Acceleration Waves in Elastic-Plastic Materials

by

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Abstract. Acceleration waves in elastic-plastic materials are studied in some detail on the basis of a nonlinear thermodynamical theory of elastic-plastic continua. Attention is confined mainly to non-conducting media, but the developments are, otherwise, general. Formulae for wave speeds are derived, fronts of plastic loading and elastic unloading are discusse and higher order discontinuities are shown to have the same characteristic speeds as those of acceleration waves. An example concerning propagation of plastic waves in a medium undergoing uni-axial motion is included.

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1. Introduction

This paper is concerned with propagation of acceleration waves in elastic-plastic materials and is based on the thermodynamical theory of such media developed by Green and Naghdi [1,2]. Attention is confined mainly to a non-conducting medium; but, otherwise, the developments are general and are carried out in the context of the nonlinear theory.

Propagation of acceleration waves in elastic materials has had a long history. A recent general development on the subject by Truesdell [3] includes historical remarks and earlier references. Some aspects of acceleration waves in nonlinear viscoelasticity, based on a mechanical theory, has been discussed by Varley [4] while Coleman and Gurtin [5] have studied acceleration waves, according to a thermodynamical theory of a class of simple materials with fading memory.[†] Among the previous works on acceleration waves in elastic-plastic media, we cite references [6-12]. However, these are confined only to mechanical aspects of the subject and either are based on a linearized theory with small deformations or if carried out in the context of the nonlinear theory employ special constitutive equations.

In the present paper, after some preliminaries in section 2, we briefly discuss the constitutive equations for elastic-plastic media and deduce some related results in section 3. Here, we recall the constitutive equations in the form in which the entropy (rather than the temperature)

[†] The results in [5] do not overlap or include those of the present paper; the thermodynamical theory on which the analysis in [5] is based and the thermodynamical theory in [1] are mutually exclusive.

is an independent thermodynamic variable and then obtain an expression for the rate of production of entropy in a non-conducting medium. It is a known result in nonlinear elasticity that in a non-conducting elastic material (and in the absence of heat supply) the time rate of entropy vanishes and the so-called acoustic tensor is homentropic. In general, as shown in sections 3 and 4, such is not the case for an elastic-plastic material.

In section 4, waves are represented as propagating surfaces in a non-conducting elastic-plastic continuum, across which there exist jumps in the values of certain mechanical and thermal variables. Acceleration waves, or singular surfaces with respect to particle acceleration, are discussed in detail and formulae for wave speeds are derived using the compatibility conditions of Hadamard [13] for jumps in piecewise continuous functions of position and time. Waves propagating through regions undergoing plastic deformation, sufficient conditions for the existence of real wave speeds, and fronts of plastic loading and elastic unloading are treated in sections 4-6. Singular surfaces with respect to any order derivatives (in time, position or mixed) of acceleration are shown in section 7 to have the same characteristic speeds as those of the acceleration discontinuities. Finally, in section 8, we consider the simple example of propagation of plastic waves in a medium undergoing uni-axial motion. Other examples, such as plastic waves in simple shear or propagation of spherically symmetric plastic waves, can be discussed in a similar fashion.

The above developments for a non-conducting medium in sections 3-7 are valid for a work-hardening elastic-plastic material which is initially anisotropic and are also applicable to the limiting case of an elastic-perfectly plastic material. Of special interest is the simple formula that the jump in the entropy production is linear in the jump of plastic strain rate. This and other results in section 4 hold also for a definite conductor, i.e., for a medium whose heat conduction vector has the form of Fourier's law.

2. Notation and Preliminaries

Let the motion of the continuum be referred to a fixed system of rectangular Cartesian axes and denote the position of a typical particle at time t by+

$$x_i = x_i(X_A, t) \quad , \tag{2.1}$$

where X_A is a reference position of the particle. We require the mapping (2.1) to be single-valued and have continuous partial derivatives with respect to its arguments, except at some singular points, curves and surfaces. We use the notation $F = F(t)$ and designate partial differentiation with respect to X_A or x_i as $(\ )_{,A}$ or $(\ )_{,i}$, respectively. Latin indices take values 1,2,3 and, except when noted otherwise, the usual summation convention for Cartesian tensors will be employed.

The components of velocity and acceleration at the point x_i at time t are, respectively,

$$\dot{x}_i = \frac{\partial}{\partial t} x_i(X_A, t) \quad , \quad \ddot{x}_i = \frac{\partial}{\partial t} \dot{x}_i(X_A, t) \quad , \tag{2.2}$$

where a superposed dot stands for differentiation with respect to t, holding X_A fixed. We define the strain tensor e_{KL} by

+We use the same symbol for a function and its value without confusion.

$$e_{KL} = \frac{1}{2} (x_{k,K}\, x_{k,L} - \delta_{KL}) \quad , \tag{2.3}$$

and note that its material time derivative is

$$\dot{e}_{KL} = d_{k\ell}\, x_{k,K}\, x_{\ell,L}$$

$$= \frac{1}{2} (x_{m,K}\, \dot{x}_{m,L} + \dot{x}_{m,K}\, x_{m,L}) \quad , \tag{2.4}$$

where δ_{KL} is the Kronecker symbol and

$$d_{k\ell} = \frac{1}{2} (\dot{x}_{k,\ell} + \dot{x}_{\ell,k}) \quad , \tag{2.5}$$

is the rate of deformation tensor.

In terms of the non-symmetric Piola-Kirchhoff stress tensor π_{Kk}, the equations of motion may be written in the form*

$$\pi_{Kk,K} + \rho_0 F_k = \rho_0 \ddot{x}_k \quad , \tag{2.6}$$

$$\pi_{Ki} = x_{i,L}\, s_{LK} \quad , \quad s_{KL} = s_{LK} \quad , \tag{2.7}$$

where ρ_0 is the mass density of the reference configuration, F_k is the externally applied body force per unit mass and s_{KL} is the symmetric Piola-Kirchhoff stress tensor. Let h_0 be the flux of heat across a surface of the continuum at time t and let Q_K be the corresponding

*The notation here is the same as that used by Green and Naghdi [1, Sec. 2].

heat flux vector, both measured per unit time and per unit area in the reference configuration. Then, the (local) energy equation and the (local) Clausius-Duhem inequality are

$$\rho_0 r - Q_{K,K} - \rho_0 \dot{U} + s_{AB} \dot{e}_{AB} = 0 \quad , \tag{2.8}$$

and

$$\rho_0 T \dot{S} - \rho_0 r + Q_{K,K} - \frac{Q_K T_{,K}}{T} \geq 0 \quad , \tag{2.9}$$

where r is the heat supply function per unit mass and per unit time, U is the internal energy per unit mass, S is the entropy per unit mass and $T(> 0)$ is the temperature.

The kinematical quantity e_{KL} is unaltered when the continuum is subjected to superposed rigid body motions at all times $t^* = (t + a)$, where a is a constant. In what follows we shall be concerned with constitutive equations which mainly involve stress, stress rate, displacement gradient, velocity gradient and such equations must remain unaltered by superposed rigid body motions. As noted in [1], the displacement gradient and the velocity gradient must be replaced by e_{KL} and $\dot{e}_{KL}$ and these will be referred to as the strain tensor and the strain rate tensor. Also the stress and the stress rate may be taken to be s_{KL} and $\dot{s}_{KL}$, both being invariant under superposed rigid body motions.

3. Constitutive Equations and Some Related Results

We summarize here the principal results from the theory of elastic-plastic continua as developed by Green and Naghdi [1,2] and also obtain some related results for later use. We limit the discussion to the material description of the theory and, for convenience, regard the entropy S (rather than the temperature) as an independent thermodynamic variable.

Let the strain tensor e_{KL} be defined at each point of the continuum by (2.3) and let e''_{KL}, a symmetric tensor with the same invariance properties as e_{KL}, denote plastic strain.[++] We introduce a constitutive assumption for s_{KL} in the form

$$s_{KL} = s_{KL}(e_{KL}, e''_{KL}, S) \quad , \tag{3.1}$$

and admit the existence of a scalar-valued continuously differentiable function $f(s_{KL}, e''_{KL}, S)$ -- called a yield or a loading function -- such that the equation

$$f(s_{KL}, e''_{KL}, S) = \kappa \quad , \tag{3.2}$$

for a fixed value of κ and e''_{KL} represents a hypersurface in seven-dimensional Euclidean space -- six components of s_{KL} and the entropy S. The scalar κ, a work-hardening parameter, depends on the past

[++]Although we assume that e''_{KL} is symmetric, the developments which follow can be readily modified to accommodate a non-symmetric plastic strain tensor included in [2,14].

history of the motion and is assumed to be initially positive. The rate of change of κ and the plastic strain rate $\dot{e}''_{KL}$ are independent of the particular time scale used to calculate the rate of change. Moreover, $\dot{\kappa}$ is a linear function of $\dot{s}_{KL}$, $\dot{e}''_{KL}$ and $\dot{S}$; $\dot{e}''_{KL}$ is a linear function of $\dot{s}_{KL}$ and $\dot{S}$; and $\dot{\kappa}$ and $\dot{e}''_{KL}$ must satisfy the requirements

$$\left.\begin{aligned} \dot{e}''_{KL} &= 0 \quad \text{when} \quad f < \kappa \quad \text{with} \quad \dot{\kappa} = 0 \quad , \\ \dot{e}''_{KL} &= 0 \quad \text{when} \quad f = \kappa \quad \text{with} \quad \dot{\kappa} = 0\ ,\ \hat{f} < 0 \quad , \\ \dot{e}''_{KL} &= 0 \quad \text{when} \quad f = \kappa \quad \text{with} \quad \dot{\kappa} = 0\ ,\ \hat{f} = 0 \quad , \\ \dot{e}''_{KL} &\neq 0 \quad \text{when} \quad f = \kappa \quad \text{and} \quad \hat{f} > 0 \quad , \end{aligned}\right\} \tag{3.3}$$

and

$$\dot{\kappa} = 0 \quad \text{when} \quad \dot{e}''_{KL} = 0 \quad , \tag{3.4}$$

where

$$\hat{f} = \frac{\partial f}{\partial s_{KL}} \dot{s}_{KL} + \frac{\partial f}{\partial S} \dot{S} \quad , \tag{3.5}$$

and where the partial derivative $\frac{\partial f}{\partial s_{KL}}$ stands for the symmetric form $\frac{1}{2}\left(\frac{\partial f}{\partial s_{KL}} + \frac{\partial f}{\partial s_{LK}}\right)$. Using the conventional terminology, the four conditions in (3.3) in the order listed correspond to an elastic state, unloading from an elastic-plastic state, neutral loading, and loading from an elastic-plastic state. Supplementary to the above, we need constitutive postulates for the internal energy U, the temperature T and the heat flux vector Q_K. Thus, we introduce*

$$U = U(e_{KL}, e''_{KL}, S) \quad , \tag{3.6}$$

and also assume that T and Q_K are functions of e_{KL}, e''_{KL}, S and that Q_K depends in addition on $T_{,M}$.

With the foregoing background, we now record certain additional results from the theory of elastic-plastic continua [1,2] which will be utilized subsequently. The constitutive equations for $\dot{e}''_{KL}$ and κ are

$$\dot{e}''_{KL} = \lambda\, \beta_{KL} \hat{f} \quad , \tag{3.7}$$

$$\dot{\kappa} = h_{KL}\, \dot{e}''_{KL} \quad , \tag{3.8}$$

both of which hold during loading, i.e., when $f = \kappa$, $\hat{f} > 0$ or during neutral loading, i.e., when $f = \kappa$, $\dot{\kappa} = 0$ and $\hat{f} = 0$. In (3.7) and (3.8), h_{KL} and β_{KL} are tensor functions and λ is a positive scalar function of s_{ML}, e''_{ML} and S. In addition, we have

*As in [2], we may allow U, T, Q_K, as well as the stress tensor s_{KL}, to depend also on κ whose rate is specified by (3.8) below. For simplicity, we retain here the constitutive assumptions of the forms (3.6) and (3.1) but note that the inclusion of κ as an independent variable will not alter the structure of most of the results in later sections of the paper.

$$\lambda\, \beta_{KL}(h_{KL} - \frac{\partial f}{\partial e''_{KL}}) = 1 \quad . \tag{3.9}$$

We also recall the relations[+]

$$T = \frac{\partial U}{\partial S} \quad , \quad s_{KL} = \rho_0 \frac{\partial U}{\partial e_{KL}} \quad , \tag{3.10}$$

as well as the inequality

$$- Q_K\, T_{,K} \geqq 0 \quad , \tag{3.11}$$

which are independent of rates and are valid both during loading and unloading.[++] Moreover, when $f = \kappa$ and $\lambda \hat{f} > 0$,

$$\rho_0 r - Q_{K,K} - \rho_0 T \dot{S} - \rho_0 \frac{\partial U}{\partial e''_{KL}} \dot{e}''_{KL} = 0 \quad , \tag{3.12}$$

$$- \rho_0 \lambda \hat{f} \frac{\partial U}{\partial e''_{KL}} \beta_{KL} - \frac{1}{T} Q_K\, T_{,K} \geqq 0 \quad , \tag{3.13}$$

and

[+] The partial derivative $\frac{\partial U}{\partial e_{KL}}$ is understood to have the symmetric form $\frac{1}{2}(\frac{\partial U}{\partial e_{KL}} + \frac{\partial U}{\partial e_{LK}})$.

[++] See Green and Naghdi [1] for details. Although the main developments in [1] are carried out in terms of the Helmholtz free energy function (with T as an independent thermodynamic variable), results corresponding to (3.6) and (3.10) are also discussed [1; section 6]. In the present paper, as in [2], the basic kinematic variables are e''_{KL} and e_{KL} which is defined by (2.3), while the original form of the theory in [1] employed e''_{KL} and a variable defined by $e'_{KL} = e_{KL} - e''_{KL}$. The two forms of the nonlinear theory in [1,2] are entirely equivalent, although in general the latter [2] is preferable. As emphasized in [1], the variable e'_{KL} is not an "elastic" strain in the usual context of elasticity; only in restrictive cases or in the infinitesimal theory of elastic-plastic continua, e'_{KL} is an elastic strain tensor and is independent of e''_{KL} .

$$\beta_{KL} \frac{\partial U}{\partial e''_{KL}} \leqq 0 \quad , \tag{3.14}$$

where (3.13) is deduced from (3.12) by considering an arbitrary homogeneous temperature distribution for which $T_{,M} = 0$ and recalling that $\lambda > 0$.

The above results are valid for any elastic-plastic continuum. For later reference, we note that a medium is called <u>elastic-perfectly plastic</u> if the loading function f and the tensor function β_{KL} reduce to the forms[++]

$$f(s_{MN}, S) = \kappa_0 \quad , \quad \beta_{KL} = \beta_{KL}(s_{MN}, S) \quad , \tag{3.15}$$

where κ_0 is a real constant. Since f is now independent of e''_{KL} , the loading surface is always stationary. In this case, all terms involving $\frac{\partial f}{\partial e''_{KL}}$ vanish and h_{KL} in (3.8) must also vanish. Neutral loading no longer exists and the condition for loading reduces to $\hat{f} = 0$. Furthermore, it follows from (3.9) that for an elastic-perfectly plastic medium $\lambda \to \infty$ with β_{KL} remaining finite and, in place of (3.7), we have

$$\dot{e}''_{KL} = \bar{\lambda}\, \beta_{KL}(s_{MN}, S) \quad , \tag{3.16}$$

$$\bar{\lambda} = \frac{\dot{e}''_{KL}\, \beta_{KL}}{\beta_{PQ}\, \beta_{PQ}} \quad . \tag{3.17}$$

[++] The definition of an elastic-perfectly plastic material here is not the same as that in [1; section 9], obtained by specialization of the spatial form of the theory. A number of different definitions of elastic-perfectly plastic materials are possible in the context of the nonlinear theory, but they coincide for a linearized theory.

The results (3.12) and (3.14) remain valid for an elastic-perfectly plastic medium, but the inequality (3.13) must be replaced by

$$- \rho_0 \, \bar{\lambda} \, \frac{\partial U}{\partial e''_{KL}} \, \beta_{KL} - \frac{1}{T} \, Q_K \, T_{,K} \geqq 0 \quad . \tag{3.18}$$

In this paper, we restrict our attention mainly to a medium which is a non-conductor and also exclude the heat supply arising from external heat sources and energy losses due to radiation,* i.e.,

$$Q_K \equiv 0 \quad , \quad r \equiv 0 \quad . \tag{3.19}$$

Then, from (3.12), we get

$$T\dot{S} = - \frac{\partial U}{\partial e''_{KL}} \, \dot{e}''_{KL} \quad . \tag{3.20}$$

Also, by (2.9),

$$\dot{S} \geqq 0 \quad . \tag{3.21}$$

Remembering the constitutive assumption (3.6), we differentiate $(3.10)_2$ to obtain

*It is not essential to assume $(3.19)_2$ which is introduced here for simplicity of subsequent formulae. Further comment on this is made in section 4.

$$\dot{s}_{IJ} = \rho_0 \left\{ \frac{\partial^2 \psi}{\partial e_{IJ}\,\partial e_{KL}}\, \dot{e}_{KL} + \frac{\partial^2 \psi}{\partial e_{IJ}\,\partial e^p_{KL}}\, \dot{e}^p_{KL} - \frac{\partial^2 \psi}{\partial e_{IJ}\,\partial \theta}\, \dot{\theta} \right\} . \tag{3.22}$$

With the help of (3.25), equations (3.22) and (3.7) become

$$\frac{1}{\rho_0}\, \dot{s}_{IJ} = A_{IJKL}\, \dot{e}_{KL} - B_{IJKL}\, \dot{e}^p_{KL} , \tag{3.23}$$

$$\dot{e}^p_{IJ} = \lambda\, \beta_{IJ} \left\{ \frac{1}{\rho_0}\, C_{KL}\, \dot{s}_{KL} - D_{KL}\, \dot{e}^p_{KL} \right\} , \tag{3.24}$$

where

$$A_{IJKL} = \frac{\partial^2 \psi}{\partial e_{IJ}\,\partial e_{KL}} ,$$

$$B_{IJKL} = \frac{\partial^2 \psi}{\partial e_{IJ}\,\partial e^p_{KL}} - \frac{1}{\gamma}\, \frac{\partial g}{\partial e^p_{KL}}\, \frac{\partial^2 \psi}{\partial e_{IJ}\,\partial \theta} , \tag{3.25}$$

$$C_{KL} = \rho_0\, \frac{\partial g}{\partial s_{KL}} , \quad D_{KL} = -\frac{1}{\gamma}\, \frac{\partial g}{\partial e^p_{KL}}\, \frac{\partial g}{\partial \theta} .$$

The derivatives in (3.25) are evaluated in a usual manner to render the coefficients $A_{IJKL}, \ldots, D_{KL}$ symmetric functions in the indices I, J and in K, L. Also, A_{IJKL} is symmetric upon interchange of the pair I, J with the pair K, L.

Multiplying (3.24) by D_{IJ} and after changing the dummy indices and substituting back into (3.24) enables one to obtain an expression for $\dot{e}^p_{IJ}$ in terms of $\dot{s}_{KL}$, namely

$$\dot{e}^p_{IJ} = \frac{\lambda \beta_{IJ}}{\rho_0}\, \frac{C_{KL}\, \dot{s}_{KL}}{1 + \lambda \beta_{MN} D_{MN}} . \tag{3.26}$$

provided $1 - \lambda\hat{\beta}_{PQ} D_{PQ}$ is non-zero. Next, substitute (3.26) into (3.23) and in a manner similar to that which led to (3.26) solve for $\dot{s}_{KL}$ in the form

$$\dot{s}_{KL} = \rho_0 (A_{KLMN} - G_{KL} H_{MN}) \dot{e}_{MN} \quad , \tag{3.27}$$

where

$$\begin{aligned} H_{MN} &= C_{PQ} A_{PQMN} \quad ; \\ G_{KL} &= \frac{-\lambda B_{KLMN} \hat{\beta}_{MN}}{1 - \lambda\hat{\beta}_{PQ} D_{PQ} - \lambda C_{AB} B_{ABRS} \hat{\beta}_{RS}} \quad ; \end{aligned} \tag{3.28}$$

provided G_{KL} is finite. Equation (3.27) holds during loading whereas during unloading or neutral loading we have

$$\dot{s}_{KL} = \rho_0 A_{KLMN} \dot{e}_{MN} \quad . \tag{3.29}$$

For later use we also record the result

$$\dot{e}^{p}_{KL} = \frac{\lambda\hat{\beta}_{KL}}{1 - \lambda\hat{\beta}_{PQ} D_{PQ} - \lambda C_{AB} B_{ABRS} \hat{\beta}_{RS}} H_{MN} \dot{e}_{MN} \quad ; \tag{3.30}$$

which holds during loading and is obtained from (3.26) and (3.27).

4. Acceleration Discontinuities

Let $x_i = x_i(X_A, t)$ denote the position at time t of the material point whose position in the reference configuration is X_A and consider a moving surface in the continuum, a smooth one-parameter family of points, across which certain derivatives of $x_i(X_A, t)$ have jump discontinuities. Such a surface is called a singular surface or a wave and may be assigned the material description $\Sigma(t) : X_A = Y_A(\theta^1, \theta^2, t)$, where θ^1 and θ^2 are parameters. Alternatively, by elimination of θ^1 and θ^2, the smooth surface $\Sigma(t)$ may be represented as

$$\Sigma(t) \quad : \quad \phi(X_A, t) = 0 \quad . \tag{4.1}$$

Equation (4.1) locates the surface as a function of time in the reference configuration.

The unit normal to $\Sigma(t)$ will be denoted by

$$N_A = \frac{\partial \phi}{\partial X_A} \Big/ |\nabla \phi| \quad , \tag{4.2}$$

where

$$|\nabla \phi| = (\phi_{,A}\, \phi_{,A})^{1/2} > 0 \quad . \tag{4.3}$$

The speed of propagation of the surface (4.1) is

$$V = - \frac{\partial \phi}{\partial t} \Big/ |\nabla \phi| \quad . \tag{4.4}$$

Let R^- and R^+ be one sided adjacent neighborhoods partitioned by $\Sigma(t)$ and let $F(X_A, t)$, a function of X_A , be continuous in R^- and R^+ but have a jump across $\Sigma(t)$. We define the jump in $F(X_A, t)$ across Σ at time t by

$$\llbracket F \rrbracket = F^+ - F^-$$

$$= F(Y^+(\theta^\alpha,t),t) - F(Y^-(\theta^\alpha,t),t) \quad , \tag{4.5}$$

where F^+ and F^- designate the values of F on the two sides of $\Sigma(t)$ approached from the neighborhoods R^+ and R^- , respectively.

In what follows, we mainly consider a singular surface of order 2, i.e., an acceleration wave but also briefly discuss singular surfaces of order* $m > 2$. We call $\Sigma(t)$ an acceleration wave in elastic-plastic continua if $\dot{x}_i$, $x_{i,A}$ (and therefore e_{AB}), e''_{AB} and S (or T) are continuous functions of X_A and t, while their derivatives (with respect to X_A or t) have at most jump discontinuities across $\Sigma(t)$ but are continuous in R^+ and R^- . We also assume that the externally applied body force is so assigned that F_i is a continuous function of X_A for all t . It then follows that the kinematical conditions of compatibility are (see, e.g., Truesdell and Toupin [15, section 190])

$$\llbracket x_{i,AB} \rrbracket = N_A N_B \lambda_i \quad , \quad \llbracket \dot{x}_{i,A} \rrbracket = - V N_A \lambda_i \quad , \quad \llbracket \ddot{x}_i \rrbracket = V^2 \lambda_i \quad , \tag{4.6}$$

*A singular surface of order 0 is a stationary surface and a singular surface of order 1 is called a shock wave. The terminology used here is due to Hadamard [13]; see also Truesdell and Toupin [15] or Thomas [16].

where λ_i is an arbitrary vector and may be called the amplitude. More generally, when $\Sigma(t)$ is a singular surface of order m with respect to $M(X_A, t)$, the jumps in mth partial derivatives of $M(X_A, t)$ are given by

$$\left[\!\left[\frac{\partial^{m-r}}{\partial t^{m-r}} M_{,A_1 \ldots A_r} \right]\!\right] = (-V)^{m-r} N_{A_1} \cdots N_{A_r} \mu \ , \quad (r = 0, 1, \ldots, m \text{ and } m = 1, 2, \ldots) . \qquad (4.7)$$

In (4.7), M may be regarded as tensor of any order and μ is an arbitrary tensor of the same order.

For an acceleration wave, in view of our constitutive assumptions in section 3,

$$\begin{gathered} U \ , \ S \ , \ T \ , \ \lambda \ , \ \beta_{KL} \ , \\ A_{KLMN} \ , \ B_{KLMN} \ , \ C_{MN} \ , \ D_{MN} \ , \ H_{MN} \ , \ G_{KL} \ , \end{gathered} \qquad (4.8)$$

must be continuous across a singular surface for all X_A and t. The dynamical equations (2.6) hold on either side of the wave, but at a singular surface of order 2 they yield

$$x_{i,A} \left[\!\left[s_{AB,B} \right]\!\right] + \left[\!\left[x_{i,AB} \right]\!\right] s_{AB} = \rho_o \left[\!\left[\ddot{x}_i \right]\!\right] \ . \qquad (4.9)$$

Recalling (4.6), application of (4.7) to the first derivatives of the stress gives

$$V \left[\!\left[s_{KL,L} \right]\!\right] = - N_L \left[\!\left[\dot{s}_{KL} \right]\!\right] \ , \qquad (4.10)$$

and equation (4.9) becomes

$$- N_B x_{i,A} \left[\!\left[\dot{s}_{AB} \right]\!\right] + V(N_A N_B s_{AB} - \rho_0 V^2)\lambda_i = 0 \quad , \tag{4.11}$$

where (4.6) and (4.10) have been used. Also, from (3.20), we have*

$$\left[\!\left[\dot{S} \right]\!\right] = - \frac{1}{T} \frac{\partial U}{\partial e''_{KL}} \left[\!\left[\dot{e}''_{KL} \right]\!\right] \quad . \tag{4.12}$$

Provided $\partial U / \partial e''_{KL}$ does not vanish, according to (4.12), the jump in the entropy production is linear in the jump of plastic strain rate. It is therefore clear that acceleration waves (with $V \neq 0$), in a non-conducting elastic-plastic medium, are not homentropic.+

In the rest of this section, we consider three types of wave propagation as follows:

(i) In the absence of loading from an elastic-plastic state, on either side of the singular surface $\Sigma(t)$, we have either a state of unloading or neutral loading. Recalling (3.29), we then write

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*It is not difficult to see that the result (4.12) holds even without the assumption $(3.19)_2$; it is only necessary to assume that r is a continuous function of X_A for all t .

+An acceleration wave is called <u>homentropic</u> if $\left[\!\left[\dot{S} \right]\!\right] = \left[\!\left[S_{,M} \right]\!\right] = 0$. (Recall that if $\left[\!\left[\dot{S} \right]\!\right] = 0$, it follows from $\left[\!\left[S \right]\!\right] = 0$ that $\left[\!\left[S_{,M} \right]\!\right] = 0$). The relation (4.12) may be contrasted with the corresponding result obtained by Coleman and Gurtin [5]. For the class of materials with memory discussed in [5], they found that every acceleration wave (with $V \neq 0$) in a non-conductor is homentropic.

$$[\![\dot{s}_{KL}]\!] = \rho_o A_{KLMN} [\![\dot{e}_{MN}]\!] \quad , \tag{4.13}$$

and with the help of (2.4) and (4.6) this becomes

$$[\![\dot{s}_{KL}]\!] = - \rho_o V A_{KLMN} N_N x_{k,M} \lambda_k \quad . \tag{4.14}$$

Using the notation $V = V_{(e)}$ for this case, substitution of (4.14) into (4.11) results in

$$\{A_{ij} - (V_{(e)}^2 - \frac{1}{\rho_o} s_{(N)})\delta_{ij}\}\lambda_j = 0 \quad , \tag{4.15}$$

and the associated eigenvalue equation is

$$\det \{A_{ij} - (V_{(e)}^2 - \frac{1}{\rho_o} s_{(N)})\delta_{ij}\} = 0 \quad , \tag{4.16}$$

where we have defined

$$\begin{gathered} s_{(N)} = s_{KL} N_K N_L \quad , \\ A_{ij} = A_{ji} = x_{i,L} x_{j,N} N_K N_M A_{KLMN} \quad . \end{gathered} \tag{4.17}$$

Mathematically, this case is similar to that for nonlinear elasticity with A_{ij} playing the role of the homentropic acoustic tensor and with $[\![\dot{S}]\!] = 0$. Here, however, the tensor A_{ij} depends on the current state of plastic strain e''_{KL} , as well as on e_{KL} and S ; it can be identified with the homentropic acoustic tensor of elasticity only when the medium has no prior history of plastic deformation.

(ii) When loading is taking place on both sides of the wave front, then by (3.27) we have

$$[\![\dot{s}_{KL}]\!] = \rho_o (A_{KLMN} - G_{KL} H_{MN}) [\![\dot{e}_{MN}]\!] \quad ,$$

or

$$[\![\dot{s}_{KL}]\!] = - \rho_o V(A_{KLMN} - G_{KL} H_{MN}) N_N x_{k,M} \lambda_k \quad , \tag{4.18}$$

if we also use (2.4) and (4.6). Then, with the notation $V = V_{(p)}$ for this case, from (4.11) and (4.18) we obtain

$$\{A_{ij} - G_i H_j - (V^2_{(p)} - \frac{1}{\rho_o} s_{(N)}) \delta_{ij}\} \lambda_j = 0 \quad , \tag{4.19}$$

and the associated eigenvalue equation is now

$$\det \{A_{ij} - G_i H_j - (V^2_{(p)} - \frac{1}{\rho_o} s_{(N)}) \delta_{ij}\} = 0 \quad , \tag{4.20}$$

where

$$\begin{aligned} G_i &= x_{i,L} N_K G_{KL} \quad , \\ H_j &= x_{j,N} N_M H_{MN} \quad . \end{aligned} \tag{4.21}$$

This describes the propagation of an acceleration wave through a region undergoing plastic deformation, i.e., a plastic wave,* and will be further discussed in the next section.

(iii) Finally we use the notation $V = V_{(u)}$ to correspond to the case when the region R^+ is in a state of loading and the region R^- is in a state of unloading or neutral loading, with $V = V_{(\ell)}$ corresponding to the reverse of this situation. In the first of these cases, it follows from (3.27) and (3.29) that

$$\begin{aligned}[\![\dot{s}_{KL}]\!] &= \rho_o A_{KLMN} [\![\dot{e}_{MN}]\!] - \rho_o G_{KL} H_{MN} \dot{e}^+_{MN} \\ &= -\rho_o V_{(u)} A_{KLMN} N_N x_{k,M} \lambda_k - \rho_o G_{KL} H_{MN} \dot{e}^+_{MN}\end{aligned} \tag{4.22}$$

Hence, with the help of (4.11), we obtain

$$\{A_{ij} - (V^2_{(u)} - \frac{1}{\rho_o} s_{(N)})\delta_{ij}\}\lambda_j = -\frac{1}{V_{(u)}} G_i H_{MN} \dot{e}^+_{MN} \quad . \tag{4.23}$$

On the other hand, when R^+ is in a state of unloading or neutral loading and R^- is in a state of loading, by (3.27) and (3.29) we have

$$\begin{aligned}[\![\dot{s}_{KL}]\!] &= \rho_o A_{KLMN} [\![\dot{e}_{MN}]\!] + \rho_o G_{KL} H_{MN} \dot{e}^-_{MN} \\ &= \rho_o (A_{KLMN} - G_{KL} H_{MN}) [\![\dot{e}_{MN}]\!] + \rho_o G_{KL} H_{MN} \dot{e}_{N} \\ &= -\rho_o V_{(\ell)} (A_{KLMN} - G_{KL} H_{MN}) N_N x_{k,M} \lambda_k + \rho_o G_K H_{MN} \dot{e}^+_{MN} \quad .\end{aligned} \tag{4.24}$$

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*In the literature, the term "plastic wave" is often used to denote a moving elastic-plastic interface (see, e.g., Hopkins [7]).

From (4.11) and (4.24) follows the equation

$$\{(A_{ij} - G_i\, E_j) - (V^2_{(\ell)} - \frac{1}{\rho_0}\, S_{(N)})\delta_{ij}\}\lambda_j = \frac{1}{V_{(\ell)}}\, G_i\, H_{MN}\, \dot{e}^{+}_{MN} \quad . \tag{4.25}$$

The last two cases correspond to a front of unloading propagating into a plastically stressed region and a plastic loading front, respectively, and do not give rise to eigenvalue equations. These will be discussed in section 6.

The above developments are carried out for a work-hardening elastic-plastic continuum which initially is anisotropic. We now note that the results (4.17), (4.21), (4.23) and (4.25) hold also in the case of an elastic-perfectly plastic continuum. Recalling (3.16) and the fact that in this case $\hat{f}$ vanishes while λ becomes infinite as we approach this limiting case, and observing that the only term in the above formulae which depends on λ is G_i during loading, it follows that as $\lambda \to \infty$,

$$G_{KL} = \frac{B_{KLMN}\, \beta_{MN}}{\beta_{PQ}(C_{AB}\, B_{ABPQ} + D_{PQ})} \quad , \tag{4.26}$$

for an elastic-perfectly plastic medium.

The results obtained in this section are valid for an elastic-plastic material which is a non-conductor. However, it is also possible to discuss a parallel development for a definite conductor, i.e., for a medium whose heat conduction vector h the form[+]

[+]Equation (4.27) is in the form of Fourier's law and the heat conduction tensor K_{KL} a function of e_{MN}, e''_{MN} and T, is positive definite.

$$Q_K = - K_{KL} T_{,L} \quad . \tag{4.27}$$

In this case, it is more convenient to choose T (rather than S) as an independent thermodynamic variable and to use the form of the theory in which the constitutive equations are expressed in terms of the Helmholtz free energy function

$$A = U - TS \quad , \tag{4.28}$$

together with appropriate changes in section[++] 3. In view of the constitutive and smoothness assumptions, it follows from the integral form of the equation of balance of energy that every acceleration wave, in an elastic-plastic material subject to Fourier's law of heat conduction, is homothermal, i.e.,[*]

$$[\![\dot{T}]\!] = [\![T_{,M}]\!] = 0 \quad . \tag{4.29}$$

Moreover, we now obtain

$$[\![\dot{S}]\!] = - \frac{1}{T} \frac{\partial A}{\partial e''_{KL}} [\![\dot{e}''_{KL}]\!] \quad , \tag{4.30}$$

in place of (4.12). It should now be clear that results parallel to those between (4.13) and (4.26) can also be deduced in this case, but we do not pursue the matter further.

[++] For details, see [1,2]. In this form of the theory, A, s_{KL} and S, as well as f, h_{KL} and β_{KL}, are functions of e_{MN}, e''_{MN} and T. Also, instead of $(3.19)_2$, we only assume that the heat supply is so assigned that r is a continuous function of X_A for all t.

[*] This conclusion parallels the corresponding result for elastic materials [3] and for the class of materials with memory considered in [5] with the heat flux vector in the form (4.27). For a definite conductor, the heat conduction tensor in [5] is given a more general definition than that in (4.27).

5. Plastic Waves

We have shown that the speeds $V_{(p)}$ at which plastic waves propagate must identically satisfy the eigenvalue equation (4.21) for given values of e_{KL}, e''_{KL} and S. Of course, the converse is not true, i.e., every solution $V_{(p)}$ to (4.21), for given values of e_{KL}, e''_{KL}, S, does not necessarily represent the speed of an acceleration wave. However, the <u>real</u> values of $V_{(p)}$ are the <u>only</u> speeds at which such waves may propagate.

To investigate sufficient conditions for real wave speeds, we appeal to known properties of matrices. Consider A_{ij}, defined by (4.16), where A_{KLMN} is given by (3.25). If P_i is an arbitrary vector, then

$$A_{ij}\, P_i\, P_j = (P_i\, x_{i,L})(P_j\, x_{j,N}) N_K\, N_M\, A_{KLMN} \quad ,$$

which, in view of the symmetry properties of A_{KLMN}, may be rewritten as

$$A_{ij}\, P_i\, P_j = A_{KLMN}\, W_{KL}\, W_{MN} \quad , \tag{5.1}$$

W_{KL} being a symmetric tensor defined by

$$W_{KL} = \frac{1}{2}\, P_i (N_K\, x_{i,L} + N_L\, x_{i,K}) \quad . \tag{5.2}$$

From (5.1), one concludes that A_{ij} is a positive definite matrix if A_{KLMN} is positive definite in the sense that

$$A_{KLMN}\, W_{KL}\, W_{MN} \geqq 0 \quad , \tag{5.3}$$

for all arbitrary symmetric W_{KL} , with the equality holding if and only if W_{KL} is identically zero.

If (5.3) holds, then the matrix A_{ij} has three real eigenvalues, which may be ordered numerically as

$$A_1 \geqq A_2 \geqq A_3 \quad , \tag{5.4}$$

and which identically satisfy the equation

$$\det |A_{ij} - A_\alpha \delta_{ij}| = 0 \qquad (\alpha = 1,2,3) \quad . \tag{5.5}$$

Associated with these eigenvalues are a set of three eigenvectors $\phi_i^{(\alpha)}$ $(\alpha = 1,2,3)$ which are found to within a scalar magnitude by the homogeneous system of equations

$$\{A_{ij} - A_\alpha \delta_{ij}\} \phi_j^{(\alpha)} = 0 \quad . \tag{5.6}$$

Now, following the technique used by Mandel [10], let the eigenvectors $\phi_i^{(\alpha)}$ be referred to the principal directions of the matrix A_{ij} and write

$$A_{ij} = A_{\underline{i}} \delta_{\underline{i}j} \quad , \tag{5.7}$$

where a bar below an index signifies suspension of the summation convention for that index. Then, the eigenvalue equation (4.19) for plastic waves becomes

$$\det \left|(A_{\underline{i}} - Y)\delta_{\underline{i}j} - G_i H_j\right| = 0 \quad , \tag{5.8}$$

which when expanded has the form

$$P(Y) \equiv (A_1 - Y)(A_2 - Y)(A_3 - Y) - G_1 H_1(A_2 - Y)(A_3 - Y)$$

$$- G_2 H_2(A_3 - Y)(A_1 - Y) - G_3 H_3(A_1 - Y)(A_2 - Y) = 0 \quad , \tag{5.9}$$

where

$$Y = V^2_{(p)} - \frac{1}{\rho_0} s_{(N)} \quad . \tag{5.10}$$

Since (5.9) is a cubic polynomial, at least one of the roots Y_α ($\alpha = 1,2,3$) must be real. Of course, with the help of (5.10), it is seen that the wave speed $V^{(\alpha)}_{(p)}$ corresponding to Y_α will be real only if

$$Y_\alpha + \frac{1}{\rho_0} s_{(N)} \geq 0 \quad . \tag{5.11}$$

We also find from (5.9) that

$$\begin{aligned} P(A_1) &= - G_1 H_1(A_1 - A_2)(A_1 - A_3) \quad , \\ P(A_2) &= G_2 H_2(A_1 - A_2)(A_2 - A_3) \quad , \\ P(A_3) &= - G_3 H_3(A_1 - A_3)(A_2 - A_3) \quad . \end{aligned} \tag{5.12}$$

In view of (5.4), we observe from (5.12) that $P(A_1)$, $P(A_2)$ and $P(A_3)$ have the signs of $-G_1 H_1$, $+G_2 H_2$ and $-G_3 H_3$, respectively. Noting that $P(Y) \to -\infty$ as $Y \to +\infty$ while $P(Y) \to +\infty$ as $Y \to -\infty$, we deduce the following information about the wave speeds:

If $G_1 H_1$, $G_2 H_2$ and $G_3 H_3$ all have the same sign, then Y_1, Y_2 and Y_3 are real and distinct; if this sign is positive, then

$$A_1 \geqq Y_1 \geqq A_2 \geqq Y_2 \geqq A_3 \geqq Y_3 \quad , \tag{5.13}$$

while if it is negative,

$$Y_1 \geqq A_1 \geqq Y_2 \geqq A_2 \geqq Y_3 \geqq A_3 \quad . \tag{5.14}$$

It is clear, however, that if $V^{(\alpha)}_{(p)}$ is to be a real wave speed, the inequality (5.11) must hold.

By comparing (5.6) with (4.15), we further observe that A_α are precisely the eigenvalues for the case where there is no loading (i.e., additional plastic deformation) on either side of the wave front. It would, therefore, be of physical interest to compare A_1, the speed of the fastest "elastic" wave, with the real roots among Y_1, Y_2 and Y_3. Similar comparisons for A_2 and A_3 may be of interest.

6. Fronts of Loading and Unloading

Any surface in the medium which separates a region of loading from one of unloading or neutral loading is called an elastic-plastic interface. With reference to case (iii) of section 4, suppose such a surface is moving through the medium with an acceleration discontinuity coinciding with it. For a specified history of loading and for known conditions ahead of the front, formally, we have

$$\begin{aligned} e_{KL} &= e_{KL}(X_M, t) \quad , \quad e''_{KL} = e''_{KL}(X_M, t) \quad , \\ s_{KL} &= s_{KL}(X_M, t) \quad , \quad S = S(X_M, t) \quad , \end{aligned} \tag{6.1}$$

these quantities being known functions of position and time. From (6.1), we can compute the strain rate ahead of the wave front, i.e.,

$$\dot{e}^{+}_{KL} = \dot{e}^{+}_{KL}(X_M, t) \quad . \tag{6.2}$$

Using (6.1) in the left-hand side of (3.2), we may then define

$$\psi(X_M, t) \equiv f(s_{KL}(X_M, t), e''_{KL}(X_M, t), S(X_M, t)) - \kappa \quad , \tag{6.3}$$

where κ is a known function of X_M, t . The velocity V of the elastic-plastic interface, and hence of the surface of acceleration discontinuity, may be computed from (6.3) by a formula of the type (4.4).

Since $\dot{e}^{+}_{KL}$ and V are determined as functions of X_A and t, equations (4.23) or (4.25) each constitute a set of three linear inhomogeneous equations in the discontinuity amplitudes λ_i . It follows that λ_i can be determined uniquely for both cases, except when the determinants of the coefficients of λ_i vanish. However, these correspond to cases in which the speeds are precisely $V_{(e)}$ and $V_{(p)}$ characterized by (4.16) and (4.19), respectively. For example, suppose the elastic-plastic interface advances into a plastic region with the speed $V_{(e)}$ satisfying equation (4.16). Then, provided $G_i \neq 0$, it follows from (4.23) that $H_{MN}\, \dot{e}^{+}_{MN} = 0$. Hence, using (3.30),

$$\dot{e}''^{+}_{KL} = 0 \quad , \tag{6.4}$$

so that we no longer have a region of loading just ahead of the wave -- it can be one of neutral or unloading region.

On the other hand, suppose the elastic-plastic interface advances into a region of unloading or neutral loading with the speed $V_{(p)}$ satisfying equation (4.20). Then, provided $G_i \neq 0$, it follows from (4.25) that

$$H_{MN}\, \dot{e}^{+}_{MN} = 0 \quad . \tag{6.5}$$

This corresponds to propagation into a region of neutral loading, if we recall (3.30). Some of the results of W. A. Green [12] obtained under more restrictive conditions are compatible with those given here.

7. Higher Order Waves

It is a known result that discontinuities of all orders higher than 2 propagate with the same characteristic wave speeds through a nonlinear elastic material as do acceleration discontinuities.+ We shall now show that identical results hold for plastic waves. Since by (3.25) and (3.28), A_{KLMN}, G_{KL} and H_{KL} are functions of e_{AB}, e''_{AB} and S, it follows from time differentiation of (3.27) that

$$\ddot{s}_{KL} = \rho_0 \dot{e}_{MN}\{\dot{e}_{AB}\frac{\partial}{\partial e_{AB}} + \dot{e}''_{AB}\frac{\partial}{\partial e''_{AB}} + \dot{S}\frac{\partial}{\partial S}\}(A_{KLMN} - G_{KL}H_{MN})$$

$$+ \rho_0(A_{KLMN} - G_{KL}F_{MN})(\dot{x}_{m,M}\dot{x}_{m,N} + x_{m,M}\ddot{x}_{m,N}) \quad , \tag{7.1}$$

during loading. Similarly, from (3.29),

$$\ddot{s}_{KL} = \rho_0 \dot{e}_{MN}\{\dot{e}_{AB}\frac{\partial}{\partial e_{AB}} + \dot{e}''_{AB}\frac{\partial}{\partial e''_{AB}} + \dot{S}\frac{\partial}{\partial S}\} A_{KLMN}$$

$$+ \rho_0 A_{KLMN}(\dot{x}_{m,M}\dot{x}_{m,N} + x_{m,M}\ddot{x}_{m,N}) \quad , \tag{7.2}$$

during unloading or neutral loading.

Consider now a singular surface $\Sigma(t)$ of order 3 with respect to displacement.++ Since $x_{m,M}$, $\dot{x}_m$, e_{KL}, e''_{KL}, S and their first partial

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+See Truesdell [3] for a general derivation.

++For a singular surface of order m, we need to make the additional assumption that the (m-2)th time derivative of F_i is a continuous function of X_A for all t.

derivatives are continuous at such a surface, the jump of the time derivative of the equations of motion (2.6) takes the form

$$x_{i,A} [\![\dot{s}_{AB,B}]\!] + [\![\dot{x}_{i,AB}]\!] s_{AB} = \rho_o [\![\dddot{x}_i]\!] \quad . \tag{7.3}$$

If either side of $\Sigma(t)$ is in a state of loading, then the jump of $\ddot{s}_{KL}$ across $\Sigma(t)$ is

$$[\![\ddot{s}_{KL}]\!] = \rho_o (A_{KLMN} - G_{KL} H_{MN}) x_{m,M} [\![\ddot{x}_{m,N}]\!] \quad , \tag{7.4}$$

by (7.1). This corresponds to the case (ii) of section 4 and the results corresponding to (i) and (iii) can be discussed in a similar manner. Making use of (4.7), we find the jump relations for 3rd order partial derivatives of displacement and 2nd order partial derivatives of stress. They are

$$\begin{gathered} [\![x_{i,ABC}]\!] = N_A N_B N_C \zeta_i \quad ; \quad [\![\dot{x}_{i,AB}]\!] = - V N_A N_B \zeta_i \quad , \\ [\![\ddot{x}_{i,A}]\!] = V^2 N_A \zeta_i \quad , \quad [\![\dddot{x}_i]\!] = - V^3 \zeta_i \quad , \end{gathered} \tag{7.5}$$

and

$$\begin{gathered} [\![s_{KL,PQ}]\!] = N_P N_Q \nu_{KL} \quad , \quad [\![\dot{s}_{KL,Q}]\!] = - V N_Q \nu_{KL} \quad , \\ [\![\ddot{s}_{KL}]\!] = V^2 \nu_{KL} \quad , \end{gathered} \tag{7.6}$$

where ζ_i is an arbitrary vector and ν_{KL} is an arbitrary tensor. Applications of (7.5) and (7.6) to (7.3) and (7.4) give

$$V^2 \nu_{KL} = \rho_o(A_{KLPQ} - G_{KL} H_{PQ})x_{m,P} V^2 N_Q \zeta_m \quad , \tag{7.7}$$

$$- V x_{i,A} N_B \nu_{AB} - V s_{AB} N_A N_B \zeta_i = - \rho_o V^3 \zeta_i \quad . \tag{7.8}$$

After eliminating ν_{AB} between (7.7) and (7.8), with the help of (4.17) and (4.21), we obtain the equation

$$\{(A_{ij} - G_i H_j) - (V^2 - \frac{1}{\rho_o} s_{(N)})\delta_{ij}\}\zeta_j = 0 \quad . \tag{7.9}$$

Comparison of (7.9) with (4.19) gives the desired result. In an analogous manner we may show, by differentiating (7.1) and (7.3) a sufficient number of times, that singular surfaces of higher order have the same plastic wave speeds as those of order 2.

8. An Example: Plastic Waves in a Uni-Axial Motion

We consider here the simple example of plastic waves, in an initially homogeneous and isotropic material, corresponding to the uni-axial motion

$$x_1 = x_1(X_1, t) \quad , \quad x_2 = X_2 \quad , \quad x_3 = X_3 \quad . \tag{8.1}$$

By (2.3) and (8.1), we have

$$e_{11} = \frac{1}{2}(\gamma^2 - 1) \quad , \quad \gamma = \frac{\partial x_1}{\partial X_1} \quad , \quad \text{all other } e_{KL} = 0 \quad , \tag{8.2}$$

throughout the history of deformation and we also assume

$$e''_{11} \neq 0 \quad , \quad \text{all other } e''_{KL} = 0 \quad . \tag{8.3}$$

Here, we concern ourselves with only one aspect of the problem, namely the determination of velocities of propagating waves in an isotropic material, using the results of section 4. Thus, we assume that a state of plastic deformation corresponding to (8.2) and (8.3) is compatible with the field equations which must also be used in a complete analysis of the problem.

Before proceeding further, we need to recall certain results concerning the forms of the constitutive functions of section 3. For an initially isotropic material, the internal energy function U, the loading function f, β_{KL} and the work-hardening tensor h_{KL} are isotropic functions of their arguments.[+] Then, U is expressible as a function of the entropy S and the ten joint invariants

$$I_1 = e_{KK} \; , \quad I_2 = e_{KL}\, e_{KL} \; , \quad I_3 = e_{KL}\, e_{LM}\, e_{MK} \; , \tag{8.4}$$

$$I''_1 = e''_{KK} \; , \quad I''_2 = e''_{KL}\, e''_{KL} \; , \quad I''_3 = e''_{KL}\, e''_{LM}\, e''_{MK} \; ,$$

$$J_1 = e_{KL}\, e''_{LK} \; , \quad J_2 = e_{KL}\, e_{LM}\, e''_{MK} \; , \tag{8.5}$$

$$J_3 = e_{KL}\, e''_{LM}\, e''_{MK} \; , \quad J_4 = e_{KL}\, e_{LM}\, e''_{MN}\, e''_{NK} \; .$$

Similarly, f must be expressible as a function of S and the ten joint invariants consisting of I''_1, I''_2, I''_3 and

$$K_1 = s_{KK} \; , \quad K_2 = s_{KL}\, s_{KL} \; , \quad K_3 = s_{KL}\, s_{LM}\, s_{MK} \; , \tag{8.6}$$

$$L_1 = e''_{KL}\, s_{LK} \; , \quad L_2 = e''_{KL}\, e''_{LM}\, s_{MK} \; , \tag{8.7}$$

$$L_3 = e''_{KL}\, s_{LM}\, s_{MK} \; , \quad L_4 = e''_{KL}\, e''_{LM}\, s_{MN}\, s_{NK} \; .$$

The constitutive equation for β_{KL} has the form [1]

[+] Our discussion here parallels that in section 7 of [1], where further details can be found.

$$\beta_{KL} = \beta_0\, \delta_{KL} + \beta_1\, s_{KL} + \beta_2\, s_{KM}\, s_{ML} + \beta_3\, e''_{KL}$$

$$+ \beta_4\, e''_{KM}\, e''_{ML} + \beta_5 (s_{KM}\, e''_{ML} + e''_{KM}\, s_{ML})$$

$$+ \beta_6 (s_{KM}\, s_{MN}\, e''_{NL} + e''_{KM}\, s_{MN}\, s_{NL})$$

$$+ \beta_7 (s_{KM}\, e''_{MN}\, e''_{NL} + e''_{KM}\, e''_{MN}\, s_{NL})$$

$$+ \beta_8 (s_{KM}\, s_{MN}\, e''_{NR}\, e''_{RL} + e''_{KM}\, e''_{MN}\, s_{NR}\, s_{RL}) \quad , \qquad (8.8)$$

with a similar expression for h_{KL}, where the coefficients $\beta_0, \beta_1, \ldots, \beta_8$ and $h_0, h_1, \ldots, h_8$ are single-valued functions of S and the invariants given by $(8.4)_{4,5,6}$, (8.6) and (8.7).

As we need to calculate the coefficients in $(4.17)_2$ and (4.21), we now record briefly certain partial derivatives which occur in (3.25). Thus,

$$\frac{\partial U}{\partial e_{11}} = \frac{\partial U}{\partial I_1} + 2\frac{\partial U}{\partial I_2} e_{11} + 3\frac{\partial U}{\partial I_3} (e_{11})^2$$

$$+ \frac{\partial U}{\partial J_1} e''_{11} + 2\frac{\partial U}{\partial J_2} e_{11}\, e''_{11} + \frac{\partial U}{\partial J_3} (e''_{11})^2 + 2\frac{\partial U}{\partial J_4} e_{11} (e''_{11})^2 \ ,$$

$$\frac{\partial U}{\partial e_{22}} = \frac{\partial U}{\partial I_1} \quad , \quad \frac{\partial U}{\partial e_{33}} = \frac{\partial U}{\partial I_1} \quad , \qquad (8.9)$$

$$\frac{\partial U}{\partial e_{12}} = \frac{\partial U}{\partial e_{23}} = \frac{\partial U}{\partial e_{31}} = 0 \quad ,$$

with analogous results holding for $\frac{\partial U}{\partial e''_{KL}}$. Also,

$$\frac{\partial^2 U}{\partial e_{11}\partial S} = \frac{\partial^2 U}{\partial I_1 \partial S} + 2\,\frac{\partial^2 U}{\partial I_2 \partial S}\, e_{11} + 3\,\frac{\partial^2 U}{\partial I_3 \partial S}\,(e_{11})^2$$

$$+ \frac{\partial^2 U}{\partial J_1 \partial S}\, e^*_{11} + 2\,\frac{\partial^2 U}{\partial J_2 \partial S}\, e_{11}\, e^*_{11} + \frac{\partial^2 U}{\partial J_3 \partial S}\,(e^*_{11})^2 + 2\,\frac{\partial^2 U}{\partial J_4 \partial S}\, e_{11}\,(e^*_{11})^2 \; .$$

$$\frac{\partial^2 U}{\partial e_{22}\partial S} = \frac{\partial^2 U}{\partial I_1 \partial S} \quad , \quad \frac{\partial^2 U}{\partial e_{33}\partial S} = \frac{\partial^2 U}{\partial I_1 \partial S} \quad ;$$

$$\frac{\partial^2 U}{\partial e_{12}\partial S} = \frac{\partial^2 U}{\partial e_{13}\partial S} = \frac{\partial^2 U}{\partial e_{23}\partial S} = 0 \quad ; \qquad (8.10)$$

$$\frac{\partial^2 U}{\partial e_{11}\partial e_{12}} = \frac{\partial^2 U}{\partial e_{11}\partial e_{13}} = \frac{\partial^2 U}{\partial e_{11}\partial e_{23}} = \frac{\partial^2 U}{\partial e_{12}\partial e_{13}} = \frac{\partial^2 U}{\partial e_{12}\partial e_{22}}$$

$$= \frac{\partial^2 U}{\partial e_{12}\partial e_{23}} = \frac{\partial^2 U}{\partial e_{12}\partial e_{33}} = \frac{\partial^2 U}{\partial e_{13}\partial e_{22}} = \frac{\partial^2 U}{\partial e_{13}\partial e_{23}}$$

$$= \frac{\partial^2 U}{\partial e_{22}\partial e_{23}} = \frac{\partial^2 U}{\partial e_{23}\partial e_{33}} = \frac{\partial^2 U}{\partial e_{13}\partial e_{33}} = 0 \quad ; \qquad (8.11)$$

with analogous results holding for $\dfrac{\partial^2 U}{\partial e_{KL}\partial e^*_{KL}}$.

From $(3.10)_2$ and (8.9), we have

$$s_{12} = s_{23} = s_{13} = 0 \quad ; \quad s_{22} = s_{33} \quad . \qquad (8.12)$$

Then, the components of β_{KL} and h_{KL} are

$$\begin{Bmatrix} \beta_{11} \\ h_{11} \end{Bmatrix} = \begin{Bmatrix} \beta_o \\ h_o \end{Bmatrix} + \begin{Bmatrix} \beta_1 \\ h_1 \end{Bmatrix} s_{11} + \begin{Bmatrix} \beta_2 \\ h_2 \end{Bmatrix} (s_{11})^2 + \begin{Bmatrix} \beta_3 \\ h_3 \end{Bmatrix} e''_{11}$$

$$+ \begin{Bmatrix} \beta_4 \\ h_4 \end{Bmatrix} (e''_{11})^2 + 2 \begin{Bmatrix} \beta_5 \\ h_5 \end{Bmatrix} e''_{11} s_{11}$$

$$+ 2 \begin{Bmatrix} \beta_6 \\ h_6 \end{Bmatrix} (s_{11})^2 e''_{11} + 2 \begin{Bmatrix} \beta_7 \\ h_7 \end{Bmatrix} (e''_{11})^2 s_{11} + 2 \begin{Bmatrix} \beta_8 \\ h_8 \end{Bmatrix} (s_{11})^2 (e''_{11})^2 \ ,$$

(8.13)

$$\begin{Bmatrix} \beta_{22} = \beta_{33} \\ h_{22} = h_{33} \end{Bmatrix} = \begin{Bmatrix} \beta_o \\ h_o \end{Bmatrix} + \begin{Bmatrix} \beta_1 \\ h_1 \end{Bmatrix} s_{22} + \begin{Bmatrix} \beta_2 \\ h_2 \end{Bmatrix} (s_{22})^2 \ ,$$

$$\beta_{12} = \beta_{23} = \beta_{31} = 0 \ ,$$

$$h_{12} = h_{23} = h_{31} = 0 \ ,$$

and the components of $\dfrac{\partial f}{\partial s_{KL}}$ become

$$\frac{\partial f}{\partial s_{11}} = \frac{\partial f}{\partial K_1} + 2\, s_{11} \frac{\partial f}{\partial K_2} + 3(s_{11})^2 \frac{\partial f}{\partial K_3} + e''_{11} \frac{\partial f}{\partial L_1} + (e''_{11})^2 \frac{\partial f}{\partial L_2}$$

$$+ 2\, e''_{11}\, s_{11} \frac{\partial f}{\partial L_3} + 2(e''_{11})^2 s_{11} \frac{\partial f}{\partial L_4} \ ,$$

(8.14)

$$\frac{\partial f}{\partial s_{22}} = \frac{\partial f}{\partial s_{33}} = \frac{\partial f}{\partial K_1} + 2\, s_{22} \frac{\partial f}{\partial K_2} + 3(s_{22})^2 \frac{\partial f}{\partial K_3} \ ,$$

$$\frac{\partial f}{\partial s_{12}} = \frac{\partial f}{\partial s_{23}} = \frac{\partial f}{\partial s_{31}} = 0 \ .$$

From the results (8.9) to (8.14), we can now obtain the following information about the coefficients on the left-hand sides of (3.25) and (3.28), i.e.,

$$A_{1112} = A_{1113} = A_{1123} = A_{1213} = A_{1222} = A_{1223} = A_{1233}$$

$$= A_{1322} = A_{1323} = A_{1333} = A_{2223} = A_{2333} = 0 \quad , \tag{8.15}$$

$$C_{12} = C_{23} = C_{31} = 0 \quad , \quad D_{12} = D_{23} = D_{31} = 0 \quad ,$$

$$H_{12} = H_{23} = H_{31} = 0 \quad , \quad G_{12} = G_{23} = G_{31} = 0 \quad ,$$

and

$$A_{1212} = A_{1313} = A_{2323} \quad ,$$

$$A_{1122} = A_{1133} \quad ,$$

$$A_{2222} = A_{2233} = A_{3333} \quad ,$$

$$C_{22} = C_{33} \quad , \quad D_{22} = D_{33} \quad ,$$

$$H_{22} = H_{33} \quad , \quad G_{22} = G_{33} \quad , \tag{8.16}$$

$$B_{1212} = B_{1313} = B_{2323} \quad ,$$

$$B_{1122} = B_{1133} \quad ,$$

$$B_{2211} = B_{3311} \quad ,$$

$$B_{2222} = B_{2233} = B_{3322} = B_{3333} \quad ,$$

together with results analogous to the first of (8.15) for B_{KLMN}. With the help of (8.1), (8.15) and (8.16), the value of $s_{(N)}$ and the non-vanishing components of A_{ij}, G_i and H_j in $(4.17)_2$ and (4.21) are

$$
\begin{aligned}
s_{(N)} &= s_{11}\, N_1^2 + s_{22}(N_2^2 + N_3^2) \quad , \\
A_{11} &= \gamma^2[N_1^2\, A_{1111} + (N_2^2 + N_3^2)A_{1212}] \quad , \\
A_{12} &= \gamma\, N_1\, N_2(A_{1122} + A_{1212}) \quad , \\
A_{13} &= \gamma\, N_1\, N_3(A_{1122} + A_{1212}) \quad , \\
A_{22} &= N_1^2\, A_{1212} + (N_2^2 + N_3^2)A_{2222} \quad , \\
A_{23} &= N_2\, N_3(A_{2222} + A_{1212}) \quad , \\
A_{33} &= (N_1^2 + N_2^2)A_{1212} + N_3^2\, A_{3333} \quad ,
\end{aligned}
\tag{8.17}
$$

and

$$
\begin{aligned}
&G_1 = \gamma\, N_1\, G_{11} \quad , \quad G_2 = N_2\, G_{22} \quad , \quad G_3 = N_3\, G_{22} \quad , \\
&H_1 = \gamma\, N_1\, H_{11} \quad , \quad H_2 = N_2\, H_{22} \quad , \quad H_3 = N_3\, H_{22} \quad .
\end{aligned}
\tag{8.18}
$$

For a prescribed wave front which is specified by its normal N_K (generally a function of position), we may substitute (8.17) and (8.18) into the eigenvalue equation (4.20) and thereby determine the wave speeds.

Consider now a plane wave front whose normal is at an arbitrary (oblique) angle to the X_1-axis. Without loss in generality, we may choose the direction of the X_3-axis to be perpendicular to the plane of the X_1-axis and the normal to the wave front. In this way, the problem is reduced to a two dimensional one, so that

$$N_K = (N_1, N_2, 0) \quad , \tag{8.19}$$

where N_1, N_2 are constants (for plane waves) satisfying

$$N_1^2 + N_2^2 = 1 \quad . \tag{8.20}$$

In view of (8.19), (8.17) and (8.18) simplify and the non-vanishing components of (4.19) become

$$\begin{aligned} (A_{11} - G_1 H_1 - Y)\lambda_1 + (A_{12} - G_1 H_2)\lambda_2 = 0 \quad , \\ (A_{12} - G_2 H_1)\lambda_1 + (A_{22} - G_2 H_2 - Y)\lambda_2 = 0 \quad , \\ (A_{22} - Y)\lambda_3 = 0 \quad , \end{aligned} \tag{8.21}$$

where the coefficients in (8.21) are obtained from those in (8.17) and (8.18) after setting $N_3 = 0$ and where Y is defined by (5.10).

Corresponding to the eigenvalue

$$Y = A_{22} \quad ; \tag{8.22}$$

we have λ_3 arbitrary and $\lambda_1 = \lambda_2 = 0$. The other two eigenvalues are the roots of the equation

$$Y^2 - (A_{11} - G_1 H_1 + A_{22} - G_2 H_2)Y$$

$$+ (A_{11} - G_1 H_1)(A_{22} - G_2 H_2) - (A_{12} - G_1 H_2)(A_{12} - G_2 H_1) = 0 \quad , \tag{8.23}$$

with $\lambda_3 = 0$ and with λ_1 arbitrary and λ_2 satisfying (8.21) or vice versa. For a plane wave whose direction of propagation is parallel to the direction of the strain, we have

$$N_X = (1, 0\ 0) \quad . \tag{8.24}$$

Then, (8.17) and (8.18) reduce to

$$A_{11} = \gamma^2 A_{1111} \quad , \quad A_{22} = A_{33} = A_{1212} \quad , \quad A_{12} = A_{23} = A_{31} = 0 \quad ,$$

$$G_1 = \gamma G_{11} \quad , \quad G_2 = G_3 = 0 \quad , \tag{8.25}$$

$$H_1 = \gamma H_{11} \quad , \quad H_2 = H_3 = 0 \quad ,$$

and the components of (4.19) become

$$(A_{11} - G_1 H_1 - Y)\lambda_1 = 0 \quad , \quad (A_{22} - Y)\lambda_2 = 0 \quad , \quad (A_{22} - Y)\lambda_3 = 0 \quad . \tag{8.26}$$

The solutions of (8.26) correspond to one longitudinal wave for which

$$Y = A_{11} - G_1 H_1 \quad , \quad \lambda_1 \text{ arbitrary} \quad , \tag{8.27}$$

and to two transverse waves for which

$$Y = A_{22} \quad , \quad \lambda_2 \text{ and } \lambda_3 \text{ arbitrary} \quad . \tag{8.28}$$

The case of a plane wave whose direction of propagation is transverse to that of the strain, namely

$$N_K = (0, 1, 0) \quad , \tag{8.29}$$

can be discussed similarly with results corresponding to (8.27) and (8.28) in the form

$$\begin{aligned} Y &= A_{22} - G_2 H_2 \quad , \quad \lambda_2 \text{ arbitrary} \quad , \\ Y &= A_{11} \quad , \quad \lambda_1 \text{ arbitrary} \quad , \\ Y &= A_{22} \quad , \quad \lambda_3 \text{ arbitrary} \quad . \end{aligned} \tag{8.30}$$

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