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A RECURSIVE METHOD FOR DETERMINING TRANSIENT TEMPERATURE DISTRIBUTIONS IN A HOLLOW CYLINDER WITH NONSTEADY BOUNDARY CONDITIONS

Percy B. Carter, Jr.
ARO, Inc.

July 1969

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FOREWORD

The work reported herein was done at the request of Arnold Engineering Development Center (AEDC), Air Force Systems Command (AFSC), under Program Element 65401F.

The results of tests (or research) presented were obtained by ARO, Inc. (a subsidiary of Sverdrup & Parcel and Associates, Inc.), contract operator of AEDC, AFSC, Arnold Air Force Station, Tennessee, under Contract F40600-69-C-0001. The research was conducted from April through December, 1968, under ARO Project No. VT8002, and the manuscript was submitted for publication on March 13, 1969.

This technical report has been reviewed and is approved.

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ABSTRACT

Recursion equations are developed for solution of transient temperature distributions in an infinite hollow cylinder with nonsteady boundary conditions. The solution is shown to be applicable to any imposed boundary condition and is also shown to be able to handle the special case of the solid cylinder. A computer program is written and applied to two examples. A comparison of the numerical results with classical exact solutions reveals close agreement between the two types of solutions for the particular cases considered.

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NOMENCLATURE

A	Area
A	Convergence indicator
C	Specific heat of pipe metal
C_p	Gas specific heat at constant pressure
D	Diameter
h	Convection heat-transfer film coefficient
K	Thermal conductivity
k	Element index number
M	Mass flow rate
N	Number of elements
P	Pressure
R	Radius
r	Radius ratio - R_i/R_o
T	Temperature
t	Thickness of elemental cylindrical shell
V	Volume
Z	Axial coordinate
α	Thermal diffusivity
β	R_o/K
θ	Time
μ	Viscosity
ρ	Density
ϕ	Azimuthal angle
ψ	$2N^2 \Delta\theta \alpha / R_o^2$

SUBSCRIPTS

1, 2, ... k	Element index number
g	Gas
i	Inside pipe
N	Number of elements
o	Outside pipe

SECTION I INTRODUCTION

The design of industrial processes frequently requires a knowledge of the transient heat-transfer phenomena within infinite cylinders. The heat treatment of solid, long steel billets, or the analysis of thermal stresses in pipelines carrying hot or cold fluids and subject to nonsteady flow conditions are examples where an analysis of transient heat transfer is desired.

The derivation of the governing differential equation for transient heat flow in a cylinder is well known and can be found in texts such as Schneider (Ref. 1) or Eckert and Drake (Ref. 2). In the most general case, the cylinder is hollow, has an arbitrary internal distribution of temperature, and is subject to nonsteady environments, not necessarily the same, at both the inside and outside surfaces. Appendix I contains the derivation of the governing differential equation, together with a formal statement of this general set of boundary conditions.

Solutions to the general equation subject to a more restrictive set of boundary conditions have been developed by several authors. Schneider (Ref. 1) develops solutions for solid cylinders with arbitrarily specified initial radial temperature distribution and with a temperature, T_o , suddenly applied to the outside face. Schneider also develops solutions for the solid cylinder subject to a uniform initial temperature and suddenly immersed in a fluid at temperature T_g . Boelter, Cherry, et al. (Ref. 3) also present solutions for transient temperature distributions in solid cylinders, with arbitrary initial temperature distributions, that are suddenly immersed in a constant temperature fluid. However, analytical solutions of the differential equation subject to the boundary conditions of Appendix I do not appear to be available in the literature.

This document addresses itself to the problem of developing a numerical technique for solving the general differential equation subject to the general set of boundary conditions.

SECTION II DERIVATION OF THE RECURSION EQUATIONS

The recursive technique is introduced by dividing the cylinder into N cylindrical shells, each of equal thickness, $2\Delta R$. Next the assumption is made that, for sufficiently small increments of time, the problem

approaches a quasi-steady state process for each of the incremental elements. Consequently, by making successive energy balances on small elements of the pipe for small time periods, a steady-state approximation of the actual process can be made.

Figure 1 (Appendix II) shows a unit length of cylinder divided into N concentric elements of equal thickness. Choosing N large enough allows each elemental shell to be considered isothermal. If the cylinder is considered a composite of N isothermal shells, each at its own specific temperature, then over short periods of time the temperature change of any element will be a function only of the temperature of that element and the temperature of adjacent elements.

Referring to Fig. 1, the geometrical aspects of the pipe indicate that the thickness, t , of each shell is given by Eq. (1)

$$t = (R_o - R_i) \div N = 2 \Delta R \quad (1)$$

where R_i and R_o are the inside and outside pipe radii, respectively, and N is the chosen number of concentric shells.

An energy balance is now made on the first cylindrical shell:

Rate of Energy In (from Gas) - Rate of Energy Out (to Element 2) = Time Rate of Change in Energy Stored in Element 1.

Expressed mathematically the energy balance becomes

$$\begin{aligned} h_i(\theta) A_i (T_{gi}(\theta) - T_1) + \frac{2\pi K(T_2 - T_1)}{\ln R_2/R_1} &= C \rho V_1 \frac{\Delta T_1}{\Delta \theta} = C \rho V_1 \frac{(T'_1 - T_1)}{\Delta \theta} \\ h_i(\theta) 2\pi R_i (T_{gi}(\theta) - T_1) + \frac{2\pi K(T_2 - T_1)}{\ln R_2/R_1} &= C \rho \pi [(R_i + 2\Delta R)^2 - R_i^2] \frac{(T'_1 - T_1)}{\Delta \theta} \end{aligned} \quad (2)$$

Here T'_1 is taken to represent the new temperature of Element 1 after exchanging energy with adjacent elements during the time period $\Delta \theta$. C is the pipe metal specific heat, K is the pipe metal thermal conductivity, ρ is the pipe metal density, V_1 is the volume per unit length of the first (inside) cylindrical shell, and A_i is the inside area of the cylinder per unit length.

Recalling the definition of ΔR allows R_1 and R_2 to be expressed as follows:

$$\begin{aligned} R_1 &= R_i + \Delta R = R_i + (R_o - R_i) / 2N = \left(\frac{2N-1}{2N} \right) R_i + \frac{1}{2N} R_o \\ R_2 &= R_i + 3 \Delta R = R_i + \frac{3(R_o - R_i)}{2N} = \left(\frac{2N-3}{2N} \right) R_i + \frac{3}{2N} R_o \end{aligned} \quad (3)$$

Substituting Eqs. (1) and (3) into Eq. (2) gives:

$$\left\{ h_i(\theta) 2\pi R_i (T_{gi}(\theta) - T_i) + 2\pi K (T_2 - T_1)/\ell_n \left[\frac{R_i + \frac{3}{2} \left(\frac{R_o - R_i}{N} \right)}{R_i + \frac{1}{2} \left(\frac{R_o - R_i}{N} \right)} \right] \right\} \Delta \theta$$

$$= C \rho \pi \left[\left(R_i + \frac{R_o - R_i}{N} \right)^2 - R_i^2 \right] (T'_1 - T_1)$$

Expanding the squared terms and simplifying yields

$$\left\{ 2\pi h_i(\theta) R_i (T_{gi}(\theta) - T_1) + 2\pi K (T_2 - T_1)/\ell_n \left[\frac{(2N-3) R_i + 3 R_o}{(2N-1) R_i + R_o} \right] \right\} \Delta \theta$$

$$= C \rho \pi \frac{[R_o^2 - (2N-2) R_o R_i + (1-2N) R_i^2] (T'_1 - T_1)}{N^2} \quad (4)$$

It now becomes convenient to redefine certain parameters into groups. Let $\beta = R_o/K$, let r represent the ratio of inside radius to outside radius (R_i/R_o), and let ψ be as defined below:

$$\psi = \frac{2 K N^2 \Delta \theta}{C \rho R_o^2} = \frac{2 N^2 \Delta \theta \alpha}{R_o^2}$$

where α is the thermal diffusivity.

Substituting these parameters into Eq. (4) and solving for the new element temperature, T'_1 , gives

$$T'_1 = T_2 + \psi \left\{ \frac{r\beta h_i(\theta) [T_{gi}(\theta) - T_1] + (T_2 - T_1)/\ell_n \left[\frac{(2N-3)r+3}{(2N-1)r+1} \right]}{(1-r)[(2N-1)r+1]} \right\} \quad (5)$$

Looking now at the inner elements of the cylinder and writing an energy balance for Element 2 gives

$$\left[\frac{2\pi K (T_1 - T_2)}{\ell_n (R_2/R_1)} + \frac{2\pi K (T_3 - T_2)}{\ell_n (R_3/R_2)} \right] \Delta \theta = C \rho V_2 (T'_2 - T_2) \quad (6)$$

Generalizing Eq. (6) to the k th cylindrical element gives

$$\left[\frac{2\pi K (T_{k-1} - T_k)}{\ell_n (R_k/R_{k-1})} + \frac{2\pi K (T_{k+1} - T_k)}{\ell_n (R_{k+1}/R_k)} \right] \Delta \theta = C \rho V_k (T'_k - T_k) \quad (7)$$

Solving for T'_k gives

$$T'_k = T_k + \frac{2\pi\Delta\theta K}{C\rho V_k} \left[\frac{T_{k-1} - T_k}{\ln(R_k/R_{k-1})} + \frac{T_{k+1} - T_k}{\ln(R_{k+1}/R_k)} \right] \quad (8)$$

Substitution for V_k , $\ln(R_k/R_{k-1})$, $\ln(R_{k+1}/R_k)$, and applying the definition of ψ gives

$$T'_k = T_k + \psi \left\{ \frac{\frac{T_{k-1} - T_k}{\ln \left[\frac{(2k-1) - (2N-2k+1)r}{(2k-3) + (2N-2k+3)r} \right]} + \frac{T_{k+1} - T_k}{\ln \left[\frac{(2k+1) + (2N-2k-1)r}{(2k-1) + (2N-2k+1)r} \right]}}{(1-r)[(2N-2k+1)r + (2k-1)]} \right\} \quad (9)$$

Next a look is taken at the outermost cylindrical element. The energy balance gives

$$\left\{ \frac{2\pi K (T_{N-1} - T_N)}{\ln \left[\frac{R_o - \Delta R}{R_o - 3\Delta R} \right]} + 2\pi h_o(\theta) R_o (T_{go}(\theta) - T_N) \right\} \Delta\theta = C\rho V_N (T'_N - T_N) \quad (10)$$

$$\left\{ \frac{2\pi K (T_{N-1} - T_N)}{\ln \left[\frac{R_o (2N-1) + R_i}{R_o (2N-3) + 3R_i} \right]} + 2\pi h_o(\theta) R_o (T_{go}(\theta) - T_N) \right\} \Delta\theta = C\rho\pi \left\{ R_o^2 - \left[R_o - \left(\frac{R_o - R_i}{N} \right)^2 \right] (T'_N - T_N) \right\} \quad (11)$$

Solving for T'_N gives

$$T'_N = T_N + \psi \left\{ \frac{\frac{T_{N-1} - T_N}{\ln \left[\frac{2N-1+r}{2N-3+3r} \right]} + \beta h_o(\theta) [T_{go}(\theta) - T_N]}{(1-r)[r + 2N-1]} \right\} \quad (12)$$

Equations (5), (9), and (12) will yield the new temperature of the inner element, the middle elements, and the outer element, respectively, after a given time interval $\Delta\theta$. The information required to solve for the new element temperatures is as follows:

$$m = \frac{0.4V}{RT}$$

$$\frac{P}{RT}$$

1. The previous element temperature as well as previous temperatures of adjacent elements.
2. Time-dependent functions for inside and outside surface film coefficients (if applicable).
3. A specification of the outside radius of the cylinder and the radius ratio.
4. A specification of the material properties of the cylinder.

SECTION III CONVERGENCE AND STABILITY

3.1 CONVERGENCE

In a qualitative sense, convergence is a measure of the ability of the numerical approach to yield results which agree with those results that could hypothetically be determined from an exact solution. If in Eqs. (5), (9), and (12) the value of $\Delta\theta$ was allowed to approach zero and the value of N was allowed to approach infinity, then the numerical solution would converge to exactly the same results as the exact solution. However, it is impossible to allow the limiting cases for $\Delta\theta$ and N to exist or even to be realistically approached. Hence, convergence is the measure of how close the exact solution can be approached using finite values of $\Delta\theta$ and N . In the interest of time, it behooves us to use the largest value of $\Delta\theta$ and the smallest value of N that will give convergent results.

Convergence is difficult to study analytically. However, a simplified approach can lead to criteria for establishing limits to the constants $\Delta\theta$ and N beyond which the solution will begin to diverge.

Equation (5) may be rewritten as follows:

$$T_1' = T_1 \left\{ 1 - \psi \left[\frac{r \beta h_i(\theta) + \frac{1}{\ln \left[\frac{(2N-3)r-3}{(2N-1)r+1} \right]}}{(1-r) [(2N-1)r+1]} \right] \right\} + \psi \left\{ \frac{r \beta h_i(\theta) T_{gi}(\theta) + \frac{T_2}{\ln \left[\frac{(2N-3)r+3}{(2N-1)r+1} \right]}}{(1-r) [(2N-1)r+1]} \right\} \quad (13)$$

Inspection of Eq. (13) indicates that it can be expressed in the following form:

$$T'_1 = T_1(1 - A_1) + B \quad (14)$$

where the terms A_1 and B are complicated terms involving the parameters ψ , r , β , $h_i(\theta)$, $T_{gi}(\theta)$, T_2 , and N . It is evident that the term B in Eq. (14) will always be positive while the term $(1 - A_1)$ can be either positive or negative depending on the value of A_1 . If A_1 is greater than unity, then $(1 - A_1)$ becomes negative, and the solution diverges. Solution divergence occurs because for $(1 - A_1) < 0$, with the value of B fixed, any new estimate of T'_1 is inversely related to T_1 . Hence, higher values of T_1 call for lower values of T'_1 and vice versa, which is a violation of intuitive reasoning and in some instances could entail a violation of the second law of thermodynamics. Thus, for the first incremental element, a convergence criterion would restrict values of A_1 to $A_1 \leq 1$. Equation (15) is the formal statement of convergence criterion based upon the first element.

$$A_1 = \psi \left\{ \frac{r\beta h_i(\theta) + \frac{1}{\ln \left[\frac{(2N-3)r+3}{(2N-1)r+1} \right]}}{(1-r)[(2N-1)r+1]} \right\} \leq 1 \quad (15)$$

Following similar reasoning, convergence criteria can be developed for the k th and N th elements. Equations (16) and (17) are formal statements of the convergence criteria for the k th and N th elements, respectively.

$$A_k = \psi \left\{ \frac{\frac{1}{\ln \left[\frac{(2k-1) + (2N-2k+1)r}{(2k-3) + (2N-2k+3)r} \right]} + \frac{1}{\ln \left[\frac{(2k+1) + (2N-2k-1)r}{(2k-1) + (2N-2k+1)r} \right]}}{(1-r)[(2N-2k+1)r + (2k-1)]} \right\} \leq 1 \quad (16)$$

$$A_N = \psi \left\{ \frac{\frac{1}{\ln \left[\frac{2N+r-1}{2N-3r-3} \right]} - (\beta h_o(\theta))}{(1-r)(2N+r-1)} \right\} \leq 1 \quad (17)$$

Figure 2 shows a typical application of the convergence criteria. The recursion equations of Section II were applied to a situation where a solid cylinder was suddenly immersed in a fluid at a constant temperature T_g . The film coefficient was held constant. For this particular application, the inside radius in the recursion equations, along with the inside film coefficient, were simply set equal to zero. Plotted on Fig. 2

is the exact solution of the problem along with the results of two recursive solutions — one in which the convergence criteria is satisfied and one in which the convergence criteria is not satisfied. Inspection of Fig. 2 indicates that the recursive solution in which the convergence criteria is satisfied converges to the exact solution. However, for the case of nonsatisfaction of the convergence criteria, the recursive solution is highly divergent.

3.2 STABILITY

Stability is a measure of the ability of the numerical system to absorb systematic errors caused by roundoff and by the finite element assumption. A stable system damps out errors and distributes them evenly throughout the system. Like convergence, stability is not easy to analyze. However, Smith (Ref. 4) gives a well-developed analysis of stability for certain classes of numerical problems. Smith indicates that one method of gaining insight into the stability of a numerical solution is to apply the numerical equations to the errors themselves. Hence, by assuming, for example, a unit error somewhere in the system and watching the propagation of this error as the solution progresses, an indication of system stability becomes available. Table I (Appendix IV) is a tabulation of the propagation of a unit error that is assumed to occur at time equal zero, in the tenth element, of a twenty-element system.

Table I reveals that errors in the system do indeed tend to distribute themselves and to diminish with time. Although a check of this sort is not an all-conclusive proof of stability, it does indicate that the system tends to be stable, provided roundoff error is not excessive and can be expected to be random with regard to sign.

SECTION IV COMPUTER PROGRAM

Equations (5), (9), and (12) were programmed for solution on the IBM 360/50 computer. Programming was carried out in G Level FORTRAN.

The program is set up to take any type of boundary condition or initial condition. The program is expected to be called on to solve the problem for two general types of boundary conditions:

1. Type I boundary conditions involve an arbitrary specification of initial temperature distribution within the cylinder wall, together with the specification of a suddenly applied temperature to either or both the inside and outside cylindrical surfaces. The initial temperature distribution is input as data, while the suddenly applied surface temperature is included in the program as an arithmetic statement function. The suddenly applied surface temperature can, in the general case, be time variant.
2. Type II boundary conditions also involve an arbitrary specification of initial temperature distribution within the cylinder. However, Type II problems differ from Type I problems in that the boundary conditions are specified in terms of a time-varying surface film coefficient and fluid temperatures adjacent to each surface of the cylinder. Both the inside and outside surface film coefficients are included in the program as time-dependent arithmetic statement functions. Likewise, the inside and outside fluid temperatures are included as time-dependent arithmetic statement functions. Film coefficients and fluid temperatures can, as a special case, be considered constant. The surface film coefficients can also be considered as functions of other parameters such as fluid temperature and the thermodynamic and transport properties of the fluid, if desirable. The thermodynamic and transport properties of the fluid must then be specified and included in the program as functions of time or temperature.

A combination of Type I and Type II boundary conditions can also be handled by the program.

The special case of the solid cylinder is handled readily by the program if the radius ratio is simply set equal to zero.

Appendix III gives a FORTRAN listing of the program as applied to Sample Problem Two of the following section.

SECTION V

APPLICATION OF THE RECURSIVE METHOD

The application of the numerical method will be illustrated by two sample problems. Sample Problem One involves transient heat transfer in a solid cylinder with Type II boundary conditions at the outside surface.

The results of the recursive solution are compared with exact solutions from the literature for the same problem.

Sample Problem Two is a typical example of the application of the method to a hollow cylinder with time-dependent, Type II, boundary conditions on both the inside and outside surfaces.

5.1 SAMPLE PROBLEM ONE

Schneider (Ref. 1) develops the exact solution for the transient temperature distribution in a solid cylinder with Type II boundary conditions and with a uniform initial temperature. Schneider's solution is developed in terms of zero and first order Bessel Functions of the first kind. The results are expressed as dimensionless temperature ratios with the Fourier modulus, $\alpha\theta/R_0^2$, and the Biot modulus, hR_0/K , as parameters.

The present method was used to generate comparative results that were then nondimensionalized and compared with Schneider's results.

The outside surface film coefficient, $h_o(\theta)$, and outside fluid temperature, $T_{go}(\theta)$, were specified as arithmetic statement functions in the program. The outside film coefficient was held constant for a given run but was varied between runs so as to allow variations in the dimensionless Biot modulus, hR_0/K . A constant fluid temperature of 800°R was used in the calculations.

A comparison of the present method with the exact method is presented in Table II. The results of Schneider's analysis for selected values of the Biot and Fourier moduli are presented along with the results of the present numerical solution. It is apparent from Table II that, for the cases studied, a close degree of correspondence exists between the exact and the recursive solutions.

5.2 SAMPLE PROBLEM TWO

In Sample Problem Two application is made to an interesting problem which occurs in a pipeline carrying gas that is being discharged from a pressurized storage vessel. For a gas stored at high pressure, for example, $P = 2500$ psia, and at ambient temperature and with the pipeline existing initially at ambient temperature, it is possible to induce severe temperature gradients in the pipeline because of the cooling effect the drawdown process has on the stored gas. The combined conditions of high pressure and severe thermal gradients could cause

serious stress problems within the pipe wall. Therefore, the unsteady thermal gradients which would exist in the pipe wall become significant.

The problem of determining the transient thermal gradients was investigated using the current numerical method for the case of a 14-in. - diam steel pipe carrying air that was being discharged from a storage vessel at a constant mass flow rate.

5.2.1 Program Inputs

The initial storage pressure and temperature of the air was 3600 psia and 576°R, respectively. The temperature of the air was found to be a function of the vessel volume, initial storage temperature, rate of mass withdrawal from the tank, and the time lapse since initiation of drawdown. Assuming the air in the tank expanded isentropically gave a relationship for the air temperature is a function of time as follows:

$$T_{gi}(\theta) = 576[1 - .006179 \theta]^{.4} \text{ } ^\circ\text{R} \quad (18)$$

The derivation of Eq. (18) assumes that the section of pipeline to be investigated is located immediately downstream of the storage vessel at a point where the gas has not had time to exchange heat with the pipeline. The air temperature surrounding the pipe was assumed to remain constant at 560°R. The inside film coefficient was assumed to follow the relationship given by Eq. (19).

$$h_i(\theta) = 0.0279 \left(\frac{K}{2 r R_o} \right) \left(\frac{\dot{M}}{2 \mu r R_o} \right)^{.8} \left(\frac{C_p \mu}{k} \right)^{.35} \frac{\text{Btu}}{\text{ft}^2 \text{-sec-}^\circ\text{R}} \quad (19)$$

*Re = $\frac{\rho v D}{\mu}$
in D
in ft*

For the particular blowdown rate ($\dot{M} = 805 \text{ lb}_m/\text{sec}$), Eq. (19) becomes

$$h_i(\theta) = 1.6919 (r R_o)^{-1.8} (K)^{.65} (\mu)^{-.45} (C_p)^{.35} \frac{\text{Btu}}{\text{ft}^2 \text{-sec-}^\circ\text{R}} \quad (20)$$

For the particular temperature range of interest, 300 to 800°R, the specific heat, viscosity, and thermal conductivity of the air have been found to obey the following relationships:

Specific Heat

$$C_p = 0.2431 - .00001919T + .0000000244T^2 \left(\frac{\text{Btu}}{\text{lb}_m \text{-}^\circ\text{R}} \right) (T = ^\circ\text{R}) \quad (21)$$

Viscosity

$$\mu = \frac{7.3026 T^{3/2}}{(T + 198.6)} \times 10^{-7} \frac{\text{lb}_m}{\text{sec-ft}} (T = ^\circ\text{R}) \quad (22)$$

*nearly independent of pressure
NACA 1135
12-19*

Thermal Conductivity

$$K = 0.003716 + .00001676T + 0.000000008T^2 \frac{\text{Btu}}{\text{hr-ft-}^\circ\text{R}} \quad (T = ^\circ\text{R}) \quad (23)$$

The outside film coefficient is assumed to correspond to natural convection from a horizontal cylinder and is given by Eq. (24).

$$h_o(\theta) = 0.00005 (T_{go} - T_n)^{1/3} \frac{\text{Btu}}{\text{ft}^2\text{-sec-}^\circ\text{R}} \quad (24)$$

Equations (18) through (24) were input into the computer program as arithmetic statement functions. Figure 3 shows the input cards for the program constants, while Fig. 4 shows the input cards for the initial temperature distribution. T_1 corresponds to the innermost element.

5.2.2 Results

The results of the study are shown in Fig. 5, where the temperature as a function of location within the pipe wall is plotted for selected time intervals.

SECTION VI DISCUSSION

The preceding sections have presented the development of the recursive method, its application, and a comparison of typical results with a corresponding exact solution. The comparison indicated close agreement between the recursive method and the exact solution for the particular cases that were investigated.

The recursive technique presented is particularly useful in cases involving complicated or noncontinuous boundary conditions, since the technique will handle any boundary condition that can be expressed as either a function of time or a function of the surface temperature. The technique also allows any type of initial temperature distribution to be specified.

Computer time required is not excessive. On the average, a nominal time of 0.15 sec was required to compute the temperature distribution over a single time increment in a system composed of ten elements.

Since the recursive technique is designed to step through the solution in discrete time intervals, a certain inaccuracy is inherent in the

early stages of the solution. These inaccuracies disappear as the solution progresses. In general, accurate results can be expected at any time, θ , provided the condition of Eq. (25) is met.

$$\theta \geq N\Delta\theta \quad (25)$$

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APPENDIXES

- I. DERIVATION OF THE GOVERNING DIFFERENTIAL EQUATION**
- II. ILLUSTRATIONS**
- III. COMPUTER PROGRAM FORTRAN LISTING**
- IV. TABLES**

APPENDIX I

DERIVATION OF THE GOVERNING DIFFERENTIAL EQUATION

Figure 6 shows the geometrical aspects of the mathematical model. The differential equation of the problem is developed by making an energy balance on a typical differential volume element of the pipe wall.

The following assumptions are made concerning the problem:

1. All energy transport occurs through thermal conduction except at the boundaries.
2. Energy transport occurs in the radial direction only.
3. Pipe thermal conductivity is constant.

The energy balance is expressed in simple terms as

$$\text{Energy In} - \text{Energy Out} = \text{Energy Stored}$$

The energy conducted into the volume element is given by

$$\text{Conduction In} = - \left(KA \frac{\partial T}{\partial R} \right) = - \left(KRd\phi dZ \frac{\partial T}{\partial R} \right) \quad (\text{I-1})$$

The energy conducted out of the volume element is given by

$$\begin{aligned} \text{Conduction Out} &= - \left(KA \frac{\partial T}{\partial R} + \frac{\partial}{\partial R} \left[KA \frac{\partial T}{\partial R} \right] dR \right) \\ &= - \left(KRd\phi dZ \frac{\partial T}{\partial R} + \frac{\partial}{\partial R} \left[KRd\phi dZ \frac{\partial T}{\partial R} \right] dR \right) \\ &= - KRd\phi dZ \frac{\partial T}{\partial R} - Kd\phi dZ \frac{\partial}{\partial R} \left[R \frac{\partial T}{\partial R} \right] dR \end{aligned} \quad (\text{I-2})$$

Combining Eqs. (I-1) and (I-2) gives

$$\text{Conduction In} - \text{Conduction Out} = Kd\phi dZ dR \left[R \frac{\partial^2 T}{\partial R^2} + \frac{\partial T}{\partial R} \right] \quad (\text{I-3})$$

The energy stored is given by

$$\text{Energy Stored} = \rho V C \frac{\partial T}{\partial \theta} = \rho R d\phi dZ dR C \frac{\partial T}{\partial \theta} \quad (\text{I-4})$$

Equating Eqs. (I-3) and (I-4) gives the governing differential equation:

$$\frac{\partial^2 T}{\partial R^2} + \frac{1}{R} \frac{\partial T}{\partial R} = \frac{1}{\alpha} \frac{\partial T}{\partial \theta} \quad (\text{I-5})$$

where $\alpha = \frac{K}{\rho C}$ and is known as the thermal diffusivity.

Associated with Eq. (I-5) are the following initial and boundary conditions:

(1) Initial Condition

$$\text{at } \theta = 0, T(R, \theta) = T(R, 0)$$

(2) Boundary Condition 1 (Inside the Pipe)

$$\begin{aligned} \text{at } R = R_i, K \frac{\partial T(R, \theta)}{\partial R} &= K \frac{\partial T(R_i, \theta)}{\partial R} \\ &= h_i(\theta) (T(R_i, \theta) - T_{gi}(\theta)) \end{aligned}$$

Here $h_i(\theta)$ and $T_{gi}(\theta)$ are the time-dependent inside fluid heat-transfer coefficient and inside fluid temperature, respectively. Rewriting this boundary condition gives:

$$\frac{\partial T(R_i, \theta)}{\partial R} - \frac{h_i(\theta)}{K} T(R_i, \theta) = - \frac{h_i(\theta)}{K} T_{gi}(\theta)$$

(3) Boundary Condition 2 (Outside the Pipe)

$$\begin{aligned} \text{at } R = R_o, K \frac{\partial T(R, \theta)}{\partial R} &= K \frac{\partial T(R_o, \theta)}{\partial R} \\ &= h_o(\theta) (T_{go}(\theta) - T(R_o, \theta)) \end{aligned}$$

Here $h_o(\theta)$ and $T_{go}(\theta)$ are the time-dependent outside fluid heat-transfer coefficient and outside fluid temperature, respectively. Rewriting this boundary condition gives

$$\frac{\partial T(R_o, \theta)}{\partial R} + \frac{h_o(\theta)}{K} T(R_o, \theta) = \frac{h_o(\theta)}{K} T_{go}(\theta)$$

Inspection of the boundary conditions reveals that they are not homogeneous; consequently, the path to an exact solution does not readily present itself. Recourse must therefore be made to approximate numerical techniques.

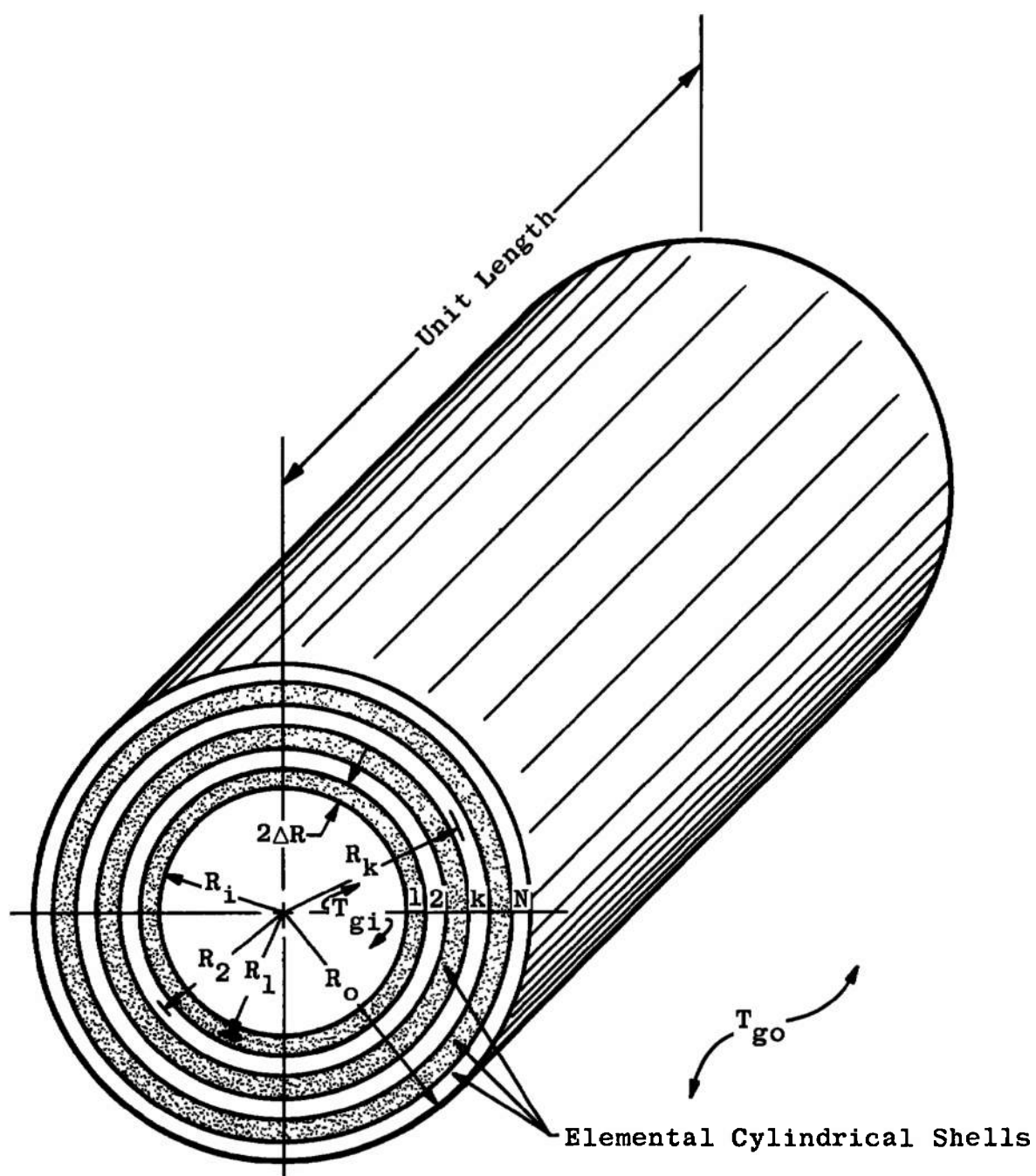


Fig. 1 Numerical Model

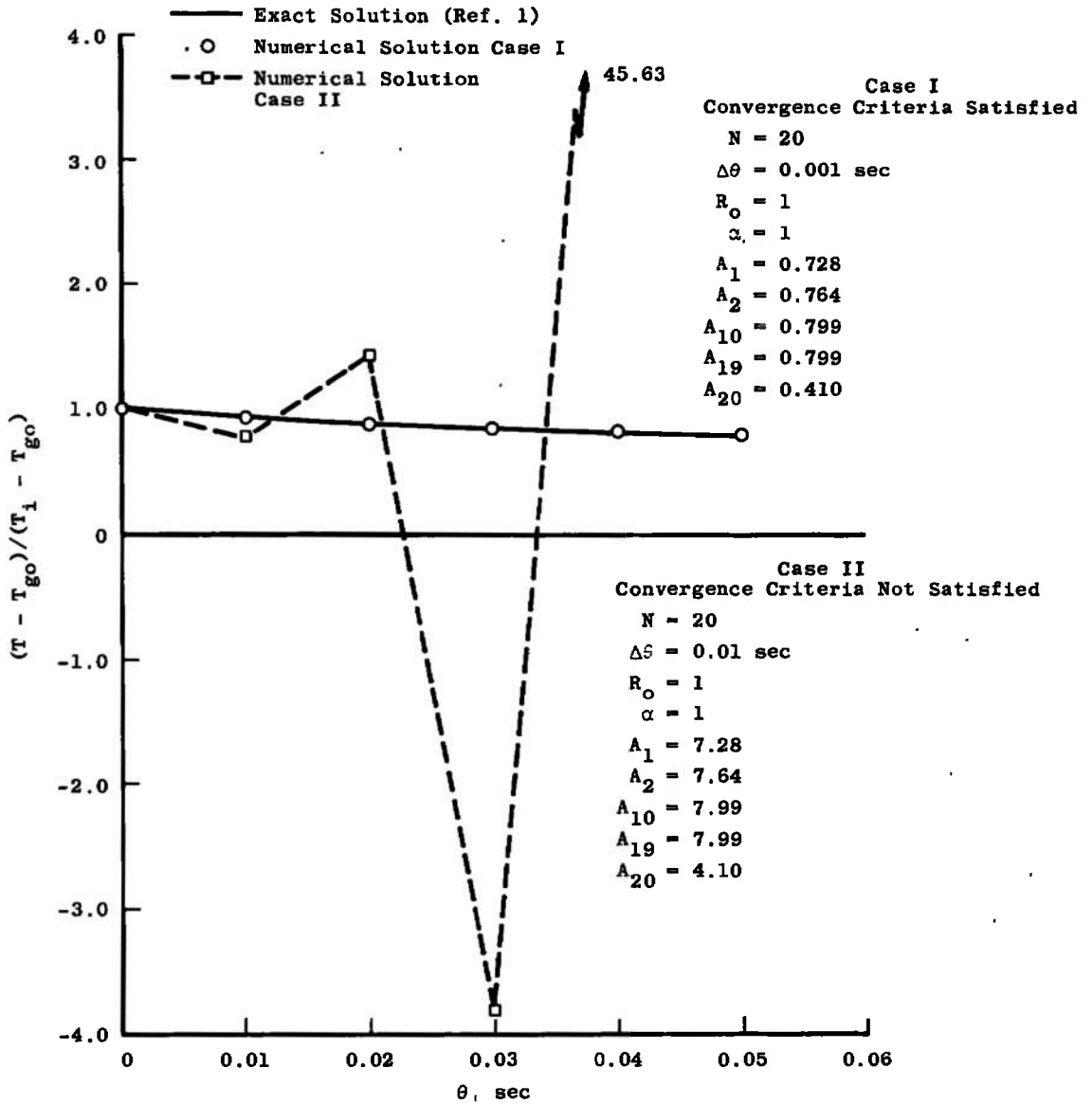


Fig. 2 Effects of Convergence Criteria on the Convergence of the Numerical Solution

0000	80.771	0.50	103.0	0.0001185	.729	0.583	10
0000	Beta (R_o/K)						
0000	(ft ² -sec-or/Btu)						
0000	Format (F10.0)						
0000	Delta Time, sec						
0000	Format (F10.0)						
0000	Total Run Time, sec						
0000	Format (F10.0)						
0000	Thermal Diffusivity, ft ² /sec						
0000	Format (F10.0)						
0000	Radius Ratio (R_1/R_o)						
0000	Format (F10.0)						
0000	Outside Radius (R_o), ft						
0000	Format (F10.0)						
0000	Number of Elements						
0000	Format (I2)						

Fig. 3 Sample Problem Two - Program Constant Inputs

512.40	511.52	511.08	510.64	510.20
T ₁	T ₂	T ₃	T ₄	T ₅
00000000	0000000000	000000 000	0000 00000	0000 00 00
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80				
111 111111	111 11111	111 11111	111 11111	111 11111
2222 21222	22222222	222222222222	2222222222	222222 22/
33333 3333	33333 3333	33333 3333	33333 3333	33333 3333
444444 444	444444444444	4444444	Initial Temperature Distribution, OR	
55 55555555	55 555 5555	55 5555	Format (F10.0)	
6666666666	666666666666	666666666666	66666666 6666	666666666666
7777777777	7777777777	7777777777	7777777777	7777777777
88080 8886	88888 0088	88888 8 80	00888 8888	80888 8808
9999999999	9999999999	9999999999	9999999999	9999999999

[illegible]

Fig. 4 Sample Problem Two - Initial Temperature Distribution

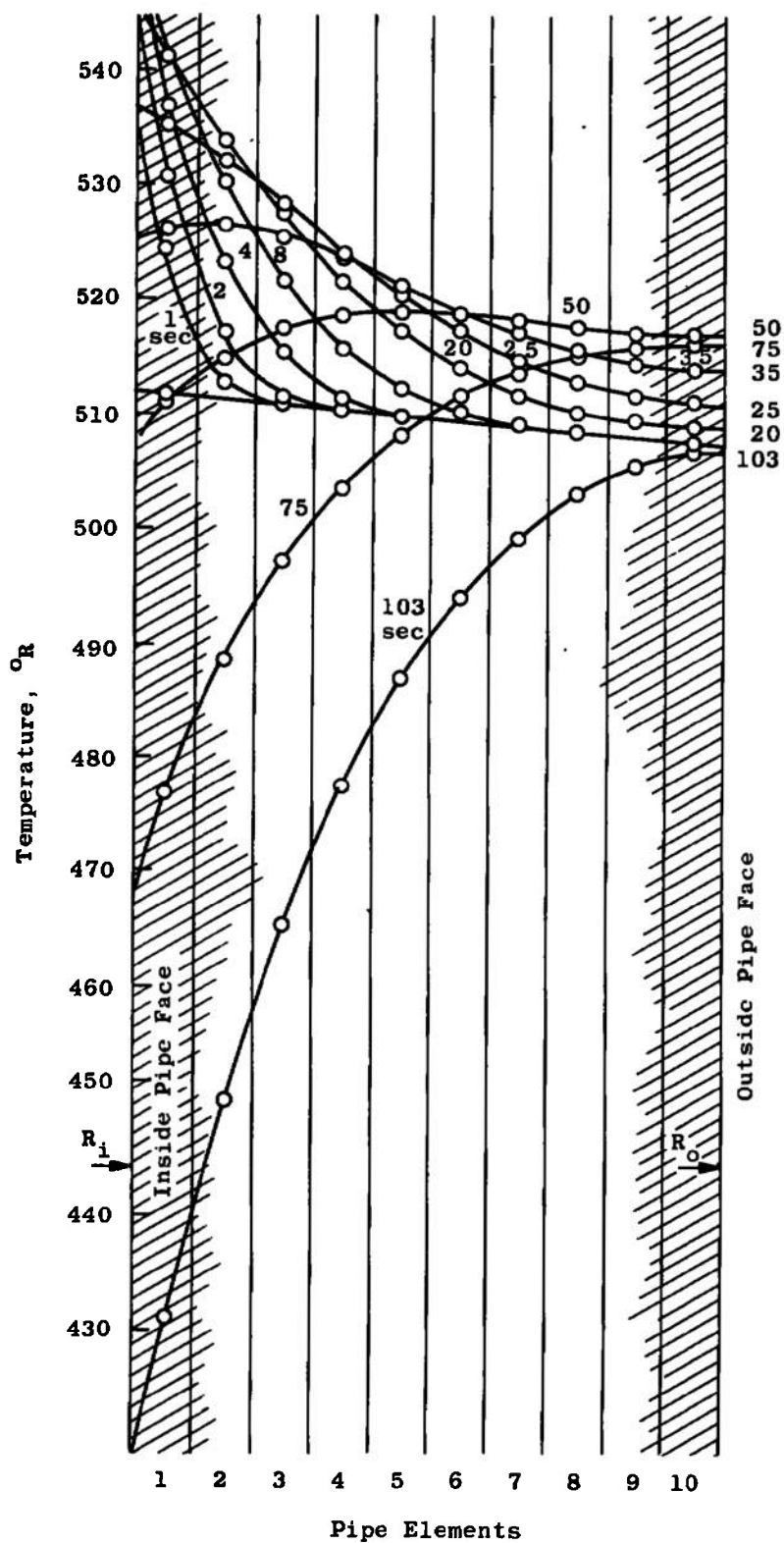


Fig. 5 Transient Temperature Distributions in a Pipe with Nonsteady Boundary Conditions

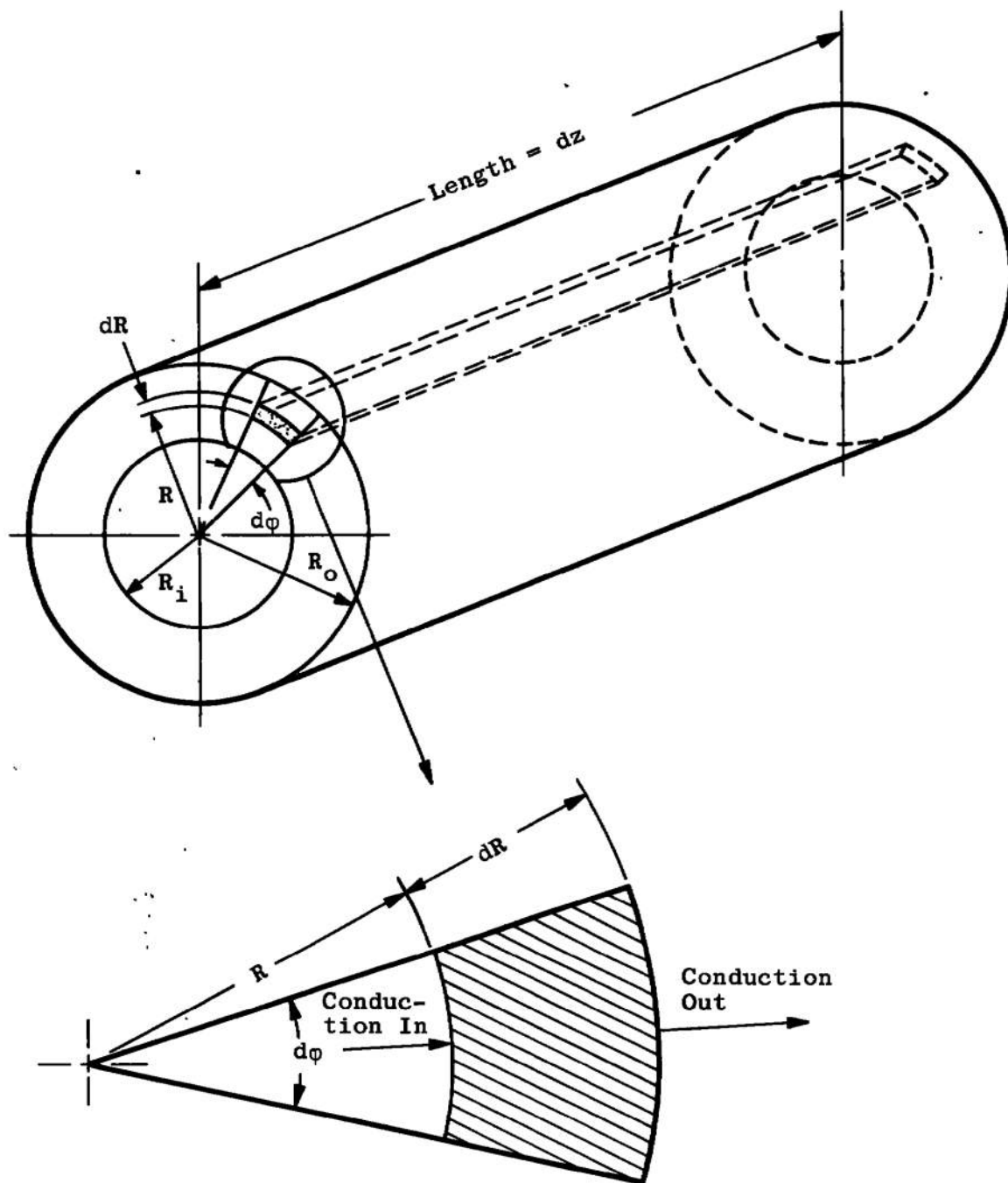


Fig. 6 Exact Mathematical Model

APPENDIX III COMPUTER PROGRAM FORTRAN LISTING

```

C*****
C*****
C*****
C FORTRAN LISTING OF PROGRAM AS APPLIED TO SAMPLE PROBLEM
C TWO.  SEE SECTION IV.
C*****
C*****
C ANALYSIS OF TRANSIENT TEMPERATURE DISTRIBUTION IN A HOLLOW
C CYLINDER WITH NON-STEADY BOUNDARY CONDITIONS.
C*****
C INSIDE FILM COEFFICIENT (HI) IS FOR AIR- TURBULENT FORCED
C CONVECTION WITH VARIABLE PROPERTIES.
C*****
C OUTSIDE FILM COEFFICIENT (HO) IS FOR AIR -NATURAL
C TURBULENT CONVECTION OVER A HORIZONTAL CYLINDER.
C*****
C*****
      DIMENSION A(20),B(20),TITLE(20)
      TGU(E)= 560.+1.*E-1.*E
      TGI(E)= 576.*((1.-.006179*E)**.4)
      CPGAS(TGI)=.2431-.00001919*TGI+.0000000244*(TGI**2)
      AMU(TGI)=7.3026*(TGI**(3./2.))*0.000001/(TGI+198.6)
      AKGAS(TGI)=(.003716+.00001676*TGI+.00008*(TGI**2)*
      1.0001)/3600.
      HI(CPGAS,AMU,AKGAS)=(1.6919/((R*RO)**1.8))*(AKGAS**
      1.65)*(CPGAS**.35)/(AMU**.45)
      HO(TGO,TN)=.00005*((ABS(TGO-TN))**(1./3.))
      F1(U,V)=(U-V)/ALOG(((2.*K+1.)+(2.*N-2.*K-1.)*R)/((2.*K
      1-1.)+(2.*N-2.*K+1.)*R))
      F2(U,V)=(U-V)/ALOG(((2.*K-1.)+(2.*N-2.*K+1.)*R)/((2.*K
      1-3.)+(2.*N-2.*K+3.)*R))
      F3(U)=(1.-U)*((2.*N-2.*K+1.)*R+(2.*K-1.))
      READ(5,1) TITLE
      1 FORMAT(20A4)
      WRITE (6,1) TITLE
      READ(5,2)BETA,D,F,ALP,R,RO,N
      2 FORMAT(6F10.0,12)
      WRITE(6,3)D,F,ALP,BETA,R,RO,N
      3 FORMAT('///// INPUT CONSTANTS'//// DELTA TIME = 'F6.3
      1' SEC.'/' TOTAL RUN TIME = 'F7.3' SEC.'/' THERMAL DIFF'
      2,'USIVITY = 'F10.8'SQ.FT./SEC.'/' BETA = 'F10.4' SQ.',
      3'FT.-SEC.-R/BTU'/' RADIUS RATIO (RI/RO) = 'F5.3/
      4' OUTSIDE RADIUS (RO) = 'F6.3'FT.'/' THE NUMBER OF',
      5' ELEMENTAL CYLINDRICAL SHELLS IS ('I2')'////)
      READ(5,4)(A(J),J=1,N)
      4 FORMAT(5F10.0)
      WRITE (6,5)
      5 FORMAT(3X'TIME'4X'T(1)'4X'T(2)'4X'T(3)'4X'T(4)'4X

```

```

1'T(5)'4X'T(6)'4X'T(7)'4X'T(8)'4X'T(9)'4X'T(10)'4X
2'TGASI'3X'TGASO')
CHI=2.*(N**2)*D*ALP/(RJ**2)
E=0.0
6 TGASO=TGO(F)
TGASI=TGI(E)
WRITE(6,7) E,(A(J),J=1,N),TGASI,TGASO
7 FORMAT(/13F8.2)
K=1
B(1)=CHI*((R*BETA*HI(CPGAS(TGI(E)),AMU(TGI(E)),AKGAS(
1TGI(E)))*(TGI(E)-A(1)))+F1(A(2),A(1)))/F3(R) + A(1)
NN=N-1
DO 8 K=2,NN
B(K)= CHI*(F2(A(K-1),A(K))+ F1(A(K+1),A(K)))/F3(R)
1+A(K)
8 CONTINUE
K=N
B(N)=CHI*(F2(A(N-1),A(N))+BETA*HC(TGO(E),A(N))*(TGO(E)
1-A(N)))/F3(R)+A(N)
DO 9 J=1,N
A(J)=B(J)
9 CONTINUE
E=E+0
IF(E-F)6,6,10
10 STOP
END

```

TABLE I
ERROR PROPAGATION IN A TWENTY-ELEMENT NUMERICAL SYSTEM

Element No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Time (sec)																				
0.0	0	0	0	0	0	0	0	0	0	1.0	0	0	0	0	0	0	0	0	0	0
0.001	0	0	0	0.01	0.02	0.05	0.09	0.14	0.16	0.18	0.15	0.12	0.06	0.03	0.01	0	0	0	0	0
0.002	0.02	0.03	0.04	0.05	0.07	0.08	0.10	0.11	0.11	0.11	0.10	0.09	0.07	0.05	0.04	0.02	0.01	0.01	0	0
0.004	0.07	0.07	0.07	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.07	0.07	0.06	0.05	0.04	0.03	0.03	0.02	0.02	0.02
0.008	0.07	0.07	0.07	0.07	0.07	0.07	0.07	0.06	0.06	0.06	0.06	0.05	0.05	0.04	0.04	0.04	0.03	0.03	0.03	0.03
0.015	0.06	0.06	0.06	0.06	0.06	0.05	0.05	0.05	0.05	0.05	0.05	0.04	0.04	0.04	0.04	0.04	0.04	0.03	0.03	0.03
0.030	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.04	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03
0.045	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.02	0.02	0.02	0.02	0.02
0.060	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
0.080	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.01	0.01	0.01	0.01	0.01	0.01	0.01
0.100	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01

$\Delta\theta = .001$ sec
 $N = 20$

TABLE II
COMPARISON OF RECURSIVE SOLUTION WITH EXACT SOLUTION
FOR TRANSIENT TEMPERATURE DISTRIBUTION IN A SOLID CYLINDER

Biot Modulus, $\frac{hR_o}{K}$	Fourier Modulus, $\frac{\alpha\theta}{R_o^2}$	Radius Ratio, R/R_o	Dimensionless Temperature Ratio, $T/T_{initial}$	
			Exact Solution	Recursive Solution
0.1	0.1	0	1.00088	1.00085
0.1	0.1	1	1.01352	1.012767
0.1	1.0	0	1.05234	1.05227
0.1	1.0	1	1.06588	1.06518
1.0	0.1	0	1.00798	1.00803
1.0	0.1	1	1.10515	1.10202
1.0	1.0	0	1.25021	1.25238
1.0	1.0	1	1.27989	1.28050
10.0	0.1	0	1.03331	1.03618
10.0	0.1	1	1.28946	1.28987
10.0	1.0	0	1.32881	1.32967
10.0	1.0	1	1.33278	1.33288

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13. ABSTRACT

Recursion equations are developed for solution of transient temperature distributions in an infinite hollow cylinder with nonsteady boundary conditions. The solution is shown to be applicable to any imposed boundary condition and is also shown to be able to handle the special case of the solid cylinder. A computer program is written and applied to two examples. A comparison of the numerical results with classical exact solutions reveals close agreement between the two types of solutions for the particular cases considered.

14.

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