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**ELECTROMAGNETIC RADIATION
AND HEAT CONDUCTION FIELDS
NEAR COAXIAL CONICAL STRUCTURES**

By

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August 1969

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ELECTROMAGNETIC RADIATION AND HEAT CONDUCTION FIELDS NEAR COAXIAL CONICAL STRUCTURES

I. INTRODUCTION

The investigation of the field induced by a "vibrating" point or line source in the presence of conical structures has been carried on for many years. Most of the investigations were concerned with semi-infinite cones and various approximations or very special cases were taken to discuss a finite cone. The purpose of this thesis is to investigate the finite conical problem with a double cone structure involved. The single finite cone and semi-infinite cones are obtained as special cases.

One of the first investigations of the field induced by a vibrating point source in the presence of a conical structure was performed by H. S. Carslaw (1914). He divided the field into two parts, a primary and a secondary, where one part satisfied the free-space conditions and the second part satisfied the boundary conditions. The primary field obtained involved evaluating contour integrals using residue theory. The integrands contained conical functions. The secondary field was assumed to be of the same form as the primary only containing an unknown function to be determined.

H. Buchholz (1940) examined the semi-infinite ideally conducting cone using the same procedures as Carslaw used. The field was

excited in this investigation by an electric source ring placed axially symmetric to the cone's axis.

The problem did not have much appeal until the advent of the space age. At that time the problem of the finite cone became of importance.

When a re-entry vehicle re-enters the atmosphere at hypersonic speeds a double conical structure of finite length is formed. The interior cone is the re-entry vehicle itself. The outer cone is a plasma sheath formed from a surface induced electromagnetic wave.

An electric dipole antennae radiates most of its power along this sheath. Thus transmission of electromagnetic waves through this conical sheath of plasma is highly attenuated. This attenuation phenomenon is exhibited by a marked reduction in the strength of the radio signals from the re-entry vehicle to the earth's surface.

There have been many investigations of the field in the vicinity of semi-infinite cones. For some of the recent investigations one should look at L. B. Felsen (1957, 1959a, 1959b) or S. Adachi (1960). However, it is clear that the electric dipole results would be greatly modified if a semi-infinite conical structure were replaced by a finite conical structure.

J. B. Keller (1960) investigated the backscattering of an electromagnetic wave incident upon a perfectly-conducting finite cone. The theory was based upon geometrical theory of diffraction. He thus

assumes the results are " . . . probably valid for wavelengths as large as cone dimensions or smaller. "

From 1964 to 1968 several contributions were made that are all directly related to this problem.

D. C. Pridemore-Brown (1964) investigated the radiation pattern of a small-loop antenna situated on the axis of a thin conical sheet of plasma. He used the same basic concept used by H. S. Carslaw (1914) and H. Buchholz (1940). He took as his secondary field that induced by a surface current on the plasma sheath. The eddy current distribution had to be obtained as the solution of an integral equation. By assuming the cone was highly reflecting, perturbation techniques were developed and applied to the special case of a cone with apex angle 10° and the wavelengths of 2π times the distance from the antenna to the cone apex.

This was not too fruitful so he developed a different perturbation method (D. C. Pridemore-Brown, 1966) and applied it to the integral equation for the eddy current distribution. He developed the equations further using the Kantorovich-Lebedev transform theory to obtain an infinite system of equations involving unknown transforms instead of an integral equation. This process was first employed by A. Leitner and C. P. Wells (1956) for a circular disk. Under the assumption the slant height of the cone is small compared to the wavelength, D. C. Pridemore-Brown obtained a solution to this infinite

system (1968).

A. Baños et al. investigated this problem along the same lines as discussed above (1964). Their solution had essentially the same restrictions.

The purpose of this investigation is to obtain the character of the field of electromagnetic waves in the conical sheath of the re-entering space vehicle in a "closed form." The conical sheath will be represented as the space between two finite coaxial cones with coincident apexes and bases. The base of the outer cone, and thus the inner one as well, is taken to be a spherical cap of radius equal to the slant height of the cones. The base and outer cone have a smooth joining surface. This geometry approximates the actual re-entry flow geometry experienced by the space vehicle at hypersonic speeds. If we assume the conical shells are perfectly conducting, the field may be investigated by placing a ring source of time harmonic character between the walls and located axially symmetric to the conical axis.

As may be seen from the above, prior investigations have been concerned with determining induced eddy currents, secondary fields, etc. The approach taken here is completely different.

Herein a very simple method is utilized. The Green's function is determined by using the well known Green's function representation as a series involving the eigenvalues and eigenfunctions of the problem (A. Sommerfeld, 1964). This approach leads successfully to a

solution for the finite conical structure with no restrictions made upon the wavelengths, body slant height, or apex angles.

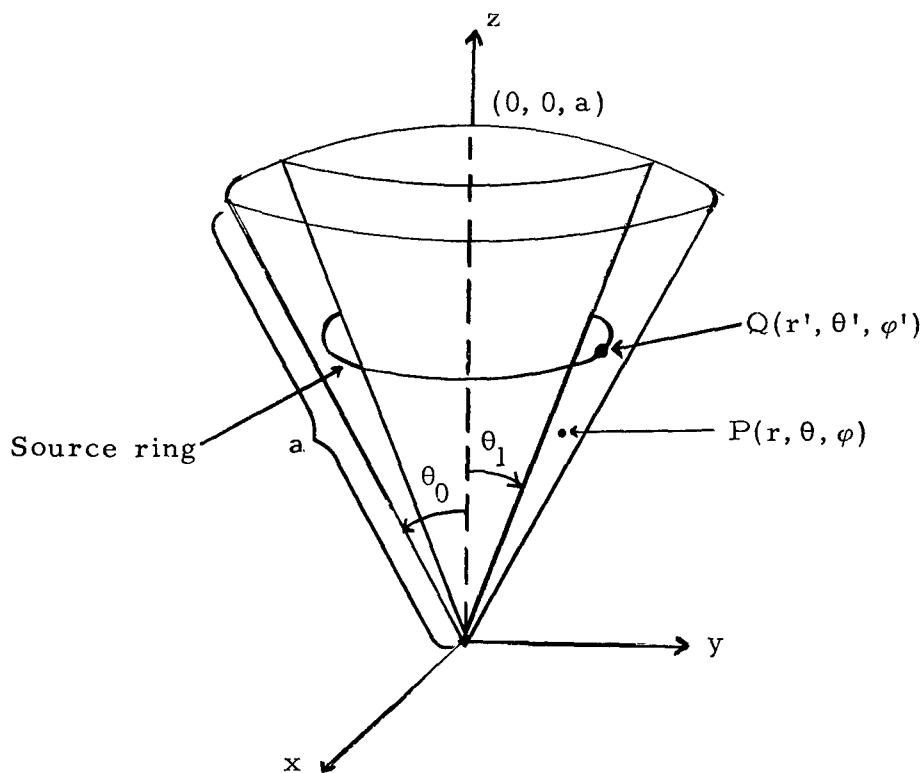
Chapter II is concerned with the derivation of the eigenvalues, eigenfunctions, normalization procedure, and the determination of the vector potential field. The electric and magnetic components of the field are obtained.

Chapter III examines several cases of the solution. Since no prior solution has been obtained, some of the cases must resort to semi-infinite conical forms to compare with previous results. Included are the single finite cone, single semi-infinite cone, semi-infinite double conical wave guide, and a bi-conical antenna.

The solution of the problem for a time harmonic electric source ring also provides the basis for the examination of the heat conduction inside the sheath. Thus it was felt that this result should be presented as Chapter IV and a few degenerate cases mentioned. This solution is obtained by the use of the inverse Laplace Transform.

II. ELECTRIC SOURCE RING FIELD

The geometry of the boundary value problem is determined by approximating the actual plasma sheath about a re-entry vehicle. The Apollo space vehicle is taken in this investigation. Thus the below figure depicts the geometry which must be used.



The base of the two cones is formed by a spherical cap of radius equal to the slant height of both cones. The cap is fitted smoothly to the outer cone.

The field between the cones is excited by a time-harmonic electric source ring of frequency ω . The ring is placed

symmetrically about the axis of the cones.

Let $P = P(r, \theta, \varphi)$ be any point of observation between the two cones and $Q = Q(r', \theta', \varphi')$ be a source point on the source ring. The source element at $Q(r', \theta', \varphi')$ produces a vector potential which is tangent to the source ring. This tangent vector potential \vec{A} has the three components A_r, A_θ, A_φ . But when we integrate about the ring with respect to φ , and the ring is placed symmetrically to the double conical structure's axis, the A_φ component becomes independent of φ and $A_r = A_\theta = 0$.

Maxwell's equations can be reduced for a time-harmonic electrical source to a form that says the vector potential field u must satisfy Helmholtz's equation

$$\Delta u + k^2 u = 0$$

where

$$k = \omega/c$$

$$c = \text{speed of light}$$

Since the vector potential has only the A_φ component, we may write

$$\Delta A_\varphi + k^2 A_\varphi = 0$$

The first form will be used for the analysis and the last form when the field components are desired.

The electric source ring induces a spherical field that has both electric and magnetic components. From Maxwell's equations

$$\begin{aligned}\vec{E} &= \nabla(\nabla \cdot \vec{A}) + k^2 \vec{A}, & \text{electric components} \\ \vec{H} &= \nabla \times \vec{A}, & \text{magnetic components}\end{aligned}$$

Introducing spherical coordinates and noting $\vec{A} = (0, 0, A_\varphi)$ we obtain

$$\begin{aligned}E_r &= E_\theta = 0; & E_\varphi &= k^2 A_\varphi \\ H_r &= \frac{1}{r} \left[\frac{\partial A_\varphi}{\partial \theta} + A_\varphi \cot \theta \right] \\ H_\theta &= -\frac{1}{r} A_\varphi - \frac{\partial A_\varphi}{\partial r} \\ H_\varphi &= 0\end{aligned}\tag{1}$$

The geometry, physics, and above equations all suggest the boundary conditions are expressed as:

$$\begin{aligned}\theta = \theta_0, \theta_1 &: E_r = E_\varphi = 0 \\ r = a &: E_\varphi = E_\theta = 0\end{aligned}\tag{2}$$

These suggest that $A_\varphi = 0$ for $\theta = \theta_0, \theta_1$ and $r = a$. Hence we wish to obtain the first Green's function for this field and geometry due to a point source located at Q .

The first Green's function will be obtained by using the well

known expression of it as a series expansion involving the eigenvalues and normalized eigenfunctions (see, for instance, A. Sommerfeld 1964, p. 183):

$$(3) \quad G(P, Q) = \sum_{(j)} \frac{u_j(P)u_j^*(Q)}{k_j^2 - k_j^2}$$

A remark on the notation should be made. In general this summation is usually over three indices. The $u_j(P)$ are the normalized eigenfunctions corresponding to the eigenvalue k_j evaluated at the point of observation P . The $u_j^*(Q)$ is the complex conjugate of $u_j(P)$, only evaluated at the source element Q .

Since we are in a spherical coordinate system, the Helmholtz equation can be examined using separation of variables. Letting $u(r, \theta, \varphi) = f_1(r)f_2(\theta)f_3(\varphi)$ and substituting into Helmholtz's equation yields the three ordinary differential equations

$$\frac{1}{2} \frac{d^2}{dr^2} (rf_1) + (k^2 - \frac{\nu(\nu+1)}{r^2})f_1 = 0$$

$$\frac{d^2 f_2}{d\theta^2} + \cot \theta \frac{df_2}{d\theta} + (\nu(\nu+1) - \frac{\mu^2}{\sin^2 \theta})f_2 = 0$$

$$\frac{d^2 f_3}{d\varphi^2} + \mu^2 f_3 = 0$$

where ν and μ are separation parameters to be determined later.

We have taken the Q coordinates as $(r', \varphi', 0)$.

The differential equation for $f_1(r)$ is recognizable as a form of Bessel's differential equation. Its solution is written as

$$f_1(r) = \frac{1}{r^{\frac{1}{2}}} C_{\nu+\frac{1}{2}}(kr)$$

where $C_{\nu+\frac{1}{2}}(kr)$ is any cylindrical function of order $\nu+\frac{1}{2}$. Since we request u be finite at $r = 0$, we select the solution to be

$$f_1(r) = r^{-\frac{1}{2}} J_{\nu+\frac{1}{2}}(kr)$$

where $J_{\nu+\frac{1}{2}}(kr)$ is the ordinary Bessel function of order $\nu+\frac{1}{2}$ and argument kr .

The differential equation for $f_2(\theta)$ may be recognized as Legendre's differential equation. It has the solution of any spherical harmonic function, or any linear combination of them. Solutions for θ real which are single-valued and regular are limited to the associated Legendre functions of the first and second kinds of degree ν and order μ , $P_{\nu}^{\mu}(\cos \theta)$ and $Q_{\nu}^{\mu}(\cos \theta)$ respectively. Thus a linear combination of these two functions would be a solution for $f_2(\theta)$.

However, a useful substitution is made. The $Q_{\nu}^{\mu}(\cos \theta)$ can be written in the form (Magnus and Oberhettinger, 1949, p. 63)

$$-\frac{2}{\pi} Q_{\nu}^{\mu}(\cos \theta) = \csc[\pi(\nu+\mu)] P_{\nu}^{\mu}(-\cos \theta) - \cot[\pi(\nu+\mu)] P_{\nu}^{\mu}(\cos \theta)$$

Thus

$$f_2(\theta) = a_1 P_{\nu}^{\mu}(\cos \theta) + a_2 P_{\nu}^{\mu}(-\cos \theta)$$

The equation for $f_3(\varphi)$ is solved immediately to be

$$f_3(\varphi) = e^{\pm i\mu\varphi}$$

We desire the vector potential field to be single-valued in φ . Furthermore, it must be even in φ . Thus we take $\mu = m = 0, 1, 2, \dots$ and

$$f_3(\varphi) = \cos(m\varphi)$$

Combining the respective equations for $f_1(r)$, $f_2(\theta)$, $f_3(\varphi)$, we obtain

$$(4) \quad u = \frac{\cos(m\varphi)}{r^{\frac{1}{2}}} J_{\nu}(kr) [a_1 P_{\nu-\frac{1}{2}}^m(\cos \theta) + a_2 P_{\nu-\frac{1}{2}}^m(-\cos \theta)]$$

The substitution process has involved the changing of the $(\nu+\frac{1}{2})$ to ν on the Bessel function. This, of course, modifies the degree of the associated Legendre functions.

We are now ready to determine the eigenvalues and eigenfunctions. This is accomplished by requiring (4) to satisfy the boundary conditions posed by (2).

Consider first the boundary condition of $u = 0$ at $r = 0$.

Since only $J_\nu(kr)$ depends upon r , $u = 0$ for all values of ν such that $J_\nu(ka) = 0$. If we denote the n th positive root of $J_\nu(ka) = 0$ by $\tau_{n,\nu}$, we have the eigenvalues

$$k_{n,\nu} = \frac{\tau_{n,\nu}}{a}$$

We need only take the positive roots as $J_{-\nu}(\zeta)$ has the same roots for real ζ .

The ν -values are obtained from the boundary conditions on θ .

We obtain the simultaneous equations of

$$a_1 P_{\nu-\frac{1}{2}}^m(\cos \theta_0) + a_2 P_{\nu-\frac{1}{2}}^m(-\cos \theta_0) = 0$$

$$a_1 P_{\nu-\frac{1}{2}}^m(\cos \theta_1) + a_2 P_{\nu-\frac{1}{2}}^m(-\cos \theta_1) = 0$$

These equations have a non-trivial solution for a_1 and a_2 only if the below determinantal equation is satisfied.

$$(5) \quad M_{\nu-\frac{1}{2}}^m(\theta_0, \theta_1) = \begin{vmatrix} P_{\nu-\frac{1}{2}}^m(\cos \theta_0) & P_{\nu-\frac{1}{2}}^m(-\cos \theta_0) \\ P_{\nu-\frac{1}{2}}^m(\cos \theta_1) & P_{\nu-\frac{1}{2}}^m(-\cos \theta_1) \end{vmatrix} = 0$$

Since m , θ_0 , and θ_1 are fixed quantities, this is a transcendental equation in ν .

All the roots are real as they represent eigenvalues of Helmholtz's equation. Furthermore since $P_{-\nu-\frac{1}{2}}^m(\zeta) = P_{\nu-\frac{1}{2}}^m(\zeta)$ we need only examine the positive roots.

As the first positive root is $\nu > 1/2$, $r^{-\frac{1}{2}}J_{\nu}(dr)$ remains finite for (kr) tending to zero. This substantiates the use of $r^{-\frac{1}{2}}J_{\nu}(kr)$ as the solution to the differential equation in $f_1(r)$.

Let us denote the l th positive root of $M_{\nu-\frac{1}{2}}^m(\theta_0, \theta_1) = 0$ by $a_{m, l}$.

Then the eigenvalues for the boundary value problem are found to be

$$(6) \quad k_{n, m, l} = \frac{\tau_{n, \nu}}{a} \quad \text{with} \quad \nu = a_{m, l}$$

The values of n, m and l are defined by

$$m = 0, 1, 2, 3, \dots \quad \text{and} \quad n, l = 1, 2, 3, \dots$$

With these eigenvalues, the non-normalized eigenfunctions are written as

$$(7) \quad u_{n, m, l} = r^{-\frac{1}{2}} \cos(m\varphi) J_{\nu} \left(\frac{r}{a} \tau_{n, \nu} \right) L_{\nu}^m(\theta)$$

where

$$(8) \quad L_{\nu}^m(\theta) = a_1 P_{\nu-\frac{1}{2}}^m(\cos \theta) + a_2 P_{\nu-\frac{1}{2}}^m(-\cos \theta)$$

and ν is determined by (5) and (6).

In order to normalize the eigenfunctions we define a numerical quantity N by the equation

$$N^2 = \int_{\varphi=0}^{2\pi} \int_{r=0}^a \int_{\theta=\theta_1}^{\theta=\theta_0} u_{n,m,l}^2 r^2 \sin \theta d\theta dr d\varphi$$

Then we define the normalized eigenfunctions by

$$v_{n,m,l} = \frac{u_{n,m,l}}{N}$$

We consider now the determination of the normalization factor N . The triple integral representation of N^2 for (7) can be written as the product of three integrals, namely:

$$N^2 = \int_0^{2\pi} \cos^2(m\varphi) d\varphi \int_{\theta_1}^{\theta_0} [L_\nu^m(\theta)]^2 \sin \theta d\theta \int_0^a [J_\nu(\frac{r}{a} \tau_{n,\nu})]^2 r dr$$

But

$$\int_0^{2\pi} \cos^2(m\varphi) d\varphi = \frac{2\pi}{\epsilon_m}$$

where ϵ_m is Neumann's number defined by

$$\epsilon_m = \begin{cases} 1, & m = 0 \\ 2, & m = 1, 2, 3, \dots \end{cases}$$

The last integral-factor is evaluated using from Erdélyi (1953, Vol. 2, p. 90)

$$\int \zeta J_{\nu}^2(a\zeta) d\zeta = \frac{\zeta^2}{2} [J_{\nu}^2(a\zeta) - J_{\nu+1}(a\zeta)J_{\nu-1}(a\zeta)]$$

to obtain

$$\begin{aligned} \int_0^a r J_{\nu}^2(rk_{n,\nu}) dr &= \frac{a^2}{2} [J_{\nu}^2(\tau_{n,\nu}) - J_{\nu+1}(\tau_{n,\nu})J_{\nu-1}(\tau_{n,\nu})] \\ &= \frac{-a^2}{2} J_{\nu+1}(\tau_{n,\nu})J_{\nu-1}(\tau_{n,\nu}) \end{aligned}$$

From Erdélyi (1953, Vol. 2, p. 12) this can be reduced by the equation

$$\zeta J_{\nu-1}(\zeta) + \zeta J_{\nu+1}(\zeta) = 2\nu J_{\nu}(\zeta)$$

to yield the final result of

$$\int_0^a r J_{\nu}^2(rk_{n,\nu}) dr = \frac{a^2}{2} J_{\nu+1}^2(\tau_{n,\nu})$$

The middle integral-factor may be evaluated also. The function $L_{\nu}^m(\theta)$ is a solution to Legendre's differential equation. Thus using the formula derived in Appendix A.

$$\begin{aligned}
& \int_{\theta_1}^{\theta_0} [L_\nu^m(\theta)]^2 \sin \theta d\theta \\
&= \frac{-1}{2\nu} \left[\left(\frac{1}{2} - \nu \right) \{ L_\nu^m(\theta_0) L_{\nu-1}^{m-1}(\theta_0) \sin \theta_0 - L_\nu^m(\theta_1) L_{\nu-1}^{m-1}(\theta_1) \sin \theta_1 \} \right. \\
(9) \quad & \left. \left(\nu - \frac{1}{2} + m \right) \{ L_{\nu-1}^m(\theta_0) \frac{\partial}{\partial \nu} (L_\nu^m(\theta_0)) - L_\nu^m(\theta_0) \frac{\partial}{\partial \nu} (L_{\nu-1}^m(\theta_0)) \right. \right. \\
& \quad \left. \left. - L_{\nu-1}^m(\theta_1) \frac{\partial}{\partial \nu} (L_\nu^m(\theta_1)) + L_\nu^m(\theta_1) \frac{\partial}{\partial \nu} (L_{\nu-1}^m(\theta_1)) \right\} \right] \\
&= D(m, \nu, \theta_0, \theta_1)
\end{aligned}$$

where $\frac{\partial}{\partial \nu} (L_\nu^m(\theta_0))$ is interpreted as $\left[\frac{\partial}{\partial \nu} (L_\nu^m(\theta_0)) \right]_{\nu=\alpha_{m,\ell}}$

For simplicity, the notation of $D(m, \nu, \theta_0, \theta_1)$ will be used for most of the below discussion.

Substituting these integral factors,

$$N^2 = \frac{\pi a^2}{\epsilon_m} J_{\nu+1}^2(\tau_{n,\nu}) D(m, \nu, \theta_0, \theta_1)$$

with $\nu = \alpha_{m,\ell}$. Therefore the normalized eigenfunctions are obtained as

$$v_{n,m,\ell} = \frac{\epsilon_m^{\frac{1}{2}} \cos(m\phi) J_\nu \left(\frac{r}{a} \tau_{n,\nu} \right) L_\nu^m(\theta)}{\left[r \pi D(m, \nu, \theta_0, \theta_1) \right]^{\frac{1}{2}} a J_{\nu+1}(\tau_{n,\nu})}$$

with $\nu = a_{m, \ell} = \ell$ th positive root of the equation

$$M_{\nu-\frac{1}{2}}^m(\theta_0, \theta_1) = P_{\nu-\frac{1}{2}}^m(\cos \theta_0)P_{\nu-\frac{1}{2}}^m(-\cos \theta_1) - P_{\nu-\frac{1}{2}}^m(\cos \theta_1)P_{\nu-\frac{1}{2}}^m(-\cos \theta_0) = 0$$

Substituting into (3) and now making the coordinate of Q to be any triple (r', θ', φ') we get the first Green's function for a finite double cone due to a point source at Q .

$$G(P, Q) = \frac{1}{(rr')^{\frac{1}{2}} a^2} \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{\epsilon_m \cos[m(\varphi-\varphi')] J_{\nu}(rk_{n,\nu}) L_{\nu}^m(\theta) L_{\nu}^m(\theta') J_{\nu}(r'h_{n,\nu})}{J_{\nu+1}^2(\tau_{n,\nu}) D(m, \nu, \theta_0, \theta_1) [k^2 - (\frac{\tau_{n,\nu}}{a})^2]}$$

In order to describe an axisymmetric source ring we integrate the Green's function over all source points on the ring. Integrating and noting that

$$\int_0^{2\pi} \cos [m(\varphi-\varphi')] \cos (\varphi-\varphi') d\varphi' = \pi \delta_{1m}$$

we obtain

$$A_{\varphi} = \frac{2}{(rr')^{\frac{1}{2}} a^2} \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \frac{J_{\nu}(r'k_{n,\nu}) J_{\nu}(rk_{n,\nu}) L_{\nu}^1(\theta) L_{\nu}^1(\theta')}{J_{\nu+1}^2(\tau_{n,\nu}) D(1, \nu, \theta_0, \theta_1) [k^2 - (\frac{\tau_{n,\nu}}{a})^2]}$$

The series in n is a Fourier-Bessel series and in this case can be summed. From Erdélyi (Vol. 2, p. 104)

$$\sum_{n=1}^{\infty} \frac{J_{\nu}(r'k_{n,\nu})J_{\nu}(rk_{n,\nu})}{J_{\nu+1}^2(\tau_{n,\nu})[k^2 - \tau_{n,\nu}^2]}$$

$$= \frac{\pi}{4J_{\nu}(ka)} \begin{cases} J_{\nu}(kr)[J_{\nu}(ka)Y_{\nu}(kr') - Y_{\nu}(ka)J_{\nu}(kr')]; & r < r' \\ J_{\nu}(kr')[J_{\nu}(ka)Y_{\nu}(kr) - Y_{\nu}(ka)J_{\nu}(kr)]; & r > r' \end{cases}$$

where $Y_{\nu}(\zeta)$ is Neumann's function defined by

$$Y_{\nu}(\zeta) = \csc(\nu\pi)[J_{\nu}(\zeta)\cos(\nu\pi) - J_{-\nu}(\zeta)]$$

It is more convenient to introduce the second Hankel function $H_{\nu}^{(2)}(\zeta)$ than to work with the Neumann function. Using the definition of

$$H_{\nu}^{(2)}(\zeta) = J_{\nu}(\zeta) - iY_{\nu}(\zeta)$$

we obtain for arbitrary parameters ζ and η

$$(10) \quad J_{\nu}(\zeta)Y_{\nu}(\eta) - J_{\nu}(\eta)Y_{\nu}(\zeta) = i[J_{\nu}(\zeta)H_{\nu}^{(2)}(\eta) - J_{\nu}(\eta)H_{\nu}^{(2)}(\zeta)]$$

Substituting we obtain for the vector potential field

$$(11) \quad A_{\varphi} = \frac{i\pi}{(rr')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \frac{L_{\nu}^1(\theta)L_{\nu}^1(\theta')}{D(1,\nu,\theta_0,\theta_1)} {}_1,2F_{\nu}(kr',kr)$$

where

$$L_{\nu}^1(\theta) = a_1 P_{\nu-\frac{1}{2}}^1(\cos \theta) + a_2 P_{\nu-\frac{1}{2}}^1(-\cos \theta)$$

(12)

$${}_1F_{\nu}^1(kr', kr) = \frac{J_{\nu}(kr)}{J_{\nu}(ka)} [J_{\nu}(ka)H_{\nu}^{(2)}(kr') - J_{\nu}(kr')H_{\nu}^{(2)}(ka)]$$

and ${}_2F_{\nu}^1(kr', kr)$ is obtained from ${}_1F_{\nu}^1(kr', kr)$ by interchanging r and r' . The constant $D(1, \nu, \theta_0, \theta_1)$ may be evaluated from (9).

The summation over ℓ is imbedded in the determination of the ℓ th positive root $a_{1, \ell}$ of the equation

$$(13) \quad P_{\nu-\frac{1}{2}}^1(\cos \theta_0)P_{\nu-\frac{1}{2}}^1(-\cos \theta_1) - P_{\nu-\frac{1}{2}}^1(\cos \theta_1)P_{\nu-\frac{1}{2}}^1(-\cos \theta_0) = 0$$

Using the above results, the electric and magnetic components of the electromagnetic field may be obtained from (1). Substituting we obtain for the electric components

$$E_{\varphi} = k^2 A_{\varphi}$$

$$E_r = E_{\theta} = 0$$

and for the magnetic components

$$H_r = \frac{\pi i}{2(r^3 r')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \frac{L_{\nu}^1(\theta') {}_1, 2F_{\nu}^1(kr', kr)}{D(1, \nu, \theta_0, \theta_1)} \left\{ (\nu - \frac{1}{2}) \tan(\frac{\theta}{2}) L_{\nu}^1(\theta) \right.$$

$$\left. + \left[\nu \left(\frac{\cos \theta - 2}{\sin \theta} \right) + \cot(\frac{\theta}{2}) \right] a_2 P_{\nu-\frac{1}{2}}^1(-\cos \theta) \right\}$$

$$H_{\theta} = \frac{-\pi i}{4(r^3 r')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \frac{L_{\nu}^1(\theta)L_{\nu}^1(\theta')}{D(1, \nu, \theta_0, \theta_1)} \{ {}_{1,2}F_{\nu}(kr', kr) + 2 {}_{1,2}H_{\nu}(kr', kr) \}$$

with

$${}_{1,2}H_{\nu}(kr', kr) = r \frac{\partial}{\partial r} [{}_{1,2}F_{\nu}(kr', kr)]$$

and finally

$$H_{\varphi} = 0$$

The field components are all real. This can be seen from (10) and (12).

In what follows, the vector potential A_{φ} will be used. If the magnetic and electric field components are desired, the immediately above equations can be used directly.

III. SPECIAL CASES

The vector potential field representation obtained in the previous chapter is new. Thus a direct comparison with previous solutions can not be made directly. However, by taking some special situations, a few cases have been solved in the past that allow a comparison to be made.

The majority of cases that exist for comparison are single semi-infinite conical structures. Thus direct comparisons will be made with those few cases available.

The cases of finite single cone and a semi-infinite double conical structure are presented not for comparison purposes but for completeness. Up to this time these solutions were not known.

Before considering each of the special cases, we must introduce a different form of the solution. This is due to the behavior of the Bessel and Neumann functions for large arguments.

In order to investigate the cases of the semi-infinite conical structures, the radius "a" must be taken to infinity. We must transform our equations to a monotonically behaving form.

This is accomplished by transforming the solution of Helmholtz's equation to the solution of the modified Helmholtz equation

$$\Delta u - \gamma^2 u = 0$$

This is obviously accomplished by the transformation

$$(14) \quad \gamma = ik \quad \text{or} \quad k = -i\gamma$$

Examination of (11) and (12) show that substitution of (14) affects the ${}_{1,2}F_\nu(kr', kr)$ term only. From Magnus and Oberhettinger (1949, p. 17 and 19)

$$(15) \quad J_\nu(-i\gamma\zeta) = e^{i\pi\nu} J_\nu(i\gamma\zeta) = e^{i\nu 3\pi/2} I_\nu(\gamma\zeta)$$

$$H_\nu^{(2)}(-i\gamma\zeta) = -e^{i\nu\pi} H_\nu^{(1)}(i\gamma\zeta) = \frac{2i}{\pi} e^{i\nu\pi/2} K_\nu(\gamma\zeta)$$

Substitution into ${}_{1,2}F_\nu(kr', kr)$ yields

$$(16) \quad {}_1F_\nu(-i\gamma r', -i\gamma r) = \frac{2ie^{i2\pi\nu}}{\pi} \frac{I_\nu(\gamma r)}{I_\nu(\gamma a)} \{I_\nu(\gamma a)K_\nu(\gamma r') - I_\nu(\gamma r')K_\nu(\gamma a)\}$$

$$= {}_1R_\nu(\gamma r', \gamma r), \quad r < r'$$

we obtain ${}_2R_\nu(\gamma r', \gamma r)$ from ${}_1R_\nu(\gamma r', \gamma r)$ by interchanging r and r' .

Thus we have the vector potential field with an imaginary wave number γ

$$(17) \quad \bar{A}_\varphi = \frac{i\pi}{2(rr')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \frac{L_\nu^1(\theta)L_\nu^1(\theta')}{D(1, \nu, \theta_0, \theta_1)} {}_{1,2}R_\nu(\gamma r', \gamma r)$$

Now from G. N. Watson (1958, p. 202, 203) for $|\zeta| \gg 1$,

$$K_\nu(\zeta) \approx \left(\frac{\pi}{2\zeta}\right)^{\frac{1}{2}} e^{-\zeta} \sum_{m=0}^{\infty} \frac{(\nu, m)}{(2\zeta)^m}, \quad \text{for } |\arg z| < 3\pi/2$$

and

$$I_\nu(\zeta) \approx \frac{e^\zeta}{(2\zeta\pi)^{\frac{1}{2}}} \sum_{m=0}^{\infty} \frac{(-)^m (\nu, m)}{(2\zeta)^m} + \frac{e^{-\zeta + (\nu + \frac{1}{2})\pi i}}{(2\zeta\pi)^{\frac{1}{2}}} \sum_{m=0}^{\infty} \frac{(\nu, m)}{(2\zeta)^m}$$

for

$$|\arg \zeta| < \pi/2 \quad \text{and} \quad (\nu, m) = \frac{\Gamma(\nu + m + \frac{1}{2})}{\Gamma(\nu - m + \frac{1}{2})}.$$

Therefore, for a given γ ,

$$(18) \quad \lim_{a \rightarrow \infty} I_\nu(\gamma a) = \infty$$

$$\lim_{a \rightarrow \infty} K_\nu(\gamma a) = 0$$

Now letting "a" tend toward infinity in (17) and utilizing (18) we obtain the result of

$$[{}_1R_\nu(\gamma r', \gamma r)]_\infty = \frac{2ie^{i2\pi\nu}}{\pi} I_\nu(\gamma r) K_\nu(\gamma r'), \quad r < r'$$

and

$$[{}_2R_\nu(\gamma r', \gamma r)]_\infty = \frac{2ie^{i2\pi\nu}}{\pi} I_\nu(\gamma r') K_\nu(\gamma r), \quad r > r'$$

Utilizing (15) again we obtain

$$(19) \quad [{}_1F_\nu(kr', kr)]_\infty = J_\nu(kr)H_\nu^{(2)}(kr'), \quad r < r'$$

$$[{}_2F_\nu(kr', kr)]_\infty = J_\nu(kr')H_\nu^{(2)}(kr), \quad r > r'$$

We are now ready to investigate the special cases.

Case 1

The first case we will investigate does not have a known existing solution with which we can compare. However by taking a special case of this case a comparison can be made. The two situations are treated separately as they are both interesting and useful results in their own right.

We let $\theta_1 = 0$. The finite double-cone structure then behaves as a finite single-cone structure. From (11) we have

$$A_\varphi = \frac{\pi i}{2(rr')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \frac{L_\nu^1(\theta)L_\nu^1(\theta')}{D(1, \nu, \theta_0, 0)} {}_{1,2}F_\nu(kr', kr)$$

As can be seen, the only term apparently modified is the term $D(1, \nu, \theta_0, 0)$. However $\theta_1 = 0$ implies only one boundary is present and thus $a_2 = 0$. Hence, from (8),

$$[L_\nu^1(\theta)]_{\theta_1=0} = P_{\nu-\frac{1}{2}}^1(\cos \theta)$$

Now $a_2 = 0$ and $\theta_1 = 0$ provide from (A.3) in Appendix A the result of

$$(20) \quad D(1, \nu, \theta_0, 0) \\ = \frac{-1}{2\nu} \left[\left(\frac{1}{2} - \nu \right) P_{\nu-\frac{1}{2}}^1(\cos \theta_0) P_{\nu-\frac{1}{2}}(\cos \theta_0) \sin \theta_0 \right. \\ \left. + \left(\frac{1}{2} + \nu \right) \left\{ P_{\nu-\frac{3}{2}}^1(\cos \theta_0) \frac{\partial}{\partial \nu} (P_{\nu-\frac{1}{2}}^1(\cos \theta_0)) - P_{\nu-\frac{1}{2}}^1(\cos \theta_0) \frac{\partial}{\partial \nu} (P_{\nu-\frac{3}{2}}^1(\cos \theta_0)) \right\} \right]$$

But from (13), as $a_2 = 0$, the values of ν are now determined from

$$(21) \quad P_{\nu-\frac{1}{2}}^1(\cos \theta_0) = 0$$

Substituting (21) into (20) we obtain

$$D(1, \nu, \theta_0, 0) = \frac{-1}{2\nu} \left(\frac{1}{2} + \nu \right) P_{\nu-\frac{3}{2}}^1(\cos \theta_0) \frac{\partial}{\partial \nu} (P_{\nu-\frac{1}{2}}^1(\cos \theta_0))$$

where $\frac{\partial}{\partial \nu} (P_{\nu-\frac{1}{2}}^1(\cos \theta_0))$ is understood to mean

$$\left[\frac{\partial}{\partial \nu} (P_{\nu-\frac{1}{2}}^1(\cos \theta_0)) \right]_{\nu=a_{1, \ell}}$$

We shall use this convention in the below analysis. But from Erdélyi (1953, Vol. 1, p. 161) and from (21)

$$\begin{aligned}
 -\left(\frac{1}{2}+\nu\right)P_{\nu-\frac{3}{2}}^1(\cos\theta_0) &= \sin\theta_0 P_{\nu-\frac{1}{2}}^2(\cos\theta_0) - \left(\nu-\frac{3}{2}\right)\cos\theta_0 P_{\nu-\frac{1}{2}}^1(\cos\theta_0) \\
 &= \sin\theta_0 P_{\nu-\frac{1}{2}}^2(\cos\theta_0)
 \end{aligned}$$

where the superscript 2 denotes second order and not a squared quantity. Therefore we obtain for the vector potential field

$$(22) \quad (A_\varphi)_{\theta_1=0} = \frac{\pi i}{(rr')^2 \sin\theta_0} \sum_{\ell=1}^{\infty} \nu U_\nu(\theta, \theta', \theta_0) {}_{1,2}F_\nu(kr', kr)$$

where

$$(23) \quad U_\nu(\theta, \theta', \theta_0) = \frac{P_{\nu-\frac{1}{2}}^1(\cos\theta)P_{\nu-\frac{1}{2}}^1(\cos\theta')}{P_{\nu-\frac{1}{2}}^2(\cos\theta_0)\frac{\partial}{\partial\nu}P_{\nu-\frac{1}{2}}^1(\cos\theta_0)}$$

$${}_{1,2}F_\nu(kr', kr) = \begin{cases} \frac{J_\nu(kr)}{J_\nu(ka)} \{J_\nu(ka)H_\nu^{(2)}(kr') - J_\nu(kr')H_\nu^{(2)}(ka)\}; r < r' \text{ i.e., } {}_1F_\nu \\ \frac{J_\nu(kr')}{J_\nu(ka)} \{J_\nu(ka)H_\nu^{(2)}(kr) - J_\nu(kr)H_\nu^{(2)}(ka)\}; r > r' \text{ i.e., } {}_2F_\nu \end{cases}$$

The index ℓ on the summation is determined as the ℓ th positive root of

$$P_{\nu-\frac{1}{2}}^1(\cos\theta_0) = 0$$

with respect to ν .

Now using a contiguous relation of the associated Legendre functions (Erdélyi, 1953, Vol. 1, p. 161, (11)) the equation for $U_\nu(\theta, \theta', \theta_0)$ may be rewritten as

$$U_\nu(\theta, \theta', \theta_0) = \frac{P_{\nu-\frac{1}{2}}^{-1}(\cos \theta) P_{\nu-\frac{1}{2}}^{-1}(\cos \theta')}{P_{\nu-\frac{1}{2}}^1(\cos \theta_0) \frac{\partial}{\partial \nu} (P_{\nu-\frac{1}{2}}^{-1}(\cos \theta_0))}$$

Case 2

This case is a further specialization of the first case. Here $\theta_1 = 0$ and $a = \infty$. We obtain this case immediately upon using (19) in (22). The vector potential field becomes for the semi-infinite single cone

$$(A_\varphi)_\infty = \frac{\pi i}{(rr')^{\frac{1}{2}} \sin \theta_0} \sum_{\ell=1}^{\infty} \nu U_\nu(\theta, \theta', \theta_0) \begin{cases} J_\nu(kr) H_\nu^{(2)}(kr'), & r < r' \\ J_\nu(kr') H_\nu^{(2)}(kr), & r > r' \end{cases}$$

where $U_\nu(\theta, \theta', \theta_0)$ is given by (23).

This is the identical result obtained by H. Buchholtz (1940).

The electric and magnetic components of the vector field are obtained by substitution into (1). The results compare with those found by F. E. Bornis and C. H. Papas (1958, p. 356-358). Bornis and Papas discuss this particular field in detail, especially its physical characteristics.

Case 3

The two cases above may be described as internal problems, i. e., the source ring is located inside its prescribed boundaries.

The case of an external source ring is obtained simply by letting $\theta_1 = 0$ and $a = \infty$, as in Case 2, only now letting θ_0 be greater than $\pi/2$. This case was investigated by L. B. Felsen (1959a) but his solution is in such a complicated form it is not possible to make a direct comparison. He has relied upon the geometrical optics, selected contour paths and use of asymptotics applied to the positioning of the source ring, observation point, and conical structure.

Case 4

We consider now the double cone with $a = \infty$. This corresponds to a semi-infinite double conical wave guide.

The vector potential field is now obtained using (11) and substituting (19) instead of (12). This process yields

$$(24) \quad (A_\varphi)_\infty = \frac{\pi i}{2(rr')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \frac{L_\nu^1(\theta)L_\nu^1(\theta')}{D(1, \nu, \theta_0, \theta_1)} \begin{cases} J_\nu(kr)H_\nu^{(2)}(kr'); & r < r' \\ J_\nu(kr')H_\nu^{(2)}(kr); & r > r' \end{cases}$$

If the normalization factor $D(1, \nu, \theta_0, \theta_1)$ were to be ignored, this result would compare favorably with that which would be obtained

by R. F. Harrington if his solution was completed (1961, p. 281).

Again, the values of ν are the l th positive roots to the transcendental equation in ν of

(13)

$$M_{\nu-\frac{1}{2}}^1(\theta_0, \theta_1) = P_{\nu-\frac{1}{2}}^1(\cos \theta_0) P_{\nu-\frac{1}{2}}^1(-\cos \theta_1) - P_{\nu-\frac{1}{2}}^1(-\cos \theta_0) P_{\nu-\frac{1}{2}}^1(\cos \theta_1) = 0$$

Case 5

As a final case, the biconical antenna is examined. This special case is obtained by letting $\theta_0 = \pi - \theta_1$ and $a = \infty$. From the geometry it is apparent we have again an external vector potential field.

The vector potential can be written almost immediately if one accepts a simple substitution of $\theta_0 = \pi - \theta_1$ into $D(1, \nu, \theta_0, \theta_1)$ and the deterministic equation (13) for determining the ν -values. However, care must be taken. Since $\theta_0 = \pi - \theta_1$, we have

$$\cos \theta_0 = -\cos \theta_1 \quad \text{and} \quad \sin \theta_0 = \sin \theta_1$$

Therefore

$$M_{\nu-\frac{1}{2}}^1(\theta_0, \theta_1) = [P_{\nu-\frac{1}{2}}^1(-\cos \theta_1)]^2 - [P_{\nu-\frac{1}{2}}^1(\cos \theta_1)] = 0$$

or, the ν -values are the l th positive roots of the transcendental equation

$$(25) \quad P_{\nu-\frac{1}{2}}^1(-\cos \theta_1) = P_{\nu-\frac{1}{2}}^1(\cos \theta_1)$$

Now substituting (25) into (A. 3) of Appendix A we obtain

$$\begin{aligned} D(1, \nu, \pi-\theta_1, \theta_1) &= \frac{(a_1^2 - a_2^2)}{2\nu} \left[2\left(\frac{1}{2} - \nu\right) P_{\nu-\frac{1}{2}}^1(\cos \theta_1) P_{\nu-\frac{1}{2}}^1(\cos \theta_1) \sin \theta_1 \right. \\ &\quad \left. - \left(\frac{1}{2} + \nu\right) \left\{ P_{\nu-\frac{1}{2}}^1(\cos \theta_1) \right\}^2 \frac{\partial}{\partial \nu} \left\{ \frac{P_{\nu-\frac{1}{2}}^1(\cos \theta_1) - P_{\nu-\frac{1}{2}}^1(-\cos \theta_1)}{P_{\nu-\frac{1}{2}}^1(\cos \theta_1)} \right\} \right] \\ &= E(\nu, \theta_1) \end{aligned}$$

This expression is derived in Appendix B.

Substituting $E(\nu, \theta_1)$ into (24) in place of $D(1, \nu, \theta_0, \theta_1)$, we obtain the vector potential

$$(A_\varphi)_\infty = \frac{\pi i}{2(rr')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \frac{L_\nu^1(\theta) L_\nu^1(\theta')}{E(\nu, \theta_1)} \begin{cases} J_\nu(kr') H_\nu^{(2)}(kr); & r' < r \\ J_\nu(kr) H_\nu^{(2)}(kr'); & r' > r \end{cases}$$

where now ν is selected, and thus ℓ , as to (25) instead of (13).

The biconical antenna is discussed in S. Adachi (1960). His results are restricted to very thin and very wide-angle cones.

We have investigated several cases, however many more can be chosen. But these cases chosen seem to provide a spectrum of the more interesting situations and/or situations which provide a comparison of the main results of this investigation.

where ${}_{1,2}R_\nu(r'\gamma^{\frac{1}{2}}, r\gamma^{\frac{1}{2}})$ is defined by

$$(27) \quad {}_1R_\nu(r'\gamma^{\frac{1}{2}}, r\gamma^{\frac{1}{2}}) = \frac{2ie^{2\pi i\nu}}{\pi} \cdot \frac{I_\nu(\gamma^{\frac{1}{2}}r)}{I_\nu(\gamma^{\frac{1}{2}}a)} \{I_\nu(a\gamma^{\frac{1}{2}})K_\nu(r'\gamma^{\frac{1}{2}}) - I_\nu(r'\gamma^{\frac{1}{2}})K_\nu(a\gamma^{\frac{1}{2}})\}$$

for $r < r'$. We interchange r and r' to obtain for $r > r'$ ${}_2R_\nu(\gamma^{\frac{1}{2}}r', \gamma^{\frac{1}{2}}r)$. As is easily seen this approach in solving the heat conduction problem does not seem to be fruitful.

However if we take two simple cases the method can be used successfully. We divide the class of problems into the classes of "a" finite and "a" infinite.

"a" Infinite

In the case of $a = \infty$ the function ${}_{1,2}R_\nu(r'\gamma^{\frac{1}{2}}, r\gamma^{\frac{1}{2}})$ as expressed by (27) reduces to

$$[{}_1R_\nu(r'\gamma^{\frac{1}{2}}, r\gamma^{\frac{1}{2}})]_\infty = \frac{2ie^{i2\pi\nu}}{\pi} I_\nu(r\gamma^{\frac{1}{2}})K_\nu(r'\gamma^{\frac{1}{2}}); \quad r < r'$$

$$[{}_2R_\nu(r'\gamma^{\frac{1}{2}}, r\gamma^{\frac{1}{2}})]_\infty = \frac{2ie^{i2\pi\nu}}{\pi} I_\nu(r'\gamma^{\frac{1}{2}})K_\nu(r\gamma^{\frac{1}{2}}); \quad r > r'$$

The parameter γ enters the vector potential A_φ only in these terms. Thus we need to obtain $\mathcal{L}^{-1}\{R_\nu(p\gamma^{\frac{1}{2}}, q\gamma^{\frac{1}{2}}), \gamma\}$ for p and q arbitrary real positive quantities.

We consider the transformation of the form

$$p = a^{\frac{1}{2}} - \beta^{\frac{1}{2}} \quad \text{and} \quad q = a^{\frac{1}{2}} + \beta^{\frac{1}{2}}$$

Thus we now require $p > q$. Solving for a and β we obtain

$$a = \frac{1}{4}(p+q)^2 > 0 \quad \text{and} \quad \beta = \frac{1}{4}(p-q)^2 > 0,$$

as p and q are both real positive quantities. Since a and β are both positive, we obtain from Erdélyi (1954, p. 284, (56))

$$\begin{aligned} & \mathcal{L}^{-1} \{ I_{\nu}(p\gamma^{\frac{1}{2}}) K_{\nu}(q\gamma^{\frac{1}{2}}); \gamma \} \\ & \qquad \qquad \qquad t \rightarrow \kappa t \\ &= \mathcal{L}^{-1} \{ I_{\nu}[(a\gamma)^{\frac{1}{2}} - (\beta\gamma)^{\frac{1}{2}}] K_{\nu}[(a\gamma)^{\frac{1}{2}} + (\beta\gamma)^{\frac{1}{2}}]; \gamma \} \\ & \qquad \qquad \qquad t \rightarrow \kappa t \\ &= \frac{I_{\nu}(\frac{a-\beta}{2\kappa t})}{2\kappa t} e^{-\frac{(a+\beta)}{2\kappa t}} = \frac{1}{2\kappa t} I_{\nu}(\frac{pq}{4\kappa t}) e^{-\frac{(p^2+q^2)}{4\kappa t}} \end{aligned}$$

We note the resulting inverse Laplace transform is symmetric in p and q . Thus the first Green's function in a semi-infinite double-conical structure may be written as

$$\begin{aligned} (28) \quad G(P, Q, t) &= \frac{-\pi i}{2(rr')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \frac{L_{\nu}^1(\theta) L_{\nu}^1(\theta')}{D(1, \nu, \theta_0, \theta_1)} \cdot \frac{2ie^{2\pi i \nu}}{2\kappa t} I_{\nu}(\frac{rr'}{4\kappa t}) e^{-\frac{(r^2+r'^2)}{4\kappa t}} \\ &= \frac{\pi e^{-\frac{(r^2+r'^2)}{4\kappa t}}}{2\kappa t (rr')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \frac{L_{\nu}^1(\theta) L_{\nu}^1(\theta)}{D(1, \nu, \theta_0, \theta_1)} e^{2\pi i \nu} I_{\nu}(\frac{rr'}{4\kappa t}) \end{aligned}$$

where the index ℓ is connected with the ν -values as the ℓ th positive root of the transcendental equation in ν of

$$(13) \quad P_{\nu-\frac{1}{2}}^1(\cos \theta_0) P_{\nu-\frac{1}{2}}^1(-\cos \theta_1) - P_{\nu-\frac{1}{2}}^1(-\cos \theta_0) P_{\nu-\frac{1}{2}}^1(\cos \theta_1) = 0$$

The quantities $L_{\nu}^1(\theta)$ and $D(1, \nu, \theta_0, \theta_1)$ are determined, as before, from (12) and (9). We see that the solution has the typical exponential decay behavior.

The cases examined in Chapter III for the semi-infinite conical structures may be re-examined again only for the heat conduction instead of the vector potential due to a time harmonic electric source ring. The only difference would lie in use of (28) instead of (11).

The specific case of a single semi-infinite cone (Case 2 of Chapter III) was investigated by R. Muki and E. Sternberg (1960). They approached solving the problem by using the Mellin transform and integration by parts to reduce the partial differential equation to an ordinary differential equation of the Sturm-Liouville type in terms of the transformation. The solution obtained is in the form of evaluating the inverse transformation. This leads to an improper integral whose integrand contains the ratio of two conical functions whose degree involves the variable of integration. The solution must be obtained by using numerical integration techniques of a result obtained from numerical integrations of the conical functions. This is quite

cumbersome.

Application of the same processes used in Chapter III provide us with the solution in closed form.

Letting $\theta_1 = 0$, the below equation is valid:

$$\begin{aligned} \frac{L_\nu^1(\theta)L_\nu^1(\theta')}{D(1,\nu,\theta_0,\theta_1)} &= \frac{P_{\nu-\frac{1}{2}}^{-1}(\cos \theta)P_{\nu-\frac{1}{2}}^{-1}(\cos \theta')}{P_{\nu-\frac{1}{2}}^{-1}(\cos \theta_0)\frac{d}{d\nu}[P_{\nu-\frac{1}{2}}^{-1}(\cos \theta_0)]} \cdot \frac{2\nu}{\sin \theta_0} \\ &= \frac{2\nu}{\sin \theta_0} U_\nu(\theta, \theta', \theta_0) \end{aligned}$$

Substituting into (28) we have the representation of the heat conduction in a semi-infinite single cone expressed by

$$G(P, Q, t) = \frac{\pi e^{-\frac{(r^2+r'^2)}{4\kappa t}}}{\kappa t (rr')^{\frac{1}{2}} \sin \theta_0} \sum_{\ell=1}^{\infty} \nu U_\nu(\theta, \theta', \theta_0) e^{2\pi i \nu} I_\nu\left(\frac{rr'}{4\kappa t}\right)$$

where under these conditions the correlation between ℓ and ν is given by the ℓ being the ℓ th positive root of the transcendental equation in ν of

$$P_{\nu-\frac{1}{2}}^1(\cos \theta_0) = 0$$

The other cases examined in Chapter III for $a = \infty$ can be expressed similarly for the heat conduction problem.

"a" Finite

The problem of obtaining an inverse Laplace transform of (27) was mentioned earlier. But upon investigation of where (27) originates shows that we obtain it by transforming the summed series in n to a function of γ instead of k . Returning to this point we have

$$A_{\varphi} = \frac{2}{\pi a^2 (rr')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \frac{L_{\nu}^1(\theta) L_{\nu}^1(\theta')}{D(1, \nu, \theta_0, \theta_1)} \cdot \frac{J_{\nu}(r'k_{n,\nu}) J_{\nu}(rk_{n,\nu})}{J_{\nu+1}^2(\tau_{n,\nu}) [k_{n,\nu}^2 - k_{n,\nu}^2]}$$

where

$$k_{n,\nu} = \tau_{n,\nu}/a$$

and ν and ℓ are related by ℓ being the ℓ th positive root of (13).

Substitution of $k = -i\gamma$ and using (28) gives the first Green's function for the heat conduction between two coaxial finite conical structures as

$$G(P, Q, t) = \frac{+2}{\pi a^2 (rr')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \frac{L_{\nu}^1(\theta) L_{\nu}^1(\theta')}{D(1, \nu, \theta_0, \theta_1)} \frac{J_{\nu}(r'k_{n,\nu}) J_{\nu}(rk_{n,\nu})}{J_{\nu+1}^2(\tau_{n,\nu})} \\ \times \mathcal{L}^{-1} \left\{ \frac{1}{\gamma + k_{n,\nu}^2} ; \gamma \right\}_{t \rightarrow \kappa t}$$

From Erdélyi (1954, P. 229, (1))

$$\mathcal{L}^{-1} \left\{ \frac{1}{\gamma + k_{n,\nu}^2}; \gamma \right\}_{t \rightarrow \kappa t} = e^{-\kappa t k_{n,\nu}^2}$$

Thus

$$G(P, Q, t) = \frac{2}{\pi a^2 (rr')^{\frac{1}{2}}} \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \frac{L_{\nu}^1(\theta) L_{\nu}^1(\theta') J_{\nu}(r' k_{n,\nu}) J_{\nu}(r k_{n,\nu}) e^{-\kappa t k_{n,\nu}^2}}{D(1, \nu, \theta_0, \theta_1) J_{\nu+1}^2(\tau_{n,\nu})}$$

where the index ℓ is the ℓ th positive root of the transcendental equation in ν expressed by (13).

The special cases for finite "a" examined in Chapter III can be treated easily using the above equation.

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APPENDIX A

The normalization factor for determining the normalized eigenfunctions involves the evaluation of a definite integral with the products of associated Legendre functions of the same order and degree. These relationships are not available in the literature and thus are derived in this appendix.

The Legendre differential equation is written as

$$(A. 1) \quad (1-x^2) \frac{d^2 u}{dx^2} - 2x \frac{du}{dx} \left[\nu(\nu+1) - \frac{\mu^2}{1-x^2} \right] u = 0$$

and has the solutions of $P_\nu^\mu(x)$, Q_ν^μ , or any linear combination of these two linearly independent transcendental functions over the real interval $(-1, 1)$. We take μ and ν as completely arbitrary parameters.

Let $w_\nu^\mu(x)$ and $v_\sigma^\rho(x)$ be any solutions to the differential equation (A. 1), with ν and μ replaced by σ , ρ respectively for $v_\sigma^\rho(x)$. Then

$$(1-x^2) \frac{d^2 w_\nu^\mu}{dx^2} - 2x \frac{dw_\nu^\mu}{dx} + [\nu(\nu+1) - \mu^2 (1-x^2)^{-1}] w_\nu^\mu = 0$$

$$(1-x^2) \frac{d^2 v_\sigma^\rho}{dx^2} - 2x \frac{dv_\sigma^\rho}{dx} + [\sigma(\sigma+1) - \rho^2 (1-x^2)^{-1}] v_\sigma^\rho = 0$$

Multiplying the first equation by v_{σ}^{ρ} and the second by w_{ν}^{μ} and then subtracting the resulting equations gives the single equation

$$v_{\sigma}^{\rho} \frac{d}{dx} \left[(1-x^2) \frac{dw_{\nu}^{\mu}}{dx} \right] - w_{\nu}^{\mu} \frac{d}{dx} \left[(1-x^2) \frac{dv_{\sigma}^{\rho}}{dx} \right] + [(\nu+1)-\sigma(\sigma+1) + (1-x^2)^{-1}(\rho^2 - \mu^2)] w_{\nu}^{\mu} v_{\sigma}^{\rho} = 0$$

This can be rewritten as

$$-\frac{d}{dx} (1-x^2) \left[v_{\sigma}^{\rho} \frac{dw_{\nu}^{\mu}}{dx} - w_{\nu}^{\mu} \frac{dv_{\sigma}^{\rho}}{dx} \right] = [(\nu-\sigma)(\nu+\sigma+1) + (\rho^2 - \mu^2)(1-x^2)^{-1}] w_{\nu}^{\mu} v_{\sigma}^{\rho}$$

Integrating both sides we obtain, for $-1 < a < b < 1$,

$$\begin{aligned} & \int_a^b [(\nu-\sigma)(\nu+\sigma+1) + (\rho^2 - \mu^2)(1-x^2)^{-1}] w_{\nu}^{\mu}(x) v_{\sigma}^{\rho}(x) dx \\ &= (1-x^2) \left[w_{\nu}^{\mu} \frac{dv_{\sigma}^{\rho}}{dx} - v_{\sigma}^{\rho} \frac{dw_{\nu}^{\mu}}{dx} \right] \Bigg|_a^b \\ &= \left[x(\nu-\sigma) w_{\nu}^{\mu} v_{\sigma}^{\rho} + (\sigma+\rho) w_{\nu}^{\mu} v_{\sigma-1}^{\rho} - (\nu+\mu) v_{\sigma}^{\rho} w_{\nu-1}^{\mu} \right] \Bigg|_a^b \end{aligned}$$

using the recurrence formula from Erdélyi (1953, Vol. 1, p. 161).

Now letting $\rho = \mu$ and $v_{\sigma}^{\rho} = w_{\sigma}^{\rho}$ we obtain

$$(A. 2) \quad \int_a^b w_\nu^\rho w_\sigma^\rho dx = \frac{x w_\nu^\rho w_\sigma^\rho}{(\nu + \sigma + 1)} + \frac{(\sigma + \rho) w_\nu^\rho w_{\sigma-1}^\rho - (\nu + \rho) w_{\nu-1}^\rho w_\sigma^\rho}{(\nu - \sigma)(\nu + \sigma + 1)} \Bigg|_a^b$$

If $\nu = \sigma$, the second term has an indeterminate form. Thus, for $\sigma \neq -1/2$, we have from l'Hospital's rule

$$\begin{aligned} \lim_{\nu \rightarrow \sigma} \frac{(\sigma + \rho) w_\nu^\rho w_{\sigma-1}^\rho - (\nu + \rho) w_{\nu-1}^\rho w_\sigma^\rho}{(\nu - \sigma)(\nu + \sigma + 1)} \\ = \frac{(\sigma + \rho)}{(2\sigma + 1)} \left[w_{\sigma-1}^\rho \frac{\partial w_\sigma^\rho}{\partial \sigma} - w_\sigma^\rho \frac{\partial w_{\sigma-1}^\rho}{\partial \sigma} \right] - \frac{w_\sigma^\rho w_{\sigma-1}^\rho}{2\sigma + 1} \end{aligned}$$

Combining this limit with (A. 2) for $\nu = \sigma$ we obtain

$$\begin{aligned} \int_a^b [w_\sigma^\rho(x)]^2 dx = \frac{1}{2\sigma + 1} \left\{ x (w_\sigma^\rho(x))^2 - w_\sigma^\rho(x) w_{\sigma-1}^\rho(x) \right. \\ \left. + (\sigma + \rho) \left[w_{\sigma-1}^\rho(x) \frac{\partial w_\sigma^\rho(x)}{\partial \sigma} - w_\sigma^\rho(x) \frac{\partial w_{\sigma-1}^\rho(x)}{\partial \sigma} \right] \right\} \Bigg|_a^b \end{aligned}$$

where $w_\sigma^\rho(x)$ is any solution of Legendre's differential equation on the cut $-1 < x < 1$.

Chapter II required the evaluation of an integral of this form. Since (8) is a solution of Legendre's differential equation, we can substitute into the above result to obtain

$$\int_{\theta_1}^{\theta_0} [L_{\nu}^1(\theta)]^2 \sin \theta d\theta$$

$$= \frac{-1}{2\nu} \left\{ [L_{\nu}^1(\theta)]^2 \cos \theta - L_{\nu}^1(\theta) L_{\nu-1}^1(\theta) + \left(\frac{1}{2} + \nu\right) \left[L_{\nu-1}^1(\theta) \frac{\partial L_{\nu}^1(\theta)}{\partial \nu} - L_{\nu}^1(\theta) \frac{\partial L_{\nu-1}^1(\theta)}{\partial \nu} \right] \right\} \Bigg|_{\theta_1}^{\theta_0}$$

Now

$$\cos \theta [L_{\nu}^1(\theta)]^2 - L_{\nu}^1(\theta) L_{\nu-1}^1(\theta) = \left(\frac{1}{2} - \nu\right) \sin \theta L_{\nu}^1(\theta) L_{\nu}^0(\theta)$$

Thus

$$\int_{\theta_1}^{\theta_0} [L_{\nu}^1(\theta)]^2 \sin \theta d\theta$$

$$= \frac{-1}{2\nu} \left\{ \left(\frac{1}{2} - \nu\right) [L_{\nu}^1(\theta_0) L_{\nu}^0(\theta_0) \sin \theta_0 - L_{\nu}^1(\theta_1) L_{\nu}^0(\theta_1) \sin \theta_1] \right.$$

$$\left. + \left(\frac{1}{2} + \nu\right) \left[L_{\nu-1}^1(\theta_0) \frac{\partial L_{\nu}^1(\theta_0)}{\partial \nu} - L_{\nu}^1(\theta_0) \frac{\partial L_{\nu-1}^1(\theta_0)}{\partial \nu} - L_{\nu-1}^1(\theta_1) \frac{\partial L_{\nu}^1(\theta_1)}{\partial \nu} + L_{\nu}^1(\theta_1) \frac{\partial L_{\nu-1}^1(\theta_1)}{\partial \nu} \right] \right\}$$

From (8) we obtain the desired integral solution. By making the substitutions of

$$P_{\nu-\frac{1}{2}}^1(\cos \theta) = P_{\mu}^1(\theta)$$

we obtain the formula expressed as (9) only in its expanded form

(A. 3)

$$\begin{aligned}
& -2\nu D(1, \nu, \theta_0, \theta_1) \\
&= -2\nu \int_{\theta_1}^{\theta_0} [L_\nu^1(\theta)]^2 \sin \theta d\theta = -2\nu \int_{\theta_1}^{\theta_0} [a_1 P_\mu^1(\theta) + a_2 P_\mu^1(-\theta)]^2 \sin \theta d\theta \\
&= a_1^2 \left\{ (-\mu) [P_\mu^1(\theta_0) P_\mu^1(\theta_0) \sin \theta_0 - P_\mu^1(\theta_1) P_\mu^1(\theta_1) \sin \theta_1] \right. \\
&\quad \left. + (1+\mu) \left[P_{\mu-1}^1(\theta_0) \frac{\partial P_\mu^1(\theta_0)}{\partial \nu} - P_\mu^1(\theta_0) \frac{\partial P_{\mu-1}^1(\theta_0)}{\partial \nu} - P_{\mu-1}^1(\theta_1) \frac{\partial P_\mu^1(\theta_1)}{\partial \nu} + P_\mu^1(\theta_1) \frac{\partial P_{\mu-1}^1(\theta_1)}{\partial \nu} \right] \right\} \\
&\quad + a_1 a_2 \left\{ (-\mu) [P_\mu^1(\theta_0) P_\mu^1(-\theta_0) + P_\mu^1(-\theta_0) P_\mu^1(\theta_0) - P_\mu^1(\theta_1) P_\mu^1(-\theta_1) - P_\mu^1(-\theta_1) P_\mu^1(\theta_1)] \right. \\
&\quad \left. + (1+\mu) \left[P_{\mu-1}^1(\theta_0) \frac{\partial P_\mu^1(-\theta_0)}{\partial \nu} + P_\mu^1(-\theta_0) \frac{\partial P_{\mu-1}^1(\theta_0)}{\partial \nu} - P_{\mu-1}^1(\theta_1) \frac{\partial P_\mu^1(-\theta_1)}{\partial \nu} - P_\mu^1(-\theta_1) \frac{\partial P_{\mu-1}^1(\theta_1)}{\partial \nu} \right. \right. \\
&\quad \left. \left. - P_\mu^1(\theta_0) \frac{\partial P_{\mu-1}^1(-\theta_0)}{\partial \nu} - P_{\mu-1}^1(-\theta_0) \frac{\partial P_\mu^1(\theta_0)}{\partial \nu} + P_\mu^1(\theta_1) \frac{\partial P_{\mu-1}^1(-\theta_1)}{\partial \nu} + P_{\mu-1}^1(-\theta_1) \frac{\partial P_\mu^1(\theta_1)}{\partial \nu} \right] \right\} \\
&\quad + a_2^2 \left\{ (-\mu) [P_\mu^1(-\theta_0) P_\mu^1(-\theta_0) \sin \theta_0 - P_\mu^1(-\theta_1) P_\mu^1(-\theta_1) \sin \theta_1] \right. \\
&\quad \left. + (1+\mu) \left[P_{\mu-1}^1(-\theta_0) \frac{\partial P_\mu^1(-\theta_0)}{\partial \nu} - P_\mu^1(-\theta_0) \frac{\partial P_{\mu-1}^1(-\theta_0)}{\partial \nu} - P_{\mu-1}^1(-\theta_1) \frac{\partial P_\mu^1(-\theta_1)}{\partial \nu} + P_\mu^1(-\theta_1) \frac{\partial P_{\mu-1}^1(-\theta_1)}{\partial \nu} \right] \right\}
\end{aligned}$$

This expanded form is quite useful for examining degenerate cases.

APPENDIX B

The biconical antenna case for the comparison of the main results of this investigation with known results requires the evaluation of $D(1, \nu, \pi - \theta_1, \theta_1)$, which was redefined as $E(\nu, \theta_1)$.

Substituting $\theta_0 = \pi - \theta_1$ provides the equations of

$$(B. 1) \quad \begin{aligned} \cos \theta_0 &= -\cos \theta_1 \\ \sin \theta_0 &= \sin \theta_1 \end{aligned}$$

Thus from these equations, (13) is rewritten as

$$[P_{\nu-\frac{1}{2}}^1(-\cos \theta_1)]^2 - [P_{\nu-\frac{1}{2}}^1(\cos \theta_1)]^2 = 0$$

Thus the roots are obtained from

$$(B. 2) \quad P_{\nu-\frac{1}{2}}^1(-\cos \theta_1) = P_{\nu-\frac{1}{2}}^1(\cos \theta_1)$$

Using the notation introduced in the previous appendix, i. e. ,

$$P_{\nu-\frac{1}{2}}^1(\cos \theta) = P_{\mu}^1(\theta),$$

substitution of (B. 1) into (A. 3) provides the equation

$$\begin{aligned}
-2\nu E(\nu, \theta_1) = & a_1^2 \left\{ (-\mu) [P_\mu^1(-\theta_1)P_\mu^1(-\theta_1)\sin\theta_1 - P_\mu^1(\theta_1)P_\mu^1(\theta_1)\sin\theta_1] \right. \\
& + (1+\mu) \left[P_{\mu-1}^1(-\theta_1) \frac{\partial P_\mu^1(-\theta_1)}{\partial \nu} - P_\mu^1(-\theta_1) \frac{\partial P_{\mu-1}^1(-\theta_1)}{\partial \nu} \right. \\
& \quad \left. \left. - P_{\mu-1}^1(\theta_1) \frac{\partial P_\mu^1(\theta_1)}{\partial \nu} + P_\mu^1(\theta_1) \frac{\partial P_{\mu-1}^1(\theta_1)}{\partial \nu} \right] \right\} \\
& + a_2^2 \left\{ (-\mu) [P_\mu^1(\theta_1)P_\mu^1(\theta_1)\sin\theta_1 - P_\mu^1(-\theta_1)P_\mu^1(-\theta_1)\sin\theta_1] \right. \\
& \quad + (1+\mu) \left[P_{\mu-1}^1(\theta_1) \frac{\partial P_\mu^1(\theta_1)}{\partial \nu} - P_\mu^1(\theta_1) \frac{\partial P_{\mu-1}^1(\theta_1)}{\partial \nu} \right. \\
& \quad \left. \left. - P_{\mu-1}^1(-\theta_1) \frac{\partial P_\mu^1(-\theta_1)}{\partial \nu} - P_\mu^1(-\theta_1) \frac{\partial P_{\mu-1}^1(-\theta_1)}{\partial \nu} \right] \right\}
\end{aligned}$$

Substituting (B. 2) and combining terms one obtains

$$\begin{aligned}
E(\nu, \theta_1) = & \frac{(a_2^2 - a_1^2)}{2\nu} \left\{ (-\mu) P_\mu^1(\theta_1) \sin\theta_1 (P_\mu^1(-\theta_1) - P_\mu^1(\theta_1)) \right. \\
& \left. + (1+\mu) (P_\mu^1(\theta_1))^2 \frac{\partial}{\partial \nu} \left[\frac{P_{\mu-1}^1(\theta_1) - P_{\mu-1}^1(-\theta_1)}{P_\mu^1(\theta_1)} \right] \right\}
\end{aligned}$$

Now from (B. 2) and Erdélyi (1953, Vol. 1, p. 148)

$$P_\mu^1(\cos\theta_1) = -\sin\theta_1 \frac{\partial P_\mu(\cos\theta_1)}{\partial(\cos\theta_1)} = \sin\theta_1 \frac{\partial P_\mu(-\cos\theta_1)}{\partial(\cos\theta_1)} = P_\mu^1(-\cos\theta_1)$$

Thus

$$\frac{\partial}{\partial(\cos \theta_1)} [P_\mu(\cos \theta_1) + P_\mu(-\cos \theta_1)] = 0$$

This implies either

$$P_\mu(\cos \theta_1) + P_\mu(-\cos \theta_1) = \text{constant}$$

for all μ for any given θ_1 or

$$P_\mu(\cos \theta_1) + P_\mu(-\cos \theta_1) = 0$$

Since the first case is absurd, thus we obtain

$$P_\mu(\cos \theta_1) = -P_\mu(-\cos \theta_1)$$

Substituting we obtain the desired result of

$$E(\nu, \theta_1) = \frac{(a_1^2 - a_2^2)}{2\nu} \left\{ 2(-\mu)P_\mu^1(\theta_1)P_\mu(\theta_1)\sin \theta_1 - (1+\mu)(P_\mu^1(\theta_1))^2 \right. \\ \left. \times \frac{\partial}{\partial \nu} \left[\frac{P_{\mu-1}^1(\theta_1) - P_{\mu-1}^1(-\theta_1)}{P_\mu^1(\theta_1)} \right] \right\}$$

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