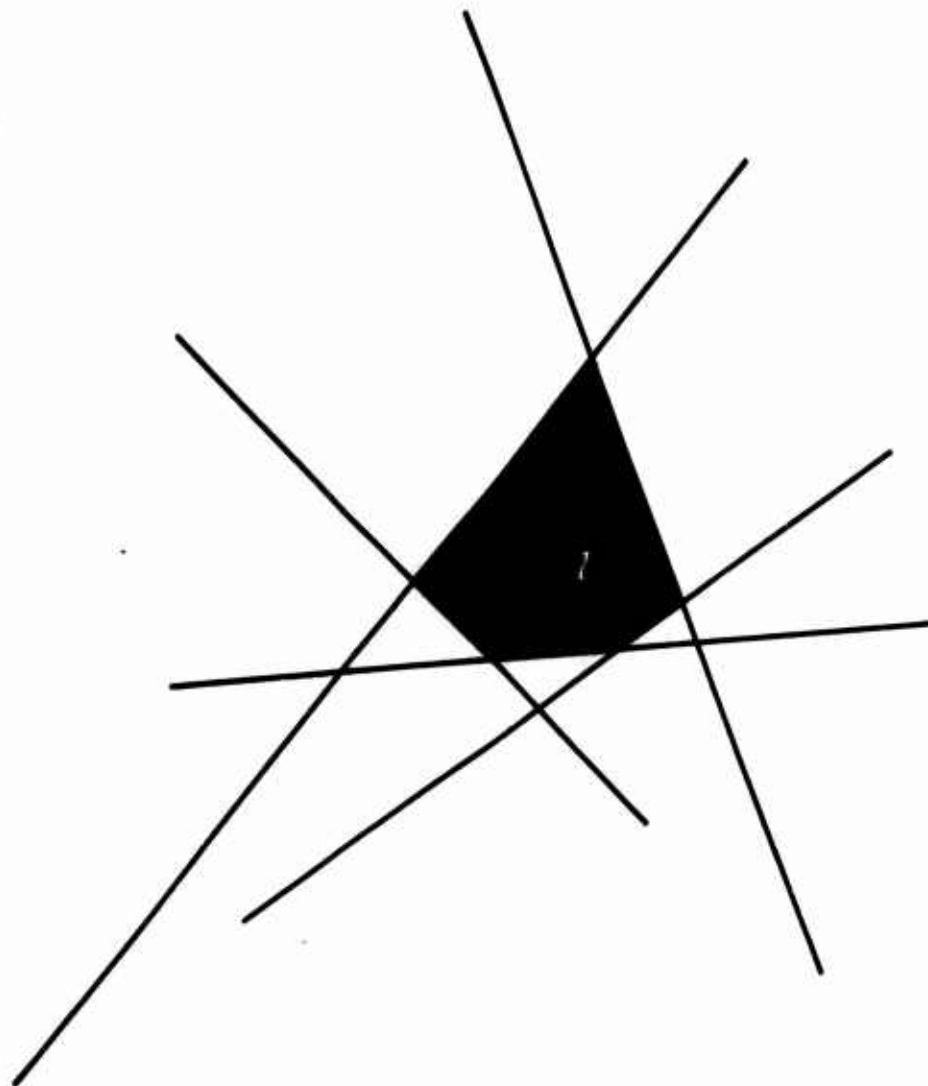


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ESTIMATION OF THE CHANGE POINT OF THE GENERALIZED FAILURE RATE FUNCTION

by
SUBRAMANI ARUNKUMAR

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Subramani Arunkumar
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University of California, Berkeley

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I dedicate this thesis to my father, [REDACTED], whose constant encouragement and endless sacrifices have made my higher education possible.

ABSTRACT

The change point of a function is defined to be the point (assumed unique) that minimizes or maximizes the function.

Fixed and narrow "window" estimators are proposed and studied for the change point of the generalized failure rate function $r(x) = \frac{f(x)}{g[G^{-1}F(x)]}$ where F and G are distributions with densities f and g , respectively. For a given G and an unknown F , the change point is estimated by (1) estimating $r(x)$, relaxing the assumption of complete sample; and (2) minimizing the estimator of $r(x)$ over $x \in \Omega_n$ with Ω_n a grid on $(-\infty, \infty)$. The estimators are shown to be consistent and their asymptotic distributions are derived using theorems on the convergence of distributions of stochastic processes. When G is the uniform distribution on $[0,1]$, estimation of the mode of a density falls out as a special case; and, by virtue of (1) and (2), the asymptotic results are shown to hold in this case under conditions more general than assumed by Chernoff (1964) and Venter (1967). Estimators have also been proposed when $r(x)$ is known to be U-shaped.

A computer program has been written in FORTRAN IV to obtain estimates of the change point of density and failure rate functions. Several numerical investigations have indicated the superiority of a particular estimator in the case of small samples.

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CHAPTER I

INTRODUCTION

1. Literature Review

Estimators for the mode of a probability density function have been considered by Chernoff [5], Parzen [12] and Venter [18]. In each case, it has been assumed that there is a sample of n independent observations from a distribution F , the mode of whose density we wish to estimate. Consistency and asymptotic distribution of the estimators have been dealt with in each of the above papers. The companion problems of estimating probability density and failure rate functions have been considered by many authors. See, for example, Bray, Crawford and Proschan [4], Parzen [12], Rao [13]. Estimation of the monotone generalized failure rate function (defined in Section 3) has recently been dealt with by Barlow and van Zwet [1,2], which subsumes the cases of estimating monotone density and failure rate functions.

2. The Estimation Problem

Definition:

The *change point* of a function is the point (assumed unique) at which the function attains its global minimum (or its global maximum).

This thesis formulates estimation of the mode of a probability density function and of the point which minimizes a failure rate function, as a special case of a larger class of problems viz. estimation of the change point of the generalized failure rate function. The second generalization is the choice of grid points for observing and/or analyzing the data. Whereas both Chernoff and Venter assume a complete sample and use a grid based on order statistics for analyzing the data, it is shown that the

asymptotic results are not changed by choosing wider grids - an important fact for implementing the estimation procedure in real life.

3. Notation and Preliminaries

Let F be the unknown distribution function, with density f , from which the sample of observations is drawn and G be a known distribution function with density g . Let s_1, s_2 be the left and right end points of the support of F .

We define two transformations $\phi_F(x)$ and $\phi_F(y)$ as follows:

$$(3.1) \quad \phi_F(x) = G^{-1}F(x) ; \quad \phi_F(y) = \int_{s_1}^{F^{-1}(y)} g[G^{-1}F(u)] du \quad 0 \leq y \leq 1 .$$

Then

$$(3.2) \quad \frac{d\phi_F(x)}{dx} = \frac{f(x)}{g[G^{-1}F(x)]} ; \quad \left. \frac{d}{dy} \phi_F(y) \right|_{y=F(x)} = \frac{g[G^{-1}F(x)]}{f(x)}$$

when the derivatives exist.

Definition:

$r(x) = \frac{f(x)}{g[G^{-1}F(x)]}$ is called the *generalized failure rate function*.

The following two important cases are worth noting. Let

(i) G be the uniform distribution on $[0,1]$. Then $r(x) = f(x)$.

(ii) G be the exponential distribution on $[0,\infty)$ with mean 1. Then

$$r(x) = \frac{f(x)}{1 - F(x)}, \text{ the failure rate function of } F .$$

If F_n is the empirical distribution function for F , based on a sample of size n , the natural estimators of $\phi_F(x)$ and $\phi_F(y)$ are given by

$$(3.3) \quad \phi_{F_n}(x) = G^{-1}F_n(x) ; \quad \phi_{F_n}(y) = \int_{s_1}^{F_n^{-1}(y)} g[G^{-1}F_n(u)] du .$$

When $y = \frac{i}{n}$ and $X_1, \dots, X_n$ are the order statistics from F ,

$$\begin{aligned} \phi_{F_n}\left(\frac{i}{n}\right) &= \sum_{j=0}^{i-1} g[G^{-1}F_n(X_j)](X_{j+1} - X_j) \quad \text{where } X_0 = s_1 \\ &= \frac{1}{n} \cdot [\text{Generalized total time on test statistic}] . \end{aligned}$$

Of special interest are estimators based on grids with wider spacings than those provided by order statistics. We, therefore, define an analogue of the empirical distribution for more general grids. Let $\{w_{i,n}\}_{i=0}^{\infty}$ be a subdivision of $(-\infty, \infty)$ and define

$$(3.4) \quad F_n^*(x) = F_n(w_{i,n}) \quad w_{i,n} \leq x < w_{i,n+1} .$$

Remark 3.1:

If $\{w_{i,n}\}_{i=0}^{\infty}$ becomes dense on the support of F with probability 1, it can be proved along the lines of the proof of the Glivenko-Cantelli theorem that

$$(3.5) \quad P\left[\sup_{-\infty < x < \infty} |F_n^*(x) - F(x)| \rightarrow 0\right] = 1 .$$

Note that the grid $\{w_{i,n}\}_{i=0}^{\infty}$ (*observer's grid*) is the grid used in observing the data and, in general, need not be the same as the set $\Omega_n \equiv \{w_{i,n}\}_{i=0}^{\infty}$ (*analyzer's grid*) used in analyzing the data. With F_n^*

defined in (3.4), $\phi_{F_n^*}(x) = G^{-1}F_n^*(x)$ and $\phi_{F_n^*}(y) = \int_{s_1}^{F_n^{*-1}(y)} g[G^{-1}F_n^*(u)] du$

are well defined and estimate $\phi_F(x)$ and $\phi_F(y)$ respectively.

Within parenthesis is given the meaning of abbreviations used throughout this thesis.

a.s. (almost sure, almost surely)

w.p.1. (with probability 1)

[] ([a] is the largest integer less than or equal to a)

$\equiv$ (is defined to be)

a.s. (almost sure convergence or convergence w.p.1.)

$\xrightarrow{P}$ (convergence in probability)

$\xrightarrow{D}$ (convergence in distribution)

$O_p(\cdot)$ ($X_n \equiv O_p(x_n)$ if X_n/x_n is bounded in probability, i.e., if for each $\epsilon > 0$, there is an M_ϵ and an N_ϵ $\ni$

$$P(|X_n| \geq M_\epsilon x_n) \leq \epsilon \quad \forall n \geq N_\epsilon .)$$

$o_p(\cdot)$ ($X_n \equiv o_p(x_n)$ if $|X_n|/x_n \xrightarrow{P} 0$ as $n \rightarrow \infty$, i.e., if

$$P(|X_n| \geq \epsilon x_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{for each } \epsilon > 0 .)$$

|| (indicates end of a proof).

A final note on notation: all equations, lemmas, theorems etc. are numbered on their respective scales. The number referring to the chapter is dropped if the reference is made within the chapter.

4. Overview of Chapters

Some useful convergence properties of:

(i) a class of estimators for density functions based on F_n^* , and

(ii) the estimators $\phi_{F_n^*}$ and $\phi_{F_n^*}$

as well as conditions for the consistency of an estimator of the change point of a function are given in Chapter II. These results, besides being used in subsequent chapters, are interesting in their own right and useful in solving allied problems. See references [1], [2]. Also given are some known results on the weak convergence of probability measures.

The motivation for the proposed estimators stems from (3.2). In the derivation of asymptotic distributions, we place the following restrictions on the observer's and analyzer's grids:

$$(i) \quad \Omega_n \equiv \left\{ w_{i,n} \right\}_{i=0}^{\infty}$$

$$(ii) \quad w_{i+1,n} - w_{i,n} = O_p(cn^{-\alpha}) \quad \text{for all } i \text{ and } c > 0, \alpha > 0.$$

The grid Ω_n is said to be "wide" for $0 < \alpha < 1/3$ and "narrow" for $1/3 \leq \alpha \leq 1$. Of special interest will be the rate at which the grid spacings are required to converge to zero.

In Chapters III and IV, we confine ourselves to estimators based on the ϕ transformation. The value of x , not necessarily unique, which minimizes $\left[\phi_{F_n^*}^*(x+a) - \phi_{F_n^*}^*(x-a) \right]$ among all $x \in \Omega_n$ is said to estimate the change point. The interval " $2a$ " is called a *window* and the estimator, a *window estimator*. When a is fixed, the estimator, termed *fixed window estimator*, is considered in Chapter III and when $a \rightarrow 0$ as $n \rightarrow \infty$, the corresponding estimator, termed *narrow window estimator*, is considered in Chapter IV. Consistency and asymptotic distribution are dealt with in each case.

Estimators derived from the ϕ transformation are considered in Chapters V and VI. The value of x , not necessarily unique, which maximizes $\left[\phi_{F_n^*}^*(F_n^*(x) + b) - \phi_{F_n^*}^*(F_n^*(x) - b) \right]$ among all $x \in \Omega_n$ is said to estimate the change point. The case of fixed b is considered in Chapter V and

$b \rightarrow 0$ as $n \rightarrow \infty$ in Chapter VI. It is interesting to note that to insure the existence of the asymptotic distributions, if the window is fixed (narrow), the grid Ω_n is required to be narrow (wide). A comparison of the two narrow window estimators is made at the end of Chapter VI.

In conclusion, other estimators of the change point as well as a discussion of computational aspects are given in Chapter VII. Bray, Crawford and Proschan [4] deal with the maximum likelihood estimation of a U-shaped failure rate function and, as a by-product, estimate the change point. Analogously, the U-shaped generalized failure rate function may be estimated by methods similar to Barlow and van Zwet [1,2] and hence estimate the change point. The development of the estimation problem in this thesis presents a natural way of obtaining consistent estimates of the change point. Some recommendations regarding the choice of the windows and the grid Ω_n , and results of Monte Carlo investigations are included at the end of Chapter VII.

A computer program has been written in FORTRAN IV to obtain the narrow window estimates of the change point. A discussion of the program, along with its listing, is given in the Appendix.

CHAPTER II

SOME CONVERGENCE THEOREMS

1. A Class of Estimators for Density Functions

Let $\{w_{i,n}\}_{i=0}^{\infty}$ be a grid which becomes dense on the support of F w.p.l. as $n \rightarrow \infty$. We saw in Remark I.3.1 that $F_n^*(x)$, defined by

$$F_n^*(x) = F_n(w_{i,n}) \quad w_{i,n} \leq x < w_{i,n+1}$$

tends to $F(x)$ w.p.l. uniformly in x , where $F_n(x)$ is the empirical distribution function for $F(x)$.

To estimate $f(x)$, the density of $F(x)$, we consider a statistic of the form:

$$(1.1) \quad f_n^*(x) = \frac{1}{h} \int_{-\infty}^x K\left(\frac{x-u}{h}\right) dF_n^*(u)$$

where $K(x)$ is a certain density function and $h \rightarrow 0$ as $n \rightarrow \infty$. Such estimates have been studied by Nadaraya [11] and Parzen [12] when $F_n^* \equiv F_n$.

Theorem 1.1:

Let the following assumptions hold:

(1.A1) $K(x)$ is a function of bounded variation (with bound μ).

(1.A2) $f(x)$ is uniformly continuous.

(1.A3) $\sum_{n=1}^{\infty} e^{-\gamma n h^2} < \infty$ for every positive γ .

(1.A4) $w_{i+1,n} - w_{i,n} = o(h)$ for all i .

Then

$$(1.2) \quad P \left[\lim_{n \rightarrow \infty} \sup_{-\infty < x < \infty} |f_n^*(x) - f(x)| = 0 \right] = 1 .$$

Proof:

$$\text{Let } f_n(x) = \frac{1}{h} \int_{-\infty}^{\infty} K\left(\frac{x-u}{h}\right) dF_n(u)$$

$$\begin{aligned} \sup_{-\infty < x < \infty} |f_n^*(x) - f_n(x)| &= \sup_{-\infty < x < \infty} \left| \frac{1}{h} \int_{-\infty}^{\infty} K\left(\frac{x-u}{h}\right) dF_n^*(u) \right. \\ &\quad \left. - \frac{1}{h} \int_{-\infty}^{\infty} K\left(\frac{x-u}{h}\right) dF_n(u) \right| \\ &\leq \sup_{-\infty < x < \infty} \left[\frac{1}{h} \int_{-\infty}^{\infty} |F_n^*(u) - F_n(u)| \left| dK\left(\frac{x-u}{h}\right) \right| \right] \\ &\leq \sup_{-\infty < x < \infty} |F_n^*(x) - F_n(x)| \frac{\mu}{h} \\ &\leq \sup_1 \frac{F_n(w_{i+1,n}) - F_n(w_{i,n})}{h} \cdot \mu \end{aligned}$$

$\rightarrow 0$ w.p.1. by (1.A4) and Theorem 1, [11].

From Nadaraya [11], $f_n(x) \xrightarrow{a.s.} f(x)$ uniformly in x . Hence $f_n^*(x) \xrightarrow{a.s.} f(x)$ uniformly in x . ||

2. Properties of the $\phi_{F_n^*}$ and ϕ_{F_n} Transformations

Theorem 2.1:

If the support of G is an interval, then

$$(2.1) \quad P \left[\left| \phi_{F_n^*}(x) - \phi_F(x) \right| \rightarrow 0 \right] = 1 \quad \text{for each } x .$$

In addition, if the support of G is bounded, then

$$(2.2) \quad P \left[\sup_{-\infty < x < \infty} |\phi_{F_n}^*(x) - \phi_F(x)| \rightarrow 0 \right] = 1 .$$

Proof:

Since G is strictly increasing, $G^{-1}(y)$ is continuous in y , $0 \leq y \leq 1$. By Remark I.3.1, $F_n^*(x) \xrightarrow{a.s.} F(x)$ uniformly in x and (2.1) follows from the continuity of G^{-1} .

If the support of G is finite, by Proposition 16, p. 164, Royden [15], $G^{-1}(y)$ is uniformly continuous in y , $0 \leq y \leq 1$. (2.2) now follows from Remark I.3.1. ||

The following lemma is any easy consequence of Proposition 6f.2(i), p. 355, Rao [14] and the Glivenko-Cantelli Theorem.

Lemma 2.1:

Let the support of F be an interval.

(2.3) If the support of F is not bounded, then

$$P \left[|F_n^{-1}(y) - F^{-1}(y)| \rightarrow 0 \right] = 1 \quad \text{for } 0 < y < 1 .$$

(2.4) If the support of F is bounded, then

$$P \left[\sup_{0 \leq y \leq 1} |F_n^{-1}(y) - F^{-1}(y)| \rightarrow 0 \right] = 1 .$$

We give below conditions for the strong uniform convergence of $\phi_{F_n}^*(y)$ to $\phi_F(y)$. Weak consistency of $\phi_{F_n}^*$ is shown in Theorem (2.3) under less stringent assumptions on the grid $\{w_{i,n}\}_{i=0}^{\infty}$.

The following theorem is due to Barlow and van Zwet [1].

Theorem 2.2:

Let conditions (2.A1), (2.A2) and either (2.A3) or (2.A4) and (2.A5) be satisfied.

(2.A1) The support of F is an interval.

(2.A2) Either $F_n^* \equiv F_n$ or the grid $\{w_{i,n}\}_{i=0}^{\infty}$ becomes dense w.p.1. on

$$(-\infty, \infty) \text{ and } \frac{w_{i,n}}{w_{i-1,n}} \leq M \text{ w.p.1. } \forall i, \text{ for some } M < \infty.$$

(2.A3) $F^{-1}(0) > -\infty$ and $F^{-1}(1) < \infty$.

(2.A4) $F^{-1}(0) > -\infty$, $F^{-1}(1) = \infty$, $\int_{s_1}^{\infty} x dF(x) < \infty$, $\int_{s_1}^{\infty} g[G^{-1}F(x)] dx < \infty$.

(2.A5) $gG^{-1}(\cdot)$ has a continuous derivative ψ on $[0,1]$.

Then

$$(2.5) \quad P \left[\sup_{0 \leq y \leq 1} |\phi_{F_n^*}(y) - \phi_F(y)| \rightarrow 0 \right] = 1.$$

Theorem 2.3:

Let the following conditions be satisfied.

(2.A6) $G(x)$ has a continuous derivative $g(x)$ in the interior of its support.

(2.A7) The support of F is a finite interval.

(2.A8) The probability that the grid $\{w_{i,n}\}_{i=0}^{\infty}$ becomes dense on $[s_1, s_2]$ approaches 1 as $n \rightarrow \infty$.

Then, for any $\varepsilon > 0$

$$(2.6) \quad \lim_{n \rightarrow \infty} P \left[\left| \phi_{F_n}^*(y) - \phi_F(y) \right| < \varepsilon \right] = 1 \quad 0 \leq y \leq 1 .$$

Proof:

$$\phi_{F_n}^*(y) - \phi_F(y) = \int_{s_1}^{F_n^{*-1}(y)} g \left[G^{-1} F_n^*(u) \right] du - \int_{s_1}^{F^{-1}(y)} g \left[G^{-1} F(u) \right] du .$$

Expanding in Taylor's series about $F^{-1}(y)$,

$$(2.7) \quad \phi_{F_n}^*(y) - \phi_F(y) = \left[F_n^{*-1}(y) - F^{-1}(y) \right] g \left[G^{-1} F(x_n) \right]$$

where x_n lies between $F_n^{*-1}(y)$ and $F^{-1}(y)$.

$$\left| F_n^{*-1}(y) - F^{-1}(y) \right| \leq |w_{i+1,n} - w_{i,n}|$$

$$+ \left| F_n^{-1}(y) - F^{-1}(y) \right|$$

$\xrightarrow{P} 0$, by (2.A8) and Lemma 2.1.

This proves the theorem. ||

3. A Consistency Condition

We give below a strong consistency condition for the estimator of the change point of an arbitrary function ψ defined on some interval $[a,b]$.

Let θ be the change point of $\Psi(x)$, i.e., θ minimizes $\Psi(x)$,
 and $\Psi_n(x)$ estimates $\Psi(x)$. $\hat{\theta}_n$ minimizes $\Psi_n(x)$ among all
 $x \in \Omega_n = \left\{ \omega_{i,n} \right\}_{i=0}^{\infty}$.

Theorem 3.1:

Assumptions:

- (3.A1) $\Psi_n(x) \xrightarrow{a.s.} \Psi(x)$ uniformly in $x \in [a,b]$.
- (3.A2) θ , assumed unique, minimizes $\Psi(x)$.
- (3.A3) $\hat{\theta}_n$ minimizes $\Psi_n(x)$ where x is confined to Ω_n .
- (3.A4) Ω_n is a grid on $[a,b]$ such that w.p.1. it becomes dense in some neighborhood of θ .
- (3.A5) For all δ small enough $\alpha(\delta) > 0$ where

$$\alpha(\delta) = \alpha_2(\delta) - \alpha_1(\delta)$$

$$\alpha_1(\delta) = \max \{ \Psi(x) : \theta - \delta \leq x \leq \theta + \delta \}$$

$$\alpha_2(\delta) = \min \{ \Psi(x) : a < x \leq \theta - 2\delta, \theta + 2\delta \leq x < b \}.$$

Then

$$(3.1) \quad \hat{\theta}_n \xrightarrow{a.s.} \theta.$$

Proof:

For δ arbitrary, but fixed, choose $\varepsilon = \alpha(\delta)/2$. Then

$\exists n_0 \equiv n_0(\varepsilon) \ni$ for all $n > n_0$

$$|\Psi_n(x) - \Psi(x)| < \varepsilon \quad \forall x.$$

By (3.A4), w.p.1. $\exists n_1 \geq n_0 \ni$ for all $n > n_1$, $|\omega_{k_n, n} - \theta| < \delta$ for some k_n . For $a < x \leq \theta - 2\delta$ or $\theta + 2\delta \leq x < b$, and $n > n_1$

$$\begin{aligned} \psi_n(x) - \psi(\omega_{k_n, n}) &> \psi(x) - \psi(\omega_{k_n, n}) - 2\epsilon \\ &\geq \alpha_2(\delta) - \alpha_1(\delta) - 2\epsilon \\ &= \alpha(\delta) - 2\epsilon \\ &= 0. \end{aligned}$$

But $\hat{\theta}_n$ minimizes $\psi_n(x) \Rightarrow \psi_n(\hat{\theta}_n) - \psi_n(\omega_{k_n, n}) \leq 0$. Hence

$\theta - 2\delta < \hat{\theta}_n < \theta + 2\delta$. Since δ may be arbitrarily small, it follows that

$$\hat{\theta}_n \xrightarrow{\text{a.s.}} \theta. \quad ||$$

Remark 3.1:

It is obvious that a corresponding result can be proved when the change point is defined to be the maximizing point of ψ .

Remark 3.2:

Let Ω_n be a grid determined by the order statistics from the underlying distribution F ; $\omega_{i, n} = X_i$ where X_i is the i th order statistic from F . If F is strictly increasing in a neighborhood of θ , then Ω_n becomes dense w.p.1. around θ .

4. Some Theorems on the Weak Convergence of Probability Measures

We give below some known results on weak convergence in the space of functions with at most discontinuities of the first kind.

4.1 Weak Convergence in $D[a,b]$

Let $C[a,b]$ denote the space of continuous functions on $[a,b]$ and $D[a,b]$ denote the space of functions on $[a,b]$ that are right-continuous and have a left-hand limit. We induce convergence in $D[a,b]$ by Skorokhod's J_1 -topology. It is well known that $C[a,b]$ with the supremum norm topology is a closed subset of $D[a,b]$ with J_1 -topology.

A sequence of stochastic processes X_n with trajectories in $D[a,b]$ a.s. is said to converge in distribution to another process X with trajectories in $D[a,b]$ a.s. if the measures ν_n induced by X_n on $D[a,b]$ converge weakly to the measure ν induced by X on $D[a,b]$.

Weak convergence in $D[a,b]$, when $[a,b]$ is a compact interval, is given in detail in Billingsley [3]. Following Stone [17], we extend this concept to $D^*(-\infty, \infty)$.

4.2 Weak Convergence in $D^*(-\infty, \infty)$

Let R be a complete, separable, metric space, with metric ρ . We denote by $D^*(-\infty, \infty)$ the space of all R -valued functions $x(t)$, $-\infty < t < \infty$, which have a limit from the left and are continuous from the right. Define on D^* the topology J_1 : a sequence $x_n(t)$ is said to be J_1 -convergent to $x(t)$ if there exists a sequence of continuous one-to-one mappings $\lambda_n(t)$ of the interval $(-\infty, \infty)$ onto itself such that for each $N > 0$

$$\sup_{-N \leq t \leq N} |\lambda_n(t) - t| \rightarrow 0 \quad \text{and} \quad \sup_{-N \leq t \leq N} \rho(x_n(t), x(\lambda_n(t))) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Note

that for continuous $x(t)$, $x_n(t)$ converges to $x(t)$ in the J_1 -topology if and only if for each $N > 0$

$$\sup_{-N \leq t \leq N} \rho(x_n(t), x(t)) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

A stochastic process W on $(-\infty, \infty)$ is said to be a two-sided *Wiener-Lévy process* if it is a Gaussian process with stationary independent increments with (i) $W(0) = 0$ (ii) $E[W(t)] = 0$ for $|t| < \infty$ (iii) $\text{Var}[W(t)] = |t|$. Further, from the law of iterated logarithm for a Wiener process, $W(t) \in C^*(-\infty, \infty)$ w.p.1., where $C^*(-\infty, \infty)$ is the space of all continuous function on $(-\infty, \infty)$ and hence, in particular, $W(t) \in D^*(-\infty, \infty)$.

From Stone [17] and problem 1, §15, Billingsley [3], we get necessary and sufficient conditions for the weak convergence of a sequence of random variables $X_n(t)$ to $X(t)$.

Theorem 4.1:

The sequence $X_n(t)$ is weakly convergent to $X(t)$ if and only if

(4.1) the finite dimensional distributions of $X_n(t)$ converge weakly to the finite dimensional distributions of $X(t)$ as $n \rightarrow \infty$ for t in some set everywhere dense on $(-\infty, \infty)$; and

(4.2) for $\epsilon > 0$ and $N > 0$

$$\lim_{\substack{n \rightarrow \infty \\ c \rightarrow 0}} P \left\{ \sup_{\substack{t-c < t_1 < t_2 < t+c \\ -N \leq t_1 < t_2 \leq N}} \min [\rho(X_n(t_1), X_n(t)) ; \rho(X_n(t), X_n(t_2))] > \epsilon \right\} = 0.$$

Further, if almost all the paths of $X(t)$ are continuous, then (4.2) may be replaced by the following simpler condition:

$$(4.3) \quad \lim_{n \rightarrow \infty; c \rightarrow 0} P \left\{ \sup_{|t_1 - t_2| \leq c; -N \leq t_1 < t_2 \leq N} \rho(X_n(t_1), X_n(t_2)) > \epsilon \right\} = 0.$$

(4.2) is the condition for the sequence of probability measures $\{P_n\}$ corresponding to $\{X_n\}$ to be relatively compact (Cf. Billingsley [3]).

(4.2) is equivalent to the following two conditions, either of which may be used to verify relative compactness of $\{P_n\}$.

(4.4) Condition: [Theorem 15.6, Billingsley [3]]

For each $N > 0$,

$$E \left\{ |X_n(t) - X_n(t_1)|^\gamma |X_n(t_2) - X_n(t)|^\gamma \right\} \leq [B(t_2) - B(t_1)]^{2\alpha}$$

for $-N \leq t_1 \leq t \leq t_2$ and $n \geq 1$ where $\gamma \geq 0$, $\alpha > \frac{1}{2}$ and B is a non-decreasing continuous function on $(-\infty, \infty)$.

(4.5) Condition: [Theorem 2.5.4, Rao [13]]

For each $N > 0$, there exist constants $\gamma_N > 0$, $C_N > 0$ independent of n such that for every $t_1, t_2 \in [-N, N]$

$$E \left\{ |X_n(t_1) - X_n(t_2)|^{\gamma_N} \right\} \leq C_N |t_1 - t_2|^2 + o(1) |t_1 - t_2|.$$

CHAPTER III

FIXED WINDOW ESTIMATORS BASED ON THE ϕ TRANSFORMATION

1. The Estimator $\hat{x}_a$

In this chapter, we shall be concerned with fixed window estimators using the ϕ transformation and "a" is the fixed window. Recall that $\phi_F(x) = G^{-1}F(x)$.

$$r(x) = \frac{d}{dx} \phi_F(x) = \frac{f(x)}{g[G^{-1}F(x)]}.$$

Hence $\frac{\phi_F(x+a) - \phi_F(x-a)}{2a}$ approximates $r(x)$. Define Ω_n as in Chapter I; i.e., let $-\infty < \omega_{0,n} < \omega_{1,n} < \dots < \omega_{i,n} < \dots < \infty$ be a subdivision of $(-\infty, \infty)$ and $\Omega_n = \{\omega_{i,n}\}_{i=0}^{\infty}$.

Definition:

The *pseudo change point* of $r(x)$ is given by $\tilde{x}_a$, assumed unique, minimizing $[\phi_F(x+a) - \phi_F(x-a)]$. $\hat{x}_a$, estimating $\tilde{x}_a$, minimizes

$$\left[\phi_{F_n}^*(x+a) - \phi_{F_n}^*(x-a) \right]$$

where x is restricted to Ω_n .

2. Consistency

When the support of G (an interval) is finite, we show in Theorem 2.1 that $\hat{x}_a$ is a strongly consistent estimator of $\tilde{x}_a$. When the support of G is infinite, it is shown in Theorem 2.2 that $\hat{x}_a$ converges to $\tilde{x}_a$ in probability.

Theorem 2.1:

Let the following assumptions be satisfied.

(2.A1) The support of G is a bounded interval.

(2.A2) Ω_n becomes dense w.p.1. in the neighborhood of $\tilde{x}_a$.

(2.A3) For all δ small enough, $\alpha(\delta) > 0$ where

$$\alpha(\delta) = \alpha_2(\delta) - \alpha_1(\delta)$$

$$\alpha_1(\delta) = \max \left\{ \phi_F(x+a) - \phi_F(x-a) : \tilde{x}_a - \delta \leq x \leq \tilde{x}_a + \delta \right\}$$

$$\alpha_2(\delta) = \min \left\{ \phi_F(x+a) - \phi_F(x-a) : s_1 < x \leq \tilde{x}_a - 2\delta, \tilde{x}_a + 2\delta \leq x < s_2 \right\}.$$

Then

$$(2.1) \quad \hat{x}_a \xrightarrow{a.s.} \tilde{x}_a.$$

Proof:

From Theorem II.2.1, $\phi_{F_n}^*(x) \xrightarrow{a.s.} \phi_F(x)$ uniformly in x . In Theorem

II.3.1, make the following identification:

$$\theta = \tilde{x}_a, \quad \hat{\theta}_n = \hat{x}_a$$

$$\Psi(x) = \phi_F(x+a) - \phi_F(x-a), \quad \Psi_n(x) = \phi_{F_n}^*(x+a) - \phi_{F_n}^*(x-a).$$

From Theorem II.3.1, $\hat{x}_a \xrightarrow{a.s.} \tilde{x}_a$. ||

Remark 2.1:

When G is the uniform distribution on $[0,1]$, $r(x) = f(x)$ and we get a stronger version of Theorem 1, Section 5, Chernoff [5].

Convergence of $\hat{x}_a$ When the Support of G is Not Finite.

Assumptions:

(2.A4) $[\phi_F(x+a) - \phi_F(x-a)]$ is continuous at $\tilde{x}_a$.

(2.A5) The support of G is an interval.

(2.A6) The probability that the set Ω_n becomes dense on the support of F approaches 1 as $n \rightarrow \infty$.

Lemma 2.1:

$$(2.2) \quad \min_{x \in \Omega_n} \left[\phi_{F_n}^*(x+a) - \phi_{F_n}^*(x-a) \right] \xrightarrow{P} \left[\phi_F(\tilde{x}_a + a) - \phi_F(\tilde{x}_a - a) \right].$$

Proof:

From Theorem II.2.1, $\phi_{F_n}^*(x) \xrightarrow{P} \phi_F(x)$ for each x .

$$\begin{aligned} \min_{x \in \Omega_n} \left[\phi_{F_n}^*(x+a) - \phi_{F_n}^*(x-a) \right] &\xrightarrow{P} \min_x \left[\phi_{F_n}^*(x+a) - \phi_{F_n}^*(x-a) \right] \quad \text{by (2.A6)} \\ &\xrightarrow{P} \min_x [\phi_F(x+a) - \phi_F(x-a)] \end{aligned}$$

since

$$(i) \quad \left[\phi_{F_n}^*(x+a) - \phi_{F_n}^*(x-a) \right] \xrightarrow{P} [\phi_F(x+a) - \phi_F(x-a)] \quad \text{for all } x, \text{ and}$$

(ii) $[\phi_F(x+a) - \phi_F(x-a)]$ is continuous at $\tilde{x}_a$ by (2.A4), the last step follows from Corollary 1 to Theorem 5.1, Billingsley [3]. ||

Theorem 2.2:

Under Assumptions (2.A4) - (2.A6),

$$(2.3) \quad \hat{x}_a \xrightarrow{P} \tilde{x}_a.$$

Proof:

From problem 40, p. 180, Royden [15] and by (II.2.1), $\phi_{F_n}^*$ converges continuously to ϕ_F w.p.1. Hence for sequence $\{x_n\} \rightarrow x$, $\left[\phi_{F_n}^*(x_n + a) - \phi_{F_n}^*(x_n - a) \right]$ converges continuously to $[\phi_F(x + a) - \phi_F(x - a)]$ w.p.1. The function $\phi_{F_n}^*$ and ϕ_F are measurable mappings from R to R .

Hence by Theorem 5.5, Billingsley [3], letting

$$h_n(x) = \phi_{F_n}^*(x + a) - \phi_{F_n}^*(x - a)$$

$$h(x) = \phi_F(x + a) - \phi_F(x - a)$$

we get by Lemma 2.1

$$h_n^{-1} \left\{ \min_{x \in \Omega_n} \left[\phi_{F_n}^*(x + a) - \phi_{F_n}^*(x - a) \right] \right\} \rightarrow h^{-1} \left\{ \min_x [\phi_F(x + a) - \phi_F(x - a)] \right\}$$

$$\text{i.e., } \hat{x}_a \xrightarrow{P} \tilde{x}_a . \quad ||$$

3. Asymptotic DistributionAssumptions:

(3.A1) $F(x)$ is continuous with density $f(x)$.

(3.A2) $r(\tilde{x}_a + a) > 0$; $r(\tilde{x}_a - a) > 0$.

(3.A3) $g'(x)/g^3(x)$ is bounded for x in the support of G (an interval).

(3.A4) $[\phi_F(x + a) - \phi_F(x - a)]$ is differentiable at $\tilde{x}_a$.

(3.A5) $\omega_{i+1,n} - \omega_{i,n} = o_p(n^{-1/3})$ (i.e., Ω_n is a narrow grid).

It is easy to see that these assumptions, which are necessary in the derivation of the asymptotic distribution, are sufficient to insure consistency of $\hat{x}_a$.

Since $\tilde{x}_a$ minimizes $[\phi_F(x+a) - \phi_F(x-a)]$, we have

$$(3.1) \quad r(\tilde{x}_a + a) = r(\tilde{x}_a - a).$$

Let

$$(3.2) \quad h_n(x) = [\phi_{F_n}(x+a) - \phi_{F_n}(x-a)].$$

$\tilde{x}_a$ minimizes $h_n(x)$ and hence minimizes

$$(3.3) \quad h_n(x) - h_n(\tilde{x}_a) = [\phi_{F_n}(x+a) - \phi_{F_n}(\tilde{x}_a+a)] - [\phi_{F_n}(x-a) - \phi_{F_n}(\tilde{x}_a-a)] \\ = Y_n + u$$

where

$$(3.4) \quad Y_n = \left\{ [\phi_{F_n}(x+a) - \phi_{F_n}(\tilde{x}_a+a)] - [\phi_F(x+a) - \phi_F(\tilde{x}_a+a)] \right\} \\ - \left\{ [\phi_{F_n}(x-a) - \phi_{F_n}(\tilde{x}_a-a)] - [\phi_F(x-a) - \phi_F(\tilde{x}_a-a)] \right\}$$

and

$$(3.5) \quad u = [\phi_F(x+a) - \phi_F(\tilde{x}_a+a)] - [\phi_F(x-a) - \phi_F(\tilde{x}_a-a)].$$

Let $\delta = x - \tilde{x}_a$. Note that $x \in \Omega_n$. Expanding in Taylor's series,

$$Y_n = \left\{ \frac{F_n(x+a) - F_n(\tilde{x}_a+a)}{g[G^{-1}F_n(\tilde{x}_a+a)]} - \frac{F(x+a) - F(\tilde{x}_a+a)}{g[G^{-1}F(\tilde{x}_a+a)]} \right\} \\ - \left\{ \frac{F_n(x-a) - F_n(\tilde{x}_a-a)}{g[G^{-1}F_n(\tilde{x}_a-a)]} - \frac{F(x-a) - F(\tilde{x}_a-a)}{g[G^{-1}F(\tilde{x}_a-a)]} \right\} \\ + O_p(n^{-1}\delta)$$

$$\begin{aligned}
&= \frac{r(\tilde{x}_a + a)}{f(\tilde{x}_a + a)} \left\{ \left[F_n(x + a) - F_n(\tilde{x}_a + a) \right] - \left[F(x + a) - F(\tilde{x}_a + a) \right] \right\} \\
&- \frac{r(\tilde{x}_a - a)}{f(\tilde{x}_a - a)} \left\{ \left[F_n(x - a) - F_n(\tilde{x}_a - a) \right] - \left[F(x - a) - F(\tilde{x}_a - a) \right] \right\} \\
&+ O_p(n^{-1/2}\delta) .
\end{aligned}$$

$$(3.6) \quad Y_n = \frac{r(\tilde{x}_a + a)}{f(\tilde{x}_a + a)} V_{n1}(\delta) - \frac{r(\tilde{x}_a - a)}{f(\tilde{x}_a - a)} V_{n2}(\delta) + O_p(n^{-1/2}\delta)$$

where

$$(3.7) \quad V_{n1}(\delta) = \left[F_n(\tilde{x}_a + a + \delta) - F_n(\tilde{x}_a + a) \right] - \left[F(\tilde{x}_a + a + \delta) - F(\tilde{x}_a + a) \right]$$

and

$$(3.8) \quad V_{n2}(\delta) = \left[F_n(\tilde{x}_a - a + \delta) - F_n(\tilde{x}_a - a) \right] - \left[F(\tilde{x}_a - a + \delta) - F(\tilde{x}_a - a) \right] .$$

Expanding u by Taylor's series, the first term about $(\tilde{x}_a + a)$, the second term about $(\tilde{x}_a - a)$ and noting that $r(\tilde{x}_a + a) = r(\tilde{x}_a - a)$, we see that

$$(3.9) \quad u = \frac{1}{2}\gamma\delta^2 + O_p(\delta^3)$$

where

$$(3.10) \quad \gamma = r'(\tilde{x}_a + a) - r'(\tilde{x}_a - a) .$$

$$\begin{aligned}
(3.11) \quad \therefore h_n(x) - h_n(\tilde{x}_a) &= \frac{r(\tilde{x}_a + a)}{f(\tilde{x}_a + a)} V_{n1}(\delta) - \frac{r(\tilde{x}_a - a)}{f(\tilde{x}_a - a)} V_{n2}(\delta) \\
&+ \frac{1}{2}\gamma\delta^2 + O_p(n^{-1/2}\delta) .
\end{aligned}$$

We note that

$$(3.12) \quad E[V_{n1}(\delta)] = E[V_{n2}(\delta)] = 0$$

$$(3.13) \quad \text{Var}[V_{n1}(\delta)] = \frac{f(\tilde{x}_a + a)|\delta|}{n} + O_p(n^{-1} \cdot \delta^2)$$

$$(3.14) \quad \text{Var}[V_{n2}(\delta)] = \frac{f(\tilde{x}_a - a)|\delta|}{n} + O_p(n^{-1} \cdot \delta^2) .$$

The correlation coefficient between $V_{n1}(\delta)$ and $V_{n2}(\delta)$ is given by

$$(3.15) \quad \rho_{12} = \frac{\text{Cov}[V_{n1}(\delta), V_{n2}(\delta)]}{\sqrt{\text{Var}[V_{n1}(\delta)] \cdot \text{Var}[V_{n2}(\delta)]}} \sim -\delta = O_p(\delta) .$$

Since $\hat{x}_a$ minimizes $h_n(x) - h_n(\tilde{x}_a)$, $\hat{\delta} = \hat{x}_a - \tilde{x}_a$ minimizes (3.11).

Let

$$(3.16) \quad \delta = \lambda t$$

$$(3.17) \quad r = r(\tilde{x}_a + a) = r(\tilde{x}_a - a)$$

$$(3.18) \quad W_n(t) = \left[\frac{\lambda r^2}{n} \left(\frac{1}{f(\tilde{x}_a + a)} + \frac{1}{f(\tilde{x}_a - a)} \right) \right]^{-\frac{1}{2}} \cdot \left[\frac{r}{f(\tilde{x}_a + a)} V_{n1}(\lambda t) - \frac{r}{f(\tilde{x}_a - a)} V_{n2}(\lambda t) \right] .$$

Then $\hat{t} = \lambda^{-1} \hat{\delta}$ minimizes

$$(3.19) \quad Z_n(t) = W_n(t) + \left[\frac{\lambda r^2}{n} \left(\frac{1}{f(\tilde{x}_a + a)} + \frac{1}{f(\tilde{x}_a - a)} \right) \right]^{-\frac{1}{2}} \left[\frac{1}{2} \lambda^2 t^2 + O_p(n^{-\frac{1}{2}} \delta) \right] .$$

Choose λ in (3.19) so that the coefficient of t^2 is one, i.e.,

$$\frac{1}{2} \left[\frac{\lambda r^2}{n} \left(\frac{1}{f(\tilde{x}_a + a)} + \frac{1}{f(\tilde{x}_a - a)} \right) \right]^{-1/2} \lambda^2 \gamma = 1.$$

$$(3.20) \quad \therefore \lambda = \left\{ \frac{4r^2}{n\gamma^2} \left[\frac{1}{f(\tilde{x}_a + a)} + \frac{1}{f(\tilde{x}_a - a)} \right] \right\}^{1/3}.$$

From (3.19), $\hat{t} = \lambda^{-1}(\hat{x}_a - \tilde{x}_a)$ minimizes

$$(3.21) \quad Z_n(t) = W_n(t) + t^2 + o_p(n^{-1/6}t).$$

Reduction to a Problem in Stochastic Processes

Lemma 3.1:

$W_n(t)$ is asymptotically normal with mean 0 and variance $|t|$, for all t .

Proof:

From (3.18) and (3.12), $W_n(0) = 0$ and $E[W_n(t)] = 0$ for all t . Since $\rho_{12} = o_p(n^{-1/3}t)$, $V_{n1}(\lambda t)$ and $V_{n2}(\lambda t)$ are asymptotically uncorrelated.

$$\therefore \text{Var} [W_n(t)] = |t| + o_p(n^{-1/3}t).$$

From DeMoivre-Laplace Theorem

$$\left(\frac{\lambda}{n} \right)^{-1/2} V_{n1}(\lambda t) \xrightarrow{D} N(0, f(\tilde{x}_a + a)|t|)$$

$$\left(\frac{\lambda}{n} \right)^{-1/2} V_{n2}(\lambda t) \xrightarrow{D} N(0, f(\tilde{x}_a - a)|t|)$$

and since $V_{n1}(\lambda t)$ and $V_{n2}(\lambda t)$ are asymptotically uncorrelated, we have

$$W_n(t) \xrightarrow{D} W(t) \sim N(0, |t|) . ||$$

Remark 3.1:

After a tedious calculation, it can be shown that for any collection of t_i , $t_1 \leq t_2 \leq \dots \leq t_k$ with $|t_i| < \infty$ for $1 \leq i \leq k$, the joint distribution of $[W_n(t_1), W_n(t_2), \dots, W_n(t_k)]$ converges to the multivariate normal distribution with mean 0 and variance-covariance matrix given by

$$(\delta(t_i, t_j) \min(|t_i|, |t_j|))$$

where

$$\delta(c, d) = \begin{cases} 1 & \text{if } c \text{ and } d \text{ are of the same sign} \\ 0 & \text{otherwise} \end{cases} .$$

With the above results, the main result of this section, Theorem 3.2, can be proved by arguments identical to those given in Sethuraman [16], pp. 112-117. We shall give an alternate proof using Theorem II.4.1.

Lemma 3.2:

For each $K > 0$ and for all $-N \leq t_1 \leq t_2 \leq N$, there exists a constant $C_N > 0$ independent of n , t_1 , t_2 such that

$$(3.22) \quad E\{|W_n(t_2) - W_n(t_1)|^4\} \leq C_N |t_2 - t_1|^2 + o(1) |t_2 - t_1| .$$

Proof:

$$\text{Let } c_2 = \frac{1}{f(\bar{x}_a + a)} ; c_3 = \frac{1}{f(\bar{x}_a - a)} ; c_1 = c_2 + c_3 . \text{ Then}$$

$$W_n(t) = \left(\frac{\lambda}{n} c_1\right)^{-\frac{1}{2}} [c_2 v_{n1}(\delta) - c_3 v_{n2}(\delta)] .$$

Let $t_1 \leq t_2$.

$$W_n(t_2) - W_n(t_1) = \left(\frac{\lambda}{n} c_1\right)^{-\frac{1}{2}} \{c_2[v_{n1}(\delta_2) - v_{n1}(\delta_1)] - c_3[v_{n2}(\delta_2) - v_{n2}(\delta_1)]\} .$$

From the elementary inequality $(x + y)^4 \leq 8x^4 + 8y^4$

$$(3.23) \quad E\{[W_n(t_2) - W_n(t_1)]^4\} \leq 8c_1^{-2}\lambda^{-2}n^2 \left\{c_2^4 E[v_{n1}(\delta_2) - v_{n1}(\delta_1)]^4 + c_3^4 E[v_{n2}(\delta_2) - v_{n2}(\delta_1)]^4\right\} .$$

If f is the value of the density at the mode, we note, after expanding in Taylor's series, that

$$F(\tilde{x}_a + a + \delta_2) - F(\tilde{x}_a + a + \delta_1) \leq f \cdot (\delta_2 - \delta_1)$$

$$F(\tilde{x}_a - a + \delta_2) - F(\tilde{x}_a - a + \delta_1) \leq f \cdot (\delta_2 - \delta_1) .$$

After a tedious calculation, we note that

$$n^2 E\{[v_{ni}(\delta_2) - v_{ni}(\delta_1)]^4\} \leq 18f^2(\delta_2 - \delta_1)^2 + \frac{f}{n}(\delta_2 - \delta_1) , \quad i = 1, 2$$

and hence

$$(3.24) \quad 8c_1^{-2}\lambda^{-2}n^2 \left\{c_2^4 E[v_{n1}(\delta_2) - v_{n1}(\delta_1)]^4 + c_3^4 E[v_{n2}(\delta_2) - v_{n2}(\delta_1)]^4\right\} \\ \leq C_N(t_2 - t_1)^2 + \frac{D}{n^{2/3}}(t_2 - t_1)$$

where

$$C_N = 144c_1^{-2}f^2(c_2^4 + c_3^4) > 0$$

$$D = 8c_1^{-2}f(c_2^4 + c_3^4) > 0 .$$

From (3.23) and (3.24)

$$E\{[W_n(t_2) - W_n(t_1)]^4\} \leq C_N |t_2 - t_1|^2 + o(1) |t_2 - t_1| \quad . \quad ||$$

By Remark 3.1 and Lemma 3.2, conditions (4.1) and (4.5) of Theorem II.4.1 are satisfied for the sequence $\{W_n(t)\}$ and hence

$$W_n(t) \xrightarrow{D} W(t) \\ t^2 + o_p(n^{-1/6}t) \xrightarrow{P} t^2 \quad .$$

Hence by an application of Slutsky's Theorem [Cf. Cramer [6], p. 254], we get

Theorem 3.1:

The distribution of $W_n(t) + t^2 + o_p(n^{-1/6}t)$ converges to the distribution of $W(t) + t^2$ where $W(t)$ is a two-sided Wiener-Lévy process with mean 0, variance 1 per unit t and $W(0) = 0$.

The Asymptotic Distribution of $\hat{x}_a$

Theorem 3.2:

$$(3.25) \quad \left\{ \frac{4r^2(\tilde{x}_a + a)}{n\gamma^2} \left[\frac{1}{f(\tilde{x}_a + a)} + \frac{1}{f(\tilde{x}_a - a)} \right] \right\}^{-1/3} (\hat{x}_a - \tilde{x}_a)$$

is asymptotically distributed as the value of t which minimizes the stochastic process $Z(t) = W(t) + t^2$, where $W(t)$ is a two-sided Wiener-Lévy process with mean 0 and variance 1 per unit t and $W(0) = 0$.

Proof:

Let $z(t) \in C^*(-\infty, \infty)$ and $k(z)$ be the value of t that minimizes $z(t)$.

- (i) $W(t) = O[2|t| \log \log |t|]^{\frac{1}{2}}$ as $|t| \rightarrow \infty$ (Cf. Chernoff [5]) and hence

$$W(t) + t^2 \approx t^2.$$

Therefore, $k[Z(t)]$ is bounded w.p.1.

- (ii) Since the distribution of $Z(t)$ has a nonzero density on $(-\infty, \infty)$ for each t , $Z(t)$ has a unique minimum w.p.1.
- (iii) Since all the trajectories of $W(t)$ are in $C^*(-\infty, \infty)$ w.p.1., the subset in $C^*(-\infty, \infty)$ on which k is continuous has probability 1 for the process $Z(t)$.

From Corollary 1 to Theorem 5.1, Billingsley [3], we see that

$$k[Z_n(t)] \xrightarrow{D} k[Z(t)].$$

Further, we need to impose the following restriction on $\{\omega_{i,n}\}_{i=0}^{\infty}$.
 Defining $\delta_i = \omega_{i,n} - \tilde{x}_a$, since $t = \lambda^{-1}\delta$ we have

$$t_{i+1} - t_i = \frac{\omega_{i+1,n} - \omega_{i,n}}{\lambda} \quad i = 0, 1, \dots$$

$$\xrightarrow{P} 0 \text{ uniformly in } i$$

if

$$\omega_{i+1,n} - \omega_{i,n} = o_p(n^{-1/3}) \quad \text{for } i = 0, 1, 2, \dots$$

Hence $\hat{t} = \lambda^{-1}(\hat{x}_a - \tilde{x}_a)$ is asymptotically distributed as the random variable which minimizes $W(t) + t^2$. ||

Remark 3.2:

Let ψ be the density of the random variable which minimizes $Z(t)$.

Chernoff [5] proves that

$$\psi(t) = \frac{1}{2} U_x(t^2, t) U_x(t^2, -t)$$

where $U(\cdot, \cdot)$ is the solution of the heat equation

$$\frac{1}{2} U_{xx} = -U_z$$

subject to the boundary conditions (i) $U(x, z) = 1$ for $x \geq z^2$ and
(ii) $U(x, z) \rightarrow 0$ as $x \rightarrow \infty$. Here U_x denotes the partial derivative of $U(x, z)$ with respect to x .

Remark 3.3:

If $\omega_{i+1, n} - \omega_{i, n} = cn^{-1/3}$, asymptotically one looks at the stochastic process $Z(t)$ only at certain fixed points t_i , with spacings given by

$$t_{i+1} - t_i = \frac{\gamma^{2/3} c}{\left[4r^2 \left(\frac{1}{f(\tilde{x}_a + a)} + \frac{1}{f(\tilde{x}_a - a)} \right) \right]^{1/3}} \quad i = 0, 1, \dots$$

and $t_0 > -\infty$, is arbitrary.

Remark 3.4:

When G is the uniform distribution on $[0, 1]$, $r(x) = f(x)$ and

$$\lambda = \left[\frac{8f(\tilde{x}_a + a)}{n\gamma^2} \right]^{1/3}$$

which is the same as Equation (3.10), Chernoff [5].

CHAPTER IV

NARROW WINDOW ESTIMATORS BASED ON THE ϕ TRANSFORMATION

1. The Estimator $\hat{x}_{a_n}$

Since $\phi_{F_n}^*(x) \xrightarrow{a.s.} \phi^*(x)$, a natural estimator for $r(x)$ is given by

$$(1.1) \quad \hat{r}_n(x) = \frac{\phi_{F_n}^*(x + a_n) - \phi_{F_n}^*(x - a_n)}{2a_n}$$

where $a_n \rightarrow 0$ as $n \rightarrow \infty$. We shall refer to $2a_n$ as a narrow window.

Definition:

The change point $\tilde{x}_0$, assumed unique, minimizes $r(x)$.

$\hat{x}_{a_n}$, the estimator of $\tilde{x}_0$, minimizes $\hat{r}_n(x)$ among all $x \in \Omega_n$.

2. Strong Consistency

If in (II.1.1)

$$K(y) = \begin{cases} \frac{1}{2} & |y| \leq 1 \\ 0 & |y| > 1 \end{cases}$$

we get

$$(2.1) \quad f_n^*(x) = \frac{F_n^*(x + h) - F_n^*(x - h)}{2h}.$$

Let $\delta > 0$ and

$$\alpha_1(\delta) = \max \left\{ r(x) : \tilde{x}_0 - \delta \leq x \leq \tilde{x}_0 + \delta \right\}$$

$$\alpha_2(\delta) = \min \left\{ r(x) : s_1 < x \leq \tilde{x}_0 - 2\delta, \tilde{x}_0 + 2\delta \leq x < s_2 \right\}$$

$$\alpha(\delta) = \alpha_1(\delta)/\alpha_2(\delta).$$

Theorem 2.1:

Suppose that the following assumptions hold.

(2.A1) F has a uniformly continuous density f .

(2.A2) $\sum_{n=1}^{\infty} e^{-\gamma n a_n^2}$ converges for every positive γ .

(2.A3) Either $F_n^* \equiv F_n$ or the grid $\{w_{i,n}\}_{i=0}^{\infty}$ is chosen such that $w_{i+1,n} - w_{i,n} = o(a_n)$.

(2.A4) The grid Ω_n becomes dense w.p.1. in a neighborhood of $\tilde{x}_0$.

(2.A5) For all δ small enough, $\alpha(\delta) < 1$.

Then

$$(2.2) \quad \hat{x}_{a_n} \xrightarrow{a_n s.} \tilde{x}_0.$$

Proof:

The proof is similar to the proof of Theorem 1, Venter [18]. For δ arbitrary, but satisfying (2.A5), $\exists n_0 \ni$ for $n > n_0$, $|\omega_{k_n,n} - \tilde{x}_0| < \delta$ for some sequence $\{k_n\}$ w.p.1., by (2.A4). Let

$$\theta_n = \omega_{k_n,n}$$

$$(2.3) \quad h_n(x) = \frac{G^{-1} F_n^*(x + a_n) - G^{-1} F_n^*(x - a_n)}{2a_n}.$$

Then

$$h_n(x) = \frac{F_n^*(x + a_n) - F_n^*(x - a_n)}{2a_n} \cdot \frac{1}{g[G^{-1}\beta_n(x)]}$$

where

$$F_n^*(x - a_n) \leq \beta_n(x) \leq F_n^*(x + a_n) .$$

Since $a_n \rightarrow 0$, by Remark I.3.1, $\beta_n(x) \xrightarrow{a.s.} F(x)$ uniformly in x .

$$h_n(\theta_n) = \frac{F_n^*(\theta_n + a_n) - F_n^*(\theta_n - a_n)}{2a_n} \cdot \frac{1}{g[G^{-1}\beta_n(\theta_n)]}$$

where $F_n^*(\theta_n - a_n) \leq \beta_n(\theta_n) \leq F_n^*(\theta_n + a_n)$. Choose $x \ni x \leq \tilde{x}_0 - 3\delta$ or $x \geq \tilde{x}_0 + 3\delta$. Then w.p.1., $\exists n_1 \geq n_0$ independent of x such that for all $n > n_1$

$$(2.4) \quad F(\tilde{x}_0 - \delta) \leq \beta_n(\theta_n) \leq F(\tilde{x}_0 + \delta)$$

and

$$(2.5) \quad \text{either } \beta_n(x) \leq F(\tilde{x}_0 - \delta) \text{ or } \beta_n(x) \geq F(\tilde{x}_0 + 2\delta) .$$

$$(2.6) \quad \frac{h_n(x)}{h_n(\theta_n)} = \frac{F_n^*(x + a_n) - F_n^*(x - a_n)}{F_n^*(\theta_n + a_n) - F_n^*(\theta_n - a_n)} \cdot \frac{f[F^{-1}\beta_n(\theta_n)]}{f[F^{-1}\beta_n(x)]} \cdot \frac{r[F^{-1}\beta_n(x)]}{r[F^{-1}\beta_n(\theta_n)]} .$$

Assumptions (2.A1), (2.A2) and (2.A3) imply, by Theorem II.1.1 when

$F_n^* \neq F_n$ and from Nadaraya [11] when $F_n^* \equiv F_n$, that

$$(2.7) \quad \frac{F_n^*(x + a_n) - F_n^*(x - a_n)}{2a_n} \xrightarrow{a.s.} f(x) ,$$

uniformly in x . Hence w.p.1., $\exists n_2 \geq n_1$, independent of x , $\exists$ for any two points x_1, x_2 and $n > n_2$

$$(2.8) \quad \frac{F_n^*(x_1 + a_n) - F_n^*(x_1 - a_n)}{F_n^*(x_2 + a_n) - F_n^*(x_2 - a_n)} \cdot \frac{f[F_n^{-1}(x_2)]}{f[F_n^{-1}(x_1)]} > \alpha(\delta).$$

Choose $n^* > n_2$ and let $\theta = \theta_n^*$. From (2.4) and (2.5), for $n \geq n^*$

$$(2.9) \quad \frac{r[F_n^{-1}(x)]}{r[F_n^{-1}(\theta)]} \geq \frac{\alpha_2(\delta)}{\alpha_1(\delta)} = \frac{1}{\alpha(\delta)} \quad \text{w.p.1.}$$

Hence, from (2.6), (2.8) and (2.9), we see that

$$\frac{h_n(x)}{h_n(\theta)} > \frac{\alpha(\delta)}{\alpha(\delta)} = 1 \quad \text{w.p.1.}$$

Therefore $\hat{x}_{a_n}$ minimizes $h_n(x) \Rightarrow h_n(\hat{x}_{a_n})/h_n(\theta) \leq 1$.

$$\therefore \bar{x}_0 - 3\delta < \hat{x}_{a_n} < \bar{x}_0 + 3\delta \quad \text{w.p.1.}$$

Since δ may be made arbitrarily small, it follows that

$$\hat{x}_{a_n} \xrightarrow{\text{a.s.}} \bar{x}_0. \quad ||$$

3. Asymptotic Distributions

Assumptions:

(3.A1) $F(x)$ has a uniformly continuous density $f(x)$.

(3.A2) $r(\bar{x}_0) > 0$ and $r(x)$ is thrice differentiable in a neighborhood of $\bar{x}_0$.

(3.A3) $g'(x)/g^3(x)$ is bounded for x in the support of G .

(3.A4) $a_n = Cn^{-\alpha}$ for some $C > 0$ and $1/8 < \alpha \leq 1/5$. Note that (2.A2)

requires α to be less than $1/2$.

(3.A5) $\omega_{i+1,n} - \omega_{i,n} = o_p\left(n^{-\frac{1-2\alpha}{3}}\right)$, i.e., Ω_n is a wide grid. Let Ω_n also satisfy (2.A4).

We shall see, in the sequel, that the bounds on α arise naturally and that similar results can be obtained for $0 < \alpha \leq 1/8$. Methods used in this section are similar to those in Section III.3 and hence proofs are given only at places where they seem to be necessary. Assumptions (3.A1) - (3.A5) insure strong consistency of $\hat{x}_{a_n}$.

Since $\tilde{x}_0$ minimizes $r(x)$, we have

$$(3.1) \quad r'(\tilde{x}_0) = 0.$$

Let

$$(3.2) \quad h_n(x) = \phi_{F_n}(x + a_n) - \phi_{F_n}(x - a_n)$$

$$(3.3) \quad Y_n = \left\{ \left[\phi_{F_n}(x + a_n) - \phi_{F_n}(\tilde{x}_0 + a_n) \right] - \left[\phi_F(x + a_n) - \phi_F(\tilde{x}_0 + a_n) \right] \right\} \\ - \left\{ \left[\phi_{F_n}(x - a_n) - \phi_{F_n}(\tilde{x}_0 - a_n) \right] - \left[\phi_F(x - a_n) - \phi_F(\tilde{x}_0 - a_n) \right] \right\}$$

$$(3.4) \quad u = \left[\phi_F(x + a_n) - \phi_F(\tilde{x}_0 + a_n) \right] - \left[\phi_F(x - a_n) - \phi_F(\tilde{x}_0 - a_n) \right]$$

and

$$\delta = x - \tilde{x}_0, \quad x \in \Omega_n.$$

$\hat{x}_{a_n}$ minimizes $h_n(x)$ and hence minimizes

$$h_n(x) - h_n(\tilde{x}_0) = Y_n + u.$$

Expanding in Taylor's series, Y_n as in Section III.3 and u about $\tilde{x}_0$, we see that

$$\begin{aligned}
Y_n = & \frac{1}{g[G^{-1}F(\tilde{x}_0 + a_n)]} \left\{ [F_n(x + a_n) - F_n(\tilde{x}_0 + a_n)] - [F(x + a_n) - F(\tilde{x}_0 + a_n)] \right\} \\
& - \frac{1}{g[G^{-1}F(\tilde{x}_0 - a_n)]} \left\{ [F_n(x - a_n) - F_n(\tilde{x}_0 - a_n)] - [F(x - a_n) - F(\tilde{x}_0 - a_n)] \right\} \\
& + O_p(n^{-\frac{1}{2}} \cdot \delta)
\end{aligned}$$

$$\begin{aligned}
(3.5) \quad Y_n = & \frac{r(\tilde{x}_0)}{f(\tilde{x}_0)} \left\{ [F_n(x + a_n) - F_n(\tilde{x}_0 + a_n)] - [F(x + a_n) - F(\tilde{x}_0 + a_n)] \right. \\
& \left. - [F_n(x - a_n) - F_n(\tilde{x}_0 - a_n)] + [F(x - a_n) - F(\tilde{x}_0 - a_n)] \right\} \\
& + O_p(n^{-\frac{1}{2}} \cdot \delta + n^{-\frac{1}{2}} \cdot a_n \cdot \delta^{\frac{1}{2}})
\end{aligned}$$

$$(3.6) \quad u = a_n r''(\tilde{x}_0) \delta^2 + O_p(n^{-\alpha} \cdot \delta^3 + n^{-3\alpha} \delta) .$$

Therefore $\hat{\delta} = (\hat{x}_{a_n} - \tilde{x}_0)$ minimizes

$$\begin{aligned}
(3.7) \quad h_n(x) - h_n(\tilde{x}_0) = & \frac{r(\tilde{x}_0)}{f(\tilde{x}_0)} \cdot v_n(\delta) + a_n r''(\tilde{x}_0) \delta^2 \\
& + O_p(n^{-\frac{1}{2}} \cdot \delta + n^{-\frac{1}{2}} \cdot a_n \cdot \delta^{\frac{1}{2}} + \delta^3 a_n + \delta a_n^3)
\end{aligned}$$

where

$$\begin{aligned}
(3.8) \quad v_n(\delta) = & \left\{ [F_n(x + a_n) - F_n(\tilde{x}_0 + a_n)] - [F(x + a_n) - F(\tilde{x}_0 + a_n)] \right\} \\
& - \left\{ [F_n(x - a_n) - F_n(\tilde{x}_0 - a_n)] - [F(x - a_n) - F(\tilde{x}_0 - a_n)] \right\} .
\end{aligned}$$

Let

$$(3.9) \quad \delta = \lambda t$$

$$(3.10) \quad W_n(t) = \left[\frac{2\lambda}{n} f(\tilde{x}_0) \right]^{-1/2} V_n(\lambda t) .$$

Multiplying (3.8) by $\left[\frac{2\lambda}{n} f(\tilde{x}_0) \right]^{-1/2} \frac{f(\tilde{x}_0)}{r(\tilde{x}_0)}$ and noting that $a_n = Cn^{-\alpha}$, we see that $\hat{t} = \lambda^{-1} \hat{\delta}$ minimizes

$$(3.11) \quad Z_n(t) = W_n(t) + \left[\frac{2\lambda}{n} f(\tilde{x}_0) \right]^{-1/2} \frac{f(\tilde{x}_0)}{r(\tilde{x}_0)} \lambda^2 r''(\tilde{x}_0) Cn^{-\alpha} t^2 \\ + O_p \left(\delta^{1/2} + n^{-\alpha} + n^{\frac{2-4\alpha}{3}} \delta^3 + n^{\frac{2-10\alpha}{3}} \delta \right) .$$

Choose λ such that the coefficient of t^2 in (3.11) is one, i.e.,

$$\left[\frac{2\lambda}{n} f(\tilde{x}_0) \right]^{-1/2} \frac{f(\tilde{x}_0)}{r(\tilde{x}_0)} \lambda^2 r''(\tilde{x}_0) Cn^{-\alpha} = 1 .$$

Hence

$$(3.12) \quad \lambda = 2^{1/3} C^{-2/3} f^{-1/3}(\tilde{x}_0) r^{2/3}(\tilde{x}_0) r''^{-2/3}(\tilde{x}_0) n^{-\frac{1-2\alpha}{3}} .$$

Therefore, $\delta = O_p \left(n^{-\frac{1-2\alpha}{3}} t \right)$. From (3.11) and (3.12), $\hat{t} = \lambda^{-1} (\hat{x}_{a_n} - \tilde{x}_0)$ minimizes

$$(3.13) \quad Z_n(t) = W_n(t) + t^2 + O_p \left(n^{-\frac{8\alpha-1}{3}} t \right) .$$

From (3.10),

$$(3.14) \quad W_n(0) = 0 \quad \text{and} \quad E[W_n(t)] = 0 \quad \text{for all } t .$$

By a straightforward but tedious calculation it can be shown that (Cf. Venter [18])

$$(3.15) \quad \text{Cov} \left\{ W_n(t), W_n(t^*) \right\} \rightarrow \frac{1}{2} \{ \min(|t|, 2B) + \min(|t^*|, 2B) - \min(|t - t^*|, 2B) \}$$

where

$$(3.16) \quad B = \lim_{n \rightarrow \infty} \frac{a_n}{\lambda}.$$

In particular,

$$(3.17) \quad \text{Var} [W_n(t)] \rightarrow \min(|t|, 2B).$$

Note that for $\alpha > 1/5$, $B = 0$.

Reduction to a Problem in Stochastic Processes

Hence for each t , $W_n(t)$ is asymptotically distributed as a normal random variable with mean 0 and variance given by (3.17). By arguments similar to those in Chapter III, we see that $W_n(t)$ is asymptotically distributed as $W(t)$, a Gaussian process with mean 0 and covariance function given by (3.15) and for $\alpha > 1/8$, by a simple extension of Slutsky's Theorem for processes,

$$Z_n(t) \xrightarrow{D} Z(t) = W(t) + t^2.$$

The Asymptotic Distribution of $\hat{x}_{a_n}$

Theorem 3.1:

The random variable

$$(3.18) \quad 2^{-1/3} c^{2/3} f^{1/3}(\tilde{x}_0) r^{-2/3}(\tilde{x}_0) r^{2/3}(\tilde{x}_0) n^{\frac{1-2\alpha}{3}} \left(\hat{x}_{a_n} - \tilde{x}_0 \right)$$

is asymptotically distributed as the variable t which minimizes

$$Z(t) = W(t) + t^2 \quad \text{where}$$

- (i) for $\alpha = 1/5$, $W(t)$ is a Gaussian process with $W(0) \equiv 0$, $E[W(t)] = 0$ for all t and covariance function given by the limit in (3.15); and
- (ii) for $1/8 < \alpha < 1/5$, $W(t)$ is a two-sided Wiener-Lévy process with $W(0) \equiv 0$, $E[W(t)] = 0$ for all t and variance 1 per unit t .

The grid Ω_n has to satisfy (3.A5), viz.

$$(3.19) \quad \omega_{i+1,n} - \omega_{i,n} = o_p \left(n^{-\frac{1-2\alpha}{3}} \right), \quad i = 0, 1, \dots$$

Proof:

By virtue of preceding arguments, it is left to show that (3.19) has to be satisfied. Since $t_{i+1} - t_i = \frac{\omega_{i+1,n} - \omega_{i,n}}{\lambda}$, in order to look at all the points of the process $Z(t)$, asymptotically, we need

$$\omega_{i+1,n} - \omega_{i,n} = o_p \left(n^{-\frac{1-2\alpha}{3}} \right); \text{ if } \omega_{i+1,n} - \omega_{i,n} = O_p \left(n^{-\frac{1-2\alpha}{3}} \right), \text{ we look at}$$

$Z(t)$ only at certain fixed intervals given by $\lim_{n \rightarrow \infty} \frac{\omega_{i+1,n} - \omega_{i,n}}{\lambda}$ and for a

grid whose spacings are even wider, $\lim_{n \rightarrow \infty} P[t_{i+1} - t_i > M] = 1 \quad \forall \quad |M| < \infty.$

Remark 3.1:

If $A = Cf(\tilde{x}_0)$ and G the uniform distribution on $[0,1]$, Theorem 3.1 yields Theorems 3a and 3b of Venter [18].

Remark 3.2:

Theorem 3.1 could have been derived by the following intuitive approach using Theorem III.3.2.

Since $\tilde{x}_{a_n}$ minimizes $\frac{\phi_F(x + a_n) - \phi_F(x - a_n)}{2a_n}$, it minimizes $r(x)$ as $n \rightarrow \infty$; i.e., $\tilde{x}_{a_n} \rightarrow \tilde{x}_0$. Expanding the terms in λ^{-1} in Taylor's series about $\tilde{x}_{a_n}$,

$$\lambda = \left\{ \frac{4r^2(\tilde{x} + a_n)}{n \left(r'(\tilde{x}_{a_n} + a_n) - r'(\tilde{x}_{a_n} - a_n) \right)^2 \left[\frac{1}{f(\tilde{x}_{a_n} + a_n)} + \frac{1}{f(\tilde{x}_{a_n} - a_n)} \right]} \right\}^{1/3}$$

as defined in (III.3.20), we get noting that $\tilde{x}_{a_n} \rightarrow \tilde{x}_0$

$$\lambda^{-1} = 2^{-1/3} \cdot c^{2/3} f^{1/3}(\tilde{x}_0) r^{-2/3}(\tilde{x}_0) r''^{2/3}(\tilde{x}_0) n^{\frac{1-2\alpha}{3}} [1 + o(n^{-2\alpha})]$$

→ Normalizing constant for $(\hat{x}_{a_n} - \tilde{x}_0)$ in Theorem 3.1.

CHAPTER V

FIXED WINDOW ESTIMATORS BASED ON THE ϕ TRANSFORMATION

1. The Estimator $\hat{x}_b$

In this chapter, we shall be concerned with fixed window estimators using the ϕ transformation and "2b" is a fixed window.

$$(1.1) \quad \phi_F(y) = \int_{s_1}^{F^{-1}(y)} g[G^{-1}F(u)] du = \int_{s_1}^{F^{-1}(y)} \frac{1}{r(u)} dF(u)$$

$$(1.2) \quad \frac{1}{r(x)} = \left. \frac{d}{dy} \phi_F(y) \right|_{y=F(x)} = \frac{g[G^{-1}F(x)]}{f(x)}$$

$$(1.3) \quad \phi_{F_n^*}(y) = \int_{s_1}^{F_n^{*-1}(y)} g[G^{-1}F_n^*(u)] du .$$

Definition:

$\tilde{x}_b$, assumed unique, is said to be the pseudo change point of $r(x)$ if it maximizes $[\phi_F(F(x) + b) - \phi_F(F(x) - b)]$. Let $\tilde{y}_b = F(\tilde{x}_b)$.

$\hat{x}_b$ is an estimator of $\tilde{x}_b$ if it maximizes $\left[\phi_{F_n^*}(F_n^*(x) + b) - \phi_{F_n^*}(F_n^*(x) - b) \right]$ among all $x \in \Omega_n$. Then $\hat{y}_b = F_n^*(\hat{x}_b)$ estimates $\tilde{y}_b$.

Remark 1.1:

Define the set $\Lambda_n = \{F_n^*(\omega_{i,n})\}_{i=0}^{\infty}$. Then, $\hat{y}_b$ maximizes $\left[\phi_{F_n^*}(y + b) - \phi_{F_n^*}(y - b) \right]$ among all $y \in \Lambda_n$ and let $\omega_{k,n}$ be a corresponding grid point such that $\hat{y}_b = F_n^*(\omega_{k,n})$. Then $\hat{x}_b$ is defined to be $\omega_{k,n}$.

Further, if F is continuous at $\tilde{x}_b$ and Ω_n becomes dense w.p.1. in a neighborhood of $\tilde{x}_b$, it is easy to see that Λ_n becomes dense in a neighborhood of $\tilde{y}_b$.

2. Consistency

Theorem 2.1:

Suppose that the following assumptions hold.

(2.A1) In the neighborhood of $\tilde{x}_b$, $F(x)$ is continuous and Ω_n becomes dense w.p.1.

(2.A2) The assumptions in Theorem II.2.2 hold; i.e.,

$$\phi_{F_n}^*(y) \xrightarrow{a.s.} \phi_F(y) \text{ uniformly for } 0 \leq y \leq 1.$$

(2.A3) For all δ small enough, $\alpha(\delta) > 0$ where

$$\alpha(\delta) = \alpha_1(\delta) - \alpha_2(\delta)$$

$$\alpha_1(\delta) = \min \left\{ \phi_F(y+b) - \phi_F(y-b) : \tilde{y}_b - \delta \leq y \leq \tilde{y}_b + \delta \right\}$$

$$\alpha_2(\delta) = \max \left\{ \phi_F(y+b) - \phi_F(y-b) : 0 < y \leq \tilde{y}_b - 2\delta, \right. \\ \left. \tilde{y}_b + 2\delta \leq y < 1 \right\}.$$

Then

$$(2.1) \quad \hat{y}_b \xrightarrow{a.s.} \tilde{y}_b.$$

From Lemma II.2.1 it follows that

$$(2.2) \quad \hat{x}_b \xrightarrow{a.s.} \tilde{x}_b.$$

Proof:

By Remark (1.1), Λ_n becomes dense w.p.1. in the neighborhood of $\tilde{y}_b$. The theorem now follows from Theorem II.3.1 and Remark II.3.1. ||

Theorem 2.2:

Let the following assumptions be satisfied.

(2.A4) The assumptions in Theorem II.2.3 hold; i.e., $\phi_{F_n^*}(y) \xrightarrow{P} \phi_F(y)$ for $0 \leq y \leq 1$.

(2.A5) The probability that Ω_n becomes dense on $(-\infty, \infty)$ approaches 1 as $n \rightarrow \infty$.

Then

$$(2.3) \quad \hat{x}_b \xrightarrow{P} \tilde{x}_b.$$

Proof:

By Theorem II.2.3, (2.A4) implies weak consistency of $\phi_{F_n^*}$. The rest of the proof is similar to the arguments involved in proving Theorem III.2.2. ||

3. Asymptotic DistributionAssumptions:

(3.A1) Let (2.A4) hold.

(3.A2) For x in the interval $[F^{-1}(\tilde{y}_b - b), F^{-1}(\tilde{y}_b + b)]$ and in a neighborhood of $F^{-1}(\tilde{y}_b - b)$ and $F^{-1}(\tilde{y}_b + b)$,

- (i) $r(x) > 0$ and continuously differentiable, and
- (ii) $f(x)$ and $f'(x)/f^3(x)$ are bounded.

Either (2.A4) ($g(x)$ is continuous in x) or (i) and (ii) imply

(iii) $gG^{-1}F(x)$ is bounded.

(3.A3) $\omega_{i+1,n} - \omega_{i,n} = o_p(n^{-1/3})$, i.e., Ω_n is a narrow grid.

(3.A1) and (3.A3) guarantee consistency of $\hat{x}_b$.

Since $\tilde{y}_b$ maximizes $[\phi_F(y+b) - \phi_F(y-b)]$ we have

$$(3.1) \quad \frac{1}{r[F^{-1}(\tilde{y}_b + b)]} = \frac{1}{r[F^{-1}(\tilde{y}_b - b)]} = \frac{1}{r} \quad (\text{say}) .$$

Let

$$(3.2) \quad h_n(y) = [\phi_{F_n}(y+b) - \phi_{F_n}(y-b)] .$$

$\hat{y}_b$ maximizes $h_n(y)$ among all $y \in \Lambda_n$ and hence maximizes

$$(3.3) \quad h_n(y) - h_n(\tilde{y}_b) = [\phi_{F_n}(y+b) - \phi_{F_n}(y-b)] - [\phi_{F_n}(\tilde{y}_b + b) - \phi_{F_n}(\tilde{y}_b - b)] .$$

Let

$$(3.4) \quad \delta = y - \tilde{y}_b$$

$$(3.5) \quad \eta_1 = \tilde{y}_b + b \quad ; \quad \eta_2 = \tilde{y}_b - b$$

$$(3.6) \quad \xi_1 = F^{-1}(\eta_1) \quad ; \quad \xi_2 = F^{-1}(\eta_2)$$

$$(3.7) \quad a_1 = [n\eta_1] \quad ; \quad a_2 = [n\eta_2]$$

$$(3.8) \quad b_1 = [n(\eta_1 + \delta)] \quad ; \quad b_2 = [n(\eta_2 + \delta)] .$$

$X_1, X_2, \dots, X_n$ are the order statistics from F .

Lemma 3.1:

$$(3.9) \quad \phi_{F_n}\left(\frac{b_1}{n}\right) - \phi_{F_n}\left(\frac{a_1}{n}\right) = \sum_{j=a_1}^{b_1-1} \frac{g[G^{-1}(j/n)]}{f(X_j)} [F(X_{j+1}) - F(X_j)] + o_p(n^{-1\delta}) .$$

Proof:

$$(3.10) \quad \phi_{F_n}\left(\frac{b_1}{n}\right) - \phi_{F_n}\left(\frac{a_1}{n}\right) = \sum_{j=a_1}^{b_1-1} g[G^{-1}(j/n)] (X_{j+1} - X_j) .$$

$$(3.11) \quad \begin{aligned} X_{j+1} - X_j &= F^{-1}F(X_{j+1}) - F^{-1}F(X_j) \\ &= \frac{F(X_{j+1}) - F(X_j)}{f(X_j)} - \frac{1}{2} [F(X_{j+1}) - F(X_j)]^2 \frac{f'(\zeta_j)}{f^3(\zeta_j)} \end{aligned}$$

where $\zeta_j = F^{-1}(\theta_j)$ and $F(X_j) \leq \theta_j \leq F(X_{j+1})$.

$$E\left\{[F(X_{j+1}) - F(X_j)]^2\right\} = \frac{2}{(n+1)(n+2)} = O(n^{-2}) .$$

Further, the fourth moment of $[F(X_{j+1}) - F(X_j)]$ is given by

$$E\left\{[F(X_{j+1}) - F(X_j)]^4\right\} = \frac{24}{(n+1)(n+2)(n+3)(n+4)} = O(n^{-4})$$

so that $\text{Var}\left\{[F(X_{j+1}) - F(X_j)]^2\right\} \rightarrow 0$ as $n \rightarrow \infty$. Therefore

$$(3.12) \quad [F(X_{j+1}) - F(X_j)]^2 = o_p(n^{-2}) .$$

By assumption, $g[G^{-1}(j/n)]$, $f'(\zeta_j)/f^3(\zeta_j)$ are bounded for $j \in [a_1, b_1 - 1]$.

Therefore

$$(3.13) \quad \frac{1}{2} \sum_{j=a_1}^{b_1-1} [F(X_{j+1}) - F(X_j)]^2 \frac{f'(\zeta_j)}{f^3(\zeta_j)} = o_p(n^{-2}) \cdot n\delta = o_p(n^{-1\delta}) .$$

Lemma 3.1 follows from (3.11), (3.12) and (3.13). ||

Lemma 3.2:

$$(3.14) \quad \phi_{F_n}\left(\frac{b_1}{n}\right) - \phi_{F_n}\left(\frac{a_1}{n}\right) = \frac{1}{r} \sum_{j=a_1}^{b_1-1} \left(Y_{j+1} - \frac{1}{n+1} \right) + \frac{\delta}{r} \\ - \frac{r'(\xi_1)}{2r^2 f(\xi_1)} \delta^2 + O_p(n^{-1/2}\delta)$$

where $\{Y_i\}_{i=1}^{n+1}$ are independent and exponentially distributed random variables

with mean $\frac{1}{n+1}$ and $r = r(\xi_1) = r(\xi_2)$.

Proof:

$F(X_j)$ and $F(X_{j+1})$ are respectively the j th and $(j+1)$ th order statistic from the uniform distribution on $[0,1]$. Let $S_j = \sum_{i=1}^j Y_i$. Then $F(X_j) = S_j/S_{n+1}$. By Strong Law of Large Numbers, $S_{n+1} \xrightarrow{a.s.} 1$. From (3.9)

$$(3.15) \quad \phi_{F_n}\left(\frac{b_1}{n}\right) - \phi_{F_n}\left(\frac{a_1}{n}\right) = \sum_{j=a_1}^{b_1-1} \frac{g[G^{-1}(j/n)]}{f(X_j)} Y_{j+1} + O_p(n^{-1}\delta) \\ = \sum_{j=a_1}^{b_1-1} \frac{g[G^{-1}(j/n)]}{f(X_j)} \left(Y_{j+1} - \frac{1}{n+1} \right) \\ + \sum_{j=a_1}^{b_1-1} \frac{g[G^{-1}(j/n)]}{f(X_j)} \frac{1}{n+1} + O_p(n^{-1}\delta).$$

Define Z_j by $F(Z_j) = F_n(X_j) = j/n$. By Kolmogorov's Theorem and the fact that f is bounded away from zero, we have

$$\sup_{a_1 \leq j \leq b_1-1} |X_j - Z_j| = O_p(n^{-1/2}).$$

4.

$$f^{-1}(X_j) = f^{-1}(Z_j) - \frac{f'(\zeta_j)}{f^2(\zeta_j)} (X_j - Z_j)$$

where ζ_j lies between X_j and Z_j . By Assumption (3.A2), $f'(\zeta_j)/f^2(\zeta_j)$ is bounded and hence

$$f^{-1}(X_j) = f^{-1}(Z_j) + O_p(n^{-1/2}).$$

$$(3.16) \quad \sum_{j=a_1}^{b_1-1} \frac{g[G^{-1}(j/n)]}{f(X_j)} \left(Y_{j+1} - \frac{1}{n+1} \right) = \sum_{j=a_1}^{b_1-1} \frac{g[G^{-1}F(Z_j)]}{f(Z_j)} \left(Y_{j+1} - \frac{1}{n+1} \right) + O_p(n^{-1/2}\delta).$$

$$(3.17) \quad \sum_{j=a_1}^{b_1-1} \frac{g[G^{-1}(j/n)]}{f(X_j)} \frac{1}{n+1} = \sum_{j=a_1}^{b_1-1} \frac{g[G^{-1}F(Z_j)]}{f(Z_j)} \cdot \frac{1}{n+1} + O_p(n^{-1/2}\delta).$$

Expanding in Taylor's series, $\frac{1}{r(Z_j)} = \frac{1}{r[F^{-1}(j/n)]}$ about η_1 , we get

$$(3.18) \quad \frac{1}{r(Z_j)} = \frac{1}{r(\xi_1)} - \frac{(j/n - \eta_1)r'(\xi_1)}{r^2(\xi_1)f(\xi_1)} + O\left(\frac{j}{n} - \eta_1\right)^2.$$

Let

$$A_n = \frac{1}{n} \sum_{j=a_1}^{b_1-1} \left(Y_{j+1} - \frac{1}{n+1} \right) (j - a_1).$$

$$E(A_n) = 0$$

$$\begin{aligned} \text{Var}(A_n) &= \frac{1}{n^2} \sum_{j=a_1}^{b_1-1} \frac{(j - a_1)^2}{(n+1)^2} \approx \frac{1}{n^4} \sum_{j=a_1}^{b_1-1} (j - a_1)^2 \\ &\approx \frac{1}{n^4} (n\delta)^3 = O(n^{-1}\delta^3). \end{aligned}$$

Therefore

$$(3.19) \quad A_n = O_p(n^{-1/2} \delta^{3/2}) .$$

Note that $r'(\xi_1)/r^2(\xi_1)f(\xi_1)$ is bounded, by (3.A2), and that in (3.16) and (3.17), the terms with $(j/n - \eta_1)$ raised to a power greater than 1 are of a still smaller order of n .

From (3.15) - (3.17)

$$\begin{aligned} \phi_{F_n}\left(\frac{b_1}{n}\right) - \phi_{F_n}\left(\frac{a_1}{n}\right) &= \sum_{j=a_1}^{b_1-1} \frac{1}{r(z_j)} \left(y_{j+1} - \frac{1}{n+1}\right) + \sum_{j=a_1}^{b_1-1} \frac{1}{r(z_j)} \frac{1}{n+1} \\ &\quad + O_p(n^{-1/2} \delta) \\ &= \sum_{j=a_1}^{b_1-1} \frac{1}{r(\xi_1)} \left(y_{j+1} - \frac{1}{n+1}\right) + \sum_{j=a_1}^{b_1-1} \frac{1}{r(\xi_1)} \frac{1}{n+1} \\ &\quad - \frac{1}{n+1} \sum_{j=a_1}^{b_1-1} \frac{(j/n - \eta_1)r'(\xi_1)}{r^2(\xi_1)f(\xi_1)} + O_p(n^{-1/2} \delta) \text{ by (3.18) and (3.19)} \\ &= \frac{1}{r} \sum_{j=a_1}^{b_1-1} \left(y_{j+1} - \frac{1}{n+1}\right) + \frac{\delta}{r} - \frac{r'(\xi_1)}{2r^2f(\xi_1)} \delta^2 + O_p(n^{-1/2} \delta) . \quad || \end{aligned}$$

Replacing subscripts 1 by 2 in Lemmas 3.1 and 3.2, we get

Lemma 3.3:

$$\begin{aligned} (3.20) \quad \phi_{F_n}\left(\frac{b_2}{n}\right) - \phi_{F_n}\left(\frac{a_2}{n}\right) &= \frac{1}{r} \sum_{j=a_2}^{b_2-1} \left(y_{j+1} - \frac{1}{n+1}\right) + \frac{\delta}{r} \\ &\quad - \frac{r'(\xi_2)}{2r^2f(\xi_2)} \delta^2 + O_p(n^{-1/2} \delta) . \end{aligned}$$

Remark 3.1:

As $n \rightarrow \infty$, $[ny]/n \rightarrow y$ uniformly in y and hence finding $\hat{y}_b$ maximizing $h_n(y) - h_n(\hat{y}_b)$ is equivalent to the problem of finding $\delta = \hat{\delta}$ which maximizes

$$\left[\phi_{F_n}\left(\frac{b_1}{n}\right) - \phi_{F_n}\left(\frac{a_1}{n}\right) \right] - \left[\phi_{F_n}\left(\frac{b_2}{n}\right) - \phi_{F_n}\left(\frac{a_2}{n}\right) \right].$$

Reduction to a Problem in Stochastic Processes

From Lemmas 3.2 and 3.3

$$(3.21) \quad \left[\phi_{F_n}\left(\frac{b_1}{n}\right) - \phi_{F_n}\left(\frac{a_1}{n}\right) \right] - \left[\phi_{F_n}\left(\frac{b_2}{n}\right) - \phi_{F_n}\left(\frac{a_2}{n}\right) \right] \\ = \frac{1}{r} v_n(\delta) - \frac{1}{2r^2} \left[\frac{r'(\xi_1)}{f(\xi_1)} - \frac{r'(\xi_2)}{f(\xi_2)} \right] \delta^2 + o_p(n^{-1/2}\delta)$$

where

$$(3.22) \quad v_n(\delta) = v_{n1}(\delta) + v_{n2}(\delta)$$

$$(3.23) \quad v_{n1}(\delta) = \sum_{j=a_1}^{b_1-1} \left(Y_{j+1} - \frac{1}{n+1} \right); \quad v_{n2}(\delta) = \sum_{j=a_2}^{b_2-1} \left(-Y_{j+1} + \frac{1}{n+1} \right).$$

Let

$$(3.24) \quad \delta = vt$$

$$(3.25) \quad W_n(t) = \left(\frac{2v}{n} \right)^{-1/2} v_n(\delta).$$

Then $\hat{t} = v^{-1}\hat{\delta}$ maximizes

$$(3.26) \quad Z_n(t) = W_n(t) - \left(\frac{2v}{n}\right)^{-1/2} \cdot \frac{1}{2r^2} \left[\frac{r'(\xi_1)}{f(\xi_1)} - \frac{r'(\xi_2)}{f(\xi_2)} \right] v^2 t^2 + o_p(\delta^{1/2}).$$

Choose v such that the coefficient of $(-t^2)$ in (3.26) is one, i.e.

$$\left(\frac{2v}{n}\right)^{-1/2} \frac{v^2}{2r^2} \left[\frac{r'(\xi_1)}{f(\xi_1)} - \frac{r'(\xi_2)}{f(\xi_2)} \right] = 1.$$

Therefore

$$(3.27) \quad v = \left[\frac{8r^2}{n \left(\frac{r'(\xi_1)}{f(\xi_1)} - \frac{r'(\xi_2)}{f(\xi_2)} \right)^2} \right]^{-1/3}.$$

Hence $\delta = o_p(n^{-1/3}t)$. Hence $\hat{t} = v^{-1}(\hat{y}_b - \bar{y}_b)$ maximizes

$$(3.28) \quad Z_n(t) = W_n(t) - t^2 + o_p(n^{-1/6}t).$$

Lemma 3.4:

$W_n(t)$ is asymptotically normal with mean 0 and variance $|t|$, for all t .

Proof:

From (3.25), (3.22) and (3.23), $W_n(0) = 0$ and $E[W_n(t)] = 0$ for all t . By the Lindberg-Lévy Theorem (Cf. Fisz [8], p. 197)

$$\left(\frac{2v}{n}\right)^{-1/2} V_{n1}(\lambda t) \xrightarrow{D} N\left(0, \frac{|t|}{2}\right)$$

$$\left(\frac{2v}{n}\right)^{-1/2} V_{n2}(\lambda t) \xrightarrow{D} N\left(0, \frac{|t|}{2}\right).$$

Since $V_{n1}(\lambda t)$ and $V_{n2}(\lambda t)$ are asymptotically independent, we have

$$W_n(t) \xrightarrow{D} N(0, |t|) \quad \text{for all } t.$$

Remark 3.2:

For any collection of $t_1, \dots, t_k$, it can be shown that the joint distribution of $[W_n(t_1), \dots, W_n(t_k)]$ converges to the multivariate normal distribution with mean 0 and variance-covariance matrix given by

$$(\delta(t_i, t_j) \min(|t_i|, |t_j|))$$

where

$$\delta(c, d) = \begin{cases} 1 & \text{if } c \text{ and } d \text{ are of the same sign} \\ 0 & \text{otherwise.} \end{cases}$$

By limiting arguments similar to Section III.3 or Section 5, Chapter 2, Rao [13], we get

Theorem 3.1:

The distribution of $Z_n(t) = W_n(t) - t^2 + O_p(n^{-1/6}t)$ converges to the distribution of $W(t) - t^2$ where $W(t)$ is a two-sided Wiener-Lévy process with mean 0 and variance 1 per unit t and $W(0) = 0$.

The Asymptotic Distributions of $\hat{y}_b$ and $\hat{x}_b$ Theorem 3.2:

The asymptotic distributions of

$$(3.29) \quad v^{-1}(\hat{y}_b - \tilde{y}_b)$$

$$(3.30) \quad \lambda^{-1}(\hat{x}_b - \tilde{x}_b)$$

have density $\psi(t)$ where ψ is the density of the value of t maximizing

$Z(t) = W(t) - t^2$, $W(t)$ is a two-sided Wiener-Lévy process with $W(0) = 0$, $E[W(t)] = 0$ and $\text{Var}[W(t)] = |t|$, for all t where

$$v = \left[\frac{8r^2}{n \left(\frac{r'(\xi_1)}{f(\xi_1)} - \frac{r'(\xi_2)}{f(\xi_2)} \right)^2} \right]^{1/3}$$

$$(3.31) \quad \lambda = \frac{v}{f(\tilde{x}_b)}.$$

Proof:

Since $\hat{t} = v^{-1}(\hat{y}_b - \tilde{y}_b)$ maximizes $Z_n(t)$ and by Theorem 3.1, $Z_n(t) \xrightarrow{D} Z(t)$, we only have to show that the assumption on grid points, viz. $\omega_{i+1,n} - \omega_{i,n} = o_p(n^{-1/3})$, is necessary.

Since $t = v^{-1}(y - \tilde{y}_b)$ with $y \in \Lambda_n$,

$$t_{i+1} - t_i = \frac{F_n(\omega_{i+1,n}) - F_n(\omega_{i,n})}{v}$$

$$\xrightarrow{P} 0 \text{ if } \omega_{i+1,n} - \omega_{i,n} = o_p(n^{-1/3}).$$

Proof of (3.30) and (3.31):

$$\begin{aligned} \hat{y}_b - \tilde{y}_b &= F_n(\hat{x}_b) - F(\tilde{x}_b) \\ &= F_n(\hat{x}_b) - F(\hat{x}_b) + (\hat{x}_b - \tilde{x}_b)f(\zeta) \end{aligned}$$

where ζ lies between $\hat{x}_b$ and $\tilde{x}_b$.

$$\therefore (\hat{x}_b - \tilde{x}_b)f(\zeta) = (\hat{y}_b - \tilde{y}_b) + o_p(n^{-1/2}),$$

by Kolmogorov's Theorem. Since, $\hat{x}_b \xrightarrow{P} \tilde{x}_b$ and $f(x)$ is continuous at

$\tilde{x}_b$, $f(\tilde{x}_b) \stackrel{D}{\rightarrow} f(\tilde{x}_b)$ by Corollary 2 to Theorem 5.1, Billingsley [3]. Hence, by Slutsky's Theorem (Cf. Cramer [6], p. 254), we get

$$\sqrt{v}^{-1} f(\tilde{x}_b) (\hat{x}_b - \tilde{x}_b) \stackrel{D}{\rightarrow} \sqrt{v}^{-1} \delta.$$

Let $\lambda = v/f(\tilde{x}_b)$ and the theorem follows. ||

An Alternate Definition of the Pseudo Change Point

The pseudo change point may be alternately defined as

$$x_b^* = \frac{F^{-1}(\tilde{y}_b + b) + F^{-1}(\tilde{y}_b - b)}{2} \quad \text{where } \tilde{y}_b \text{ maximizes } \phi_F(y + b) - \phi_F(y - b).$$

Let Λ_n be any set in $[0,1]$ and containing the points 0 and 1.

Then $\hat{x}_b^* = F_n^{*-1}(\hat{y}_b)$ is said to estimate x_b^* when $\hat{y}_b$ maximizes $\left[\phi_{F_n^*}(y + b) - \phi_{F_n^*}(y - b) \right]$, where y is restricted to Λ_n .

Under regularity conditions similar to those in Section 2, it can be shown that $\hat{y}_b$ and $\hat{x}_b^*$ are consistent estimator of $\tilde{y}_b$ and x_b^* respectively. Further, it can be shown that

$$\left[\frac{1}{f(\xi_1)} + \frac{1}{f(\xi_2)} \right]^{-1} \sqrt{v}^{-1} (\hat{x}_b^* - x_b^*)$$

has density $\psi(t)$ where v and ψ are defined in Theorem 3.2.

Remark 3.3:

If b satisfies $\frac{F^{-1}(\tilde{y}_b + b) + F^{-1}(\tilde{y}_b - b)}{2} = \tilde{x}_a$ and $a = \frac{F^{-1}(\tilde{y}_b + b) - F^{-1}(\tilde{y}_b - b)}{2}$, where a and $\tilde{x}_a$ are as defined in Chapter III, then it is clear that $\hat{x}_a$ and $\hat{x}_b^*$ are asymptotically equivalent.

CHAPTER VI

NARROW WINDOW ESTIMATORS BASED ON THE ϕ TRANSFORMATION

1. The Estimator $\hat{x}_{b_n}$

A natural window estimator of $1/r(x)$, using the ϕ transformation is given by

$$(1.1) \quad \frac{1}{r_n^*(x)} = \frac{\phi_{F_n^*}(F_n^*(x) + b_n) - \phi_{F_n^*}(F_n^*(x) - b_n)}{2b_n}$$

where $2b_n$ is a narrow window.

Definition:

$\tilde{x}_0$ is said to be the change point if it maximizes $1/r(x)$. Let $\tilde{y}_0 = F(\tilde{x}_0)$.

$\hat{x}_{b_n}$ is said to estimate $\tilde{x}_0$ if it maximizes $1/r_n^*(x)$ among $x \in \Omega_n$.

Let $\hat{y}_{b_n} = F_n^*(\hat{x}_{b_n})$ and $\Lambda_n = \{F_n^*(\omega_{i,n})\}_{i=0}^\infty$.

We shall obtain the analog of the results in Chapter IV and this will enable us to obtain the asymptotic efficiency of $\hat{x}_{a_n}$ relative to $\hat{x}_{b_n}$.

2. Strong Consistency

Let $\delta > 0$ and

$$\alpha_1(\delta) = \min \left\{ \frac{1}{r(x)} : \tilde{x}_0 - \delta \leq x \leq \tilde{x}_0 + \delta \right\}$$

$$\alpha_2(\delta) = \max \left\{ \frac{1}{r(x)} : s_1 < x \leq \tilde{x}_0 - 2\delta, \tilde{x}_0 + 2\delta \leq x < s_2 \right\}$$

$$\alpha(\delta) = \alpha_1(\delta)/\alpha_2(\delta).$$

Theorem 2.1:

If

- (2.A1) either $F_n^* \equiv F_n$ or the grid $\{w_{i,n}\}_{i=0}^{\infty}$ becomes dense w.p.l. on the support of F as $n \rightarrow \infty$ and $\sup_i |w_{i+1,n} - w_{i,n}| \xrightarrow{a.s.} 0$;
- (2.A2) Ω_n becomes dense w.p.l. in the neighborhood of $\tilde{x}_0$;
- (2.A3) the support of F is an interval and F has a uniformly continuous density f ; and
- (2.A4) for all δ small enough, $\alpha(\delta) > 1$,

then

$$(2.1) \quad \hat{y}_{b_n} \xrightarrow{a.s.} \tilde{y}_0 \quad \text{and} \quad \hat{x}_{b_n} \xrightarrow{a.s.} \tilde{x}_0 .$$

Proof:

Let

$$\begin{aligned} h_n(y) &= \frac{\phi_{F_n^*}(y + b_n) - \phi_{F_n^*}(y - b_n)}{2b_n} \\ &= \frac{F_n^{*-1}(y + b_n) - F_n^{*-1}(y - b_n)}{2b_n} \cdot g[G^{-1}\beta_n(y)] \end{aligned}$$

where

$$F_n^{*-1}(y - b_n) \leq F_n^{*-1}(\beta_n(y)) \leq F_n^{*-1}(y + b_n) ,$$

i.e.,

$$(2.2) \quad y - b_n \leq \beta_n(y) \leq y + b_n .$$

Clearly,

$$(2.3) \quad \beta_n(y) \xrightarrow{a.s.} y$$

uniformly in y . Let $X_1, \dots, X_n$ be the order statistics from F . Then, $F(X_1) = \frac{S_1}{S_{n+1}}$ where $S_1 = \sum_{j=1}^1 Y_j$ and $Y_1, \dots, Y_{n+1}$ are independent random variables and have exponential distribution with mean $1/(n+1)$.

$$F_n^{*-1}(y + b_n) - F_n^{*-1}(y - b_n) = X_{[n(y+b_n)]} - X_{[n(y-b_n)]} + O(w_n)$$

where $w_n = 0$ when $F_n^* \equiv F_n$ and $w_n = \sup_i |w_{i+1,n} - w_{i,n}|$ when $F_n^* \neq F_n$.

By (2.A1), in the latter case, $w_n \xrightarrow{a.s.} 0$.

$$\begin{aligned} F_n^{*-1}(y + b_n) - F_n^{*-1}(y - b_n) &= F^{-1}\left(\frac{S_{[n(y+b_n)]}}{S_{n+1}}\right) - F^{-1}\left(\frac{S_{[n(y-b_n)]}}{S_{n+1}}\right) + O(w_n) \\ &= \frac{S_{[n(y+b_n)]} - S_{[n(y-b_n)]}}{S_{n+1}} \cdot \frac{1}{f(F^{-1}Y_n(y))} + O(w_n) \end{aligned}$$

where

$$\frac{S_{[n(y-b_n)]}}{S_{n+1}} \leq \gamma_n(y) \leq \frac{S_{[n(y+b_n)]}}{S_{n+1}}.$$

By proof along the lines of Venter [18], it can be shown that

$$(2.4) \quad \gamma_n(y) \xrightarrow{a.s.} y \quad \text{uniformly in } y$$

and

$$(2.5) \quad \frac{S_{[n(y+b_n)]} - S_{[n(y-b_n)]}}{2b_n} \xrightarrow{a.s.} 1 \quad \text{uniformly in } y.$$

By (2.A2), w.p.1. $\exists n_0 \ni$ for all $n \geq n_0$, $\exists$ an integer k_n satisfying $|\omega_{k_n, n} - \tilde{x}_0| < \delta$ and for which $|F_n^*(\omega_{k_n, n}) - \tilde{y}_0| < \epsilon$ for some $\epsilon > 0$. For $n = n_0$, define $x_0 = \omega_{k_n, n}$ and $y_0 = F_n^*(x_0)$.

$$\begin{aligned} \frac{h_n(y)}{h_n(y_0)} &= \frac{\left[\frac{S[n(y+b_n)] - S[n(y-b_n)]}{2b_n} \right] \frac{1}{f(F_n^{-1}\gamma_n(y))} + o(w_n)}{\left[\frac{S[n(y_0+b_n)] - S[n(y_0-b_n)]}{2b_n} \right] \frac{1}{f(F_n^{-1}\gamma_n(y_0))} + o(w_n)} \cdot \frac{g[G_n^{-1}\beta_n(y)]}{g[G_n^{-1}\beta_n(y_0)]} \\ (2.6) \quad \frac{h_n(y)}{h_n(y_0)} &= \frac{S[n(y+b_n)] - S[n(y-b_n)]}{S[n(y_0+b_n)] - S[n(y_0-b_n)]} \cdot \frac{f(F_n^{-1}\gamma_n(y_0))}{f(F_n^{-1}\gamma_n(y))} \cdot \frac{f(F_n^{-1}\beta_n(y))}{f(F_n^{-1}\beta_n(y_0))} \\ &\quad \cdot \frac{r(F_n^{-1}\beta_n(y_0))}{r(F_n^{-1}\beta_n(y))} + o(w_n). \end{aligned}$$

Choose $\delta \ni y \leq F(\tilde{x}_0 - 3\delta)$ or $y \geq F(\tilde{x}_0 + 3\delta)$. From (2.3), w.p.1.

$\exists n_1 \geq n_0$ independent of $y \ni$ for all $n \geq n_1$, $\beta_n(y) \leq F(\tilde{x}_0 - 2\delta)$ or $\beta_n(y) \geq F(\tilde{x}_0 + 2\delta)$. Hence for all $n \geq n_1$

$$(2.7) \quad \frac{r[F_n^{-1}\beta_n(y_0)]}{r[F_n^{-1}\beta_n(y)]} \leq \frac{\alpha_2(\delta)}{\alpha_1(\delta)} = \frac{1}{\alpha(\delta)}.$$

From (2.3), (2.4), (2.5) and uniform continuity of $f(x)$ ((2.A3)), w.p.1.

$\exists n_2 \geq n_1$ independent of $y \ni$ for all $n \geq n_2$

$$(2.8) \quad \frac{\left[\frac{S[n(y+b_n)] - S[n(y-b_n)]}{2b_n} \right]}{\left[\frac{S[n(y_0+b_n)] - S[n(y_0-b_n)]}{2b_n} \right]} \cdot \frac{f(F_n^{-1}\gamma_n(y_0))}{f(F_n^{-1}\gamma_n(y))} \cdot \frac{f(F_n^{-1}\beta_n(y))}{f(F_n^{-1}\beta_n(y_0))} < \alpha(\delta)$$

and, hence, from (2.6) - (2.8),

$$(2.9) \quad \frac{h_n(y)}{h_n(y_0)} < \alpha(\delta) + O(w_n) .$$

If $w_n \neq 0$, i.e., $F_n^* \neq F_n$, w.p.1. $\exists n_3 \geq n_2$ independent of y for all $n \geq n_3$

$$\alpha(\delta) + O(w_n) < 1$$

and from (2.9)

$$\frac{h_n(y)}{h_n(y_0)} < 1 .$$

But $\hat{y}_{b_n}$ maximizes $h_n(y) \Rightarrow \frac{h_n(\hat{y}_{b_n})}{h_n(y_0)} \geq 1$. Hence $F(\tilde{x}_0 - 3\delta) < \hat{y}_{b_n} < F(\tilde{x}_0 + 3\delta)$ w.p.1. Since δ is arbitrary, $\hat{y}_{b_n} \xrightarrow{a.s.} \tilde{y}_0$. By Lemma II.2.1 $\hat{x}_{b_n} \xrightarrow{a.s.} \tilde{x}_0$. ||

3. Asymptotic Distributions

Assumptions:

(3.A1) Conditions (2.A2) and (2.A3) hold.

(3.A2) In the neighborhood of $\tilde{x}_0$

(i) $r(x)$ is thrice differentiable, and

(ii) $f'(x)/f^3(x)$ is bounded.

Note that $\min r(x) = r(\tilde{x}_0) < \infty$ implies

(iii) $g[G^{-1}F(x)] > 0$.

(3.A3) $b_n = An^{-\alpha}$; $A > 0$, $1/8 < \alpha \leq 1/5$.

(3.A4) $\omega_{i+1,n} - \omega_{i,n} = o_p\left(n^{-\frac{1-2\alpha}{3}}\right)$.

$\tilde{x}_0$ minimizes $r(x) \Rightarrow r'(\tilde{x}_0) = 0$. As in Section V.3, define

$$\delta = y - \tilde{y}_0$$

$$\eta_1 = \tilde{y}_0 + b_n \quad ; \quad \eta_2 = \tilde{y}_0 - b_n$$

$$\xi_1 = F^{-1}(\eta_1) \quad ; \quad \xi_2 = F^{-1}(\eta_2)$$

$$a_1 = [n\eta_1] \quad ; \quad a_2 = [n\eta_2]$$

$$b_1 = [n(\eta_1 + \delta)] \quad ; \quad b_2 = [n(\eta_2 + \delta)] .$$

Lemma 3.1:

$$\begin{aligned}
 (3.1) \quad & \left[\phi_{F_n}\left(\frac{b_1}{n}\right) - \phi_{F_n}\left(\frac{a_1}{n}\right) \right] - \left[\phi_{F_n}\left(\frac{b_2}{n}\right) - \phi_{F_n}\left(\frac{a_2}{n}\right) \right] = \frac{1}{r(\tilde{x}_0)} \left[\sum_{j=a_1}^{b_1-1} \left(Y_{j+1} - \frac{1}{n+1} \right) \right. \\
 & \quad \left. + \sum_{j=a_2}^{b_2-1} \left(-Y_{j+1} + \frac{1}{n+1} \right) \right] \\
 & \quad - \frac{r''(\tilde{x}_0)b_n}{r^2(\tilde{x}_0)f^2(\tilde{x}_0)} \delta^2 \\
 & \quad + O_p(\delta b_n^3 + \delta^3 b_n)
 \end{aligned}$$

where $\{Y_i\}_{i=1}^{n+1}$ are independent and identically distributed, having the exponential distribution with mean $1/(n+1)$.

Proof:

From Lemmas V.3.2 and V.3.3

$$(3.2) \quad \phi_{F_n}\left(\frac{b_1}{n}\right) - \phi_{F_n}\left(\frac{a_1}{n}\right) = \frac{1}{r(\xi_1)} \sum_{j=a_1}^{b_1-1} \left(y_{j+1} - \frac{1}{n+1}\right) + \frac{\delta}{r(\xi_1)} - \frac{r'(\xi_1)}{2r^2(\xi_1)f(\xi_1)} \delta^2 \\ + O_p(n^{-1/2}\delta)$$

$$(3.3) \quad \phi_{F_n}\left(\frac{b_2}{n}\right) - \phi_{F_n}\left(\frac{a_2}{n}\right) = \frac{1}{r(\xi_2)} \sum_{j=a_2}^{b_2-1} \left(-y_{j+1} + \frac{1}{n+1}\right) + \frac{\delta}{r(\xi_2)} - \frac{r'(\xi_2)}{2r^2(\xi_2)f(\xi_2)} \delta^2 \\ + O_p(n^{-1/2}\delta).$$

From Taylor Series expansion

$$(3.4) \quad \frac{1}{r[F^{-1}(\tilde{y}_o \pm b_n)]} = \frac{1}{r(\tilde{x}_o)} - \frac{1}{2} \frac{r''(\tilde{x}_o)}{r^2(\tilde{x}_o)f^2(\tilde{x}_o)} b_n^2 + O_p(b_n^3).$$

The lemma is immediate from the above relations. ||

Reduction to a Problem in Stochastic Processes

$$\text{Let } v_n(\delta) = \sum_{j=a_1}^{b_1-1} \left(y_{j+1} - \frac{1}{n+1}\right) + \sum_{j=a_2}^{b_2-1} \left(-y_{j+1} + \frac{1}{n+1}\right). \text{ Then}$$

$$\hat{\delta} = \left(\hat{y}_{b_n} - \tilde{y}_o\right) \text{ maximizes}$$

$$\frac{v_n(\delta)}{r(\tilde{x}_o)} - \frac{r''(\tilde{x}_o)}{r^2(\tilde{x}_o)f^2(\tilde{x}_o)} \delta^2 + O_p(\delta b_n^3 + \delta^3 b_n).$$

$$\text{Let } \delta = vt \text{ and } W_n(t) = \left(\frac{2v}{n}\right)^{-1/2} v_n(\delta). \text{ Then } \hat{t} = v^{-1} \hat{\delta} \text{ maximizes}$$

$$(3.5) \quad Z_n(t) = W_n(t) - \left(\frac{2v}{n}\right)^{-1/2} \frac{r''(\tilde{x}_o)}{r(\tilde{x}_o)f^2(\tilde{x}_o)} v^2 t^2 + O_p(\delta^{1/2} \cdot n^{1/2-3\alpha} + \delta^{5/2} \cdot n^{1/2-\alpha}).$$

Choose v such that the coefficient of $(-t^2)$ in (3.5) is one, i.e.,

$$\left(\frac{2v}{n}\right)^{-1/2} \cdot v^2 \frac{r''(\tilde{x}_0) A n^{-\alpha}}{r(\tilde{x}_0) f^2(\tilde{x}_0)} = 1.$$

Therefore

$$(3.6) \quad v = 2^{1/3} A^{-2/3} f^{4/3}(\tilde{x}_0) r^{2/3}(\tilde{x}_0) r''^{-2/3}(\tilde{x}_0) n^{-\frac{1-2\alpha}{3}}.$$

Therefore, $\delta = O_p\left(n^{-\frac{1-2\alpha}{3}} t\right)$. Hence $\hat{t} = \lambda^{-1}(\hat{y}_{b_n} - \tilde{y}_0)$ maximizes

$$(3.7) \quad Z_n(t) = W_n(t) - t^2 + O_p\left(n^{-\frac{8\alpha-1}{3}} t\right).$$

Lemma 3.2:

$W_n(t)$ is asymptotically normal with mean 0 and variance $\min(|t|, 2B)$, for all t where

$$(3.8) \quad B = \lim_{n \rightarrow \infty} \frac{b_n}{v}.$$

Proof:

Note that $W_n(0) = 0$ and $E[W_n(t)] = 0$ for all t . By a straightforward but tedious calculation (Cf. Venter [18]), it can be shown that

$$(3.9) \quad \text{Cov} \{W_n(t), W_n(t^*)\} \rightarrow \frac{1}{2} \{ \min(|t|, 2B) + \min(|t - t^*|) - \min(|t - t^*|, 2B) \}.$$

In particular, for $t = t^*$,

$$(3.10) \quad \text{Var } [W_n(t)] = \min(|t|, 2B).$$

Since all the conditions of L\'evy's Theorem, p. 203, Fisz [8], are satisfied, the lemma follows. ||

Remark 3.1:

The distribution of $W_n(t)$ tends to a Gaussian process with mean 0 and variance-covariance function given by (3.9) and hence, by an application of Slutsky's Theorem,

$$Z_n(t) \xrightarrow{D} Z(t) = W(t) - t^2.$$

The Asymptotic Distribution of $\hat{y}_{b_n}$ and $\hat{x}_{b_n}$

Theorem 3.1:

The random variables

$$(3.11) \quad \nu^{-1/2} (\hat{y}_{b_n} - \tilde{y}_0)$$

$$(3.12) \quad \lambda^{-1/2} (\hat{x}_{b_n} - \tilde{x}_0)$$

are asymptotically distributed as the variable t which maximizes $Z(t) = W(t) - t^2$ where

- (i) for $\alpha = 1/5$, $W(t)$ is a Gaussian process with $W(0) = 0$, $E[W(t)] = 0$ for all t and covariance function given by (3.9); and
- (ii) for $1/8 < \alpha < 1/5$, $W(t)$ is a two-sided Wiener-L\'evy process with $W(0) = 0$, $E[W(t)] = 0$ for all t and variance 1 per unit t .

$\lambda = v/f(\tilde{x}_0)$ with v as defined in (3.6) and Ω_n has to satisfy (3.A4).

Proof:

$$t_{i+1} - t_i = \frac{F_n(\omega_{i+1,n}) - F_n(\omega_{i,n})}{v} \xrightarrow{P} 0 \text{ for}$$

$\omega_{i+1,n} - \omega_{i,n} = o_p\left(n^{-\frac{1-2\alpha}{3}}\right)$. Also, for $\alpha < 1/5$, $B = \infty$ and hence $W_n(t)$ tends to a two-sided Wiener process for $1/8 < \alpha < 1/5$. $\hat{t} = v^{-1}(\hat{y}_{b_n} - \tilde{y}_0)$ maximizes $Z_n(t)$ and since $Z_n(t) \rightarrow Z(t)$, (3.11) is immediate.

Proof of (3.12):

$$\begin{aligned} \hat{y}_{b_n} - \tilde{y}_0 &= F_n(\hat{x}_{b_n}) - F(\tilde{x}_0) \\ &= F_n(\hat{x}_{b_n}) - F(\hat{x}_{b_n}) + (\hat{x}_{b_n} - \tilde{x}_0)f(\zeta) \end{aligned}$$

where ζ lies between $\hat{x}_{b_n}$ and $\tilde{x}_0$. Therefore $(\hat{x}_{b_n} - \tilde{x}_0)f(\zeta) = (\hat{y}_{b_n} - \tilde{y}_0) + O_p(n^{-1/2})$, by Kolmogorov's Theorem. $\hat{x}_{b_n} \xrightarrow{a.s.} \tilde{x}_0 \Rightarrow \zeta \xrightarrow{a.s.} \tilde{x}_0$

and the result follows from Slutsky's Theorem. ||

Remark 3.2:

For the special case of estimation of the mode of a density, the above theorem reduces to Theorems 3a and 3b of Venter [18]. It is interesting to note that in all the four estimators discussed, for the fixed (narrow) window, the grid is required to be narrow (wide).

4. Asymptotic Efficiency of Narrow Window Estimators

In this section, we obtain the asymptotic efficiency of $\hat{x}_{a_n}$ relative to $\hat{x}_{b_n}$.

Let $\hat{Z}_n(t)$ and $\hat{Z}_n^*(t)$ be two consistent estimators of $Z(t)$ such that

$$(4.1) \quad n^\gamma (\hat{Z}_n(t) - Z(t)) \xrightarrow{D} H_1(x)$$

and

$$(4.2) \quad n^\gamma (Z_n^*(t) - Z(t)) \xrightarrow{D} H_2(x)$$

for some $\gamma > 0$, where H_1, H_2 depend, in general, on $Z(t)$.

Definition:

Kolmogorov-Smirnov distance

$$(4.3) \quad d[H_1(x), H_2(x)] = \sup_{-\infty < x < \infty} |H_1(x) - H_2(x)|.$$

Following Hodges and Lehmann [9], we define the asymptotic efficiency of $\hat{Z}_n(t)$ and $Z_n^*(t)$ as follows:

Definition:

The asymptotic efficiency of $\hat{Z}_n(t)$ relative to $Z_n^*(t)$ is

$$(4.4) \quad e(\hat{Z}_n(t), Z_n^*(t)) = \sigma_0^2$$

where σ_0 satisfies

$$(4.5) \quad \inf_{\sigma} d[H_1(x), H_2(x/\sigma)] = d[H_1(x), H_2(x/\sigma_0)].$$

In particular, if $H_1(x) = H(\sigma_1 x)$ and $H_2(x) = H(\sigma_2 x)$, then it easily follows that

$$(4.6) \quad e(\hat{Z}_n(t), Z_n^*(t)) = \frac{\sigma_2^2}{\sigma_1^2}.$$

If $e(\hat{Z}_n(t), Z_n^*(t)) = 1$, the two estimators are said to be asymptotically equivalent. From Theorems IV.3.1 and Theorem 3.1, by definition (4.6),

$$(4.7) \quad e(\hat{x}_{a_n}, \hat{x}_{b_n}) = \left(\frac{Cf(\tilde{x}_o)}{A} \right)^{4/3}.$$

Remark 4.1:

If we choose $A = Cf(\tilde{x}_o) + O_p(n^{-\gamma})$ for some $\gamma > 0$, it can be easily seen that the two estimators are asymptotically equivalent for

$$\gamma > \frac{1 - 2\alpha}{2} \quad \text{and} \quad 1/8 < \alpha \leq 1/5.$$

CHAPTER VII

OTHER ESTIMATORS AND COMPUTATIONAL ASPECTS

1. Other Estimators

1.1 The Naive Estimator

$$(1.1) \quad \check{r}_n(x) = \frac{f_n^*(x)}{g[G^{-1}F_n^*(x)]} = \frac{F_n^*(x + a_n) - F_n^*(x - a_n)}{2a_n \cdot g[G^{-1}F_n^*(x)]}$$

is called the *naive window estimator* of $r(x)$ and $\check{x}_n$ minimizing $\check{r}_n(x)$ estimates the change point $\tilde{x}_0$.

Under the conditions of Theorem IV.2.1, it follows from that theorem that $\check{x}_n$ is a strongly consistent estimator of $\tilde{x}_0$. Furthermore, if the assumptions in Section IV.3 are satisfied, it can be shown, by a proof similar to the proof of Theorem IV.3.1, that $\check{x}_n$ and $\hat{x}_{a_n}$ have the same asymptotic distribution, i.e., they are asymptotically equivalent.

1.2 A Family of Estimators of the Generalized Failure Rate Function and the Change Point

To estimate the unknown generalized failure rate function $r(x)$, consider statistics of the form:

$$(1.2) \quad \hat{r}_n(x) = \frac{1}{a_n} \int_{-\infty}^{\infty} K\left(\frac{x-u}{a_n}\right) d\phi_{F_n^*}(u)$$

and

$$(1.3) \quad \frac{1}{r_n^*(x)} = \frac{1}{b_n} \int_{-\infty}^{\infty} K\left(\frac{y-u}{b_n}\right) d\phi_{F_n^*}(u), \text{ at } x = F_n^{*-1}(y)$$

where $K(x)$ is a certain density function and a_n, b_n tend to 0 as $n \rightarrow \infty$. When G is the uniform distribution on $[0,1]$, (1.2) reduces to the statistic considered by Nadaraya [11], Parzen [12] and others for estimating a density function and mode. Note that when

$$(1.4) \quad K(u) = \begin{cases} \frac{1}{2} & |u| \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

we get the estimators considered in Chapters IV and VI. One can now define estimators of the change point with respect to the smoothing function K . It would seem possible to obtain the analogue of the results of Nadaraya and Parzen to this more general case.

1.3 Estimation of the U-Shaped Generalized Failure Rate Function and the Change Point

Suppose we know, apriori, that $r(x)$ is U-shaped. Using the approach of Barlow and van Zwet [1,2], estimators are suggested for the change point. In this subsection, we assume the case of complete sample; i.e., $F_n^* \equiv F_n$.

Assume initially that $\omega_{k,n} \leq \tilde{x}_0 < \omega_{k+1,n}$ for some k . Let r_n be an initial or basic estimator for r and $\hat{x}_n$ minimize r_n . Consider the following regression of r_n with respect the discrete measure μ_n :

$$\begin{aligned}
 (1.5) \quad \check{r}_n(x, k) = & \left[\begin{array}{ll}
 \sup_{k+1 \geq \underline{t} \geq \underline{i}+1} \inf_{s \leq \underline{i}} \frac{\sum_{j=s}^{t-1} r_n(\omega_{j,n}) \mu_n\{\omega_{j,n}\}}{\sum_{j=s}^{t-1} \mu_n\{\omega_{j,n}\}} & \omega_{i,n} \leq x < \omega_{i+1,n} \\
 & i = 0, 1, \dots, k-1 \\
 \inf_{s \leq k} \frac{\sum_{j=s}^k r_n(\omega_{j,n}) \mu_n\{\omega_{j,n}\}}{\sum_{j=s}^k \mu_n\{\omega_{j,n}\}} & \omega_{k,n} \leq x < \frac{\omega_{k,n} + \omega_{k+1,n}}{2} \\
 \inf_{\underline{t} \geq k+1} \frac{\sum_{j=k}^{t-1} r_n(\omega_{j,n}) \mu_n\{\omega_{j,n}\}}{\sum_{j=k}^{t-1} r_n(\omega_{j,n}) \mu_n\{\omega_{j,n}\}} & \frac{\omega_{k,n} + \omega_{k+1,n}}{2} \leq x < \omega_{k+1,n} \\
 \inf_{\underline{t} \geq i+1} \sup_{k \leq s \leq \underline{i}} \frac{\sum_{j=s}^{t-1} r_n(\omega_{j,n}) \mu_n\{\omega_{j,n}\}}{\sum_{j=s}^{t-1} \mu_n\{\omega_{j,n}\}} & \omega_{i,n} \leq x < \omega_{i+1,n} \\
 & i = k+1, \dots
 \end{array} \right]
 \end{aligned}$$

Note that $\check{r}_n(x, k)$ is a step function and decreases till $\omega_{k,n}$ and increases after $\omega_{k+1,n}$. The following criteria, for example, may be chosen to obtain the optimum value of k , viz. $k^* \equiv k^*(n)$.

- (i) Choose $k = k^*$ such that k^* minimizes

$$\sup_x |\check{r}_n(x, k) - r_n(x)|$$

- (ii) Choose $k = k^*$ such that k^* minimizes

$$\sum_{i=0}^{\infty} |\check{r}_n(\omega_{i,n}, k) - r_n(\omega_{i,n})|^2.$$

Note that k^* is, in general, not unique. While one could modify the definition to make k^* unique, this will not be necessary, since these intervals typically all lie within a range which is small compared to the variability of k^* . Then

$$(1.6) \quad x_n^* = \frac{\omega_{k^*,n} + \omega_{k^*+1,n}}{2} \text{ estimates } \tilde{x}_0.$$

Barlow and van Zwet suggest the following basic estimators r_n and discrete measures μ_n :

$$(1.7) \quad r_n(x) = \frac{F_n(\omega_{i+1,n}) - F_n(\omega_{i,n})}{(\omega_{i+1,n} - \omega_{i,n})g[G^{-1}F_n(\xi_i)]} \quad \begin{aligned} \omega_{i,n} &\leq x < \omega_{i+1,n} \\ \xi_i &= \frac{1}{2}(\omega_{i,n} + \omega_{i+1,n}) \end{aligned}$$

$$\mu_n(\omega_{i,n}) = (\omega_{i+1,n} - \omega_{i,n})g[G^{-1}F_n(\xi_i)].$$

Call the estimator, obtained from substituting (1.7) in (1.5), $\hat{r}_n(x, k)$. For the case of complete sample, $\omega_{i,n} = X_i$ for all i , G assumed to be the exponential distribution with mean 1 and k^* chosen to maximize the likelihood of the sample, $\hat{r}_n(x, k^*)$ is the same as the maximum likelihood estimator of a U-shaped failure rate function considered by Bray, Crawford and Proschan [4].

$$(1.8) \quad r_n(x) \text{ as defined in (1.7)}$$

$$\mu_n(\omega_{i,n}) = \omega_{i+1,n} - \omega_{i,n}$$

and substituting (1.8) in (1.5), we get $\hat{r}_n^*(x, k)$, the corresponding "smoothed" estimator.

$$r_n(x) = \frac{G_n^{-1}(\omega_{i+1,n}) - G_n^{-1}(\omega_{i,n})}{\omega_{i+1,n} - \omega_{i,n}} \quad \omega_{i,n} \leq x < \omega_{i+1,n}$$

(1.9)

$$\mu_n\{\omega_{i,n}\} = \omega_{i+1,n} - \omega_{i,n}$$

and $\tilde{r}_n(x, k)$ is obtained by substituting (1.9) in (1.5).

Let $I_n^* = [\omega_{k,n}^*, \omega_{k+1,n}^*]$. Based on the results of Barlow and van

Zwet [2], we make the following

Conjecture:

If

(1.A1) $\tilde{x}_0$ is unique;

(1.A2) r is continuously differentiable and f'' exists;

(1.A3) $r'(x) < 0$ for $x < \tilde{x}_0$ and $r'(x) > 0$ for $x > \tilde{x}_0$;

(1.A4) $r_n(x)$ is a consistent estimator of $r(x)$; and

(1.A5) $\omega_{i+1,n} - \omega_{i,n} = cn^{-\alpha}$ $0 < \alpha < 1/3$ and $c > 0$,

then

$$(1.10) \quad \lim_{n \rightarrow \infty} P \left[|x_n^* - \tilde{x}_0| \neq 0 \right] = 0$$

$$(1.11) \quad \lim_{n \rightarrow \infty} P \left[\sup_{x \notin I_n^*} |\check{r}_n(x, k^*) - r_n(x)| \neq 0 \right] = 0$$

where $\check{r}_n(x, k)$ is equal to $\hat{r}_n(x, k)$ or $r_n^*(x, k)$ or $\tilde{r}_n(x, k)$.

Analogous to the estimator based on total time on test measure given in Section 5, [2], we can define the following basic estimator and discrete measure:

$$r_n(x) = \frac{F_n(\omega_{i+1,n}) - F_n(\omega_{i,n})}{\phi_{F_n}[F_n(\omega_{i+1,n})] - \phi_{F_n}[F_n(\omega_{i,n})]} \quad \omega_{i,n} \leq x < \omega_{i+1,n}$$

(1.12)

$$\mu_n\{\omega_{i,n}\} = \phi_{F_n}[F_n(\omega_{i+1,n})] - \phi_{F_n}[F_n(\omega_{i,n})] .$$

Thus any one of the four basic estimators suggested above may be used to estimate $r(x)$ and $\hat{x}_0$. Mathematical analysis of such estimators of the change point, obtained from smoothing a basic estimator of $r(x)$ and based one of two criteria suggested above, seems intractable to obtain meaningful asymptotic results. Computational results are inconclusive to suggest that any one criterion or any one version of $\hat{r}_n(x,k)$ is superior to the rest.

1.4 Estimation of the Change Point - The Case of Incomplete Data

Items on test may possibly be of different ages. Further, an item may be removed from test by one of two ways - failure or truncation. Truncation is the action of summarily removing an item from test. Truncation times may or may not be known.

In this case, the maximum likelihood estimator of F , when no assumptions are made concerning the distribution, has been obtained by Kaplan and Meier [10]. This can be used to estimate the ϕ_F and $\phi_{\bar{F}}$ transformations and hence the change point.

2. Computational Aspects

2.1 Recommendations

We shall restrict ourselves to the strongly consistent estimators $\hat{x}_{a_n}$ and $\hat{x}_{b_n}$, discussed in Chapters IV and VI respectively. In order to

correspond to the asymptotic theory developed earlier, the windows are required to satisfy the relations: $a_n = Cn^{-\alpha}$, $b_n = An^{-\alpha}$, where A , C and α are positive constants. $\omega_{i+1,n} - \omega_{i,n} = cn^{-\frac{1-2\alpha}{3}}$, for all i and c a positive constant, is a convenient choice for the grid Ω_n and $\omega_{0,n}$ is determined by the left end point of the support F .

(1) Choice of α :

The estimator $\hat{r}_n(x)$ defined in Chapter IV is asymptotically the same as the basic estimator defined in (1.9). Barlow and van Zwet have shown that if $r(x)$ is twice differentiable, the mean square error of the basic estimator is minimized for $\alpha = 1/5$ (Cf. [2], p. 7). Hence $\alpha = 1/5$ is recommended. For this choice of α , $(\hat{x}_{a_n} - \tilde{x}_0) = O_p(n^{-1/5})$ and $(\hat{x}_{b_n} - \tilde{x}_0) = O_p(n^{-1/5})$.

(2) Choice of c :

By Theorems IV.3.1 and VI.3.1, Ω_n must satisfy the condition,

$\omega_{i+1,n} - \omega_{i,n} = o_p\left(n^{-\frac{1-2\alpha}{3}}\right)$ for all i . For $\alpha = 1/5$, this reduces to the condition $\omega_{i+1,n} - \omega_{i,n} = o_p(n^{-1/5})$ for all i , which may be satisfied in practice by choosing c less than both A and C .

(3) Choice of A and C :

The preference of one narrow window estimator over another as well as the "optimal" values of A and C depend on the particular distribution function, which is of course unknown. However on the basis of Monte Carlo simulations, some important conclusions are noteworthy for estimating the change point of probability density and failure rate functions.

- (i) $\hat{x}_{a_n}$ was noted to be sensitive to the choice of C for small samples (up to 3000). Improper choice of C could lead to estimates of $\tilde{x}_0$ and $r(\tilde{x}_0)$ well away from the true values.
- (ii) In contrast to the above, $\hat{x}_{b_n}$ and $r_n^*(\hat{x}_{b_n})$ were seen to be very good estimators if the criterion is to minimize the maximum error. Furthermore, they were relatively insensitive to the choice of A . It should be noted that a check has been built into the computer program to reduce the value of A if it is found to be large. It has not been possible to include a corresponding check for C .

Hence, though $\hat{x}_{a_n}$ and $\hat{x}_{b_n}$ are strongly consistent (under the assumptions in Chapters IV and VI), the estimator based on the ϕ transformation is recommended for small samples.

2.2 Numerical Results

Monte Carlo simulations were conducted in the following two cases:

- (i) change point (mode) of the $N(0,1)$ density (Table 2.1);
- (ii) change point (maximizing point) of the failure rate of a parallel structure composed of two independent components having exponential failure distributions with mean lifetimes $\frac{1}{2}$ and 1 (Table 2.2).

In Tables 2.1 and 2.2, for each sample size n , values computed from the ϕ transformation are given below the corresponding values from the ϕ transformation. These estimates are the average of values obtained in 25 simulations; there was no significant variation in the results when the number of simulations was increased. Estimates were obtained in both cases for a number of values of A and C . In case (i), A was chosen to make

the two estimators asymptotically equivalent (i.e., $A = Cf(\hat{x}_0)$). Simulations were conducted in both cases for n ranging from 50 to 3000. Some typical results are shown in the tables.

Several numerical investigations for estimating density and failure rate functions have been conducted by Watson and Leadbetter [19] and [20]. They obtained the best results, in the case of estimating a failure rate function, from a "heuristic graphical estimator" (Cf. [19], p. 180). To obtain this estimator, $-\log [1 - F_n(x)]$ is plotted against x and a smooth curve is drawn through the points by any reasonable method. The slope of the curve at any point x , say $\hat{r}_g(x)$, estimates the failure rate at that point. Since $-\log [1 - F_n(x)]$ is infinite for x equal to the last observation, no interpolation is possible between $(n-1)$ th and n th sample points. The change point can now be estimated by determining the point, say $\hat{x}_g$, (not necessarily unique) at which $\hat{r}_g(x)$ is minimum. This estimator, by its construction, does not come with formulae for its mean and variance.

The computer program was also applied to actual life data on two types of very expensive radar tubes. The data was in the form of failure times of 19 tubes of type 1 and 25 tubes of type 2. In each case, the change point was also estimated graphically (Figure 2.1) and the results are summarized in Tables 2.3 and 2.4.

The numerical investigations were carried out on a CDC-6400 computer at the Computer Center, University of California, Berkeley. The execution time for the case mentioned above was approximately 30 seconds for each tube type.

Details of the computer program are given in the Appendix.

TABLE 2.1

ESTIMATION OF THE MODE OF THE $N(0,1)$ DENSITY

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2} dt \quad -\infty < x < \infty$$

$$r(x) = f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad -\infty < x < \infty$$

$$\tilde{x}_0 = 0.0 \quad r(\tilde{x}_0) = 0.398916$$

$$A = Cf(\tilde{x}_0), \text{ i.e., } e(\hat{x}_{a_n}, \hat{x}_{b_n}) = 1$$

$$\alpha = 1/5$$

$$c = 1/4$$

Number of Simulations = 25

n	c	MEAN VALUE	MEAN SQUARE	MEAN VALUE	MEAN SQUARE
		OF $\hat{x}_{a_n}$ $\hat{x}_{b_n}$	ERROR OF $\hat{x}_{a_n}$ $\hat{x}_{b_n}$	OF $\hat{r}_n(\hat{x}_{a_n})$ $r_n^*(\hat{x}_{b_n})$	ERROR OF $\hat{r}_n(\hat{x}_{a_n})$ $r_n^*(\hat{x}_{b_n})$
50	0.4	-0.028814 0.094659	0.830221E-03 0.896029E-02	0.605723 0.684200	0.427688E-01 0.813866E-01
100	0.4	-0.013206 0.082339	0.174405E-03 0.677979E-02	0.552615 0.592910	0.236233E-01 0.376335E-01
250	0.4	-0.005406 0.097342	0.292295E-04 0.947540E-02	0.492992 0.498336	0.835025E-02 0.988418E-02
50	2.8	-0.037960 0.066669	0.144094E-02 0.444469E-02	0.320511 0.267973	0.614737E-02 0.171463E-01
100	2.8	0.034567 0.030586	0.119485E-02 0.935476E-03	0.341437 0.289064	0.330387E-02 0.120675E-01
250	2.8	-0.015350 -0.005406	0.235616E-03 0.292295E-04	0.356965 0.335935	0.175995E-02 0.396660E-02

TABLE 2.2

ESTIMATION OF THE MAXIMIZING POINT OF $r(x) = \frac{f(x)}{1 - F(x)}$

$$F(x)^{\dagger} = \begin{cases} 1 - e^{-x} - e^{-2x} + e^{-3x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$r(x) = \frac{f(x)}{1 - F(x)} = \frac{1 + 2e^{-x} - 3e^{-2x}}{1 + e^{-x} - e^{-2x}}$$

$$\tilde{x}_0 = 1.443635 \qquad r(\tilde{x}_0) = 1.105573$$

$$\alpha = 1/5 \qquad c = 1/4$$

Number of Simulations = 25

n	C	A	MEAN VALUE OF $\hat{x}_{a_n}$ $\hat{x}_{b_n}$	MEAN SQUARE ERROR OF $\hat{x}_{a_n}$ $\hat{x}_{b_n}$	MEAN VALUE OF $\hat{r}_n(\hat{x}_{a_n})$ $\hat{r}_n^*(\hat{x}_{b_n})$	MEAN SQUARE ERROR OF $\hat{r}_n(\hat{x}_{a_n})$ $\hat{r}_n^*(\hat{x}_{b_n})$
50	8.6	0.55	0.187495 1.166128	0.157789E+01 0.770105E-01	0.338594 1.261499	0.588256E+00 0.243129E-01
100	8.6	0.55	0.744460 1.337640	0.488846E+00 0.112350E-01	0.668490 1.236343	0.191042E+00 0.171007E-01
250	8.6	0.55	2.094735 1.365555	0.423931E+00 0.609655E-02	0.941717 1.196830	0.268488E-01 0.832793E-02
50	9.6	0.9	0.077742 1.015217	0.186567E+01 0.183542E+00	0.214144 1.080619	0.794646E+00 0.622694E-03
100	9.6	0.9	0.418013 1.086833	0.105190E+01 0.127308E+00	0.574817 1.099600	0.281701E+00 0.356783E-04
250	9.6	0.9	1.822950 1.166688	0.143879E+00 0.767000E-01	0.848998 1.109128	0.658305E-01 0.126385E-04

[†] Such a distribution describes the failure law of a parallel structure composed of two independent components having exponential failure distributions with mean lifetimes $\frac{1}{2}$ and 1.

TABLE 2.3
RADAR TUBE - TYPE 1

Data

- 1. n = 19
- 2. Observed Failure Times in Hours

533
827
877
1007
1271
2394
2741
3244
4130
4368
4744
7253
7705
9482
11813
12317
12563
14977
16713

Estimates

$\alpha = 1/5$ $c = 1/4$

C	$\hat{x}_{a_n}$ in Hrs.	$\hat{r}_n(\hat{x}_{a_n})$	A	$\hat{x}_{b_n}$ in Hrs.	$r_n^*(\hat{x}_{b_n})$
10	527.45	0.0	0.3	4130.01	0.102284E-03
30	516.35	0.0	0.4	4368.08	0.111463E-03
50	505.25	0.0	0.5	4368.08	0.115497E-03
100	477.51	0.0	0.6	4368.08	0.132645E-03
500	255.53	0.0	0.7	3244.04	0.134635E-03
1000	1826.02	0.0	0.8	4130.01	0.150098E-03
1500	5576.47	0.0	0.9	4368.08	0.159543E-03

$\hat{x}_g = 5750$ Hrs.

From Figure 2.1,

$\hat{r}_g(\hat{x}_g) = 0.250000E-04$

TABLE 2.4
RADAR TUBE - TYPE 2

Data

- 1. n = 25
- 2. Observed Failure Times in Hours

44
384
548
1172
1373
1527
1611
1614
1634
1873
2249
2892
3100
5160
5468
5531
5809
6631
7368
7511
8611
10847
10920
11546
11567

Estimates

$\alpha = 1/5$ $c = 1/4$

C	$\hat{x}_{a_n}$ in Hrs.	$\hat{r}_n(\hat{x}_{a_n})$	A	$\hat{x}_{b_n}$ in Hrs.	$r_n^*(\hat{x}_{b_n})$
10	38.75	0.0	0.3	3100.10	0.157882E-03
30	28.24	0.0	0.4	5809.10	0.150124E-03
50	70.40	0.0	0.5	1614.01	0.162450E-03
100	96.66	0.0	0.6	1614.01	0.184687E-03
500	810.68	0.0	0.7	3100.10	0.179432E-03
1000	3625.40	0.0	0.8	2892.08	0.183676E-03
1500	3880.05	0.0	0.9	3100.10	0.204013E-03

$\hat{x}_g = 4000$ Hrs.

From Figure 2.1,

$\hat{r}_g(\hat{x}_g) = 0.145833E-04$

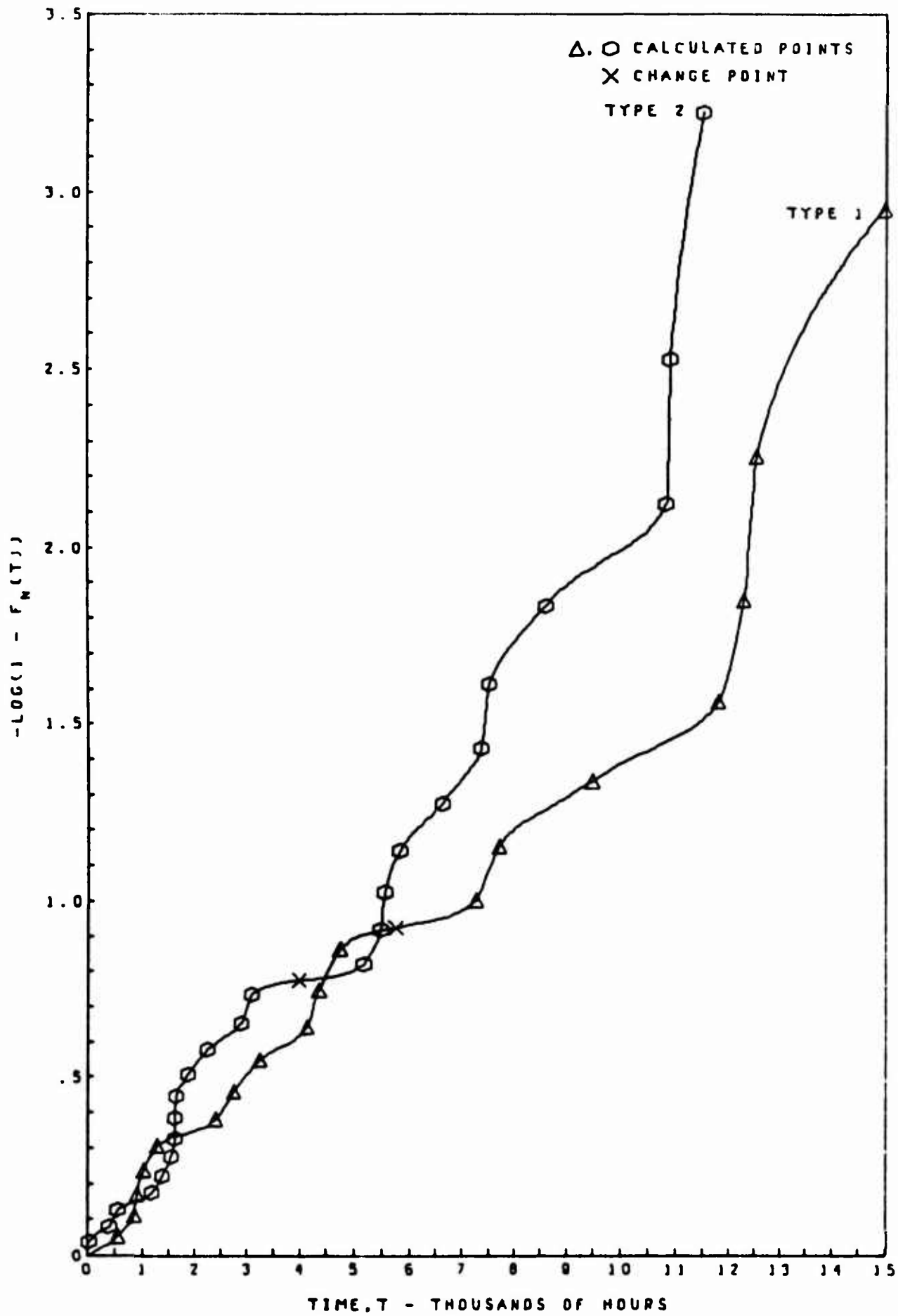


FIGURE 2.1 - GRAPHICAL ESTIMATION OF THE CHANGE POINT AND FAILURE RATE FOR TWO TYPES OF RADAR TUBES.

REFERENCES

- [1] Barlow, R. E. and W. R. van Zwet, "Asymptotic Properties of Isotonic Estimators for the Generalized Failure Rate Function," Proceedings of the First International Symposium on Nonparametric Techniques in Statistical Inference, Indiana University, Bloomington, Indiana, (June 1-6, 1969).
- [2] Barlow, R. E. and W. R. van Zwet, "Asymptotic Properties of Isotonic Estimators for the Generalized Failure Rate Function - Part II: Asymptotic Distributions," ORC 69-10, Operations Research Center, University of California, Berkeley, (1969).
- [3] Billingsley, P., CONVERGENCE OF PROBABILITY MEASURES, John Wiley & Sons, New York, (1968).
- [4] Bray, T. A., G. B. Crawford and F. Proschan, "Maximum Likelihood Estimation of a U-Shaped Failure Rate Function," Boeing Scientific Research Laboratories, Document D1-82-0660, (1967).
- [5] Chernoff, H., "Estimation of the Mode," Annals of the Institute of Statistical Mathematics, Vol. 16, pp. 31-41, (1964).
- [6] Cramér, H., MATHEMATICAL METHODS OF STATISTICS, Princeton University Press, Princeton, New Jersey, (1946).
- [7] Doob, J. L., STOCHASTIC PROCESSES, John Wiley & Sons, New York, (1953).
- [8] Fisz, M., PROBABILITY THEORY AND MATHEMATICAL STATISTICS, Third Edition, John Wiley & Sons, New York, (1963).
- [9] Hodges, J. L., Jr. and E. L. Lehmann, "Deficiency," Submitted to The Annals of Mathematical Statistics in 1969.
- [10] Kaplan, E. L. and P. Meier, "Nonparametric Estimation from Incomplete Observations," Journal of the American Statistical Association, Vol. 53, pp. 457-481, (1958).
- [11] Nadaraya, E. A., "On Non-Parametric Estimates of Density Functions and Regression Curves," Theory of Probability and Its Applications, Vol. 10, pp. 186-190, (1965).
- [12] Parzen, E., "On Estimation of a Probability Density Function and Mode," The Annals of Mathematical Statistics, Vol. 33, pp. 1065-1076, (1962).
- [13] Rao, B. L. S. P., "Asymptotic Distributions in Some Non-Regular Statistical Problems," Technical Report No. 9, Department of Statistics and Probability, Michigan State University, East Lansing, Michigan, (1966).
- [14] Rao, C. R., LINEAR STATISTICAL INFERENCE AND ITS APPLICATIONS, John Wiley & Sons, New York, (1965).

- [15] Royden, H. L., REAL ANALYSIS, Second Edition, The Macmillan Company, New York, (1968).
- [16] Sethuraman, J., "Limit Theorems for Stochastic Processes," Technical Report No. 10, Department of Statistics, Stanford University, Stanford, (1965).
- [17] Stone, C., "Weak Convergence of Stochastic Processes Defined on Semi-Infinite Time Intervals," Proceedings of The American Mathematical Society, Vol. 14, pp. 694-696, (1963).
- [18] Venter, J. H., "On Estimation of the Mode," The Annals of Mathematical Statistics, Vol. 38, pp. 1446-1455, (1967).
- [19] Watson, G. S. and M. R. Leadbetter, "Hazard Analysis I," Biometrika, Vol. 51, Parts 1 and 2, pp. 175-184, (1964).
- [20] Watson, G. S. and M. R. Leadbetter, "Hazard Analysis II," Sankhyā, Series A, Vol. 26, pp. 101-116, (1964).

APPENDIX

COMPUTER PROGRAM

The following program, written in FORTRAN IV, computes $\hat{x}_{a_n}$ and $\hat{r}_n(\hat{x}_{a_n})$, $\hat{x}_{b_n}$ and $r_n^*(\hat{x}_{b_n})$, for probability density and failure rate functions, under the assumption of complete data. The program comprises of a main routine and four subroutines as follows.

Main Routine

CPOINT : Controls the over-all computation and calculates the final results. A user has only to provide the input data specified by this routine.

Subroutines

ORSTAT : Sorts the failure times in ascending order.

EMP : Computes the empirical distribution function.

GINVF : Computes the value of $\phi_{F_n}(x)$ at any specified point x .

GGINVF : Computes $g\left[G^{-1}\left(\frac{i}{n}\right)\right]$ for $i = 0, 1, \dots, n$. These values are necessary to calculate $\phi_{F_n}(x)$.

A listing of the program is given on the following pages and, for each routine, the listing includes comments regarding the pertinent quantities used by that routine.

A.2

J7186,7,50,50000,30.7186,S. ARUNKUMAR,CHANGE POINT ESTIMATION
RUN,S,,,,,50000.

LGO.

```
PROGRAM CPOINT(INPUT,OUTPUT)
C
C MAIN ROUTINE - PROCESSES THE DATA AND OBTAINS TWO
C ESTIMATES OF THE CHANGE POINT AND THE VALUE OF THE
C GENERALIZED FAILURE RATE FUNCTION AT THE CHANGE POINT.
C
C INPUT REQUIRED BY THE PROGRAM.
C
C N = NUMBER OF FAILURE DATA.
C NF - NF IS EQUAL TO ZERO IF R(X) IS A PROBABILITY
C DENSITY FUNCTION AND EQUAL TO ONE IF R(X) IS A
C FAILURE RATE FUNCTION.
C NM - NM IS EQUAL TO ZERO IF THE CHANGE POINT IS THE
C MINIMIZING POINT AND EQUAL TO ONE IF IT IS THE
C MAXIMIZING POINT.
C XZERO = LEFT HAND END POINT OF THE SUPPORT OF F.
C N,NF,NM,XZERO SHOULD BE INPUT ACCORDING TO FORMAT 1000.
C X(I),I = 1,N, ARE THE N FAILURE TIMES. THEY SHOULD BE
C READ IN ACCORDING TO FORMAT 1010.
C
C OUTPUT FROM THE PROGRAM.
C
C X(AN) = ESTIMATE OF THE CHANGE POINT FROM THE LITTLE
C PHI TRANSFORMATION.
C R(AN) = ESTIMATE OF THE GENERALIZED FAILURE RATE
C FUNCTION AT THE CHANGE POINT FROM THE LITTLE
C PHI TRANSFORMATION.
C X(BN) = ESTIMATE OF THE CHANGE POINT FROM THE CAPITAL
C PHI TRANSFORMATION.
C R(BN) = ESTIMATE OF THE GENERALIZED FAILURE RATE
C FUNCTION AT THE CHANGE POINT FROM THE CAPITAL
C PHI TRANSFORMATION.
C
COMMON N,NF,NF,NM,GGINV0,GGINV(1000),IN,X(1000)
C SPECIFY A,ALPHA,C - CONSTANT FOR WINDOW,CL - CONSTANT
C FOR GRID.
A = 0.5
ALPHA = 0.2
C = 10.0
CL = 0.25
READ 1000,N,NF,NM,XZERO
1000 FORMAT(I4,2I1,F20.10)
READ 1010,(X(I),I=1,N)
1010 FORMAT(4F20.8)
FN = N
CALL ORSTAT
CALL GGINVF
1 GRID = FN**(-ALPHA)
AN = C*GRID
BN = A*GRID
GRID = CL*GRID
```

```

    IST1 = 1
    IST2 = 1
    IST3 = 1
    PE1 = -1.0
    PE2 = -1.0
    PE = -1.0
    P = X(1)-AN
    NCOUNT = 1
10  WMIN = P-AN
    WMAX = P+AN
    IN = IST1
    CALL EMP(WMIN,E1)
    IST1 = IN
    IN = IST2
    CALL EMP(WMAX,E2)
    IF (PE1 .LT. E1)GO TO 11
    IF (PE2 .EQ. E2)GO TO 60
    GO TO 12
11  CALL GINV(F(E1,A1))
12  IST2 = IN
    PE1 = E1
    PE2 = E2
    IF (E2 .LT. 1.0)GO TO 20
    IF (NF .EQ. 1)GO TO 60
20  CALL GINV(F(E2,A2))
    OF = A2-A1
    IF (NM .EQ. 1)GO TO 50
    IF (NCOUNT .EQ. 1)GO TO 30
    IF (OF .GE. FR1)GO TO 60
30  FR1 = OF
    CP1 = P
    GO TO 60
40  INDEX = 1
    GO TO 110
50  IF (NCOUNT .EQ. 1)GO TO 30
    IF (OF .LE. FR1)GO TO 60
    GO TO 30
60  IN = IST3
    CALL EMP(P,E)
    IF (PE .EQ. E)GO TO 110
    IST3 = IN
    PE = E
    MIN = FN*(E-BN)
    IF (MIN)40,70,70
70  MAX = FN*(E+BN)-1.0
    IF (MAX .GE. N)GO TO 110
    SUM = 0.0
    DO 90 I = MIN,MAX
    IF (I .GT. 0)GO TO 80
    SUM = SUM+GGINV(I)*(X(I+1)-X(I))
    GO TO 90
80  SUM = SUM+GGINV(I)*(X(I+1)-X(I))
90  CONTINUE
    OF = SUM
    IF (NM .EQ. 1)GO TO 120

```

```

      IF (NCOUNT .EQ. 1)GO TO 100
      IF (INDEX .EQ. 1)GO TO 100
      IF (OF .LE. FR2)GO TO 110
100  FR2 = OF
      CP2 = P
      INDEX = 0
110  NCOUNT = NCOUNT+1
111  IF (E .EQ. 1.0)GO TO 130
      P = P+GRID
      IF (E2 .EQ. 1.0)GO TO 60
      GO TO 10
120  IF (NCOUNT .EQ. 1)GO TO 100
      IF (INDEX .EQ. 1)GO TO 100
      IF (OF .GE. FR2)GO TO 110
      GO TO 100
130  FR1 = FR1/(2.0*AN)
      IF (INDEX .EQ. 1)GO TO 160
      IF (FR2 .EQ. 0.0)GO TO 140
      FR2 = (2.0*BN)/FR2
      GO TO 150
140  FR2 = 999999999.99999
150  PRINT 1020,N,A,C,CP1,FR1,CP2,FR2
1020 FORMAT(4X,*N =*,I4,3X,*A =*,F5.2,3X,*C =*,F5.1,3X,
1*X(AN) =*,E14.6,3X,*R(AN) =*,E14.6,3X,*X(BN) =*,E14.6,
23X,*R(BN) =*,E14.6)
      GO TO 170
160  A = A-0.05
      IF (A .GT. 0.0)GO TO 1
      PRINT 1030,N,A,C,CP1,FR1
1030 FORMAT(4X,*N =*,I4,3X,*A =*,F5.2,3X,*C =*,F5.1,3X,
1*X(AN) =*,E14.6,3X,*R(AN) =*,E14.6,3X,
2*DECREASE A BY A SMALLER AMOUNT IN STATEMENT 160*)
170  STOP
      END

```

```

SUBROUTINE ORSTAT
C
C   SORTS THE FAILURE TIMES IN ASCENDING ORDER TO OBTAIN
C   ORDER STATISTICS.
C
COMMON N,FN,NF,NM,GGINV0,GGINV(1000),IN,X(1000)
NN = N-1
10 IND = 0
DO 20 I = 1,NN
  J = I+1
  IF (X(I) .LE. X(J))GO TO 20
  S = X(I)
  X(I) = X(J)
  X(J) = S
  IND = I
20 CONTINUE
NN = IND-1
IF (IND .GE. 2)GO TO 10
RETURN
END

```

```

SUBROUTINE EMP(XX,FNX)
C
C   FN = VALUE OF THE EMPIRICAL DISTRIBUTION FUNCTION AT
C   XX.
C
COMMON N,FN,NF,NM,GGINV0,GGINV(1000),IN,X(1000)
DO 10 I = IN,N
  IF (XX .LT. X(I))GO TO 20
10 CONTINUE
  FI = FN
  GO TO 40
20 IF (I .GT. 1)GO TO 30
  FI = I
  FN = 0.0
  GO TO 50
30 FI = I-1
40 FN = FI/FN
50 IN = FI
RETURN
END

```

```

SUBROUTINE GINVF(Y,VALUE)
C
C  VALUE = GINVERSE FUNCTION COMPUTED AT Y.
C
COMMON N,FN,NF,NM,GGINVO,GGINV(1000),IN,X(1000)
IF (NF .EQ. 1)GO TO 10
VALUE = Y
GO TO 20
10 VALUE = -ALOG(1.0-Y)
20 RETURN
END

```

```

SUBROUTINE GGINVF
C
C  COMPUTES THE GGINVERSE FUNCTION.
C
COMMON N,FN,NF,NM,GGINVO,GGINV(1000),IN,X(1000)
IF (NF .EQ. 1)GO TO 20
DO 10 I =1,N
GGINV(I) = 1.0
10 CONTINUE
GO TO 40
20 DO 30 I = 1,N
F = I
GGINV(I) = (FN-F)/FN
30 CONTINUE
40 GGINVO = 1.0
RETURN
END

```

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