

# COMPARISON OF SEVERAL NONPARAMETRIC ESTIMATORS OF THE FAILURE RATE FUNCTION

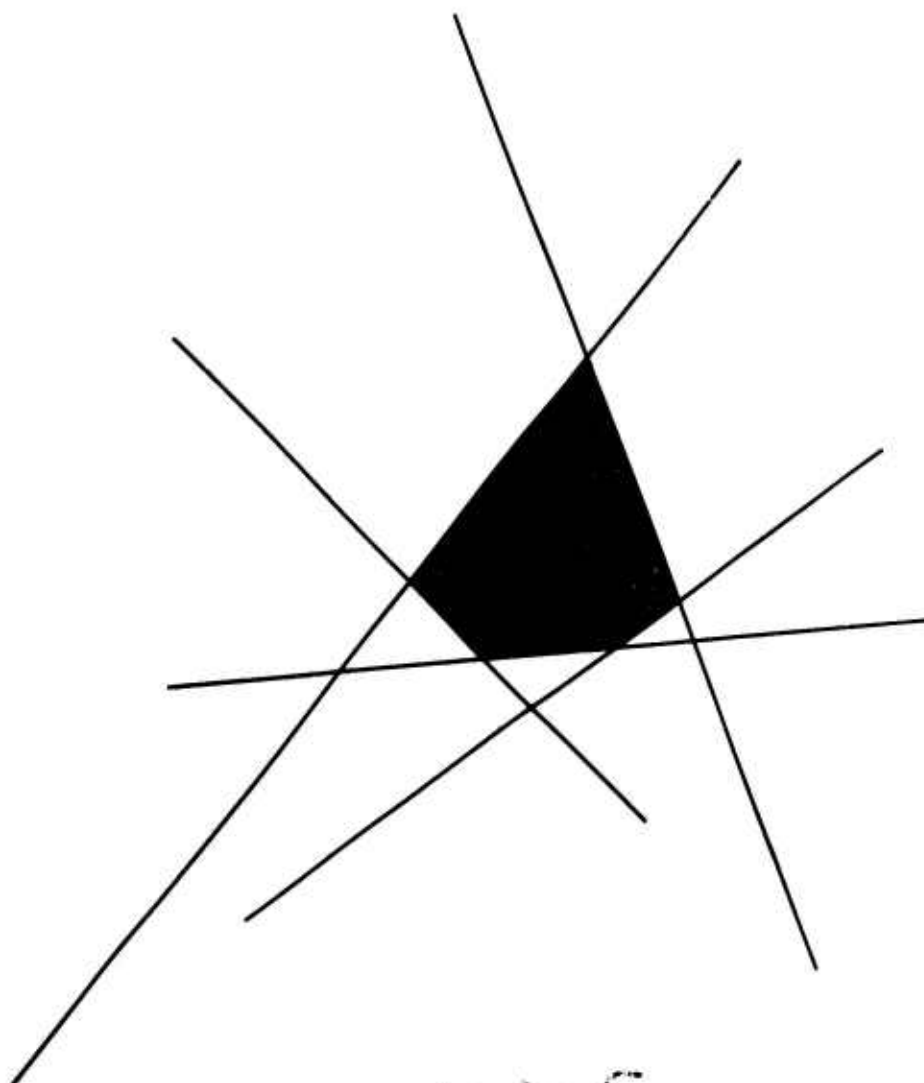
by

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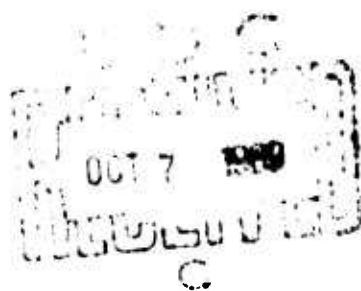
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COMPARISON OF SEVERAL NONPARAMETRIC  
ESTIMATORS OF THE FAILURE RATE FUNCTION<sup>+</sup>

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# ABSTRACT

→ Several nonparametric estimators of the failure rate function, assuming that the true unknown failure rate is increasing, are proposed and their asymptotic properties compared. Finite sample comparisons are made using Monte Carlo methods. ( ) <— Recommendations are made for the benefit of potential users. ( )

## 1. INTRODUCTION

In many statistical studies involving failure data, biometric mortality data, and actuarial mortality data, the failure rate  $r(x) = f(x)/[1 - F(x)]$ , (corresponding to a lifetime distribution with density  $f(x)$  and distribution function  $F(x)$ ), is of prime importance. A problem of considerable interest, therefore, is the estimation of the failure rate function from a sample of  $n$  independent identically distributed lifetimes. Many observed failure rate function estimators seem to first decrease and then increase (are U-shaped) or simply increase or decrease. For simplicity, we will confine attention to *monotone failure rate functions*. (The U-shaped case can be estimated by suitably modifying the *monotonic* estimators, see for example Bray, Crawford and Proschan [4] and Barlow [1, pp. 60-61].)

In most cases, of course, the mortality data will be incomplete. Hence, we require estimators which can cope with incomplete data and which will, in some sense, be efficient. The efficiency of a method may be expressed by its mean square error, and one wants naturally an estimate of high efficiency. We will compare several nonparametric estimators of the failure rate function on the basis of their asymptotic mean square error for large sample sizes, as well as their Monte Carlo mean square error computed for small sample sizes.

Parametric and nonparametric methods for estimating the failure rate function are discussed in detail by Grenander [5]. He also characterizes the maximum likelihood estimate (MLE) of the failure rate function under the increasing failure rate (IFR) assumption. The MLE can be easily computed even for very incomplete data (withdrawals may be allowed for example). If a total of  $n$  items are exposed to risk, failures are observed at times

$$Z_1 \leq Z_2 \leq \dots \leq Z_k \quad (k \leq n)$$

and  $n(u)$  is the number of items exposed to risk at time  $u$ , then the MLE estimate for the failure rate,  $r(t)$ , can be expressed as a step function, where

$$(1.1) \quad \hat{r}_n(t) = \begin{cases} 0 & 0 \leq t < Z_1 \\ \hat{r}_n(Z_i) & Z_i \leq t < Z_{i+1} \\ \infty & t \geq Z_k \end{cases}$$

and

$$(1.2) \quad \hat{r}_n(Z_i) = \max_{s \leq i} \min_{t \geq i+1} \left\{ \frac{t-s}{\sum_{j=s}^{t-1} \int_{Z_j}^{Z_{j+1}} n(u) du} \right\}.$$

Marshall and Proschan [6] proved that  $\hat{r}_n(t)$  is strongly consistent in the complete sample case.

Prakasa Rao [7] has characterized the limiting distribution of  $\hat{r}_n(t)$  assuming  $r(t)$  increasing and  $r'(t) > 0$ . He shows that

$$(1.3) \quad \mathcal{L} \left[ n^{\frac{1}{3}} \left\{ \frac{2f(t)}{r'(t)r^2(t)} \right\}^{\frac{1}{3}} \left\{ \hat{r}_n(t) - r(t) \right\} \right] \rightarrow H(x)$$

where  $H(x)$  is a distribution whose density is determined implicitly as a solution to the heat equation. This result enables us to make asymptotic comparisons with other nonparametric estimators of the failure rate function.

The main deficiencies of the maximum likelihood estimate as defined in (1.1) and (1.2) are:

- (1) According to (1.3) the MLE converges like  $n^{-\frac{1}{3}}$  while many estimators for parametric models converge much faster; e.g., like  $n^{-\frac{1}{2}}$ .

- (2) Some failure times must be observed to use (1.2) whereas in practice we often know only the number of failures within prescribed intervals. However, this criticism is partially unfair since the MLE estimate for discrete IFR distributions could be used in this situation (see Marshall and Proschan [6], pp. 76-77).

Watson and Leadbetter [9] study nonparametric estimators for the failure rate function which perform better, asymptotically, than the IFR maximum likelihood estimator. However, their estimator will in general be very "ragged" and certainly not monotonic even though we may feel that the true failure rate function is monotonic. Let  $0 \equiv w_{0,n} < w_{1,n} < w_{2,n} < \dots$  be a subdivision of  $[0, \infty)$ . Let  $F_n$  be the empirical distribution function corresponding to an ordered sample

$$0 \equiv X_0 < X_1 < X_2 < \dots < X_n$$

from  $F$ ; i.e.,

$$F_n(x) = \begin{cases} 0 & x < X_1 \\ \frac{i}{n} & X_i \leq x < X_{i+1} \\ 1 & x \geq X_n \end{cases}$$

Watson and Leadbetter [9] study the estimator

$$r_n(x) = \frac{F_n(w_{i+1,n}) - F_n(w_{i,n})}{(w_{i+1,n} - w_{i,n})[1 - F_n(w_{i,n})]}$$

where  $w_{i,n} \leq x < w_{i+1,n}$ . Let  $w_{i+1,n} - w_{i,n} = cn^{-\alpha}$  for  $\frac{1}{5} < \alpha < 1$ . They show that asymptotically, as  $n \rightarrow \infty$

$$(1.4) \quad \mathcal{L} \left\{ \left[ \frac{cf(x)}{r(x)} \right]^{\frac{1-\alpha}{2}} n^{\frac{1-\alpha}{2}} [r_n(x) - r(x)] \right\} \rightarrow N(0,1)$$

where  $N(0,1)$  stands for the normal distribution with mean 0 and variance 1.

If we compare (1.3) and (1.4), we see that  $r_n(x)$  will be asymptotically more efficient than the MLE for IFR failure rate functions when  $\alpha < \frac{1}{3}$ .

Intuitively, it seems reasonable that if we can "smooth"  $r_n(x)$  so that it is monotonic, we will be able to improve on the IFR maximum likelihood estimate.

## 2. MONOTONIC REGRESSION ESTIMATORS

Let  $0 \equiv w_{0,n} < w_{1,n} < \dots < w_{i,n} < \dots$  be a subdivision of  $[0, \infty)$  and  $\mu_n\{w_{j,n}\}$  a sequence of weights on  $\{w_{j,n}\}_{j=0}^{\infty}$ . Assume  $w_{i,n} \leq x < w_{i+1,n}$ . We consider two basic or initial estimators for the failure rate function, namely,

$$(2.1) \quad r_n(x) = \frac{F_n(w_{i+1,n}) - F_n(w_{i,n})}{(w_{i+1,n} - w_{i,n})[1 - F_n(w_{i,n})]}$$

and

$$(2.2) \quad \phi_n(x) = \frac{-\log [1 - F_n(w_{i,n})] + \log [1 - F_n(w_{i+1,n})]}{w_{i+1,n} - w_{i,n}}.$$

We call  $\phi_n(x)$  the "graphical" estimator since it is motivated by the fact that

$$\int_{w_{i,n}}^{w_{i+1,n}} r(u) du = -\log [1 - F(w_{i+1,n})] + \log [1 - F(w_{i,n})].$$

The monotone increasing regression estimators corresponding to (2.1) and (2.2) and weights  $\mu\{w_{i,n}\} \geq 0$  for  $i = 1, 2, \dots$  are

$$(2.3) \quad \hat{r}_n(x) = \min_{t \geq i+1} \max_{s \leq i} \frac{\sum_{j=s}^{t-1} r_n(w_{j,n}) \mu_n\{w_{j,n}\}}{\sum_{j=s}^{t-1} \mu_n\{w_{j,n}\}}$$

and

$$(2.4) \quad \hat{\phi}_n(x) = \min_{t \geq i+1} \max_{s \leq i} \frac{\sum_{j=s}^{t-1} \phi_n(w_{j,n}) \mu_n\{w_{j,n}\}}{\sum_{j=s}^{t-1} \mu_n\{w_{j,n}\}}.$$

(Monotone decreasing regression estimators correspond to interchanging the Min and Max operations.) It is easy to verify that both  $\hat{r}_n(x)$  and  $\hat{\phi}_n(x)$  are nondecreasing functions of  $x$ . It can also be shown that if  $r(x)$  is the true nondecreasing failure rate function, then, in a certain sense,  $\hat{r}_n$  is closer than  $r_n$  to  $r$  and  $\hat{\phi}_n$  is closer than  $\phi_n$  to  $r$ .

We shall see that the maximum likelihood estimate for  $r(x)$  suggests the *total time on test* weights. Suppose we take as our grid, the order statistics  $0 \equiv X_0 < X_1 < \dots < X_n$  and

$$\mu_n\{X_i\} = \frac{1}{n} (n - i)(X_{i+1} - X_i) = [1 - F_n(X_i)](X_{i+1} - X_i) .$$

Then  $\hat{r}_n(x)$  becomes, for  $X_i \leq x < X_{i+1}$ ,

$$(2.5) \quad \hat{r}_n(x) = \min_{t \geq i+1} \max_{s \leq i} \frac{\sum_{j=s}^{t-1} r_n(X_j) [1 - F_n(X_j)] (X_{j+1} - X_j)}{\sum_{j=s}^{t-1} [1 - F_n(X_j)] (X_{j+1} - X_j)}$$

or

$$(2.6) \quad \hat{r}_n(x) = \min_{t \geq i+1} \max_{s \leq i} \frac{t - s}{\sum_{j=s}^{t-1} (n - j)(X_{j+1} - X_j)} .$$

We recognize (2.6) as the IFR MLE (1.2) when we have a complete sample. Thus, the IFR MLE is a special case of a monotonic regression estimator. Note that  $(n - i)(X_{i+1} - X_i)$  is the total time on test between the  $i$ th and  $(i + 1)$ st failure. For arbitrary grids (2.5) becomes

$$(2.7) \quad \hat{r}_n(x) = \min_{t \geq i+1} \max_{s \leq i} \frac{F_n(w_{t,n}) - F_n(w_{s,n})}{\sum_{j=s}^{t-1} [1 - F_n(w_{j,n})] (w_{j+1,n} - w_{j,n})}.$$

If we choose  $\mu_n\{w_{i,n}\} = (w_{i+1,n} - w_{i,n})$  and use the graphical estimator, then (2.4) becomes

$$(2.8) \quad \hat{\phi}_n(x) = \min_{t \geq i+1} \max_{s \leq i} \frac{-\log [1 - F_n(w_{t,n})] + \log [1 - F_n(w_{s,n})]}{w_{t,n} - w_{s,n}}$$

Although we consider only monotonic regression estimators of the form (2.7) or (2.8) it is clear that we have a whole zoo of possible estimators. We are at liberty to choose the basic estimators,  $r_n(x)$ , the grid  $\{w_{i,n}\}_{i=1}^{\infty}$  and the weights  $\mu_n\{w_{i,n}\} \geq 0$ . In [2] it is shown that  $\hat{r}_n(x)$  and  $\hat{\phi}_n(x)$  converge to  $r(x)$  as  $n \rightarrow \infty$  under very general conditions on the grid size, etc.

### 3. ASYMPTOTIC RESULTS

In order to make asymptotic comparisons, we consider grids with spacings of the type  $w_{i+1,n} - w_{i,n} = cn^{-\alpha}$  for  $c > 0$  and  $0 < \alpha < 1$ .

It can be shown that the estimator,  $r_n(x)$ , given by (2.1) is asymptotically normal with mean

$$(3.1) \quad r(x) + \frac{1}{24} \frac{r(x)f''(x)}{f(x)} c^2 n^{-2\alpha} + o(n^{-3\alpha})$$

and variance

$$(3.2) \quad \frac{r^2(x)n^{\alpha-1}}{f(x)} + o(n^{-1}).$$

Hence, the mean square error (MSE) of  $r_n(x)$  is approximately

$$(3.3) \quad \text{MSE} [r_n(x)] = \frac{r^2(x)n^{\alpha-1}}{cf(x)} + \left[ \frac{c^2 r(x)f''(x)}{24f(x)} \right]^2 n^{-4\alpha}$$

For  $1/5 < \alpha < 1$

$$(3.4) \quad \frac{\sqrt{cf(x)}}{r(x)} n^{\frac{1-\alpha}{2}} [r_n(x) - r(x)]$$

has asymptotically a  $N(0,1)$  distribution, while for  $1/7 < \alpha < 1/5$

$$(3.5) \quad \frac{\sqrt{cf(x)}}{r(x)} n^{\frac{1-\alpha}{2}} \left[ r_n(x) - r(x) - \frac{c^2 n^{-2\alpha} r(x)f''(x)}{24} \right]$$

is asymptotically  $N(0,1)$ .

It is shown in [3] that for  $0 < \alpha < 1/3$  and  $r'(x) > 0$

$$\text{MSE} [\hat{r}_n(x)] = \text{MSE} [r_n(x)]$$

asymptotically so that asymptotically the monotonic regression estimator is no more efficient than the basic estimator,  $r_n(x)$ . However, as we pointed out earlier, if the true failure rate is nondecreasing, the monotonic regression estimator will be "closer" to the true failure rate, at least for small sample sizes, than the basic estimator. However, if the true distribution is *exponential*, the monotonic regression estimator will converge like  $n^{-1/2}$  for any choice of  $\alpha$  ( $0 < \alpha \leq 1$ ) [see [3]]. Note that in this case even the IFR MLE is superior to the basic estimator.

It is shown in [3], that the estimator  $\phi_n(x)$  is asymptotically normal with mean

$$(3.6) \quad r(x) + \frac{1}{24} r''(x) c^2 n^{-2\alpha} + o(n^{-3\alpha})$$

and variance

$$(3.7) \quad \frac{r^2(x) n^{\alpha-1}}{n c n^{-\alpha} f(x)} + o(n^{-1}) .$$

For  $1/5 < \alpha < 1$ ,  $\phi_n(x)$  is asymptotically equivalent to  $r_n(x)$ . However, for  $1/7 < \alpha \leq 1/5$

$$(3.8) \quad \frac{\sqrt{c f(x)}}{r(x)} n^{\frac{1-\alpha}{2}} \left[ \phi_n(x) - r(x) - \frac{c^2 r''(x)}{24} n^{-2\alpha} \right]$$

is asymptotically  $N(0,1)$ . Note that (3.5) and (3.8) differ in the bias term.

Asymptotically, for  $0 < \alpha < 1/3$ ,  $\hat{\phi}_n(x)$  and  $\phi_n(x)$  are equivalent. However, the same remarks concerning  $\hat{r}_n(x)$  apply to  $\hat{\phi}_n(x)$ . Therefore, if the true failure rate is nondecreasing, we would prefer  $\hat{\phi}_n(x)$  to  $\phi_n(x)$  with  $0 < \alpha < 1/3$ .

The mean square error of both  $r_n(x)$  and  $\phi_n(x)$  are minimized when we choose  $\alpha = 1/5$ . Therefore, on the basis of asymptotic considerations, we would recommend using either  $\hat{r}_n(x)$  or  $\hat{\phi}_n(x)$  with  $\alpha = 1/5$ . However, this is not

too helpful since the optimum choice of  $c$  would still depend on the true failure rate function which is of course unknown. The MLE takes care of this problem by choosing  $w_{i,n} = X_i$ , the  $i$ th order statistic from our sample. One might modify the MLE by choosing  $w_{i,n} = X_{[in^\beta]}$  where  $[ ]$  denotes the greatest integer in the quantity within the brackets and  $\beta = 1 - \alpha$ . Hence,

$$X_{[(i+1)n^\beta]} - X_{[in^\beta]} = O_p\left(\frac{n^\beta}{n}\right) = O_p(n^{-\alpha}),$$

and by choosing  $\alpha = 1/5$ , we realize the recommended requirement.

#### 4. NUMERICAL RESULTS

Asymptotic investigations indicate that we should use either  $\hat{\phi}_n(x)$  or  $\hat{r}_n(x)$  with  $\alpha = 1/5$ . However, for this choice of  $\alpha$ ,  $\hat{\phi}_n(x)$  and  $\hat{r}_n(x)$  are not asymptotically equivalent. They differ in their bias terms. In general, we will prefer  $\hat{\phi}_n(x)$  if  $r$  is "nearly" linear in a neighborhood of  $x$  and  $\hat{r}_n(x)$  if  $f$  is "nearly" linear in a neighborhood of  $x$ .

In order to choose between  $\hat{\phi}_n(x)$  and  $\hat{r}_n(x)$ , several numerical investigations have been made. The most extensive numerical investigations have been conducted by Watson and Leadbetter [8] and [9]. They did not consider monotonic regression estimators except for the IFR MLE estimator. They obtained the best results from a "heuristic graphical" estimator. To obtain this estimator, they plot  $-\log [1 - F_n(x)]$  against  $x$ . A smooth curve is then drawn through these points and its slope determined, perhaps just with a straight edge. This estimator, by its construction, does not come with formulae for its mean and variance. The estimator  $\hat{\phi}_n(x)$  is similar to this estimator except that the slopes of the tangent lines are now required to be increasing. Because of this similarity, we might expect the good results obtained for their heuristic estimator to carry over for  $\hat{\phi}_n(x)$ .

Tables I and II present the mean square error of the IFR MLE,  $\hat{r}_n(x)$ , and its mean value when the underlying failure distribution is exponential with unit mean. Since the estimator is always infinite beyond the largest observation, we have also included the number of infinities obtained in the simulations.

Tables III and IV present the mean square error and mean value of the MLE estimator assuming a U-shaped failure rate function when the underlying failure distribution is exponential with unit mean. In the case of an infinite estimator within the range of  $x$  values recorded, the estimator was assumed constant (not infinite) beyond the last order statistic for which the estimator was finite. This probably accounts for the large mean square error values for large  $x$  values.

Tables V through XIV compare the graphical estimator and the maximum likelihood estimator using the grid determined by order statistics when the true distribution is a Weibull distribution. On the basis of mean square error calculations, we can detect little difference in the two estimators.

These computer computations were performed by Tom Bray of the Boeing Scientific Research Laboratories.

TABLE I

MLE FOR INCREASING FAILURE RATE WHEN TRUE  
DISTRIBUTION IS THE UNIT EXPONENTIAL DISTRIBUTION

Number of Simulations = 3000

Sample Size =  $n = 25$

| X    | MEAN VALUE<br>OF $\hat{r}_n(x)$ | MEAN SQUARE<br>ERROR OF $\hat{r}_n(x)$ | NUMBER OF<br>INFINITIES |
|------|---------------------------------|----------------------------------------|-------------------------|
| 0.10 | 0.606914                        | 0.210603                               | 0                       |
| 0.20 | 0.814584                        | 0.117653                               | 0                       |
| 0.30 | 0.886149                        | 0.902205E-01                           | 0                       |
| 0.40 | 0.927219                        | 0.835712E-01                           | 0                       |
| 0.50 | 0.962705                        | 0.841033E-01                           | 0                       |
| 0.60 | 0.993960                        | 0.869380E-01                           | 0                       |
| 0.70 | 1.02032                         | 0.918217E-01                           | 0                       |
| 0.80 | 1.04681                         | 0.101941                               | 0                       |
| 0.90 | 1.07076                         | 0.112549                               | 0                       |
| 1.00 | 1.09374                         | 0.126824                               | 0                       |
| 1.10 | 1.12219                         | 0.157527                               | 0                       |
| 1.20 | 1.15558                         | 0.239723                               | 0                       |
| 1.30 | 1.19133                         | 0.428387                               | 1                       |
| 1.40 | 1.25097                         | 3.72891                                | 2                       |
| 1.50 | 1.24791                         | 0.392483                               | 6                       |

TABLE II

MLE FOR INCREASING FAILURE RATE WHEN TRUE  
DISTRIBUTION IS THE UNIT EXPONENTIAL DISTRIBUTION

Number of Simulations = 250

Sample Size =  $n = 100$

| X    | MEAN VALUE<br>OF $\hat{F}_n(x)$ | MEAN SQUARE<br>ERROR OF $\hat{F}_n(x)$ | NUMBER OF<br>INFINITIES |
|------|---------------------------------|----------------------------------------|-------------------------|
| 0.20 | 0.919915                        | 0.268983E-01                           | 0                       |
| 0.40 | 0.974432                        | 0.176583E-01                           | 0                       |
| 0.60 | 1.01091                         | 0.151868E-01                           | 0                       |
| 0.80 | 1.03226                         | 0.199563E-01                           | 0                       |
| 1.00 | 1.05027                         | 0.241104E-01                           | 0                       |
| 1.20 | 1.06319                         | 0.265414E-01                           | 0                       |
| 1.40 | 1.07543                         | 0.307885E-01                           | 0                       |
| 1.60 | 1.09109                         | 0.364013E-01                           | 0                       |
| 1.80 | 1.12102                         | 0.581159E-01                           | 0                       |
| 2.00 | 1.14981                         | 0.797719E-01                           | 0                       |
| 2.20 | 1.18126                         | 0.106373                               | 0                       |
| 2.40 | 1.22572                         | 0.161607                               | 0                       |
| 2.60 | 1.27003                         | 0.236557                               | 0                       |
| 2.80 | 1.30381                         | 0.259973                               | 2                       |
| 3.00 | 1.43084                         | 0.686589                               | 2                       |

TABLE III  
MLE FOR U-SHAPED FAILURE RATE WHEN TRUE DISTRIBUTION  
IS THE UNIT EXPONENTIAL DISTRIBUTION

Number of Simulations = 150

Sample Size =  $n = 25$

| X    | MEAN VALUE<br>OF $\hat{r}_n(x)$ | MEAN SQUARE<br>ERROR OF $\hat{r}_n(x)$ | NUMBER OF<br>INFINITIES |
|------|---------------------------------|----------------------------------------|-------------------------|
| 0.05 | 1.29297                         | 0.549797                               | 0                       |
| 0.10 | 1.08526                         | 0.381039                               | 0                       |
| 0.15 | 0.959894                        | 0.314845                               | 0                       |
| 0.20 | 0.937388                        | 0.302800                               | 0                       |
| 0.25 | 0.890439                        | 0.273362                               | 0                       |
| 0.30 | 0.877699                        | 0.253101                               | 0                       |
| 0.35 | 0.864808                        | 0.232931                               | 0                       |
| 0.40 | 0.835557                        | 0.228444                               | 0                       |
| 0.45 | 0.828066                        | 0.220882                               | 0                       |
| 0.50 | 0.788042                        | 0.249237                               | 0                       |
| 0.60 | 0.758739                        | 0.266121                               | 0                       |
| 0.70 | 0.782102                        | 0.257277                               | 0                       |
| 0.80 | 0.836567                        | 0.215058                               | 0                       |
| 0.90 | 0.820652                        | 0.253918                               | 0                       |
| 1.00 | 0.824013                        | 0.265950                               | 0                       |
| 1.20 | 0.922699                        | 0.324050                               | 0                       |
| 1.40 | 1.04247                         | 0.439205                               | 0                       |
| 1.60 | 1.17449                         | 0.861991                               | 0                       |
| 1.80 | 1.63692                         | 17.6990                                | 2                       |
| 2.00 | 1.83803                         | 18.7647                                | 3                       |
| 2.20 | 4.56656                         | 502.373                                | 10                      |
| 2.40 | 5.06498                         | 514.575                                | 17                      |
| 2.60 | 5.74322                         | 549.582                                | 23                      |
| 2.80 | 8.96341                         | 1544.13                                | 31                      |
| 3.00 | 9.17613                         | 1545.75                                | 50                      |
| 3.20 | 9.51944                         | 1550.91                                | 55                      |
| 3.40 | 9.77481                         | 1556.98                                | 66                      |
| 3.60 | 9.80845                         | 1557.00                                | 76                      |
| 3.80 | 9.80993                         | 1557.00                                | 87                      |
| 4.00 | 9.81614                         | 1557.01                                | 96                      |

TABLE IV

MLE FOR U-SHAPED FAILURE RATE WHEN TRUE  
DISTRIBUTION IS THE UNIT EXPONENTIAL DISTRIBUTION

Number of Simulations = 100

Sample Size =  $n = 50$

| $x$  | MEAN VALUE<br>OF $\hat{r}_n(x)$ | MEAN SQUARE<br>ERROR OF $\hat{r}_n(x)$ | NUMBER OF<br>INFINITIES |
|------|---------------------------------|----------------------------------------|-------------------------|
| 0.05 | 1.16594                         | 0.265775                               | 0                       |
| 0.10 | 1.04175                         | 0.164489                               | 0                       |
| 0.15 | 1.00883                         | 0.148645                               | 0                       |
| 0.20 | 0.994515                        | 0.108994                               | 0                       |
| 0.25 | 0.961543                        | 0.128702                               | 0                       |
| 0.30 | 0.956456                        | 0.124807                               | 0                       |
| 0.35 | 0.888189                        | 0.164545                               | 0                       |
| 0.40 | 0.860720                        | 0.174514                               | 0                       |
| 0.45 | 0.852992                        | 0.187309                               | 0                       |
| 0.50 | 0.845861                        | 0.205879                               | 0                       |
| 0.60 | 0.849351                        | 0.219402                               | 0                       |
| 0.70 | 0.940997                        | 0.122661                               | 0                       |
| 0.80 | 0.943531                        | 0.127397                               | 0                       |
| 0.90 | 0.915578                        | 0.159765                               | 0                       |
| 1.00 | 0.910690                        | 0.162759                               | 0                       |
| 1.20 | 0.936932                        | 0.143381                               | 0                       |
| 1.40 | 0.952359                        | 0.165201                               | 0                       |
| 1.60 | 1.00287                         | 0.168174                               | 0                       |
| 1.80 | 1.06294                         | 0.212498                               | 0                       |
| 2.00 | 1.10732                         | 0.328394                               | 0                       |
| 2.20 | 1.13930                         | 0.389655                               | 0                       |
| 2.40 | 1.46696                         | 5.32901                                | 2                       |
| 2.60 | 1.50744                         | 5.38283                                | 2                       |
| 2.80 | 1.70013                         | 6.04579                                | 2                       |
| 3.00 | 1.97122                         | 8.75948                                | 6                       |
| 3.20 | 2.65745                         | 23.2704                                | 14                      |
| 3.40 | 2.77963                         | 23.8924                                | 21                      |
| 3.60 | 2.86158                         | 24.2406                                | 27                      |
| 3.80 | 2.91245                         | 24.3000                                | 29                      |
| 4.00 | 2.98504                         | 24.4582                                | 34                      |

TABLE V

$$\alpha = 1$$

MLE FOR INCREASING FAILURE RATE WHEN TRUE  
DISTRIBUTION IS THE WEIBULL DISTRIBUTION

$$F(t) = 1 - e^{-x^\alpha}$$

Number of Simulations = 500

Sample Size = n = 20

| x   | MEAN<br>BIAS | MEAN SQUARE<br>ERROR OF $\hat{r}_n(x)$ | NUMBER OF<br>INFINITIES |
|-----|--------------|----------------------------------------|-------------------------|
| 0.0 | + 0.00000    | 0.00000                                | 0                       |
| 0.2 | - 0.17781    | 0.11695                                | 0                       |
| 0.4 | + 0.08053    | 0.08095                                | 0                       |
| 0.6 | + 0.00269    | 0.10185                                | 0                       |
| 0.8 | + 0.06714    | 0.13693                                | 0                       |
| 1.0 | + 0.12675    | 0.17096                                | 0                       |
| 1.2 | + 0.18478    | 0.26036                                | 0                       |
| 1.4 | + 0.23401    | 0.32004                                | 1                       |
| 1.6 | + 0.37319    | 0.90744                                | 1                       |
| 1.8 | + 0.48808    | 1.54212                                | 7                       |
| 2.0 | + 0.62947    | 4.67073                                | 23                      |

TABLE VI

$$\alpha = 1$$

GRAPHICAL ESTIMATOR OF FAILURE RATE FUNCTION WHEN TRUE DISTRIBUTION IS  
THE WEIBULL DISTRIBUTION

$$F(t) = 1 - e^{-x^\alpha}$$

Number of Simulations = 500

Sample Size = n = 20

| x   | MEAN<br>BIAS | MEAN SQUARED<br>ERROR OF $r_n(x)$ | NUMBER OF<br>INFINITIES |
|-----|--------------|-----------------------------------|-------------------------|
| 0.0 | + 0.00000    | 0.00000                           | 0                       |
| 0.2 | - 0.22220    | 0.11886                           | 0                       |
| 0.4 | - 0.14343    | 0.08153                           | 0                       |
| 0.6 | - 0.06969    | 0.09808                           | 0                       |
| 0.8 | - 0.01284    | 0.12645                           | 0                       |
| 1.0 | + 0.04850    | 0.16330                           | 0                       |
| 1.2 | + 0.11063    | 0.25271                           | 0                       |
| 1.4 | + 0.17110    | 0.32801                           | 1                       |
| 1.6 | + 0.32720    | 0.96436                           | 1                       |
| 1.8 | + 0.45788    | 1.63002                           | 7                       |
| 2.0 | + 0.60703    | 4.76836                           | 23                      |

TABLE VII

$$\alpha = 1.5$$

MLE FOR INCREASING FAILURE RATE WHEN TRUE  
DISTRIBUTION IS THE WEIBULL DISTRIBUTION

$$F(t) = 1 - e^{-t^\alpha}$$

Number of Simulations

Sample Size =  $n = 20$

| x   | MEAN<br>BIAS | MEAN SQUARE<br>ERROR OF $r_n(x)$ | NUMBER OF<br>INFINITIES |
|-----|--------------|----------------------------------|-------------------------|
| 0.0 | + 0.00000    | 0.00000                          | 0                       |
| 0.2 | + 0.04326    | 0.05232                          | 0                       |
| 0.4 | + 0.00724    | 0.09421                          | 0                       |
| 0.6 | + 0.04241    | 0.14567                          | 0                       |
| 0.8 | + 0.11443    | 0.25389                          | 0                       |
| 1.0 | + 0.21644    | 0.43585                          | 0                       |
| 1.2 | + 0.35952    | 1.06299                          | 0                       |
| 1.4 | + 0.68761    | 3.39264                          | 2                       |
| 1.6 | + 1.15985    | 23.00837                         | 24                      |
| 1.8 | + 1.02670    | 9.71217                          | 86                      |
| 2.0 | + 1.24338    | 103.99067                        | 159                     |

TABLE VIII

$$\alpha = 1.5$$

GRAPHICAL ESTIMATOR OF FAILURE RATE FUNCTION WHEN TRUE DISTRIBUTION IS  
THE WEIBULL DISTRIBUTION

$$F(t) = 1 - e^{-t^\alpha}$$

Number of Simulations = 500

Sample Size =  $n = 20$

| $x$ | MEAN<br>BIAS | MEAN SQUARE<br>ERROR OF $r_n(x)$ | NUMBER OF<br>INFINITIES |
|-----|--------------|----------------------------------|-------------------------|
| 0.0 | + 0.00000    | 0.00000                          | 0                       |
| 0.2 | + 0.04180    | 0.04764                          | 0                       |
| 0.4 | - 0.01285    | 0.08097                          | 0                       |
| 0.6 | + 0.00005    | 0.12260                          | 0                       |
| 0.8 | + 0.05764    | 0.22584                          | 0                       |
| 1.0 | + 0.15529    | 0.41857                          | 0                       |
| 1.2 | + 0.31463    | 1.08971                          | 0                       |
| 1.4 | + 0.68591    | 3.67826                          | 2                       |
| 1.6 | + 1.18606    | 23.28767                         | 24                      |
| 1.8 | + 1.04786    | 9.86327                          | 86                      |
| 2.0 | + 1.26325    | 104.11156                        | 159                     |

TABLE IX

$$\alpha = 2.0$$

MLE FOR INCREASING FAILURE RATE WHEN TRUE  
DISTRIBUTION IS THE WEIBULL DISTRIBUTION

$$F(t) = 1 - e^{-t^\alpha}$$

Number of Simulations = 500

Sample Size = n = 20

| x   | MEAN<br>BIAS | MEAN SQUARE<br>ERROR OF $r_n(x)$ | NUMBER OF<br>INFINITIES |
|-----|--------------|----------------------------------|-------------------------|
| 0.0 | + 0.00000    | 0.00000                          | 0                       |
| 0.2 | + 0.08906    | 0.03585                          | 0                       |
| 0.4 | + 0.07154    | 0.09127                          | 0                       |
| 0.6 | + 0.07192    | 0.18479                          | 0                       |
| 0.8 | + 0.15319    | 0.36775                          | 0                       |
| 1.0 | + 0.32118    | 0.84297                          | 0                       |
| 1.2 | + 0.57333    | 2.19050                          | 1                       |
| 1.4 | + 1.45527    | 26.21926                         | 17                      |
| 1.6 | + 1.16589    | 9.95119                          | 113                     |
| 1.8 | + 1.52419    | 39.64151                         | 223                     |
| 2.0 | + 0.12816    | 2.52929                          | 342                     |

TABLE X

$$\alpha = 2.0$$

GRAPHICAL ESTIMATOR OF FAILURE RATE FUNCTION WHEN TRUE DISTRIBUTION IS  
THE WEIBULL DISTRIBUTION

$$F(t) = 1 - e^{-x^\alpha}$$

Number of Simulations = 500

Sample Size = n = 20

| x   | MEAN<br>BIAS | MEAN SQUARE<br>ERROR OF $\hat{r}_n(x)$ | NUMBER OF<br>INFINITIES |
|-----|--------------|----------------------------------------|-------------------------|
| 0.0 | + 0.00000    | 0.00000                                | 0                       |
| 0.2 | + 0.09201    | 0.03649                                | 0                       |
| 0.4 | + 0.06967    | 0.08389                                | 0                       |
| 0.6 | + 0.04531    | 0.15483                                | 0                       |
| 0.8 | + 0.10356    | 0.31729                                | 0                       |
| 1.0 | + 0.26524    | 0.81702                                | 0                       |
| 1.2 | + 0.54895    | 2.30813                                | 1                       |
| 1.4 | + 1.57024    | 28.33272                               | 17                      |
| 1.6 | + 1.22317    | 10.29591                               | 113                     |
| 1.8 | + 1.58241    | 39.89294                               | 223                     |
| 2.0 | + 0.13982    | 2.62531                                | 342                     |

TABLE XI

$$\alpha = 2.5$$

MLE FOR INCREASING FAILURE RATE WHEN TRUE  
DISTRIBUTION IS THE WEIBULL DISTRIBUTION

$$F(t) = 1 - e^{-x^\alpha}$$

Number of Simulations = 500

Sample Size = n = 20

| x   | MEAN<br>BIAS | MEAN SQUARE<br>ERROR OF $r_n(x)$ | NUMBER OF<br>INFINITIES |
|-----|--------------|----------------------------------|-------------------------|
| 0.0 | + 0.00000    | 0.00000                          | 0                       |
| 0.2 | + 0.06873    | 0.02436                          | 0                       |
| 0.4 | + 0.11064    | 0.07822                          | 0                       |
| 0.6 | + 0.09774    | 0.18706                          | 0                       |
| 0.8 | + 0.19814    | 0.49999                          | 0                       |
| 1.0 | + 0.43034    | 1.39276                          | 0                       |
| 1.2 | + 1.08370    | 8.84637                          | 1                       |
| 1.4 | + 2.47167    | 58.14368                         | 68                      |
| 1.6 | + 2.12447    | 78.06147                         | 222                     |
| 1.8 | - 0.17729    | 7.53517                          | 385                     |
| 2.0 | - 1.62648    | 4.42413                          | 467                     |

TABLE XII

$$\alpha = 2.5$$

GRAPHICAL ESTIMATOR OF FAILURE RATE FUNCTION WHEN TRUE DISTRIBUTION IS  
THE WEIBULL DISTRIBUTION

$$F(t) = 1 - e^{-x^\alpha}$$

Number of Simulations = 500

Sample Size =  $n = 20$

| x   | MEAN<br>BIAS | MEAN SQUARE<br>ERROR OF $r_n(x)$ | NUMBER OF<br>INFINITIES |
|-----|--------------|----------------------------------|-------------------------|
| 0.0 | + 0.00000    | 0.00000                          | 0                       |
| 0.2 | + 0.07079    | 0.02546                          | 0                       |
| 0.4 | + 0.11355    | 0.07525                          | 0                       |
| 0.6 | + 0.08569    | 0.16751                          | 0                       |
| 0.8 | + 0.15261    | 0.42956                          | 0                       |
| 1.0 | + 0.37411    | 1.35641                          | 0                       |
| 1.2 | + 1.10588    | 9.49925                          | 1                       |
| 1.4 | + 2.62338    | 69.11603                         | 68                      |
| 1.6 | + 2.22374    | 78.52692                         | 222                     |
| 1.8 | - 0.16100    | 7.47486                          | 385                     |
| 2.0 | - 1.60758    | 4.27771                          | 467                     |

TABLE XIII

$$\alpha = 3.0$$

MLE FOR INCREASING FAILURE RATE WHEN TRUE  
DISTRIBUTION IS THE WEIBULL DISTRIBUTION

$$F(t) = 1 - e^{-x^\alpha}$$

Number of Simulations = 500

Sample Size =  $n = 20$

| x   | MEAN<br>BIAS | MEAN SQUARE<br>ERROR OF $r_n(x)$ | NUMBER OF<br>INFINITIES |
|-----|--------------|----------------------------------|-------------------------|
| 0.0 | + 0.00000    | 0.00000                          | 0                       |
| 0.2 | + 0.03376    | 0.01045                          | 0                       |
| 0.4 | + 0.11782    | 0.05387                          | 0                       |
| 0.6 | + 0.13871    | 0.21125                          | 0                       |
| 0.8 | + 0.20063    | 0.55209                          | 0                       |
| 1.0 | + 0.53841    | 2.08249                          | 0                       |
| 1.2 | + 1.85227    | 21.50619                         | 5                       |
| 1.4 | + 2.00208    | 25.27304                         | 140                     |
| 1.6 | + 0.89666    | 62.18391                         | 354                     |
| 1.8 | + 0.20132    | 100.25058                        | 471                     |
| 2.0 | - 2.28445    | 6.30387                          | 498                     |

TABLE XIV

$$\alpha = 3.0$$

GRAPHICAL ESTIMATOR OF FAILURE RATE FUNCTION WHEN TRUE DISTRIBUTION IS  
THE WEIBULL DISTRIBUTION

$$F(t) = 1 - e^{-x^\alpha}$$

Number of Simulations = 500

Sample Size =  $n = 20$

| x   | MEAN<br>BIAS | MEAN SQUARE<br>ERROR OF $r_n(x)$ | NUMBER OF<br>INFINITIES |
|-----|--------------|----------------------------------|-------------------------|
| 0.0 | + 0.00000    | 0.00000                          | 0                       |
| 0.2 | + 0.03467    | 0.01093                          | 0                       |
| 0.4 | + 0.12249    | 0.05486                          | 0                       |
| 0.6 | + 0.13366    | 0.19345                          | 0                       |
| 0.8 | + 0.16158    | 0.48076                          | 0                       |
| 1.0 | + 0.48233    | 2.03612                          | 0                       |
| 1.2 | + 1.95784    | 23.51841                         | 5                       |
| 1.4 | + 2.17050    | 27.00391                         | 140                     |
| 1.6 | + 0.93123    | 62.01564                         | 354                     |
| 1.8 | + 0.19225    | 100.29024                        | 471                     |
| 2.0 | - 2.28445    | 6.30387                          | 498                     |

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|                            | ROLE   | WT | ROLE   | WT | ROLE   | WT |
| Estimation                 |        |    |        |    |        |    |
| Monotone Failure Rate      |        |    |        |    |        |    |
| Monotonic Regression       |        |    |        |    |        |    |
| Total Time on Test Weights |        |    |        |    |        |    |