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FOR MINIMUM COMPLIANCE

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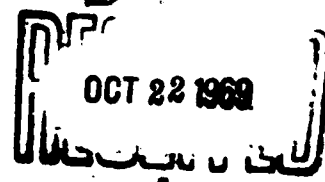
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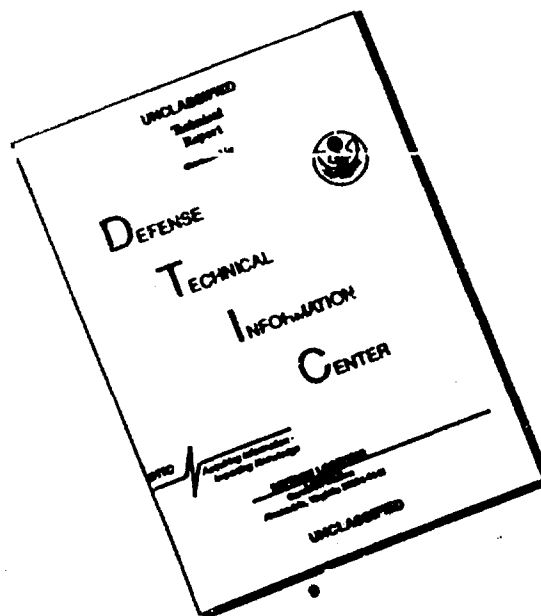
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OPTIMAL DESIGN OF ELASTIC CIRCULAR SANDWICH BEAMS FOR MINIMUM COMPLIANCE¹

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ABSTRACT

Elastic circular sandwich beams are designed for minimum compliance and given total weight. To treat the problem in a more realistic manner, the beams are regarded as extensible. Examples are given for the optimal design of circular rings and semicircular arches with different end conditions. The calculated optimal compliance is compared with the corresponding compliance of a uniform beam with identical weight. Finally, the optimal design with stress bounds is also investigated for the ring problem.

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1. INTRODUCTION

In the analysis of the deformation of the curved beams such as rings and arches, we usually assume that the center line of the beam is inextensible and the deflection of the beam is entirely due to flexure. Such an assumption could simplify the analysis considerably and the error introduced by the assumption is very small if the ratio of the thickness of the beam to the radius of curvature of the center line of the beam is much less than unity. However, in the optimal elastic design of structures for given total weight and minimum compliance, the design variable is proportional to the strain energy density [1]². Therefore, the assumption of inextensibility would lead to a design with zero cross sections at those locations of vanishing bending moment. The fiber stress would become infinite there when the axial force does not vanish. To remedy this deficiency one may stipulate a minimum cross section constraint [2], or set bounds on the fiber stress, or remove the assumption of inextensibility.

The present paper illustrates how the last amendment is applied to a realistic design. For simplicity, our investigation is restricted to the design of sandwich beams for which the bending stiffness is regarded as proportional to the extensional stiffness.

² Numbers in brackets designate References at end of paper.

2. BASIC EQUATIONS

Consider the in-plane infinitesimal deformation of a circular, curved beam with symmetrical cross sections. The thickness of the beam is assumed to be much smaller than the radius R of the center line of the beam. The stress-displacement relations of the curved beam can be derived based on the Euler-Bernoulli assumption [3]. The stress-displacement relations thus obtained are too complicated to be applied to practical problems unless further approximations or linearizations are imposed. In the following, we shall postulate that the fiber stress at any point in the beam with a distance z from the central surface can be expressed as

$$\tilde{\sigma} = \frac{F}{A} + \frac{Mz}{I} \quad (1)$$

where F is the fiber stress resultant, M is the bending moment, A is the cross-sectional area and I is the moment of inertia of the cross-sectional area about the centroidal axis. In fact, Eq. (1) can be derived based on Euler-Bernoulli assumption if terms of the order of r/R are neglected as compared with unity where r is the radius of gyration of the cross section of the beam. The stress energy per unit length of the beam can be expressed by the following area integral over the cross section:

$$U = \frac{1}{2E} \int \tilde{\sigma}^2 dA = \frac{1}{2} \left(\frac{F^2}{AE} + \frac{M^2}{EI} \right) \quad (2)$$

Let the inward normal distributed load be $p(s)$, the counterclockwise tangential distributed load $t(s)$, the inward normal displacement w and the counterclockwise tangential displacement v . By considering the equilibrium in moment, radial and tangential forces acting on an infinitesimal element of the beam as shown in Fig. 1, we obtain the following equations of equilibrium:

$$\frac{F}{R} + M'' + p = 0 \quad , \quad (3)$$

$$F' - \frac{M'}{R} + t = 0 \quad . \quad (4)$$

The stress displacement relations can be derived from the principle of minimum complementary energy with Eqs. (3) and (4) as constraint conditions while the displacements v and w are considered to be prescribed. We obtain

$$\frac{F}{AE} = -\frac{1}{R} (\dot{v} + w) \quad , \quad (5)$$

$$\frac{M}{EI} = \frac{1}{R^2} (\dot{v} - \ddot{w}) \quad (6)$$

where the dot represents the differentiation with respect to the polar angle θ . Note that $(\dot{}) \equiv -R()'$.

Next, let us consider the optimal design of circular curved beams. The beam is designed to minimize total compliance due to a concentrated force of given magnitude while the total weight of the beam is prescribed. The necessary and sufficient condition of local

optimum is given in [1]. It is

$$U/B = \text{constant} \quad (7)$$

where U is the strain (or stress) energy per unit length of the beam given by Eq. (2) and B is the design variable. In the case of optimal design of sandwich beams with given height of the core, we shall choose the area of the cover sheets A as the design variable. In this case, the stiffness per unit length of the beam is proportional to A and Eq. (7) becomes the necessary and sufficient condition of global optimum. The application of Eq. (7) to the optimal design of straight sandwich beams is given in [4].

If we assume that the curved beam is inextensible, then the first term in Eq. (2) is dropped. In this case, the design variable A would vanish whenever the bending moment M vanishes. Note that at any section with zero value of A , the fiber stress becomes infinity. Thus the assumption of inextensibility is no longer valid. In order to treat our problem in a more realistic manner, we shall regard the beam as extensible and use the expression of strain energy, Eq. (2).

We shall assume the cross-sectional areas of the top and bottom cover sheets to be identical. The core is regarded as perfectly soft and the whole fiber stress is carried by the cover sheets. Since the thickness of both cover sheets is much smaller than the height of the core T , the moment of inertia of the cover sheets can be expressed approximately as

$$I = \beta^2 R^2 A \quad (8)$$

where $\beta = T/(2R)$ is a constant. The curved beam is said to be slender if $\beta \ll 1$. From Eqs. (2), (7), (5), (6), and (8), the optimality condition can be expressed in terms of displacement components as

$$\beta^2 (\dot{v} - \ddot{w})^2 + (\dot{v} + w)^2 = c \quad (9)$$

where c is a positive constant. Note that Eq. (9) is a nonlinear differential equation of v and w .

3. OPTIMAL DESIGN OF CIRCULAR RINGS AND ARCHES

A circular elastic slender sandwich ring of radius R is deformed by two equal and opposite forces P as shown in Fig. 2a. For given total weight of the ring, we shall find the variation of thickness of the cover sheets such that the deflections at a and b are minimized. A similar problem of optimal design of plastic rings is investigated by Prager [2].

Since the optimum ring is symmetrical with respect to the diameters $\theta = 0$ and $\theta = \pi/2$, we need only consider one-quarter of the ring in the first quadrant $0 \leq \theta \leq \pi/2$. At $\theta = 0$, the axial tension is $P/2$ and the transverse shear is zero. Note that the bending moment at $\theta = 0$ remains unknown and we shall denote it by M_1 . At any section within $0 \leq \theta \leq \pi/2$, the bending moment and the axial tension are respectively

$$M = M_1 - \frac{PR}{2} (1 - \cos \theta) , \quad (10)$$

$$F = \frac{P}{2} \cos \theta . \quad (11)$$

After elimination of M and F from Eqs. (5), (6), (10), and (11), we obtain

$$-A(\dot{v} + w) = \frac{PR}{2E} \cos \theta , \quad (12)$$

$$A(\ddot{w} - \dot{v}) = \frac{PR}{2E\beta^2} (k - \cos \theta) \quad (13)$$

where $k = 1 - \frac{2M_1}{PR}$ is a constant to be determined. From Eqs. (9), (12), and (13), the design variable A can be expressed as

$$A = \frac{PR}{2E} c^{-1/2} \left[\frac{1}{\beta^2} (k - \cos \theta)^2 + \cos^2 \theta \right]^{-1/2} . \quad (14)$$

Since the total weight of the core is fixed, to prescribe the total weight of the beam is equivalent to assigning the total weight of the cover sheets. Let the density of the cover sheets be ρ . The total weight of the cover sheets is

$$W = 4\rho R \int_0^{\pi/2} A d\theta = \frac{2P\rho R^2}{E} c^{-1/2} f \quad (15)$$

where

$$f = \int_0^{\pi/2} \left[\frac{1}{\beta^2} (k - \cos \theta)^2 + \cos^2 \theta \right]^{-1/2} d\theta . \quad (16)$$

From Eqs. (14) and (15) the cross-sectional area within $0 \leq \theta \leq \pi/2$ can be written as

$$A = \frac{W}{4\rho R f} \left[\frac{1}{\beta^2} (k - \cos \theta)^2 + \cos^2 \theta \right]^{1/2} . \quad (17)$$

Since the value of k is found to be nonzero, the value of A would never vanish.

In view of Eq. (17), the general solutions of Eqs. (12) and (13) are found to be

$$w = c_1 \sin \theta + c_2 \cos \theta + \frac{2P\rho R^2 f}{EW\beta} \int_0^\theta [k - (1 + \beta^2) \cos \varphi] [(k - \cos \varphi)^2 + \beta^2 \cos^2 \varphi]^{-1/2} \sin(\theta - \varphi) d\varphi, \quad (18)$$

and

$$v = c_1 \cos \theta - c_2 \sin \theta + c_3 - \frac{2P\rho R^2 f}{EW\beta} \int_0^\theta \left\{ k - \cos \varphi - [k - (1 + \beta^2) \cos \varphi] \cos(\theta - \varphi) \right\} [(k - \cos \varphi)^2 + \beta^2 \cos^2 \varphi]^{-1/2} d\varphi. \quad (19)$$

The integration constants c_1 , c_2 , c_3 and the value of k can be determined by the boundary conditions

$$\dot{w}(0) = v(0) = \dot{w}(\pi/2) = v(\pi/2) = 0 . \quad (20)$$

It is found that

$$c_1 = c_2 = 0 , \quad (21)$$

$$c_2 = \frac{2P\rho R^2 f}{EW\beta} \int_0^{\pi/2} \sin \theta [k - (1+\beta^2) \cos \theta] [(k - \cos \theta)^2 + \beta^2 \cos^2 \theta]^{-1/2} d\theta , \quad (22)$$

and the constant k satisfies the following equation:

$$\int_0^{\pi/2} (k - \cos \theta) [(k - \cos \theta)^2 + \beta^2 \cos^2 \theta]^{-1/2} d\theta = 0 . \quad (23)$$

Note that Eq. (23) can also be obtained directly by integrating Eq. (13) and using Eqs. (17) and (20). After the value k is determined from Eq. (23), f , and hence $A(\theta)$, can be evaluated from Eqs. (16) and (17) respectively. In Fig. 3, the ratio of A to its average value is plotted against θ in solid lines for $\beta = 0.05, 0.1$, and 0.15 .

It is found that when β approaches zero the minimum cross-sectional area also approaches zero and its limiting position is at $\theta = \frac{\pi}{4}$. This is not surprising because in this limiting case the first term in the bracket of Eq. (14) becomes dominant and the design is actually equivalent to that of an inextensible ring. One can easily show that for the optimal design of inextensible rings, $k = \cos^{-1}(\pi/4)$ and $A(\theta)$ is proportional to $|k - \cos \theta|$. Accordingly, we conclude that for a very small value of β , the optimal design of an inextensible ring is a good approximate design except in the region near the minimum cross section. We shall discuss this point further.

To show the efficiency of the optimal design, the deflection

δ_o at the point of the application of the force P is compared with that of a ring of uniform cross section and equal weight. For optimum rings, we have

$$\frac{EW}{2P\rho R^3} \delta_o = -\frac{f}{\beta} \int_0^{\pi/2} [k - (1 + \beta^2) \cos \theta] [(k - \cos \theta)^2 + \beta^2 \cos^2 \theta]^{-1/2} \cos \theta d\theta. \quad (24)$$

For uniform rings, the deflection δ_u can be obtained from Castigliano's theorem as

$$\frac{EW}{2P\rho R^3} \delta_u = \pi \left[\frac{1}{\beta^2} \left(\frac{\pi}{8} - \frac{1}{\pi} \right) + \frac{\pi}{8} \right]. \quad (25)$$

These compliances and their ratio are plotted against β in Fig. 4. It is found that the compliance is reduced by 18-25% between $\beta = 0.15$ and 0.05 due to the optimum design.

In the foregoing discussion, we have considered the design of the circular rings. The optimal design of other types of the circular slender beams can also be analyzed in a similar manner. Let us consider the optimal designs of semicircular arches deformed by a vertical force P at the crown, as shown in Figs. 2b and 2c. For two hinged arches, the bending moment and the axial tension can be expressed in terms of the horizontal outward reaction H at $\theta = 0$. For clamped arches, the bending moment and the axial tension can be expressed in terms of two redundant quantities M_1 and H at the end of the arch.

Using appropriate boundary conditions we can easily derive the following governing equations for the optimal design of semicircular arches:

(a) Two hinged arches

$$A = \frac{W}{2\rho R g_1} \left[\frac{1}{\beta^2} (\cos \theta + h \sin \theta - 1)^2 + (\cos \theta + h \sin \theta)^2 \right]^{1/2}, \quad (26)$$

$$g_1 = \int_0^{\pi/2} \left[\frac{1}{\beta^2} (\cos \theta + h \sin \theta - 1)^2 + (\cos \theta + h \sin \theta)^2 \right]^{1/2} d\theta, \quad (27)$$

$$\int_0^{\pi/2} [(1+\beta^2)(\cos \theta + h \sin \theta) - 1] [(\cos \theta + h \sin \theta - 1)^2 + \beta^2 (\cos \theta + h \sin \theta)^2]^{-1/2} \sin \theta d\theta = 0, \quad (28)$$

$$\begin{aligned} \frac{EW\delta}{P\rho R^3} = & -\frac{g_1}{\beta} \int_0^{\pi/2} [(\cos \theta + h \sin \theta - 1)(1 - \cos \theta) \\ & - \beta^2 \cos \theta (\cos \theta + h \sin \theta)] [(\cos \theta + h \sin \theta - 1)^2 + \beta^2 (\cos \theta + h \sin \theta)^2]^{-1/2} d\theta, \end{aligned} \quad (29)$$

$$\frac{EW\delta_u}{P\rho R^3} = \frac{\pi}{2} \left[\frac{1}{\beta^2} \left(\frac{3}{4} \pi - 2 - \frac{1}{\pi} \frac{1-\beta^2}{1+\beta^2} \right) + \frac{\pi}{4} \frac{1-\beta^2}{1+\beta^2} \right]. \quad (30)$$

(b) Clamped arches

$$A = \frac{W}{2\rho R g_2} \left[\frac{1}{\beta^2} (\cos \theta + h \sin \theta + n)^2 + (\cos \theta + h \sin \theta)^2 \right]^{1/2}, \quad (31)$$

$$g_2 = \int_0^{\pi/2} \left[\frac{1}{\beta^2} (\cos \theta + h \sin \theta + n)^2 + (\cos \theta + h \sin \theta)^2 \right]^{1/2} d\theta. \quad (32)$$

$$\int_0^{\pi/2} [(1 + \beta^2)(\cos \theta + h \sin \theta) + n] [(\cos \theta + h \sin \theta + n)^2 + \beta^2 (\cos \theta + h \sin \theta)^2]^{-1/2} \sin \theta d\theta = 0, \quad (33)$$

$$\int_0^{\pi/2} (\cos \theta + h \sin \theta + n) [(\cos \theta + h \sin \theta + n)^2 + \beta^2 (\cos \theta + h \sin \theta)^2]^{-1/2} d\theta = 0, \quad (34)$$

$$\frac{EW\delta_o}{P\rho R} = \frac{g_2}{\beta} \int_0^{\pi/2} [(1 + \beta^2)(\cos \theta + h \sin \theta) + n] [(\cos \theta + h \sin \theta + n)^2 + \beta^2 (\cos \theta + h \sin \theta)^2]^{-1/2} \cos \theta d\theta, \quad (35)$$

$$\frac{EW\delta_u}{P\rho R^2} = \frac{1 + \beta^2}{8\beta^2} \pi(\pi - 2) [(1 + \beta^2) \pi(\pi + 2) - 16] / [(1 + \beta^2) \pi^2 - 8]. \quad (36)$$

In Eqs. (26 - 36), $h = \frac{2H}{P}$ and $n = \frac{2M_1}{PR} - 1$. The distributions of the cross-sectional area A of these two arches are shown in Fig. 5 and 6 respectively. The comparison of the compliance of the optimum arch with that of a corresponding uniform arch is shown in Figs. 7 and 8.

By comparison of our design with the design of an inextensible beam, it is found that the variations of A in these two designs are similar, particularly when the values of β is very small. When the curved beam is very thin the design is primarily governed by the flexure, except in the vicinities of $M = 0$ where the axial force dominates and the value of A attains a local minimum. This phenomenon can be visualized by examining the expressions of A , Eqs. (17), (26), and (31). In these equations the first term in the bracket is due to flexure

and the second is due to extension. The first term is much larger than the second when β is small and θ is not close to the value where $M = 0$. When β increases, the influence of the axial force becomes more evident, the cross section becomes more uniform, and hence the efficiency of the optimal design compared with the uniform beam decreases.

4. OPTIMAL DESIGN OF CIRCULAR RINGS WITH AN UPPER BOUND ON STRESSES

Unlike the case of an optimal straight beam, the magnitude of extreme fiber stresses in an optimal ring with cross section as shown by Eq. (17) is no longer uniform. Instead, it can be written as

$$\begin{aligned}\sigma_o &\equiv \max_z |\tilde{\sigma}| = \left| \frac{F}{A} \right| + \left| \frac{MT}{2I} \right| \\ &= \frac{2PR\rho f}{W} \left\{ 1 + 2\beta^{-1} |k - \cos \theta| \left[\frac{1}{\beta^2} (k - \cos \theta)^2 + \cos^2 \theta \right]^{-1} \cos \theta \right\}^{-1/2}.\end{aligned}\quad (37)$$

In Fig. 9, σ_o is plotted against θ for $\beta = 0.05$. It is found that σ_o reaches its maximum value at $\theta = \cos^{-1} \frac{k}{1+\beta}$ and $\cos^{-1} \frac{k}{1-\beta}$, and minimum value at $\theta = \cos^{-1} k$ and $\pi/2$. The magnitude of these extreme values increases rapidly as β decreases. Therefore, to make the design more realistic it is necessary to set an upper bound on the fiber stress.

Therefore the problem is to minimize the overall compliance

$$C = \alpha \int e A d\theta \quad (38)$$

under the condition of prescribed weight

$$W = c \int A d\theta \quad (39)$$

where α and c are given constants and e is the strain energy per unit volume. In Eqs. (38) and (39) integrals are taken over the entire beam. Since σ_o is not greater than its upper bound σ_c , we have

$$\sigma_c - \sigma_o = \eta^2(\theta) \quad (40)$$

where $\eta(\theta)$ is an unknown slack variable function. By introducing the Lagrange multipliers λ_1 and $\lambda_2(\theta)$ we have the following variational equation for optimality,

$$\delta \left\{ \alpha \int e A d\theta - \lambda_1 (c \int A d\theta - W) - \int \lambda_2(\theta) [\sigma_c - \sigma_o - \eta^2(\theta)] d\theta \right\} = 0. \quad (41)$$

The variations with respect to λ_1 , λ_2 , η , and A furnish Eqs. (39) and (40),

$$\lambda_2 \eta = 0 \quad (42)$$

as the Euler equations and

$$\int (\alpha e - \lambda_1 c) \delta A d\theta + \delta_A \int \lambda_2 (\sigma_c - \sigma_o - \eta^2) d\theta = 0 \quad (43)$$

In deriving Eq. (43), the dependence of e on A is disregarded because of the principle of minimum potential energy. In view of Eqs. (40),

(42), and (43), we conclude that when $\eta = 0$, we have

$$\sigma_c = \sigma_o \quad (44)$$

and when $\eta \neq 0$, $\lambda_2 = 0$ and Eq. (43) reduces to Eq. (7) as the optimality condition. Therefore, for the optimal design of rings, if the stress bound σ_c is greater than the maximum value of σ_o , i. e.,

$$\sigma_c > (\sigma_o)_{\max} = 2^{3/2} PR\rho f/W \equiv (\sigma_c)_u, \quad (45)$$

then the stress constraint is ineffective and the optimality conditions Eq. (9) governs the design of the entire ring. Hence, $(\sigma_c)_u$ is an upper bound of σ_c below which the stress constraint will influence the design. On the other hand, if σ_c is sufficiently small, we have either the case that the given weight is not large enough for designing a ring satisfying the stress constraint $\sigma_c \geq \sigma_o$ in all sections or the case that the stress constraint governs the entire design of the ring. In the latter case, $\sigma_o = \sigma_c$ and

$$A = \frac{P}{2\sigma_c} \left[\frac{1}{\beta} |k - \cos \theta| + \cos \theta \right]. \quad (46)$$

From the condition of the given weight, we find that

$$\sigma_c = \frac{2PR\rho}{W} \int_0^{\pi/2} \left[\frac{1}{\beta} |k - \cos \theta| + \cos \theta \right] d\theta = (\sigma_c)_t. \quad (47)$$

The equation for k can be obtained from Eq. (23) with a modification.

It is

$$\int_0^{\pi/2} (k - \cos \theta) \left[\frac{1}{\beta} |k - \cos \theta| + \cos \theta \right]^{-1} d\theta = 0 \quad (48)$$

When $\sigma_c = (\sigma_c)_l$, the design is entirely governed by the stress constraint. When $\sigma_c < (\sigma_c)_l$, the given weight of the beam is not sufficient to fulfill the condition of stress constraint. Finally, when $\sigma_c > (\sigma_c)_l$, the stress constraint does not govern the design of the whole ring and the optimality condition, Eq. (9), has to be taken into consideration. Therefore, $(\sigma_c)_l$ is a lower bound above which both Eqs. (9) and (44) have to be used to determine the optimal design. In Fig. 10, the dimensionless $(\sigma_c)_u$ and $(\sigma_c)_l$ are plotted against β . It is found that the values of $(\sigma_c)_u$ and $(\sigma_c)_l$ increase rapidly as β decreases.

In the following, we are interested only in the problem with σ_c in the range $(\sigma_c)_l < \sigma_c < (\sigma_c)_u$. In this case, each of Eqs. (9) and (44) governs the design in some intervals. In view of Fig. 10, it is reasonable to assume that $\sigma_o = \sigma_c$ in two intervals $\theta_1 \leq \theta \leq \theta_2$ and $\theta_3 \leq \theta \leq \theta_4$ where $\theta_3 \geq \theta_2$, and Eq. (9) holds in the remaining intervals of $0 \leq \theta \leq \frac{\pi}{2}$. Accordingly, the design variable A will take the form of Eq. (46) in the former intervals and the form of Eq. (14) in the remaining intervals. The total weight of the cover sheets is thus

$$W = 4\rho R \left\{ \left(\int_0^{\theta_1} + \int_{\theta_2}^{\theta_3} + \int_{\theta_4}^{\pi/2} \right) \frac{PR}{2E} c^{-1/2} \left[\frac{1}{\beta^2} (k - \cos \theta)^2 + \cos^2 \theta \right]^{1/2} d\theta \right.$$

$$+ \left(\int_{\theta_1}^{\theta_2} + \int_{\theta_3}^{\theta_4} \right) \frac{P}{2\sigma_c} \left[\frac{1}{\beta} |k - \cos \theta| + \cos \theta \right] d\theta \} . \quad (49)$$

Integrating Eqs. (12) and (13) and using the boundary conditions

Eq. (20) and the continuity conditions

$$w(\theta_i^-) = w(\theta_i^+), \quad v(\theta_i^-) = v(\theta_i^+) \quad \text{and} \quad \dot{w}(\theta_i^-) = \dot{w}(\theta_i^+),$$

$$i = 1, 2, 3, 4, \quad (50)$$

we finally obtain the deflection at $\theta = \frac{\pi}{2}$ as

$$\begin{aligned} \delta_0 = & - \left\{ c^{1/2} \left(\int_0^{\theta_1} + \int_{\theta_2}^{\theta_3} + \int_{\theta_4}^{\pi/2} \right) \left[\frac{1}{\beta^2} (k - \cos \theta) - \cos \theta \right] \left[\frac{1}{\beta^2} (k - \cos \theta)^2 \right. \right. \\ & \left. \left. + \cos^2 \theta \right]^{-1/2} \cos \theta d\theta \right. \\ & + \frac{\sigma_c R}{E} \left(\int_{\theta_1}^{\theta_2} + \int_{\theta_3}^{\theta_4} \right) \left[\frac{1}{\beta^2} (k - \cos \theta) - \cos \theta \right] \left[\frac{1}{\beta} |k - \cos \theta| \right. \\ & \left. \left. + \cos \theta \right]^{-1} \cos \theta d\theta \right\}, \quad (51) \end{aligned}$$

and

$$\begin{aligned} c^{1/2} \left(\int_0^{\theta_1} + \int_{\theta_2}^{\theta_3} + \int_{\theta_4}^{\pi/2} \right) \frac{1}{\beta^2} (k - \cos \theta) \left[\frac{1}{\beta^2} (k - \cos \theta)^2 \right. \\ \left. + \cos^2 \theta \right]^{-1/2} d\theta \\ + \frac{\sigma_c R}{E} \left(\int_{\theta_1}^{\theta_2} + \int_{\theta_3}^{\theta_4} \right) \frac{1}{\beta^2} (k - \cos \theta) \left[\frac{1}{\beta} |k - \cos \theta| + \cos \theta \right]^{-1} d\theta = 0. \end{aligned} \quad (52)$$

The unknowns c and k can be solved from Eqs. (49) and (52). However, θ_1 , θ_2 , θ_3 , and θ_4 are still undetermined. If these values are prescribed, the constants c and k can be found from Eqs. (49) and (52). Hence the design is determined by Eq. (46) in the intervals $\theta_1 \leq \theta \leq \theta_2$ and $\theta_3 \leq \theta \leq \theta_4$, and by Eq. (14) in the remaining intervals $0 \leq \theta \leq \frac{\pi}{2}$. The design of this type is optimal among all designs with $\sigma_0 = \sigma_c$ in the intervals $\theta_1 \leq \theta \leq \theta_2$ and $\theta_3 \leq \theta \leq \theta_4$. Note that in this design the value of A is not necessarily continuous at $\theta = \theta_1$, θ_2 , θ_3 , θ_4 . For the optimal design, however, the values of θ_1 to θ_4 should make the right-hand side of Eq. (51) a minimum with respect to all admissible values. This criterion finally leads to a condition that A must be continuous at $\theta = \theta_1$, θ_2 , θ_3 , and θ_4 [see Appendix]. Hence, we have

$$\frac{PR}{2E} c^{-1/2} \left[\frac{1}{\beta^2} (k - \cos \theta_i)^2 + \cos^2 \theta_i \right]^{-1/2} = \frac{P}{2\sigma_c} \left[\frac{1}{\beta} |k \cos \theta_i| + \cos \theta_i \right],$$

$$i = 1, 2, 3, 4 \quad (53)$$

Equations (49), (52), and (53) can be used to determine c , k , and θ_i ($i = 1, 2, 3, 4$).

A slightly modified Newton-Raphson iteration method is used successfully in solving these equations. The broken line in Fig. 3 shows the value of A with the stress constraint $\sigma_c = 0.85 (\sigma_c)_u$ for $\beta = 0.05$. It is found that the deviation of A from the corresponding

value in the design without the stress constraint is very small. However, the difference becomes much larger when σ_c approaches the value $(\sigma_c)_l \approx 0.78 (\sigma_c)_u$. The extreme fiber stress σ_o in the optimal beam is shown by the broken line in Fig. 9. It is seen that the considerable reduction of σ_o in $\theta_1 \leq \theta \leq \theta_2$ and $\theta_3 \leq \theta \leq \theta_4$ does not lead to a significant increase of σ_o in the remaining regions.

When β is sufficiently small, say less than 0.1, the distance $\Delta\theta$ between two maxima of σ_o in the design without stress constraint, is approximately equal to 2β (Fig. 9). In the design with stress constraint, the interval between two regions governed by the stress constraint is much smaller than $\Delta\theta$ as shown by the broken line. Therefore, if the value of β is very small, it is reasonable to approximate the design by assuming that there is only one interval, say $\theta_1 \leq \theta \leq \theta_4$, governed by the stress constraint.

As we have pointed out, the extreme fiber stress becomes infinity at the point of zero moment in the optimal inextensible beam. Therefore, to impose the stress bound constraint, one may assume that there is only one continuous interval, in the neighborhood of zero moment, governed by the stress constraint. However, this assumption would lead to a design somewhat equivalent to that of the one-interval-approximation for the extensible beam discussed in the last paragraph. To obtain more precise results, one has to assume that there are two separated

intervals governed by the stress constraint even for the inextensible beam. This is due to the fact that the variation of the extreme fiber stress σ_o in the inextensible beam is similar to the solid curve in Fig. 9 for any nonzero value of A in the vicinity of zero moment.

5. CONCLUDING REMARKS

From the above investigation, we have the following conclusions:

- (1) A more realistic optimal design can be achieved by considering the extensibility of the beam. When the thickness-diameter ratio β of the beam is very small, the influence of extensibility to the design is restricted to the vicinity of minimum moment.
- (2) The efficiency of optimal design defined by comparing its compliance with that of a uniform beam with the same weight depends on β . Smaller value of β leads to a higher efficiency.
- (3) The stress bound condition would influence the design only if the value of allowable stress is within a certain range. For large value of allowable stress, the design is entirely governed by the optimality condition. For small value of allowable stress, the prescribed weight of the beam is not large enough to ensure the satisfaction of the stress bound condition everywhere.

- (4) From the stress distribution in the ring, designed without the stress bound condition, we may find approximately the regions where material must be added to satisfy the stress bound condition. However, in the regions close to the points of application of the load, material can be deducted to fulfill the optimality condition.
- (5) By imposing the condition of stress bound, the stress in the regions of increased cross section can be reduced considerably. However, the stress in the remaining regions of the beam only increases slightly.

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APPENDIX

A necessary condition that the deflection δ given by Eq. (51) has a minimum with respect to θ_i ($i = 1, 2, 3, 4$) is

$$\frac{\partial \delta_o}{\partial \theta_i} = \left(\frac{\partial \delta_o}{\partial \theta_i} \right)_{c,k} + \frac{\partial \delta_o}{\partial k} \frac{\partial k}{\partial \theta_i} + \frac{\partial \delta_o}{\partial c} \frac{\partial c}{\partial \theta_i} = 0, \quad i = 1, 2, 3, 4 \quad (54)$$

where the subscripts c and k indicate the variable held constant in partial differentiations, and $\frac{\partial k}{\partial \theta_i}$, $\frac{\partial c}{\partial \theta_i}$ can be found from differentiating Eqs. (49) and (52) with respect to θ_i . After substitution, Eq. (54) yields

$$\begin{aligned} & \left\{ e \left[\frac{1}{\beta^2} (k - \cos \theta_i)^2 + \cos^2 \theta_i \right]^{1/2} - \left[\frac{1}{\beta} |k - \cos \theta_i| + \cos \theta_i \right] \right\} \\ & \times \left\{ \cos \theta_i \left[\frac{1}{\beta^2} (k - \cos \theta_i) - \cos \theta_i \right] \left[e a_1^2 + \frac{1}{\beta} (\theta_1 - \theta_2 - \theta_3 + \theta_4) a_1 \right. \right. \\ & \quad \left. \left. + e a_2 (a_3 + e b_1) \right] \right. \\ & + \left[\frac{1}{\beta^2} (k - \cos \theta_i)^2 + \cos^2 \theta_i \right]^{1/2} \left[\frac{1}{\beta} |k - \cos \theta_i| + \cos \theta_i \right] \left[-a_4 (a_3 \right. \\ & \quad \left. + e b_1) + k a_1 a_3 + e a_1 b_2 \right] \\ & - \frac{1}{\beta^2} (k - \cos \theta_i) \left[e k a_2 a_3 + e^2 a_2 b_2 + e a_4 a_1 \right. \\ & \quad \left. + \frac{1}{\beta} (\theta_1 - \theta_2 - \theta_3 + \theta_4) a_4 \right] \Big\} = 0 \quad (55) \end{aligned}$$

where the following notations are used

$$e = R\sigma_c / (Ec^{1/2}) ,$$

$$a_1 = \left(\int_0^{\theta_1} + \int_{\theta_2}^{\theta_3} + \int_{\theta_4}^{\pi/2} \right) \frac{1}{\beta^2} (k - \cos \theta) \left[\frac{1}{\beta^2} (k - \cos \theta)^2 + \cos^2 \theta \right]^{-1/2} d\theta ,$$

$$a_2 = \left(\int_0^{\theta_1} + \int_{\theta_2}^{\theta_3} + \int_{\theta_4}^{\pi/2} \right) \left[\frac{1}{\beta^2} (k - \cos \theta)^2 + \cos^2 \theta \right]^{1/2} d\theta ,$$

$$a_3 = \left(\int_0^{\theta_1} + \int_{\theta_2}^{\theta_3} + \int_{\theta_4}^{\pi/2} \right) \frac{1}{\beta^2} \cos^2 \theta \left[\frac{1}{\beta^2} (k - \cos \theta)^2 + \cos^2 \theta \right]^{-3/2} d\theta ,$$

(56)

$$a_4 = \left(\int_0^{\theta_1} + \int_{\theta_2}^{\theta_3} + \int_{\theta_4}^{\pi/2} \right) \left[\frac{1}{\beta^2} (k - \cos \theta) - \cos \theta \right] \left[\frac{1}{\beta^2} (k - \cos \theta)^2 + \cos^2 \theta \right]^{-1/2} \cos \theta d\theta ,$$

$$b_1 = \left(\int_{\theta_1}^{\theta_2} + \int_{\theta_3}^{\theta_4} \right) \frac{1}{\beta^2} \cos \theta \left[\frac{1}{\beta} |k - \cos \theta| + \cos \theta \right]^{-2} d\theta ,$$

$$b_2 = \int_{\theta_1}^{\theta_2} \frac{1}{\beta^2} (1 - \beta) \cos^2 \theta \left[\frac{1}{\beta} |k - \cos \theta| + \cos \theta \right]^{-2} d\theta \\ + \int_{\theta_3}^{\theta_4} \frac{1}{\beta^2} (1 + \beta) \cos^2 \theta \left[\frac{1}{\beta} |k - \cos \theta| + \cos \theta \right]^{-2} d\theta ,$$

and $\theta_2 < \cos^{-1} k < \theta_3$ is assumed. The second brace of Eq. (55)

can be easily shown to be nonvanishing in the limiting

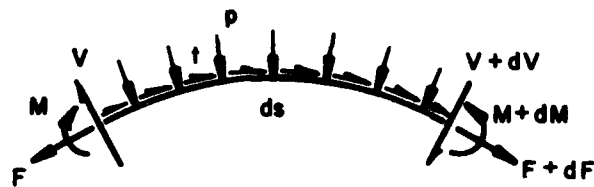
case $\sigma_c \rightarrow (\sigma_c)_u$. In this case, $\theta_1 = \theta_2 = \cos^{-1} \frac{k}{1 - \beta}$,

$\theta_3 = \theta_4 = \cos^{-1} \frac{k}{1 + \beta}$, $a_1 = b_1 = b_2 = 0$, $e = (2)^{1/2}$, and the second

brace in Eq. (55) becomes

$$- 4(2)^{1/2} \left(\frac{k}{1-\beta} \right)^2 \tilde{a}_2 \tilde{a}_3 \quad (57)$$

where $\tilde{a}_2$ and $\tilde{a}_3$ are the limiting values of a_2 and a_3 respectively, which are obviously nonvanishing. Therefore if we set the second brace in Eq. (55) zero, we shall not obtain the optimal design. Otherwise, in the limiting case, the expression (57) would vanish. Hence we conclude that the first brace in Eq. (55) must vanish and A is continuous at $\theta = \theta_i$ ($i = 1, 2, 3, 4$) for the optimal design.



F : Axial tension

M : Bending moment

V : Transverse shear

Fig. 1. Forces and moments on an infinitesimal arc of the beam.

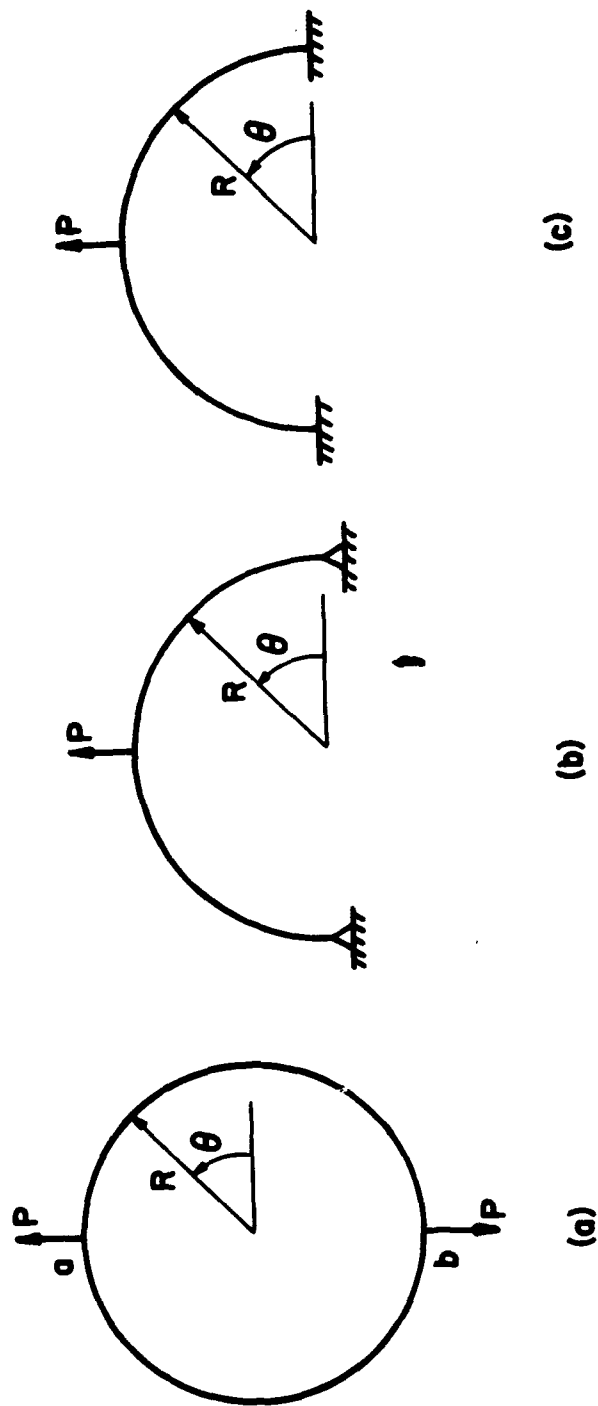


Fig. 2. Geometry of Circular Ring and Semicircular Arches.

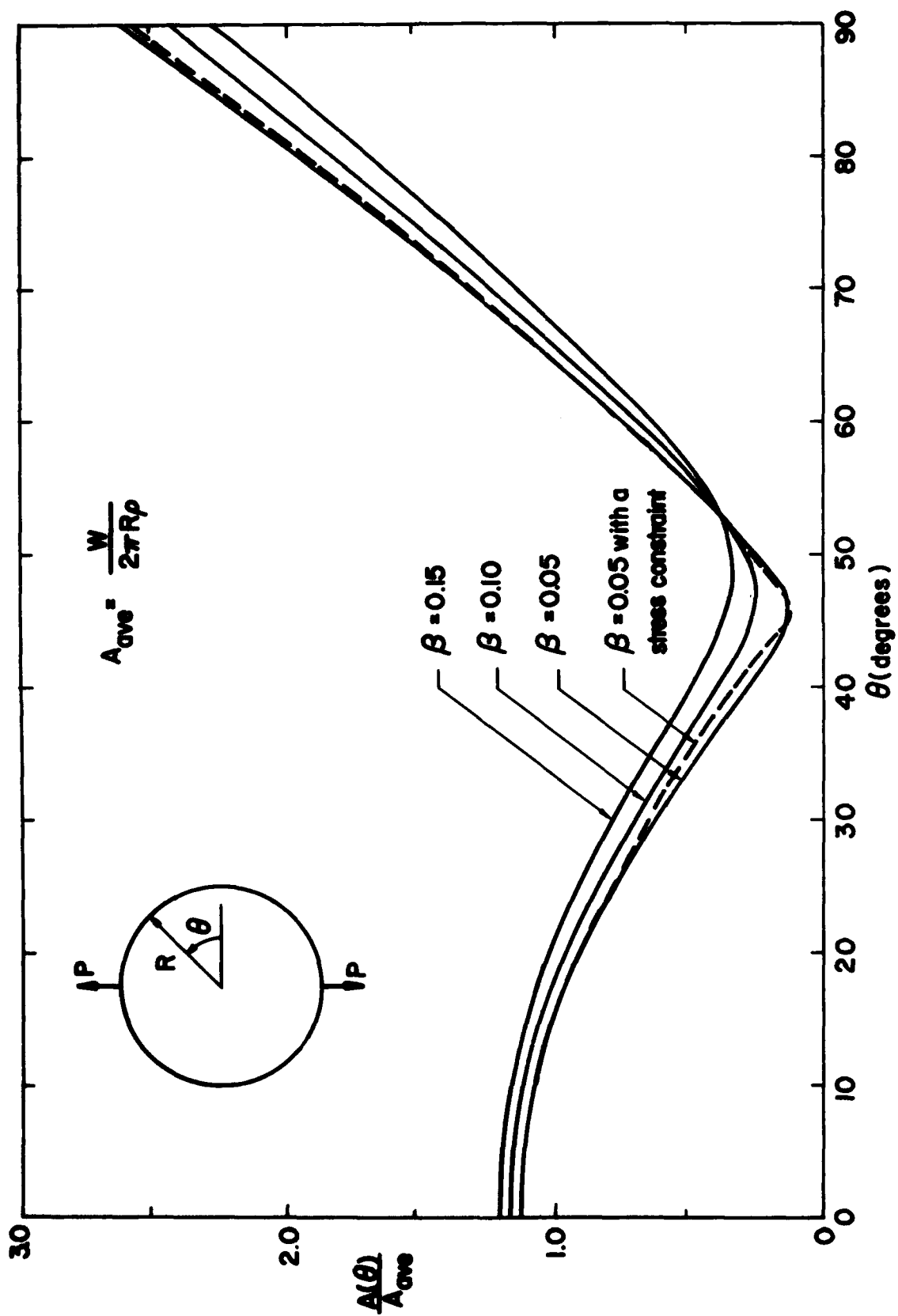


Fig. 3. $A(\theta)/A_{ave}$ Curves for Optimal Circular Rings.

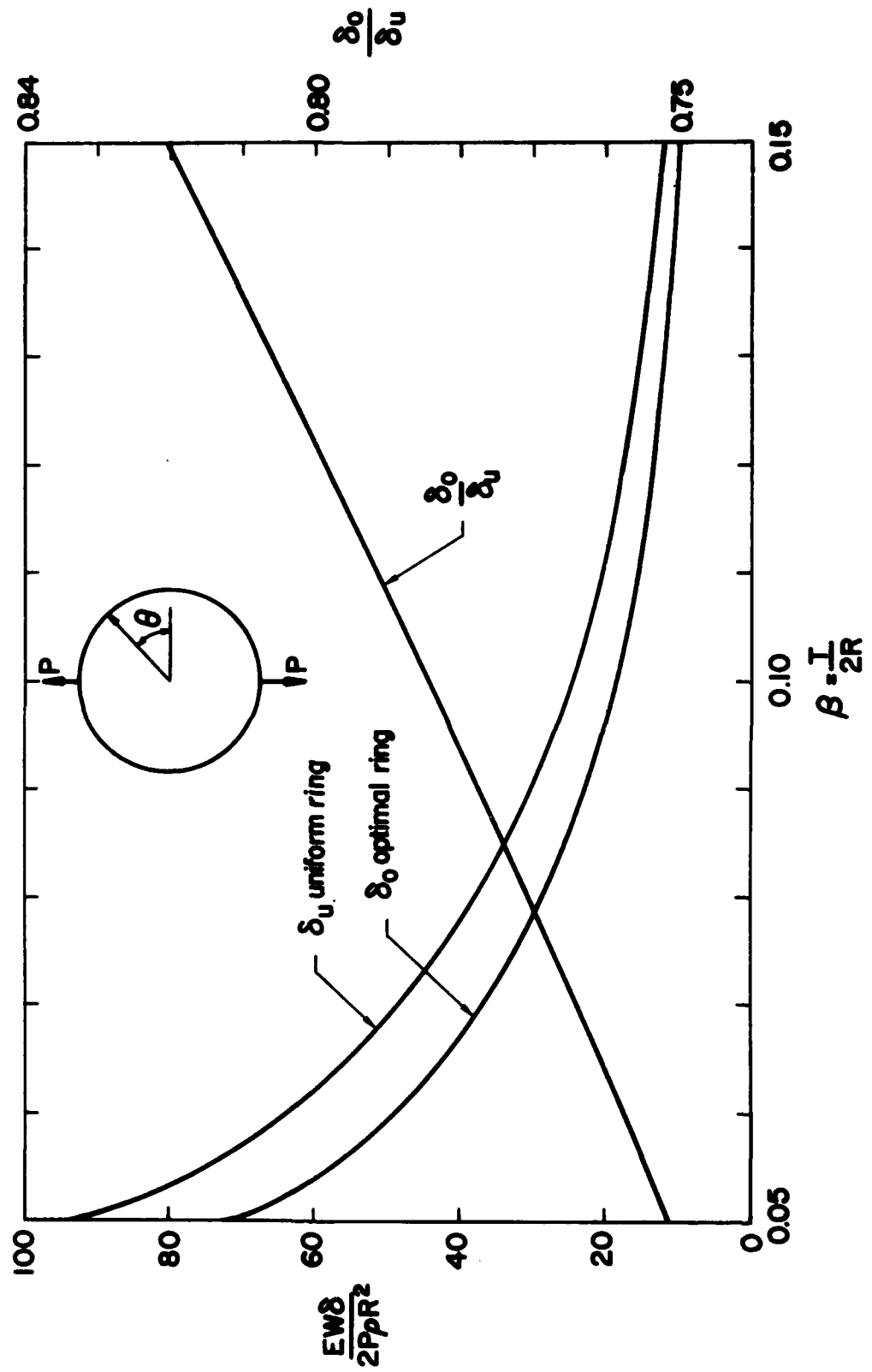


Fig. 4. Comparison of Optimal Rings with Rings of Uniform Cross Section.

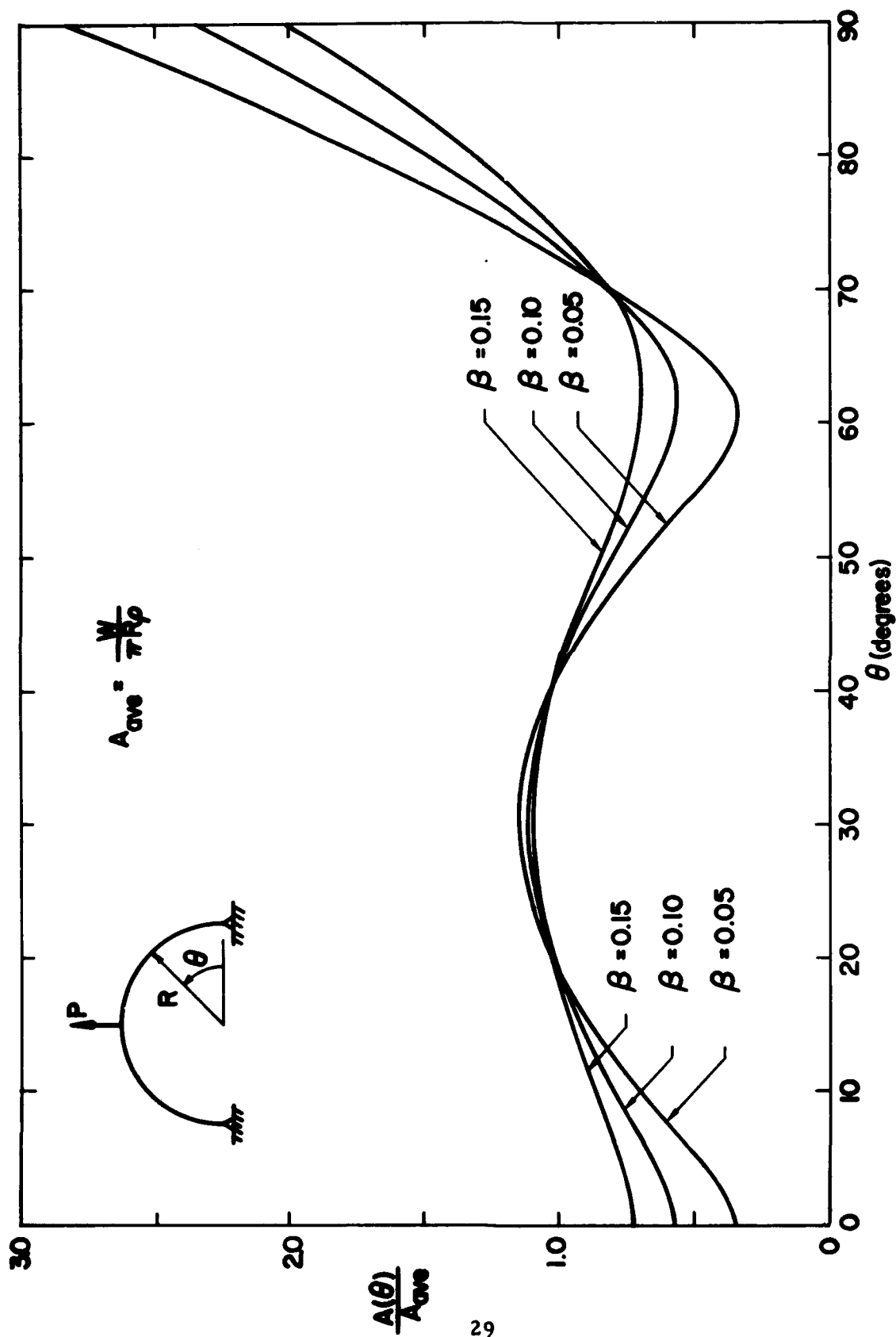


Fig. 5. $A(\theta)/A_{ave}$ Curves for Optimal Two-hinged Semicircular Arches.

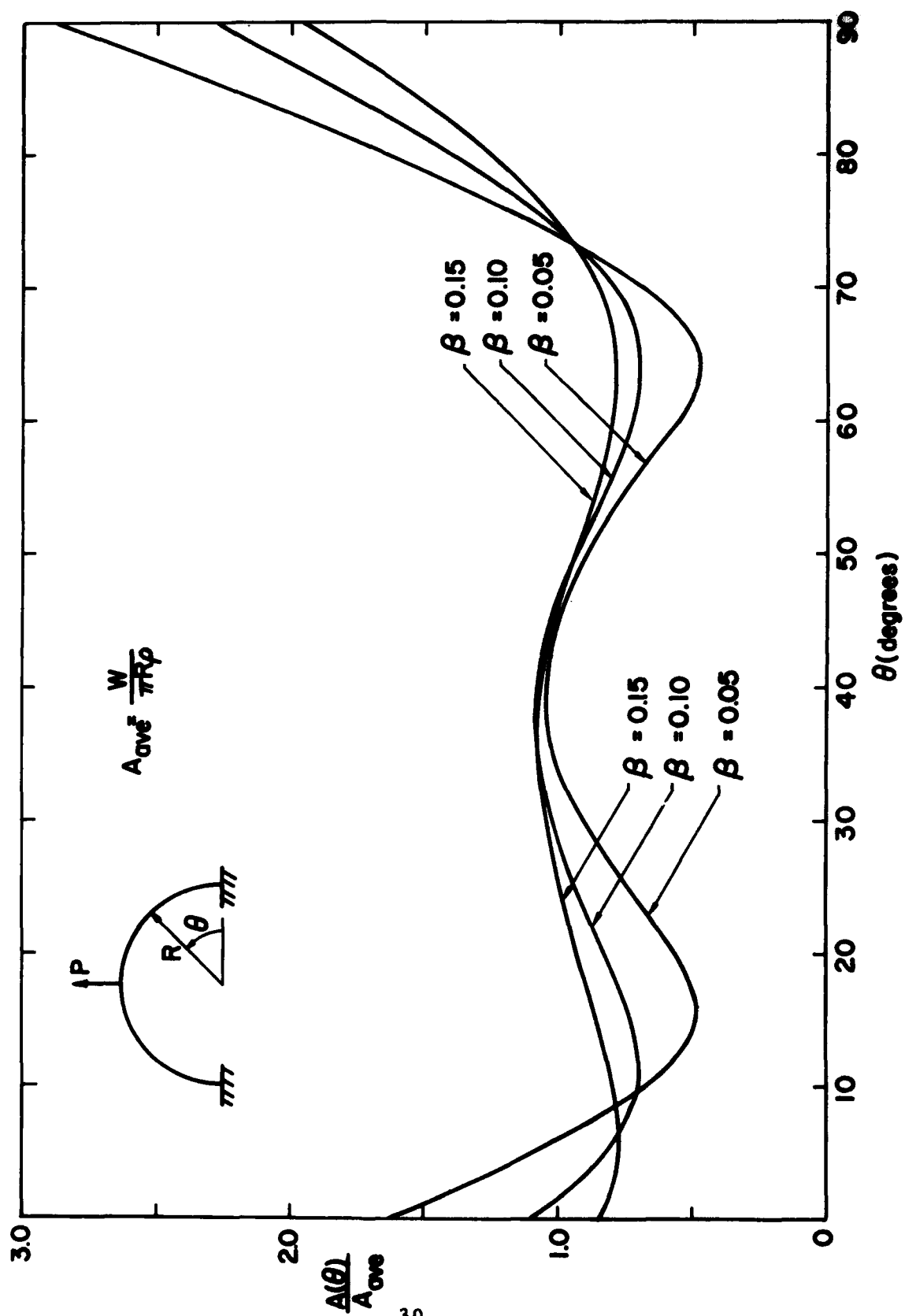


Fig. 6. $A(\theta)/A_{ave}$ Curves for Optimal Clamped Semicircular Arches.

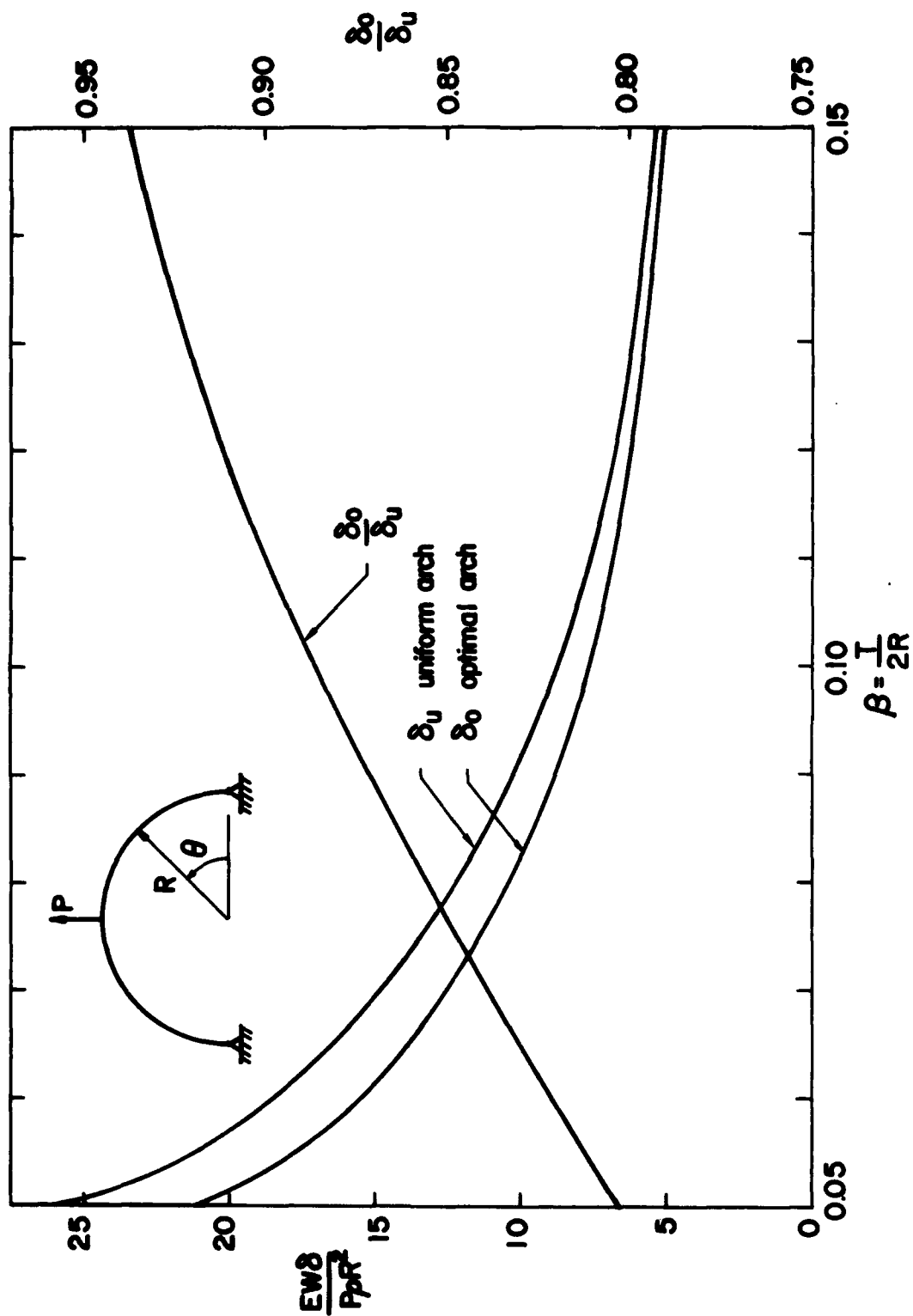


Fig. 7. Comparison of Optimal Two-hinged Arches with Arches of Uniform Cross Section.

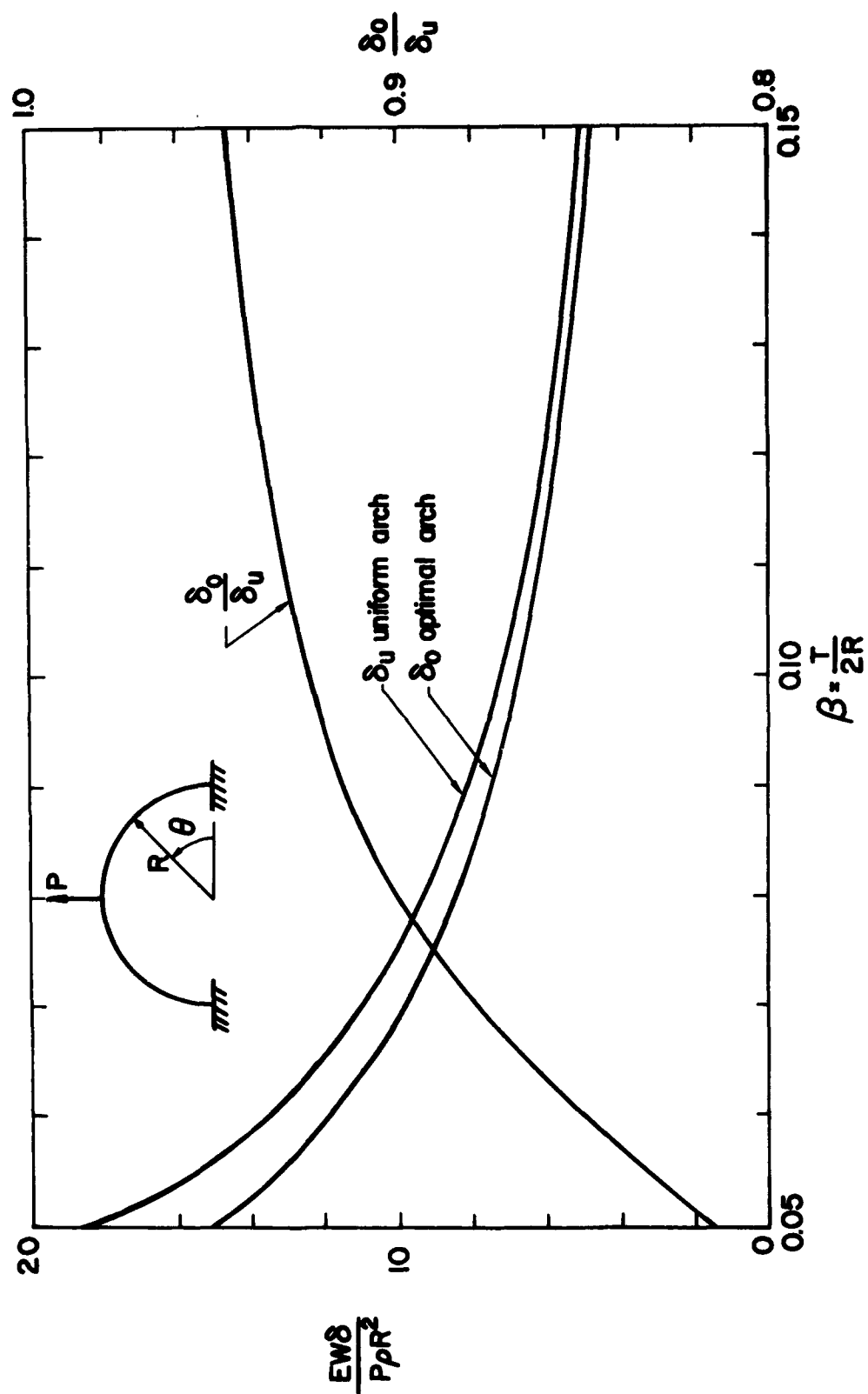


Fig. 8. Comparison of Optimal Clamped Arches with Arches of Uniform Cross Section.

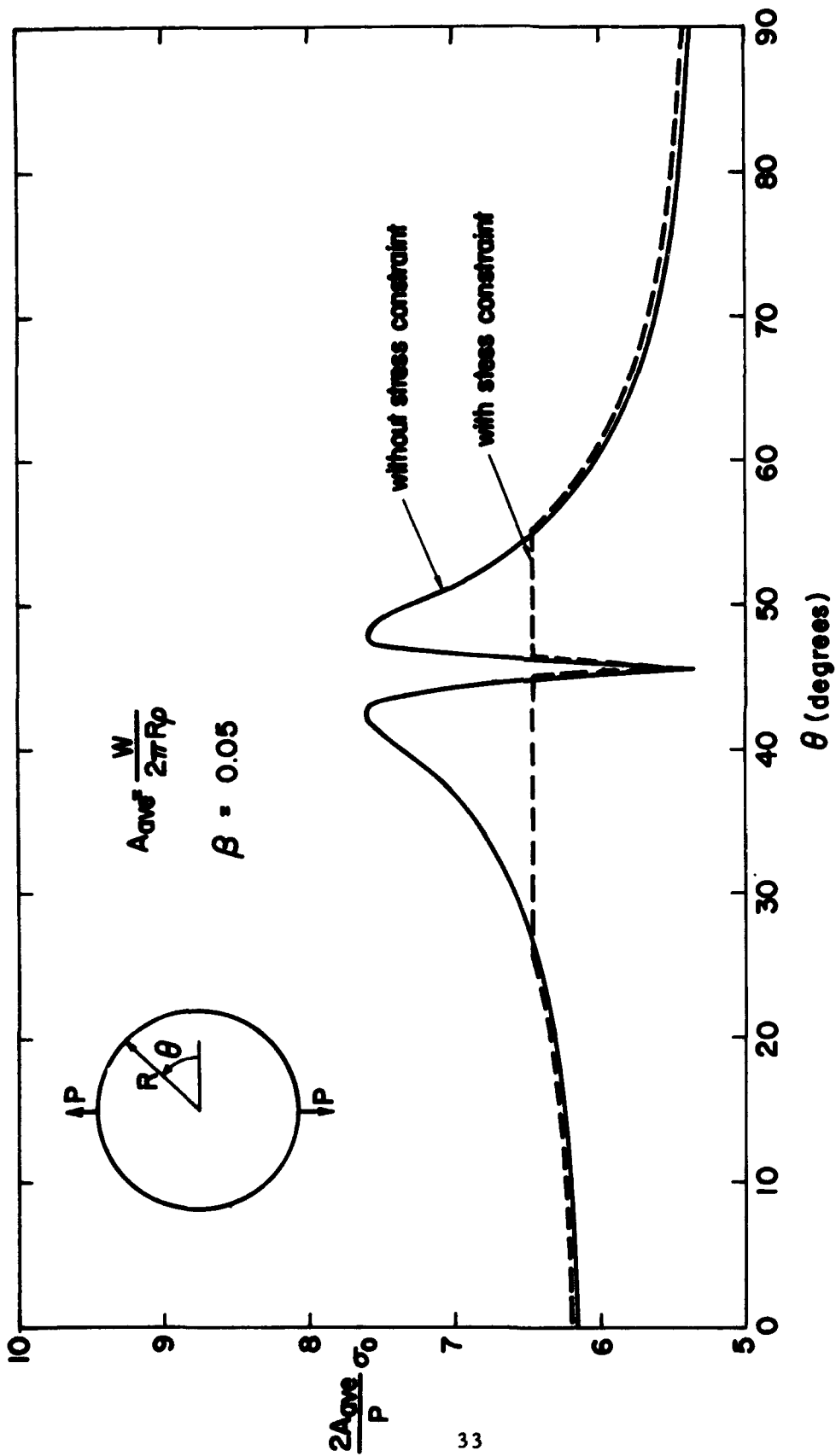


Fig. 9. Extreme fiber stress in the Optimal Rings with and without Stress Constraint.

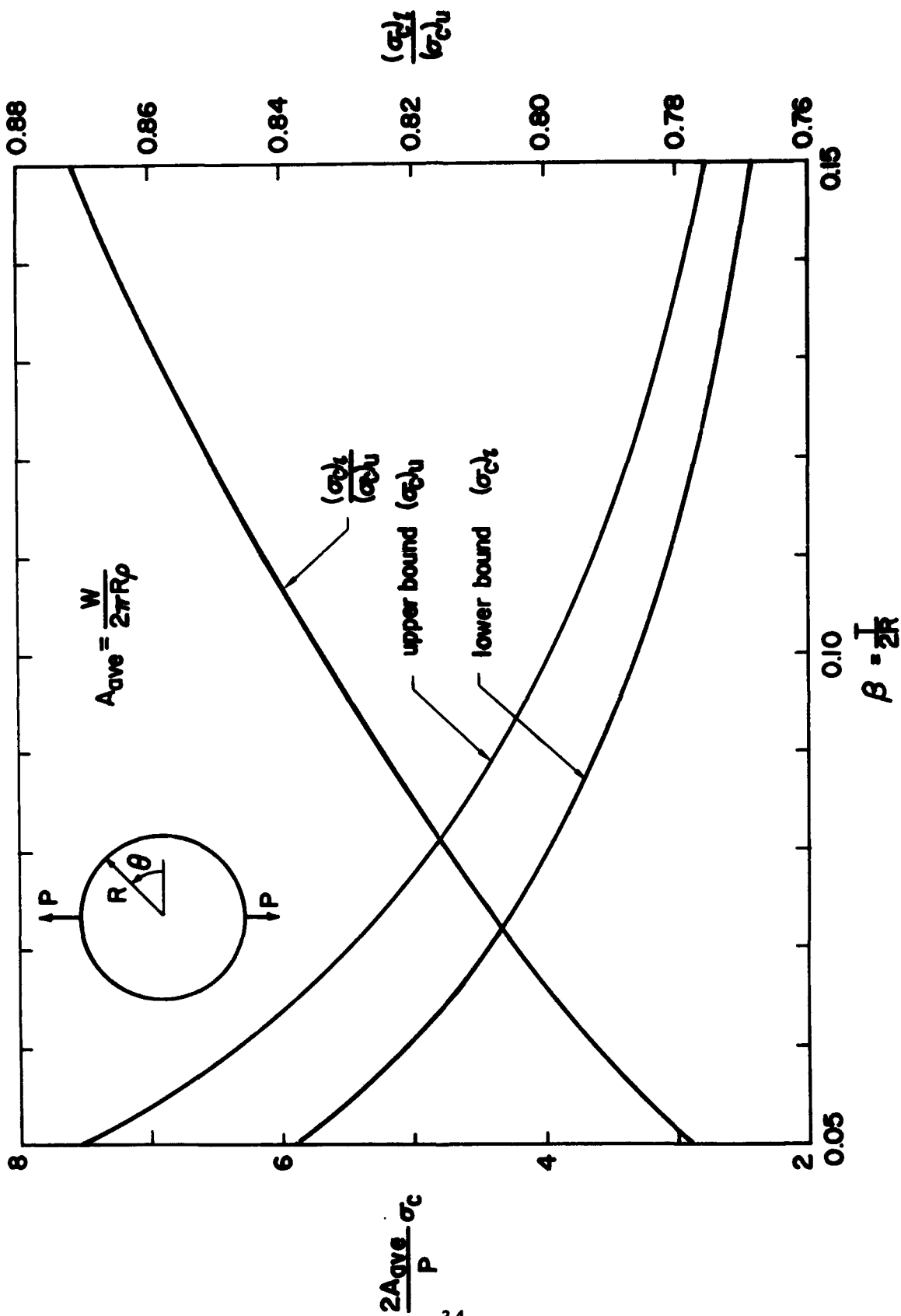


Fig. 10. Upper and Lower Bound of $\sigma_c / \left(\frac{P}{2A_{gve}} \right)$.

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