an Integration Method (a_i, b_i): The coefficients of an integration method are the fixed values, a_i and b_i, in the expression

$$y_n = \sum_{i=1}^{K} a_i y_{n-i} + h \sum_{i=0}^{K} b_i \dot{y}_{n-i}$$

Critical Eigenvalue (g, λ): The critical eigenvalue, g (or λ), of a system of first-order differential equations is the eigenvalue of the corresponding linearized system which has the maximum modulus.

Integration Interval (h, Δ t): The integration interval used with an integration method is the step size used in the integration.

Nonclassical: An integration method is said to be nonclassical if its coefficients are not uniquely determined by the Taylor Series matching method.

Open (O_{ij}): An integration method is said to be open and is denoted O_{ij} if it uses i past values of the dependent variable and j past values of its derivative to compute the present value of the dependent variable.

Order (K): An O_{ij} or C_{ij} integration method is said to be of order K, where K = max (i, j).

EXPLANATION OF TERMS (CONT)

Propagated Error (e_n): The propagated error at the nth step in an integration is the difference between the true and calculated solutions at the nth step. It is formed by a step-by-step propagation of the sum of local truncation error and local round-off error.

Round-Off Error (E): The round-off error associated with an integration process is the quantity which must be added to a finite representation of the computed number in order to yield an exact representation of that number. Round-off error generally increases as the computer word-length decreases.

Stability Interval (Stability Range; $|hg|$): The stability interval of an integration method is the set of values of the product (hg) of the integration interval and the critical eigenvalue for which the method produces a solution which is locally stable. In practice, general comparisons of integration methods on the basis of their stability intervals limit consideration to real eigenvalues, and the stability interval is expressed as a segment of the real axis along which hg may lie. For flight simulation applications, this segment normally begins at $hg = 0$ and extends in the negative direction, and the stability interval is often designated by max $|hg|$.

Truncation Error (T_n): The truncation error at the nth step in an integration is the quantity T_n which must be added to the computed number in order to produce the theoretically true number, ignoring round-off error. The truncation error results from the approximation of the solution of a differential equation using a finite difference equation.

SUMMARY AND CONCLUSIONS

PROBLEM

A significant task in flight simulation for training is to numerically solve the differential equations of motion for the vehicle of concern. For this task, a method of numerical integration is required. Integration methods currently used are not optimal, i.e. other methods can be derived permitting greater accuracy and stability using larger integration step-sizes. Also, currently used methods are not guaranteed adequate theoretically. A previous study resulted in a computer program which derives optimal integration methods for any simulation problem and predicts the solution characteristics to be expected in practical applications. The problem of the present study was to evaluate the program's utility empirically.

APPROACH

The approach was to compare the program's predictions about solution characteristics with those actually observed using various integration methods. The experimental problems included one first- and one second-order differential system. The program was also used to evaluate known integration methods and derive new methods for application to the F-100A simulation problem, for which an abundance of comparative data exists.

RESULTS

Correlations and regression plots of actual vs estimated percent errors show excellent prediction accuracy of the program. The computed stability weight for each integration method is shown to be a reliable predictor of error propagation in the solution. Computed merit-figures agree with intuitive judgments of the methods' relative adequacies. Two new integration methods were derived for the F-100A problem which should perform acceptably at twice the step-size popularly used for this problem. A list of over 70 new methods with details of their characteristics was compiled.

SUMMARY AND CONCLUSIONS (CONT)

CONCLUSIONS

The computer program evaluated in this study appears to be a powerful tool for flight simulation work. Empirical results have proved the program's capability for accurately deriving or comparing integration methods for application to specific problems.

SECTION I

INTRODUCTION

Although a theoretically infinite number of numerical integration techniques exists, fewer than a half-dozen are popularly chosen as candidates each time a flight simulator is developed. The final selection of one method of integration is normally based on two considerations: (1) the apparent past success of the method for other simulations with which the simulator manufacturer is familiar, and (2) the apparent accuracy of the method for the present simulation as judged by several empirical tests. Until very recently, the state of the art has simply not permitted any rigorous theoretical justification for selecting one method over another; nor has it permitted the characteristics of integration methods to be effectively compared so that an optimal derivation might easily be made of a method best suited for the problem at hand.

In 1966 a study[1] was begun to develop methods for automatically deriving optimal numerical integration techniques for flight simulation problems. The study was conducted in two phases over a two-year period, one phase being devoted exclusively to theoretical research and mathematical analysis and the second to numerical methods and the implementation of all results in a FORTRAN computer program.

During the first phase, a study was made of possible approaches to deriving optimal integration methods (Nigro, 1967). Both the stability chart approach and the Z-transform approach were examined. The approach finally selected and developed was a modified classical approach, in the framework of which both classical and nonclassical methods of integration may be examined. Techniques were derived for computing, predicting, and controlling specific characteristics of integration methods such as stability, truncation error, round-off-error, propagated error, percent accuracy, and required computing time.

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[1] This study was sponsored by the Simulation Techniques Branch, Training Research Division, Air Force Human Resources Laboratory, Wright-Patterson Air Force Base, Ohio. The work was conducted by Bell Aerosystems Co., Buffalo, New York.

During the second phase, a family of two-step integration methods with arbitrarily large stability intervals was derived. Also, a family of three-step methods was developed with arbitrarily large stability intervals and arbitrarily small truncation error coefficients. A FORTRAN program was then developed to implement all of the theoretical work performed during the study (Nigro, et al, 1968). The resulting program is capable of either deriving or evaluating integration methods for application to specified simulation problems. The success of the computer program in accurately predicting the solution characteristics and overall adequacies of integration methods for various problems has been examined from an empirical standpoint to provide demonstrations of the program's utility.

SECTION II

THE SIMULATION ANALYZER PROGRAM

The FORTRAN program, Simulation Analyzer Program (SAP), developed in the above-referenced studies is the focal point for the present discussion. Originally programmed and debugged in FORTRAN IV on an IBM 360/50 computer, the program is now operational on a Raytheon 440 computer.[2] Memory requirements for the program on the IBM 360/50 total approximately 9000 32-bit words. On the Raytheon 440, the program is chained in two segments of 5050_{10} and 4130_{10} 24-bit words, respectively.

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[2] The Raytheon 440 is part of the Real Time Simulation Research System of the Simulation Techniques Branch, Training Research Division, Air Force Human Resources Laboratory, Wright-Patterson Air Force Base, Ohio.

1. USER INPUTS

In using the program, one must supply certain input data which describes both the simulation problem of concern and the user-imposed cycle-time limits and accuracy tolerances for the integrated solution. A list and brief descriptions of the required inputs are provided in Table I.

TABLE I

USER INPUTS REQUIRED FOR THE SIMULATION ANALYZER PROGRAM

No.	Input	Description
1	Option-Code	Tells whether option 1 (derivation of best methods) or option 2 (evaluation of candidate methods) is desired.
2	Methods	(Option 2 only): Names and descriptions of candidate integration methods.
3	No. of equations	The number of differential equations in the system to be solved.
4	Degrees	Greatest degree of the dependent variable in each differential equation in the system.
5	Operations	Average number of arithmetic operations required to evaluate a derivative in the simulation program.
6	Percent error	The percent inaccuracy the user will tolerate in the integrated quantities.
7	Critical eigenfrequency	The real and imaginary parts of the most critical eigenfrequency of the system to be solved.
8	Particular eigenfrequency	The real and imaginary parts of a particular eigenfrequency, if any, at which the user wants the computed solution to be examined.
9	Minimum cycle time	The minimum number of times per time unit (eg, seconds) that the simulation variables must be updated.
10	Weights	The weights to be applied to individual merit figures in computing an overall figure of merit for each integration method.

Of the inputs required, the critical eigenfrequency (eigenvalue) is perhaps the most difficult to obtain (see Table I, No. 7). For linear, constant-coefficient systems, the derivation of the critical eigenfrequency is straightforward. However, the derivation of eigenfrequencies for the nonlinear, variable-coefficient systems with which we are concerned in flight simulation is somewhat of a task. The role played by this variable in the derivation of optimal integration methods and the techniques available for its computation are discussed in detail by Nigro (1967), Knoop (1967), and Nigro, et al (1968) and are summarized in Section IV of this report.

The user is required to specify the minimum cycle time of the simulation program (see Table I, No. 9). The main reason for and use of this information is to prohibit the Simulation Analyzer Program from deriving a numerical integration method and a corresponding integration interval (h or Δt) which, though ideal theoretically, is worthless in training simulation from a practical standpoint. For example, the program may easily derive a method of integration satisfying user requirements with an integration rate of once per second, but this would not be a usable rate for updating cockpit instruments realistically.

The user's inputs may be prepared beforehand on cards or paper tape or may be supplied via the on-line typewriter at the time of running.

2. PROGRAM OUTPUTS

For each integration method derived by the program (option 1) or supplied by the user (option 2), the program computes and outputs key characteristics. From these, the user can easily discern the advantages and disadvantages afforded by the use of each method for the particular simulation problem of concern. In addition, the program outputs can be used to accurately predict the behavior and accuracy of the solutions generated by each integration method.

a. Maximum Stability Interval

The program computes the maximum stability interval (or stability range) for each integration technique. In order for an integration method to produce a stable solution, the propagated error generated during the integration must be bounded. To establish necessary and sufficient conditions for stability, the general solution to the propagated error difference equation was computed (Nigro, 1967). It is seen that the condition for boundedness of the propagated error and, subsequently, for solution stability is that the roots of the characteristic equation of the difference equation for propagated error be less than or equal to one if simple and strictly less than one if multiple.

The characteristic equation of the difference equation for propagated error is

$$(1 - hgb_0) M^K - \sum_{i=1}^{K} (a_i + hgb_i) M^{K-i} = \phi \qquad (1)$$

where h = integration interval

g = the critical eigenfrequency for the system

and a_i and b_i are the coefficients of the integration method

$$y_{n+k} = \sum_{i=1}^{K} a_i y_{n+k-i} + h \sum_{i=0}^{K} b_i \dot{y}_{n+k-i} \qquad (2)$$

It is the roots, M, of Equation 1 which must be less than (or ≤) unity. The maximum stability interval is the maximum real value of the quantity hg for which the roots of Equation 1 remain less than unity.

For a given simulation problem, g, the critical eigenvalue, is fixed. However h, the integration interval, is free to vary. Thus, the maximum stability interval tells us how large we may allow the integration interval to become and still produce a stable solution, i.e. a solution in which the propagated errors are bounded.

A method was devised (Knoop, 1967) for deriving the maximum stability interval for any integration technique using a system of inequalities. This method is used by SAP, and a summary of derived stability intervals for some well-known integration methods is provided in Table II.

TABLE II

STABILITY INTERVALS FOR SOME WELL-KNOWN INTEGRATION METHODS

Method	Stability Range	
	Lower	Upper
Forward Rectangular	-2	0
Backward Rectangular*	hg ≤ 0	hg ≥ 2
2nd Order Adams	-1	0
Trapezoidal*	- ∞	0
O_{33} Mod Gurk	-.821	0
Classical O_{14}	-.286	0

*For flight simulation applications, closed integration methods are not solely applicable. They are presented here, however, as a matter of interest.

b. Coefficient of Truncation Error

With rare exception (see Nigro, et al, 1968), the truncation error for the types of integration methods of concern in flight simulation is of the form

$$T = A_p h^p y^{(p)}(\xi)$$

where A_p is the truncation error coefficient, p is the degree to which the integration method is classical, and $y^{(p)}(\xi)$ is the pth derivative of the dependent variable evaluated at a point, ξ, within the interval of concern. The Simulation Analyzer Program computes and outputs A_p and p for each integration method. Table III gives the truncation error coefficients and the degree to which the method is classical for several well-known methods.

An integration method which is classical to the pth degree is one which produces an exact solution for the linear equation

$$\dot{y} = (p-1)x^{p-2}$$

Therefore, if y is a polynomial of degree p-1 or less and if the integration method is classical to the pth degree, the solution for $\dot{y}$ will be exact.

TABLE III

TRUNCATION ERROR TERMS FOR SOME WELL-KNOWN INTEGRATION METHODS

Method	A_p*	p*, **
Forward Rectangular	.5	2
Backward Rectangular	-.5	2
2nd Order Adams	.417	3
Trapezoidal	-.083	3
O_{33} Mod Gurk	.033	2
Classical O_{14}	-.042	4

*Truncation Error = $A_p h^p y^{(p)}(\xi)$

**Since h is rarely larger than 1.0 in flight simulation, larger values of p normally provide smaller truncation error terms due to the contribution of h^p.

Henrici (1962) has shown that all integration methods must be classical to at least the second degree in order to be consistent. The criterion for consistency for the integration methods of concern (Equation 2) is that the following equations be satisfied:

$$\sum_{i=1}^{K} a_i = 1$$

$$\sum_{i=1}^{K} (K-i)\, a_i + \sum_{i=1}^{K} b_i = K - b_0 - b_K$$

Techniques for deriving methods classical to any desired degree are presented by Nigro (1967).

c. Maximum h for Stability

The program computes the largest integration interval, h, which may be used with each method of integration while retaining stability in the computed solution. If the critical eigenfrequency is real, this is merely the maximum stability interval divided by the critical eigenfrequency for the problem, with the result adjusted slightly to assure that all roots of Equation 1 lie well within the unit circle. If the critical eigenfrequency is complex, h is systematically reduced by SAP until all roots of Equation 1, as computed, lie within the unit circle. The maximum h values for several integration methods for application to the F-100A simulation problem are given in Table IV.

TABLE IV

MAXIMUM INTEGRATION INTERVALS FOR SELECTED METHODS FOR THE F-100A SIMULATION PROBLEM*

Method	Description**	Maximum h†
O_{33} Mod Gurk	3 step, nonclassical	.0699
2nd order Adams	2 step, classical	.0852
Euler	1 step, classical	.1704
MTH.03-10	3 step, nonclassical	.8521
MTH.03-.8	3 step, nonclassical	.0698

*Based on a critical eigenfrequency of g = -11.5, as reported by the University of Pennsylvania (1955).

**See the Appendix for more complete description.

† This is the largest h which may be used while still maintaining stability in the computed solution.

d. Maximum h for Accuracy

The program computes the largest integration interval which may be used while still satisfying the user's accuracy tolerances. This is possible because of the successful derivation (Nigro, 1967; and Nigro, et al, 1968) of an exact expression for propagated error, which led to the derivation of a closed-form expression for percentage of error in the numerical solution. A detailed account of the derivation of the percentage-of-error scheme is given by Chen (August 1967 and September 1967).

The user specifies the percent error he will tolerate at the start of the solution. Assuming that the integration methods usable for flight simulation are at least stable, we are assured that this error will not propagate without bound. The program estimates the percent error at the (K + 2)th step in the integration, where K is the order of the integration method. Thus, the resulting figures accurately predict the solution accuracy early in the integration and are quite useful in comparing integration methods for application to a given problem. Table V shows the maximum h usable for several integration methods to maintain an approximate 3% inaccuracy at the start of the solution of the F-100A simulation problem.

TABLE V

MAXIMUM INTEGRATION INTERVALS FOR MAINTAINING AN APPROXIMATE 3% INACCURACY AT THE START OF THE SOLUTION OF THE F-100A SIMULATION PROBLEM

Method	Description	Maximum h
O_{33} Mod Gurk	3 step, nonclassical	.0394
2nd order Adams	2 step, classical	.0215
MTH.03-10	3 step, nonclassical	.0227
MTH.03-.8	3 step, nonclassical	.0493
2nd step	2 step, nonclassical	.0083

e. Maximum h for the User

The program records the largest integration interval which may be used, as specified by the user. This is merely the reciprocal of the user's specified update rate (eg, 20/second yields 1/20 or .05 for the maximum users's h).

f. Optimum h for the Problem and Associated Percent Error

Having computed the maximum values for h which satisfy (1) the user's update-rate requirements, (2) the user's accuracy requirements, and (3) the problem's stability requirements, the program then selects the smallest of these to arrive at an optimum integration interval for the simulation. This, of course, is accomplished for each integration method. For this optimum h, the program then computes the approximate percent error to be expected at the start of the solution.

g. Figures of Merit

Five individual and one total figure of merit are computed for each integration method.

(1) Stability. The Simulation Analyzer Program checks to see that all integration methods either derived by it or supplied to it are stable for the problem. Methods which cannot be forced to produce a stable solution while satisfying all other user requirements are discarded by the program early in the analysis. Therefore, at the point where figures of merit are computed, all remaining candidate methods are stable.

A measure of the degree of stability is computed as a figure of merit by examining the propagated error term for the method and evaluating the rapidity with which its dominating terms go to zero. The propagated error is

$$e_n = \sum_{j=1}^{q} \sum_{i=1}^{S_J} A_J^i \, n^{i-1} \, (M_J)^n - \frac{\phi}{hg \sum_{i=0}^{K} b_i} \tag{3}$$

where A_J^i = constants to be determined from initial conditions

S_J = the multiplicities of the roots M_J

h = the integration interval

g = the critical eigenfrequency

b_i = the coefficients of the integration method (Equation 2)

and ϕ = truncation error minus round-off error

A measure of the rapidity with which the double-summation term in Equation 3 goes to zero is obtained by examining the moduli of the roots M_J. The stability figure of merit is therefore

W_1 = 1.0 - (average distance of roots M_J from the origin)

(2) Truncation Error. The truncation error is assumed to be of the form

$$T_n = A_p h^p y^{(p)}(\xi) \tag{4}$$

A figure of merit for truncation error is computed as a function of the first two terms in Equation 4. To penalize methods for which the term $A_p h^p$ is large as well as to produce an easily discernable spread of merit figures, the program computes

$$W_2 = 0.10 - 0.0651 \ln(A)$$

where

$$A = \min(1, |A_p h^p|)$$

This may be interpreted as (approximately)

$$W_2 = 0.10 + n/6.7$$

where n is the number of decimal places of accuracy in the solution, assuming no other errors.

(3) Round-Off Error. A thorough discussion is provided on computing an upper bound for local round-off error by Nigro (1967) and Nigro, et al (1968). Computation of this bound requires that certain assumptions be made. Since the purpose of the merit figures is to permit comparative (rather than absolute) evaluations to be made of the integration methods, we may assume the following without jeopardizing the value of the results (Nigro, et al, 1968):

a. Computations are performed in single-precision, floating point

b. The computer arithmetic uses a base B number system and carries a D-digit mantissa

c. The computer accumulator carries 2D digits

d. Sums and products are rounded to D-digit accuracy

e. h is fixed and absorbed into the coefficients b_i of the integration method

Analysis shows that an upper bound on the round-off error, E, is

$$|E| \leq \sum_{i=2}^{2K} |X_i||Z_i|(2K+2-i)\left(\frac{1}{2}\right)(1.06)\,B^{1-D} + X_1 Z_1 (2K)\left(\frac{1}{2}\right)(1.06)\,B^{1-D} \qquad (5)$$

where X_i and Z_i are the a_i (or b_i) and y_i (or $\dot{y}_i$) of the problem. In the event some of the a_i or b_i are unity and no multiplication is required, an adjustment is made in Equation 5 to correctly reflect this in the results.

(4) Propagated Error. Our figure of merit for stability reflects the rapidity with which propagated errors tend to decay in the computed solution. We derive the propagated error figure of merit to compute an additional important characteristic of integration methods closely related to error decay. The purpose of this merit figure is similar to the objectives sought by Gray (1953) in his development of the well known stability chart techniques.

In the solution of a linear, first-order differential equation, two different integration methods may possess propagated error characteristics which cause unwanted errors to damp-out at approximately the same rate but in different manners. An example of such a situation is represented in Figure 1. The propagated error merit figure is intended to reflect the superiority of Method 1 over Method 2 (see Figure 1) in a solution of this type.

The computed solution of an equation such as $\dot{y} = -\lambda y$ $(\lambda > 0)$ may be written

$$y_n = \sum_{j=1}^{K} C_j (\omega_j)^n \qquad (6)$$

where ω_j are the roots of the characteristic equation of the difference equation for propagated error (Equation 1). The true solution may be written

$$y_{n(\text{true})} = \bar{C}\,(e^{-\lambda h})^n \qquad (7)$$

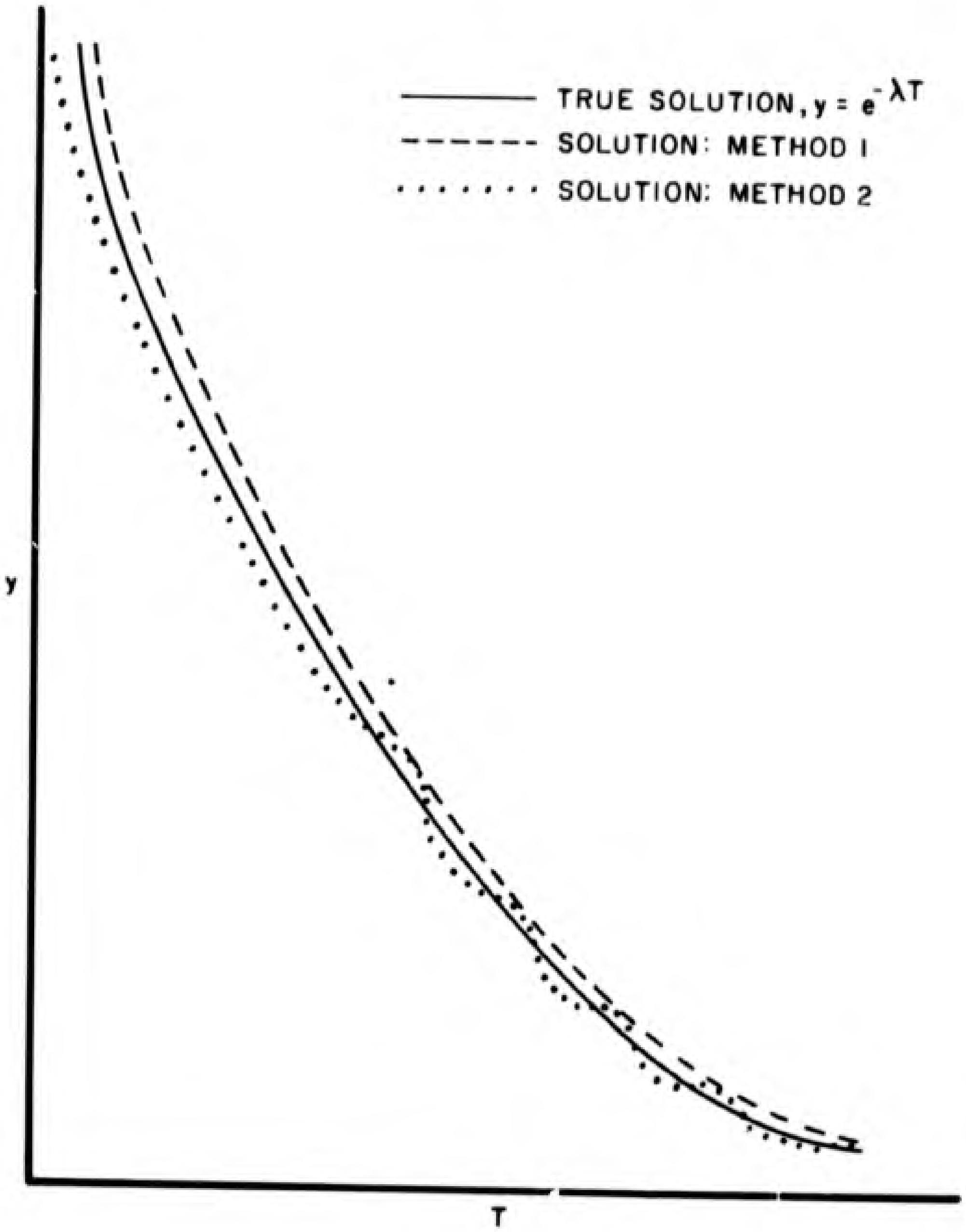

Figure 1. Representative Solutions for $y = e^{-\lambda t}$ Using Two Methods of Integration

As $t \to \infty$, the calculated solution (Equation 6) will tend to

$$\sum_{i=1}^{m} c_i \omega_i^n$$

where $\omega_1, \omega_2, \ldots \omega_m$ are roots of equal modulus. Therefore, an integration method for which $\omega_1 = e^{-\lambda h}$ $(m = 1)$ is preferred, where λ is the system-eigenfrequency with the greatest real part.

Unfortunately, λ is seldom known or easily derived. However, we may compare ω_1 instead with a series of λ_i's which lie in a region in which the true λ must lie (Nigro, et al, 1968). This is accomplished by the Simulation Analyzer Program for each of 15 λ_i values lying within the computed λ-region. A propagated error merit figure is then computed as an average "matching-potential" of ω_1 with each λ_i. If ω_1 is not unique, appropriate alterations are made in the calculations.

(5) Required Computing Time. The program also computes a merit figure for all methods so they may be compared on the basis of required computing time. Obviously a one- or two-step method, like Euler or 2nd Order Adams, would be preferred to a three-step method such as O_{33} Mod Gurk, all other considerations being equal.

The merit figure is a scaled weight reflecting the number of seconds of real time required to compute one second of simulated time using the integration method in question. The formula for this is

$$\text{MERIT} = 0.1 - 0.097716 \ln(D)$$

where D = $1.6\,(10^{-5})\,K(N + N_F)(1/h)$

and K = number of equations in the system

N = number of arithmetic operations in the integration method

N_F = average number of arithmetic operations per differential equation

h = integration interval

It is assumed that the average arithmetic-command execution time in the computer to be used is 16 microseconds. Since our measure is a relative one, this does not affect the significance of the merit figure in general, regardless of the specific computer to be used.

(6) Total Merit Figure. A total figure of merit for each integration method is computed by combining the five individual merit figures in a linear weighting scheme which may be specified by the user. The user may specify that each linear weight factor, W_i ($i = 1,5$), be unity, or he may supply as inputs the value of each factor. It is difficult to determine in an a priori sense how each consideration (i.e. stability, truncation error, etc.) should be weighted to affect a choice of a "best" method satisfactory for all users in all cases. Many times the user may desire accuracy in the solution at the possible expense of solution stability; other times he may be willing to sacrifice accuracy for a highly stable solution with a low required computing time. The ultimate decision, in short, must be the user's. Table VI shows the figures of merit for a second-order problem (critical eigenfrequency of $-.785 + 3.042i$) using several methods of integration.

h. Output Format

The format in which SAP provides outputs consists of nine sections of information. Some information is provided in table format and other as text-like prose. The information given is:

1. A listing of the names, order (K), and coefficients of all candidate or derived methods of integration.

2. A listing of any methods which should be rejected from consideration due to inconsistency or lack of stability.

3. A discussion of the problem which has been described by the user.

4. A table (labeled Table 1 in the SAP output) giving each method's maximum stability interval, truncation error coefficient, degree classical, and the largest possible Δt usable on the basis of stability alone.

TABLE VI

FIGURES OF MERIT FOR SOME INTEGRATION METHODS FOR APPLICATION TO A SECOND-ORDER PROBLEM

Method*	Stability	Truncation Error	Round-Off Error	Propagated Error	Computing Time	Total
Euler	.027	.535	.858	.995	.644	3.059
O_{33} Mod Gurk	.512	.712	.106	.998	.548	2.876
Parabolic	.501	.944	.168	.998	.584	3.195
2nd Ord. Adams	.479	.742	.288	.998	.604	3.111
O_{14}	.438	1.000	.112	.959	.568	3.078
MTH.03-.8	.066	.712	.167	.333	.548	1.825
5-Step	.132	.464	.103	.360	.584	1.643

*All methods are evaluated for an integration interval of h = .05.

5. A table (labeled Table 2 in the SAP output) giving each method's maximum Δt for accuracy, maximum Δt for user requirements, maximum Δt for stability, optimum Δt, and the percent error at the start of the solution using the optimum Δt.

6. A Figure of Merit scheme based on the use of the optimum integration interval for each method.

7. A selection of a "best" method of integration based on the merit scheme for the optimum Δt.

8. A figure of merit scheme based on the use of the maximum Δt for stability for each method.

9. A selection of a "best" method of integration based on the merit scheme for the maximum Δt for stability.

SECTION III

THEORETICAL BASES FOR THE PROGRAM

The theoretical work in analysis which forms a basis for the computer program is well documented by Nigro (1967) and Nigro, et al (1968). A very brief review of some of the key results of the analysis is presented here in summary and is taken from the above documents.

1. CLASSICAL METHODS OF INTEGRATION

Any classical integration method may be derived by solving the following matrix equation:

$$\begin{bmatrix} 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ K-1 & \cdots & 1 & 0 & 1 & \cdots & 1 \\ \frac{(K-1)^2}{2!} & & \frac{1}{2!} & 0 & K-1 & & 1 \\ \vdots & & \vdots & \vdots & & & \vdots \\ \frac{(K-1)^{2K-2}}{(2K-2)!} & \cdots & \frac{1}{(2K-2)!} & 0 & \frac{(K-1)^{2K-3}}{(2K-3)!} & \cdots & \frac{1}{(2K-3)!} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_K \\ b_1 \\ b_2 \\ \vdots \\ b_{K-1} \end{bmatrix} = \begin{bmatrix} 1 \\ K - b_0 - b_K \\ \frac{K^2}{2!} - Kb_0 \\ \frac{K^3}{3!} - \frac{K^2}{2!} b_0 \\ \vdots \\ \frac{K^{(2K-2)}}{(2K-2)!} - \frac{K^{(2K-3)}}{(2K-3)!} b_0 \end{bmatrix} \qquad (8)$$

where K is the order of the integration method and $a_1, a_2, \ldots a_K, b_0, b_1, \ldots b_K$ are the coefficients of the integration method

$$y_n = \sum_{i=1}^{K} a_i y_{n-i} + h \sum_{i=0}^{K} b_i \dot{y}_{n-i} \qquad (9)$$

All integration methods, whether or not classical, must satisfy at least the first two equations in Equation 8 to be consistent.

2. NONCLASSICAL METHODS OF INTEGRATION

Nonclassical methods are nth order methods which do not satisfy all of the first m equations in Equation 8, where m is the number of coefficients in the method. The O_{33} Mod Gurk method is an example. Any nonclassical method may be formed by assuring that the coefficients satisfy the first two equations in

Equation 8 to assure consistency and freely picking the remaining coefficients. As will be shown below, other criteria besides classicalness may be used to determine these remaining coefficients.

3. STABILITY OF AN INTEGRATION METHOD

An integration method (such as Equation 9) will be stable; i.e., errors in the solution will not accumulate without bound, if the roots of

$$(1 - hgb_0)p^K - (a_1 + hgb_1)p^{K-1} \cdots - (a_K + hgb_K) = 0, \qquad (10)$$

where h = the integration interval

and g = the critical eigenfrequency of the differential system, all lie within or on the unit circle, with roots of modulus one simple.

4. FINDING THE STABILITY RANGE OF AN INTEGRATION METHOD

A detailed analysis is given by Knoop (1967) of a method whereby the stability range may be determined for any method of integration. (Recall that the stability range is the largest real value of hg such that all roots of Equation 10 lie within the unit circle.)

Let $$C^* = (1 - hgb_0)p^K - (a_1 + hgb_1)p^{K-1} \cdots - (a_K + hgb_K)$$

$$= A_0 p^K + A_1 p^{K-1} + \cdots + A_K$$

and

$$C = C(p) = A_K p^K + A_{K-1} p^{K-1} + \cdots + A_0$$

where $A_K \neq 0$ and $A_0 \neq 0$.

Let

$$T_1(C) = A_0 C - A_K C^*$$

$$T_2(C) = T_1\left[T_1(C)\right]$$

$$\vdots$$

$$T_n(C) = T_1\left[T_{n-1}(C)\right].$$

Now let C have no root on the unit circle Γ and let $C(0) \neq 0$. If, for some $i > 0$, we have $T_i\left[C(\phi)\right] < \phi$

then C* has at least one root outside Γ. If instead $T_i\left[C(\phi)\right] > \phi$ for all $1 \leq i \leq n$, where $T_n(C) = 0$ and $T_{n-1}(C) = \text{constant}$, then no root of C* lies outside Γ. (The equation C* will be recognized as the characteristic equation of the difference equation for propagated error.)

5. TWO-STEP METHODS WITH ARBITRARILY LARGE STABILITY INTERVALS

The stability interval for a method is the largest that the quantity $|hg|$ may become while stability is maintained. Since, for any given problem, g is fixed, a large stability interval allows us to use larger values of h. So it is to advantage to find a method by which the stability interval may be forced to become as large as we wish.

This has been accomplished by Nigro (1968). The method is described as follows:

a. Select the maximum hg desired, a negative number less than -2. Call it K.

b. Compute

$$a_2 = \frac{-4 - K}{K}$$

$$a_1 = 1 - a_2$$

$$b_1 = \frac{3}{4} + \frac{1}{2}a_2 - \frac{1}{4}a_2^2$$

$$b_2 = \frac{1}{4} + \frac{1}{2}a_2 + \frac{1}{4}a_2^2$$

For example, if we desire a two-step method with a stability interval of -8,

we set

$$K = -8$$

and compute

$$a_2 = -1/2$$

$$a_1 = 3/2$$

$$b_1 = 7/16$$

$$b_2 = 1/16$$

and our derived method becomes

$$y_n = \frac{3}{2} y_{n-1} - \frac{1}{2} y_{n-2} + h \left[\frac{7}{16} \dot{y}_{n-1} + \frac{1}{16} \dot{y}_{n-2} \right]$$

6. THREE-STEP METHODS WITH ARBITRARILY LARGE STABILITY INTERVALS

The following procedure was derived by Nigro (1968) for finding three-step methods with arbitrarily large stability intervals:

a. Select the maximum hg desired. Call it K.

b. Solve for a_1 and b_1 (i.e., by Newton Raphson) using the equations

$$\left\{ \begin{aligned} & 16 b_1^3 - 96 b_1^2 - 4(5a_1^2 + 6a_1 - 51) b_1 - (a_1^4 + 4a_1^3 - 18a_1^2 \\ & \qquad\qquad - 60a_1 + 153) = 0 \\ & -4K b_1^2 - (4a_1 - 12K + 12) b_1 + (K a_1^2 - 9K + 24) = 0 \end{aligned} \right\}$$

c. Solve for a_2, a_3, b_2, and b_3 by the formulas:

$$a_2 = \frac{10b_1 + 4a_1 b_1 + a_1 - 21}{a_1 - 2b_1 + 3}$$

$$a_3 = -\frac{a_1^2 + 2a_1 b_1 + 3a_1 + 12b_1 - 24}{a_1 - 2b_1 + 3}$$

$$b_2 = -\frac{2a_1^2 - 4b_1^2 + 2a_1 + 20b_1 - 24}{a_1 - 2b_1 + 3}$$

$$b_3 = -\frac{2b_1^2 + a_1 b_1 + 2a_1 - b_1 - 6}{a_1 - 2b_1 + 3}$$

d. If $(a_1 - 1)^2 < 4a_3$, check to see that

$$0 < a_3 < 1.$$

If it is not, return to step B above, using a different starting-guess for a_1 and b_1.

e. If $(a_1 - 1)^2 \geq 4a_3$, check to see that

$$-1 < a_1 < 3$$

and

$$\mathrm{MAX}\left[(-1 - a_1),(a_1 - 3)\right] < \sqrt{(a_1 - 1)^2 - 4a_3} < \mathrm{MIN}\left[(3 - a_1),(a_1 + 1)\right]$$

If it is not, return to step B above, using a different starting-guess for a_1 and b_1.

7. THREE-STEP METHODS WITH ARBITRARILY LARGE STABILITY INTERVALS AND ARBITRARILY SMALL TRUNCATION ERROR COEFFICIENTS

As noted in the preceding discussions, a two-step method is uniquely determined by the conditions for consistency and the specification of a desired stability interval. This is not true of three-step methods however, for there are many three-step methods with the same stability interval. In short, another degree of freedom exists with three-step methods once the desired stability interval is specified. This suggests that we use this degree of freedom to attempt to control the magnitude of the truncation error coefficient.

Nigro (1968) has derived the following method for so doing:

a. Select a desired C = stability interval (C negative)

b. Select a desired D = truncation error coefficient

c. Using a Newton-Raphson or similar scheme, solve for a_1 and b_1 by the following equations:

$$a_1 - 2b_1 + 3 \neq 0$$

$$Ca_1^2 - 4a_1b_1 - \left[4Cb_1^2 - 12(C-1)b_1 + 9C - 24\right] + D\left[(a_1 - 2b_1 + 3)C - 2(a_1 + 5)\right] = 0$$

$$16b_1^3 + 16(D-6)b_1^2 - 4\left[5a_1^2 + 6(D+1)a_1 + 22D - 51\right]b_1 - \left[a_1^4 + 2(D+2)a_1^3 + (D^2 + 14D - 18)a_1^2 + (10D^2 - 2D - 60)a_1 + (25D^2 - 126D + 153)\right] = 0$$

d. Calculate a_2, a_3, b_2, and b_3 by the following equations:

$$a_2 = \frac{a_1 + 10b_1 + 4a_1b_1 - 21 + 2D(a_1 + 5)}{a_1 - 2b_1 + 3}$$

$$a_3 = -\frac{a_1^2 + 2a_1b_1 + 3a_1 + 12b_1 - 24 + 2D(a_1 + 5)}{a_1 - 2b_1 + 3}$$

$$b_2 = -\frac{2a_1^2 - 4b_1^2 + 2a_1 + 20b_1 - 24 + 2D(a_1 - b_1 + 4)}{a_1 - 2b_1 + 3}$$

$$b_3 = -\frac{2b_1^2 + a_1b_1 + 2a_1 - b_1 + 2D(b_1 + 1) - 6}{a_1 - 2b_1 + 3}$$

It should be noted that when D, the desired truncation error coefficient, is set to zero, the method thus computed by the above equations is classical to the third degree rather than the second (i.e., an additional equation in Equation 8 is satisfied). In this case, the truncation error term is

$$\text{T.E.} = \frac{-a_1^2 + 6b_1^2 + 4a_1 - 20b_1 + 15}{3(a_1 - 2b_1 + 3)} \, h^3 y'''$$

rather than $Dh^2(\ddot{y})$. Table VII presents several derived methods with varying stability intervals and the O_{33} Mod Gurk truncation error ($.032967\, h^2 \ddot{y}$).

8. LOCAL ESTIMATE OF PERCENT ERROR IN THE COMPUTED SOLUTION

Nigro (1967 and 1968) has developed a method for estimating the percent error to be expected at the start of the solution using given integration methods.

TABLE VII

DERIVED THREE-STEP METHODS WITH THE O_{33} MOD GURK TRUNCATION ERROR

Stability Interval	a_1 b_1	a_2 b_2	a_3 b_3
-.1	-.816711 2.785948	.821185 .117968	.995526 .908322
-.82	.070352 1.993749	.105529 .286065	.824118 .473952
-2.0	.967033 1.372879	-.491514 .032967	.524481 .151603
-10.0	2.509498 .386909	-2.162834 -.244364	.653336 .001293
-40.0	2.861014 .099693	-2.734700 -.087032	.873686 .000010
-120.0	2.932631 .033316	-2.867436 -.031141	.934805 .295 (10^{-6})

Let

e_n = exact propagated error at the nth step of the integration

$$\delta^n y(\xi_J) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} y\left(\xi_{J+\frac{n}{2}-i}\right)$$

h = the integration interval (Δt)

$A_p{}^*$ = the truncation error coefficient

T_{n+K} = truncation error

K = the order of the integration method

Assume the following:

a. The solution of $\dot{y} = f(t, y)$ is generated by

$$y_{n+k} = \sum_{i=1}^{K} a_i y_{n+k-i} + h \sum_{i=1}^{K} b_i \dot{y}_{n+k-i}$$

b. Local round-off error in the numerical solution is negligible when compared with local truncation error.

c. $e_0 = e_1 = \cdots = e_{K-1} = 0$

d. $y^{(p^*)}(\xi_J) = \delta^{p^*} y(\xi_J)/h^{p^*}$

e. $y(\xi_{\frac{n+p^*}{2}-i}) = y(\xi_{\frac{n+p^*-1}{2}-i}) = \cdots = y(\xi_{\frac{n+2p^*+K}{4}-i})$

f. $\dot{y} = Ly + \epsilon$, where L is chosen (in a small interval) so that ϵ is small.

Let

$$\zeta = C_{11} + C_{12} + \cdots + C_{1,n-K+1}$$

where

$$\begin{aligned}
C_{11} &= 1 \\
C_{12} &= \gamma_{1,n-1} \\
C_{13} &= \gamma_{1,n-2} C_{12} + \gamma_{2,n-2} \\
&\vdots \\
C_{1,K+1} &= \gamma_{1,n-K} C_{1K} + \cdots + \gamma_{K-1} C_{12} + \gamma_{K,n-K} \\
C_{1,K+2} &= \gamma_{1,n-K-1} C_{1,K+1} + \cdots + \gamma_{K,n-K-1} C_{12} \\
&\vdots \\
C_{1,n-K+1} &= \gamma_{1,K} C_{1,n-K} + \cdots + \gamma_{K,K} C_{1,n-2K+1}
\end{aligned}$$

and

$$\gamma_{i,J} = a_i + b_i h g_J$$

where $g_J = \frac{\partial f}{\partial y}(\xi)$ at the jth step of the integration.

Then $\frac{e_n}{y_n}$ = decimal equivalent of local percent error at nth step

$$\doteq A_{p^*} \zeta e^{-\frac{Lh}{2}(n-K-p^*)} (1 - e^{-Lh})^{p^*}$$

The above results are derived and proved by Nigro (1967 and 1968) and Chen (1967). Nigro (1968) points out that the assumption of a single differential equation does not seriously impair the value of the error estimate, because, in a local growth study, an arbitrary system of differential equations may be equated in behavior to an uncoupled system. Table VIII presents comparisons of the exact and estimated percent errors for several problems and integration methods.

TABLE VIII

COMPARISON OF ACTUAL AND ESTIMATED PERCENT ERRORS FOR SELECTED PROBLEMS*

Method	h	Differential** Equation	n	Estimated % Error	Actual % Error
O_{33} Mod Gurk	.1	$\dot{y} = -5y$	5	3.08	2.81
Euler	.1	$\dot{y} = x + y$	4	1.74	1.65
Euler	.1	$\dot{y} = x + y$	5	5.97	4.25
Euler	.1	$\ddot{y} + y = o$	5	4.98	2.47
Point-Slope[†]	.1	$\dot{y} = -2xy^2$	5	.28	.25
Point-Slope	.1	$\dot{y} = -2xy^2$	6	.36	.27

*From Nigro, et al (1968)

**All initial conditions are 1.0

[†] $y_{n+1} = y_{n-1} + 2h\dot{y}_n$

SECTION IV

APPLICATIONS AND EMPIRICAL RESULTS

The Simulation Analyzer Program has been used to evaluate and derive integration methods for application to selected problems. The problems were subsequently solved using various of these methods and results were compared with those predicted by SAP. So far, the problems examined have consisted of one first- and one second-order differential system. Eventually, similar empirical studies will be conducted for total existing flight simulation problems, but for this initial empirical study it was desired to use problems whose analytical solution and behavior are well known.

1. A FIRST-ORDER PROBLEM

Consider the following first-order, linear problem:

$$\dot{y} = -5y\,, \qquad y(o) = 1 \tag{11}$$

The true solution of Equation 11 is

$$y = e^{-5T}$$

and the only eigenvalue is

$$\lambda = -5.$$

Twenty-six different integration methods were used to solve Equation 11 at an integration interval of h = .05. At each step of the solution, the following quantities were computed and printed:

1. Computed solution
2. True solution
3. Error
4. Percent error
5. Cumulative average error

The computations were performed in FORTRAN II using a Raytheon 440 computer.

The integration methods which were used are listed in Table IX. Nineteen are nonclassical, second-order methods which were derived using the SAP program and which have stability intervals ranging from -.29 to -113. One method is a new three-step method with the same truncation error and stability interval as the O_{33} Mod Gurk method. The remaining six methods are composed of a five-step method, a four-step method, and four well-known simulation integration techniques.

TABLE IX

METHODS OF INTEGRATION* FOR EXPERIMENTS WITH A FIRST-ORDER PROBLEM

Name of Method	Order	Stability Range	Truncation Error Coefficient	Degree to Which Classical
SR.29	2	-.286	-2.5	2
SR.33	2	-.333	-2.0	2
SR.40	2	-.400	-1.5	2
SR.50	2	-.500	-1.0	2
SR.67	2	-.667	-.50	2
SR2.5	2	-2.5	.84	2
SR3.0	2	-3.0	.78	2
SR4.0	2	-4.0	.75	2
SR5.0	2	-5.0	.76	2
SR8.0	2	-8.0	.813	2
SR10.0	2	-10.0	.84	2
SR20.0	2	-20.0	.91	2
SR25.0	2	-25.0	.926	2
SR35.0	2	-35.0	.946	2
SR40.0	2	-40.0	.953	2
SR50.0	2	-50.0	.962	2
SR73.5	2	-73.46	.974	2
SR100.0	2	-100.0	.980	2
SR113.8	2	-113.76	.984	2
Euler	1	-2.0	.5	2
2nd Ord. Adams	2	-1.0	.417	3
O_{33} Mod Gurk	3	-.821	.033	2
MTH.03-.8	3	-.820	.033	2
Shift. Trap.	2	-2.0	1.0	2
Open 5 Step	5	-1.223	1.5	2
O_{14}	4	-.286	-.042	4

*See the Appendix for the coefficients of the above integration methods.

Each integration method used to solve Equation 11 was also submitted to SAP for analysis and evaluation. To provide a consistent basis for comparison of the methods, SAP was forced, by the nature of our specified inputs at the time of running[3], to select .05 as the optimum integration interval. Therefore, SAP provided the expected percent error at the start of the solution using $\Delta t = .05$, and our actual integrations of Equation 11 at the same Δt could then be compared with SAP's predictions.

Table X shows the estimated and actual percent errors at the (K + 2)th step in each integration (K is the order of the integration method). The estimated errors were provided by SAP and the actual errors were computed during the integrations by the formula

$$\text{Percent Error} = \frac{\text{True - Computed}}{\text{True}} (100)$$

The remarkably close agreement between the values is apparent in Table X. A simple correlation coefficient of .997 was computed between the actual percent errors and the estimations. A regression plot of the results is provided in Figure 2.

The stability weight derived by SAP for each integration method can be very useful in predicting solution behavior. Stability is a difficult phenomenon to measure empirically; however, two observations are presented here to illustrate the utility of the derived stability weight.

Both the O_{33} Mod Gurk method and the MTH.03-.8 method (see Table IX) have the same stability interval. This tells us that both will produce completely unstable solutions, i.e., solutions in which errors propagate without bound, as Δt is increased to the same value for each method (assuming a problem with a real critical eigenvalue).

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[3] To force SAP to derive .05 as an optimum integration interval for every method, we specified the inaccuracy tolerance as 98%. Thus, the Δt required to achieve this level of accuracy was large for each method (eg, on the order of .1 to .2). We specified 20/second as our required minimum update rate. Thus, .05 (1/20) was the largest Δt satisfying specified requirements.

TABLE X

ESTIMATED AND ACTUAL PERCENT ERRORS OF SOLUTIONS TO A FIRST-ORDER PROBLEM

Integration Method	K	Estimated % Error at K + 2	Actual % Error at K + 2	Estimated - Actual % Error
SR.29	2	53.55	52.90	.65
SR.33	2	43.22	42.33	.89
SR.40	2	33.28	32.68	.60
SR.50	2	23.14	23.26	-.12
SR.67	2	12.24	13.37	-1.13
SR2.5	2	16.71	13.64	3.07
SR3.0	2	17.75	14.24	3.51
SR4.0	2	22.13	18.10	4.03
SR5.0	2	26.81	22.45	4.36
SR8.0	2	37.88	33.06	4.82
SR10.0	2	42.98	38.05	4.93
SR20.0	2	55.93	50.92	5.01
SR25.0	2	59.03	54.04	4.99
SR35.0	2	62.81	57.86	4.95
SR40.0	2	64.04	59.11	4.93
SR50.0	2	65.81	60.91	4.80
SR73.5	2	68.26	63.40	4.86
SR100.0	2	69.52	64.68	4.84
SR113.8	2	70.16	65.33	4.83
Euler	1	14.15	10.69	3.46
2nd Ord. Adams	2	3.29	2.28	1.01
O_{33} Mod Gurk	3	.94	.197	.743
MTH.03-.8	3	.32	.019	.301
Shift. Trap	2	30.79	26.25	4.54
Open 5 Step	5	19.51	22.12	-2.61
O_{14}	4	.10	.25	-.15

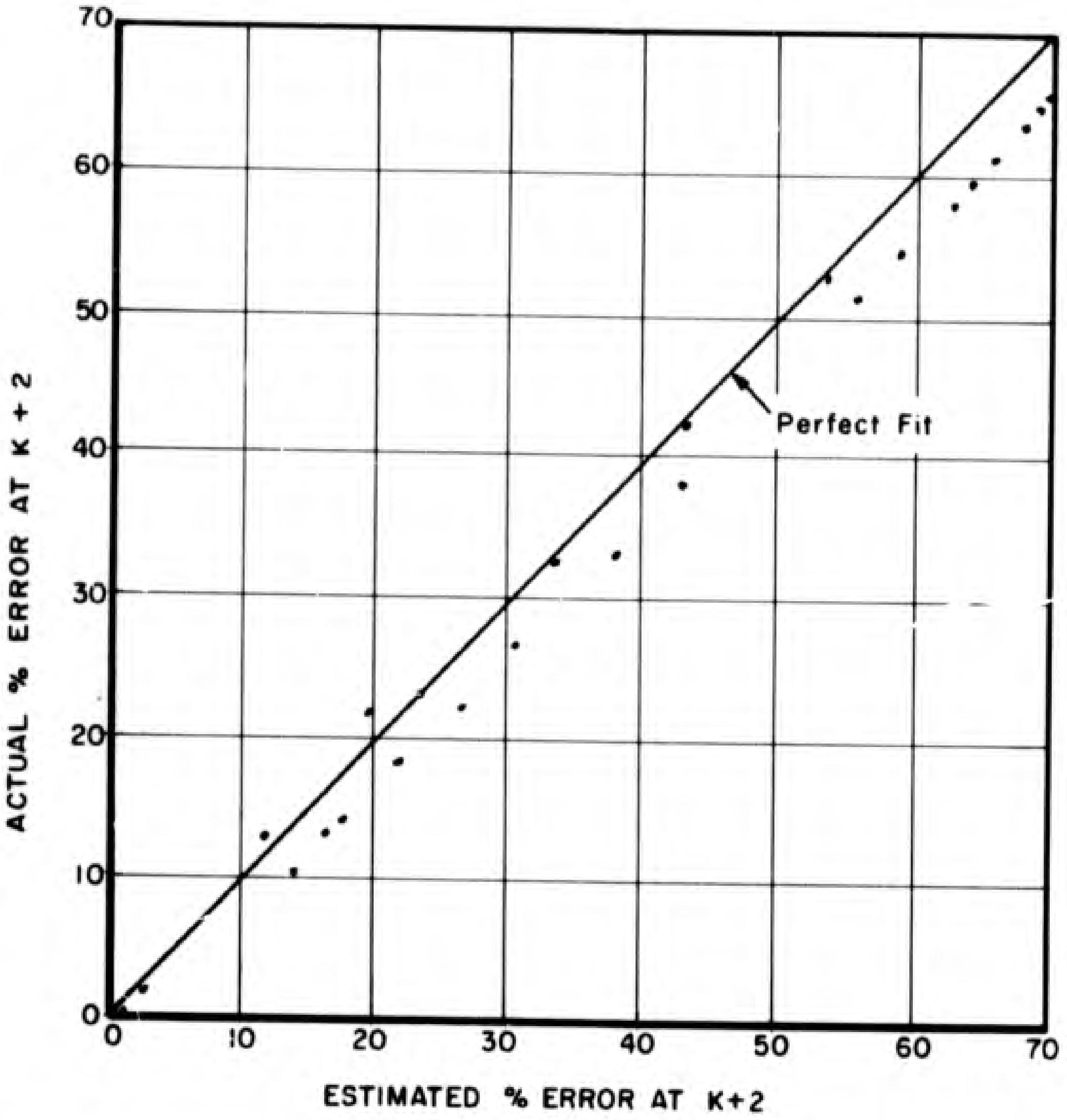

Figure 2. Plot of Actual Vs Estimated Percent Errors at (K + 2)th Step for a First-Order Problem Using Various Integration Methods

The stability weight (first-order problem) for the O_{33} Mod Gurk method is .6515, and for the MTH.03-.8 method it is .1059. This tells us that at a given Δt for which each solution remains stable, a greater degree of stability will be observed using the O_{33} Mod Gurk method.

The solutions for Equation 11 using the above two methods are summarized in Tables XI and XII. It is seen that, for the O_{33} Mod Gurk solution, the error decreases smoothly, and no oscillations are observed in the computed solution. For the MTH.03-.8 method, however, the magnitudes of the errors oscillate, and the computed solution oscillates about the true solution.

TABLE XI

O_{33} MOD GURK SOLUTION TO A FIRST-ORDER PROBLEM*

T	Computed Solution (10^4)	True Solution (10^4)	Error (10^5)
0.	10000.	10000.	0.
.2	3674.	3679.	48.21
.4	1348.	1353.	53.22
.6	494.6	497.9	32.59
.8	181.5	183.2	16.77
1.0	66.59	67.38	7.920
1.2	24.43	24.79	3.556
1.4	8.964	9.119	1.544
1.6	3.289	3.355	.6546
1.8	1.207	1.234	.2726
2.0	.4428	.4540	.1119
2.2	.1625	.1670	.0455
2.4	.0596	.0614	.0183
2.45	.0464	.0479	.0146
2.5	.0361	.0373	.0116
2.55	.0281	.0290	.0092
2.6	.0219	.0226	.0073
2.65	.0170	.0176	.0058
2.7	.0133	.0137	.0046
2.75	.0103	.0107	.0037
2.8	.0080	.0083	.0029
2.85	.0062	.0065	.0023
2.9	.0049	.0050	.0018
2.95	.0038	.0039	.0014
3.0	.0029	.0031	.0011

*An integration interval of h = .05 was used.

TABLE XII

MTH.03-.8 SOLUTION TO A FIRST-ORDER PROBLEM*

T	Computed Solution (10^4)	True Solution (10^4)	Error (10^5)
0.	10000.	10000.	0.
.2	3679.	3679.	3.742
.4	1354.	1353.	6.261
.6	498.3	497.9	3.822
.8	183.2	183.2	.0239
1.0	67.64	67.38	2.629
1.2	24.64	24.79	1.503
1.4	9.282	9.119	1.634
1.6	3.257	3.355	.972
1.8	1.292	1.234	.581
2.0	.4402	.4540	.138
2.2	.1524	.1670	.146
2.4	.0943	.0614	.329
2.45	-.0060	.0479	.539
2.5	.0729	.0373	.357
2.55	.0352	.0290	.0613
2.6	-.0168	.0226	.394
2.65	.0599	.0176	.422
2.7	-.0014	.0137	.151
2.75	-.0092	.0107	.199
2.8	.0461	.0083	.378
2.85	-.0211	.0065	.275
2.9	.0041	.0050	.0099
2.95	.0301	.0039	.262
3.0	-.0276	.0031	.307

*An integration interval of h = .05 was used.

This observation implies that the stability weight itself should correlate highly with observed degree of stability in a solution. An empirical measure of the latter was needed so an effective comparison could be made. Oscillation of the computed solution about the true solution is one sign of probable instability for this example. However, another measure which is both more easily obtained numerically and more reliable for the general problem of integration concerns the decay of errors in the solution. Highly stable methods tend to damp-out errors quickly, even if starting errors are high. Lowly or marginally stable methods tend to allow errors to propagate more readily.

A percent-error reduction in the computed solution was calculated for each of the 26 integration methods over the interval t = .3 to t = 3.0. This was accomplished by the formula

$$PER = \frac{C_{0.3} - C_{3.0}}{C_{0.3}}$$

where PER is the percent-error reduction and C_k is the cumulative average error at t = k.

These were compared with the stability weights for the methods, and the results are shown in Table XIII. The percent-error reductions and stability weights were ranked and a Spearman rank correlation computed. The high correlation of .8 shows good agreement and indicates the utility of the stability weight for this problem in providing comparative data on several integration methods. (It is hypothesized that a perfect correlation would result using a suitable combination of the stability and propagated error weights rather than only the former. Unfortunately, the nature of this combination is yet unknown.) Figure 3 shows a regression plot of the results.

2. A SECOND-ORDER PROBLEM

Consider the following second-order, linear differential equation:

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = F(t) \tag{12}$$

TABLE XIII

COMPARISON OF PERCENT-ERROR REDUCTION AND STABILITY WEIGHT FOR VARIOUS INTEGRATION METHODS

Method	Cumulative Error T = .3	Cumulative Error T = 3.0	% Error* Reduction	Rank**	Stability Weight	Rank**
SR.29	.1600	.04395	72.53	18	.1340	17
SR.33	.1346	.03549	73.63	17	.2070	16
SR.40	.1076	.02702	74.89	16	.2819	12
SR.50	.0780	.01855	76.22	13	.3596	9
SR.67	.0449	.01007	77.57	12	.4410	7
SR2.5	.04426	.008031	81.86	2	.3319	10
SR3.0	.04753	.008776	81.53	4	.4788	6
SR4.0	.05846	.01111	81.00	5	.5935	2
SR5.0	.07145	.01401	80.39	7	.4800	5
SR8.0	.1082	.02399	77.83	10	.2819	11
SR10.0	.1279	.03108	75.70	15	.2190	15
SR20.0	.1847	.0860	53.44	19	.1042	19
SR25.0	.1997	.1219	38.96	20	.0826	20
SR35.0	.2187	.2048	6.36	21	.0585	21
SR40.0	.2250	.2495	-10.89	22	.0510	22
SR50.0	.2343	.3369	-43.79	23	.0406	23
SR73.5	.2474	.5576	-125.38	24	.0269	24
SR100.0	.2542	.7850	-208.81	25	.0202	25
SR113.8	.2577	.9274	-259.88	26	.0168	26
Euler	.04486	.00868	80.65	6	.2500	13
2nd Ord. Adams	.7588E-2	.0016	78.91	8	.5280	4
O_{33} Mod Gurk	.4874E-3	.1092E-3	77.60	11	.6515	1
MTH.03-.8	.7655E-4	.1655E-4	78.38	9	.1059	18
Shift. Trap.	.08385	.01535	81.69	3	.5625	3
Open 5 Step	.06353	.006742	89.39	1	.2422	14
O_{14}	.4279E-3	.1033E-3	75.86	14	.3864	8

$$\text{*\% Error Reduction} = \frac{100\left[(\text{Cum Error at } T = .3) - (\text{Cum Error at } T = 3.0)\right]}{\text{Cum Error at } T = .3}$$

**Spearman Rank Correlation = .8

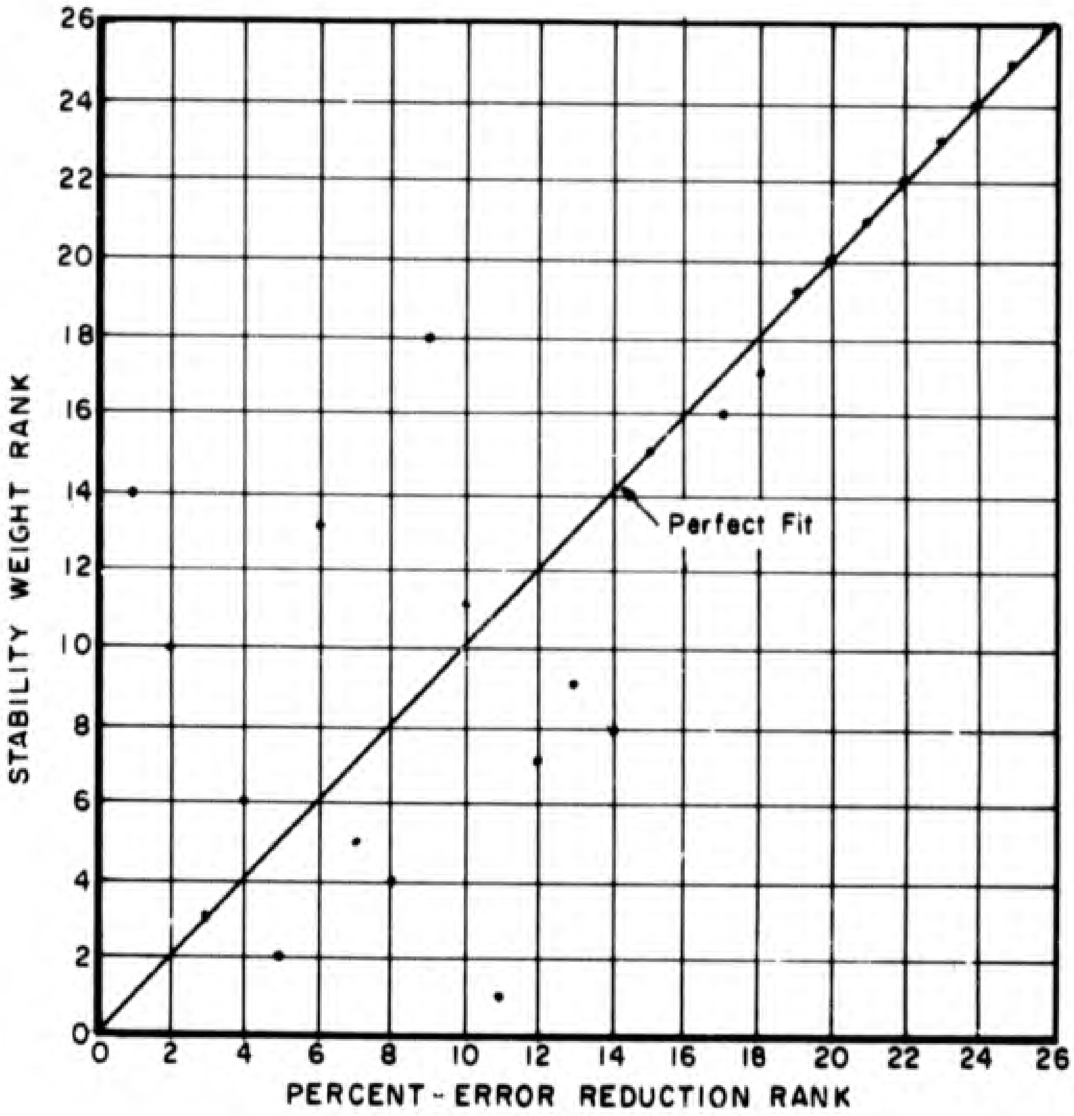

Figure 3. Plot of Percent Error Reduction Rank Vs Stability Weight Rank for a First-Order Problem Using Various Integration Methods

where

$$X(0) = \dot{X}(0) = 0$$

In Equation 12, ξ represents the damping ratio and ω_n is the undamped natural frequency (radians/second) of the system.

From classical theories of ordinary differential equations, we know that the solution of Equation 12 is of the form

$$X = A_1 e^{-\xi \omega_n t} \sin\left[wt + A_2\right]$$

This represents a sine wave whose amplitude decays exponentially. The quantity ($\xi\,\omega_n$) denotes the rate of decay, or the damping characteristics associated with the solution. The quantity w represents the actual angular frequency of oscillation of the solution in radians/second. The actual and undamped natural frequencies are related by the expression

$$w = \omega_n \sqrt{1 - \xi^2}$$

Note that if $\xi = 0$, i.e., if no damping is present, then $w = \omega_n$ as expected.

Equation 12 may be rewritten in the form of two first-order differential equations by setting a new parameter

$$Z = \dot{X}$$

We obtain

$$\left\{\begin{array}{l} \dot{Z} = F(t) - \omega_n^2 X - 2\xi\omega_n Z \\ \dot{X} = Z \end{array}\right\} \qquad (13)$$

where $X(0) = Z(0) = 0$.

The system Equation 13 is equivalent to Equation 12 and will be the form used in the following discussions.

a. Eigenvalue Analysis

We will now derive the critical eigenvalue for system Equation 13, since this is required as an input to SAP. To do this, we form the characteristic equation for Equation 13 and solve for its roots. The root with the maximum modulus is the critical eigenvalue we seek.

The characteristic equation for Equation 13 is

$$\begin{vmatrix} \dfrac{\partial \dot{Z}}{\partial Z} - \lambda & \dfrac{\partial \dot{Z}}{\partial X} \\ \dfrac{\partial \dot{X}}{\partial Z} & \dfrac{\partial \dot{X}}{\partial X} - \lambda \end{vmatrix} = 0 \qquad (14)$$

Solving Equation 13 for the partial derivatives and substituting them into Equation 14, we obtain

$$\begin{vmatrix} -2\xi\omega_n - \lambda & -\omega_n^2 \\ 1 & -\lambda \end{vmatrix} = 0$$

or

$$\lambda^2 + 2\xi\omega_n\lambda + \omega_n^2 = 0$$

the solution of which is

$$\lambda = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$\lambda = -\xi\omega_n \pm \omega_n\sqrt{1-\xi^2}\ i \qquad (15)$$

Since both values for λ in Equation 15 have the same modulus, we may select either for our critical eigenvalue. Let us choose

$$\lambda_1 = -\xi\omega_n + \omega_n\sqrt{1-\xi^2}\ i$$

From this, we know that in the most critical area of the solution, the true frequency is $\omega_n\sqrt{1-\xi^2}$ and the damping factor is $e^{-\xi\omega_n t}$.

b. True Solution

For this second-order problem, we may proceed to calculate the analytic solution. We may do this using LaPlace transforms or using the eigenvalues we have just derived. The solution will be of the form

$$x = e^{-\xi\omega_n t}\left[C_1\cos(\omega_n\sqrt{1-\xi^2}\,t) + C_2\sin(\omega_n\sqrt{1-\xi^2}\,t)\right] + C_3 \qquad (16)$$

where C_1, C_2, and C_3 are yet-unknown coefficients to be determined from initial conditions.

Differentiating Equation 16 twice and substituting initial conditions for X, $\dot{X}$, and $\ddot{X}$, we obtain

$$C_1 + C_3 = 0$$

$$-\xi C_1 + \sqrt{1+\xi^2}\; C_2 = 0$$

$$(2\xi^2-1)\, C_1 - 2\xi\sqrt{1-\xi^2}\; C_2 = \frac{F(t)}{\omega_n^2}$$

Solving, we obtain

$$C_1 = -\frac{F(t)}{\omega_n^2}$$

$$C_2 = -\frac{\xi F(t)}{\omega_n^2\sqrt{1-\xi^2}}$$

$$C_3 = \frac{F(t)}{\omega_n^2}$$

Therefore,

$$x = \frac{F(t)}{\omega_n^2} - e^{-\xi\omega_n t}\left\{\frac{F(t)}{\omega_n^2}\left[\cos(\omega_n\sqrt{1-\xi^2}\; t) + \frac{\xi}{\sqrt{1-\xi^2}}\sin(\omega_n\sqrt{1-\xi^2}\; t)\right]\right\}$$

c. Empirical Tests

To submit the second-order problem to SAP for analysis, we must first establish values for ξ, ω_n, and F(t) and solve Equation 15 for the (numerical) critical eigenvalue.

Let

$$\xi = 0.25$$

$$\omega_n = \pi$$

$$F(t) = \pi^2$$

Then

$$\lambda_1 = -0.78539 + 3.0417\, i$$

This information along with a total of 16 integration methods (see Table XIV) were submitted to SAP for analysis. The integration methods include one five-step, one four-step, six three-step, seven two-step, and one one-step methods. The stability intervals of these methods range from -.28 to -231.48.

TABLE XIV

METHODS OF INTEGRATION* FOR EXPERIMENTS WITH A SECOND-ORDER PROBLEM

Integration Method	Order	Stability Range	Truncation Error Coefficient	Degree to Which Classical
Euler	1	-2.0	.5	2
O_{33} Mod Gurk	3	-.8213	.03297	2
MTH.03-.8	3	-.820	.03297	2
2-Step	2	-4.0	.75	2
5-Step	5	-1.2227	1.5	2
O_{14}	4	-.2857	-.04167	4
Parabolic	3	-.545	.375	4
SR.29	2	-.2857	-2.5	2
SR.40	2	-.4	-1.5	2
SR10.0	2	-10.0	.84	2
SR50.0	2	-50.0	.9616	2
3-Step	3	-.8236	.0328	2
M.01-231	3	-231.48	.00602	2
M.01-15	3	-15.75	.00792	2
Shift Trap.	2	-2.0	1.0	2
2nd Ord. Adams	2	-1.0	.4167	3

*See the Appendix for the coefficients of the above integration methods

Each integration method was first submitted to SAP along with a minimum update rate of 20/second and inaccuracy tolerance of 98%, thus forcing the selection of an optimum h of .05 for most methods. For a second run, an update-rate of 10/second was specified, permitting the program to choose .1 as the optimum integration interval for many of the methods. Table XV presents the actual output from the Simulation Analyzer Program for these runs. (Note that to save execution time, the figure of merit schemes for HMAX were not generated in the output.)

TABLE XV

SAP OUTPUT FOR A SECOND-ORDER PROBLEM

The user has selected the second option in the program which evaluates candidate methods which have been supplied.

The candidate methods are

```
Euler
                 K = 1          A(1) =  1.00000000E 00
                                B(0) =  0.00000000E-01
                                B(1) =  1.00000000E 00

Mod Gurk
                 K = 3          A(1) =  1.14620800E 00
                                A(2) = -2.01087000E-01
                                A(3) =  5.48790000E-02
                                B(0) =  0.00000000E-01
                                B(1) =  1.64158600E 00
                                B(2) = -1.00801300E 00
                                B(3) =  2.75097000E-01

M.03-.8
                 K = 3          A(1) =  7.03520000E-02
                                A(2) =  1.05529000E-01
                                A(3) =  8.24113000E-01
                                B(0) =  0.00000000E-01
                                B(1) =  1.99374900E 00
                                B(2) =  2.86065000E-01
                                B(3) =  4.73952000E-01

Two Step
                 K = 2          A(1) =  1.00000000E 00
                                A(2) =  0.00000000E-01
                                B(0) =  0.00000000E-01
                                B(1) =  7.50000000E-01
                                B(2) =  2.50000000E-01

Five Step
                 K = 5          A(1) =  5.00000000E-01
                                A(2) =  0.00000000E-01
                                A(3) =  0.00000000E-01
                                A(4) =  0.00000000E-01
                                A(5) =  5.00000000E-01
                                B(0) =  0.00000000E-01
                                B(1) =  1.00000000E 00
                                B(2) =  1.00000000E 00
                                B(3) =  0.00000000E-01
                                B(4) =  0.00000000E-01
                                B(5) =  1.00000000E 00
```

TABLE XV (CONT)

Open 14

K = 4

A(1) = 1.00000000E 00
A(2) = 0.00000000E-01
A(3) = 0.00000000E-01
A(4) = 0.00000000E-01
B(0) = 0.00000000E-01
B(1) = 2.33333300E 00
B(2) = -2.58333300E 00
B(3) = 1.66666700E 00
B(4) = -4.16667000E-01

Parabolic

K = 3

A(1) = 1.00000000E 00
A(2) = 0.00000000E-01
A(3) = 0.00000000E-01
B(0) = 0.00000000E-01
B(1) = 1.91666700E 00
B(2) = -1.33333300E 00
B(3) = 4.16667000E-01

SR.29

K = 2

A(1) = 1.00000000E 00
A(2) = 0.00000000E-01
B(0) = 0.00000000E-01
B(1) = 4.00000000E 00
B(2) = -3.00000000E 00

SR.40

K = 2

A(1) = 1.00000000E 00
A(2) = 0.00000000E-01
B(0) = 0.00000000E-01
B(1) = 3.00000000E 00
B(2) = -2.00000000E 00

SR10.0

K = 2

A(1) = 1.60000000E 00
A(2) = -6.00000000E-01
B(0) = 0.00000000E-01
B(1) = 3.60000000E-01
B(2) = 4.00000000E-02

The following methods have been rejected since they are inconsistent.

None

The following methods have been rejected since they have stability intervals of zero length.

None

TABLE XV (CONT)

The candidate methods have been examined within the framework of the user-s specifications which are given below.

The differential system which defines our simulation has as the most critical eigenfrequency, the complex quantity -0.7853900 + I (3.042) which has modulus 3.14146086.

In order to adequately describe the physical system, the dependent variables must be updated 20 times per time unit, which is seconds.

The percentage of error which will be tolerated in the integrated quantities is 98.00 percent.

In order to aid in our examination of the candidate methods, the following parameters have been computed and are given in Table 1.

(A) Stability interval. The number N1 so that N1.LT. HG .LE. 0.0

(B) Truncation error, T. (That is the numbers N2 and N3 such that T = (N2) (H superscript N3) (N3-th derivative of the dependent variable)

(C) The maximum integration interval (N4) possible for each method. (An interval greater than N4 will produce instability in the computed solution.)

In order to update our dependent variables 20 times per time unit, the maximum integration interval which may be used is 0.05000. Call it Hmax. Distinguish it from N4 (below). Clearly, a workable integration interval (H) must be less than Hmax and less than or equal to N4.

TABLE 1

Method	N1 Maximum Stability Interval	N2 Coefficient of Truncation Error	N3 Degree Classical	N4 Hmax
Euler	-2.00000	0.50000	2.00000	0.15794
Mod Gurk	-0.82134	0.03297	2.00000	0.25110
M.03-.8	-0.82000	0.03297	2.00000	0.05185
Two Step	-4.00000	0.75000	2.00000	0.10610
Five Step	-1.22271	1.50000	2.00000	0.10900
Open 14	-0.28571	-0.04167	4.00000	0.08913
Parabolic	-0.54545	0.37500	4.00000	0.17016
SR.29	-0.28571	-2.50000	2.00000	0.08913
SR.40	-0.40000	-1.50000	2.00000	0.12478
SR10.0	-10.00000	0.84000	2.00000	0.03971

Our analysis of the candidate methods reveals that the following integration intervals H(ACC) are required to maintain a 98.000 percent error in the computed solutions (Table 2).

TABLE XV (CONT)

TABLE 2

Method	H (Accuracy)	Users-s Hmax	Method-s Hmax	H (Optimum)	Percent Error by Using H
Euler	0.1796358	0.0500000	0.1579420	0.0500000	6.758
Mod Gurk	0.3854973	0.0500000	0.2510986	0.0500000	0.459
M.03-.8	0.3416472	0.0500000	0.0518523	0.0500000	0.153
Two Step	0.1436122	0.0500000	0.1061050	0.0500000	10.406
Five Step	0.2845107	0.0500000	0.1090017	0.0500000	10.349
Open 14	0.2794744	0.0500000	0.0891305	0.0500000	0.023
Parabolic	0.2097606	0.0500000	0.1701582	0.0500000	0.204
SR.29	0.0935942	0.0500000	0.0891305	0.0500000	26.216
SR.40	0.1158257	0.0500000	0.1247827	0.0500000	16.852
SR10.0	0.1022053	0.0500000	0.0397125	0.0397125	11.726

In Table 2 (above), a difference between HACC and H indicates that HACC is too large an H-value to maintain either stability and/or the required update rate.

We conclude our analysis of the candidate methods by computing a figure of merit for each. A breakdown by selected categories as well as total figures is given in the figure of merit schemes (below).

FIGURE OF MERIT SCHEME (HOPT)

Method	Stability	Truncation Error	Round-Off Error	Propagated Error	Required Computing Time	Total
Euler	2.731E-02	5.352E-01	8.576E-01	9.949E-01	6.435E-01	3.059E-00
Mod Gurk	5.118E-01	7.122E-01	1.064E-01	9.981E-01	5.477E-01	2.876E-00
M.03-.8	6.607E-02	7.122E-01	1.665E-01	3.327E-01	5.477E-01	1.825E-00
Two Step	4.905E-01	5.088E-01	3.007E-01	9.926E-01	6.039E-01	2.896E-00
Five Step	1.320E-01	4.636E-01	1.031E-01	3.598E-01	5.843E-01	1.643E-00
Open 14	4.383E-01	1.000E 00	1.124E-01	9.592E-01	5.680E-01	3.078E-00
Parabolic	5.012E-01	9.439E-01	1.677E-01	9.981E-01	5.843E-01	3.195E-00
SR.29	2.837E-01	4.304E-01	2.419E-01	9.855E-01	6.039E-01	2.545E-00
SR.40	3.657E-01	4.636E-01	2.585E-01	9.898E-01	6.039E-01	2.682E-00
SR10.0	1.994E-01	5.314E-01	1.066E-01	9.857E-01	5.533E-01	2.376E-00

In our figure of merit evaluation of the candidate methods, stability, truncation error, round-off error, propagated error, and required computing time have been weighted 1.000, 1.000, 1.000, 1.000 and 1.000, respectively (FPER = 0.000E-01, FPEI = 0.000E-01)

Based on this figure of merit, we conclude that the best integration scheme is the Parabolic method.

TABLE XV (CONT)

The user has selected the second option in the program which evaluates candidate methods which have been supplied.

The candidate methods are

```
SR50.
                    K = 2           A(1) =  1.92000000E 00
                                    A(2) = -9.20000000E 01
                                    B(0) =  0.00000000E-01
                                    B(1) =  7.84000000E-02
                                    B(2) =  1.60000000E-03

Three Stp
                    K = 3           A(1) =  1.13927800E 00
                                    A(2) = -1.99971600E-01
                                    A(3) =  6.06935000E-02
                                    B(0) =  0.00000000E-01
                                    B(1) =  1.64241200E 00
                                    B(2) = -9.96196000E-01
                                    B(3) =  2.75199700E-01

M.01-231
                    K = 3           A(1) =  2.97328600E 00
                                    A(2) = -2.95568000E 00
                                    A(3) =  9.82394000E-01
                                    B(0) =  0.00000000E-01
                                    B(1) =  1.88910000E-02
                                    B(2) = -1.25350000E-02
                                    B(3) =  2.75100000E-03

M.01-15
                    K = 3           A(1) =  2.72477600E 00
                                    A(2) = -2.53639500E 00
                                    A(3) =  8.11620000E-01
                                    B(0) =  0.00000000E-01
                                    B(1) =  2.45898000E-01
                                    B(2) = -1.81075000E-01
                                    B(3) =  2.20210000E-02

Shft Trap
                    K = 2           A(1) =  1.00000000E 00
                                    A(2) =  0.00000000E-01
                                    B(0) =  0.00000000E-01
                                    B(1) =  5.00000000E-01
                                    B(2) =  5.00000000E-01

2nd Adams
                    K = 2           A(1) =  1.00000000E 00
                                    A(2) =  0.00000000E-01
                                    B(0) =  0.00000000E-01
                                    B(1) =  1.50000000E 00
                                    B(2) = -5.00000000E-01
```

TABLE XV (CONT)

The following methods have been rejected since they are inconsistent.

None

The following methods have been rejected since they have stability intervals of zero length.

None

The candidate methods have been examined within the framework of the user-s specifications which are given below.

The differential system which defines our simulation has as the most critical eigenfrequency the complex quantity -0.7853900 + I (3.042) which has modulus 3.14146086.

In order to adequately describe the physical system, the dependent variables must be updated 20 times per time unit, which is seconds.

The percentage of error which will be tolerated in the integrated quantities is 98.00 percent.

In order to aid in our examination of the candidate methods, the following parameters have been computed and are given in Table 1.

(A) Stability interval. The number N1 so that N1 .LT. HG .LE. 0.0

(B) Truncation error, T. (That is the numbers N2 and N3 such that T = (N2) (H superscript N3) (N3-th derivative of the dependent variable)

(C) The maximum integration interval (N4) possible for each method. (An interval greater than N4 will produce instability in the computed solution.)

In order to update our dependent variables 20 times per time unit, the maximum integration interval which may be used is 0.05000. Call it Hmax. Distinguish it from N4 (below). Clearly, a workable integration interval (H) must be less than Hmax and less than or equal to N4.

TABLE 1

Method	N1 Maximum Stability Interval	N2 Coefficient of Truncation Error	N3 Degree Classical	N4 Hmax
SR50.	-50.00000	0.96160	2.00000	0.00694
Three Stp	-0.82364	0.03280	2.00000	0.25180
M.01-231	-231.48199	0.00602	2.00000	0.00996
M.01-15	-15.75253	0.00792	2.00000	0.04262
Shft Trap	-2.00000	1.00000	2.00000	0.07947
2nd Adams	-1.00000	0.41667	3.00000	0.25489

TABLE XV (CONT)

Our analysis of the candidate methods reveals that the following integration intervals H(ACC) are required to maintain a 98.000 percent error in the computed solutions (Table 2).

TABLE 2

Method	H (Accuracy)	User-s Hmax	Method-s Hmax	H (Optimum)	Percent Error by Using H
SR50.	0.0858309	0.0500000	0.0069411	0.0069411	0.464
Three Stp	0.3857530	0.0500000	0.2518028	0.0500000	0.454
M.01-231	0.3966178	0.0500000	0.0099560	0.0099560	0.010
M.01-15	0.4011928	0.0500000	0.0426160	0.0426160	0.248
Shft Trap	0.1209663	0.0500000	0.0794665	0.0500000	14.249
2nd Adams	0.2094535	0.0500000	0.2548914	0.0500000	1.110

In Table 2 (above), a difference between HACC and H indicates that HACC is too large an H-value to maintain either stability and/or the required update rate.

We conclude our analysis of the candidate methods by computing a figure of merit for each. A breakdown by selected categories as well as total figures is given in the figure of merit schemes (below).

FIGURE OF MERIT SCHEME (HCPT)

Method	Stability	Truncation Error	Round-off Error	Propagated Error	Required Computing Time	Total
SR50.	4.000E-02	7.497E-01	8.303E-02	9.976E-01	3.828E-01	2.253E-00
Three Stp	5.044E-01	7.125E-01	1.067E-01	9.981E-01	5.477E-01	2.869E-00
M.01-231	5.902E-03	1.000E 00	2.330E-02	5.258E-01	3.900E-01	1.945E-00
M.01-15	6.643E-02	8.258E-01	2.643E-02	7.599E-01	5.321E-01	2.211E-00
Shft Trap	4.675E-01	4.900E-01	3.019E-01	9.907E-01	6.039E-01	2.854E-00
2nd Adams	4.789E-01	7.421E-01	2.881E-01	9.982E-01	6.039E-01	3.111E-00

In our figure of merit evaluation of the candidate methods, stability, truncation error, round-off error, propagated error, and required computing time have been weighted 1.000, 1.000, 1.000, 1.000 and 1.000, respectively (FPER = 0.000E-01, FPEI = 0.000E-01)

Based on this figure of merit, we conclude that the best integration scheme is the 2nd Adams method.

TABLE XV (CONT)

The user has selected the second option in the program which evaluates candidate methods which have been supplied.

The candidate methods are

```
Euler
                        K = 1         A(1) =  1.00000000E 00
                                      B(0) =  0.00000000E-01
                                      B(1) =  1.00000000E 00
Mod Gurk
                        K = 3         A(1) =  1.14620800E 00
                                      A(2) = -2.01087000E-01
                                      A(3) =  5.48790000E-02
                                      B(0) =  0.00000000E-01
                                      B(1) =  1.64158600E 00
                                      B(2) = -1.00801300E 00
                                      B(3) =  2.75097000E-01
M.03-.8
                        K = 3         A(1) =  7.03520000E-02
                                      A(2) =  1.05529000E-01
                                      A(3) =  8.24118000E-01
                                      B(0) =  0.00000000E-01
                                      B(1) =  1.99374900E 00
                                      B(2) =  2.86065000E-01
                                      B(3) =  4.73952000E-01
Two Step
                        K = 2         A(1) =  1.00000000E 00
                                      A(2) =  0.00000000E-01
                                      B(0) =  0.00000000E-01
                                      B(1) =  7.50000000E-01
                                      B(2) =  2.50000000E-01
Five Step
                        K = 5         A(1) =  5.00000000E-01
                                      A(2) =  0.00000000E-01
                                      A(3) =  0.00000000E-01
                                      A(4) =  0.00000000E-01
                                      A(5) =  5.00000000E-01
                                      B(0) =  0.00000000E-01
                                      B(1) =  1.00000000E 00
                                      B(2) =  1.00000000E 00
                                      B(3) =  0.00000000E-01
                                      B(4) =  0.00000000E-01
                                      B(5) =  1.00000000E 00
```

TABLE XV (CONT)

Open 14

K = 4

A(1) = 1.00000000E 00
A(2) = 0.00000000E-01
A(3) = 0.00000000E-01
A(4) = 0.00000000E-01
B(0) = 0.00000000E-01
B(1) = 2.33333300E 00
B(2) = -2.58333300E 00
B(3) = 1.66666700E 00
B(4) = -4.16667000E-01

Parabolic

K = 3

A(1) = 1.00000000E 00
A(2) = 0.00000000E-01
A(3) = 0.00000000E-01
B(0) = 0.00000000E-01
B(1) = 1.91666700E 00
B(2) = -1.33333300E 00
B(3) = 4.16667000E-01

SR.29

K = 2

A(1) = 1.00000000E 00
A(2) = 0.00000000E-01
B(0) = 0.00000000E-01
B(1) = 4.00000000E 00
B(2) = -3.00000000E 00

SR.40

K = 2

A(1) = 1.00000000E 00
A(2) = 0.00000000E-01
B(0) = 0.00000000E-01
B(1) = 3.00000000E 00
B(2) = -2.00000000E 00

SR10.0

K = 2

A(1) = 1.60000000E 00
A(2) = -6.00000000E-01
B(0) = 0.00000000E-01
B(1) = 3.60000000E-01
B(2) = 4.00000000E-02

The following methods have been rejected since they are inconsistent.

None

The following methods have been rejected since they have stability intervals of zero length.

None

TABLE XV (CONT)

The candidate methods have been examined within the framework of the user-s specifications which are given below.

> The differential system which defines our simulation has as the most critical eigenfrequency the complex quantity -0.7853900 + I (3.042) which has modulus 3.1146086.
>
> In order to adequately describe the physical system, the dependent variables must be updated 10 times per time unit, which is seconds.
>
> The percentage of error which will be tolerated in the integrated quantities is 98.00 percent.

In order to aid in our examination of the candidate methods, the following parameters have been computed and are given in Table 1.

(A) Stability interval. The number N1 so that N1 .LT. HG .LE. 0.0

(B) Truncation error, T. (That is the numbers N2 and N3 such that T = (N2) (H superscript N3) (N3-th derivative of the dependent variable)

(C) The maximum integration interval (N4) possible for each method. (An interval greater than N4 will produce instability in the computed solution.)

In order to update our dependent variables 10 times per time unit, the maximum integration interval which may be used is 0.10000. Call it Hmax. Distinguish it from N4 (below). Clearly, a workable integration interval (H) must be less than Hmax and less than or equal to N4.

TABLE 1

Method	N1 Maximum Stability Interval	N2 Coefficient of Truncation Error	N3 Degree Classical	N4 Hmax
Euler	-2.00000	0.50000	2.00000	0.15794
Mod Gurk	-0.82134	0.03297	2.00000	0.25110
M. 03-. 8	-0.82000	0.03297	2.00000	0.05185
Two Step	-4.00000	0.75000	2.00000	0.10610
Five Step	-1.22271	1.50000	2.00000	0.10900
Open 14	-0.28571	-0.04167	4.00000	0.08913
Parabolic	-0.54545	0.37500	4.00000	0.17016
SR. 29	-0.28571	-2.50000	2.00000	0.08913
SR. 40	-0.40000	-1.50000	2.00000	0.12478
SR10. 0	-10.00000	0.84000	2.00000	0.03971

Our analysis of the candidate methods reveals that the following integration intervals H(ACC) are required to maintain a 98.000 percent error in the computed solutions (Table 2).

TABLE XV (CONT)

TABLE 2

Method	H (Accuracy)	User-s Hmax	Method-s Hmax	H (Optimum)	Percent Error by Using H
Euler	0. 1796358	0. 1000000	0. 1579420	0. 1000000	28. 381
Mod Gurk	0. 3854973	0. 1000000	0. 2510986	0. 1000000	1. 860
M. 03-. 8	0. 3416472	0. 1000000	0. 0518523	0. 0518523	0. 164
Two Step	0. 1436122	0. 1000000	0. 1061050	0. 1000000	44. 821
Five Step	0. 2845107	0. 1000000	0. 1090017	0. 1000000	35. 785
Open 14	0. 2794744	0. 1000000	0. 0891305	0. 0891305	0. 250
Parabolic	0. 2097606	0. 1000000	0. 1701582	0. 1000000	3. 392
SR. 29	0. 0935942	0. 1000000	0. 0891305	0. 0891305	86. 434
SR. 40	0. 1158257	0. 1000000	0. 1247827	0. 1000000	66. 700
SR10. 0	0. 1022053	0. 1000000	0. 0397125	0. 0397125	11. 726

In Table 2 (above), a difference between HACC and H indicates that HACC is too large an H-value to maintain either stability and/or the required update rate.

We conclude our analysis of the candidate methods by computing a figure of merit for each. A breakdown by selected categories as well as total figures is given in the figure of merit schemes (below).

FIGURE OF MERIT SCHEME (HOPT)

Method	Stability	Truncation Error	Round-off Error	Propagated Error	Required Computing Time	Total
Euler	2. 963E-02	4. 449E-01	7. 862E-01	9. 828E-01	7. 113E-01	2. 955E-00
Mod Gurk	4. 784E-01	6. 219E-01	1. 007E-01	9. 960E-01	6. 154E-01	2. 812E-00
M. 03-. 8	6. 610E-02	7. 074E-01	1. 660E-01	3. 648E-01	5. 512E-01	1. 856E-00
Two Step	4. 633E-01	4. 185E-01	2. 881E-01	9. 693E-01	6. 716E-01	2. 811E-00
Five Step	1. 186E-01	3. 734E-01	9. 626E-02	2. 540E-01	6. 520E-01	1. 494E-00
Open 14	3. 618E-01	9. 364E-01	9. 951E-02	6. 887E-01	6. 245E-01	2. 711E-00
Parabolic	4. 278E-01	7. 634E-01	1. 509E-01	9. 962E-01	6. 520E-01	2. 990E-00
SR. 29	8. 104E-02	3. 551E-01	2. 049E-01	7. 761E-01	6. 604E-01	2. 078E-00
SR. 40	2. 066E-01	3. 734E-01	2. 194E-01	9. 605E-01	6. 716E-01	2. 432E-00
SR10. 0	1. 994E-01	5. 314E-01	1. 066E-01	9. 857E-01	5. 533E-01	2. 376E-00

In our figure of merit evaluation of the candidate methods, stability, truncation error, round-off error, propagated error, and required computing time have been weighted 1.000, 1.000, 1.000, 1.000 and 1.000, respectively (FPER = 0.000E-01, FPEI = 0.000E-01)

Based on this figure of merit, we conclude that the best integration scheme is the Parabolic method.

TABLE XV (CONT)

The user has selected the second option in the program which evaluates candidate methods which have been supplied.

The candidate methods are

```
SR50.
                    K = 2           A(1) =  1.92000000E 00
                                    A(2) = -9.20000000E-01
                                    B(0) =  0.00000000E-01
                                    B(1) =  7.84000000E-02
                                    B(2) =  1.60000000E-03
Three Stp
                    K = 3           A(1) =  1.13927800E 00
                                    A(2) = -1.99971600E-01
                                    A(3) =  6.06935000E-02
                                    B(0) =  0.00000000E-01
                                    B(1) =  1.64241200E 00
                                    B(2) = -9.96196000E-01
                                    B(3) =  2.75199700E-01
M.01-231
                    K = 3           A(1) =  2.97328600E 00
                                    A(2) = -2.95568000E 00
                                    A(3) =  9.82394000E-01
                                    B(0) =  0.00000000E-01
                                    B(1) =  1.88910000E-02
                                    B(2) = -1.25350000E-02
                                    B(3) =  2.75100000E-03
M.01-15
                    K = 3           A(1) =  2.72477600E 00
                                    A(2) = -2.53639500E 00
                                    A(3) =  8.11620000E-01
                                    B(0) =  0.00000000E-01
                                    B(1) =  2.45898000E-01
                                    B(2) = -1.81075000E-01
                                    B(3) =  2.20210000E-02
Shft Trap
                    K = 2           A(1) =  1.00000000E 00
                                    A(2) =  0.00000000E-01
                                    B(0) =  0.00000000E-01
                                    B(1) =  5.00000000E-01
                                    B(2) =  5.00000000E-01
2nd Adams
                    K = 2           A(1) =  1.00000000E 00
                                    A(2) =  0.00000000E-01
                                    B(0) =  0.00000000E-01
                                    B(1) =  1.50000000E 00
                                    B(2) = -5.00000000E-01
```

TABLE XV (CONT)

The following methods have been rejected since they are inconsistent.

None

The following methods have been rejected since they have stability intervals of zero length.

None

The candidate methods have been examined within the framework of the user-s specifications which are given below.

The differential system which defines our simulation has as the most critical eigenfrequency the complex quantity -0.7853900 + I (3.042) which has modulus 3.14146086.

In order to adequately describe the physical system, the dependent variables must be updated 10 times per time unit, which is seconds.

The percentage of error which will be tolerated in the integrated quantities is 98.00 percent.

In order to aid in our examination of the candidate methods, the following parameters have been computed and are given in Table 1.

(A) Stability interval. The number N1 so that N1 .LT. HG .LE. 0.0

(B) Truncation error, T. (That is the numbers N2 and N3 such that T = (N2) (H superscript N3) (N3-th derivative of the dependent variable)

(C) The maximum integration interval (N4) possible for each method. (An interval greater than N4 will produce instability in the computed solution.)

In order to to update our dependent variables 10 times per time unit, The maximum integration interval which may be used is 0.10000. Call it Hmax. Distinguish it from N4 (below). Clearly, a workable integration interval (H) must be less than Hmax and less than or equal to N4.

TABLE 1

Method	N1 Maximum Stability Interval	N2 Coefficient of Truncation Error	N3 Degree Classical	N4 Hmax
SR50.	-50.00000	0.96160	2.00000	0.00694
Three Stp	-0.82364	0.03280	2.00000	0.25180
M.01-231	-231.48199	0.00602	2.00000	0.00996
M.01-15	-15.75253	0.00792	2.00000	0.04262
Shft Trap	-2.00000	1.00000	2.00000	0.07947
2nd Adams	-1.00000	0.41667	3.00000	0.25489

Our analysis of the candidate methods reveals that the following integration intervals H(ACC) are required to maintain a 98.000 percent error in the computed solutions (Table 2).

TABLE XV (CONT)

TABLE 2

Method	H (Accuracy)	User-s Hmax	Method-s Hmax	H (Optimum)	Percent Error by Using H
SR50.	0.0858309	0.1000000	0.0069411	0.0069411	0.464
Three Stp	0.3857530	0.1000000	0.2518028	0.1000000	1.834
M.01-231	0.3966176	0.1000000	0.0099560	0.0099560	0.010
M.01-15	0.4011928	0.1000000	0.0426160	0.0426160	0.248
Shft Trap	0.1209663	0.1000000	0.0794665	0.0794665	38.219
2nd Adams	0.2094535	0.1000000	0.2548914	0.1000000	8.992

In Table 2 (above), a difference between HACC and H indicates that HACC is too large an H-value to maintain either stability and/or the required update rate.

We conclude our analysis of the candidate methods by computing a figure of merit for each. A breakdown by selected categories as well as total figures is given in the figure of merit schemes (below).

FIGURE OF MERIT SCHEME (HOPT)

Method	Stability	Truncation Error	Round-off Error	Propagated Error	Required Computing Time	Total
SR50.	4.000E-02	7.497E-01	8.303E-02	9.976E-01	3.828E-01	2.253E-00
Three Stp	4.751E-01	6.223E-01	1.009E-01	9.960E-01	6.154E-01	2.810E-00
M.01-231	5.902E-03	1.000E 00	2.330E-02	5.258E-01	3.900E-01	1.945E-00
M.01-15	6.643E-02	8.258E-01	2.643E-02	7.599E-01	5.321E-01	2.211E-00
Shft Trap	4.380E-01	4.297E-01	2.949E-01	9.738E-01	6.492E-01	2.786E-00
2nd Adams	4.559E-01	6.067E-01	2.657E-01	9.939E-01	6.716E-01	2.994E-00

In our figure of merit evaluation of the candidate methods, stability, truncation error, round-off error, propagated error, and required computing time have been weighted 1.000, 1.000, 1.000, 1.000 and 1.000, respectively (FPER = 0.000E-01, FPEI = 0.000E-01)

Based on this figure of merit, we conclude that the best integration scheme is the 2nd Adams method.

All sixteen methods were used to solve the system Equation 13 at integration intervals of .05 and .1. At each step in the solution, the following were computed and printed:

a. True solution
b. Computed solution
c. Error
d. Percent error
e. Cumulative average error

This permitted a comparison to be made of the actual solution characteristics with those predicted by SAP.

Table XVI shows the estimated and actual percent errors at the (K + 16)th step in each integration for $h = .05$.[4] Similar data are presented in Table XVII for $h = .10$. Figures 4 and 5 show regression plots of these data.

Figure 6 shows a linearized plot of the actual percent-error functions for six of the integration methods tested. An intuitive selection of a "best" integration method based on this plot may be the following, in order of preference:

a. Parabolic
b. O_{33} Mod Gurk and Three-Step
c. Second-Order Adams
d. SR.40
e. Euler

It is also observed (see Figure 5) that errors very early in the solution are quite large for Parabolic and SR.40, although Parabolic accommodates by exhibiting excellent accuracy characteristics elsewhere in the solution.

Table XVIII summarizes the above observations and provides the overall figure of merit (exclusive of the merit-component representing required computing time) computed by SAP for each of these methods (re Table XV).

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[4] Those integration techniques which produce unstable solutions at the integration interval of interest are omitted from referenced tables and plots.

TABLE XVI

ESTIMATED AND ACTUAL PERCENT ERRORS AT K + 16 FOR A SECOND-ORDER PROBLEM (h = .05)

Integration Method	K	Actual % Error	Estimated Error	Difference*
Euler	1	8.570	6.758	-1.812
O_{33} Mod Gurk	3	1.236	.459	-.777
MTH.03-.8	3	1.402	.153	-1.249
2-Step	2	12.76	10.406	-2.354
5-Step	5	8.533	10.349	1.816
O_{14}	4	.1988	.02[illegible]	-.1758
Parabolic	3	.4429	.204	-.2389
SR.29	2	19.79	26.216	6.426
SR.40	2	13.95	[illegible].852	2.902
3-Step	3	1.226	.454	-.772
Shifted Trap.	2	16.57	14.249	-2.321
2nd Order Adams	2	1.689	1.110	-.579

*Estimated % error minus actual % error

TABLE XVII

ESTIMATED AND ACTUAL PERCENT ERRORS AT K + 8 FOR A SECOND-ORDER PROBLEM (h = .10)

Method	K	Actual % Error	Estimated % Error	Difference*
Euler	1	19.84	28.381	8.541
O_{33} Mod Gurk	3	1.25	1.86	.61
2-Step	2	31.53	44.821	13.291
5-Step	5	16.94	35.785	18.845
Parabolic	3	2.546	3.392	.846
SR.40	2	21.33	66.70	45.37
3-Step	3	1.266	1.834	.568
2nd Order Adams	2	1.056	8.992	7.936

*Estimated % error minus actual % error

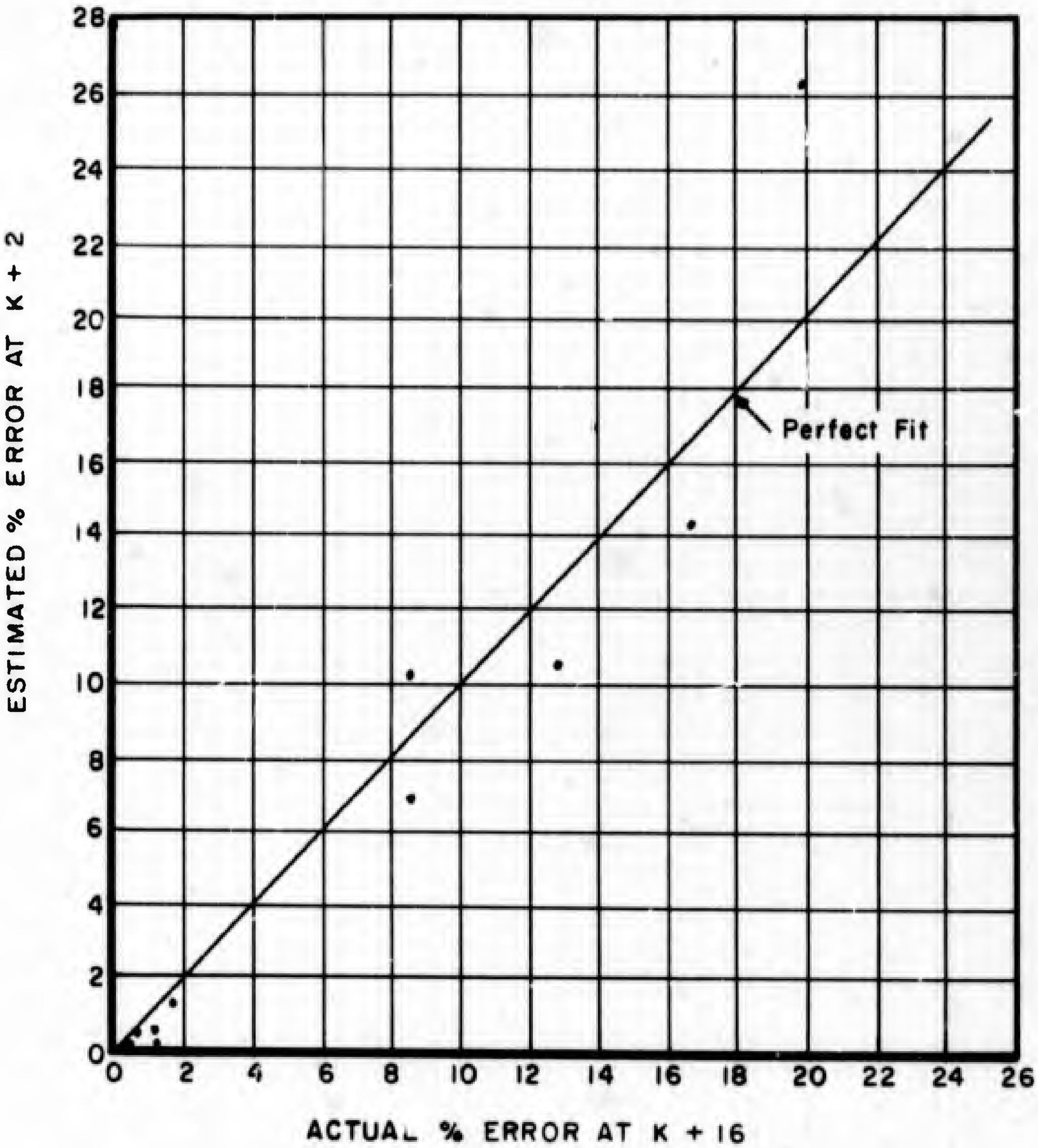

Figure 4. Plot of Estimated K + 2 Percent Errors Vs Actual K + 16 Percent Errors for a Second-Order Problem (h = .05)

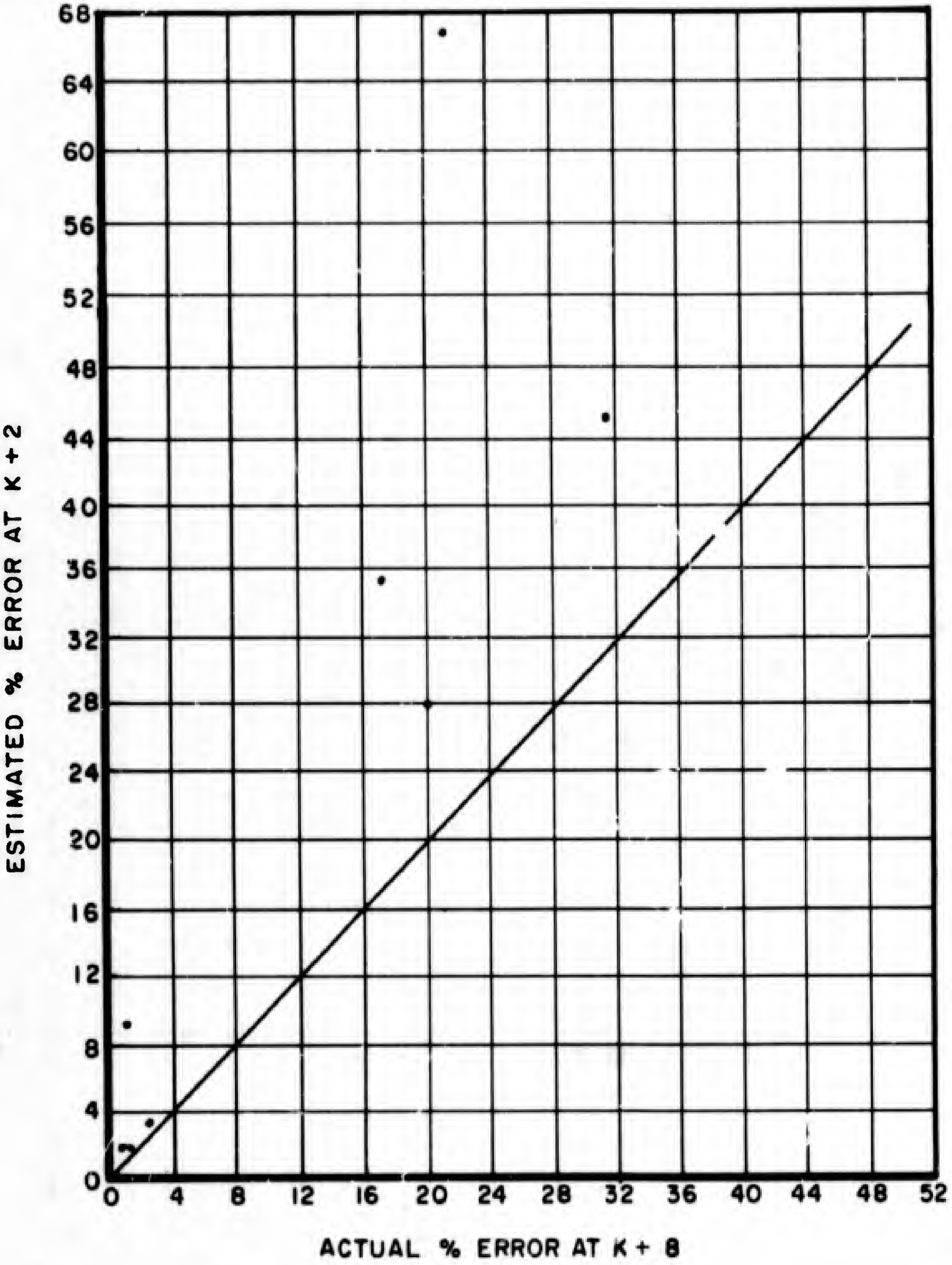

Figure 5. Plot of Estimated K + 2 Percent Errors Vs Actual K + 8 Percent Errors for a Second-Order Problem (h = .10)

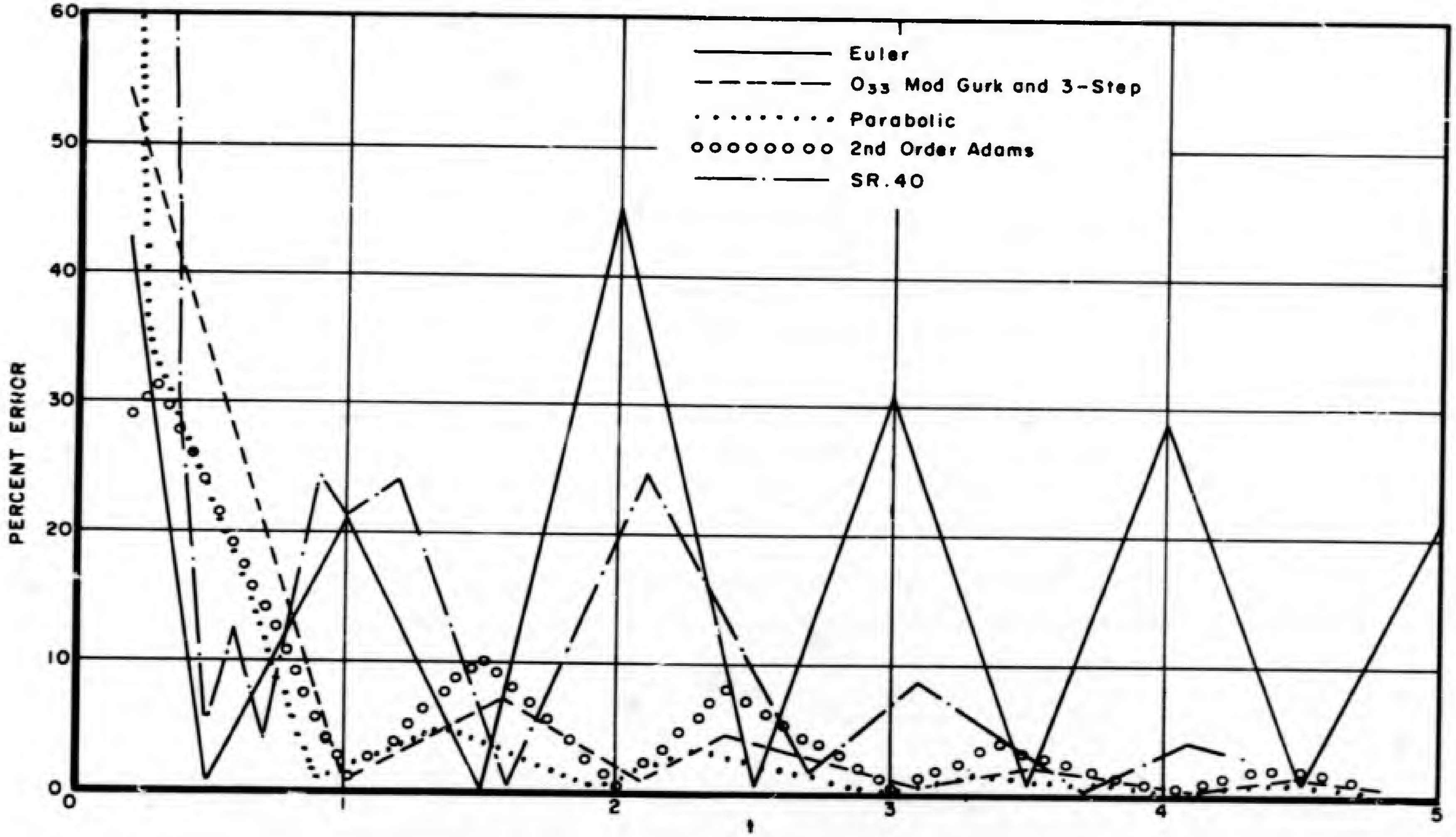

Figure 6. Linearized Plot of Percent-Error Functions for Several Integration Methods (h = .10)

The results show that the merit figures, taken as a straightforward sum, appear to weight heavily the accuracy characteristics of the methods which are reflected by percent errors early in the solution. In exception, however, is the Parabolic method which has a large percent error at t = .2 but which still receives the highest computed merit figure. Intuitively, at least, the computed figures of merit appear realistic and, in general, seem to provide reasonable indicants of the comparative merits of the methods.

TABLE XVIII

INTUITIVE OBSERVATIONS AND COMPUTED MERIT FIGURES FOR SIX INTEGRATION METHODS APPLIED TO A SECOND-ORDER PROBLEM

Intuitive Order of Preference*	% Error at T = .2	Computed Merit Figure**
1. Parabolic	110.10	2.338
2. 3-Step and O_{33} Mod Gurk	54.27 54.12	2.195 2.197
3. 2nd Order Adams	28.68	2.322
4. SR.40	414.70	1.760
5. Euler	42.81	2.244

*Based on linearized plot of percent error functions

**Merit figures do not include component representing required computing time

From the information generated by SAP (Table XV) we see that the following maximum integration intervals (for stability) are derived:

Euler	-	.158
O_{33} Mod Gurk	-	.251
Shifted Trap.	-	.079
Five-Step	-	.109

These are largest Δt's to be used with each respective method of integration if stability in the solution is to be maintained. Figures 7 through 10 show the computed solutions using each of these methods with various integration intervals and illustrate the meaning and accuracy of the computed maximum Δt's. The Euler method, for example, is plotted for $h = .05, .10$, and $.20$ (Figure 7). It is clear that the solution is stable for the first two Δt's but not for the last, which is larger than the maximum h for this problem using Euler's method. Also of interest are Figures 8 and 10 in which the solutions using O_{33} Mod Gurk and the Five-step method are shown. One can clearly see the effects of the extraneous roots of these higher-order methods using larger than optimal integration intervals.

3. FLIGHT SIMULATION PROBLEMS

Although no empirical results have yet been obtained by applying derived optimum integration methods to specific simulation problems, such problems have been submitted to SAP for analysis. At this time we are able to discuss flight simulation-integration techniques in general and present by example a practical approach to using SAP for deriving or selecting ideal integration methods for specific simulation problems.

a. Integration Methods Currently Used

There are five methods of integration known by the author to be popularly selected for use in flight simulation today. These are discussed below from both a general and a comparative point of view. Table XIX summarizes the key characteristics of these five methods.

(1) O_{33} Mod Gurk

The O_{33}[5] Mod Gurk method is so named because it is a nonclassical, open three-step method derived by H. M. Gurk specifically for the F-100A simulation problem. It is classical to the second degree, has a truncation error coefficient

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[5] The double subscript, eg. 0_{ij}, means that the method of integration uses i past values of the variable and j past values of its derivative. The "0" signifies an open integration method, i.e., one which does not use the present value of the derivative.

TABLE XIX

CHARACTERISTICS OF FIVE POPULAR SIMULATION-INTEGRATION TECHNIQUES

Integration Method	Stability Range	Truncation Error Coefficient	Degree Classical	Rank-Order of Computing Speed
O_{33} Mod Gurk	-.821	.033	2	5
2nd Order Adams	-1.00	.417	3	3
Shifted Trapezoidal	-2.00	1.000	2	2
Euler	-2.00	.500	2	1
Parabolic	-.545	.375	4	4

of .03297, and has a stability range of $(-.82134, 0]$. Its stability and propagated-error damping rate characteristics make it excellently suited for problems of the form

$$\dot{y} = -\lambda y \ (\lambda > 0)$$

Whereas many three-step methods tend to oscillate about solutions and solution components of a form which goes to zero monotonically decreasing, the O_{33} Mod Gurk method produces a smooth, nonoscillatory solution.

The O_{33} Mod Gurk method has relatively good truncation error characteristics but poor round-off error characteristics. Its stability interval is somewhat small, permitting, for example, an integration interval no larger than approximately .07 for the F-100A problem. As a three-step method with mixed coefficients, it is time-consuming to compute.

The O_{33} Mod Gurk method is possibly the most often used and probably the most over-rated method in flight simulation work. Its single exceptional characteristic is its smoothness in approximating solutions of the form

$$y = e^{-\lambda t} \ (\lambda > 0)$$

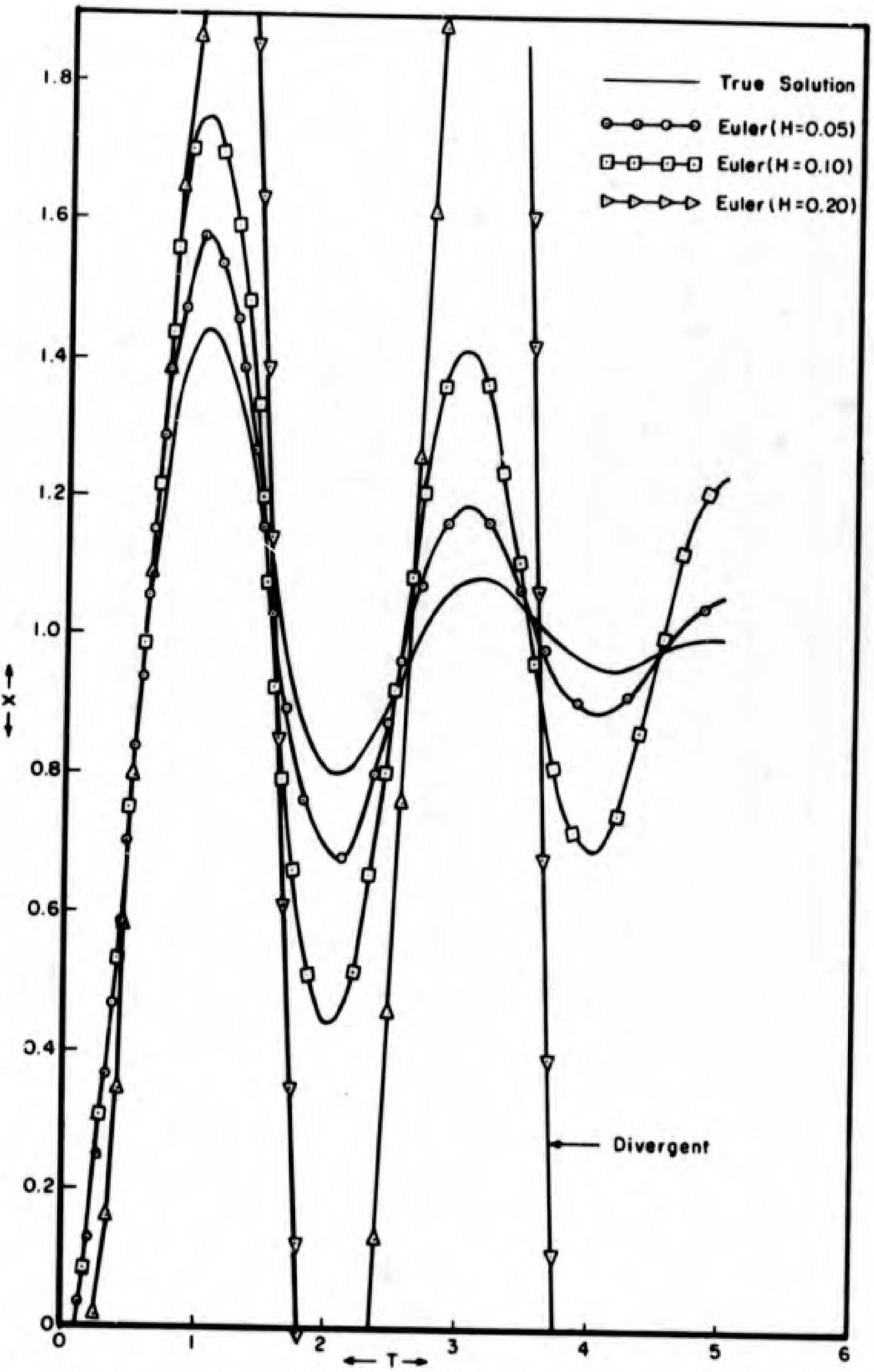

Figure 7. Plot of Euler Solution to Second-Order Problem Using Three Different Δt's

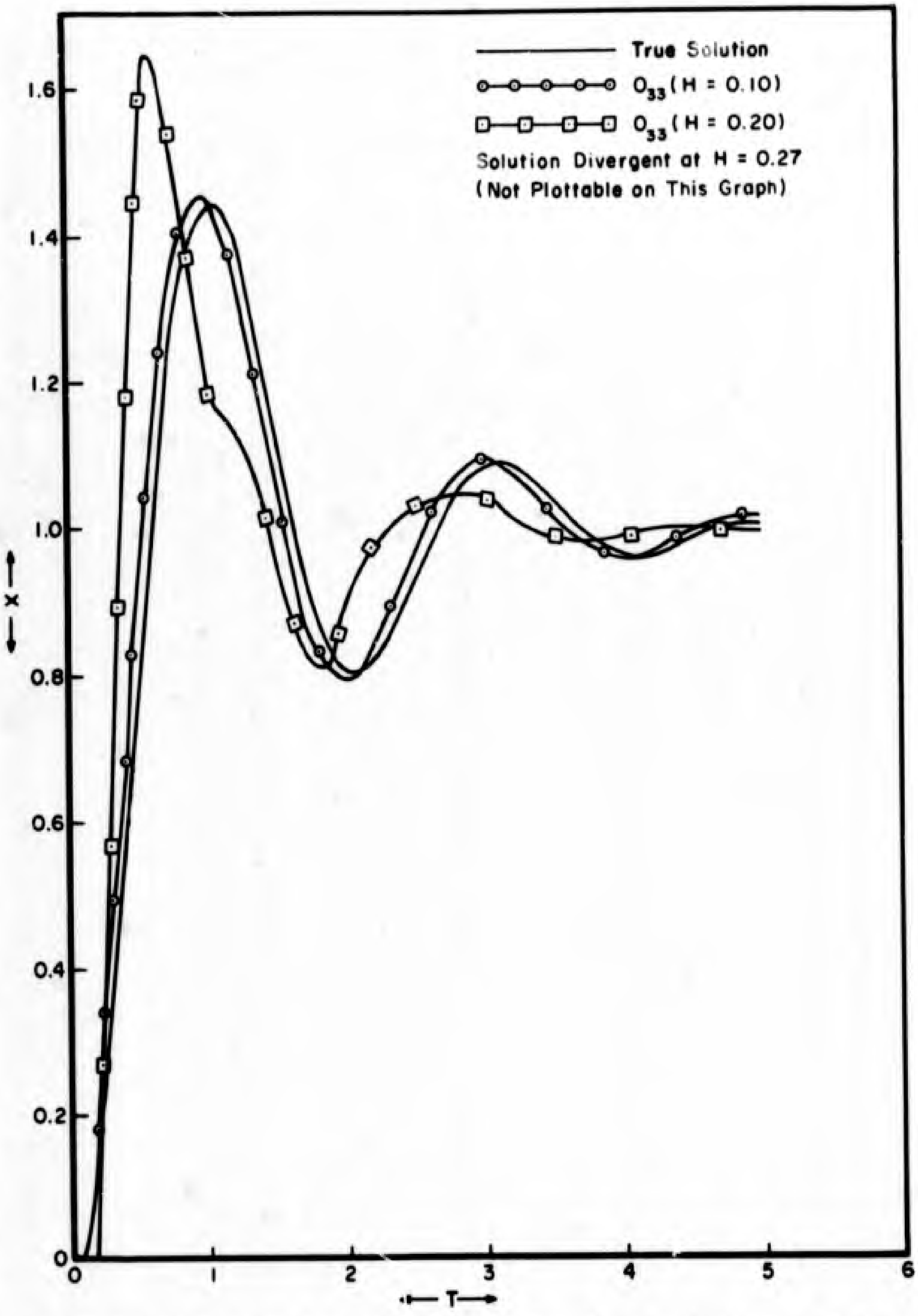

Figure 8. Plot of O_{33} Mod Gurk Solution to Second-Order Problem Using Two Different Δt's

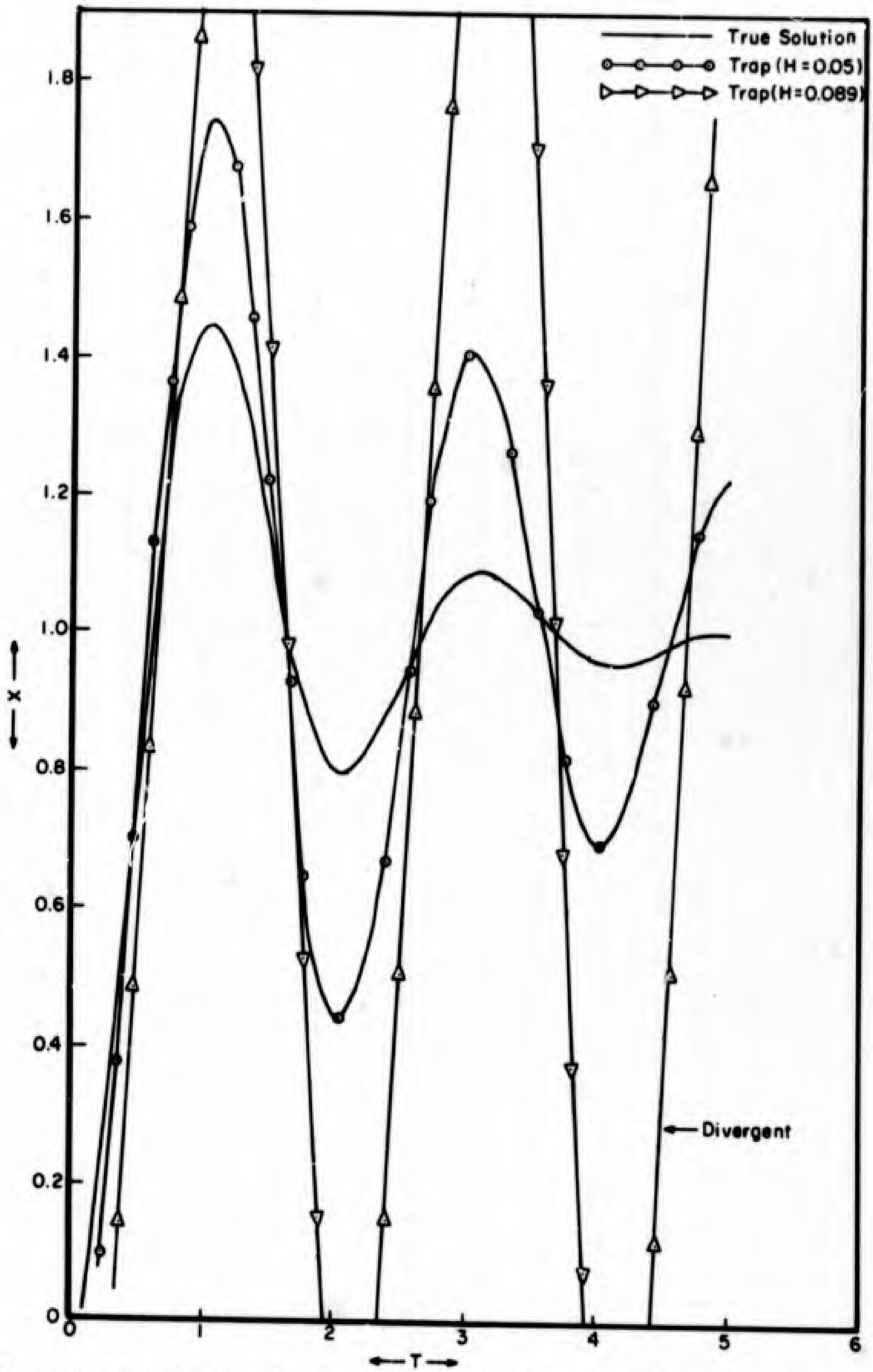

Figure 9. Plot of Shifted Trapezoidal Solution to Second-Order Problem Using Two Different Δt's

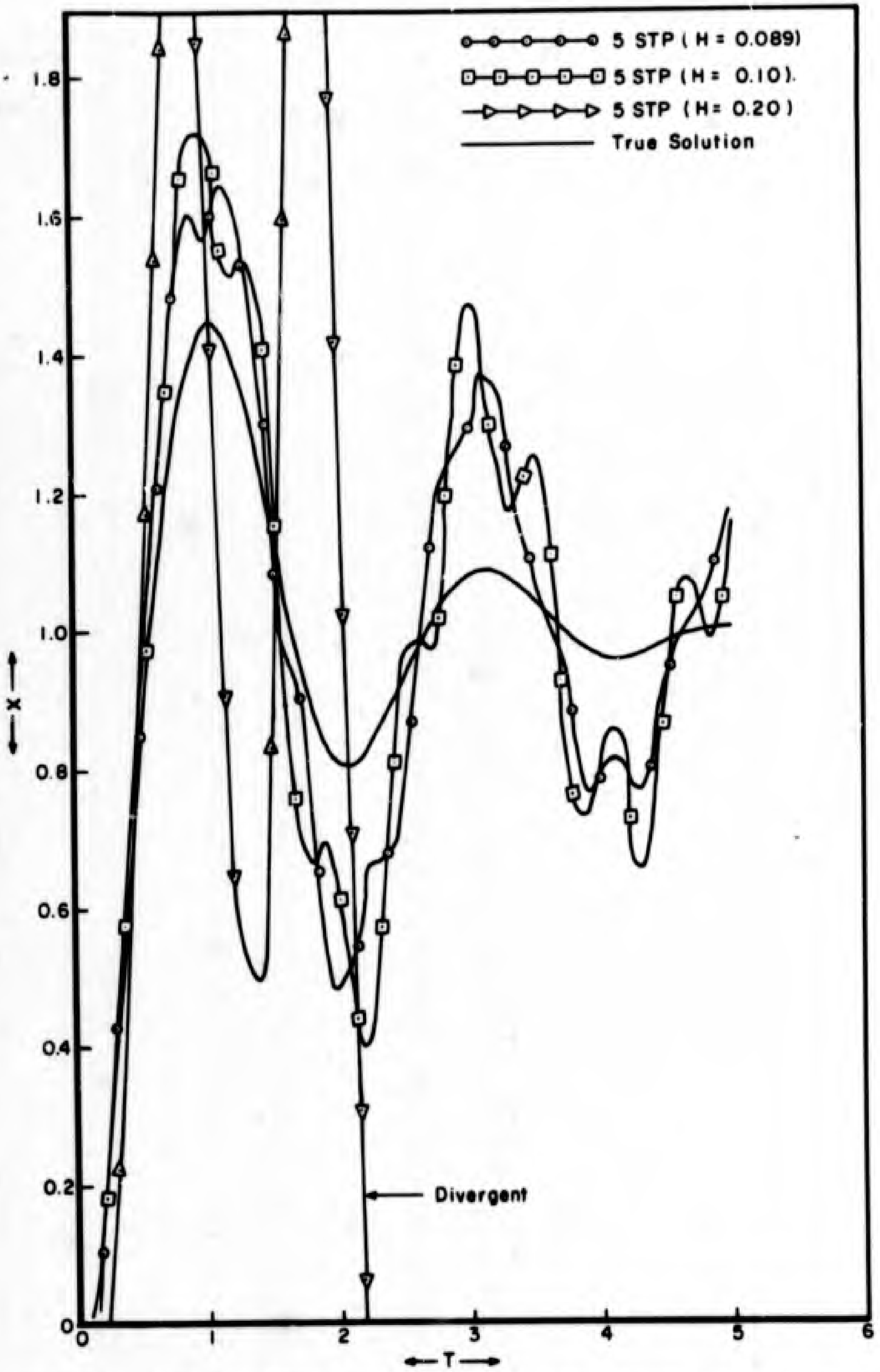

Figure 10. Plot of 5-Step Solution to Second-Order Problem Using Three Different Δt's

but other of its poorer characteristics often make its use unjustifiable theoretically. It has enjoyed great popularity primarily because (1) it is one of the only three-step methods ever suggested or recommended for use in flight simulation and (2) it appeared to produce satisfactory results for the F-100A simulation problem.

(2) Second-Order Adams

Possibly second in popularity in simulation work is the Second-Order Adams method. This is a two-step O_{12} method which, because of its simple coefficients, is extremely rapid to compute. While having a larger coefficient of truncation error (.417) than the O_{33} Mod Gurk method, it is classical to the third degree. Thus, depending on the integration interval, it could produce a smaller truncation error term and usually does with the Δ t's popularly used in simulation.

The Second-Order Adams method has a stability interval of $(-1, 0]$, slightly larger than that for Mod Gurk. The Adams method, therefore, could use an integration interval up to .085 for the F-100A problem, considering stability alone. Its round-off error characteristics are better than those for Mod Gurk. Like the O_{33} Mod Gurk method, the Second-Order Adams method has excellent propagated-error damping characteristics.

(3) Shifted Trapezoidal

This method is so named (by the author) because it is the familiar Trapezoidal method but, because of how it must be used in flight simulation work, it is effectively shifted from a closed to an open method. It is impossible (assuming theoretical accuracy) to use a closed method of integration exclusively in flight simulation. Because this issue has been the subject of some discussion, it deserves attention here.

In flight simulation, we are normally faced with problems of the form

$$\ddot{y} = f(\dot{y}, y, t)$$

Upon beginning any given cycle of the simulation program, say the nth cycle, we have available the (n-1)th (and preceding) values of $\ddot{y}$, $\dot{y}$, and y. At this time, then, it is impossible to employ a closed integration method which must use the nth value of $\dot{y}$ or $\ddot{y}$. Further, we cannot compute $\ddot{y}_n$ at this point because it is a function of $\dot{y}_n$ and y_n, which we have yet to derive via integration. The only correct approach, therefore, is:

1. Use an open method of integration to compute $\dot{y}_n$
2. Use an open or closed method of integration to compute y_n
3. Use y_n and $\dot{y}_n$ to compute $\ddot{y}_n$ by its formula

The Shifted Trapezoidal method has a truncation error coefficient of 1.0 and is classical to the second degree, resulting in very poor accuracy characteristics, comparatively speaking. It has a stability range of $(-2, 0]$, larger than that of most other popular simulation-integration methods. With it, a Δt up to .17, for example, could be used for the F-100A simulation problem and the solution (though perhaps lacking in accuracy) would remain stable. It has good stability, round-off error, and propagated error characteristics and, of course, is very rapid to compute.

(4) Euler

Euler's method is commonly used for portions of the simulation-integration task involving lower eigenfrequencies and/or less strict accuracy requirements. It has a truncation error coefficient of .5 and is classical to the second degree. Its stability range of $(-2, 0]$ is the same as that of the Shifted Trapezoidal. In most respects, in fact, it is very comparable to Shifted Trapezoidal, but it has much better round-off error properties and is slightly faster to compute.

(5) Parabolic

The Parabolic method is a three-step O_{13} method with a stability range of $(-.545, 0]$ and a truncation error coefficient of .375. It is classical to the fourth degree, resulting in exceptionally good truncation error characteristics. Its relatively small stability range limits its applicability; for example, a Δt no larger than .046 would be possible for the F-100A simulation. The Parabolic method has round-off error characteristics slightly better than those of O_{33} Mod Gurk but slightly poorer stability and propagated error damping rate characteristics.

b. Application of the Simulation Analyzer Program to a Well-Known Simulation Problem

Of all the digital flight simulation problems ever implemented, the one most often used as a research tool, most often implemented in part or whole for simulation studies, and most often alluded to in discussions about flight simulation is the F-100A supersonic fighter. As a result, an abundance of comparative data and detailed knowledge about this simulation has accumulated over the years, making it ever more apropos for isolated studies of simulation techniques in general. For example, the F-100A is the only simulated aircraft for which the author was able to find derived critical eigenvalues (University of Pennsylvania, 1955; and Morrison and Paler, 1962).

(1) Typical Procedure for Selecting an Integration Method

Typically, when confronted with an aircraft simulation problem (such as the F-100A), the approach would be to select in an a priori manner one of the five popular integration methods mentioned earlier. For example, the O_{33} Mod Gurk method might be selected. When most of the simulation programming has been completed, several empirical tests are run using the selected integration method. If tolerances are not met, possible recourses include shrinking the integration interval and/or trying a different integration method.

Several rather obvious perils exist with an approach such as this. For one, isolated empirical tests seldom, if ever, succeed in testing all possible contingencies. If the test-flight situation wherein critical eigenfrequencies are revealed is not included, chances are great that an inadequate integration method or interval will be erroneously accepted and used. Secondly, this approach does not allow one to use the largest possible Δt, which is always desirable because of resulting savings in computer time, speed, cost, etc. Third, the approach does not attempt to match the integration method to the simulation problem for optimal results. Finally, there is never any theoretical justification for the selection of one integration method or interval over another; and when conclusive empirical tests are difficult, as in flight simulation, theoretical certainty, at least, should be assured.

(2) Recommended Procedure Using the Simulation Analyzer Program

A description will now be given of how the Simulation Analyzer Program may be used to derive an optimal integration method for a specific simulation problem. This procedure will assure (theoretically) that:

a. The method of integration will produce a stable solution using the recommended Δt

b. The Δt to be used will be the largest possible for the job

c. Accuracy tolerances will be met

The procedure will be described using the F-100A problem as an example.

(a) Eigenvalue Analysis. Unfortunately, the first step in using the Simulation Analyzer Program can be the most difficult, for the most critical eigenvalue of the differential system must be derived or closely "guestimated." As formally defined, eigenvalues for nonlinear systems do not exist as discrete quantities but, instead, are functions which vary with time. Further, no one has yet determined a method for finding eigenvalues of nonlinear systems. Therefore, the only known recourse is to linearize the system of differential equations at the point where critical frequencies are expected. Then finding the critical eigenvalue of the linearized system is straightforward. Assumedly, this affords a good approximation of the true critical eigenvalue of the associated nonlinear system (see Morrison and Paler, 1962).

The general nonlinear system for which the critical eigenvalue must be found may be written

$$\begin{aligned} \dot{y}_1 &= f_1(y_1, y_2, \cdots y_m, t) \\ \dot{y}_2 &= f_2(y_1, y_2, \cdots y_m, t) \\ &\vdots \\ \dot{y}_m &= f_m(y_1, y_2, \cdots y_m, t) \end{aligned} \tag{17}$$

where $y_1(t_0)$, $y_2(t_0)$, . . . $y_m(t_0)$ are known initial conditions. If we assume that the f_i are all real functions of the real variable t, and if all partial derivatives of f_i of all orders exist for

$$y_i(t_q) \leq y_i(t) \leq y_i(t_r)$$

where $\left[t_q, t_r \right]$ is the interval of interest, then we can expand each f_i in a Taylor Series as follows:

$$f_K(y_1, y_2, \cdots y_m, t) = \sum_{J=0}^{\infty} \frac{1}{J!} \left[\Delta y_1 \frac{\partial}{\partial y_1} + \Delta y_2 \frac{\partial}{\partial y_2} + \cdots + \Delta y_m \frac{\partial}{\partial y_m} + \Delta t \frac{\partial}{\partial t} \right]^J \Bigg|_P \tag{18}$$

where

$$\begin{aligned} \Delta y_i &= y_i(t) - y_i(t_q) \\ \Delta t &= t - t_q \end{aligned}$$

and

$$P = \left[y_1(t_q), y_2(t_q), \cdots y_m(t_q), t_q \right]$$

Therefore, from Equation 18 we have

$$\begin{aligned} f_K &= f_K(y_1, y_2, \cdots y_m, t) \\ &= \left[f_K \right]_P + \Delta y_1 \left[\frac{\partial f_K}{\partial y_1} \right]_P + \Delta y_2 \left[\frac{\partial f_K}{\partial y_2} \right]_P + \cdots \\ &\quad + \Delta y_m \left[\frac{\partial f_K}{\partial y_m} \right]_P + \Delta t \left[\frac{\partial f_K}{\partial t} \right]_P + R_K \end{aligned}$$

In local linearization, the expansion (Equation 18) is truncated after the linear Δy or Δt terms. This truncated series is identical to the above equation with the R_k term neglected. If we approximate f_k by this truncated series and substitute into Equation 17, we obtain the following system:

$$\dot{y}_1 = \left[f_1\right]_p + \Delta y_1\left[\frac{\partial f_1}{\partial y_1}\right]_p + \Delta y_2\left[\frac{\partial f_2}{\partial y_2}\right]_p + \cdots$$

$$+ \Delta y_m\left[\frac{\partial f_1}{\partial y_m}\right]_p + \Delta t\left[\frac{\partial f_1}{\partial t}\right]_p$$

$$\vdots$$

$$\dot{y}_m = \left[f_m\right]_p + \Delta y_1\left[\frac{\partial f_m}{\partial y_1}\right]_p + \Delta y_2\left[\frac{\partial f_m}{\partial y_2}\right]_p + \cdots$$

$$+ \Delta y_m\left[\frac{\partial f_m}{\partial y_m}\right]_p + \Delta t\left[\frac{\partial f_m}{\partial t}\right]_p$$

Thus,

$$\dot{y}_i = \left[f_i\right]_p + \sum_{j=1}^{m}\left[\frac{\partial f_i}{\partial y_j}\right]_p \left(y_j(t) - y_j(t_Q)\right) + (t - t_Q)\left[\frac{\partial f_i}{\partial t}\right]_p$$

$$\dot{y}_i = \sum_{j=1}^{m} y_j\left[\frac{\partial f_i}{\partial y_j}\right]_p + g_i(t)$$

where

$$g_i(t) = \left[f_i\right]_p + (t - t_Q)\left[\frac{\partial f_i}{\partial t}\right]_p - \sum_{j=1}^{m} y_j(t_Q)\left[\frac{\partial f_i}{\partial y_j}\right]_p$$

Therefore we have the linearized system

$$\dot{y}_1 = \sum_{i=1}^{m} a_{1i}\, y_i + g_1(t)$$

$$\dot{y}_2 = \sum_{i=1}^{m} a_{2i}\, y_i + g_2(t)$$

$$\vdots$$

$$\dot{y}_m = \sum_{i=1}^{m} a_{mi}\, y_i + g_m(t)$$

where

$$a_{ki} = \left[\frac{\partial f_k}{\partial y_i}\right]_p$$

In matrix notation this becomes

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \vdots \\ \dot{y}_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & \cdots & & a_{2m} \\ \vdots & & & \vdots \\ a_{m1} & & & a_{mm} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} + \begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_m(t) \end{bmatrix} \qquad (19)$$

It is the eigenvalue with the maximum modulus of the a_{ij} matrix in Equation 19 which we seek as the critical eigenvalue.

Summarizing, to derive an approximation of the most critical eigenvalue of an aircraft differential system, we must:

1. Write the system as a set of first-order, nonlinear differential equations.

2. Find the first partial derivatives of each derivative with respect to each dependent variable.

3. Evaluate the partial derivatives at the point at which we suspect the critical frequencies to occur.

4. Find the eigenvalues of the a_{ij} matrix in Equation 19.

So far as the author is aware, the above analysis has only been performed for one aircraft system — the F-100A. As it is somewhat laborious, it will not be repeated here. Table XX lists the extreme eigenvalues for the F-100A (Morrison and Paler, 1962), the most critical of which is that for the Immelman Turn.

(b) Evaluation of Candidate Methods and Derivation of Optimal Methods. In addition to being both interesting and informative, the Simulation Analyzer Program's evaluation of known candidate methods is of assistance in interpreting the results of its more powerful capability of deriving new optimal methods. Particularly because the SAP program is new and experience with it is limited, it is recommended that known methods of integration be submitted to it for evaluation prior to using it for the derivation of a "best" method. The reasons for this recommendation will be made obvious in the following discussion.

TABLE XX

F-100A AIRCRAFT EIGENVALUES*

Maneuver	Eigenvalue (λ)	$\lvert\lambda\rvert$
Turn, Bank, and Peel-Off	$.65 \pm 9.97i$	9.99
Dive and Pull-Out	$-.46 \pm 6.07i$	6.08
Immelman Turn	$-11.43 \pm 0i$	11.43
Spiral Dive	$-.24 \pm 7.50i$	7.51
Steady Flight	$-.30 \pm 5.59i$	5.60

*From (Morrison and Paler, 1962)

Ten methods of integration were submitted to SAP for evaluation for application to the F-100A simulation problem. The input data required for this is given in Table XXI. It will be noted that the Open Five-Step method was deliberately inserted in error to illustrate the program's capability for eliminating from consideration any methods which are inconsistent. The percent inaccuracy tolerated was set at 98% and the minimum update rate at 20/second to force SAP to evaluate the methods at $h = .05$. A subsequent run was made with 10/second specified as a minimum update rate. Tables XXII and XXIII show the output from the simulation Analyzer Program for these runs. (The figure of merit scheme for HMAX was not generated.)

In Table XXII (Table 2 therein), the reader will note that, for three of the integration methods submitted as candidates, the program did not recommend an optimum integration interval of .05. The h possible with the Parabolic Method is limited to .046 due to the method's relatively small stability interval. The h's possible with the Shifted Trapezoidal and Second-Step methods are limited to .0379 and .0446, respectively, because of these methods' limited accuracies. Looking at Table 1 within Table XXII, we see that the truncation error coefficients for the Shifted Trapezoidal and Second-Step methods are quite large, in part explaining the program's recommendation of smaller optimum Δt's for them.

TABLE XXI

F-100A INPUT DATA FOR CANDIDATE-METHODS OPTION OF THE SIMULATION ANALYZER PROGRAM

INPUT	MEANING	INPUT	MEANING
2	Selects candidate-methods option of SAP	SR.67	
		2	
10	Number of methods	1.0	
O33 Gurk	Name of first method	0.0	
3	Order of method	0.0	
1.146208	Coefficients of method	2.0	
-.201087		-1.0	
.054879		Open 5 Stp	
0.0		5	
1.641586		.5	
-1.008013		0.0	
.275097		0.0	
2nd Adam	Name of 2nd method etc.	0.0	
2		0.0	
1.0		0.0	
0.0		1.0	
0.0		1.0	
1.5		0.0	
-.5		0.0	
Euler		1.0	
1		M.03-.8	
1.0		3	
0.0		.070352	
1.0		.105529	
Parabolic		.824118	
3		0.0	
1.0		1.993749	
0.0		.286065	
0.0		.473952	
0.0		10	No. of diff. eqns. in F-100A system
1.91667		2.0	Greatest degree of dep. var. in each eqn.
-1.33333		2.0	
.41667		2.0	
Shif Trap		2.0	
2		2.0	
1.0		2.0	
0.0		2.0	
0.0		2.0	
.5		2.0	
.5		2.0	
M.03-.82		8	Avg. no. of arith. oprns. to compute a derivative
3			
1.139289		.98	Decimal equivalent of tolerated % inaccuracy
-.189154			
.049865		-11.5	Real part of critical eigenvalue
0.0		0.0	Imaginary part of critical eigenvalue
1.64241			
-1.00162		0.0	Zeros indicate the analysis is to be performed for the critical eigenvalue and not at any other particular eigenvalue
.26979		0.0	
2nd Step			
2		20	Minimum updates per time unit
1.0		Secs.	Name of time unit
0.0		0	Means all weights on merit figure should be 1.0
0.0			
.75		1	Long form of output is requested.
.25			

TABLE XXII

SAP OUTPUT FOR THE F-100A PROBLEM (I)

OUTPUT FROM THE SIMULATION ANALYZER PROGRAM

```
The user has selected the second option in the program which evaluates candidate methods
which have been supplied.

The candidate methods are

O33 Gurk
               K = 3          A(1) =  1.14620800E 00
                              A(2) = -2.01087000E-01
                              A(3) =  5.48790000E-02
                              B(0) =  0.00000000E-01
                              B(1) =  1.64158600E 00
                              B(2) = -1.00801300E 00
                              B(3) =  2.75097000E-01
2nd Adam
               K = 2          A(1) =  1.00000000E 00
                              A(2) =  0.00000000E-01
                              B(0) =  0.00000000E-01
                              B(1) =  1.50000000E 00
                              B(2) = -5.00000000E-01
Euler
               K = 1          A(1) =  1.00000000E 00
                              B(0) =  0.00000000E-01
                              B(1) =  1.00000000E 00
Parabolic
               K = 3          A(1) =  1.00000000E 00
                              A(2) =  0.00000000E-01
                              A(3) =  0.00000000E-01
                              B(0) =  0.00000000E-01
                              B(1) =  1.91667000E 00
                              B(2) = -1.33333000E 00
                              B(3) =  4.16670000E-01
Shif Trap
               K = 2          A(1) =  1.00000000E 00
                              A(2) =  0.00000000E-01
                              B(0) =  0.00000000E-01
                              B(1) =  5.00000000E-01
                              B(2) =  5.00000000E-01
M.03-.82
               K = 3          A(1) =  1.13928900E 00
                              A(2) = -1.89154000E-01
                              A(3) =  4.98650000E-02
                              B(0) =  0.00000000E-01
                              B(1) =  1.64241000E 00
                              B(2) = -1.00162000E 00
                              B(3) =  2.69790000E-01
```

TABLE XXII (CONT)

```
2nd Step
                     K = 2          A(1) =  1.00000000E 00
                                    A(2) =  0.00000000E-01
                                    B(0) =  0.00000000E-01
                                    B(1) =  7.50000000E-01
                                    B(2) =  2.50000000E-01
SR.67
                     K = 2          A(1) =  1.00000000E 00
                                    A(2) =  0.00000000E-01
                                    B(0) =  0.00000000E-01
                                    B(1) =  2.00000000E 00
                                    B(2) = -1.00000000E 00
Open 5 Stp
                     K = 5          A(1) =  5.00000000E-01
                                    A(2) =  0.00000000E-01
                                    A(3) =  0.00000000E-01
                                    A(4) =  0.00000000E-01
                                    A(5) =  0.00000000E-01
                                    B(0) =  0.00000000E-01
                                    B(1) =  1.00000000E 00
                                    B(2) =  1.00000000E 00
                                    B(3) =  0.00000000E-01
                                    B(4) =  0.00000000E-01
                                    B(5) =  1.00000000E 00
M.03-.8
                     K = 3          A(1) =  7.03520000E-02
                                    A(2) =  1.05529000E-01
                                    A(3) =  8.24118000E-01
                                    B(0) =  0.00000000E-01
                                    B(1) =  1.99374900E 00
                                    B(2) =  2.86065000E-01
                                    B(3) =  4.73952000E-01
```

The following methods have been rejected since they are inconsistent.

Open 5 Stp

The following methods have been rejected since they have stability intervals of zero length.

None

The candidate methods have been examined within the framework of the user-s specifications which are given below.

The differential system which defines our simulation has as the most critical eigenfrequency, the complex quantity -11.50000000 + I (0.000) which has modulus 11.50000000.

In order to adequately describe the physical system, the dependent variables must be updated 20 times per time unit, which is secs.

The percentage of error which will be tolerated in the integrated quantities is 98.00 percent.

TABLE XXII (CONT)

In order to aid in our examination of the candidate methods, the following parameters have been computed and are given in Table 1.

(A) Stability interval. The number N1 so that N1 .LT. HG .LE. 0.0

(B) Truncation error, T. (That is the numbers N2 and N3 such that T = (N2) (H superscript N3) (N3-th derivative of the dependent variable)

(C) The maximum integration interval (N4) possible for each method. (An interval greater than N4 will produce instability in the computed solution.)

In order to update our dependent variables 20 times per time unit, the maximum integration interval which may be used is 0.05000. Call it Hmax. Distinguish it from N4 (below). Clearly, a workable integration interval (H) must be less than Hmax and less than or equal to N4.

TABLE 1

Method	N1 Maximum Stability Interval	N2 Coefficient of Truncation Error	N3 Degree Classical	N4 Hmax
O33 Gurk	-0. 82134	0. 03297	2. 00000	0. 06999
2nd Adam	-1. 00000	0. 41667	3. 00000	0. 08522
Euler	-2. 00000	0. 50000	2. 00000	0. 17043
Parabolic	-0. 54545	0. 37500	4. 00000	0. 04648
Shif Trap	-2. 00000	1. 00000	2. 00000	0. 17043
M. 03-. 82	-0. 81622	0. 03280	2. 00000	0. 06956
2nd Step	-4. 00000	0. 75000	2. 00000	0. 34087
SR. 67	-0. 66667	-0. 50000	2. 00000	0. 05681
M. 03-. 8	0. 82000	0. 03297	2. 00000	0. 06988

Our analysis of the candidate methods reveals that the following integration intervals H(ACC) are required to maintain a 98.000 percent error in the computed solutions (Table 2).

TABLE 2

Method	H (Accuracy)	User-s Hmax	Method-s Hmax	H (Optimum)	Percent Error by Using H
O33 Gurk	0. 1143752	0. 0500000	0. 0699926	0. 0500000	4. 9689
2nd Adam	0. 0651155	0. 0500000	0. 0852174	0. 0500000	38. 4617
Euler	0. 0572470	0. 0500000	0. 1704348	0. 0500000	74. 6101
Parabolic	0. 0639691	0. 0500000	0. 0464822	0. 0464822	19. 6342
Shif Trap	0. 0379406	0. 0500000	0. 1704348	0. 0379406	97. 3300
M. 03-. 82	0. 1143692	0. 0500000	0. 0695558	0. 0500000	4. 9371
2nd Step	0. 0445827	0. 0500000	0. 3408696	0. 0445827	95. 3798
SR. 67	0. 0566279	0. 0500000	0. 0568116	0. 0500000	67. 2624
M. 03-. 8	0. 1003379	0. 0500000	0. 0698782	0. 0500000	3. 1341

In Table 2 (above), a difference between HACC and H indicates that HACC is too large an H-value to maintain either stability and/or the required update rate.

TABLE XXII (CONT)

We conclude our analysis of the candidate methods by computing a figure of merit for each. A breakdown by selected categories as well as total figures is given in the figure of merit schemes (below).

FIGURE OF MERIT SCHEME (HOPT)

Method	Stability	Truncation Error	Round-off Error	Propagated Error	Required Computing Time	Total
O33 Gurk	5.067E-01	7.122E-01	1.064E-01	8.308E-01	3.686E-01	2.525E 00
2nd Adam	4.594E-01	7.421E-01	2.881E-01	9.702E-01	4.107E-01	2.870E 00
Euler	5.750E-01	5.352E-01	8.576E-01	9.226E-01	4.363E-01	3.327E 00
Parabolic	3.537E-01	9.629E-01	1.690E-01	6.606E-01	3.896E-01	2.536E 00
Shif Trap	5.329E-01	5.260E-01	3.048E-01	8.782E-01	3.837E-01	2.626E 00
M.03-.82	5.033E-01	7.125E-01	1.081E-01	8.557E-01	3.686E-01	2.548E 00
2nd Step	6.420E-01	5.237E-01	3.021E-01	8.874E-01	3.995E-01	2.755E 00
SR.67	2.380E-01	5.352E-01	2.775E-01	7.616E-01	4.185E-01	2.231E 00
M.03-.8	1.523E-01	7.122E-01	1.665E-01	3.317E-02	3.686E-01	1.433E 00

In our figure of merit evaluation of the candidate methods, stability, truncation error, round-off error, propagated error, and required computing time have been weighted, 1.000, 1.000, 1.000, 1.000 and 1.000, respectively (FPER = 0.000E-01, FPEI = 0.000E-01)

Based on this figure of merit, we conclude that the best integration scheme is the Euler method.

TABLE XXII (CONT)

OUTPUT FROM THE SIMULATION ANALYZER PROGRAM

The user has selected the first option in the program which derives one, two, and three step methods of integration

The derived methods are

One Step

K = 1

A(1) = 1.00000000E 00
B(0) = 0.00000000E-01
B(1) = 1.00000000E 00

Two Step

K = 2

A(1) = 1.00666667E 00
A(2) = -6.66666725E-03
B(0) = 0.00000000E-01
B(1) = 1.48805556E 00
B(2) = -4.94722223E-01

Three Step

K = 3

A(1) = 1.13928114E 00
A(2) = -1.99676248E-01
A(3) = 6.03951088E-02
B(0) = 0.00000000E-01
B(1) = 1.64240983E 00
B(2) = -9.96345894E-01
B(3) = 2.75050034E-01

The candidate methods have been examined within the framework of the user-s specifications which are given below.

The differential system which defines our simulation has as the most critical eigenfrequency the complex quantity -11.50000000 + I (0.000) which has modulus 11.50000000.

In order to adequately describe the physical system, the dependent variables must be updated 20 times per time unit, which is secs.

The percentage of error which will be tolerated in the integrated quantities is 98.00 percent.

In order to aid in our examination of the candidate methods, the following parameters have been computed and are given in Table 1.

(A) Stability interval. The number N1 so that N1 .LT. HG .LE. 0.0

(B) Truncation error, T. (That is the numbers N2 and N3 such that T = (N2) (H superscript N3) (N3-th derivative of the dependent variable)

(C) The maximum integration interval (N4) possible for each method. (An interval greater than N4 will produce instability in the computed solution.)

TABLE XXII (CONT)

In order to update our dependent variables 20 times per time unit, the maximum integration interval which may be used is 0.05000. Call it Hmax. Distinguish it from N4 (below). Clearly, a workable integration interval (H) must be less than Hmax and less than or equal to N4.

TABLE 1

Method	N1 Maximum Stability Interval	N2 Coefficient of Truncation Error	N3 Degree Classical	N4 Hmax
One Step	-2.00000	0.50000	2.00000	0.17043
Two Step	-1.01541	0.00861	2.00000	0.08653
Three Step	-0.82344	0.03280	2.00000	0.07017

Our analysis of the candidate methods reveals that the following integration intervals H(ACC) are required to maintain a 98.000 percent error in the computed solutions (Table 2).

TABLE 2

Method	H (Accuracy)	User-s Hmax	Method-s Hmax	H (Optimum)	Percent Error by Using H
One Step	0.0572470	0.0500000	0.1704348	0.0500000	74.6101
Two Step	0.1498448	0.0500000	0.0865306	0.0500000	1.1617
Three Step	0.1144654	0.0500000	0.0701717	0.0500000	4.8962

In Table 2 (above), a difference between HACC and H indicates that HACC is too large an H-value to maintain either stability and/or the required update rate.

We conclude our analysis of the candidate methods by computing a figure of merit for each. A breakdown by selected categories as well as total figures is given in the figure of merit schemes (below).

FIGURE OF MERIT SCHEME (HOPT)

Method	Stability	Truncation Error	Round-off Error	Propagated Error	Required Computing Time	Total
One Step	5.750E-01	5.352E-01	8.576E-01	9.226E-01	4.363E-01	3.327E 00
Two Step	4.676E-01	7.996E-01	2.181E-01	9.708E-01	3.904E-01	2.846E 00
Three Step	5.126E-01	7.125E-01	1.067E-01	8.308E-01	3.686E-01	2.531E 00

In our figure of merit evaluation of the candidate methods, stability, truncation error, round-off error, propagated error, and required computing time have been weighted, 1.000, 1.000, 1.000, 1.000 and 1.000, respectively (FPER = 0.000E-01, FPEI = 0.000E-01)

Based on this figure of merit, we conclude that the best integration scheme is the One Step method.

TABLE XXIII

SAP OUTPUT FOR THE F-100A PROBLEM (II)

```
OUTPUT FROM THE SIMULATION ANALYZER PROGRAM

The user has selected the second option in the program which evaluates candidate methods
which have been supplied.

The candidate methods are

O33 Gurk
                      K = 3           A(1) =  1.14620800E 00
                                      A(2) = -2.01087000E-01
                                      A(3) =  5.48790000E-02
                                      B(0) =  0.00000000E-01
                                      B(1) =  1.64158600E 00
                                      B(2) = -1.00801300E 00
                                      B(3) =  2.75097000E-01
2nd Adam
                      K = 2           A(1) =  1.00000000E 00
                                      A(2) =  0.00000000E-01
                                      B(0) =  0.00000000E-01
                                      B(1) =  1.50000000E 00
                                      B(2) = -5.00000000E-01
Euler
                      K = 1           A(1) =  1.00000000E 00
                                      B(0) =  0.00000000E-01
                                      B(1) =  1.00000000E 00
Parabolic
                      K = 3           A(1) =  1.00000000E 00
                                      A(2) =  0.00000000E-01
                                      A(3) =  0.00000000E-01
                                      B(0) =  0.00000000E-01
                                      B(1) =  1.91667000E 00
                                      B(2) = -1.33333000E 00
                                      B(3) =  4.16670000E-01
Shif Trap
                      K = 2           A(1) =  1.00000000E 00
                                      A(2) =  0.00000000E-01
                                      B(0) =  0.00000000E-01
                                      B(1) =  5.00000000E-01
                                      B(2) =  5.00000000E-01
M.03-.82
                      K = 3           A(1) =  1.13928900E 00
                                      A(2) = -1.89154000E-01
                                      A(3) =  4.98650000E-02
                                      B(0) =  0.00000000E-01
                                      B(1) =  1.64241000E 00
                                      B(2) = -1.00162000E 00
                                      B(3) =  2.69790000E-01
```

TABLE XXIII (CONT)

```
2nd Step
                    K = 2          A(1) =  1.00000000E 00
                                   A(2) =  0.00000000E-01
                                   B(0) =  0.00000000E-01
                                   B(1) =  7.50000000E-01
                                   B(2) =  2.50000000E-01
SR.67
                    K = 2          A(1) =  1.00000000E 00
                                   A(2) =  0.00000000E-01
                                   B(0) =  0.00000000E-01
                                   B(1) =  2.00000000E 00
                                   B(2) = -1.00000000E 00
Open 5 Stp
                    K = 5          A(1) =  5.00000000E-01
                                   A(2) =  0.00000000E-01
                                   A(3) =  0.00000000E-01
                                   A(4) =  0.00000000E-01
                                   A(5) =  0.00000000E-01
                                   B(0) =  0.00000000E-01
                                   B(1) =  1.00000000E 00
                                   B(2) =  1.00000000E 00
                                   B(3) =  0.00000000E-01
                                   B(4) =  0.00000000E-01
                                   B(5) =  1.00000000E 00
M.03-.8
                    K = 3          A(1) =  7.03520000E-02
                                   A(2) =  1.05529000E-01
                                   A(3) =  8.24118000E-01
                                   B(0) =  0.00000000E-01
                                   B(1) =  1.99374900E 00
                                   B(2) =  2.86065000E-01
                                   B(3) =  4.73952000E-01
```

The following methods have been rejected since they are inconsistent.

Open 5 Stp

The following methods have been rejected since they have stability intervals of zero length.

None

The candidate methods have been examined within the framework of the user-s specifications which are given below.

The differential system which defines our simulation has as the most critical eigenfrequency, the complex quantity -11.50000000 + I (0.000) which has modulus 11.50000000.

In order to adequately describe the physical system, the dependent variables must be updated 10 times per time unit, which is secs.

The percentage of error which will be tolerated in the integrated quantities is 98.00 percent.

TABLE XXIII (CONT)

In order to aid in our examination of the candidate methods, the following parameters have been computed and are given in Table 1.

(A) Stability interval. The number N1 so that N1 .LT. HG .LE. 0.0

(B) Truncation error, T. (That is the numbers N2 and N3 such that T = (N2) (H superscript N3) (N3-th derivative of the dependent variable)

(C) The maximum integration interval (N4) possible for each method. (An interval greater than N4 will produce instability in the computed solution.)

In order to update our dependent variables 10 times per time unit, the maximum integration interval which may be used is 0.10000. Call it Hmax. Distinguish it from N4 (below). Clearly, a workable integration interval (H) must be less than Hmax and less than or equal to N4.

TABLE 1

Method	N1 Maximum Stability Interval	N2 Coefficient of Truncation Error	N3 Degree Classical	N4 Hmax
O33 Gurk	-0.82134	0.03297	2.00000	0.06999
2nd Adam	-1.00000	0.41667	3.00000	0.08522
Euler	-2.00000	0.50000	2.00000	0.17043
Parabolic	-0.54545	0.37500	4.00000	0.04648
Shif Trap	-2.00000	1.00000	2.00000	0.17043
M.03-.82	-0.81622	0.03280	2.00000	0.06956
2nd Step	-4.00000	0.75000	2.00000	0.34087
SR.67	-0.66667	-0.50000	2.00000	0.05681
M.03-.8	-0.82000	0.03297	2.00000	0.06988

Our analysis of the candidate methods reveals that the following integration intervals H(ACC) are required to maintain a 98.000 percent error in the computed solutions (Table 2).

TABLE 2

Method	H (Accuracy)	User-s Hmax	Method-s Hmax	H (Optimum)	Percent Error by Using H
O33Gurk	0.1143752	0.1000000	0.0699926	0.0699926	11.9506
2nd Adam	0.0651155	0.1000000	0.0852174	0.0651155	95.0287
Euler	0.0572470	0.1000000	0.1704348	0.0572470	99.1207
Parabolic	0.0639691	0.1000000	0.0464822	0.0464822	19.6342
Shif Trap	0.0379406	0.1000000	0.1704348	0.0379406	97.3300
M.03-.82	0.1143692	0.1000000	0.0695558	0.0695558	11.6791
2nd Step	0.0445827	0.1000000	0.3408696	0.0445827	95.3798
SR.67	0.0566279	0.1000000	0.0568116	0.0566279	94.8840
M.03-.8	0.1003379	0.1000000	0.0698782	0.0698782	13.7114

TABLE XXIII (CONT)

In Table 2 (above), a difference between HACC and H indicates that HACC is too large an H-value to maintain either stability and/or the required update rate.

We conclude our analysis of the candidate methods by computing a figure of merit for each. A breakdown by selected categories as well as total figures is given in the figure of merit schemes (below).

FIGURE OF MERIT SCHEME (HOPT)

Method	Stability	Truncation Error	Round-off Error	Propagated Error	Required Computing Time	Total
O33 Gurk	3.993E-01	6.684E-01	1.040E-01	6.276E-01	4.015E-01	2.201E 00
2nd Adam	3.850E-01	6.905E-01	2.809E-01	7.990E-01	4.365E-01	2.592E 00
Euler	6.583E-01	5.175E-01	8.465E-01	8.896E-01	4.496E-01	3.362E 00
Parabolic	3.537E-01	9.629E-01	1.690E-01	6.606E-01	3.896E-01	2.536E 00
Shif Trap	5.329E-01	5.260E-01	3.048E-01	8.782E-01	3.837E-01	2.626E 00
M.03-.82	3.998E-01	6.695E-01	1.057E-01	5.924E-01	4.009E-01	2.168E 00
2nd Step	6.420E-01	5.237E-01	3.021E-01	8.874E-01	3.995E-01	2.755E 00
SR.67	1.790E-01	5.190E-01	2.732E-01	5.812E-01	4.307E-01	1.983E 00
M.03-.8	1.856E-01	6.686E-01	1.610E-01	0.000E-01	4.013E-01	1.417E 00

In our figure of merit evaluation of the candidate methods, stability, truncation error, round-off error, propagated error, and required computing time have been weighted, 1.000, 1.000, 1.000, 1.000 and 1.000, respectively (FPER = 0.000E-01, FPEI = 0.000E-01)

Based on this figure of merit, we conclude that the best integration scheme is the Euler method.

TABLE XXIII (CONT)

OUTPUT FROM THE SIMULATION ANALYZER PROGRAM

The user has selected the first option in the program which derives one, two, and three step methods of integration

The derived methods are

```
One Step
                  K = 1          A(1) =  1.00000000E 00
                                 B(0) =  0.00000000E-01
                                 B(1) =  1.00000000E 00

Two Step
                  K = 2          A(1) =  1.06000000E 00
                                 A(2) = -6.00000005E-02
                                 B(0) =  0.00000000E-01
                                 B(1) =  1.39250000E 00
                                 B(2) = -4.52500000E-01

Three Step
                  K = 3          A(1) =  1.15028654E 00
                                 A(2) = -3.61523642E-01
                                 A(3) =  2.11237107E-01
                                 B(0) =  0.00000000E-01
                                 B(1) =  1.50816705E 00
                                 B(2) = -6.63178765E-01
                                 B(3) =  2.15962287E-01
```

The candidate methods have been examined within the framework of the user-s specifications which are given below.

> The differential system which defines our simulation has as the most critical eigenfrequency the complex quantity -11.50000000 + I (0.000) which has modulus 11.50000000.
>
> In order to adequately describe the physical system, the dependent variables must be updated 10 times per time unit, which is secs.
>
> The percentage of error which will be tolerated in the integrated quantities is 98.00 percent.

In order to aid in our examination of the candidate methods, the following parameters have been computed and are given in Table 1.

(A) Stability interval. The number N1 so that N1 .LT. HG .LE. 0.0

(B) Truncation error, T. (That is the numbers N2 and N3 such that T = (N2) (H superscript N3) (N3-th derivative of the dependent variable)

(C) The maximum integration interval (N4) possible for each method. (An interval greater than N4 will produce instability in the computed solution.)

TABLE XXIII (CONT)

In order to update our dependent variables 10 times per time unit, the maximum integration interval which may be used is 0.10000. Call it Hmax. Distinguish it from N4 (below). Clearly, a workable integration interval (H) must be less than Hmax and less than or equal to N4.

TABLE 1

Method	N1 Maximum Stability Interval	N2 Coefficient of Truncation Error	N3 Degree Classical	N4 Hmax
One Step	-2.00000	0.50000	2.00000	0.17043
Two Step	-1.14905	0.07750	2.00000	0.09792
Three Step	-1.14064	0.02703	2.00000	0.09720

Our analysis of the candidate methods reveals that the following integration intervals H(ACC) are required to maintain a 98.000 percent error in the computed solutions (Table 2).

TABLE 2

Method	H (Accuracy)	User-s Hmax	Method-s Hmax	H (Optimum)	Percent Error by Using H
One Step	0.0572470	0.1000000	0.1704348	0.0572470	99.1207
Two Step	0.1049618	0.1000000	0.0979192	0.0979192	70.3378
Three Step	0.1271480	0.1000000	0.0972019	0.0972019	22.5291

In Table 2 (above), a difference between HACC and H indicates tha HACC is too large an H-value to maintain either stability and/or the required update rate.

We conclude our analysis of the candidate methods by computing a figure of merit for each. A breakdown by selected categories as well as total figures is given in the figure of merit schemes (below).

FIGURE OF MERIT SCHEME (HOPT)

Method	Stability	Truncation Error	Round-off Error	Propagated Error	Required Computing Time	Total
One Step	6.583E-01	5.175E-01	8.465E-01	8.896E-01	4.496E-01	3.362E 00
Two Step	2.830E-01	5.690E-01	1.895E-01	5.828E-01	4.561E-01	2.080E 00
Three Step	5.368E-01	6.385E-01	8.616E-02	3.748E-01	4.336E-01	2.070E 00

In our figure of merit evaluation of the candidate methods, stability, truncation error, round-off error, propagated error, and required computing time have been weighted, 1.000, 1.000, 1.000, 1.000 and 1.000, respectively (FPER = 0.000E-01, FPEI = 0.000E-01)

Based on this figure of merit, we conclude that the best integration scheme is the One Step method.

For all of the other methods, an optimum Δt of .05 was recommended as we anticipated (indeed, as we planned). The program has computed the percent error to be expected at the start of the solution using the optimal h for each method. It is seen that this percent error is nearly the same for the O_{33} Mod Gurk, M.03-.82, and M.03-.8 methods. This was expected, since their truncation error terms are nearly equal and they are all three-step methods. The slightly lower percent error for the M.03-.8 method is attributed to superior round-off error characteristics, as observed in the Figure of Merit Scheme (Table XXII).

Weights reflecting each method's relative merits on five different bases are given in the Figure of Merit Scheme. These are based on the assumed use of the computed optimum integration interval. A total Figure of Merit is given as the direct sum of the individual merit figures. As seen, Euler's method received the highest total weight, and here is where some interpretation of the data provided is essential to proper utilization of the SAP program.

The reason for the high total weight computed for Euler's method is largely due to its excellent round-off error characteristics and corresponding high round-off error weight. However, it is doubtful that round-off error should have the same level of influence on the selection of a best method as does truncation error. Nevertheless, we have weighted them equal in this run of the SAP program. In addition, required computing time should probably not have as much influence as accuracy and stability considerations, so long as all methods are one-, two-, or three-step methods. Again, we have weighted the computing-time merit figure equal to the rest.

An example of how a different weighting scheme would affect the program's (and our) choice of a single "best" method is afforded in Table XXIV. In this table, the total figure of merit has been recomputed for those methods for which an optimum h of .05 is recommended. This has been done once by omitting the computing-time weight altogether, once by omitting the round-off error weight, and a third time by omitting both computing-time and round-off error weights. It can be seen from Table XXIV that the omission of the computing-time weight does not change the rank-ordering of the methods. When round-off error alone is omitted, the Second-Order Adams method and the Euler method trade ranks, and the former becomes the "best" choice. When both round-off error and required computing time are omitted, Second-Order Adams method is still first, followed by M.03-.82, O_{33} Mod Gurk, and then Euler.

The ideal weighting scheme, if indeed there is a unique, ideal scheme, is not known. It is felt that the best weighting scheme is largely a function of user preference, and this is why he is permitted to provide inputs to SAP to define his preferred weighting system. The Figure of Merit Scheme as provided in Table XXII allows one to compare all methods on the bases of five different considerations, but the ultimate selection of a single method must be left to the user, either through intelligent examination of the SAP data or through the selection and specification of a preferred weighting scheme. It should also be noted that if the recommended optimum h is used, any of the methods which succeed in not being rejected by the program will do the job. Thus, the user's responsibility in selecting one method after analyzing the data is not serious; and if he has submitted realistic accuracy tolerances and correct problem data, he may safely select the method using the largest optimum h and having the shortest computing time requirements.

In addition to providing a basis for relative comparison of different integration methods, the Figure of Merit Scheme also allows us to search for and recognize comparable characteristics of the candidate methods. For example, a logical question at this point in the analysis of the F-100A problem is the following: "Is there an integration method with characteristics comparable to or better than those of Second-Order Adams (or M.03-.82 or O_{33} Mod Gurk) which will allow me to use an integration interval of .1?" Alternately, we may ask "Is there another two- or three-step method with better characteristics than those candidate methods of which I am aware for an integration interval of .05?" Both of these questions may be answered using the SAP program.

In the second run of data in Table XXII, SAP was asked to derive new integration methods for the F-100A problem. As in the preceding run involving candidate methods, the data provided as inputs were chosen to force SAP to select h = .05 as an optimum integration interval.

TABLE XXIV

FIGURE OF MERIT WEIGHTS FOR SELECTED METHODS FOR APPLICATION TO THE F-100A SIMULATION PROBLEM

Integration Method	Total	Rank	Total - Computing Time	Rank	Total - Round-off Error	Rank	Total* - Both	Rank
O_{33} Gurk	2.525	4	2.156	4	2.419	4	2.050	3
2nd Adam	2.870	2	2.459	2	2.582	1	2.171	1
Euler	3.327	1	2.891	1	2.469	2	2.033	4
M.03-.82	2.548	3	2.179	3	2.440	3	2.071	2
SR.67	2.231	5	1.812	5	1.953	5	1.534	5
M.03-.8	1.433	6	1.064	6	1.266	6	.897	6

*Total - both computing-time weight and round-off error weight

The derived two-step method (see Table XXII) has a percent error (using h = .05) lower than any of the candidate methods previously submitted to SAP. It also has excellent truncation error characteristics. Disregarding computing time and round-off error weights, the computed merit figure for the derived two-step method is 2.238, which is higher than any of the original candidate methods. It appears as if SAP has derived a two-step method superior to any previously known for application to the F-100A problem at a Δt of .05.

The derived three-step method is quite comparable to the M.03-.82 method previously submitted to SAP for analysis. (The latter method, incidentally, was derived by SAP during one of many miscellaneous trial runs.) The derived method has slightly better stability characteristics than any of the other known three-step methods.

In Table XXIII, the data from two more runs of SAP is given. Here, we sought data on an optimum h of .10, first submitting the candidate methods and then asking for new, derived methods.

As can be seen in Table XXIII, only two of the candidate methods will produce stable solutions at h = .10, and none of the methods will satisfy all user requirements at h = .10. Unfortunately, the two known candidate methods with relatively large stability intervals (i.e., large enough to accommodate h = .10 for the F-100A) also have poor accuracy characteristics.

The second run in Table XXIII shows data for derived two- and three-step methods using a Δt of .10. In examining these newly derived methods, we first observe that both have small truncation error coefficients, and both satisfy all user requirements at h = .10. The derived two-step method has an approximate percent error at the start of the solution for a Δt of .10 smaller than that for the Euler method at a Δt of .05. Similarly, the derived three-step method, at Δt = .10, has a percent error smaller than that of Second-Order Adams at Δt = .05.

The derived two-step method (h = .10) has good truncation error and propagated error characteristics but is given a low stability weight (see Table XXIII, second run, Figure of Merit Scheme). At h = .10, it appears comparable to the M.03-.8 or SR.67 methods at h = .05. Thus, although it will do the job at h = .10, the resulting solution will not be as good as most of the h = .05 solutions using the best known candidate methods.

The derived three-step method (h = .10) has excellent characteristics of stability and truncation error but has a poor propagated error weight. Its solution at h = .10 will be comparable to that of O_{33} Mod Gurk at h = .05, for example, with the exception that it will tend to oscillate somewhat about the true solution. Accuracies of the two methods at the two respective Δt's, however, should be quite similar.

In summary, either the derived two-step or three-step method will perform satisfactorily at h = .10, but slightly more solution oscillation and/or less stability will be noted than with the best of other methods at h = .05. The SAP program has derived for us two new integration methods which will perform acceptably at h = .10 for the F-100A simulation problem. Only two other (well-known) methods, Euler and Shifted Trapezoidal, were even stable for this problem at h = .10, and they were rejected for this Δt because of very poor accuracy characteristics. The utility of the program is thus clearly seen for application to real-time flight simulation problems.

SECTION V

CONCLUSIONS

The Simulation Analyzer Program, built on well-founded theoretical bases, appears to be a powerful tool for simulation work. Empirical results thus far obtained have proved the program's capability for accurately deriving or comparing integration methods for application to a specific problem. The data generated by SAP for each integration method permits one to confidently predict the solution characteristics to be expected. Most important, the program allows us to seek and find a unique integration method satisfying our requirements for accuracy and stability at the largest possible Δt. In real-time training simulation, this capability is of utmost importance and value. For example, it now appears possible to use a Δt twice as large as that originally used with the F-100A simulation problem and still produce an acceptable solution. It is probable that an even larger Δt may be applicable, pending subsequent runs of the SAP program and the derivation of other new integration methods.

In any application of numerical integration, particularly in real-time training simulation, the question that ultimately arises is "Can't a larger integration interval be used?" Related questions are: "With a larger Δt, how much accuracy will be lost?" "Will the solution still be stable for all regions of interest?" The SAP program answers these questions with ease.

In flight simulation, once the problem has been defined, the amount of time required to compute one cycle of a worst-case condition becomes fixed for a given computer. The variables which may be manipulated include the integration interval, which dictates the number of times per second that the program must be executed. In turn, this tells us how fast the chosen computer must be. The ability to use larger Δt's enables us to (1) use slower, less expensive computers or (2) have more spare time when all simulation computations are completed. The latter consideration may be very important now that more emphasis is being placed on the use of automatic monitoring and scoring routines in training simulation, which must be computed along with or in the spare time of the flight simulation program.

In the past, numerical integration in flight simulation has been largely a matter of faith: Faith that the chosen method (one of five or so) will work satisfactorily for the new problem, since it appeared to work satisfactorily for a previous problem; faith that a Δt of, say, .05 is adequate, since it has been adequate for other simulations; and faith that no untested situations will arise in practice that cause the solution to "blow-up." This has been understandable, because it has never been truly worthwhile or possible to attempt to perform by hand for each new simulation problem the analysis that is performed by the SAP program. Rather than create additional problems by attempting to expand Δt (which, in fact, is not possible for many problems with any of the five popular integration methods), it has always been safer to draw on past experience and use the same methods of integration and the same integration intervals that have been used since digital flight simulation was first conceived nearly 20 years ago.

With the Simulation Analyzer Program, this is no longer true. Those characteristics of integration methods previously accepted on faith may now be easily proved or disproved in theory. The methods of integration and the Δt's suitable for past simulation problems are not necessarily optimal or even adequate for present simulations. Larger integration intervals are possible with newly derived integration methods. Of most importance, empirical tests of integration methods and intervals for flight simulation are only necessary as supporting proof, not sole proof, that an adequate solution will be generated for all realms of flight, and this should be the only purpose of empirical tests, after all. Finally, with a few runs of the SAP program for the problem of concern, we can answer nearly any conceivable question about which method is best, why it is better than others, how large a Δt we may safely use, and what characteristics to expect in the effected solution.

APPENDIX

TABLES OF NUMERICAL INTEGRATION TECHNIQUES

The following tables of integration methods were compiled using the Simulation Analyzer Program. They include 77 new two- and three-step open methods which were derived by the SAP program during miscellaneous test runs using a variety of representative problems and accuracy tolerances as inputs. Also included are some familiar open and closed methods of orders 1 through 5. With the exception of the last six methods in the tables, the methods are arranged in order of increasing stability range.

Method	Order	a_1 b_1	a_2 b_2	a_3 b_3	Stability Range	Truncation Error Coefficient	Degree Classical
SR. 23	2	1. 866667 -. 434503	-. 866667 . 567835		-. 235	1. 501	2
SR. 29	2	1. 0 4. 0	-3. 0		-. 286	-2. 500	2
SR. 33	2	1. 0 3. 5	-2. 5		-. 333	-2. 000	2
SR. 40	2	1. 0 3. 0	-2. 0		-. 400	-1. 500	2
SR. 50	2	1. 0 2. 5	-1. 5		-. 500	-1. 000	2
Parabolic	3	1. 0 1. 91667	-1. 33333	. 41667	-. 545	. 375	4
SR. 67	2	1. 0 2. 0	-1. 0		-. 667	-. 500	2
MTH. 03-. 82	3	1. 139289 1. 64241	-. 189154 -1. 00162	. 049865 . 26979	-. 816	. 033	2
MTH. 03-. 8	3	. 070352 1. 993749	. 105529 . 286065	. 824118 . 473952	-. 820	. 033	2
O_{33} Mod Gurk	3	1. 146208 1. 641586	-. 201087 -1. 008013	. 054379 . 275097	-. 821	. 033	2
Third Step	3	1. 139278 1. 642412	-. 1999716 -. 996196	. 0606935 . 2751997	-. 824	. 033	2
MTH. 01-. 9	3	. 614318 1. 707964	. 0097546 -. 161177	. 375927 . 214822	-. 950	. 012	2
2nd Order Adam	2	1. 0 1. 5	-. 5		-1. 000	. 417	3
MTH. 01-1	2	1. 006667 1. 488056	-. 006667 -. 494722		-1.020	. 009	2
MTH. 03-1. 14	3	1. 150287 1. 508167	-. 361524 -. 663179	. 211237 . 215962	-1. 140	. 027	2
SR1. 15	2	1. 06 1. 3925	-. 06 -. 4525		-1. 149	. 078	2
SR1. 73	2	1. 226667 1. 093889	-. 226667 -. 320556		-1. 730	. 293	2
MTH1. 75	3	1. 441766 1. 259026	-. 718905 -. 565869	. 277139 . 142216	-1. 750	. 024	2
Euler	1	1. 0 1. 0			-2. 000	. 500	2
Shifted Trapezoidal	2	1. 0 . 5	. 5		-2. 000	1. 000	2
MTH. 4-2. 3	2	1. 239565 . 92755	-. 239565 -. 16711		-2. 265	. 453	2
SR2. 5	2	. 4 . 96	. 6 . 64		-2. 500	. 840	2
SR3. 0	2	. 666667 . 888889	. 333333 . 444445		-3. 000	. 780	2

Method	Order	a_1 b_1	a_2 b_2	a_3 b_3	Stability Range	Truncation Error Coefficient	Degree Classical
SR3.89	2	1.48 .64	-.48 -.12		-3.890	.620	2
MTH.01-3.9	3	2.06370 .78729	-1.52954 -.44895	.46584 .063798	-3.891	.012	2
2nd Step	2	1.0 .75	.25		-4.000	.750	2
MTH.01-4	3	1.82545 .86591	-1.28909 -.25318	.463637 .025455	-4.000	.015	2
MTH.01-4.3	3	2.139095 .733573	-1.634334 -.439906	.495239 .062477	-4.260	.012	2
MTH.68-5	2	1.56467 .53664	-.56467 -.10131		-4.910	.681	2
MTH.01-5	3	2.24765 .650998	-1.79240 -.41256	.54475 .058664	-4.980	.011	2
SR5.0	2	1.2 .64	-.2 .16		-5.000	.760	2
MTH5.69	3	2.326458 .580839	-1.910123 -.382453	.583664 .052820	-5.690	.011	2
SR5.73	2	1.614493 .474753	-.614493 -.089246		-5.730	.718	2
SR7.50	2	1.695484 .37830	-.695484 -.073784		-7.500	.774	2
MTH7.65	3	2.475401 .462483	-2.138286 -.316457	.662885 .041457	-7.650	.010	2
MTH.01-7.8	3	2.483438 .455894	-2.151422 -.312782	.667985 .041436	-7.780	.010	2
SR7.95	2	1.708333 .360764	-.708333 -.069097		-7.948	.785	2
SR8.0	2	1.5 .4375	-.5 .0625		-8.000	.813	2
MTH.01-10	3	2.52992 .38742	-2.19754 -.25091	.66762 .001185	-10.000	.015	2
SR10.0	2	1.6 .36	-.6 .04		-10.000	.840	2
MTH12.42	3	2.658465 .304532	-2.429079 -.219924	.770614 .027541	-12.420	.009	2
SR12.49	2	1.805882 .241657	-.805882 -.047539		-12.490	.855	2
MTH.01-15.8	3	2.724776 .245898	-2.536395 -.181075	.81162 .022021	-15.750	.008	2
SR15.79	2	1.843333 .195069	-.843333 -.038403		-15.790	.883	2
SR16.34	2	1.850892 .187810	-.850892 -.038702		-16.340	.887	2
MTH16.59	3	2.737263 .234671	-2.558841 -.172373	.821578 .022017	-16.590	.008	2

Method	Order	a_1 b_1	a_2 b_2	a_3 b_3	Stability Range	Truncation Error Coefficient	Degree Classical
SR18.91	2	1.870241 .163799	-.870241 -.034040		-18.910	.901	2
MTH19.28	3	2.771094 .204344	-2.614529 -.151265	.843435 .019262	-19.280	.008	2
SR20.0	2	1.8 .19	-.8 .01		-20.000	.910	2
SR22.04	2	1.887228 .142031	-.887228 -.029259		-22.040	.914	2
MTH22.43	3	2.800939 .177512	-2.663231 -.132668	.862292 .016509	-22.430	.007	2
SR25.0	2	1.84 .1536	-.84 .0064		-25.000	.926	2
SR27.21	2	1.908808 .115746	-.908808 -.024553		-27.210	.930	2
SR27.54	2	1.910417 .114171	-.910417 -.024588		-27.540	.931	2
MTH28.07	3	2.838012 .14380	-2.725694 -.107886	.887682 .013756	-28.070	.007	2
MTH.01-28.6	3	2.840571 .141394	-2.730506 -.105787	.889935 .013756	-28.590	.007	2
SR33.12	2	1.923165 .096481	-.923165 -.019647		-33.120	.942	2
MTH33.63	3	2.863427 .120867	-2.766602 -.092125	.903174 .011005	-33.630	.007	2
SR34.53	2	1.927799 .091931	-.927799 -.019730		-34.530	.944	2
SR35.0	2	1.885714 .111027	-.885714 .003259		-35.000	.946	2
MTH35.85	3	2.870741 .113933	-2.780470 -.085948	.909729 .011004	-35.850	.007	2
SR40.0	2	1.9 .0975	-.9 .0025		-40.000	.953	2
SR43.88	2	1.941099 .073686	-.941099 -.014785		-43.880	.956	2
MTH44.33	3	2.894413 .092576	-2.818225 -.071430	.923812 .0082532	-44.330	.007	2
SR46.71	2	1.946377 .068479	-.946377 -.014856		-46.710	.958	2
SR47.87	2	1.948354 .066525	-.948354 -.014879		-47.870	.959	2
SR48.75	2	1.949792 .065103	-.949792 -.014859		-48.750	.960	2
MTH48.81	3	2.902655 .084668	-2.834041 -.064190	.931386 .008253	-48.810	.007	2
SR48.89	2	1.950026 .064871	-.950026 -.014897		-48.890	.960	2

Method	Order	a_1 b_1	a_2 b_2	a_3 b_3	Stability Range	Truncation Error Coefficient	Degree Classical
SR50. 0	2	1. 92 . 0784	-. 92 . 0016		-50. 000	. 962	2
MTH50. 72	3	2. 905726 . 081706	-2. 839965 -. 061446	. 934239 . 008253	-50. 720	. 007	2
MTH. 01-52. 2	3	2. 907955 . 079552	-2. 844274 -. 05944	. 936319 . 008253	-52. 220	. 007	2
MTH52. 47	3	2. 908317 . 079201	-2. 844975 -. 059113	. 936658 . 008253	-52. 470	. 007	2
SR73. 5	2	1. 94667 . 052606	-. 94667 . 000726		-73. 460	. 974	2
SR78. 55	2	1. 969792 . 040182	-. 969792 -. 009973		-78. 550	. 975	2
MTH. 01-87. 9	3	2. 941818 . 048143	-2. 902133 -. 035147	. 960316 . 005502	-87. 890	. 006	2
SR100. 0	2	1. 96 . 0396	-. 96 . 0004		-100. 000	. 980	2
SR113. 8	2	1. 966667 . 033039	-. 966667 . 000293		-113. 760	. 984	2
SR183. 69	2	1. 98835 . 016649	-. 98835 -. 004999		-183. 690	. 989	2
SR197. 54	2	1. 989853 . 015147	-. 989853 -. 005		-197. 540	. 990	2
MTH. 01-231. 5	3	2. 973286 . 018891	-2. 95568 -. 012535	. 982394 . 002751	-231. 500	. 006	2
MTH. 01-266. 8	3	2. 975551 . 016636	-2. 96019 -. 010299	. 984639 . 002751	-266. 810	. 006	2

Method	Order	b_o	a_1 b_1	a_2 b_2	a_3 b_3	a_4 b_4	a_5 b_5	Stability Range	Truncation Error Coefficient	Degree Classical
O_{41}	4		-3.33333 4.0	6.0	-2.0	.33333		0	-	-
O_{14}	4		1.0 2.33333	-2.58333	1.66667	-.41667		-.286	-.042	4
Open 5-Step	5		.5 1.0	1.0			.5 1.0	-1.223	1.5	2
Trapezoidal	1	.5	1.0 .5					$-\infty$	-.083	3
Backward Rectangular	1	1.0	1.0					$-\infty$	-.5	2
Shifted 2nd Order Adam	1	1.5	1.0 -.5					$-\infty$	-1.0	2

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13. ABSTRACT

This report describes applications of a Simulation Analyzer Program for deriving or evaluating numerical integration methods for use in flight simulation for training. The FORTRAN program was developed in an earlier study, and its theoretical bases and capabilities are briefly presented in this report. The program is used to derive and evaluate optimal integration methods for application to a selected first-order and a second-order differential system. The derived methods are used to solve these systems, and actual solution characteristics are compared with those predicted beforehand by the program. Characteristics of accuracy, actual percent error, stability, and error propagation are shown to be accurately predicted, as is the largest integration interval usable for each problem with each integration method. A thorough description is provided of the five popular simulation-integration techniques in use today and of a recommended procedure for using the Simulation Analyzer Program to derive new integration methods which allow maximization of the integration interval for specific simulation problems. As an example, the program is used to evaluate known methods and derive new methods for the F-100A problem using integration intervals of 0.05 and 0.10. A list of over 70 new integration methods derived by the program, including their stability and truncation-error characteristics, is provided.

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Numerical Integration						
Flight Simulation						
Training Simulation						
Simulation Analyzer Program						
Digital Simulation						
Real-Time Simulation						
Digital Computers						
Error Analysis						
Iteration Rate						
Integration Step-Size						