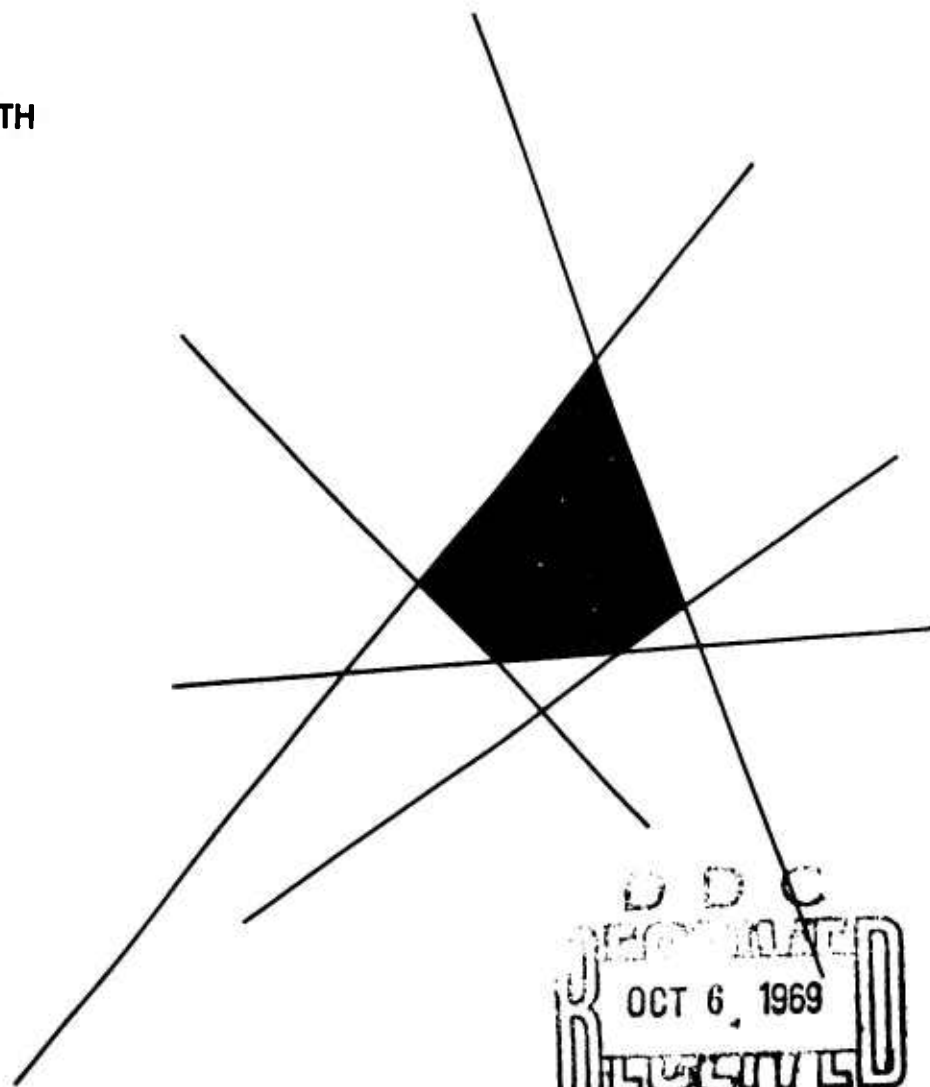


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# MODULES OF COHERENT SYSTEMS AND THEIR RELATIONSHIP TO BLOCKING SYSTEMS

by  
RICHARD W. BUTTERWORTH

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## ABSTRACT

The first section of this report deals with modules of coherent systems. We define modules in terms of their interaction with the min path sets of the coherent system. This leads to the same notion as given in [1], except we allow the set of all components to be a module, while [1] excludes this set as a module. The results concerning modules found in [1], most notable being the Three Modules Theorem, are given new proofs based on the properties of min path sets in the presence of modules. The results of [1] are slightly augmented by the two modules lemma, which gives necessary conditions for a set and its complement to be a module, and by another characterization of modules, Proposition 4, which results when blocking systems are introduced.

The second section defines blocking systems and shows that blocking systems and coherent systems are fundamentally the same, however, the emphasis of interest is different. Modules are interpreted for blocking systems as sets which can be condensed, a term made precise; Proposition 3 suggests how typical blocking systems can be decomposed. Some suggestions for further research are given.

## PREFACE

In this report, some known results concerning modules of coherent systems from reliability are given with new proofs. The relationship of coherent systems to blocking systems is pointed out, as well as an interpretation of modules for blocking systems.

MODULES OF COHERENT SYSTEMS AND THEIR  
RELATIONSHIP TO BLOCKING SYSTEMS

by

Richard Butterworth

1. MODULES OF COHERENT SYSTEMS

The goal of this section is to illustrate a proof of the "Three Modules Theorem" which is particularly suited to the identification of coherent systems with blocking systems. The proof which appears in [1] is of an algebraic nature corresponding to the functional relationship by which modules are defined there (see our Proposition 2). The proof given here is of a set-theoretic nature, giving another view to this established result.

A. Coherent Systems

Coherent systems arise in the study of reliability when one considers a physical system whose operation is classified as either functioning or failing, and when this operation is determined by the joint functioning or failing of some finite set of components  $C$ . A "coherent system" is one for which the replacement of a failed component by a functioning one will *not* cause a functioning system to fail. The system which functions if and only if all its components function is called a series system, while the system which fails if and only if all its components fail is called a parallel system. These are both special cases of the  $k$ -out-of- $n$  system, which is one of  $n$  components which functions whenever  $k$  or more of its components function, where  $1 \leq k \leq n$ .

More precisely now, let  $C$  be a finite nonempty set and  $\{0,1\}^C$  denote all functions on  $C$  to  $\{0,1\}$ . For  $X \in \{0,1\}^C$ , we say  $X$  is a *joint performance* of the components  $C$ , with the interpretation:  $\forall c \in C$ ,

$$X(c) = \begin{cases} 1 & \text{if component } c \text{ functions} \\ 0 & \text{if component } c \text{ fails.} \end{cases}$$

A system is any function  $\phi$  on  $\{0,1\}^C$  to  $\{0,1\}$ , with the interpretation that, for a joint performance  $X$ ,

$$\phi(X) = \begin{cases} 1 & \text{if the system functions} \\ 0 & \text{if the system fails.} \end{cases}$$

For joint performances  $X$  and  $Y$ , we say  $X \leq Y$  whenever for all  $c \in C$ ,  $X(c) \leq Y(c)$ . Then  $(C, \phi)$  is a *coherent system* whenever:

$$(1) \text{ if } X \leq Y, \text{ then } \phi(X) \leq \phi(Y).$$

Some components may have no effect on the system's behavior. We classify these as inessential components; all components not inessential will be called essential. Precisely, a component  $c$  is *inessential* to  $(C, \phi)$  when,  $\forall X$ ,  $\phi(0_c, X) = \phi(1_c, X)$ , where  $(1_c, X)(e) = X(e)$  if  $e \neq c$ ,  $= 1$  if  $e = c$ .

The usual definition of coherent systems requires, in addition to (1) above,

$$(2) \text{ at least one component is essential to } (C, \phi).$$

We remark this is equivalent to

$$(2') \phi(0) = 0 \text{ and } \phi(1) = 1, \text{ where } 0 \text{ is identically zero and } 1 \text{ is identically one,}$$

which is how (2) is stated in [3], [6], and elsewhere. We have not chosen to formally require (2) for coherent systems in order to simplify their identification with blocking systems. On the other hand, in our treatment of modules, as in [1], we confine our attention to coherent systems, all of whose components are essential.

Examples of coherent systems with all components essential are the series

system on  $C$ , for which  $\phi(X) = \text{Min} \{X(c) \mid c \in C\}$ , the parallel system on  $C$ , for which  $\phi(X) = \text{Max} \{X(c) \mid c \in C\}$ , and the  $k$ -out-of- $n$  system on any  $n$ -element set  $C$ , for which  $\phi(X) = 1 \Leftrightarrow \sum_{c \in C} X(c) \geq k$ .

Coherent systems are examined in [3], [6], [5] and [1], while [2] gives an excellent application of coherent systems in formulating a class of life distributions, those with increasing hazard rate average.

### B. Paths and Min Paths

The following notions are well defined for any function  $\phi$  on  $\{0,1\}^C$  to  $\{0,1\}$ . For  $A \subseteq C$ , let  $I_A \in \{0,1\}^C$  be  $= 1$  on  $A$ ,  $= 0$  on  $C - A$ . A *path* (cut) of  $(C, \phi)$  is any set  $P(K) \subseteq C$  such that  $\phi(I_P) = 1$  ( $\phi(I_{C-K}) = 0$ ). A *min path* (*min cut*) is any path (cut) which is set minimal with respect to being a path (cut). As for coherent systems, knowing all min paths or all min cuts is equivalent to knowing the function  $\phi$ , indeed  $\phi(X) = 1 \Leftrightarrow \exists$  min path  $P \ni X \geq I_P$  or  $\phi(X) = 0 \Leftrightarrow \exists$  min cut  $K \ni X \leq I_{C-K}$ .

We give a characterization of coherent systems in terms of their min path sets. It is easily proven from the definitions.

#### Proposition 1:

If  $(C, \phi)$  is a coherent system, then the family of all min paths,  $P$ , satisfies:

- (1)  $\forall P, Q \in P$ , we have  $P \not\subseteq Q$ .
- (2)  $UP =$  the set of essential components of  $(C, \phi)$ .

Conversely, if  $P$  is a family of subsets of  $C$ , a finite nonempty set, and if  $P$  satisfies (1), then there exists a uniquely determined coherent system  $(C, \phi)$  which has  $P$  as its family of min paths. It will have  $UP$  as its set of essential components. Indeed, we can define  $\phi(X) = 1 \Leftrightarrow \exists P \in P \ni X \geq I_P$ .

### C. Duality

If  $1 \in \{0,1\}^C$  is identically one on  $C$ , then for  $\phi^d(X) = 1 - \phi(1 - X)$ , where  $X \in \{0,1\}^C$ , and  $(C, \phi)$  is a coherent system,  $(C, \phi^d)$  is also a coherent system, the *dual* of  $(C, \phi)$ . The paths and cuts of  $(C, \phi)$  are the cuts and paths respectively of its dual  $(C, \phi^d)$ . This notion of duality is a reasonable one;  $(\phi^d)^d = \phi$ .

These observations imply a dual proposition to Proposition 1, in which we just replace every occurrence of the word "path" by the word "cut" and change the last line to read: "Indeed, we can define  $\phi(X) = 0 \Leftrightarrow \exists P \in \mathcal{P} \ni X \leq I_{C-P}$ ."

### D. Modules

From now on we will assume our coherent systems have no inessential components, equivalently that the union of all min paths is  $C$ . This will avoid needless complications, particularly where modules are concerned, and brings our assumptions into line with those made in [1].

A module of a coherent system is a subset of the components which functions as a coherent system itself within the given system. Let  $(C, \phi)$  be a coherent system with min path sets  $\mathcal{P}$ . We say a nonempty set  $A \subseteq C$  is a *module* of  $(C, \phi)$  if  $\forall P$  and  $Q \in \mathcal{P} \ni PA \neq \emptyset$  and  $QA \neq \emptyset$ , we have  $PA \cup Q(C - A) \in \mathcal{P}$ .

This definition is motivated by our needs. However, it does correspond to the notion of a module given in [1], except there the set  $C$  of all components is not a module, while, under our definition, it always is. The following characterization of modules is shown in [1] to be equivalent to our definition.

#### Proposition 2:

Let  $(C, \phi)$  be a coherent system with all components essential and  $A$  be a nonempty subset of  $C$ .  $A$  is a module of  $(C, \phi) \Leftrightarrow \phi(X) = \psi(\Gamma(X|_A), X|_{C-A})$ , where:

$X|_A$  is  $X$  restricted to  $A$

$(A, \Gamma)$  is a coherent system

$(\{c_A\} \cup (C - A), \psi)$  is a coherent system, and

$(\Gamma(X|_A), X|_{C-A})(c) = X(c)$  for  $c \in C - A$ ,  $= \Gamma(X|_A)$  for  $c = c_A$ .

We remark that if  $A$  is a module, then  $\Gamma$  and  $\psi$  in Proposition 2 are uniquely determined. We can therefore refer unambiguously to the coherent system  $(A, \Gamma)$  as a module of  $(C, \phi)$  whenever  $A$  is a module of  $(C, \phi)$ . Since  $C$  is always a module,  $(C, \phi)$  is always a module of itself. Also, every one element subset  $\{c\}$  of  $C$  is a module. Further consequences of these definitions are that the min paths and min cuts of  $(A, \Gamma)$ , a module of  $(C, \phi)$ , are the nonempty intersections of  $A$  with the min paths and min cuts respectively of  $(C, \phi)$ . This leads to the characterization of modules given in Proposition 4.

It is easy to see that  $(A, \Gamma)$  is a module of  $(C, \phi)$  if and only if  $(A, \Gamma^d)$  is a module of  $(C, \phi^d)$ . Using this, one can give a dual theorem for each of our theorems in which paths are replaced by cuts, similar to what was done in Part C above for Proposition 1.

### E. Three Modules Theorem

#### Intersection Lemma:

Let  $A$  and  $B$  be modules of  $(C, \phi)$ , a coherent system. Then  $AB = \emptyset$  or  $AB$  is a module of  $(C, \phi)$ .

#### Proof:

Suppose  $AB \neq \emptyset$ . If  $P$  and  $Q$  are min paths which intersect  $AB$ , then  $PA \cup Q(C - A)$  is a min path which intersects  $B$ , so  $(PA \cup Q(C - A))B \cup Q(C - B) = PAB \cup Q(C - AB)$  is a min path, showing  $AB$  is a module. ||

#### Difference Lemma:

Let  $A$  and  $B$  be modules of  $(C, \phi)$ , a coherent system with all components

essential. If  $A - B$  and  $B - A$  are both nonempty, then both are modules.

Proof:

It is sufficient to show  $A - B$  is a module whenever  $A - B$  and  $B - A$  are both nonempty. If  $AB = \emptyset$ , then  $A - B = A$ , so we assume  $AB \neq \emptyset$ . In a very general way, suppose we want to show a nonempty set  $H \subseteq C$  is a module. The assertion, if  $P$  and  $Q$  are min paths such that  $PH \neq \emptyset$  and  $QH \neq \emptyset$ , then  $PH \cup Q(C - H)$  is a path, implies (hence is equivalent to) the modularity of  $H$ . Simply show no min path is strictly contained in  $PH \cup Q(C - H)$  by assuming not and reach a contradiction by using the hypothesis about  $H$ . It then follows that  $PH \cup Q(C - H)$  is in fact a min path. Accordingly, we will show that  $R \stackrel{\Delta}{=} P(A - B) \cup Q(C - (A - B))$  is a path whenever  $P$  and  $Q$  are min paths both intersecting  $A - B$ .

Let  $E$  denote  $A \cup B$ . Suppose  $P$  and  $Q$  both intersect  $A - B$ ,  $P$  and  $Q$  being min paths, and set  $R = P(A - B) \cup Q(C - (A - B))$ . For  $R_1 = PA \cup Q(C - A)$  and  $R_2 = QA \cup P(C - A)$ , both are min paths by modularity of  $A$ . If  $PAB = \emptyset$ , then  $R_1 \subseteq R$ , showing  $R$  is a path, so we assume  $PAB \neq \emptyset$ . But then both  $R_1$  and  $P$  intersect  $B$ , so  $R_3 = R_1B \cup P(C - B) = PA \cup Q(B - A) \cup P(C - E)$  is a min path. Now, if  $QAB \neq \emptyset$ , then  $R_1$  and  $Q$  intersect  $AB$ , a module, so  $QAB \cup R_1(C - AB) = R$  is a min path, as needed. However, we show  $QAB = \emptyset$  leads to a contradiction:

either  $P(B - A) \neq \emptyset$  which  $\Rightarrow R_2$  and  $P$  intersect  $B \Rightarrow$   
 $PB \cup R_2(C - B) \supset R_2$ , a contradiction  
 or  $P(B - A) = \emptyset \Rightarrow P \subseteq R_3 \Rightarrow Q(B - A) = \emptyset$ . Now choose a min path  
 $P_0$ ,  $P_0(B - A) \neq \emptyset$  (possible because we are assuming all components essential)  
 $\Rightarrow R_4 = P_0B \cup P(C - B)$  is a min path, intersecting  $A$ , so  $QA \cup R_4(C - A)$   
 $= Q(A - B) \cup P_0(B - A) \cup P(C - E) \supset R_2 = Q(A - B) \cup P(C - E)$ , a  
 contradiction. ||

Two Modules Lemma:

Let  $A$  be a nonempty proper subset of  $C$  and  $(C, \phi)$  be a coherent system all of whose components are essential, with min paths  $P$ . If  $A$  and  $C - A$  are modules of  $(C, \phi)$ , then:

- either (1)  $\forall P \in \mathcal{P}, P \subseteq A$  or  $P \subseteq C - A$   
 or (2)  $\forall P \in \mathcal{P}, PA \neq \emptyset$  and  $P(C - A) \neq \emptyset$ .

If (1) holds, we say  $A$  and  $C - A$  are in parallel, while if (2) holds, we say  $A$  and  $C - A$  are in series.

Proof:

Suppose (1) fails while  $A$  and  $C - A$  are modules. Then  $\exists P_0 \in \mathcal{P}$  such that  $P_0 A \neq \emptyset$  and  $P_0(C - A) \neq \emptyset$ . It follows that (2) holds, for if  $Q \in \mathcal{P}$  is such that  $Q \subseteq A$ , then  $P_0 A \cup Q(C - A) = P_0 A$  is a min path, however,  $P_0 A \neq P_0$ , which is a contradiction. The same reasoning shows  $\nexists Q \in \mathcal{P}, Q \subseteq C - A$ .

The parallel and series designation is not arbitrary; indeed if  $(A, \Gamma')$  and  $(C - A, \Gamma'')$  are modules of  $(C, \phi)$ , then

$$\phi(X) = \text{Max} \{ \Gamma'(X|_A), \Gamma''(X|_{C-A}) \}$$

or

$$= \text{Min} \{ \Gamma'(X|_A), \Gamma''(X|_{C-A}) \}$$

according as (1) or (2) holds respectively. ||

Three Modules Theorem:

Let  $(C, \phi)$  be any coherent system with all components essential. If  $A$  and  $B$  are modules such that  $A - B$ ,  $AB$  and  $B - A$  are nonempty, then

- (1)  $A - B$ ,  $AB$  and  $B - A$  are modules.

(2)  $A \Delta B$  and  $A \cup B$  are modules.

(3) The three modules  $A - B$ ,  $AB$  and  $B - A$  appear in either parallel or series.

Proof:

(1) follows by applying the previous lemmas. (2) and (3) will follow by using the lemma below and repeated application of the definitions.

Lemma:

Under the hypothesis of the theorem, set  $A_1 = A - B$ ,  $A_2 = AB$ ,  $A_3 = B - A$  and  $E = A \cup B$ . Then

either (1)  $\forall$  min paths  $P \ni PE \neq \emptyset$  we have  $PE \subseteq A_1$  or  $PE \subseteq A_2$  or

$$PE \subseteq A_3$$

or (2)  $\forall$  min paths  $P \ni PE \neq \emptyset$  we have  $PA_1 \neq \emptyset$ ,  $PA_2 \neq \emptyset$  and

$$PA_3 \neq \emptyset.$$

Proof:

In general, if  $(A, \Gamma)$  is a module of  $(C, \phi)$  and  $B$  is any nonempty subset of  $A$ , then  $(B, \Gamma')$  is a module of  $(A, \Gamma)$  if and only if it is a module of  $(C, \phi)$ . This allows us to apply the two modules lemma to  $A_1 \cup A_2$  and  $A_2 \cup A_3$ . Recall we have set  $E = A_1 \cup A_2 \cup A_3$ .

Suppose  $\exists$  min path  $P_0 \ni P_0 E \neq \emptyset$  and  $P_0 E \subseteq A_2$ . Then the above remark shows (1) holds, provided  $\nexists$  min path  $R \ni RA_1 \neq \emptyset$  and  $RA_3 \neq \emptyset$ . But if  $\exists$  min path  $R_0 \ni R_0 A_1 \neq \emptyset$  and  $R_0 A_3 \neq \emptyset \Rightarrow P_0(A_1 \cup A_2) \cup R_0(A_3 \cup (C - E))$  is a min path which intersects both  $A_2$  and  $A_3$  which is a contradiction.

The above fails if and only if  $\forall$  min paths  $P \ni PE \neq \emptyset$  we have  $PE \subseteq A_2$ . Then the above remark shows (2) holds, provided  $\exists$  min path  $P \ni PA_1 \neq \emptyset$  and  $PA_3 \neq \emptyset$ . Let  $P_0$  be a min path  $\ni P_0 A_2 \neq \emptyset$ . Either  $P_0 A_1 \neq \emptyset$  or  $P_0 A_3 \neq \emptyset$ .

Without loss of generality, we assume  $P_0 A_1 \neq \emptyset$ . Let  $Q_0$  be a min path  $\ni Q_0 A_3 \neq \emptyset \Rightarrow Q_0 (A_2 \cup A_3) \cup P_0 (A_1 \cup (C - E))$  is a min path which intersects both  $A_1$  and  $A_3$ . ||

Returning to the theorem, suppose first (1) of the above lemma holds. To show  $E = A \cup B$  is modular, if  $P$  and  $Q$  are min paths  $\ni PE \neq \emptyset$  and  $QE \neq \emptyset$ , we show  $PE \cup Q(C - E)$  is a min path. This is clearly true if  $PE \subseteq A_1 \cup A_2$  and  $QE \subseteq A_1 \cup A_2$  or if  $PE \subseteq A_2 \cup A_3$  and  $QE \subseteq A_2 \cup A_3$ . One remaining possibility is  $PE \subseteq A_1$  and  $QE \subseteq A_3$ . Let  $P_0$  be a min path  $\ni P_0 A_2 \neq \emptyset$ . Then  $P_0 E \subseteq A_2$  and  $P_0 (A_2 \cup A_3) \cup Q(A_1 \cup (C - E)) = P_0 A_2 \cup Q(C - E)$  is a min path which intersects  $A_1 \cup A_2$ , so  $P(A_1 \cup A_2) \cup (P_0 A_2 \cup Q(C - E))(A_3 \cup (C - E)) = PE \cup Q(C - E)$  is a min path, which was to be shown. The only remaining possibility,  $PE \subseteq A_3$  and  $QE \subseteq A_1$ , is similar.

The modularity of  $A \cup B$  shown above allows us to apply the two modules lemma to  $A_1 \cup A_2 \cup A_3$ , then to  $A_1 \cup A_2$ . We conclude if  $(A \cup B, \Gamma)$ ,  $(A_1, \Gamma_1)$ ,  $(A_2, \Gamma_2)$  and  $(A_3, \Gamma_3)$  are the modules of  $(C, \phi)$  indicated above, when (1) of the above lemma holds, then  $\Gamma(X|_{A \cup B}) = \text{Max} \{ \Gamma_1(X|_{A_1}), \Gamma_2(X|_{A_2}), \Gamma_3(X|_{A_3}) \}$ , i.e., the modules  $A_1$ ,  $A_2$  and  $A_3$  appear in parallel. This shows  $A \triangle B = A_1 \cup A_3$  is also a module.

Now suppose (2) of the above lemma holds. We show  $E$  is a module. Let  $P$  and  $Q$  be min paths  $\ni PE \neq \emptyset$  and  $QE \neq \emptyset$ . By definition,  $P(A_2 \cup A_3) \cup Q(A_1 \cup (C - E))$  is a min path intersecting  $A_1 \cup A_2$ , so  $P(A_1 \cup A_2) \cup (P(A_2 \cup A_3) \cup Q(A_1 \cup (C - E)))(A_3 \cup (C - E)) = PE \cup Q(C - E)$  is a min path, showing  $E$  is a module. Again by using the two modules lemma, if  $(A \cup B, \Gamma)$ ,  $(A_1, \Gamma_1)$ ,  $(A_2, \Gamma_2)$  and  $(A_3, \Gamma_3)$  are modules of  $(C, \phi)$ , when (2) of the above lemma holds, then  $\Gamma(X|_{A \cup B}) = \text{Min} \{ \Gamma_1(X|_{A_1}), \Gamma_2(X|_{A_2}), \Gamma_3(X|_{A_3}) \}$ , i.e., the modules  $A_1$ ,  $A_2$  and  $A_3$  appear in series. Again, it follows  $A \triangle B$  is a module. ||

### F. Extensions to Systems with Inessential Components

We have focused our attention on coherent systems with all components essential, principally to duplicate the results of [1] and also because the statements in this case are aesthetically more pleasing. We will give a brief review of how the previous results extend without this hypothesis.

Firstly, the definition of a module remains the same in the presence of inessential components. Proposition 2 also remains the same, except that we can no longer assert uniqueness of the function  $\Gamma$  when the module  $A$  consists entirely of inessential components. In fact,  $\Gamma$  could be either identically one or identically zero on  $\{0,1\}^C$ . Either choice would satisfy the equation in Proposition 2. Of course, the assertion relating the min paths and min cuts of  $(A, \Gamma)$  and  $(C, \phi)$  is no longer valid in this case.

To maintain the difference lemma when inessential components may be present, we can replace the hypothesis that  $A - B$  and  $B - A$  are both nonempty by the statement,  $A - B$  and  $B - A$  each contain at least one essential component. The intersection lemma did not assume any components to be essential. The two modules lemma remains valid, except that some care must be taken in stating the parallel-series representation because of the ambiguity of  $\Gamma'$  and  $\Gamma''$  when  $A$  and  $C - A$  respectively consist entirely of inessential components. The three modules theorem must be altered by replacing the assumption that  $A - B$ ,  $AB$  and  $B - A$  are nonempty by the hypothesis that  $A - B$ ,  $AB$  and  $B - A$  each contain at least one essential component. The proof of each new lemma and theorem are exactly as before.

### G. An Application

An excellent application of the three modules theorem concerns "maximal" modules. We give the results here and refer the interested reader to [1] for the proof.

We will say a module  $M \neq C$  of  $(C, \phi)$  is *maximal* if it is set maximal with respect to being a module other than  $C$ .

Let  $M$  be the set of maximal modules of  $(C, \phi)$ , a coherent system with all components essential. Then

either (1)  $M$  is a partition of  $C$

or (2)  $\bar{M} \triangleq \{C - M \mid M \in M\}$  is a set of modules which partition  $C$  and which appear in either series or parallel.

## 2. BLOCKING SYSTEMS

Blocking systems are studied extensively in [7], [4] and [8]. We show below they are fundamentally the same mathematical notion as coherent systems.

A basic notion for what follows is that of a clutter. We say a family  $F$  of subsets of the set  $C$  is a clutter on  $C$  whenever no member of  $F$  contains another member of  $F$ . This is exactly the property which characterizes those families of subsets of  $C$  which are the min path sets of some coherent system with components  $C$ .

Let  $C$  be a finite nonempty set and let  $P$  and  $K$  be families of subsets of  $C$ . We say  $(C, P, K)$  is a *blocking system* when

- (1) both  $P$  and  $K$  are clutters on  $C$

and

- (2)  $\forall A \subseteq C$ , either  $\exists P \in P \ni P \subseteq A$  or  $\exists K \in K \ni K \subseteq C - A$ ,  
but not both.

Our definition follows that given in [7], except that there the set  $C$  may be empty.

If  $(C, \phi)$  is a coherent system, then for every subset  $A \subseteq C$ , either  $A$  is a path ( $\phi(1_A) = 1$ ) or  $C - A$  is a cut ( $\phi(1_A) = 0$ ), but not both. It follows that if  $P$  and  $K$  are the min paths and min cuts respectively of  $(C, \phi)$ , then  $(C, P, K)$  is a blocking system. This mapping of coherent systems to blocking systems is one-to-one since a coherent system is characterized by its min paths and min cuts. Further, we see the mapping is onto because the blocking system  $(C, P, K)$  is the image of the coherent system  $(C, \phi)$  when

$$\phi(X) \triangleq \begin{cases} 1 & \text{if } \{c \mid X(c) = 1\} \supseteq P & \text{for some } P \in P \\ 0 & \text{if } \{c \mid X(c) = 0\} \supseteq K & \text{for some } K \in K \end{cases}$$

We will express this one-to-one onto mapping between coherent systems and blocking systems by writing  $(C, \phi) \sim (C, P, K)$  whenever  $(C, \phi)$  is mapped to  $(C, P, K)$ . In [4], a correspondence from blocking systems into switching functions (all functions  $\phi$  on  $\{0,1\}^C$  to  $\{0,1\}$ ) is mentioned briefly, however, the "monotonic" switching functions (those  $\phi$  for which  $(C, \phi)$  is a coherent system) are not identified.

In [7], the blocking system  $(C, P, K)$  has as its dual  $(C, K, P)$ . This notion of duality coincides with ours since  $(C, \phi) \sim (C, P, K)$  if and only if  $(C, \phi^d) \sim (C, K, P)$ .

Some results in the study of coherent systems have relevant meaning for blocking systems. For example, our Proposition 1 says that given a nonempty set  $C$  and a clutter  $P$  on  $C$ , there exists a unique clutter  $K$  on  $C$  such that  $(C, P, K)$  is a blocking system. This result is also given in [7].

The min-max theorem given in [7] has a very intuitive interpretation and proof in terms of coherent systems. This theorem says that if  $(C, P, K)$  is a blocking system and  $f$  a real-valued function defined on  $C$ , then

$$\max_{P \in \mathcal{P}} \min_{c \in P} f(c) = \min_{K \in \mathcal{K}} \max_{c \in K} f(c).$$

To interpret this result for coherent systems, we see it is equivalent to state the theorem for functions  $f$  which are nonnegative. Given any such  $f$ , take  $f(c)$  to be the functioning lifetime of component  $c$  in the coherent system corresponding to  $(C, P, K)$ . If all components begin service simultaneously, then the system functions until the first component to fail from each min path has failed, or  $\max_{P \in \mathcal{P}} \min_{c \in P} f(c)$ . Equivalently, the system fails as soon as every component in some

min cut has failed, or  $\min_{K \in \mathcal{K}} \max_{c \in K} f(c)$ .

The notion of a module for a blocking system could be stated directly using the definition and correspondence, however, we will give another characterization and then show its equivalence to modules.

Let  $(C, P, K)$  be a blocking system. We say a nonempty set  $A \subseteq C$  can be condensed if

either (1)  $A \subseteq C - UP$

or (2)  $(A, P|_A, K|_A)$

is a blocking system, where for a family  $F$  of subsets,  $F|_A$  denotes the nonempty intersections of  $A$  with the elements of  $F$ , that is,  $\{FA \mid F \in F, FA \neq \emptyset\}$ .

We remark that condition (2) is not equivalent to  $P|_A$  and  $K|_A$  just being clutters. For example, take  $C = \{1, 2, 3, 4, 5\}$ ,  $P = \{\{1, 3, 5\}, \{2, 3, 4\}, \{2, 5\}, \{1, 4\}\}$ ,  $K = \{\{1, 3, 5\}, \{2, 3, 4\}, \{1, 2\}, \{4, 5\}\}$  and  $A = \{1, 2, 4, 5\}$ . Then  $P|_A$  and  $K|_A$  are clutters, however,  $(A, P|_A, K|_A)$  is not a blocking system.

It is true that the sets which can be condensed in a blocking system are the modules of the corresponding coherent system. Such a set  $A$  can be replaced by a pseudo-element  $c_A$  as we show in the following proposition.

Proposition 3:

Let  $(C, P, K)$  be a blocking system and  $(C, \phi) = (C, P, K)$ . For any subset  $A \subseteq C$ , we have

$$A \text{ is a module of } (C, \phi) \Leftrightarrow A \text{ can be condensed in } (C, P, K)$$

$$\Rightarrow (C', P', K') \text{ is a blocking system}$$

where

$$C' = (C - A) \cup \{c_A\}, c_A \text{ being a pseudo-element}$$

replacing the set  $A$

$$P' = \{P \mid P \in P, PA = \emptyset\} \cup \{(P - A) \cup \{c_A\} \mid P \in P, PA \neq \emptyset\}$$

$$K' = \{K \mid K \in K, KA = \emptyset\} \cup \{(K - A) \cup \{c_A\} \mid K \in K, KA \neq \emptyset\}.$$

Proof:

We dispense first with the uninteresting case in which  $A$  contains only inessential components. Since the essential components of  $(C, \phi)$  are  $UP$ , we see the proposition holds for any nonempty set  $A$  with  $A \subseteq C - UP$ , or equivalently, for  $A$  which contains only inessential components. In this case,  $P' = P$  and  $K' = K$ .

We may now assume the  $A$  at hand contains at least one essential component, or equivalently, that  $A \not\subseteq C - UP$ . Suppose  $A$  is a module of  $(C, \phi)$ . Using Proposition 2 (which remains valid in these circumstances), we have that  $(\Gamma, A) \sim (A, P|_A, K|_A)$ , showing in particular that  $A$  can be condensed. Conversely, if  $A$  can be condensed, then  $(A, P|_A, K|_A)$  is a blocking system. To show  $A$  is a module of  $(C, \phi)$ , we will show  $R = PA \cup Q(C - A)$  is a path of  $(C, \phi)$  whenever  $P$  and  $Q$  are min paths of  $(C, \phi)$  which intersect  $A$ . See the proof of the difference lemma for remarks concerning the sufficiency of this condition which appears weaker than the modularity of  $A$ . Suppose there were such a  $R = PA \cup Q(C - A)$  which is not a path. Then  $C - R$  would be a cut and hence contain at least one min cut, say  $K$ . Because in any coherent system every min path has a nonempty intersection with every min cut, it follows that  $KA \supseteq KQ \neq \emptyset$ . We also see  $(KA)(PA) = \emptyset$  because  $PA \subseteq R$  and  $K \subseteq C - R$ . This contradicts the hypothesis that  $(A, P|_A, K|_A)$  is a blocking system;  $PA$  and  $KA$  would be disjoint nonempty elements of  $P|_A$  and  $K|_A$  respectively.

The assertion that  $(C', P', K')$  is a blocking system follows from Proposition 2; a coherent system  $(C', \psi)$  is defined there for which  $(C', \psi) \sim (C', P', K')$ . An example showing the converse is false, namely, showing that  $(C', P', K')$  is a blocking system  $\not\Rightarrow A$  can be condensed, is  $C = \{1, 2, 3, 4, 5\}$ ,  $P = \{\{1, 3, 5\}, \{2, 3, 4\}, \{2, 5\}, \{1, 4\}\}$ ,  $K = \{\{1, 3, 5\}, \{2, 3, 4\}, \{1, 2\}, \{4, 5\}\}$  and  $A = \{2, 4\}$ . Then  $(C', P', K')$  is a blocking system but  $A$  can not be condensed. ||

The above proposition yields another characterization of modules in coherent systems which we believe should be isolated.

Proposition 4:

Let  $(C, \phi)$  be a coherent system. For any nonempty set  $A \subseteq C$ , we have  $A$  is a module of  $(C, \phi)$  if and only if

- either (1)  $A$  consists entirely of inessential components  
 or (2) the families  $P|_A$  and  $K|_A$  are the min paths and min cuts respectively of some coherent system with components  $A$ .

Proof:

Follows from Proposition 3. ||

The significance of Proposition 3 is that it outlines how a blocking system might be decomposed into pseudo-elements. For example, this decomposition could be the one indicated by the maximal modules partition theorem of Section 1.

As for applying the ideas surrounding blocking systems to coherent systems, it seems there might emerge a useful tool in computing system reliability for certain coherent systems, namely, those whose corresponding blocking system satisfies a length-width inequality (see [7]).

## REFERENCES

- [1] Birnbaum, Z. W. and J. D. Esary, "Modules of Coherent Binary Systems," Journal of the Society of Industrial Applied Mathematics, Vol. 13, No. 2, pp. 444-462, (June 1965).
- [2] Birnbaum, Z. W., J. D. Esary and A. W. Marshall, "A Stochastic Characterization of Wear-Out for Components and Systems," Annals of Mathematical Statistics, Vol. 37, No. 4, pp. 816-825, (August 1966).
- [3] Birnbaum, Z. W., J. D. Esary and S. C. Saunders, "Multi-Component Systems and Structures and Their Reliability," Technometrics, Vol. 3, No. 1, pp. 55-77, (February 1961).
- [4] Edmonds, Jack and D. R. Fulkerson, "Bottleneck Extrema," RAND Report RM-5375-PR, (January 1968).
- [5] Esary, J. D. and A. W. Marshall, "System Structure and the Existence of a System Life," Technometrics, Vol. 6, (1964).
- [6] Esary, J. D. and F. Proschan, "Coherent Structures of Non-Identical Components," Technometrics, Vol. 5, No. 2, pp. 191-209, (May 1963).
- [7] Fulkerson, D. R., "Networks, Frames, Blocking Systems," MATHEMATICS OF THE DECISION SCIENCES, Part I, Vol. II, pp. 303-334.
- [8] Fulkerson, D. R., "Blocking Polyhedra," RAND Report RM-5834-PR, (December 1968).

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