

AFCRL-69-0249

AD693205

Reports of the Department of Geodetic Science

Report No. 119

# FURTHER STUDIES IN THE COMBINATION OF GRAVIMETRIC AND SATELLITE DATA

by

Richard H. Rapp

Prepared for

Air Force Cambridge Research Laboratories  
Office of Aerospace Research  
United States Air Force  
Bedford, Massachusetts 01730

Contract No. AF19(628)-5701

Project No. 7600

Task No. 760002,04

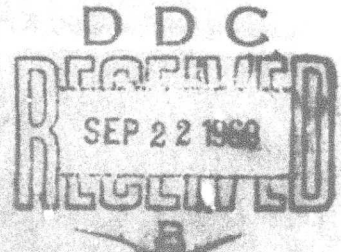
Work Unit No. 76000201, 76000401

Scientific Report No. 27

Contract Monitor: Bela Szabo  
Terrestrial Sciences Laboratory



The Ohio State University  
Research Foundation  
Columbus, Ohio 43212



December, 1968

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## ABSTRACT

This report is composed of two portions. The first is a general theoretical discussion of the problem of assuming compatibility of surface gravity data and potential coefficients derived from satellite orbital analysis. The second part of the report describes a combination of gravimetric and satellite data using  $5^{\circ} \times 5^{\circ}$  mean free air anomalies supplied by the Aeronautical Chart and Information Center, and a set of potential coefficients designated NWL 8I derived by Anderle. Several methods were applied to this data to obtain an adjusted set of potential coefficients complete to  $n = 15$  plus an additional number of higher degree resonance terms. In addition three sets of adjusted  $5^{\circ} \times 5^{\circ}$  mean free air anomalies were obtained. From these anomalies a set of potential coefficients complete to  $n = 30$  was derived by the development formulas. Analysis of the various solutions indicated that one was to be preferred over all others. Problems continue to remain with respect to proper weighting of the data.

## FOREWORD

This report was prepared by Richard H. Rapp, Associate Professor, Department of Geodetic Science, The Ohio State University, under Air Force Contract No. AF19(628)-5701, OSURF Project No. 2122. The contract covering this research is administered by the Air Force Cambridge Research Laboratories, Office of Aerospace Research, Laurence G. Hanscom Field, Bedford, Massachusetts, with Mr. Owen W. Williams and Mr. Bela Szabo, Project Scientists.

The author gratefully acknowledges the help of Mr. John M. Snowden in the computer programming involved with this study; of Mr. Gabriel T. Obenson who carried out the statistical analysis of the anomaly residuals; and Dr. Urho A. Uotila for several interesting discussions.

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## 1. Introduction

The purpose of this report is to describe a combination of satellite and gravimetric data using data supplied by the Naval Weapons Laboratory and the Aeronautical Chart and Information Center. The basic methods used in this combination study have been described in detail in two previous reports [Rapp, 1968a, 1968b].

The scope of this study is to include consideration of modern theories of gravimetric geodesy, the compatibility of spherical and spheroidal harmonics and the use of model anomalies. Future sections of this report will describe the analysis carried out in these areas.

## 2. The Basic Approaches

We define two methods for the combination of gravimetric and satellite data. The first, called method A, was proposed by Kaula (1966). In this solution a global anomaly field is established so that potential coefficients may be computed from this data and compared to satellite determined values. Functionally this may be expressed as follows:

$$(1) \quad \left\{ \begin{array}{c} \bar{C}_{n\lambda} \\ \bar{S}_{n\lambda} \end{array} \right\} - \frac{1}{4\pi(n-1)\gamma} \int \int_{\sigma} \Delta g_{\tau} \bar{P}_{n\lambda}(\sin \varphi) \begin{Bmatrix} \cos m\lambda \\ \sin m\lambda \end{Bmatrix} d\sigma = 0$$

In this expression  $\bar{C}_{n\lambda}$  and  $\bar{S}_{n\lambda}$  are potential coefficients,  $\Delta g_{\tau}$  is loosely, for now, considered to be a mean free air anomaly,  $\bar{P}_{n\lambda}$  is a fully normalized Legendre polynomial and the integration is taken over the spherical surface  $\sigma$ . An appropriate adjustment is made to obtain unique values for a set of potential coefficients and an adjusted anomaly field.

The second method, designated method B, starts from the expression for the gravitational potential  $V$  expressed as follows:

$$(2) \quad V = \frac{kM}{r} \left[ 1 + \sum_{n=2}^{\infty} \left( \frac{a}{r} \right)^n \sum_{\lambda=0}^n (\bar{C}_{n\lambda} \cos m\lambda + \bar{S}_{n\lambda} \sin m\lambda) \bar{P}_{n\lambda}(\sin \varphi) \right]$$

where  $kM$  is the geocentric gravitational constant,  $r$  is the distance from the center of the earth to the point of evaluation, and  $a$  is the equatorial radius of the earth. We may deduce from (1) the expression for a gravity anomaly on a sphere of radius  $a$  as follows:

$$(3) \quad \Delta g_H = \Delta g_0 + \gamma \sum_{n=2}^{\infty} (n-1) \sum_{\lambda=0}^n (\bar{C}_{n\lambda} \cos m\lambda + \bar{S}_{n\lambda} \sin m\lambda) \bar{P}_{n\lambda}(\sin \varphi)$$

In expression (3)  $\Delta g_0$  is the mean global anomaly with respect to a specified gravity formula, and it is understood in (3) that  $\bar{C}_{2,0}$  and  $\bar{C}_{4,0}$  have been modified by subtracting the normal values of a reference ellipsoid from the actual values of  $\bar{C}_{2,0}$  and  $\bar{C}_{4,0}$ . The model for method B is then formed by comparing the terrestrial estimate,  $\Delta g_H$ , with  $\Delta g_T$ . In general we may write:

$$(4) \quad \Delta g_T - \Delta g_H = 0$$

The adjustment is carried out so that a unique set of  $\bar{C}_{2n}$  and  $\bar{S}_{2n}$  values are determined. If a global anomaly field has been used, the adjusted anomalies may be found by imposing as conditions on this field the potential coefficients found by the adjustment carried out through equation (4).

### 3. Theoretical Considerations

The two approaches described in the previous section involve several assumptions that must be defined and their effect noted.

#### 3.1 The Anomaly at the Reference Ellipsoid

Consider first method A. The basic assumption involving the validity of equation (1) is that the reference surface is a sphere and thus the anomalies,  $\Delta g_T$ , should be referred to a spherical surface. However, in general, we must regard free air anomalies to refer to the surface of the earth. In order to obtain a consistent set of anomalies referred to a common reference surface, we may reduce all surface anomalies to mean sea level to obtain  $\Delta g_T'$ . This may be done as follows (Heiskanen and Moritz, p. 210, eq (8-66)):

$$(5) \quad \Delta g_T' = \Delta g_T - \frac{\partial \Delta g}{\partial h} h$$

where  $h$  is the height of  $\Delta g_T$  above the mean sea level. The value of the derivative required in (5) is expressed as equation (2-217) in Heiskanen and Moritz.

A more practical reduction of surface anomalies to anomalies at sea level is to write the anomaly at sea level as (Moritz, 1968, p. 41, eq (117)):

$$(6) \quad \Delta g_T' = \Delta g_T + C$$

where  $C$  is the terrain correction expressed as:

$$(7) \quad C = \frac{1}{2} k \rho R^2 \iint_{\sigma} \frac{(h - h_0)^2}{l_0^3} d\sigma$$

$h_p$  is the height of the point P at which  $\Delta g_T$  is known. In addition we have:

R mean earth radius  
 $\sigma$  unit sphere  
 $d\sigma$  element of solid angle  
k Newtonian gravitational constant  
 $\rho$  density  
 $\psi$  angular distance from P to  $d\sigma$   
 $l_o = 2R \sin \frac{\psi}{2}$

Other expressions approximating C, or for which C is an approximation are as follows:

$$(8) \quad C \approx G' = \frac{R^2}{4\pi} \iint \frac{(h-h_p)(\Delta g - \Delta g_p)}{l_o^3} d\sigma$$

which is due to Pellinen (e.g. 1966, p. 70, equation (16)). Also we have:

$$(9) \quad C \approx G_1 = \frac{R^2}{4\pi} \iint \frac{h-h_p}{l_o^3} \left( \Delta g + \frac{3G}{2R} \zeta_o \right) d\sigma$$

which is given in Heiskanen and Moritz, p. 306, equation (8-51). In equation (9), G is a mean value of gravity and  $\zeta_o$  is the zero approximation to the height anomaly.

Although we have equations (8) and (9) we continue the discussion assuming the standard terrain effect computation as expressed by equation (7). If we consider equation (6) applying to mean anomalies, a mean anomaly over an area  $d\sigma$  needed in equation (1) may be written:

$$(10) \quad \Delta g_T' = \frac{1}{A} \iint_A (\Delta g_T + C) dA$$

where A is the area corresponding to  $d\sigma$ . This value of  $\Delta g_T'$  should be used in equation (1).

The question then arises as to what happens if the terrain effect is neglected in establishing the mean anomalies? Thus for a single block we would have an error  $\epsilon_1$  defined as:

$$(11) \quad \epsilon_1 = \frac{1}{A} \iint_A (\Delta g_T + C) dA - \frac{1}{A} \iint_A \Delta g_T dA$$

or approximately:

$$(12) \quad \epsilon_1 = \frac{1}{A} \iint_A C dA$$

Equation (12) may be interpreted as the mean terrain effect in the block  $d\sigma$  or A.

Now we ask, how large is  $\epsilon_1$ ? We know individual values of  $C$  may be as much as 100 mgals in very rugged areas. However, as we increase the area under consideration, the average terrain will smooth out and consequently the larger the area, the smaller the average terrain effect will be. Tests previously carried out (Rapp, 1967) showed that for a very rugged area the mean terrain effect for a  $1^\circ \times 1^\circ$  block could be on the order of 20 mgals, but that for average topography the mean effect would only be on the order of several mgals at most. If we consider the smoothing in going to  $5^\circ \times 5^\circ$  mean anomalies, the mean terrain effect would be on the order of 5 mgals for rugged areas, reducing to negligible amounts for most areas. Such figures for  $5^\circ \times 5^\circ$  values have not been documented at this time so that further study is required. However, the procedure has been defined to yield an anomaly that will make assumption one a valid assumption. For practical purposes it is probably within accuracy limits to neglect the mean terrain effect. However, future work in the estimation of the anomalies should consider this. This will prove valuable, not only in combination solutions, but in precise evaluations of height anomalies and deflections of the vertical.

### 3.2 The Effect of the Shape of the Reference Surface

We next examine the assumption concerning the shape of the reference surface. Equation (1) applies to a spherical reference surface whereas the earth is more precisely taken in terms of an ellipsoidal reference surface. Consequently, we are applying formulas valid for a sphere to data given on an ellipsoidal surface. If no corrections are made to the given anomalies, the percentage error in the coefficients so found is on the order of  $n/f$  where  $n$  is the degree of the coefficient and  $f$  is the flattening of the ellipsoid (Ostach and Pellinen, 1966, p. 123, Pellinen, 1966, p. 69). Thus, at  $n = 10, 15, 20, 25,$  and  $30$  we have the following percentage errors: 3%, 5%, 7%, 8%, and 10%. A more precise calculation may be carried out with respect to degree and order using more detailed equations in the Ostach and Pellinen article. It thus appears that for the lower degrees ( $n \leq 30$ ) the errors caused by using a spherical reference surface are small.

The question remains, however, as to what the proper procedure is in the use of an ellipsoidal reference surface. There are several possibilities. For example we may use ellipsoidal harmonics which could be found from the gravity data. We then could transform these ellipsoidal harmonics into spherical harmonics which would be compatible with the spherical harmonic coefficients found through orbital analysis. The use of ellipsoidal (or spheroidal) harmonics has been discussed by several persons, for example: Heiskanen and Moritz (section 1-20, p. 41), Cook (1967, p. 297), and Hotine (1967).

The papers of Hotine (1967), and Ostach and Pellinen (1967) give equations that may be used to convert ellipsoidal harmonics to spherical harmonics and vice versa in regions where the spherical harmonic and ellipsoidal harmonic expansions are equally valid. To my knowledge no numerical application of these equations has been made.

### 3.3 Upward Continuation to a Minimum Sphere

An alternative to using ellipsoidal harmonics is to transform the anomalies from an ellipsoid surface to a sphere that encloses all masses of the earth. Such a sphere has been called a geosphere by Bjerhammer (1967). The problem lies in the best method for upward continuing the anomalies to the sphere, and in the exact definition of the radius of this sphere. The simplest procedure for the upward continuation appears to be to adopt the equations derived by Moritz (1966) for the downward continuation of mean gravity anomalies. We can write from Moritz (1966, p. 92):

$$(13) \quad \Delta \bar{g}_H = \Delta \bar{g}_0 - \sum_{i=0} d_i \Delta \bar{g}_i$$

where  $\Delta \bar{g}_H$  is the mean anomaly at an elevation H above the reference ellipsoid,  $\Delta \bar{g}_0$  is the mean anomaly directly below  $\bar{g}_H$ , and  $\Delta \bar{g}_i$  are anomalies surrounding and including  $\Delta \bar{g}_0$ . The value of  $d_i$  is implied from the following diagram from Moritz (1966, p. 93) where the center of the figure represents the mean anomalies positioned with respect to  $\Delta g$ . The values in the blocks are the values of  $d_i$  which are to be multiplied by  $\Delta g_i$ .

		-L		
	-L - M	-18L + 2M	-L - M	
-M	2L - 18M	38L + 38M	2L - 18M	-M
	-L - M	-18L + 2M	-L - M	
		-L		

Figure 1

Values of  $d_i$  for Equation (13)

We have:

$$(14) \quad L = \frac{H b}{32 \pi a^2} \quad \ell n \quad \frac{\sqrt{a^2 + b^2} + a}{\sqrt{a^2 + b^2} - a}$$

$$(15) \quad M = \frac{H a}{32 \pi b^2} \quad \ell n \quad \frac{\sqrt{a^2 + b^2} + b}{\sqrt{a^2 + b^2} - b}$$

In equations (14) and (15) H is the height of the elevated mean anomaly, and a and b are the linear dimensions of the blocks which are taken to be of the same size. These equations are a plane approximation and assume H is a constant for all blocks  $\Delta \bar{g}_i$ .

As previously stated the radius of the chosen sphere should enclose the entire mass of the earth. Cook (1967) has called this sphere a "minimum sphere." To determine its radius we should know the height and location in latitude of the mountain that extends farthest from the reference ellipsoid. Thus, it is not the elevation of the highest mountain that determines the radius, but a combination of the elevation

and latitude. This may be seen from the following exaggerated figure:

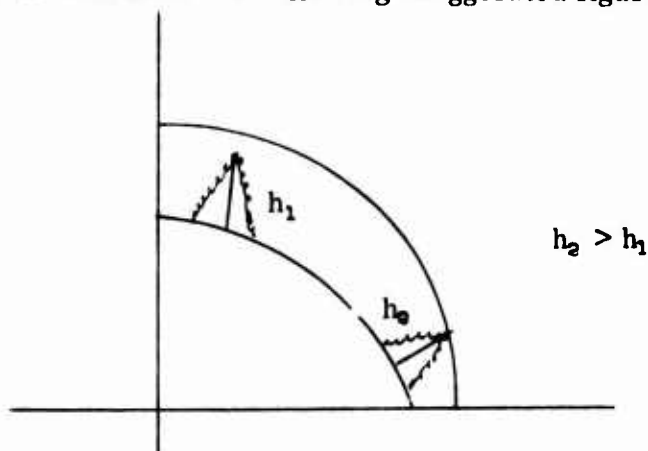


Figure 2

Radius of the Minimum Sphere

Snowden (1968) has found that an appropriate radius for the minimum sphere is 6384403 m based on an elevation of 6272 m at a latitude of  $-1^{\circ} 28'$ .

Computations were then carried out to evaluate the coefficients in Figure 1. The reference ellipsoid was taken with an equatorial radius of 6378160 m and a flattening of 1/298.25. At latitudes corresponding to the center of a  $5^{\circ} \times 5^{\circ}$  block the separation between this ellipsoid and the sphere of a defined radius were computed. This value was then used as the H required in equations (14) and (15). The linear sides of the  $5^{\circ} \times 5^{\circ}$  blocks were computed from the simplified equations:

$$(16) \quad \begin{aligned} a &= R (5^{\circ}/57.295780) \\ b &= a \cos \phi_n \end{aligned}$$

where  $\phi_n$  is mean latitude of the centermost block. We report the results for three  $d_1$  values and for two radii of the minimum sphere at three different latitudes.

Table 1  
Upward Continuation Factors

	R = 6378160m			R = 6384402m		
	$\phi$	$\phi$	$\phi$	$\phi$	$\phi$	$\phi$
	2.5°	47.5°	82.5°	2.5°	47.5°	82.5°
38L + 38M	.000	.034	.228	.015	.053	.296
-18L + 2M	-.000	-.005	.007	-.003	-.007	.009
2L - 18M	-.000	-.010	-.103	-.003	-.015	-.133

Consider the results obtained with  $R = 6384402$  m. We see the predominant effect is due to the  $(38L + 38M)$  term. If we take the anomaly corresponding to the central block on the reference surface as 40 mgals, the corrections to that anomaly to obtain the partially reduced anomaly on the sphere would be: 0.6 mgals for  $2.5^\circ$ , 2 mgals for  $47.5^\circ$ , and 12 mgals for  $82.5^\circ$ . Thus at the lower latitudes the correction is small. However at higher latitudes the corrections may be appreciable. This is due to the fact that at these higher latitudes the blocks being considered assume the shape of thin rectangles with one side being approximately 555 km long with the short side being  $555 \cos \phi$  km. The approximation made in changing a spherical trapezoid (i. e., a block bordered by meridians and parallels) to a plane rectangle causes errors in the above procedures that would be most apparent at high latitudes.

The computations with the radius of the sphere equal to the radius of the ellipsoid (6378160 m) reveals similar features as just discussed. Of course, the corrections are not the same, but they are similar.

In order to test the above procedures on an actual  $5^\circ \times 5^\circ$  field, a test field was established and upward continued to a sphere of radius 6384402 m. Excluding the polar regions the root mean square difference between the anomalies on the reference ellipsoid and those on the reference sphere were on the order of  $\pm 0.3$  mgals. We regard this difference as negligible at this time. In the polar areas, however, the difference between the two types of anomalies could reach 10 mgals, which is significant in itself. However, individual anomalies in the polar regions have a small effect on the combination solution because they are multiplied by the cosine of the mean latitude of the block (assuming meridian and parallel blocks) which is close to zero. Consequently, we can argue that it is not necessary to consider the upward continuation in the polar areas. I feel this argument is somewhat specious, but it is reasonable at this time.

In summary of this section I have outlined a procedure for the upward continuation of  $5^\circ \times 5^\circ$  mean anomalies to our reference sphere. The difference between the anomalies on the ellipsoid and those on this sphere is small in regions to the upper middle latitudes. In the higher latitudes significant effects may be noted, but the total effect on a combination solution would be expected to be small. If we went to equal area blocks, some of these problems would be reduced. However, there would still be problems where the pole is the apex of certain mean anomaly blocks. Consequently, it may be appropriate in the future to derive the upward (or downward) continuation formulas for mean anomalies in the polar regions.

In general, though, we proceed with the solutions in this paper assuming that the anomalies on the reference ellipsoid are the same as on the reference sphere. The discussion of section 3.2 indicated we may expect percentage error in our coefficients of  $nf$ .

### 3.4 Application of Sections 3.1, 3.2, 3.3 to Method B

The previous discussion has been oriented towards forming a consistency toward the gravity data and satellite determined potential coefficients with respect to method A. Similar considerations must be carried out for method B.

In going from equation (2) to equation (3) we assumed the evaluation of this anomaly to take place on a sphere of radius  $a$ . We do not evaluate equation (3), in a theoretical sense, on the surface of the earth, or on the reference ellipsoid because of the convergence problem associated with equation (2) at and near the surface of the earth. We need then to first reduce the surface free air anomalies to the ellipsoid surface, and then upward continue these anomalies to our reference sphere. Such a procedure will maintain a consistency between the terrestrial anomalies (appropriately modified) and anomalies computed from satellite determined potential coefficients. If the radius of the reference sphere is different from  $a$ , the value of  $\Delta g_0$  and of  $\gamma$  in equation (3) should be multiplied by  $(a/R)^2$ . For most choices of  $R$  this ratio is essentially one.

We thus can argue that the procedures suggested to yield a theoretical consistency for method A will also yield consistent data for method B.

### 3.5 The Effect of Finite Block Size

The solutions for potential coefficients or for coefficients of an anomaly expansion from gravity data are based on mean anomalies determined in an  $x^\circ$  by  $x^\circ$  block. We might have, for example,  $5^\circ \times 5^\circ$  or  $10^\circ \times 10^\circ$  blocks. I have used, and intend to use in this study,  $5^\circ \times 5^\circ$  values, but other authors have used  $10^\circ \times 10^\circ$  values. Noting this finite block size we can see that we cannot extend our solutions to a high degree. The usual rule of thumb is to state that you could solve for coefficients up to  $n = 180^\circ/\theta^\circ$  where  $\theta^\circ$  is the size of the block side in degrees. For example, with  $\theta = 5^\circ$ ,  $n$  would equal 36, etc.

A more precise estimate in this area is to ask what is the expected error in the coefficients of a certain degree caused by the integration errors inherent in using a finite block size. Specifically the percentage errors in the coefficients (anomaly or potential) are given by: (Pellinen, 1966, p.76, equation (21)):

$$(17) \quad 100\% (1 - \overline{P_n(\theta)}) = \% \text{ error in degree } n$$

where  $\overline{P_n(\theta)}$  is the zonal harmonic  $P_n$  averaged over the size of the block  $\theta$  used in the computation of the coefficients. Thus:

$$(18) \quad \overline{P_n(\theta)} = \frac{1}{\theta} \int_0^\theta P_n(\cos \theta) d\theta$$

I list in Table 2 values for the percentage error at various  $n$  values for  $\theta$  equal  $10^\circ$  and  $5^\circ$ .

Table 2  
Percentage Error In Harmonic Coefficients  
Due To Finite Block Size

n	$\theta$		n	$\theta$	
	$5^\circ$	$10^\circ$		$5^\circ$	$10^\circ$
5	2%	7%	20	24%	67%
10	7	25	25	34	81
15	14	25	30	45	87
18	20	59	35	56	88

At  $n = 36$  with  $\theta = 5^\circ$  we would expect a percentage error of 58% while at  $n = 18$  with  $\theta = 10^\circ$  the percentage error is 59%. Thus the rule of thumb previously mentioned would yield a fairly large uncertainty in the found coefficients. If we would accept a 25% error in our coefficients, it would appear we should go no higher in our developments than  $n = 20$  for  $5^\circ \times 5^\circ$  anomalies, and  $n = 10$  for  $10^\circ \times 10^\circ$  anomalies.

The results in this section are not meant to be final. However, they should act as a restraint in the indiscriminate development of anomalies into higher and higher degrees.

### 3.6 Summary of Section 3

We have seen in this section the various ramifications of considering modern theories of gravimetric geodesy, and the problems of the convergence of the spherical harmonic expansions in the methods for the combination of satellite and gravimetric data. The procedures are relatively simple: 1) Reduce the surface free air anomalies to the reference ellipsoid by applying the terrain effect, and 2) Upward continue these new anomalies to the minimum sphere.

In addition it has been shown that these effects are small. However, they must be considered at some time in the future. Consequently, serious consideration must be made to evaluating mean terrain effects and to the upward continuation of mean anomalies in the polar areas. Although the use of ellipsoidal harmonics may be helpful, there may still be a convergence problem near the surface of the earth. In light of the fairly straight forward procedures described for the spherical harmonic solution, it may be questionable to pursue ellipsoidal harmonics other than as a theoretical endeavor.

#### 4. The Data Used in the New Solution

##### 4.1 Gravity Data

A set of 1323  $5^\circ \times 5^\circ$  mean free air anomalies was provided for this study by the DOD Gravity Service Branch of the Aeronautical Chart and Information Center (ACIC). These anomalies were based on  $1^\circ \times 1^\circ$  terrestrial anomalies only, and do not contain any extrapolations or geophysical predictions (Hauer, 1968). Each of these anomalies had a standard error assigned by ACIC which was dependent upon the number of  $1^\circ \times 1^\circ$  anomalies in a  $5^\circ \times 5^\circ$  block and the basic accuracy of the individual  $1^\circ \times 1^\circ$  blocks used in obtaining the mean  $5^\circ \times 5^\circ$  anomaly. The number of  $5^\circ \times 5^\circ$  anomalies having the same standard error is shown in Table 3.

Table 3

Number of  $5^\circ \times 5^\circ$  Anomalies within a Given Standard Error (m)

m	number	m	number
1	44	19	2
2	52	20	1
3	37	21	63
4	58	22	7
5	42	23	1
6	73	24	3
7	63	25	0
8	67	26	4
9	69	27	0
10	49	28	93
11	83	29	0
12	54	30	4
13	59	31	0
14	62	32	0
15	200	33	0
16	55	34	0
17	9	35	0
18	68	36	1

The reason for the obvious grouping at  $m = \pm 18, \pm 21, \text{ and } \pm 28$  mgals is not clear.

In applying method A a global estimate of the anomaly field is required. In method B, a global estimate is only required if an adjusted anomaly field is to

be found. To fill out the areas in which no observations exist we may carry out predictions from the known data; use geologic information; use model anomalies, by setting the mean anomaly in the unobserved areas zero or by some other technique.

I have chosen the use of model anomalies to fill out the unobserved areas. I feel that this adds more information to the solution than would be found through statistical prediction or by setting unobserved anomalies zero. Geologic predictions could also have been used if they were available. The important question now is: What standard errors should be assigned to the model anomalies?

In order to answer this question Obenson [1968] compared the model anomalies of Uotila [1964] (appropriately referenced to the International Gravity Formula) with unclassified  $5^{\circ} \times 5^{\circ}$  mean values based on terrestrial data alone. Using 891 blocks where the estimated standard error of the terrestrial estimates varied from  $\pm 2$  to  $\pm 13$  mgals, a root mean square difference between the terrestrial anomalies and the model anomalies was  $\pm 17$  mgals. Considering the accuracy figures of the terrestrial estimates Obenson concluded that an accuracy estimate of  $\pm 19$  mgals was reasonable for the model anomalies of Uotila.

Now we note that previous computations performed using model anomalies [Rapp, 1968a] assumed a model anomaly accuracy of  $\pm 20$  mgals. In order to have some consistency with previous work, and because the difference between  $\pm 19$  mgals and  $\pm 20$  mgals is small, we shall adopt for the model anomalies a standard error of  $\pm 20$  mgals.

Next we ask: Where should we use the model anomalies? The obvious answer is: Where there was no terrestrial data. However, we are then faced with a minor inconsistency because for some of the observed anomalies we should have standard errors greater than the  $\pm 20$  mgals of the model anomalies. An alternative is to replace the terrestrial anomalies having a standard error greater than  $\pm 20$  mgals with the model anomalies. This approach can be criticized because we would have to neglect certain observed data. On the other hand assume we have based an estimate of a  $5^{\circ} \times 5^{\circ}$  anomaly on single point anomaly. Then the standard error of that estimate would be (Heiskanen and Moritz, p. 279. Table 7-4):

$\sqrt{763} \pm 28$  mgals. In this case I would still prefer to accept the model anomaly estimate than one based on a single point value.

In light of this discussion a global  $5^{\circ} \times 5^{\circ}$  anomaly field was constructed based on the terrestrial estimates of ACIC and the model anomalies of Uotila in those unestimated (by ACIC) blocks and in those blocks where the standard error estimate of ACIC was greater than 20 mgals. This global set then consisted of 2592 mean anomalies, 1147 of which were based on actual gravity observations. For a special test to be described later, an additional deck of anomalies was established based on only the ACIC supplied anomalies.

I should note that the original published model anomalies were assumed to refer to the best available gravity formula, which was taken as the WGS66 gravity

formula. These anomalies were then converted to the International Gravity Formula to which the terrestrial anomaly estimates were referred.

#### 4.2 The Satellite Determined Potential Coefficients

##### 4.21 The Selection of a Satellite Set

Anderle (1968) supplied three sets of potential coefficients for possible use in this study. These sets were designated NWL 8D, NWL 8H, and NWL 8I. Anderle indicated that the 8I set was to be preferred. These solutions were compared to the ACIC gravity data by computing anomalies at the center of  $5^\circ \times 5^\circ$  blocks with respect to a flattening of 1/298.25. These anomalies were assumed to refer to the WGS66 gravity formula. They were then converted to the International Gravity Formula for the comparisons. The equations used for this comparison were given by Kaula (1966) and by Rapp (1968a) in previous solution comparisons. We first define the following quantities:

$E((g_T - g_s)^2)$	=	mean square difference between the terrestrial anomaly and that computed from the potential coefficients
$E(g_T^2)$	=	variance of the terrestrial sample anomalies
$E(g_s^2)$	=	variance of the anomalies computed from the potential coefficients
$E(g_H^2)$	=	mean square true contribution to $g_s$ set
$E(\epsilon_T^2)$	=	mean square value of terrestrial anomaly error
$E(\delta_g^2)$	=	mean square value of neglected higher order terms in the $g_s$ set
$E(\epsilon_s^2)$	=	mean square error due to errors in the potential coefficients

The results of these computations are given in Table 4 for the three NWL sets and as a comparison, the coefficients to  $n = 14$  from Table 1 of my recent report [Rapp, 1968a]. These values were based on 691 terrestrial anomalies whose standard error was  $\pm 12$  mgals or less. The results were typical of computations made with different terrestrial sets within the basic ACIC set.

Table 4  
 Comparison of Anomalies from Potential  
 Coefficient Solutions and Terrestrial Estimates  
 (mgal<sup>2</sup>)

	NWL 8D	NWL 8H	NWL 8I	Rapp [1968 a]
$E((g_T - g_S)^2)$	246	226	222	170
$E(g_T^2)$	352	352	352	352
$E(g_S^2)$	277	255	254	186
$E(g_M^2)$	191	191	192	185
$E(\epsilon_T^2)$	58	58	58	58
$E(\delta_g^2)$	103	103	102	110
$E(\epsilon_S^2)$	85	64	62	2
$E(\delta_g^2) + E(\epsilon_S^2)$	188	168	164	111
$r_T$	0.61	0.64	0.64	0.72
$r_L$	0.69	0.75	0.76	0.99

The value of  $r_T$  and  $r_L$  are as defined in Kaula (1966). Of the three NWL solutions we can see that 8I agrees best with the terrestrial data as judged by  $E((g_T - g_S)^2)$ . However 8I is only slightly better than 8H. With respect to the errors in the coefficients, 8I again is superior to the other two NWL solutions. 8H, however, is only slightly poorer than 8I. We can conclude from this comparison that NWL 8I is the best of the NWL solutions and therefore, I have chosen this set as the basis for the satellite determined harmonic coefficients to be used in the combination study.

In passing we note that the Rapp [1968a] solution shows the best agreement with the terrestrial data and has the most accurate coefficients with respect to this gravity test. This fact is primarily due to the use of satellite and gravimetric data in establishing this set of coefficients.

#### 4.22 The Standard Errors for the Satellite Determined Potential Coefficients

We must now estimate how accurate are the potential coefficients that we have accepted to be used. This is most difficult and is the weakest link in the solution. In Kaula's (1966) combination standard errors for a mean set of coefficients

were established as one quarter of the range of the solutions used in forming the mean set of coefficients. This method is reasonable if several coefficient solutions exist that use about the same amount of data. However, it is of no help when we are attempting to evaluate the accuracy of a coefficient reported in a single solution. A second method was used by Rapp (1968b) where an external accuracy estimate of mean anomalies computed from a set of potential coefficients was related to an internal accuracy estimate of the anomalies, so that a scaling factor for the solution standard deviations could be obtained. For example, we estimated that the standard deviations should be multiplied by four to determine standard error estimates in the SAO Standard Earth Coefficients.

In the present situation we have neither a complete comparison set nor internal or external anomaly accuracy estimates. However, we do have the square root of the diagonal element of the inverse matrix of the normal equations developed in the 8I solution [Anderle, 1968]. There should be some information within this data that should provide some guidelines for standard error estimates. Examining the diagonal elements (regarded loosely as standard deviations) I noted six separate groupings based on the type of coefficients being considered. In Table 5 I give the mean value of the standard deviation for the indicated groups.

Table 5  
Grouped Standard Deviations of NWL 8I

Group	Mean Standard Deviation ( $\times 10^6$ )
1. Even zonal harmonics	.00059
2. Odd zonal harmonics	.00035
3. (n,1) harmonics, n even	.00378
4. Tesseral harmonics, (12, 12) and higher (excluding (n, 11) terms)	.00191
5. All remaining tesseral harmonics	.00064

We would expect from Table 5 that this grouping would indicate the coefficients more accurately determined than others by groups. In order to check this idea, the groups of harmonics were compared where possible to the most current realistic solutions. In these tests I would take one-half the maximum range of a group of solutions from NWL 8I and consider this as a tentative standard error estimate. Then I computed the ratio of the half range to the standard deviation for that particular estimate. These ratios were then meant to form an overall scale factor by which an individual standard deviation, by group, could be multiplied to yield a standard error estimate. Designating this ratio as K, the values obtained are given in Table 6, column 2.

Now we note that the value of K changes considerably from group to group indicating that in a general sense one group is not determined better than another. We thus thought that perhaps the scale factor K should be established for the group of coefficients having the most comparisons with other potential coefficients. Then the value of K could be assigned for the other groups on the basis of the ratio of the mean standard deviation of the master group to the mean standard deviation of the specified group. Using group 5 where K = 10 as the master group additional values of K were computed. The results are given in column three of Table 6. It can be seen that columns two and three agree fairly well. Because of this agreement, we have decided to adopt as the scale factors to convert standard deviations to standard errors the values of K given in column three of Table 6.

Table 6  
Scale Factors, K, By Groups

Group (Table 5)	K (from individual comparisons)	K (based on group 5)
1	34	32
2	65	55
3	5	5
4	40	30
5	10	10

The following data was used for comparison to NWL 8I:

- Group 1: Kozai (1968), Smith (1965)
- Group 2: Kozai (1968), King-Hele, Cook and Scott (1966)
- Group 3: SAO (1966)
- Group 4: SAO (1966), Gaposchkin and Veis (1967), Kaula (1968)
- Group 5: SAO (1966)

This weighting scheme may seem somewhat arbitrary and not sufficiently justified. However, it is only presented as an attempt to establish some feel for the coefficient accuracy. Until more precise estimates of coefficient standard errors become available, we will always be dealing with a somewhat nebulous state of affairs.

The standard errors finally computed for the coefficients of 8I solution are given in Table 1A to be found in the appendix of this report.

##### 5. How High Do We Go?

An important question is posed by the title of this section. We must put

some guidelines on the degree to which the solution is carried. As a guide we note that the NWL 8I solution is carried to  $n = 12$  in full with selected terms to  $n = 22$ . Thus we should go as high as  $n = 12$  and possibly  $n = 22$  for the direct coefficient determinations.

In order to obtain an estimate of how high we should go we pose the question in the following form: At what degree may the coefficient be hypothesized to be zero considering the accuracy of the coefficient determinations from gravity data done?

To answer this question we apply a single parameter hypothesis test as outlined in Hamilton [1964, p. 140-141]. We first establish a theoretical value of the F statistic written as  $F_{1,d}$  where  $d$  is the degrees of freedom in this solution. Using a  $d$  typical of the contemplated solutions,  $F_{1,d} = F_{\tau} = 3.9$ , the computed values would be:

$$(19) \quad F_c = \frac{x_i}{m_i^2}$$

where  $x_i$  is the value of the parameter and  $m_i$  is the standard error of its determination. If  $F_c < F_{\tau}$ , the hypothesis that the coefficients are zero is accepted. We would then say that at this  $n$  (or the previous  $n$ ) we should go no higher.

The value of  $x_i$  is estimated as the root mean square variation of the  $\bar{C}_{nn}$ ,  $\bar{S}_{nn}$  coefficients.

$$(20) \quad x_i \approx \frac{10 \cdot 10^{-6}}{n^2}$$

For the standard error,  $m$ , we use the value estimated by Rapp (1968c) for coefficients found from gravimetric data. This is:

$$(21) \quad m_i = \frac{0.338 \cdot 10^{-6}}{(n-1)}$$

Inserting (20) and (21) into (19) we have:

$$(22) \quad F_c = \frac{1}{(0.338(n+1))^2}$$

Then we ask at what  $n$  is  $F_c < 3.9$ . This occurs at  $n = 15$  which would indicate we should carry the general solutions to  $n = 14$ .

On the other hand this  $n$  value is close to  $n = 15$  adopted by NASA (1967) as a goal to achieve in their satellite programs. We thus can argue that we might as well extend the direct solution to  $n = 15$  so that it will be compatible to NASA's goal. In addition to the complete coefficient solution to  $n = 15$ , we will

also solve in the direct solution for the special resonance coefficients that appeared in the NWL 8I solution. We thus have agreed to solve for the following coefficients in a direct solution: complete to  $n = 15$ , (16,0), (16,12), (16,13), (16,14), (16,15), (17,12), (17,13), (17,14), (18,13), (18,14), (19,0), (19,13), (19,14), (20,13), (21,13), (22,13). In addition we will include, where appropriate, conditions enforcing the mean adjusted global anomaly to be a certain value, and other conditions as the forbidden coefficients.

In the previous paragraphs I have used the word direct solution to indicate that solution made a part of the general least squares adjustment. After this adjustment is completed we could use the development formulas applied to the adjusted anomalies to obtain a set of coefficients, consistent with the specifically adjusted coefficients, but extended to a higher degree. We shall do this where appropriate. Based on the figures given in Table 2, we somewhat arbitrarily take  $n = 30$  to be that degree to which the coefficients will be found by the summation process.

## 6. Outline of Proposed Solutions

We now describe the solutions to be made and analyzed for this report. The results of these computations will be described in a later section.

### 6.1 Solution A

The first solution will impose as constraints on the anomaly field the NWL 8I coefficient. We thus use equation (1) as a set of condition equations. Besides the 217 8I coefficients we will force the (1,0), (1,1) and (2,1) terms in the anomaly field to be zero. However the mean global anomaly  $\Delta g_0$  will be set to be 1.4 mgals. This value has been computed using equation (12) of Snowden and Rapp (1968) based on the best gravity formula being that of WGS 66 and the referenced gravity formula to be the International Gravity Formula. In total, then, there will be a total of 223 conditions imposed on 2592 mean free air gravity anomalies.

The main purpose of this solution is to obtain a set of adjusted  $5^\circ \times 5^\circ$  anomalies that are completely compatible with the 8I set of coefficients. In addition we may apply the usual summation formulas (e.g. equation (1)) to the adjusted anomaly field to obtain estimates of the behavior of the higher order coefficients or more specifically those coefficients not present in the 8I solution.

The number of degrees of freedom of this solution is taken to be the total number of condition equations used in the solution. Thus the number of degrees of freedom is (223).

### 6.2 Solution B

The second solution is the general least squares solution according to

method B as outlined near equation (2), using the 2592 mean anomalies described in section 4.1 and the 8I coefficients. All weighting was done according to the standard errors obtained through the procedures previously described. The solution was made for all coefficients to  $n = 15$  (excluding  $\bar{C}_{21}$  and  $\bar{S}_{21}$ ) plus assorted coefficients as outlined in section 5. There were thus a total of 280 potential coefficients being sought. Of this number 217 had a priori values and weights of the NWL 8I solution. The remaining 63 coefficients were a priori estimated to be zero with an infinite standard error. In other words no a priori weight was attached to the zero estimate for the completely unknown potential coefficients. An alternative to this is to attach a standard error to the zero estimate based on the RMS coefficient variation as expressed, for example, by equation (20). Although I have done this previously [Rapp, 1968a], I felt that the inconsistencies which could arise in such an assignment were undesirable. These inconsistencies occur when the standard errors actually calculated by the procedures described in section 4.22 are larger than those based on the RMS coefficient variation.

After the main solution for the coefficients is completed the adjusted  $5^\circ \times 5^\circ$  anomalies were obtained by forcing these coefficients on the original estimate of the 2592 mean anomalies. This was done in a procedure similar to that described in section 6.1. Thus in this case we have 286 condition equations. (280 from the coefficients of Solution B; 5 from forcing the anomaly coefficients (1,0), (1,1), and (2,1) to be zero, and 1 from forcing the mean anomaly to be the value specified in section 6.1.

After the adjusted anomalies are obtained they can be developed to  $n = 30$  for further analysis of the coefficients.

The calculation of the degrees of freedom for this solution is not so clear cut as in section 6.1. We first total the number of observations. This will include 2592 anomaly estimates and 217 coefficients for a total of 2809 observations. The total number of unknown potential coefficients in the solution is 280. Thus the excess number of observations is 2809 minus 280 which equals 2529. The degrees of freedom will be regarded as 2529 for the original adjustment where the coefficients are found to  $n = 15$  plus additional coefficients.

### 6.3 Solution C

The third solution is the general least squares solution according to method B as outlined near equation (2), using the 1323 ACIC determined mean anomalies. Thus in this solution no model anomaly data is included. The main purpose of this solution is to ascertain if the exclusion of the model anomalies appreciably changes this type of solution over the case when they are included. The analysis of the difference between solution B and C will be of interest in this regards.

The coefficients to be solved for directly (i. e. in the least squares adjustment) will correspond exactly to the direct solution described for method B.

Since we do not have a complete global anomaly field as a starting point for this solution it is not possible to obtain an adjusted set of  $5^\circ \times 5^\circ$  anomalies by imposing the coefficient solution on a complete global field. Of course, a set of anomalies based on the adjusted coefficients could be computed.

The calculation of the degrees of freedom for this solution is similar to that of section 6.2 except now the number of observed anomalies is 1323. Thus the number of degrees of freedom is 1260.

#### 6.4 Solution D

The fourth solution made for this report is that associated with method A described near equation (1). This solution has the same input data as solution B of section 6.2. In addition to the 280 coefficients described in solution B, we have the additional unknowns of (0,0), (1,0), (1,1), and (2,1). The (0,0) term was forced to be 1.4 mgals (see section 6.1) by using an a priori standard error of 0.01 mgals. The (1,0) and (1,1) coefficients were a priori estimated as 0.0 with a standard error of  $\pm 0.01$  mgals for the estimate of 0.0. There were thus a total of 286 unknowns for this solution.

A direct result of the method applied in this solution is a set of adjusted  $5^\circ \times 5^\circ$  mean anomalies. These anomalies may be developed into potential coefficients to  $n = 30$  as previously described.

The number of degrees of freedom in this solution is not clear. If we regard it as the number of equations (of condition) used we will have 286. On the other hand, if we assume that there were no a priori coefficient estimates the number of degrees of freedom would be zero since a global  $5^\circ \times 5^\circ$  field is required to define a single coefficient through the summation formula. On the other hand we do have estimates of 223 coefficients (217 of NWL 8I plus the six special coefficients) which may be regarded as excess information. With this somewhat weak argument we will regard the degrees of freedom of solution D as 223.

### 7. Results from the Computations

The four solutions described in section 6 have been made on the IBM 360/75 using computer programs similar to that described by Snowden and Rapp (1968). The main difference between the IBM 7094 programs of Snowden and Rapp and the 360/75 programs is that the latter allow the solution of potential coefficients directly to (20,20) or approximately 441 unknowns.

#### 7.1 The Potential Coefficients

The potential coefficients from the four solutions are given in a single table that may be found in the appendix (Table 2A). In this table we list the potential

coefficient and where that coefficient has been the result of a direct solution the square root of the diagonal element corresponding to that coefficient in the inverse matrix. This latter value, when multiplied by the standard error of unit weight will give the standard deviation of that coefficient. Since we have assigned weights with an assumed standard error of unit weight equal to one, the inverse weight elements are an estimate of the standard deviation of the coefficients.

When no inverse weight element is given after a solution, we have made no direct solution for that coefficient. The value of such a coefficient is then the result of applying the summation formula to the adjusted  $5^\circ \times 5^\circ$  anomalies of that solution.

A comparison of these solutions will be made in later sections of the report. However a few comments are in order here. First I give in Table 7 the coefficients receiving largest corrections found in the three last solutions (i. e. B, C, and D). These are indicated as first, the largest corrections to the 8I coefficients, and second the largest corrections to a coefficient assumed zero.

Table 7

Coefficients Receiving the Largest Corrections to Assumed Coefficients

	Solution B	Solution C	Solution D
To 8I Set	$\bar{S}_{8,1}$	$\bar{S}_{8,2}$	$\bar{S}_{8,2}$
To Zero Estimate	$\bar{S}_{15,2}$	$\bar{C}_{13,2}$	$\bar{C}_{14,0}$

We need to note here that the corrections to the assumed coefficients are highly sensitive to the weights assigned to these coefficients. This is especially true when a coefficient is being given a high weight relative to other coefficients. For this reason we have obtained some consistency among the three solutions in that the largest correction to the 8I set is  $\bar{S}_{8,1}$  or  $\bar{S}_{8,2}$ . The main reason for this, I suspect, is that these coefficients have been given a standard error higher than most of the coefficients in the 8I set.

We also note some large coefficients in solutions B and C. For example, in solution B the coefficients  $\bar{C}_{15,8}$ ,  $\bar{C}_{15,8}$ ,  $\bar{S}_{15,2}$ ,  $\bar{S}_{15,3}$ , and  $\bar{S}_{15,8}$  are approximately three times what we might expect from the rule of thumb expressed in equation (20). Large coefficients are also found in solution C for degrees 13, 14, and 15. Among the large coefficients in this set are  $\bar{C}_{13,2}$ ,  $\bar{C}_{15,8}$ ,  $\bar{C}_{15,8}$ ,  $\bar{S}_{14,9}$ ,  $\bar{S}_{15,2}$ , and  $\bar{S}_{15,8}$ . No large values of a similar nature were found in solution D.

The question is: are these large coefficients real or simply the product of an inadequate mathematical model. This point will be discussed after further comparisons have been made.

### 7.11 Potential Coefficient Comparisons

One way to compare the various solutions is to compute the root mean square (RMS) differences between quantities pertinent to this solution. For this section we give the RMS differences between the coefficients of each solution, and the RMS undulation difference. These results compare only the common coefficients of the direct solution (i.e. to  $n = 15$  plus additional coefficients). Table 8 presents this data for the four solutions previously outlined and the WGS 66 set of coefficients carried to  $n = 8$ .

Table 8

RMS Coefficient Difference ( $\times 10^6$ ) and RMS Undulation Difference (m)  
for the Direct Solution Coefficients

	Solution B	Solution C	Solution D	WGS 66 (to $n = 8$ )
NWL 81	$\pm 0.027$ $\pm 2.6m$	$\pm 0.028$ $\pm 2.6m$	$\pm 0.021$ $\pm 2.0m$	$\pm 0.126$ $\pm 7.0m$
Solution A	$\pm 0.033$ $\pm 3.5m$	$\pm 0.044$ $\pm 4.7m$	$\pm 0.022$ $\pm 2.3m$	$\pm 0.126$
Solution B		$\pm 0.022$ $\pm 2.4m$	$\pm 0.026$ $\pm 2.7m$	$\pm 0.115$
Solution C			$\pm 0.042$ $\pm 4.5m$	$\pm 0.178$ $\pm 6.5m$

We can see that Solution D agrees best with 81 coefficients. This can be also interpreted in the form that solution D yielded the smallest corrections to the 81 set. The differences between solutions B, C, D are on the order of  $(\pm 0.02$  to  $\pm 0.04) \times 10^{-6}$  in the potential coefficients and  $\pm 2$  to  $\pm 5$  meters in geoid heights.

We have three sets of coefficients generated to  $n = 30$ . These three sets are compared in terms of RMS coefficient and undulation differences in Table 9.

Table 9

RMS Coefficient Difference ( $\times 10^6$ ) and RMS Undulation Differences (m)  
for the Solutions Carried to  $n = 30$

	Solution A	Solution B
Solution D	$\pm 0.012$ $\pm 2.4m$	$\pm 0.015$ $\pm 3.0m$

### 7.2 The $5^\circ \times 5^\circ$ Anomaly Fields

There are three  $5^\circ \times 5^\circ$  anomaly fields that have been generated in the solutions under study. The first is the set found by enforcing the NWL 8I set on the estimated  $5^\circ \times 5^\circ$  anomaly field (Solution A). The second is the set formed by enforcing the coefficients to  $n = 15$  plus assorted coefficients found in Solution B. The third set of  $5^\circ \times 5^\circ$  anomalies is inherent in Solution D.

These three sets of anomalies, referred to the International Gravity Formula may be found in the appendix to this report.

We should note that it is possible to form a set of  $5^\circ \times 5^\circ$  anomalies from the potential coefficients of the 8I solution, or solution B or D. These anomalies will not be the same as those described in the first paragraph of this section. The anomalies computed directly from the potential coefficients will be considerably smoother than the adjusted anomalies of the first paragraph.

We may compare these sets of anomalies in several ways. In Table 10 we give the largest absolute difference (upper figure) and the root mean square difference between the three sets of  $5^\circ \times 5^\circ$  anomalies.

Table 10

Largest Difference and RMS Difference Between  $5^\circ \times 5^\circ$   
Anomaly Fields (mgals)

	Solution A	Solution D
Solution B	35 $\pm 7.6$	37 $\pm 6.5$
Solution D	24 $\pm 4.4$	-

We see that there can be a considerable difference between the three

solutions. However these large differences do not occur very often as could be implied by the relatively small RMS differences.

Recently Bouguer anomaly maps have been published (Department of Energy, Mines, and Resources, 1968) which enabled us to estimate 19  $5^\circ \times 5^\circ$  anomalies that were not used as observed data in the combination solutions. It was appropriate to compare the adjusted anomalies of solutions A, B, and D, and the model anomaly estimates for these blocks to the estimates based on actual data. This is in effect a check on the "predicted" anomalies of solutions A, B, and D. The root mean square differences between the new estimates and the four other anomaly sets are given in Table 11.

Table 11  
RMS Difference of New  $5^\circ \times 5^\circ$  Anomalies  
and Predicted Anomalies

	RMS Difference (mgal)
Model Anomalies	$\pm 24.8$
Solution A	$\pm 14.8$
Solution B	$\pm 17.8$
Solution D	$\pm 14.2$

The RMS value of the observed anomalies was  $\pm 21$  mgals. It is apparent that solution D has yielded the best predictions closely followed by solution A. No consideration was given to establishing if there was a "significant" difference between the solutions. Although this sample is too small to form a firm general conclusion, it would appear that solution D is somewhat better than solution B.

### 7.3 Geoid Undulations

We may compute the geoid undulations from these solutions in three ways: 1) from the direct solution coefficients (i. e. from coefficients to  $n = 15$  plus additional coefficients); 2) from the coefficients to  $n = 30$  of solution A, B, and D; and 3) from the adjusted  $5^\circ \times 5^\circ$  anomalies of solutions A, B, and D.

The third way, although of interest, is somewhat difficult numerically because of the large size of the blocks and the fast variation of the kernel in the Stokes' equation. Errors of several meters might result unless special precautions were taken.

The second way is possible but it will not yield a geoid that is as representative of the direct solution coefficients. However we shall see later that the

contribution of the higher order terms to the geoid undulations are small.

We are then left with the representation of the geoid undulations with the coefficients found in solutions B, C, and D. Of primary interest, however, is solutions B and D since solution C did not use model anomalies. We therefore present in the Appendix the geoid undulations with respect to a reference ellipsoid of flattening 1/298.25 for solutions B and D computed from those coefficients given in the appendix (Table 2A) which were the result of the direct solution.

Examination of these geoids shows only minor differences. We know, in fact, that the RMS difference between the two geoids is only  $\pm 2.7$  m (Table 8). If we would examine the undulations implied by the NWL 8I coefficient set, we would find little noticeable difference also (RMS difference equal to  $\pm 2.6$  m for solution B and  $\pm 2.0$  m for solution D). The reason for the geoid agreements stems from the fact that the contributions to the undulations comes predominately from the lower order coefficients (say to  $n = 8$ ). Since these coefficients have, for the most part, received relatively high weight which forces small corrections to these coefficients, we do not expect the lower order coefficients to change appreciably between solutions. Thus it is natural that the geoids are very similar.

An alternate argument relative to the agreement of the geoids is that we have weighted the lower order coefficients too highly. They thus have not been able to sufficiently adjust to the gravity data which may cause some geoid changes. The particular problem of coefficient weighting will be discussed later.

## 8. Analysis of the Results

### 8.1 Analysis of the Anomaly Residuals

We may define two types of anomaly residuals that occur in our present solutions. The first type is associated with method A defined near equation (1). The residuals in this method are computed according to equation (22) found in Rapp [1968a]:

$$(23) \quad V_{\ell} = P_{\ell}^{-1} B'_{\ell} P_x V_x$$

where  $P_{\ell}$  is the weight matrix of the observations,  $B_{\ell}$  is a matrix composed of the partial derivatives of the mathematical model (equation (1)) with respect to the gravity anomalies,  $P_x$  is the weight matrix for the input potential coefficients and  $V_x$  are the corrections to the input coefficients. Equation (23) is the form of the equation used in forming an adjusted set of  $5^{\circ} \times 5^{\circ}$  anomalies in solution A, B (i.e. fixing the coefficients found in the direct solution), and D.

The second type of residual is associated with solutions B and C. This residual is simply the difference between the anomaly computed from the directly

adjusted potential coefficients and the original anomaly estimate.

It is of interest to investigate the statistical behaviour of these residuals or more specifically the behaviour of the statistic  $\sqrt{p_i} v_i$  where  $p_i$  is the weight of the input anomaly  $i$ . Since  $p_i = 1/m_i^2$  we will investigate the behaviour of  $v_i/m_i$ . We cannot examine the 2592 residuals as one group because the size of the blocks changes with respect to latitude. Instead we propose to analyze the residuals of blocks of the same size (i.e. located at the same absolute latitude). Since there are 72 blocks in one latitude strip around the earth, there will be a total of 144 residuals for each group.

The basic procedure is to compute 144 values of  $v_i/m_i$  and separate them into classes. Knowing the mean value of this statistic and the RMS value we can calculate the theoretical number of residuals that would be expected in each class. We can then apply the "goodness-of-fit" test described in Hamilton (1964) to judge the distribution of these residuals. This test uses the evaluation of the following equation which yields the computed

$$(24) \quad \chi^2 = \sum_{k=1}^n \frac{(n_k - d_k)^2}{d_k}$$

where  $n$  is the number of classes,  $n_k$  is the number of actual residuals in the sample, and  $d_k$  is the theoretical number of residuals in the interval. Provided  $d_k$  is at least 5, the value found from equation (24) should be distributed as  $\chi^2_{n-1}$  (since the parameters of the theoretical values have been determined from the residual data).

A histogram and theoretical distribution curve for a set of residuals at latitudes  $5^\circ$  and  $-5^\circ$  is shown in Figure 3. The upper figure refers to the type 1 residual found from solution D, while the lower figure refers to the type 2 residual of solution B. We can see that the upper figure has a somewhat skewed shape with a strong peak at the center. The lower figure is more uniformly spread out with a lower peak than the upper figure. For the upper figure (type 1 residual, solution D) the computed  $\chi^2$  was 37.0 compared to a theoretical value of 25.2 with 10 degrees of freedom at  $\alpha = 0.005$ . Since the computed value is greater than the theoretical  $\chi^2$ , we conclude that these residuals are not normally distributed. The analysis for this type residual was carried out at latitude  $40^\circ$  and  $90^\circ$ . At  $\phi = 40^\circ$ ,  $\chi^2$  computed was 88.2 with the theoretical  $\chi^2_{3, .005} = 12.8$ ; at  $\phi = 90^\circ$ ,  $\chi^2$  computed was 152.3 while  $\chi^2_{3, .005} = 12.8$ . Both cases indicate a non-normal residual distribution. In fact, at certain latitudes, there were no negative residuals indicating that the normal distribution is not applicable to this residual.

Using the data plotted in the lower part of Figure 1 (residual type 2), the computed  $\chi^2$  was 19.8 while  $\chi^2_{3, .005} = 23.6$ . Thus this residual group is

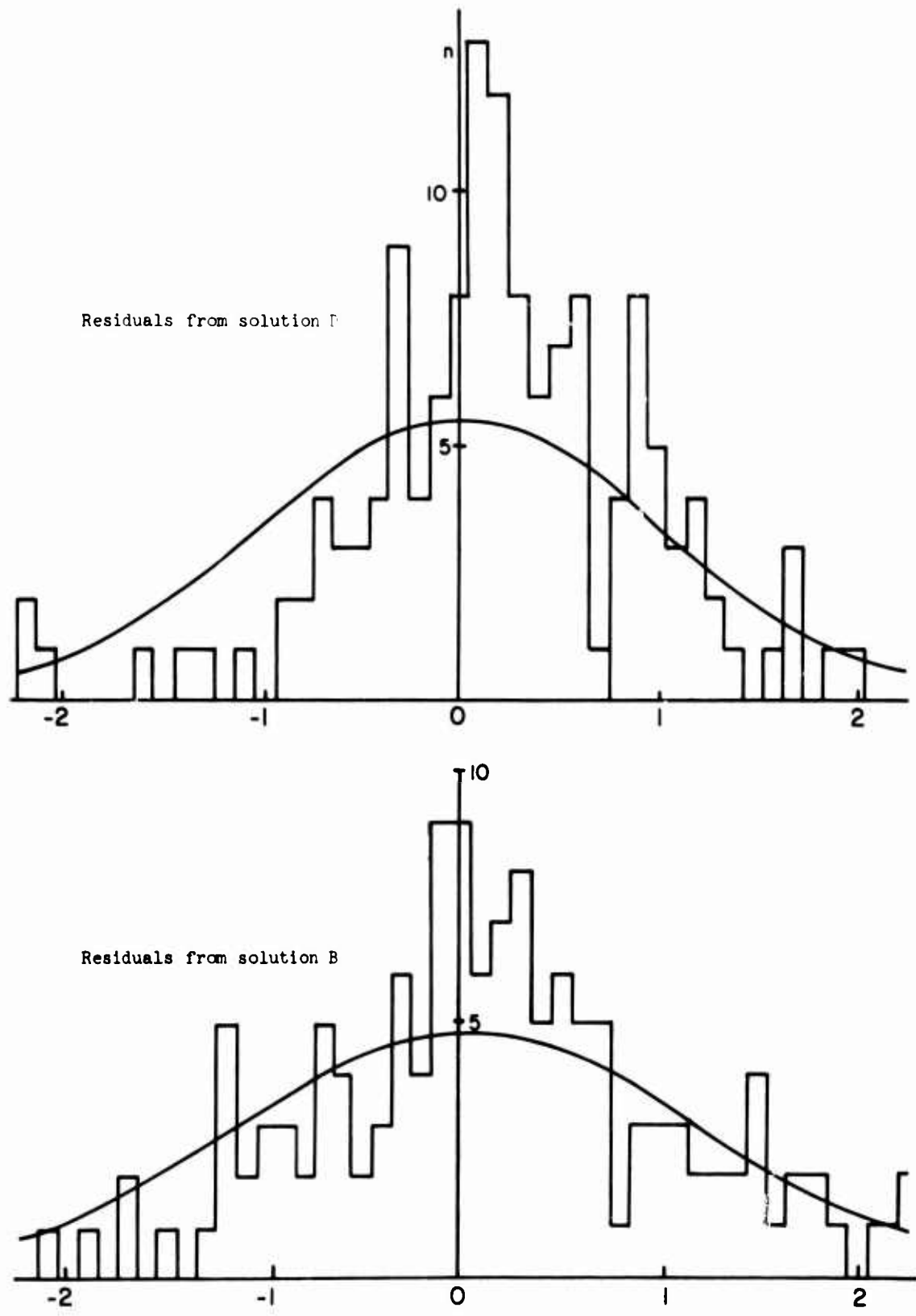


Figure 3

Histogram and Theoretical Distribution of Anomaly Residuals at Latitude  $5^{\circ}$  (U)

indicative of a normal distribution. Similar conclusions were formed at latitude  $40^\circ$  and latitude  $60^\circ$ . At  $90^\circ$ , however, there were indications of non-normality.

We thus see that generally the residuals of type 1 are not normally distributed. The reason for this is probably due to the fact that the direct solution coefficients have not been carried to a sufficiently high degree to represent the variations to be expected of  $5^\circ \times 5^\circ$  mean anomalies. The second type of residuals include the effects of errors in the original anomaly estimates, and by the error caused by the truncation of the coefficient set at a certain degree. If this truncation were causing a bias in Solution B, we might expect to see its effect in some erratic behaviour of the residuals. We might conclude that the effect of higher order terms is acting randomly so that the lower order coefficients will not be appreciably changed if the higher coefficients had been allowed in the solution. This statement is not justified at this time.

## 8.2 Comparison of the Generated Anomaly Field with the Terrestrial Data

From the adjusted coefficients of the direct solution, we can compute a set of  $5^\circ \times 5^\circ$  anomalies that may in turn be compared to the terrestrial anomalies used as input. We then use the same procedures described in section 4.21 to ascertain information on the accuracies of the various solutions. This procedure was originally applied by Kaula (1966) and later by Rapp (1968b) to their combination solutions. Numerical values for the quantities described in section 4.21 as applied to the data of solutions A, B, C, D and 8I (repeated from Table 4) are given in Table 12. The quantities are given for two data samples. The first when comparisons were made with 691 known anomalies having a standard error less than or equal to  $\pm 12$  mgals while the second set consisted of 436 anomalies having a standard error less than or equal to  $\pm 8$  mgals.

We can note from this table the following:

1. Solution A shows a slightly better agreement with the terrestrial field than does the original 8I set as judged by  $E((g_T - g_S)^2)$ . As would be expected, solution A has a higher  $E(g_M^2)$  than does 8I, and a smaller  $E(\delta_g^2)$ . However the accuracy of the coefficients as judged by  $E(\epsilon_s^2)$  is essentially the same in solution A and 8I set.
2. Solution C is slightly better than solution B as judged by  $E((g_T - g_S)^2)$  but essentially no different when judged by  $E(\epsilon_s^2)$ . The better agreement from  $E((g_T - g_S)^2)$  stems from the use of only terrestrial data in solution C. Thus the solution tends to adjust to the anomalies that are being used in the comparison test better than solution B which has to fit model anomaly data and terrestrial estimates at the same time.
3. Solutions B and C agree better with the terrestrial field as judged by  $E((g_T - g_S)^2)$  than solution D or A. Solutions B and C have a higher  $E(g_M^2)$  than solutions A

Table 12

## Comparisons of Generated Anomalies to Terrestrial Data

Number of blocks = 691,  $m \leq \pm 12$  mgals

	Solution A	Solution B	Solution C	Solution D	NWL 81
$E((g_T - g_s)^2)$	209	167	164	179	222
$E(g_T^2)$	352	352	352	352	352
$E(g_s^2)$	268	258	262	230	254
$E(g_H^2)$	206	222	225	202	192
$E(\epsilon_T^2)$	58	58	58	58	58
$E(\delta g^2)$	88	72	69	92	102
$E(\epsilon_s^2)$	63	36	37	28	62
$E(\delta g^2) + E(\epsilon_s^2)$	151	109	106	121	164
$r_T$	0.67	0.74	0.74	0.71	0.64
$r_L$	0.77	0.86	0.86	0.88	0.76

Number of blocks = 436,  $m \leq \pm 8$  mgals

$E((g_T - g_s)^2)$	176	124	120	152	193
$E(g_T^2)$	359	359	359	359	359
$E(g_s^2)$	255	244	246	221	239
$E(g_H^2)$	219	240	242	214	202
$E(\epsilon_T^2)$	29	29	29	29	29
$E(\delta g^2)$	111	90	87	115	127
$E(\epsilon_s^2)$	36	5	4	7	37
$E(\delta g^2) + E(\epsilon_s^2)$	147	95	91	122	164
$r_T$	0.72	0.81	0.82	0.76	0.69
$r_L$	0.86	0.98	0.99	0.97	0.85

or D and a smaller neglect of higher order terms ( $E(\delta_g^2)$ ) than A or D.

4. These comparison tests were also made for the following groupings of the terrestrial estimates: the standard error of the anomaly less than or equal to: 15, 12, 11, 10, 9, 8, and 7. The results indicated in Table 12 were generally found to be the same in the extended set. However, the value of  $E(\epsilon_s^2)$  did fluctuate. To form a judgment with respect to this value, the mean  $E(\epsilon_s^2)$  was computed for seven comparison sets for each solution given in Table 12. We have the following for  $E(\epsilon_s^2)$ : solution A, 52, solution B, 23, solution C, 24, solution D, 20, solution 8I, 52 (all values in  $\text{mgal}^2$ ). Thus the coefficients of solution D are judged to be slightly more accurate than other solution presented. Whether the differences found are significant is not clear at this time. We shall also recall that although  $E(\epsilon_s^2)$  is smaller for solution D, the value of  $E((g_r - g_s)^2)$  is smallest for solutions B or C.
5. Generally it has paid to make solutions B, C, and D as opposed to simply fixing the 8I set. This is inferred from the smaller value of  $E((g_r - g_s)^2)$  and  $E(\epsilon_s^2)$  for solutions B, C, and D as opposed to solution A. It could be that improvement of solutions B, C, and D over A is not significant so that for orbital prediction solution A might be preferred. This feature will be discussed in a later part of this report.

### 8.3 Standard Error of Unit Weight

The standard error of unit weight,  $m_0$ , or the variance of unit weight may be estimated from the following:

$$(24) \quad m_0^2 = \sigma^2 = \frac{V'PV}{\text{d. f.}}$$

where d. f. indicates the degrees of freedom in the solution. Knowledge of  $m_0$  is important since it is needed for the evaluation of the standard error of the adjusted potential coefficients. We have:

$$(25) \quad m \left\{ \begin{array}{c} \bar{c} \\ \bar{s} \end{array} \right\} = m_0 \sqrt{\bar{p}} \left\{ \begin{array}{c} \bar{c} \\ \bar{s} \end{array} \right\}$$

where  $\bar{p}$  is the diagonal element of the inverse normal equations. The value of  $\sqrt{\bar{p}}$  has been given in Table 2A for the coefficients of the direct solution.

The value of  $V'PV$  is equal to the sum of  $V'_\ell P_\ell V_\ell$  (i. e. the contribution from the anomaly corrections) and  $V'_x P_x V_x$  (the contribution from the corrections to the input coefficients). There are two types of residuals that arise in the computations of  $V'_\ell P_\ell V_\ell$  for solution B. In the basic solution for the potential coefficients we have a type two residual (see section 8.1) for which  $V'_\ell P_\ell V_\ell$  is to be

added to  $V' P V$ . Let us call this formulation Solution B<sub>2</sub>. When we fix the coefficients found in the basic solution B, we obtain residuals of type one again for which  $V' P V$  is to be computed. We call this computation Solution B<sub>1</sub>. The results for these computations as well as those for the other solutions are given in Table 12.

Table 12

	V' PV and the Variance of Unit Weight				
	Solution A	Solution B <sub>1</sub>	Solution B <sub>2</sub>	Solution C	Solution D
$V' P V$	1297	1154	3952	3450	833
$V' P V$	-	-	292	290	207
V'PV	1297	1154	4244	3740	1040
d. f.	223	286	2529	1260	223
m <sub>0</sub>	±2.41	±2.01	±1.35	±1.72	±2.16

Since we have estimated our weights of the input coefficients and observed anomalies as  $1/m_1^2$ , we would expect m<sub>0</sub> to be reasonably close to one. However from the last row in Table 12 we see that no actual m<sub>0</sub> is significantly close to one (This may be shown by establishing confidence intervals for the variance. (Hamilton, 1964, p. 81)).

The question is why does the estimate of m<sub>0</sub> disagree with the a priori estimate? For solutions A, B, and D we may have estimated the degrees of freedom improperly. However it would take a large increase in the number of degrees of freedom to reduce m<sub>0</sub> to be near one. A second reason could stem from the non-normal behaviour of the anomaly residuals in solution D (and A, and B.). A third reason could stem from improper assignment of the original weights (or standard errors) of the input data. Although I feel within each data type consistency has been maintained (of course it may not have been), there is no guarantee that between data types we are consistent. Perhaps we have weighted the potential coefficients too high with respect to the weighting for the gravity data which in turn causes an inconsistency in the value of m<sub>0</sub>. It perhaps would be of interest to re-run solution B, at least, with a set of weights for the potential scaled by a factor greater than one from the original weight. For example, we might try a scale factor of 1.5. We then could examine the value of m<sub>0</sub> along with other factors to see if the adjustment is more statistically reliable.

We still have the question to answer as to what m<sub>0</sub> should be used in evaluating equation (25) for the current solutions. We know that the standard errors from method B and method D should be approximately the same if m<sub>0</sub> is the same for each

method. This is borne out by the general agreement of the  $\sqrt{\bar{p}}$  given in Table 2A. It would thus appear to be undesirable to adopt different  $m_0$ 's for each solution (i. e. B or D). For an optimistic estimate of the standard errors we could choose  $m_0 = 1.00$ , the a priori value, while a somewhat pessimistic, but perhaps more realistic  $m_0$ , would be,  $m_0 = \pm 1.35$  found in solution B.

For the  $m_0 = 1.00$  choice the values of  $\sqrt{\bar{p}}$  in Table 2A may be interpreted directly as standard errors. For  $m_0 = 1.35$  we must multiply  $\sqrt{\bar{p}}$  by this quantity to obtain the estimated standard errors.

#### 8.4 Anomaly Degree Variances

The anomaly degree variances may be computed from the following expression:

$$(26) \quad \sigma_{n, \Delta g}^2 = \gamma^2 (n-1)^2 \sum_m (\bar{C}_{nm}^2 + \bar{S}_{nm}^2)$$

where  $\gamma$  is a mean value of gravity and  $\bar{C}_{2,0}$  and  $\bar{C}_{4,0}$  are referred to a reference flattening. Anomaly degree variances can be used to measure the anomaly content of each degree of the spherical harmonic expansion of the earth's gravitational potential. In addition, the anomaly degree variances may be computed directly from gravity anomaly data through the use of the anomaly covariances. We have:

$$(27) \quad \sigma_{n, \Delta g}^2 = \frac{2n+1}{2} \int_{\psi=0}^{\pi} C(\psi) P_n(\cos \psi) \sin \psi d\psi$$

In equation (27),  $C(\psi)$  is the covariance between anomaly blocks having a spherical separation of  $\psi$ . We have computed the anomaly degree variances for solutions A, B, C, and D from equation (26) and have used long range autocovariances of  $5^\circ \times 5^\circ$  mean free air gravity anomalies computed by the Aeronautical Chart and Information Center in equation (27). The results, with respect to a reference flattening of  $1/298.25$  are given in Table 13.

These values have not been corrected for the smoothing to be expected in the potential coefficients due to the finite block size in which the gravity data has been estimated. Because of this smoothing the given degree variances are slightly smaller than what we might actually have. Further discussion of this general point may be found in Pellinen [1966].

We notice for the lower degrees a good agreement within solutions A through D. This is due to the relatively high weighting of the 8I solution for the lower  $n$ . We note that at  $n = 4$  and  $5$  the degree variances from gravimetric data alone (equation (27)) are about one half that indicated from the other solutions. An even larger discrepancy is indicated at  $n = 7$  where equation (27) has resulted in  $7 \text{ mgals}^2$  while the other solutions yield approximately  $20 \text{ mgals}^2$ . These discrepancies are not

Table 13

Anomaly Degree Variances  
(mgal<sup>2</sup>)

n	Solution A	Solution B	Solution C	Solution D	By Equation (27)
2	7.7	7.7	7.7	7.7	10.9
3	34.7	34.7	34.7	34.6	32.9
4	20.1	19.8	19.9	19.8	9.7
5	21.2	19.8	20.2	19.6	10.8
6	21.8	19.5	20.0	19.6	18.7
7	21.9	19.5	20.7	19.0	7.3
8	12.8	8.9	9.2	9.1	11.1
9	11.5	9.0	9.6	8.2	3.2
10	14.6	14.0	14.5	12.2	7.0
11	11.7	8.0	10.5	7.1	6.6
12	8.0	6.3	9.5	4.2	3.1
13	7.6	9.0	22.1	4.4	5.2
14	8.0	9.3	20.0	3.9	3.8
15	8.4	20.1	45.1	2.7	6.4
16	6.8	4.2		6.1	
17	5.8	6.7		4.4	
18	5.7	6.6		6.7	
19	6.5	2.8		4.1	
20	3.9	3.7		3.5	
21	4.7	3.5		3.7	
22	6.4	5.0		4.4	
23	5.0	4.1		4.9	
24	4.5	3.2		4.0	
25	4.8	4.7		4.7	
26	6.5	5.5		5.9	
27	4.6	4.0		4.1	
28	4.5	4.7		4.5	
29	4.3	4.0		3.9	
30	5.6	5.0		5.0	

resolved at this time.

At  $n = 15$  for solution B and at  $n = 13, 14,$  and  $15$  for solution C, we see that anomaly degree variances are considerably higher than expected. Solution C may be biased by not using a complete estimate of the anomaly field. However, no similar reason for the high degree variances at  $n = 15$  for solution B is seen. The direct reason, of course, is the large value of certain potential coefficients in solutions B and C. This has been discussed in section 7.1. It would appear upon examination of Table 13 that solution B and C may not be reasonable because of these large degree variances. We might argue that because we are solving for a finite potential coefficient set solutions of the type B and C will tend to distort these solutions to fit the data more than a solution of this type carried out in solution D. There has been no previous indication of this in previous solutions.

On the other hand there may be a possibility that, in fact, there are several large coefficients in the  $n = 15$  series. It perhaps is worth while to see if the orbital analysis approach could obtain estimates for the coefficients of degree 15 that have been predicted to be larger than expected from solution B. These coefficients have been listed in section 7.1.

#### 8.5 General Accuracy Equation for Potential Coefficient Determinations from Gravimetric Data

The accuracy of the determination of potential coefficients where no satellite data existed was estimated by equation (21) in section 5. With the completion of these new solutions a re-evaluation of the constant in that equation is possible. From solution B we estimate:

$$(28) \quad m_1 = \frac{0.300m_0 \cdot 10^{-6}}{(n-1)}$$

while from solution D we have:

$$(29) \quad m_1 = \frac{0.250m_0 \cdot 10^{-6}}{(n-1)}$$

where  $m_0$  is the standard error of unit weight. Averaging (28) and (29), and using an  $m_0 = \pm 1.35$  as discussed in section 8.2 we have an estimate of the standard error of a potential coefficient as determined from gravity data alone as:

$$(30) \quad m_1 = \frac{0.371 \cdot 10^{-6}}{(n-1)}$$

This result is somewhat, but not significantly, higher than the estimate given in

equation (21).

## 9. Conclusions

This report has presented a combination of gravimetric and satellite data by several techniques. We have determined 283 potential coefficients by direct solution and corresponding adjusted  $5' \times 5''$  anomalies. The previous pages have compared and discussed these solutions. Besides that discussion we may make the following comments.

- A. The solutions requiring weighting of the potential coefficients (solutions B, C, and D) are very sensitive to that weighting. Thus the corrections to the potential coefficients determined from satellite data are highly dependent on the standard errors assigned to those coefficients. Consequently more study should be given to the proper weighting of the potential coefficients.
- B. The standard error of unit weight computed in the solutions was significantly different from the "a priori" value of one. I would suggest several different solutions be made with various weighting schemes for the potential coefficients to see the effect on the adjusted anomalies and the computed standard error of unit weight.
- C. The use of the solutions for prediction purposes indicates, from a limited test, solution D is better than solution A or solution B. This was found from the comparisons of the adjusted anomalies of solutions A, B, and D to 19  $5'' \times 5''$  anomalies recently estimated from Canadian maps and which were not used as observed data in these solutions.
- D. Judging from the degree variances there appears to be some distortion in the coefficients of solutions B and C. Although there is the explanation that certain coefficients in the  $n = 15$  series are larger than expected, it is unlikely.
- E. For best agreement of the anomalies computed from the direct solution potential coefficients, solution B or C could be preferred.
- F. The value of  $E(\epsilon_s^2)$  for solution D is slightly smaller than for solution B or C indicating that the coefficients of D are slightly better than those of B or C. This is discussed further in section 8.2.
- G. Orbital prediction tests should be carried out to determine which solution is to be preferred from that standpoint. Since solution A has fixed the 81 coefficients which in turn were based only on satellite analysis, we might expect solution A to show the best orbital prediction accuracy.
- H. Based primarily on the conclusions described in comments C, D, and F, we would choose the results of solution D as the best least squares combination solution of the given data.

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Unclassified

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1 ORIGINATING ACTIVITY (Corporate author) The Ohio State University Research Foundation Department of Geodetic Science 1314 Kinnear Road, Columbus, Ohio 43212		2a. REPORT SECURITY CLASSIFICATION Unclassified	
		2b. GROUP N/A	
3 REPORT TITLE FURTHER STUDIES IN THE COMBINATION OF GRAVIMETRIC AND SATELLITE DATA			
4 DESCRIPTIVE NOTES (Type of report and inclusive dates) Scientific Interim			
5 AUTHOR(S) (First name, middle initial, last name) Richard H. Rapp			
6 REPORT DATE December, 1968		7a. TOTAL NO OF PAGES 41	7b. NO OF REFS 27
8a. CONTRACT OR GRANT NO AF19(628)-5701		9a. ORIGINATOR'S REPORT NUMBER(S) Department of Geodetic Science Report No. 119 Scientific Report No. 27	
b. PROJECT NO Project, Task, Work Unit Nos. 7600-02, 04-01		9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) AFRL - 69 - 0249	
c. DoD Element 62405394			
d. DoD Subelement 681000			
10 DISTRIBUTION STATEMENT 1 - Distribution of this document is unlimited. It may be released to the Clearinghouse, Department of Commerce, for sale to the general public.			
11 SUPPLEMENTARY NOTES TECH, OTHER		12 SPONSORING MILITARY ACTIVITY Air Force Cambridge Research Laboratories (CRJ) L. G. Hanscom Field Bedford, Massachusetts 01730	
13 ABSTRACT <p>This report is composed of two portions. The first is a general theoretical discussion of the problem of assuming compatibility of surface gravity data and potential coefficients derived from satellite orbital analysis. The second part of the report describes a combination of gravimetric and satellite data using 5° x 5° mean free air anomalies supplied by the Aeronautical Chart and Information Center, and a set of potential coefficients designated NWL 81 derived by Anderle. Several methods were applied to this data to obtain an adjusted set of potential coefficients complete to n = 15 plus an additional number of higher degree resonance terms. In addition three sets of adjusted 5° x 5° mean free air anomalies were obtained. From these anomalies a set of potential coefficients complete to n = 30 was derived by the development formulas. Analysis of the various solutions indicated that one was to be preferred over all others. Problems continue to remain with respect to proper weighting of the data. (U)</p>			

DD FORM 1 NOV 65 1473

Unclassified

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Geodesy Gravity Satellites Gravitational Potential Earth						