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MEMORANDUM

RM-5604-1-PR

JUNE 1968

# SUBGRAPHS OF BIPARTITE AND DIRECTED GRAPHS

Jon Folkman and D. R. Fulkerson

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UNITED STATES AIR FORCE PROJECT RAND

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PREFACE

This Memorandum continues Project RAND's program of research in graph theory and other aspects of combinatorics. In particular, various known theorems about finite bipartite and directed graphs are generalized to infinite bipartite and directed graphs.

Graph theory not only finds application to transportation networks and to similar operations research problems, but constitutes a general theory of relations on finite or infinite sets.

SUMMARY

The main theorem of this Memorandum provides necessary and sufficient conditions in order that a locally finite bipartite graph have a subgraph whose valences lie in prescribed intervals. This theorem is applied to the study of integer-valued flows in locally finite directed graphs. In particular, generalizations of the max-flow min-cut theorem and of the circulation theorem are obtained.

The axiom of choice is assumed throughout.

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## SUBGRAPHS OF BIPARTITE AND DIRECTED GRAPHS

### 1. INTRODUCTION

Our object in this paper is to generalize certain known theorems about finite bipartite and directed graphs to infinite (usually locally finite) bipartite and directed graphs. In the development that follows, we have chosen as our main theorem (Theorem 1) one that gives necessary and sufficient conditions for the existence of a valence-constrained subgraph  $H$  of a bipartite graph  $G$ . Specifically, let  $G$  be a bipartite graph having vertex parts  $I, J$  and suppose that for each vertex  $i \in I$  we are given a pair of nonnegative integers  $[a_i, a'_i]$  satisfying  $a_i \leq a'_i$ , and that for each vertex  $j \in J$  we are given a pair of nonnegative integers  $[b_j, b'_j]$  satisfying  $b_j \leq b'_j$ . We also suppose that if  $a_i > 0$ , then  $G$  has finite valence at  $i$ ; similarly, if  $b_j > 0$ , we suppose that  $G$  has finite valence at  $j$ . Under these assumptions, we obtain necessary and sufficient conditions in order that  $G$  have a subgraph  $H$  whose valence at  $i \in I$  lies in the interval  $[a_i, a'_i]$  and whose valence at  $j \in J$  lies in the interval  $[b_j, b'_j]$ . If  $G$  is finite, existence conditions for such an  $H$  are known [3, 4]. In Sec. 2 we show that these conditions extend to the infinite case. Our proof invokes the Tychonoff theorem explicitly, and hence the axiom of choice implicitly.

There are a number of theorems that can be viewed as

special cases of Theorem 1. Among these are the Schröder-Bernstein theorem (or, equivalently, the Banach mapping theorem [1]), the Hall theorem on systems of distinct representatives [5, 6], a generalization of the Schröder-Bernstein theorem due to Perfect and Pym [10], and a recent generalization by Mirsky [9] of a theorem of Ford and Fulkerson [2] concerning systems of representatives with repetition. We shall discuss these briefly in Sec. 3.

The remainder of the paper deals with applications of Theorem 1 to flows in directed graphs. In particular, the max-flow min-cut theorem of Ford and Fulkerson [3] and the circulation theorem due to Hoffman [3, 8] are generalized to locally finite directed graphs via Theorem 1.

## 2. THE MAIN THEOREM

By a natural number we mean a nonnegative integer. By an extended natural number we mean a natural number or  $\infty$ . We extend the ordering of the natural numbers to the extended natural numbers by defining  $n < \infty$  to be true for every natural number  $n$ . We extend the operation of addition to the extended natural numbers by defining  $\infty + \infty = \infty$  and  $\infty + n = n + \infty = \infty$  for every natural number  $n$ .

If  $S$  is a finite set and  $x_i$  is an extended natural number for each  $i \in S$ , then  $\sum_{i \in S} x_i$  is a well-defined extended natural number. If  $S$  is a (possibly infinite) set and  $x_i$  is an extended natural number for each  $i \in S$ , let  $S^+ = \{i \in S \mid x_i > 0\}$ . If  $S^+$  is infinite, set  $\sum_{i \in S} x_i = \infty$ . If  $S^+$  is finite, set  $\sum_{i \in S} x_i = \sum_{i \in S^+} x_i$ .

Let  $I$  and  $J$  be sets. For each  $i \in I$  let  $a_i$  be a natural number and  $a'_i$  an extended natural number with  $a_i \leq a'_i$ . For each  $j \in J$  let  $b_j$  be a natural number and  $b'_j$  be an extended natural number with  $b_j \leq b'_j$ . For each  $i \in I, j \in J$ , let  $c_{ij}$  be a natural number. Suppose the following "weak local finiteness" condition is satisfied:

(W.L.F.) For each  $i \in I$  either  $a_i = 0$  or  $c_{ij} = 0$  for all but finitely many  $j \in J$ . For each  $j \in J$  either  $b_j = 0$  or  $c_{ij} = 0$  for all but finitely many  $i \in I$ .

Consider the following conditions which may or may not be satisfied by the numbers  $a_i, b_j, c_{ij}$ :

(Ia) For each finite  $N \subseteq I$ ,  $\sum_{i \in N} a_i \leq \sum_{j \in J} \min(b'_j, \sum_{i \in N} c_{ij})$ .

(Ib) For each finite  $M \subseteq J$ ,  $\sum_{j \in M} b_j \leq \sum_{i \in I} \min(a'_i, \sum_{j \in M} c_{ij})$ .

(IIa) There is a family  $x = \{x_{ij}\}_{i \in I, j \in J}$  of natural numbers such that

(IIai) for each  $i \in I$ ,  $j \in J$ ,  $x_{ij} \leq c_{ij}$ ,

(IIaia) for each  $i \in I$ ,  $a_i \leq \sum_{j \in J} x_{ij}$ , and

(IIaiai) for each  $j \in J$ ,  $\sum_{i \in I} x_{ij} \leq b'_j$ .

(IIb) There is a family  $x = \{x_{ij}\}_{i \in I, j \in J}$  of natural numbers such that

(IIbi) for each  $i \in I$ ,  $j \in J$ ,  $x_{ij} \leq c_{ij}$ ,

(IIbia) for each  $j \in J$ ,  $b_j \leq \sum_{i \in I} x_{ij}$ , and

(IIbiiai) for each  $i \in I$ ,  $\sum_{j \in J} x_{ij} \leq a'_i$ .

(III) There is a family  $x = \{x_{ij}\}_{i \in I, j \in J}$  of natural numbers such that

(IIIi) for each  $i \in I$ ,  $j \in J$ ,  $x_{ij} \leq c_{ij}$ ,

(IIIia) for each  $i \in I$ ,  $a_i \leq \sum_{j \in J} x_{ij} \leq a'_i$ , and

(IIIiaa) for each  $j \in J$ ,  $b_j \leq \sum_{i \in I} x_{ij} \leq b'_j$ .

**THEOREM 1.** Assume that condition (W.L.F.) holds. Then

(i) Condition (IIa) holds if and only if condition (Ia) holds.

(ii) Condition (IIb) holds if and only if condition

(Ib) holds.

(iii) Condition III holds if and only if conditions

(Ia) and (Ib) both hold.

(iv) Condition III holds if and only if conditions

(IIa) and (IIb) both hold.

PROOF. IIa  $\Rightarrow$  Ia. Let  $x = \{x_{ij}\}_{i \in I, j \in J}$  be a family of natural numbers satisfying IIa. Let  $N$  be a finite subset of  $I$ . For each  $j \in J$ ,  $\sum_{i \in N} x_{ij} \leq \sum_{i \in I} x_{ij} \leq b'_j$  by (IIa iii). For each  $i \in N \subseteq I$ ,  $x_{ij} \leq c_{ij}$  by (IIa i). Hence,  $\sum_{i \in N} x_{ij} \leq \sum_{i \in N} c_{ij}$ . Hence, for each  $j \in J$ ,  $\sum_{i \in N} x_{ij} \leq \min(b'_j, \sum_{i \in N} c_{ij})$ . By (IIa ii),  $a_i \leq \sum_{j \in J} x_{ij}$  for each  $i \in I$ . Therefore,

$$\sum_{i \in N} a_i \leq \sum_{i \in N} \sum_{j \in J} x_{ij} = \sum_{j \in J} \sum_{i \in N} x_{ij} \leq \sum_{j \in J} \min(b'_j, \sum_{i \in N} c_{ij}).$$

By a similar argument, interchanging the roles of  $I$  and  $J$ , IIb  $\Rightarrow$  Ib.

III  $\Rightarrow$  IIa and IIb. Let  $x = \{x_{ij}\}_{i \in I, j \in J}$  be a family of natural numbers satisfying III. Then  $x$  also satisfies IIa and IIb.

Ia and Ib  $\Rightarrow$  III. For  $i \in I$ ,  $j \in J$ , let  $X_{ij}$  be the set of integers  $n$  with  $0 \leq n \leq c_{ij}$ . With the discrete topology,  $X_{ij}$  is a compact Hausdorff space. Hence, by Tychonoff's theorem,  $X = \prod_{\substack{i \in I \\ j \in J}} X_{ij}$  is a compact Hausdorff space.

For each  $i \in I$ , let  $P_i$  be the set of  $x \in X$  such that  $a_i \leq \sum_{j \in J} x_{ij} \leq a'_i$ . For each  $j \in J$ , let  $Q_j$  be the set of  $x \in X$  such that  $b_j \leq \sum_{i \in I} x_{ij} \leq b'_j$ .

Lemma. For each  $i \in I$  and each  $j \in J$ ,  $P_i$  and  $Q_j$  are closed subsets of  $X$ .

Proof. Let  $x \in X$  with  $x \notin P_i$ . There are two possibilities:

$$(1) \sum_{j \in J} x_{ij} < a_i \text{ or}$$

$$(2) \sum_{j \in J} x_{ij} > a'_i.$$

Suppose the first possibility holds. Then  $a_i > 0$ , and so by condition (W.L.F.) the set  $M = \{j \in J | c_{ij} > 0\}$  is finite. Let  $U = \{y \in X | y_{ij} = x_{ij} \text{ for } j \in M\}$ . Then  $U$  is an open subset of  $X$  and  $x \in U$ . Let  $y \in U$ . For  $j \notin M$ ,  $y_{ij} \leq c_{ij} = 0$ , and so  $\sum_{j \in J} y_{ij} = \sum_{j \in M} y_{ij} = \sum_{j \in M} x_{ij} = \sum_{j \in J} x_{ij} < a_i$ . Hence  $y \notin P_i$ , so  $U$  is an open subset of  $X$  containing  $x$  which does not intersect  $P_i$ .

Now suppose the second possibility holds. Then there is a finite set  $N \subseteq J$  such that  $\sum_{j \in N} x_{ij} > a'_i$ . Let  $V = \{y \in X | y_{ij} = x_{ij} \text{ for } j \in N\}$ . Then  $V$  is an open subset of  $X$  and  $x \in V$ . If  $y \in V$  then  $\sum_{j \in J} y_{ij} \geq \sum_{j \in N} y_{ij} = \sum_{j \in N} x_{ij} > a'_i$ , so  $y \notin P_i$ . Therefore,  $V$  does not intersect  $P_i$ .

We have now shown that every point  $x \in X$  which is not in  $P_i$  is contained in an open set not intersecting  $P_i$ . Hence,  $P_i$  is closed. Similarly,  $Q_j$  is closed.

Now let  $N$  and  $M$  be finite sets with  $N \subseteq I$  and  $M \subseteq J$ . Let  $N^+ = \{i \in N | a_i > 0\}$  and  $M^+ = \{j \in M | b_j > 0\}$ . Let  $\bar{N} = \{i \in I | c_{ij} > 0 \text{ for some } j \in M^+\}$ , and let

$\bar{M} = \{j \in J \mid c_{ij} > 0 \text{ for some } i \in N^+\}$ . Since  $N^+ \subseteq N$  and  $M^+ \subseteq M$  are finite sets, it follows from condition (W.L.F.) that  $\bar{N}$  and  $\bar{M}$  are finite.

For each  $i \in N \cup \bar{N}$ , let  $\bar{a}_i = a_i$  if  $i \in N$  and  $\bar{a}_i = 0$  if  $i \in \bar{N} - N$ . For each  $j \in M \cup \bar{M}$  let  $\bar{b}_j = b_j$  if  $j \in M$  and  $\bar{b}_j = 0$  if  $j \in \bar{M} - M$ .

Let  $N' \subseteq N \cup \bar{N}$ . Then

$$\sum_{i \in N'} \bar{a}_i = \sum_{i \in N \cap N'} a_i = \sum_{i \in N^+ \cap N'} a_i \leq \sum_{j \in J} \min(b'_j, \sum_{i \in N^+ \cap N'} c_{ij}).$$

Now if  $j \in J - \bar{M}$ , then  $c_{ij} = 0$  for each  $i \in N^+$ , and so  $\min(b'_j, \sum_{i \in N^+ \cap N'} c_{ij}) = \min(b'_j, 0) = 0$ . Hence,

$$\begin{aligned} \sum_{i \in N'} \bar{a}_i &\leq \sum_{j \in J} \min(b'_j, \sum_{i \in N^+ \cap N'} c_{ij}) = \sum_{j \in \bar{M}} \min(b'_j, \sum_{i \in N^+ \cap N'} c_{ij}) \\ &\leq \sum_{j \in M \cup \bar{M}} \min(b'_j, \sum_{i \in N'} c_{ij}). \end{aligned}$$

Similarly, if  $M' \subseteq M \cup \bar{M}$ , then

$$\sum_{j \in M'} \bar{b}_j \leq \sum_{i \in N \cup \bar{N}} \min(a'_i, \sum_{j \in M'} c_{ij}).$$

Hence, by [4, Theorem 1 or Theorem 5], there is a family of natural numbers  $x = \{x_{ij}\}_{i \in N \cup \bar{N}, j \in M \cup \bar{M}}$  such that

$$(3) \quad \bar{a}_i \leq \sum_{j \in M \cup \bar{M}} x_{ij} \leq a'_i \quad \text{for each } i \in N \cup \bar{N},$$

$$(4) \quad \bar{b}_j \leq \sum_{i \in N \cup \bar{N}} x_{ij} \leq b'_j \quad \text{for each } j \in M \cup \bar{M}, \text{ and}$$

$$(5) \quad x_{ij} \leq c_{ij} \quad \text{for each } i \in N \cup \bar{N}, j \in M \cup \bar{M}.$$

each  $j \in J$  let  $\bar{b}_j = 0$ . Then  $\bar{b}_j \leq b'_j$ . If  $M \subseteq J$  is a finite set, then  $\sum_{j \in M} \bar{b}_j = 0 \leq \sum_{i \in I} \min(a'_i, \sum_{j \in M} c_{ij})$ . Hence the numbers  $\{a_i\}_{i \in I}$ ,  $\{a'_i\}_{i \in I}$ ,  $\{\bar{b}_j\}_{j \in J}$  and  $\{b'_j\}_{j \in J}$  satisfy condition Ib as well as condition Ia. By part (iii) of the theorem there is a family  $x = \{x_{ij}\}_{i \in I, j \in J}$  of natural numbers such that

$$\begin{aligned} a_i &\leq \sum_{j \in J} x_{ij} \leq a'_i && \text{for each } i \in I, \\ \bar{b}_j &\leq \sum_{i \in I} x_{ij} \leq b'_j && \text{for each } j \in J, \text{ and} \\ x_{ij} &\leq c_{ij} && \text{for each } i \in I, j \in J. \end{aligned}$$

The family  $x$  satisfies condition IIa. By a similar argument, Ib  $\Rightarrow$  IIb. This establishes parts (i) and (ii) of the theorem. Part (iv) follows from parts (i), (ii), and (iii).

In connection with Theorem 1, we note the following:

Remark 1. Suppose  $G = (I, J; E)$  is a bipartite graph with vertex parts  $I, J$  and edge set  $E \subseteq I \times J$ . Let  $c_{ij} = 1$  or 0 according as  $(i, j) \in E$  or  $(i, j) \notin E$ , and suppose that  $G$  has the "vertex assignment of intervals"  $[a_i, a'_i]$ ,  $i \in I$ , and  $[b_j, b'_j]$ ,  $j \in J$ . If  $G$  is locally finite at vertices  $i \in I$  ( $j \in J$ ) for which  $a_i > 0$  ( $b_j > 0$ ), then Theorem 1 gives necessary and sufficient conditions in order that  $G$  have a subgraph  $H$  whose valences lie in the prescribed intervals.

Remark 2. The assumption in Theorem 1 that  $c_{ij}$  is a natural number, rather than an extended natural number,

shows. Let  $I$  and  $J$  be the positive integers and define

$$c_{ij} = \begin{cases} 1 & \text{if either } i = 1 \text{ or } i = j + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Take  $a_i = a'_i = 1$  for all  $i \in I$ , and  $b_j = 0$ ,  $b'_j = 1$  for all  $j \in J$ . Thus (W.L.F.) fails to hold for  $i = 1$ . Conditions (Ia) and (Ib) hold, but (III) fails.

Remark 6. Another proof of Theorem 1 can be given in which the main tools are the finite case of the theorem and the following "infinity lemma," which is a consequence of Zorn's lemma.

Infinity Lemma. Let  $S$  and  $X$  be sets. Let " $\leq$ " be a partial ordering on  $S$ . Suppose that for any  $i, j \in S$  there is a  $k \in S$  with  $i \leq k$  and  $j \leq k$ . For each  $i \in S$ , let  $X_i$  be a finite nonempty subset of  $X$ . For each  $i, j \in S$  with  $i \geq j$ , let  $f_{ij}$  be a function from  $X_i$  to  $X_j$ . Suppose that  $f_{ii}(x) = x$  for each  $i \in S$  and each  $x \in X_i$ . Finally, suppose that if  $i, j, k \in S$  with  $i \geq j \geq k$ , then  $f_{jk}(f_{ij}(x)) = f_{ik}(x)$  for each  $x \in X_i$ . Then there is a function  $f$  from  $S$  to  $X$  such that  $f(i) \in X_i$  for each  $i \in S$ , and if  $i, j \in S$  with  $i \geq j$ , then  $f_{ij}(f(i)) = f(j)$ .

Use of the Tychonoff theorem instead of the above lemma shortens the proof considerably, however.

### 3. RELATED RESULTS

Theorem 1 includes a number of known results about mappings, systems of distinct representatives, systems of representatives with repetition, and so on. We discuss some of these and begin with the Schröder-Bernstein theorem. Let  $I$  and  $J$  be sets and let  $\varphi : I \rightarrow J$ ,  $\psi : J \rightarrow I$  be injective mappings. The Schröder-Bernstein theorem asserts the existence of a bijection  $\sigma : I \rightarrow J$ . Usual proofs of the theorem assert more, namely that the bijection  $\sigma$  can be viewed as a subgraph of the bipartite graph  $G = (I, J; E_1 \cup E_2)$ , where  $E_1 = \{(i, \varphi(i)) \mid i \in I\}$ ,  $E_2 = \{(\psi(j), j) \mid j \in J\}$ . In terms of Theorem 1, take  $c_{ij} = 1$  for each edge of  $G$ ,  $c_{ij} = 0$  otherwise, and let each vertex of  $G$  have the interval assignment  $[1,1]$ . Note that (W.L.F.) holds and that the hypothesis of the Schröder-Bernstein theorem implies that (IIa) and (IIb) hold. Hence (III) holds, yielding the bijection  $\sigma : I \rightarrow J$ .

Next let  $J$  be a set and let  $F = \{F_i\}_{i \in I}$  be a family of finite subsets of  $J$ . The Hall theorem [5, 6] concerns the existence of a system of distinct representatives for the family  $F$ . In terms of Theorem 1, take  $c_{ij} = 1$  or  $0$  according as  $j \in F_i$  or  $j \notin F_i$ , and take  $a_i = a'_i = 1$  for each  $i \in I$ ,  $b_j = 0$ ,  $b'_j = 1$  for each  $j \in J$ . Note that (W.L.F.) is satisfied. The Hall theorem asserts that  $F$  has a system of distinct representatives if and only if the "Hall condition" holds: For each finite subset  $N \subseteq I$ ,

the cardinality of  $N$  is less than or equal to the cardinality of  $\bigcup_{i \in N} F_i$ . In other words, (III) holds if and only if (Ia) holds. (Condition (Ib) holds automatically, since  $b_j = 0$  all  $j \in J$ .) A proof of the Hall theorem that uses the Tychonoff theorem in the infinite case has been given in [7].

A generalization of the Schröder-Bernstein theorem due to Perfect and Pym [10] runs as follows. Let  $I, J, I', J'$  be sets with  $I' \subseteq I, J' \subseteq J$ . Let  $E$  be a subset of  $I \times J$  and let  $\varphi : I' \rightarrow J, \psi : J' \rightarrow I$  be injective mappings such that  $(i, \varphi(i)) \in E$  for all  $i \in I'$  and  $(\psi(j), j) \in E$  for all  $j \in J'$ . Then there exist sets  $I_0, J_0$  with  $I' \subseteq I_0 \subseteq I, J' \subseteq J_0 \subseteq J$ , and a bijection  $\sigma : I_0 \rightarrow J_0$  such that  $(i, \sigma(i)) \in E$  for all  $i \in I_0$ . (If  $I' = I$  and  $J' = J$ , this reduces to the Schröder-Bernstein theorem.) To deduce this result from Theorem 1, take  $a_i = 0$  or 1 according as  $i \in I - I'$  or  $i \in I'$ , and take  $a_i' = 1$  for all  $i \in I$ . Similarly, take  $b_j = 0$  or 1 according as  $j \in J - J'$  or  $J \in J'$ , and take  $b_j' = 1$  for all  $j \in J$ . Define  $c_{ij} = 1$  if either  $j = \varphi(i)$  or  $i = \psi(j)$ ,  $c_{ij} = 0$  otherwise. Thus (W.L.F.) holds and (IIa) and (IIb) are satisfied by hypothesis. Hence (III) holds, producing the desired sets  $I_0, J_0$  and the bijection  $\sigma : I_0 \rightarrow J_0$ .

Mirsky [9] has recently generalized to the case of infinite families of finite sets a theorem of Ford and Fulkerson [2] for finite families concerning systems of

representatives with repetition allowed. The general result may be described as follows. Let  $I$  and  $J$  be sets and define  $a_i = a'_i = 1$  for each  $i \in I$ , but consider an arbitrary assignment of intervals  $[b_j, b'_j]$  for  $j \in J$ . Suppose  $c_{ij} = 0$  or  $1$  in such a way that (W.L.F.) holds, i.e. for each  $i \in I$ ,  $c_{ij} = 1$  for only finitely many  $j \in J$ , and if  $b_j > 0$  for  $j \in J$ , then  $c_{ij} = 1$  for only finitely many  $i \in I$ . For each finite set  $N \subseteq I$ , define

$$A(N) = \{j \in J \mid c_{ij} = 1 \text{ for some } i \in N\}.$$

Similarly, for each finite set  $M \subseteq J$ , define

$$B(M) = \{i \in I \mid c_{ij} = 1 \text{ for some } j \in M\}.$$

The theorem asserts that (III) holds if and only if:

- (a) For each finite  $N \subseteq I$ ,  $|N| \leq \sum_{j \in A(N)} b'_j$ ;
- (b) For each finite  $M \subseteq J$ ,  $\sum_{j \in M} b_j \leq |B(M)|$ .

(Here  $|S|$  denotes cardinality of a set  $S$ .) Since  $a'_i = 1$  for all  $i \in I$ , it is apparent that (b) above is equivalent to (Ib). It is also not hard to see that, since  $a_i = 1$  for all  $i \in I$ , condition (a) above is equivalent to (Ia). Mirsky's proof of this theorem in the infinite case uses two principal tools: the Hall condition and the generalized form of the Schröder-Bernstein theorem due to Perfect and Pym.

#### 4. FLOWS IN DIRECTED GRAPHS

Let  $V$  be a set and suppose that for each  $i \in V$  there are integers  $d_i, d'_i$  satisfying  $d_i \leq d'_i$ . We also suppose that for each  $(i, j) \in V \times V$  there are integers  $l_{ij}, u_{ij}$  satisfying  $0 \leq l_{ij} \leq u_{ij}$ , with  $u_{ii} = 0$ . Throughout this section we make the following local finiteness assumption:

(L.F.) For each  $i \in V$ ,  $u_{ij} = 0$  for all but finitely many  $j \in V$ , and  $u_{ji} = 0$  for all but finitely many  $j \in V$ .

We call a function  $f$  from  $V \times V$  to the natural numbers a feasible flow if and only if

$$(6) \quad d_i \leq \sum_{j \in V} f_{ij} - \sum_{j \in V} f_{ji} \leq d'_i, \text{ all } i \in V,$$

$$(7) \quad l_{ij} \leq f_{ij} \leq u_{ij}, \text{ all } (i, j) \in V \times V.$$

If we think of the directed graph  $G = (V; E)$ , with vertex set  $V$  and edge set  $E = \{(i, j) \in V \times V \mid u_{ij} > 0\}$ , then (L.F.) says that  $G$  is locally finite. The inequalities (6) stipulate that the "net flow out of vertex  $i$ " lies in the prescribed interval  $[d_i, d'_i]$ , and (7) that the "flow in edge  $(i, j)$ " lies in the prescribed interval  $[l_{ij}, u_{ij}]$ . If  $G$  is finite, necessary and sufficient conditions for the existence of a feasible flow are known [3, 8]. We can use Theorem 1 to extend these conditions to the case of locally finite infinite directed graphs.

**THEOREM 2.** Assume that condition (L.F.) holds.

Then there is a feasible flow if and only if for each finite  $X \subseteq V$ ,

$$(8) \quad \sum_{\substack{i \in X \\ j \in V-X}} t_{ij} \leq \sum_{i \in X} d'_i + \sum_{\substack{i \in V-X \\ j \in X}} u_{ij}$$

$$(9) \quad \sum_{i \in X} d_i + \sum_{\substack{i \in V-X \\ j \in X}} t_{ij} \leq \sum_{\substack{i \in X \\ j \in V-X}} u_{ij}.$$

**Proof.** The translation  $g = f - t$  shows that a feasible flow  $f$  exists if and only if an integer-valued  $g$  exists satisfying

$$(6') \quad h_i \leq \sum_{j \in V} g_{ij} - \sum_{j \in V} g_{ji} \leq h'_i, \text{ all } i \in V,$$

$$(7') \quad 0 \leq g_{ij} \leq d_{ij}, \text{ all } (i, j) \in V \times V,$$

where

$$(10) \quad h_i = d_i + \sum_{j \in V} t_{ji} - \sum_{j \in V} t_{ij},$$

$$(11) \quad h'_i = d'_i + \sum_{j \in V} t_{ji} - \sum_{j \in V} t_{ij},$$

$$(12) \quad d_{ij} = u_{ij} - t_{ij}.$$

The existence of such a  $g$  is equivalent to the existence of  $g$  and (an integer-valued)  $y$  defined on  $V \times V$  satisfying

$$(13) \quad g_{ij} + y_{ij} = d_{ij}, \text{ all } (i, j) \in V \times V,$$

$$(14) \quad \sum_{j \in V} d_{ij} - h_i' \leq \sum_{j \in V} y_{ij} + \sum_{j \in V} g_{ji} \leq \sum_{j \in V} d_{ij} - h_i,$$

all  $i \in V$ ,

$$(15) \quad g_{ij} \geq 0, y_{ij} \geq 0, \text{ all } (i, j) \in V \times V.$$

We can now apply Theorem 1 to the constraints (13), (14), (15). First note that (13), (14), and (15) are equivalent to (13), (14'), and (15) where (14') is given by

$$(14') \quad \max(0, \sum_{j \in V} d_{ij} - h_i') \leq \sum_{j \in V} y_{ij} + \sum_{j \in V} g_{ji} \leq \sum_{j \in V} d_{ij} - h_i$$

for all  $i \in V$ .

In Theorem 1 take  $I = \{(i, j) \in V \times V \mid u_{ij} > 0\}$ ,  $J = V$ , with the interval assignments

$$(16) \quad [d_{ij}, d_{ij}], \quad (i, j) \in I.$$

$$(17) \quad [\max(0, \sum_{j \in V} d_{ij} - h_i'), \sum_{j \in V} d_{ij} - h_i'], \quad i \in J,$$

and define the numbers  $c_{ijk}$ ,  $(i, j) \in I$ ,  $k \in V$  by

$$(18) \quad c_{ijk} = \begin{cases} \infty, & \text{if } i = k \text{ or } j = k \\ 0 & \text{otherwise.} \end{cases}$$

Note that (L.F.) implies that (W.L.F.) is satisfied for (16), (17) and (18); indeed for fixed  $(i, j) \in I$ ,  $c_{ijk} > 0$  for at most two  $k \in V$  and for fixed  $k \in V$ ,  $c_{ijk} > 0$  for only finitely many pairs  $(i, j) \in I$ . Also note that for  $i \in V$ ,

$$\begin{aligned} \sum_{j \in V} d_{ij} - h_i &= \sum_{j \in V} u_{ij} - \sum_{j \in V} t_{ij} - d_i - \sum_{j \in V} t_{ji} + \sum_{j \in V} t_{ij} \\ &= \sum_{j \in V} u_{ij} - d_i - \sum_{j \in V} t_{ji} \geq 0 \end{aligned}$$

by (9) with  $X = \{i\}$ . Hence, since  $h'_i \geq h_i$ , the lower bounds of the intervals (17) never exceed the upper bounds.

The existence of the family  $x = \{x_{ijk}\}_{(i,j) \in I, k \in J}$  satisfying (III) is equivalent to the existence of  $g$  and  $y$  satisfying (13), (14), (15), as one sees by putting

$$x_{ijk} = \begin{cases} g_{ij}, & \text{if } k = j, \\ y_{ij}, & \text{if } k = i, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2 now follows from Theorem 1, part (iii). The inequalities (9) are equivalent to those of (Ia) and (8) to those of (Ib). We omit a detailed proof of these assertions.

By taking  $d_i = d'_i = 0$  in Theorem 2, necessary and sufficient conditions are obtained for the existence of a feasible conservative flow in locally finite directed graphs.

Theorem 2 can be used also to prove the max-flow min-cut equality for locally finite directed graphs. Here we distinguish two vertices of  $G = (V; E)$ , say  $s, t \in V$ . We assume that each edge  $(i, j) \in E$  has an integer flow capacity  $u_{ij} \geq 0$ , and seek a maximum flow from  $s$  to  $t$ , i.e., subject to the following constraints on integers  $f_{ij}$ ,

$$(19) \quad \sum_{j \in V} f_{ij} - \sum_{j \in V} f_{ji} = \begin{cases} v, & i = s \\ -v, & i = t, \\ 0, & \text{otherwise,} \end{cases}$$

$$(20) \quad 0 \leq f_{ij} \leq u_{ij}, \quad (i, j) \in E,$$

we wish to maximize  $v$ , the amount of flow from  $s$  to  $t$ . By adding the special "return-flow edge"  $(t, s)$  to  $E$ , with its associated interval  $[v, v]$ , taking  $d_i = d_i' = 0$  all  $i \in V$ , and taking  $l_{ij} = 0$  for edges of  $E$  other than the special edge  $(t, s)$ , the problem becomes that of seeking the largest  $v$  for which there is a conservative feasible flow in the resulting graph. Theorem 2 then implies Theorem 3, below. To state Theorem 3 concisely, we make the following definitions. A finite cut separating  $s$  and  $t$  is a partition of  $V$  into two sets  $X, V - X$ , where  $s \in X$ ,  $t \in V - X$ , and one of  $X, V - X$  is a finite set. The capacity of such a cut is given by the sum

$$(21) \quad \sum_{\substack{i \in X \\ j \in V-X}} u_{ij}.$$

Theorem 3. Assume that (L.F.) holds. Then the maximum amount of flow from  $s$  to  $t$  is equal to the minimum capacity of all finite cuts separating  $s$  and  $t$ .

We conclude with the following remarks.

Remark 7. Theorem 2 is false if we allow  $u_{ij} = \infty$ . For example, take  $V$  to be the set of integers, and define

$d_i = d'_i = 0$ ,  $i \in V$ ,  $u_{i,i+1} = \infty$ ,  $u_{ij} = 0$  if  $j \neq i+1$ ,  
 $l_{i,i+1} = i$  if  $i \geq 0$ ,  $l_{ij} = 0$  otherwise. Then (8) and (9)  
are satisfied, but there is no feasible flow.

Remark 8. It is essential in Theorem 3 to restrict the class of cuts to the finite ones. For example, consider the disconnected graph composed of two disjoint uniformly directed one-way infinite paths, with  $s$  as the front end of one of these paths,  $t$  the tail-end of the other. Suppose each edge of this graph has capacity 1. Then there is a flow from  $s$  to  $t$  of amount 1, but  $s$  and  $t$  are separated by an infinite cut of capacity zero.

Remark 9. It follows from Remark 3 and the proof of Theorem 2 that Theorem 2 remains valid if the  $d_i$ ,  $d'_i$ , for  $i \in V$ , and the  $l_{ij}$ ,  $u_{ij}$ , for  $(i, j) \in V \times V$ , are assumed to be real numbers, rather than integers, and a feasible flow is defined to be a function from  $V \times V$  to the non-negative reals satisfying (6) and (7). A similar remark holds for Theorem 3.

Remark 10. Condition (8) of Theorem 2 is equivalent to the existence of an  $f$  satisfying (7) and the upper bounds  $d'_i$  on the net flow out of vertex  $i$ , all  $i \in V$ . Similarly, condition (9) is equivalent to the existence of an  $f$  satisfying (7) and the lower bounds  $d_i$  on the net flow out of vertex  $i$ , all  $i \in V$ .

Remark 11. Various known theorems about systems of representatives can be extended to the infinite case via Theorem 2. We mention one, a theorem of Ford and Fulkerson [2] that provides necessary and sufficient conditions in order that two finite families of subsets of a finite set  $E$  have a common system of distinct representatives. The infinite extension is as follows. Let  $A = \{A_i\}_{i \in I}$  and  $B = \{B_j\}_{j \in J}$  be two families of finite subsets of a set  $E$ , and suppose no element of  $E$  occurs infinitely often in either family. The families  $A$  and  $B$  have a common system of distinct representatives if and only if

$$(22) \quad |X| \leq |Y| + \left| \bigcup_{i \in I-Y} A_i \cap \bigcup_{j \in X} B_j \right|$$

$$(23) \quad |Y| \leq |X| + \left| \bigcup_{i \in X} A_i \cap \bigcup_{j \in J-X} B_j \right|$$

hold for all finite  $X \subseteq J$ , finite  $Y \subseteq I$ . Moreover, (22) holds if and only if  $B$  and a subfamily of  $A$  have a common system of distinct representatives, and (23) holds if and only if  $A$  and a subfamily of  $B$  have a common system of distinct representatives. The proof of this theorem via Theorem 2 is somewhat more delicate than the corresponding proof for the finite case.

Added note. It has recently come to our attention that Theorem 1 has been established independently by Brualdi. His method of proof appears to be quite different from ours.

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