A BASIC APPROACH TO THE EVALUATION OF RISKY INTERRELATED INVESTMENTS

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1. Introduction

When evaluating proposals for capital investment, it often is necessary to consider interrelationsh.ps of various kinds between the projects. For example, some projects may be complementary, such as those which would benefit in common from the facilities that would need to be provided for any one of them. Some proposals may even be necessarily contingent on undertaking one or more prerequisite projects or on the occurrence of particular favorable conditions. Certain proposals may instead be mutually exclusive in that they involve different ways of performing the same service. Other proposed investments may at least be competitive in the sense that undertaking one reduces the benefits that would result from also undertaking any of the others. Furthermore, some or all of the proposals may need to compete for certain limited resources such as capital, manpower, and facilities.

Fortunately, considerable progress has been made in recent years in developing ways of taking these interrelationships into account. Of particular note is Weingartner's comprehensive work [17] on applying mathematical programming to the analysis of such capital budgeting problems.[1/] All of the interrelationships discussed above usually can

[1/] Also see the survey paper by Weingartner [16].

be formulated quite conveniently in either a linear or nonlinear programming format, so this provides a powerful tool for the analysis of interrelated projects.

Another crucial factor to consider when evaluating proposed investments is the element of risk. Unfortunately, risk is a relatively difficult factor to deal with in an explicit, quantitative fashion. However, Hillier [8] has introduced the concept of analyzing the various individual determinants of project worth and estimating the most likely outcome and degree of uncertainty associated with each in order to obtain the overall probability distribution of a measure of merit for the investment (e.g., distribution of internal rate of return), and then using this distribution as a basis for management's evaluation of risk.[2/] He also presented an analytical approach for developing this distribution. Hertz [4] then pointed out that computer simulation also could be used to develop this same information. This approach to risk analysis has been adopted quite widely during the last few years.

Thus, by using the approaches mentioned above, it is now possible to deal separately with (1) the effect of interrelationships between proposed projects, and (2) the evaluation of the risk associated with individual investments. However, this does not answer the question of how to simultaneously take both considerations into account in order to choose the best overall combination from the group of proposed investments. Is it possible to combine the two approaches in some reasonable way to accomplish this?

[2/] Also see [7] and [10].

Unfortunately, this kind of extension usually would not be particularly feasible with the simulation approach to risk analysis. As the author points out elsewhere [9, pp. 84f.], this approach "has a number of serious disadvantages which make it poorly suited for the analysis of risky interrelated investments. One of the lesser of these is that simulation is inherently an imprecise technique, even with respect to the model used, since it provides only statistical estimates rather than exact results. In other applications, these estimates commonly are of the mean of a distribution, in which case the ones yielded by the usual computer run sizes tend to have a low but tolerable precision. However, estimates of probability distributions are considerably more crude, especially when a tail of the distribution is critical, as is the case here. Estimates of differences between alternatives also are at least as imprecise. Furthermore, simulation is a cumbersome way to study a problem, since it requires developing the model and input data, and then doing the computer programming and executing the computer runs. However, perhaps the most critical disadvantage is that simulation yields only numerical data about the predicted performance of investments, so that it yields no additional insight into cause-and-effect relationships. Therefore, every slightly new case must be completely rerun. This, in addition to the imprecision, tends to make it impossible to conduct a satisfactory sensitivity analysis. An even more serious implication is that a new simulation run is required for each combination of interrelated investments under consideration. Since even one run of acceptable length is expensive, and there may be thousands or even millions of feasible combinations, this too would tend to be prohibitively expensive. Finally, even if one were able to obtain a crude estimate of the probability distribution of present value for each of these feasible combinations, the simulation approach provides no guidance on how to use all of these masses of data in order to select the investments to be approved."

On the other hand, the analytical approach to risk analysis is relatively well-suited for extension to the case of interrelated investments. In fact, it can be incorporated directly into a mathematical programming format. One way of doing this is by means of a chance-constrained programming formulation. This method has been explored in some depth by Byrnes, Charnes, Cooper, and Kortanek [1,2], Naslund [13,14], Hillier [5; 9, Ch. 7], and others. An alternative approach which may provide a more fundamental and precise evaluation of the overall merit of the set of approved investments also has been developed by Hillier [9] in a companion volume. This method uses present value (treated as a random variable) and expected utility of present value as criteria in order to choose the best overall combination from the group of proposed investments.

In this companion volume [9], the investigation begins by deriving properties that characterize an optimal combination of investments. A model is then developed for describing the interrelated cash flows generated by a given set of investments. This model would be used for determining the mean, variance, and possibly the functional form of the probability distribution of present value. The question of when this distribution would be normal or approximately normal (even when the distributions of the individual cash flows are not) is explored in detail. Next, several convenient models of utility functions having desirable properties are formulated. For each, an expression is derived for calculating the expected utility of approving any particular combination of investments. Given these results, an approximate linear programming approach and an exact branch-and-bound algorithm are developed for determining the best combination. Numerous suggestions regarding the practical implementation of the theory and procedures also are given.

The purpose of this paper is to focus on a particular basic model from [9][3/] that seems to be particularly well-suited for practical use, and to elaborate further on how to formulate and apply it. However, rather than repeating most of the technical details involved in developing this model and its supporting theory, the emphasis here is on motivating, identifying, and interpreting the basic structure of the model. Thus, this is to be a relatively nontechnical expository treatment.[4/]

In pursuit of this objective, the next section develops the portion of the model designed for considering interrelationships between investments. Section 3 then introduces the evaluation of risk into this framework. Section 4 reviews solution procedures for this model that were developed in [9] for seeking the best combination of investments. Computational experience with these procedures is outlined in Section 5, with conclusions given in Section 6.

3/ See primarily Sections 1.6, 5.2, 5.3, 5.4, and 6.1.

4/ For ease of exposition, the problem is discussed in terms of a business firm which currently has a number of opportunities for major capital investments or projects to be considered by top management.

2. A Model for Considering Interrelationships

Suppose that a number of proposals for capital investment have been submitted to management for their consideration. As discussed above, it is likely that some of the proposed projects are interrelated in one or more significant ways. Therefore, it is important to recognize the nature of these interrelationships and to take their effects into account in the analysis. A model for doing this is developed below.

The first assumption is that all decisions to be made are of the "yes-or-no" rather than the "how much" type. Thus, the kind of investments being considered are capital projects to be approved or rejected rather than, for example, common stock where the decisions are how many shares of each type to purchase.[5] However, it should be emphasized that "investment decision" may be defined in a broad sense here so as to include various investment strategy possibilities. Thus, in addition to a flat approval or rejection of a project, other alternatives such as "postpone the investment for a year" or "try a pilot run first" may also be considered by treating all of the possibilities as mutually exclusive investments to be approved or rejected. Similarly, several alternative levels at which to undertake a project may be treated as mutually exclusive investments. Furthermore, alternative strategies, such as "try project A for a year and then switch to project B if the losses are more than x dollars" for several different values of x, also are mutually exclusive investments which may be

[5] The latter problem of portfolio selection has been analyzed effectively by Markowitz [12] and others.

contingent upon certain other decisions (such as not undertaking project B immediately). As these examples illustrate, confining the analysis to yes-or-not decisions need not be particularly restrictive in a capital budgeting context.

Now consider how to begin formalizing the problem mathematically. Suppose that all of the interesting possible investment decisions, in the broad sense described above, have been identified. Let the number of such investment decisions be denoted by m. Then introduce a decision variable δ_k for each of the decisions, where the variable is assigned a value of one or zero according to whether a decision is yes or no. Thus,

$$\delta_k = \begin{cases} 1, & \text{if the } k^{th} \text{ investment is approved} \\ 0, & \text{if the } k^{th} \text{ investment is rejected} \end{cases},$$

for $k = 1,2,\ldots,m$. Let $\underline{\delta} = (\delta_1,\delta_2,\ldots,\delta_m)$. Hence, a "solution" to this capital budgeting problem corresponds to a particular value of $\underline{\delta}$ where each of the components is either zero or one. However, not all combinations of zeroes and ones need be a "feasible" solution. Thus, if a particular group of investments are mutually exclusive, then the constraint,

$$\sum_{k\in K} \delta_k \leq 1 ,$$

must be satisfied, where the summation is over the indices corresponding to the set K of these mutually exclusive alternatives. One such constraint would need to be imposed for each set of mutually exclusive investments. Any other restrictions also must be taken into account.

For example, if investment j is contingent upon investment i being approved, this can be expressed by the constraint,

$$\delta_j \leq \delta_i ,$$

since then δ_j can equal one only if δ_i is equal to one. It is also common to have budget restrictions whereby the total outlay for all approved projects during a certain period or periods must fall within prescribed limits. Similarly, limitations on required resources such as labor and materials may rule out certain combinations of investments. These and similar restrictions usually can be expressed conveniently in a mathematical programming format, as described in some detail by Weingartner [17, Sections 2.3, 3.6, 3.7, 7.1, 7.2].

In order to systematically determine which particular feasible solution $\underline{\delta}$ is optimal (with respect to the model being developed), it is necessary to quantify the effect of a choice of $\underline{\delta}$. In very general terms, the basic effect of approving a particular combination of investments is to generate a stream of cash flows or imputed cash flows. Immediate cash outlays (negative cash flows) normally are required for the investments, and such outlays may need to be continued for some period of time. The payoff then comes in the form of income (positive cash flow) over some interval or instant of time, or in the form of other benefits of real value to the investors. In order to have a common measure for these consequences, the other benefits will be measured in terms of their equivalent cash value (positive imputed cash flow).[6]

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[6] Hereafter, the term "cash flow" will be taken to encompass both real cash flow and imputed cash flow.

For purposes of analysis, the future is divided into time periods of convenient length (e.g., a year). Let n be the maximum number of periods into the future in which significant cash flows may occur because of the investments. Immediate cash flows are considered to take place in period 0. Thus, it is necessary to analyze what the cash flows will be in periods $0,1,2,\ldots,n$. Let $X_j(\underline{\delta})$ be the total net cash flow (total positive cash flow minus total negative cash flow) during time period j that would result from making the particular decision $\underline{\delta}$ $(j = 0,1,\ldots,n)$.

It is important to recognize two facts about the nature of the $X_j(\underline{\delta})$. The first is that the cash flow stream resulting from a decision $\underline{\delta}$ is actually an aggregation of many distinct cash flow streams, some of which are interrelated in various ways. Each individual investment approved generates a cash flow stream, which may itself be an aggregation of cash flow streams from a number of sources representing the different kinds of outlays or incomes for the investment. However, the numerical values associated with the individual cash flow streams may be affected significantly by the others. It is the analysis of these interrelationships between the components of the aggregated cash flow stream that forms the basis for the model developed in this section.

The second crucial property of the $X_j(\underline{\delta})$ is that, when dealing with risky investments, they normally are random variables rather than constants (with the probable exception of $X_0(\underline{\delta})$). It is inherent in the nature of risky investments that there is considerable uncertainty about their outcomes. Thus, a decision $\underline{\delta}$ can result in the total net cash flow during period j being any one of many different possible values, depending on intervening circumstances. Therefore, the value that $X_j(\underline{\delta})$ will actually take on can only be described in terms of a probability distribution.

Thus, the direct impact of a decision $\underline{\delta}$ is that an aggregate cash flow stream is generated from the joint probability distribution of $X_0(\underline{\delta}), X_1(\underline{\delta}), \ldots, X_n(\underline{\delta})$. In order to evaluate a particular $\underline{\delta}$, and to choose between alternative feasible values of $\underline{\delta}$, it is therefore necessary to evaluate the desirability of different cash flow streams. Although various measures of desirability have been used in practice, there is considerable theoretical support for using discounted cash flow methods and particularly the present value method. The latter criterion is the one adopted here. Thus, let i be the rate of interest, normally the market cost of capital, which properly reflects the investor's time value of money.[7] Then let $P(\underline{\delta})$ be the total net present value of the investments approved by $\underline{\delta}$, so that

$$P(\underline{\delta}) = \sum_{j=0}^{n} \frac{X_j(\underline{\delta})}{(1+i)^j} .$$

Since the $X_j(\underline{\delta})$ are <u>net</u> cash flows (income minus outlays) for the respective periods, $P(\underline{\delta})$ thereby represents the profit (if positive) or loss (if negative) from the approved investments after discounting for the time value of the money invested.

It should be emphasized that, since the $X_j(\underline{\delta})$ are random variables, $P(\underline{\delta})$ also is a random variable. When a set of risky

[7] This interest rate may be set at different values in different periods if desired.

investments is approved, it is not definitely known in advance whether they will result in a large loss or a large profit or something in between. Therefore, "the" present value $P(\underline{\delta})$ that will result from a decision $\underline{\delta}$ can only be represented in terms of a probability distribution over its range of possible values, as shown in Figure 1.

When only a single investment is being evaluated, the essence of the risk analysis approach proposed by Hillier [8] and Hertz [4] is to develop such a probability distribution (for either present value or another measure of merit) and then to analyze this "risk profile" in order to conclude whether the amount of risk is acceptable or not. If the degree of risk represented by the probabilities and magnitudes of possible losses is sufficiently small relative to the values of these quantities for the possible gains, then the investment would be approved. For the problem now under consideration, where many interrelated investments need to be evaluated simultaneously, the purpose of the analysis changes somewhat. Rather than assessing whether a particular risk profile is "acceptable," the main objective becomes to identify the "best" obtainable risk profile. In effect, the goal is to determine which combination of investments provides the "best" probability distribution of $P(\underline{\delta})$, considering both the possibilities of losses and of gains. An approach to making this kind of choice between alternative probability distributions is developed in the next section. Meanwhile, the question remains of how to analytically determine the probability distribution of $P(\underline{\delta})$ for any given value of $\underline{\delta}$, taking into account the role played by the various kinds of interrelationships between the approved investments. This is the

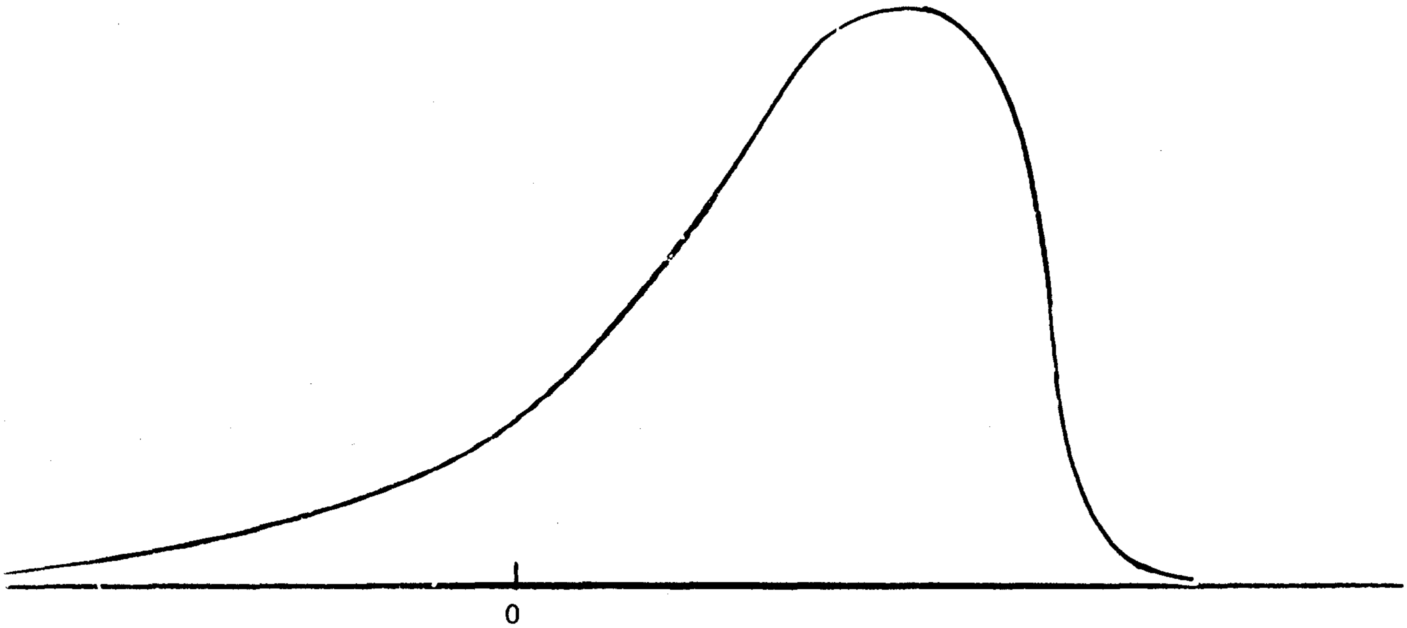

Figure 1. A typical probability distribution of $P(\underline{\delta})$.

question that will be explored throughout the remainder of this section.

Before considering interrelationships between investments, it is necessary to investigate the individual investment proposals by themselves. Thus, the first step in determining the distribution of $P(\underline{\delta})$ is to consider each investment in isolation, analyzing what its performance would be if it were the only one approved (except for any necessary predecessors). To illustrate, consider a typical investment k $(k = 1,2,\ldots,m)$ in isolation. This investment generates one or more cash flow streams from its sources of outlays or incomes. Denote the present value of the aggregation of these cash flow streams (when no other investments are made) by P_k. If this investment has any risk associated with it, then P_k actually is a random variable having some underlying probability distribution. Therefore, the cash flow streams should be analyzed in order to estimate at least the mean μ_k and variance σ_k^2 of this distribution.[8/] The expected present value μ_k is merely the sum of all of the expected discounted cash flows over the respective time periods $(0,1,\ldots,n)$. Determining the variance σ_k^2 is somewhat more difficult, particularly because some of the cash flows may be correlated. One fairly likely kind of correlation is between cash flows from the same source in different periods. For example, it is quite common that if an investment performs considerably better (or worse) than expected in the early periods, then it will tend to also perform better (or worse) in subsequent periods

8/ The estimation of these and other parameters of the model is discussed at the end of the section, and references giving practical estimating procedures are cited.

than had been expected initially. The degree of this tendency can be measured by the correlation coefficient between the cash flows in at least consecutive time periods (as described further in [9, Sec. A.2]). Some models for calculating σ_k^2 in such cases are described in [8] and illustrated in [10]. The second (and perhaps less likely) kind of correlation is between different cash flow streams. The approach in this case is completely analogous to that described subsequently for considering the correlation between the aggregate cash flow streams of different investments in order to calculate the standard deviation of $P(\underline{\delta})$.[9/]

Suppose that one has estimated the mean μ_k and variance σ_k^2 of the present value P_k of investment k, assuming it is the only investment approved, for each $k = 1,2,\ldots,m$. If there were no interrelationships of any kind between the investments, it would then be necessary only to add these values for the approved investments in order to find the mean and variance of $P(\underline{\delta})$. However, the presence of certain kinds of interrelationships can invalidate this simple additive relationship. Consider first how this can happen with the mean because of complementary or competitive effects. Two investments may be complementary because they would share common costs or be mutually reinforcing in generating income. Therefore, even if either investment by itself would be unattractive, the combination of the two investments together might be very worthwhile. Conversely, two competitive investments might be very attractive on an individual basis, but still be very undesirable in combination. In both cases, what is happening, in effect, is that the present value of both investments together is

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9/ Also see [9, Sec. A.2]

something different than the sum of the present values that each would attain if the other were not approved. Thus,

$$P(\underline{\delta}) = \sum_{k=1}^{m} P_k \delta_k + h(\underline{\delta})$$

where the function $h(\underline{\delta})$ is the net amount by which $P(\underline{\delta})$ needs to be adjusted due to complementary or competitive interactions between the approved investments. In general, it is possible to have the amount of a complementary or competitive effect be a random variable rather than a constant, and to have some of the interactions involve more than two investments simultaneously in a complicated way. However, for simplicity, the model here assumes that all these effects are both deterministic and "pairwise additive," so that any joint effect involving more than two investments is merely the cumulation of the pairwise effects. As a result,

$$h(\underline{\delta}) = \sum_{j=1}^{m} \sum_{\substack{k=1 \\ k \neq j}}^{m} \mu_{jk} \delta_j \delta_k ,$$

where $(\mu_{jk} + \mu_{kj})$ is the net addition (positive or negative) to total present value due to complementary or competitive interactions (if any) between investments j and k if both are made. Although only the sum $(\mu_{jk} + \mu_{kj})$ is relevant, the convention is adopted here that $\mu_{jk} = \mu_{kj}$, so μ_{jk} represents investment j's "equal share" of this effect. Thus, whereas μ_j or μ_k would be the individual contribution to expected total present value of investment j or k made by itself, their joint contributions become $(\mu_j + \mu_{jk})$ and $(\mu_k + \mu_{kj})$ if both are approved. Each investment j may have a nonzero complementary or competitive interaction μ_{jk} with more than one investment k. Therefore, letting $\mu(\underline{\delta})$ denote the expected value of $P(\underline{\delta})$, this quantity becomes simply

$$\mu(\underline{\delta}) = \sum_{j=1}^{m} \left[\mu_j + \sum_{\substack{k=1 \\ k \neq j}}^{m} \mu_{jk} \delta_k \right] \delta_j .$$

Now consider how to find the varianc of $P(\underline{\delta})$, which will be denoted by $\sigma^2(\underline{\delta})$. Since $h(\underline{\delta})$ is assumed to be deterministic in the expression for $P(\underline{\delta})$ given above, complementary and competitive interactions have no effect on this variance. However, there may well be other kinds of interrelationships between the investments that affect the amount by which the realized value of $P(\underline{\delta})$ deviates from its expectation. In particular, any common factors that help determine the actual performance of the investments relative to expectations are of this kind. These factors may be internal to the firm, such as the outcome of labor negotiations or the time at which resources from an ongoing project will become available for reallocation. However, the most important factors probably are exogenous, such as the general state of the economy or advances made by competitors. If there are any such factors that may exert a general influence on the performance of some of the investments, the consequence is that the P_j $(j = 1,2,\ldots,m)$ would be correlated rather than statistically independent. Let ρ_{jk} denote the correlation between P_j and P_k, and let $\sigma_{jk} = \rho_{jk}\sigma_j\sigma_k$ be the corresponding covariance. Therefore, assuming that estimates of these parameters can be obtained (as will be discussed shortly), $\sigma^2(\underline{\delta})$ can be calculated simply as

$$\sigma^2(\underline{\delta}) = \sum_{j=1}^{m} \left[\sigma_j^2 + \sum_{\substack{k=1 \\ k \neq j}}^{m} \sigma_{jk} \delta_k \right] \delta_j .$$

Given the mean $\mu(\underline{\delta})$ and variance $\sigma^2(\underline{\delta})$ of total present value $P(\underline{\delta})$, the only remaining question regarding the probability distribution of this random variable is its shape (functional form). One simple case is where the P_j are normally distributed (i.e., $P_1, P_2, \ldots, P_m$ has a multivariate normal distribution), so that the sum yielding $P(\underline{\delta})$ also has a normal distribution. Unfortunately, the distribution of the P_j often would differ significantly from normal, and most other distributions are not preserved under addition. The nature of risky investments is such that their distributions frequently are skewed to the left, as depicted in Figure 1, or even bimodal. In such cases, it is not possible to draw any conclusion as to the exact functional form of the distribution of $P(\underline{\delta})$. However, there is a good chance that an approximate answer can be given. To motivate this, recall that $P(\underline{\delta})$ is the discounted aggregation of many different cash flow streams, with perhaps several such streams being generated by each of the approved investments. Then recall that the Central Limit Theorem indicates that, when summing random variables having distributions other than normal, the distribution f this sum still approaches a normal distribution under certain conditions. The conditions required actually are not very stringent. Although the most commonly known version of the Central Limit Theorem requires that the random variables be independent and identically distributed, various other versions also exist where one or both of these assumptions can be replaced by much weaker conditions.[10] This has two important implications for the current model. First, since the present value of each individual cash flow stream is the (discounted) sum of the cash flow random variables for the respective periods, its distribution actually may be much closer to normal than would have been expected from the perhaps highly skewed or bimodal distributions for these random variables. Second, except for the constant $h(\underline{\delta})$ term, $P(\underline{\delta})$ is the sum of the present value random variables for the individual cash flow streams generated by the approved investments. Therefore, the distribution of $P(\underline{\delta})$ may itself be much closer to normal than would have been expected from the distributions of the individual present values. For these two reasons, it often would be a reasonable approximation to take the distribution of $P(\underline{\delta})$ to be normal, even when the distributions for individual cash flows are far from normal.

Therefore, if it is feasible to estimate the parameters of the model (the μ_j, μ_{jk}, σ_j^2, and σ_{jk}), then the problem of how to identify the probability distribution of present value $P(\underline{\delta})$ for any given feasible solution $\underline{\delta}$ is at least partially solved. The mean $\mu(\underline{\delta})$ and variance $\sigma^2(\underline{\delta})$ can be calculated by the expressions given above, and the form of the distribution can perhaps be concluded to be approximately normal. As will be seen later, this much information about the distribution is adequate for the required analysis.

Hence, it is crucial to be able to estimate the parameters of the model. Actually, this is not as imposing a task from a technical

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[10] These versions are presented and discussed in detail in the companion volume [9, Sec. 4.2].

standpoint as it might appear. The companion volume [9, App.] describes simple estimating procedures whereby personnel without technical backgrounds can provide the needed information, which can then be converted into the desired estimates. The basic approach is patterned after PERT. Thus, Section A.1 of [9] proposes obtaining not one but three estimates for each cash flow, namely, a "most likely" estimate, an "optimistic" estimate, and a "pessimistic" estimate. These three estimates can then be converted into estimates of the mean and variance of the cash flow. Section A.2 presents a useful model for correlation patterns between cash flows in different time periods and between different cash flow streams. Section A.4 indicates how the three-estimate approach can be extended in order to construct estimates of correlation coefficients. Combining these tools leads to the desired estimates of the μ_j, σ_j^2, and σ_{jk} by only obtaining other readily comprehendable types of estimates. Section A.6 discusses the estimation of complementary or competitive effects (for a more complicated model). For most purposes, it should be adequate to use only "most likely" estimates in order to construct the desired estimates of the μ_{jk}.

Thus, the model for considering interrelationships presented in this section allows one to systematically identify (at least in part) and compare the probability distribution of present value $P(\underline{\delta})$ for the various feasible solutions $\underline{\delta}$. The next question that needs to be addressed is how to assess the element of risk in making these comparisons. A model for this purpose is developed in the next section.

3. A Model for Considering Risk

Suppose that the probability distribution of present value $P(\underline{\delta})$ has been identified for a number of different feasible solutions $\underline{\delta}$. For example, two such distributions are shown in Figure 2, one corresponding to a fairly conservative set of investments and the other to a relatively risky set. It is somewhat more likely that the latter set will yield the larger gain, but there is also a significant probability that it will yield a large loss. Therefore, it is necessary for management to assess these risk profiles and evaluate the consequences of the different possible outcomes in order to choose between two such sets of investments.

In general, there would be many alternative sets of investments to be considered. In fact, with m individual investment proposals, there can be as many as 2^m feasible solutions $\underline{\delta}$, each with its own distribution of present value. It is necessary to choose between all of these distributions in order to determine the best combination of investments. There usually would be far too many interesting combinations for it to be feasible for management to evaluate and compare all of these alternatives explicitly. Therefore, what needs to be done is to formalize management's judgment on risk tradeoffs, expressing it quantitatively so that the selection procedure may be carried out systematically on an electronic computer.[11] Fortunately, utility theory provides a relatively satisfactory way of doing just this. The approach involves developing management's utility function $U(p)$, as

11/ In actuality, it will be management, not a computer, that will make the final decision. However, the computer is needed to identify the few best alternatives that should be explicitly evaluated and compared by management.

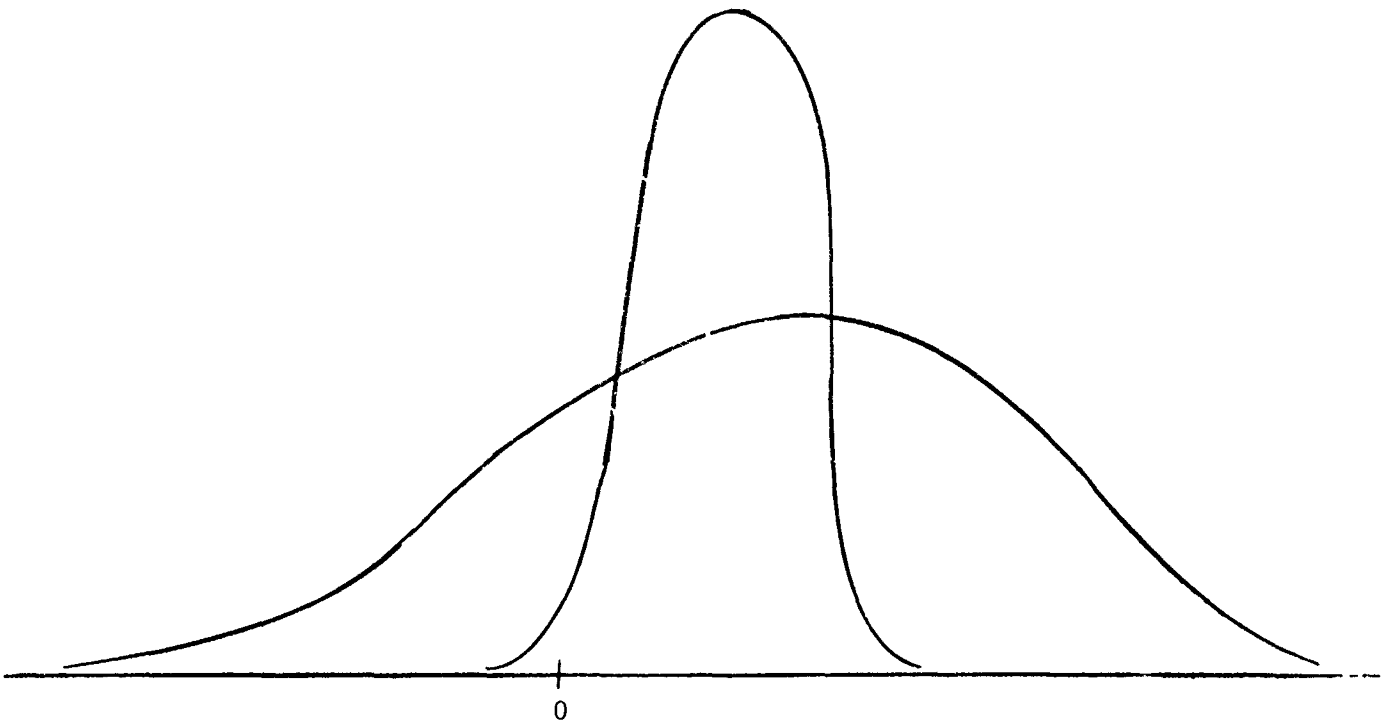

Figure 2. Comparing sets of investments.

illustrated in Figure 3, where $U(p)$ is the utility if p is the realized present value of the approved set of investments.[12] A basic interpretation of utility is that it measures the relative desirability (if positive) or undesirability (if negative) of the outcome. Thus, suppose that $p = 1$ corresponds to a present value of $1 (and that the slope of $U(p)$ remains equal to one for $0 \leq p \leq 1$).[13] For any other value of p, $U(p)$ then converts a present value of that amount to its equivalent worth relative to the first dollar of present value in terms of one's willingness to expend effort and take risks to achieve it (or to avoid it if p is negative).

This concept of $U(p)$ can be made more concrete by comparing risk-taking situations. For example, suppose that the choice is between a very safe and a relatively risky set of investments. The safe investments will yield a positive present value of exactly p_1. The second set of investments would have a 50-50 chance (probability of 1/2) of only breaking even $(p = 0)$ or of yielding a present value of exactly p_2. Then a preference for the safe set of investments implies that $\frac{1}{2} U(p_2) < U(p_1)$, whereas a preference for the second set implies that $\frac{1}{2} U(p_2) > U(p_1)$. Thus, $U(p_2) = 2U(p_1)$ is the break-even value of p_2 (as shown in Figure 3), since then the expected contribution to the firm's welfare (i.e., expected utility) would be exactly the same for the two alternatives.

[12] To fix the scale of $U(p)$, the convention is adopted here that this function passes through the origin with slope one.

[13] For ease of exposition, money often will be assumed here to be measured in units of dollars, although another denomination of meaningful size or a denomination from another monetary system could of course be used instead.

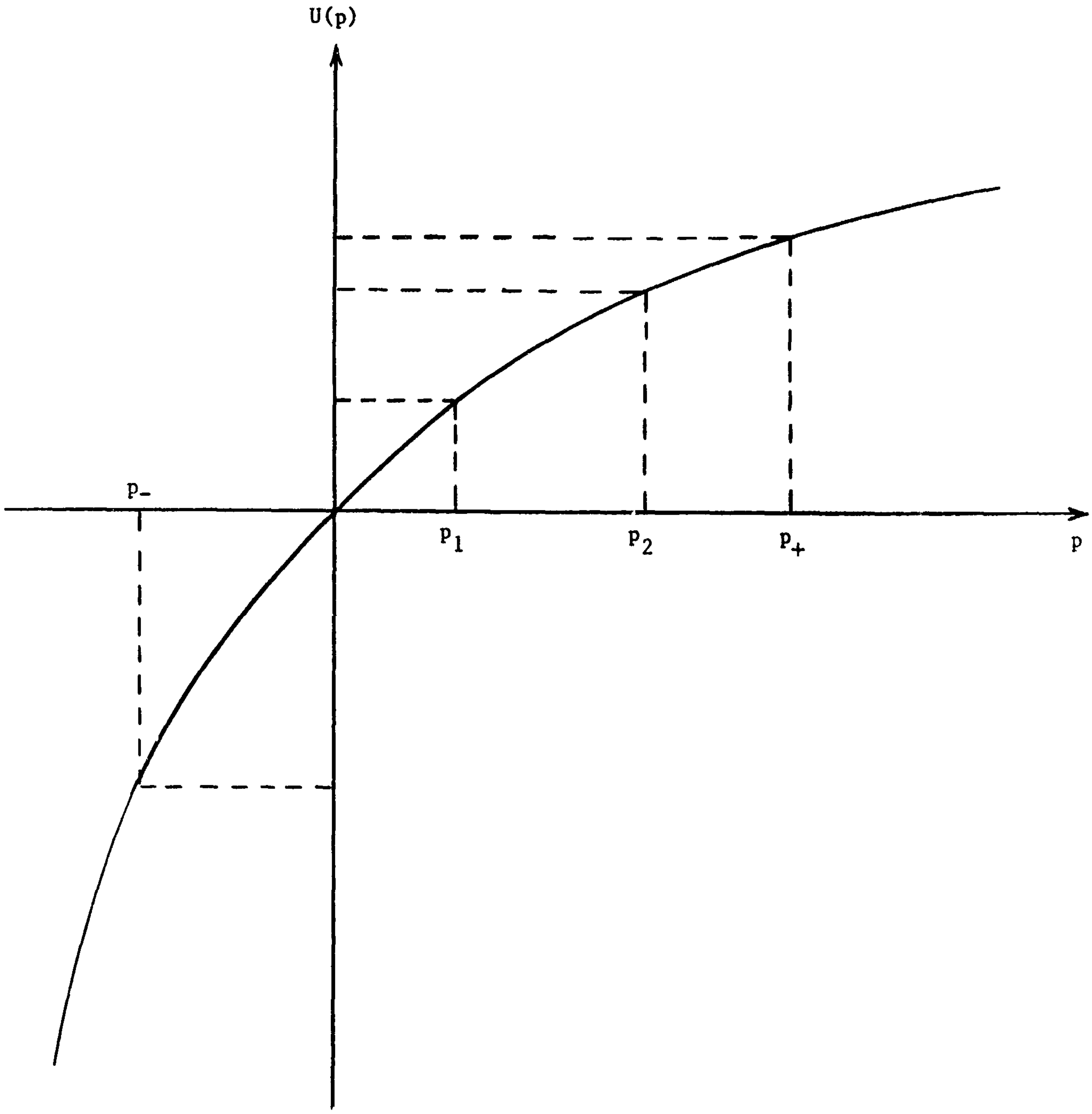

Figure 3. A typical utility function U(p).

As a second example, suppose that the choice is between doing nothing (so $p = 0$) or undertaking a risky set of investments that would yield either a negative present value of exactly p_- or a positive present value of exactly p_+, each with probability $\frac{1}{2}$. To make this decision, management would need to analyze the risk profile of the set of investments, and decide how the undesirability of the loss p_- compares with the desirability of the gain p_+. Approval of the investments implies that $U(p_+) \geq -U(p_-)$, whereas rejection implies that $U(p_+) \leq -U(p_-)$. Thus, the value of p_+ such that $U(p_+) = -U(p_-)$ is the break-even value (as shown in Figure 3) above which the risk of incurring the loss p_- is justified by the largeness of this prospective gain.

Thus, the utility function $U(p)$ merely quantifies management's judgment on risk tradeoffs. Given this function, any decision $\underline{\delta}$ can then be evaluated by calculating its expected utility, $E\{U(P(\underline{\delta}))\}$, where the expectation is taken with respect to the probability distribution of present value $P(\underline{\delta})$. This expected value may be thought of as the difference of two quantities, namely, a positive term (expected positive utility) built up from the gains that may be obtained minus an appropriate penalty for the risk of possible losses. Hence, $E\{U(P(\underline{\delta}))\}$ is a valid single-valued measure of the merit of $\underline{\delta}$ that appropriately combines information about the probable gains and the risk associated with the approved set of investments. This reduces the problem to determining which feasible combination of investments maximizes expected utility, i.e.,

$$\max E\{U(P(\underline{\delta}))\}$$

over all feasible solutions $\underline{\delta}$. Methods for doing this will be discussed in the next section. However, before turning to this topic, the crucial question of how to develop the appropriate utility function $U(p)$ will now be considered.

Although the interpretation of $U(p)$ was suggested above, it is still necessary to describe just what information is needed, how to obtain it from management in a realistic way, and how to use it to generate $U(p)$. To do this, two alternative models for the shape of the utility function are developed below which require obtaining only three relatively nontechnical decisions from management.

How should utility functions $U(p)$ ordinarily be expected to behave? Several characteristics suggest themselves as a starting point for developing a model. First, $U(p)$ should be monotone increasing, i.e., it is always better to increase present value. Second, since tiny changes in present value ordinarily should not matter much, it is likely that $U(p)$ would be continuous and possess continuous derivatives of all orders. (The n^{th} derivative will be denoted here by $U^{(n)}(p)$.) Third, it is plausible that $U(p)$ would be a concave function, since the incremental worth of the next dollar of present value should not increase as present value increases (i.e., decreasing marginal utility). Thus, recalling that $U^{(1)}(0) = 1$, the marginal utility $U^{(1)}(p)$ would always be decreasing from one toward zero as p increases over positive values, whereas it would always be increasing above one as p decreases over negative values.[14/]

[14/] "decreasing" and "increasing" are used here in the weak sense which also allows "remaining the same."

The next question is what happens when p becomes very large positively or negatively. For the positive direction, does $U^{(1)}(p)$ continue decreasing all the way to zero in the limit or does it instead converge to some strictly positive lower bound as p increases? Whichever the case, $U(p)$ thereby converges to a linear asymptote $a_1 + b_1 p$ (assuming a_1 is finite), where $b_1 \geq 0$ is this lower bound on $U^{(1)}(p)$. For p growing very large in the negative direction instead, $U^{(1)}(p)$ may either be converging to some upper bound or increasing indefinitely as p decreases. In the former case, $U(p)$ would thereby converge to a linear asymptote $a_2 + b_2 p$ (assuming a_2 is finite) as $p \rightarrow -\infty$, where the upper bound on $U^{(1)}(p)$ is b_2. For the latter possibility, $U(p)$ would decrease with p at an exponential rate.

Both models for $U(p)$ have all of the properties described above. The distinction between them lies in which possibility occurs as p grows very large in the negative direction. Thus, the first model (which will be called the "basic model" because of its flexibility) assumes that $U(p)$ converges to linear asymptotes in both the negative and positive directions, as shown in Figure 4. The well-known class of functions having such asymptotes are the hyperbolas, and it is further assumed that $U(p)$ has precisely this algebraic form. The parameters of the asymptotes are assumed to be finite, and to satisfy the conditions, $a_1 > 0$, $0 \leq b_1 < 1$, $a_2 > 0$, and $b_2 > 1$. Because of the convention that $U(0) = 0$ and $U^{(1)}(0) = 1$, these parameters are further required to satisfy the relation,

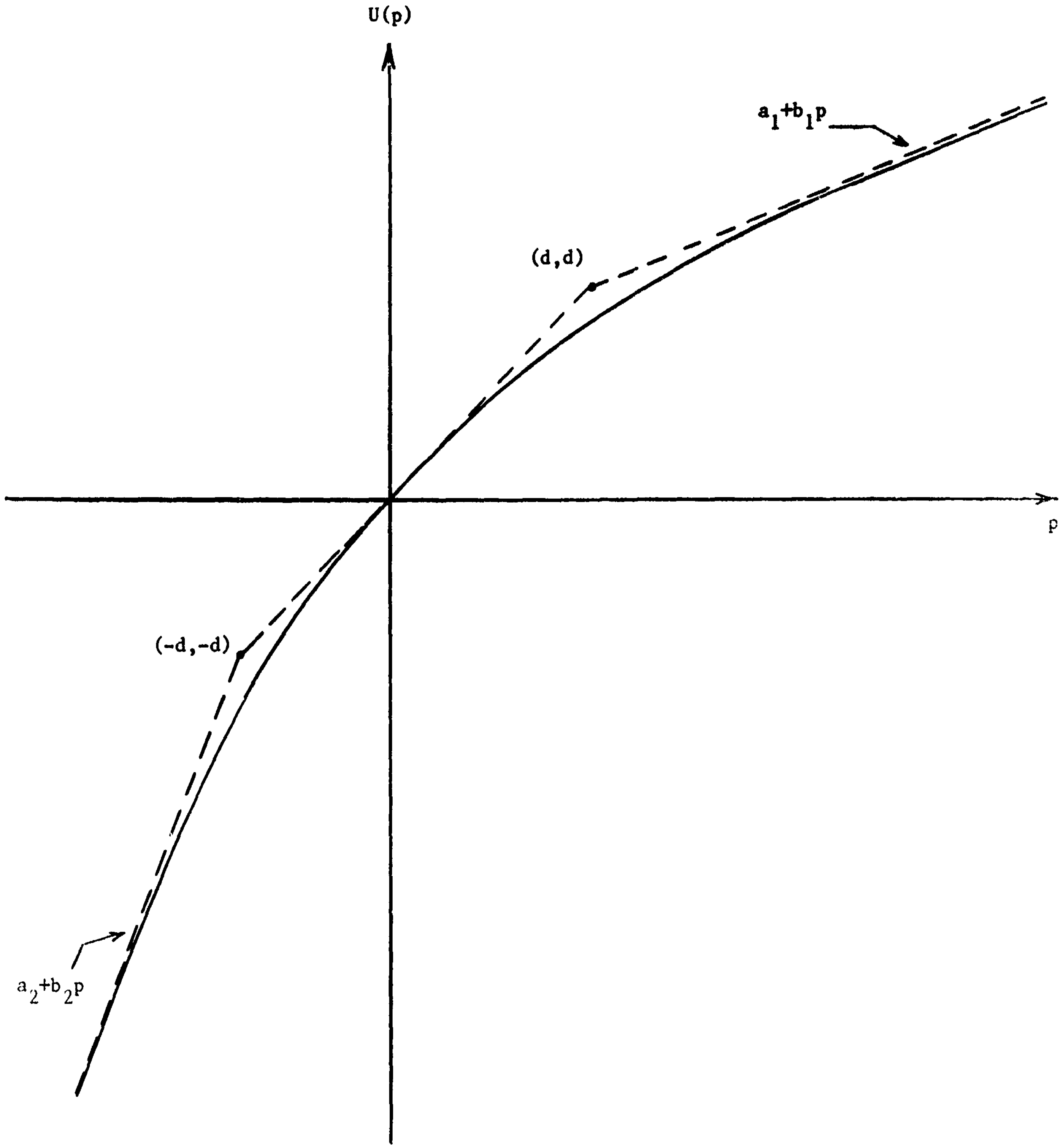

Figure 4. The basic model for U(p).

$a_1 b_2 + a_2 b_1 = a_1 + a_2$, in order for the desired hyperbola to exist.[15]
To interpret this relation, let d denote the number such that $a_2 + b_2 p$ intersects the 45° line through the origin at $(-d,-d)$, as shown in Figure 4. Thus, $a_2 = -d + b_2 d = d(b_2 - 1)$. Since the relation can also be written as $a_1 = \frac{(1-b_1)a_2}{b_2 - 1}$, it is seen that $a_1 = d - b_1 d$, so that $a_1 + b_1 p$ must intersect the 45° line at (d,d). Therefore, the model is, in effect, defined by three parameters--d, b_1, and b_2--where $b_1 (0 \leq b_1 < 1)$ is the slope of an asymptote passing through (d,d), $b_2 (b_2 > 1)$ is the slope of an asymptote passing through $(-d,-d)$, and $U(p)$ is the resulting hyperbola passing through the origin with slope one.

The expression for $U(p)$ defined by this hyperbola is derived in the companion volume [9, pp. 37-42], and is found to be

$$U(p) = \frac{(a_1 + b_1 p) + (a_2 + b_2 p) - Q}{2}$$

where

$$Q = \sqrt{[(a_1 + b_1 p) + (a_2 + b_2 p)]^2 - 4p[a_1 + b_1 b_2 p + a_2]} \ ,$$

$$a_1 = d(1 - b_1) \ ,$$

$$a_2 = d(b_2 - 1) \ .$$

In the next section, it also will be useful to know the first three derivatives of $U(p)$, which are easily found to be

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15/ This is shown in the companion volume [9, p. 41].

$$U^{(1)}(p) = \frac{b_1 + b_2}{2} - \frac{T}{2Q},$$

$$U^{(2)}(p) = \frac{(\frac{T}{Q})^2 - 2(b_2 - b_1)^2}{2Q},$$

$$U^{(3)}(p) = -\frac{3}{2} U^{(2)}(p) \frac{T}{Q^2},$$

where

$$T = (b_2 - b_1)^2 p + (a_1 + a_2)(b_1 + b_2 - 2) .$$

Thus, in order to construct $U(p)$ for the basic model, it is only necessary to obtain judgment decisions from management that lead to estimates of d, b_1, and b_2. The question to be considered now is how to interpret each of these three parameters of the model in a relatively nontechnical way, so that the needed information can be obtained from management in terms meaningful to them. To this end, note that $(-d,-d)$ is the point in Figure 4 at which the "piece-wise linear approximation" of $U(P)$, (the dashed-line function in the figure), changes from a slope of b_2 to a slope of one. Hence, roughly speaking, the additional detrimental effect of each incremental dollar (or larger denomination) of loss is comparable to that for the first dollar if the total loss is somewhat less than d, whereas it is considerably more serious if the total loss is somewhat larger than d. Thus, when examining the effect of increasingly large losses, d is the magnitude of loss above which the situation would deteriorate at an accelerated rate. This interpretation of d can be explained to management in terms of the ability to absorb additional losses. Thus, when the total loss (in present value) is relatively small, an additional dollar (or another meaningfully large denomination of money) of loss can be absorbed almost as easily as the first one. However, when the total loss is already relatively large, an additional loss would be much more serious and difficult to absorb. What is the point of demarcation between these two situations? How large can the loss be before it has become much more difficult to absorb any additional loss? After a suitable explanation, management's answer to this kind of question provides the desired estimate of d.

To interpret b_1 and b_2, recall that the marginal utility (the slope of $U(p)$) is essentially one when p is close to zero, essentially b_1 when p is very large, and essentially b_2 when p is a very large negative number. Therefore, b_1 and b_2 essentially measure the relative desirability of an incremental dollar of present value when a very large gain or loss had already been incurred, measured in "equivalent dollars" of worth corresponding to the desirability of an incremental dollar when the realized present value had been close to zero. To make this more concrete, it is suggested that the following hypothetical situation be posed to management. Suppose that a project has already been approved (to the exclusion of all other proposed investments) that will lead to one of two possible outcomes. The first possibility is that it will "break even," yielding a present value of zero. The second possible outcome is that the project will "make it big," yielding a very large gain much larger than d. Furthermore, the two outcomes are equally likely, i.e., each has a 50-50 chance of happening. Finally, a further moderating action is available that would have the effect of making this project less of an all-or-nothing proposition. In particular, if the project would have broken even, this action would add a gain of one unit of money (a meaningfully large denomination) of present value. On the other hand, if the project would have made it big, the action would decrease the very large present value by x_1 units of money. Should management approve this moderating action? This obviously will depend on the size of x_1, the price that must be paid for this "insurance." The question that needs to be answered by management is, "what is the break-even value of x_1, below which the action should be approved and above which it should not?" Given this break-even value, call it x_1^*, it is then simple to verify that the desired estimate of b_1 is

$$b_1 = 1/x_1^* .$$

To estimate b_2, management should consider a very similar hypothetical situation. The one difference is that, instead of perhaps breaking even, the first possibility for the outcome of the project is that a very large loss (d or larger) in present value will be incurred. Nevertheless, the firm is already committed to this project. However, a moderating action again can be undertaken now if desired. Thus, if the very large loss would have occurred, this action would decrease the loss by one unit of money. As before, if the very large gain would have been attained, the action would decrease the gain by some amount, which will be denoted here by x_2. The corresponding question again needs to be answered by management, namely, "what is the break-even value of x_2, below which the moderating action should be approved and above which it should not?"

Given the break-even value x_2^*, the desired estimate becomes

$$b_2 = x_2^* b_1 = x_2^*/x_1^* .$$

The second model for $U(p)$ differs from the above one only in the behavior of the utility function as p grows very large in the negative direction. Thus, rather than assuming that $U(p)$ converges to a linear asymptote $a_2 + b_2 p$ as this happens, it is instead assumed that $U(p)$ decreases exponentially as $p \to -\infty$, as shown in Figure 5. Therefore, the algebraic form of the function is

$$U(p) = a_1 + b_1 p - a_1 e^{-\left(\frac{1-b_1}{a_1}\right)p} ,$$

where the constants for the last term are implied by the convention that $U(0) = 0$ and $U^{(1)}(0) = 1$. (As before, it is assumed that $a_1 > 0$ and $0 \leq b_1 < 1$.) It is then straightforward to find the first three derivatives as

$$U^{(1)}(p) = b_1 + (1 - b_1) e^{-\left(\frac{1-b_1}{a_1}\right)p} ,$$

$$U^{(2)}(p) = -a_1 \left(\frac{1-b_1}{a_1}\right)^2 e^{-\left(\frac{1-b_1}{a_1}\right)p} ,$$

$$U^{(3)}(p) = a_1 \left(\frac{1-b_1}{a_1}\right)^3 e^{-\left(\frac{1-b_1}{a_1}\right)p} .$$

Since the marginal utility (or marginal negative utility for p decreasing) $U^{(1)}(p)$ increases indefinitely as $p \to -\infty$ rather than approaching a constant, this model exhibits an even greater aversion

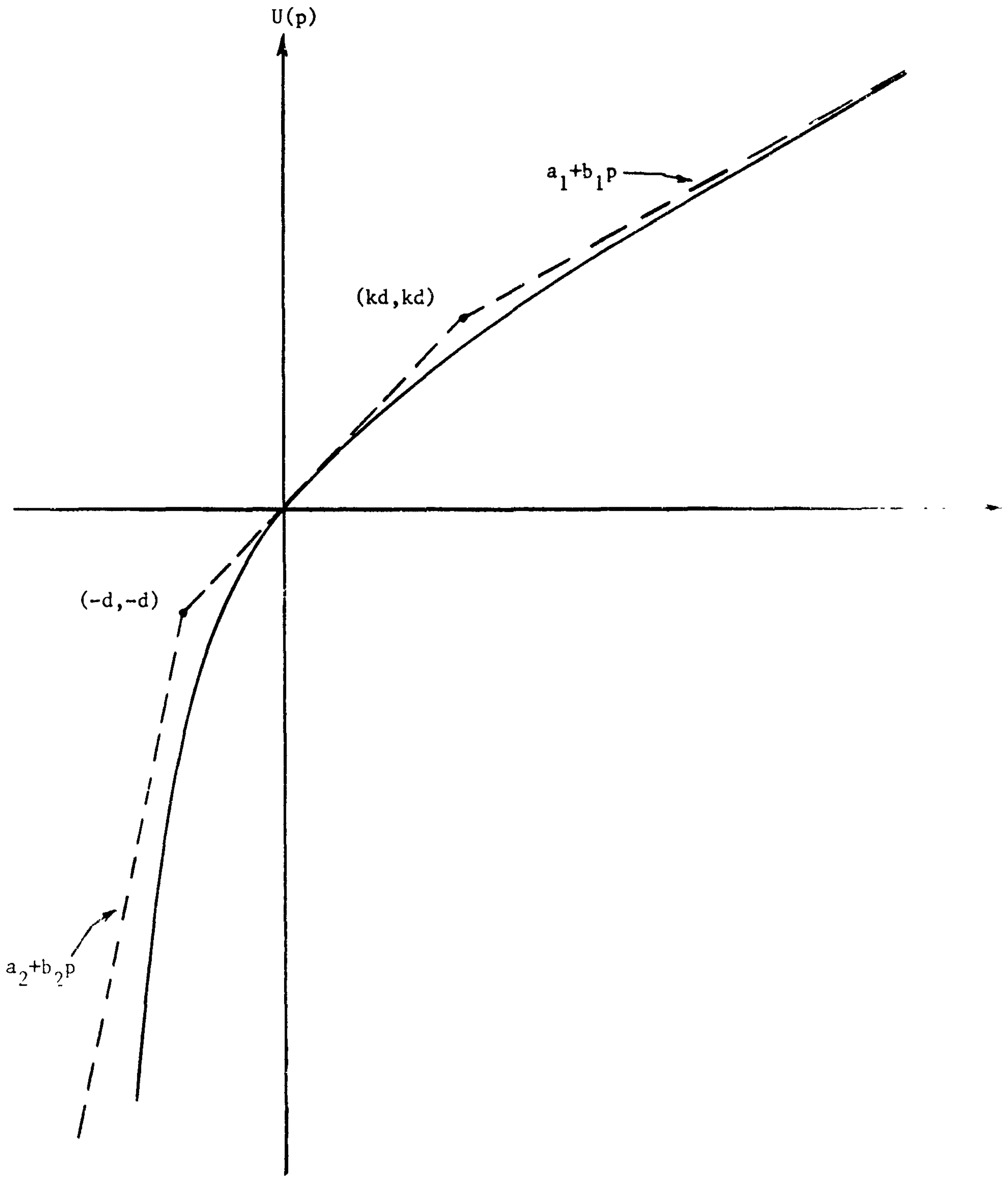

Figure 5. The high risk aversion model for U(p).

to very large losses than the basic model. Therefore, it will be referred to hereafter as the "high risk aversion" model.

Just as for the basic model, the high risk aversion model can also be expressed in terms of the three parameters, d, b_1, and b_2, although their precise meaning is somewhat different here. From a mathematical viewpoint, d now is a somewhat arbitrary large constant (although its interpretation from management's viewpoint is similar to before). b_1 is the slope of the asymptote $a_1 + b_1 p$ which $U(p)$ converges to as $p \to +\infty$, just as before, but b_2 now is defined as

$$b_2 \equiv U^{(1)}(-d) ,$$

the marginal utility at a loss in present value of d. Therefore,

$$b_2 = b_1 + (1 - b_1)e^{-\left(\frac{1-b_1}{a_1}\right)(-d)} ,$$

so that

$$\ell n\left(\frac{b_2 - b_1}{1 - b_1}\right) = \left(\frac{1 - b_1}{a_1}\right)d ,$$

where ℓn denotes the logarithm to the base e. Letting k denote

$$k = 1/\ell n\left(\frac{b_2 - b_1}{1 - b_1}\right) ,$$

a_1 thereby becomes

$$a_1 = (1 - b_1)kd .$$

Hence, the asymptote $a_1 + b_1 p$ must intersect the 45^o line through the origin at (kd, kd), as shown in Figure 5.

From the viewpoint of management, the interpretation of the judgment decisions required in order to estimate d, b_1, and b_2 is essentially the same as for the basic model. Thus, d may be interpreted to management just as before, especially if it is not clear at the outset which of the two models is appropriate. If it is already apparent that the high risk aversion model should be used, then there is some additional flexibility in how the question may be posed. For example, d may be interpreted simply as a "very large loss" above which the situation would be deteriorating disastrously. Since d is only a convenient benchmark value for this model, the important consideration is that it represent a magnitude of loss that is meaningful to management for measuring its aversion to risk (in the process of estimating b_2). The approach to estimating b_1 should be precisely the same as for the basic model. The approach for b_2 also should be the same with the one exception that the unfavorable possible outcome from the hypothetical project must now be a loss of d, rather than any very large loss (d or larger) as before.

Since the difference between the two models is in the behavior of $U^{(2)}(p)$ (which can also be interpreted as the rate of increase of negative utility when p is decreasing) as $p \to -\infty$, the choice between them would be based on an analysis of this behavior. Thus, if there exists a real disaster level where further losses cannot be absorbed reasonably, so that increasing the loss above this level increases the damage (negative utility) at an exponential rate, then the high risk aversion model should be used. On the other hand, if very large possible losses could be absorbed (albeit reluctantly), so that increasing such a loss even further would increase the detrimental effect at a nearly constant rate, then the basic model should be used. This would presumedly be the case if the total commitment for the proposed investments would be quite small relative to the corporate resources. If it is not clear which case applies, then management's attitude can be assessed in the process of obtaining the judgment decision leading to the estimate of b_2. In particular, after the hypothetical situation has been posed in terms of the two possible outcomes (a very large loss d or a very large gain), and the decision on x_2^* has been reached, then management should be asked if their answer would change much if the size of the loss for the first possible outcome were increased significantly. An answer of yes would suggest that $U^{(1)}(p)$ is continuing to grow substantially as p grows very large in the negative direction, so the high risk aversion model should be chosen. An answer of no would suggest that $U^{(1)}(p)$ is approaching a constant as $p \to -\infty$, so the basic model would be the appropriate one.

4. Finding the Best Combination of Investments

Section 2 developed a model for considering interrelationships between investments that allows one to identify the probability distribution of present value $P(\underline{\delta})$ for given feasible solutions $\underline{\delta}$. In particular, expressions were obtained for the mean $\mu(\underline{\delta})$ and variance $\sigma^2(\underline{\delta})$ of this distribution, and it was indicated that the functional form of the distribution would be at least approximately normal under certain conditions. Section 3 then described how to construct an appropriate utility function $U(p)$ (as a function of the realized present value p) which formally quantifies management's judgment on risk tradeoffs. This reduced the problem under consideration to that of determining which feasible combination of investments maximizes expected utility, i.e.,

$$\max E\{U(P(\underline{\delta}))\}$$

over all feasible solutions $\underline{\delta}$. What remains now is to indicate how $E\{U(P(\underline{\delta}))\}$ can be calculated for a given $\underline{\delta}$, and then what solution procedures can be used to find the particular feasible $\underline{\delta}$ which maximizes this quantity.

Under most circumstances, finding the exact value of $E\{U(P(\underline{\delta}))\}$ is not a straightforward task, since this requires calculating the expected value of a rather complicated function $U(\cdot)$ of a random variable $P(\underline{\delta})$ having perhaps a very complex probability distribution. However, one exception is when $P(\underline{\delta})$ has a normal distribution and $U(p)$ has the form described by the high risk aversion model. For this case, the expected utility is found in the companion volume [9, p. 43] to be

$$E\{U(P(\underline{\delta}))\} = a_1 + b_1\mu(\underline{\delta}) - a_1 e^{-\left(\frac{1-b_1}{a_1}\right)\mu(\underline{\delta})+\left(\frac{1-b_1}{a_1}\right)^2\sigma^2(\underline{\delta})/2},$$

where a_1 and b_1 are defined in the preceding section. In most other cases, it is necessary to develop an approximation of $E\{U(P(\underline{\delta}))\}$. As shown in [9, pp. 35-37], this can be done very easily by using a

Taylor series expansion and ignoring terms above the second-order,[16] which yields

$$E\{U(P(\underline{\delta}))\} \approx U(\mu(\underline{\delta})) + \frac{U^{(2)}(\mu(\underline{\delta}))}{2} \sigma^2(\underline{\delta}) .$$

This approximation is a very convenient one since it only requires knowing the mean and variance of $P(\underline{\delta})$, the utility function $U(p)$, and its second derivative $U^{(2)}(p)$, all of which have already been identified for the models considered here. The effect of using the approximation is investigated at the end of the next section.

Two solution procedures for seeking the optimal solution (i.e., the $\underline{\delta}$ corresponding to the best combination of investments) with respect to the model are developed in the companion volume [9, pp. 49-67]. Since both procedures are designed to be executed on a digital computer, only a brief introduction to their general nature will be repeated here.

One of these methods is an approximate procedure, i.e., it will find a solution $\underline{\delta}$ which should be a very good one but is not guaranteed to be actually optimal. It involves solving a sequence of five linear programming problems, where each one is constructed by introducing a suitable linear approximation to the objective function $(E\{U(P(\underline{\delta}))\})$ based on the solution to the preceding problem. For two of these problems, it is necessary to obtain an approximate integer solution, which can be done either by rounding the linear programming solution or by using suitable integer programming methods. (These methods

[16] See [9, p. 36] for this remainder term expressed in terms of the third derivative $U^{(3)}(p)$.

include the highly efficient heuristic procedures developed elsewhere by the author [6] or, if the problem is not too large, the algorithm given by Petersen [15] and Geoffrion [3].) The better of these two integer solutions (usually the second one) is the desired approximate solution.

Since computer codes for solving a linear programming problem are widely available and very efficient, the approximate procedure provides a convenient approach that can even be used on extremely large problems. Furthermore, as Weingartner [17] has found, linear programming is a powerful tool for the further analysis of the problem. For example, the dual evaluators provide vital information for studying whether it would be worthwhile to increase the initial allocation of scarce resources (such as budgeted funds) available for the proposed investments. Another crucial advantage is the ease with which an extensive sensitivity analysis of the input parameters of the model may be conducted.

The second solution procedure is an exact one, i.e., the solution $\underline{\delta}$ it obtains is definitely optimal with respect to the model and objective function used. It is based on the branch-and-bound technique,[17/] inasmuch as the procedure consists of applying a special new branch-and-bound algorithm for integer nonlinear programming[18/] to first a subproblem and then to the original problem. The objective of the subproblem is to find the feasible solution $\underline{\delta}$ which maximizes

[17/] See Hillier and Lieberman [11, pp. 565-570] for an elementary description of this technique.

[18/] This algorithm is presented in the companion volume [9, pp. 53-60].

expected present value $E\{P(\underline{\delta})\}$. This solution is used to structure the original problem in such a way that the branch-and-bound algorithm can then be applied directly to it.

Although the exact procedure is not computationally feasible for the very large problems that can be handled by the approximate procedure, it is quite efficient for problems of moderate size (approximately ten investment proposals). The branch-and-bound approach is not as well-suited as linear programming for sensitivity analysis, etc., so it may sometimes be advantageous to apply both procedures. Furthermore, the two integer solutions obtained in the course of the approximate procedure can be used to expedite the respective applications of the branch-and-bound algorithm in the exact method.

Computational experimentation measuring the efficiency of the two solution procedures, and the effectiveness of the approximate procedure in obtaining good solutions, is reported in the next section.

5. Computational Experience

In order to evaluate the two solution procedures described in the preceding section, ALGOL codes were developed for the IBM-360/67.[19/] For the two phases of the approximate procedure where an approximate integer solution is required, one of the heuristic procedures for integer programming developed by the author [6] was used.[20/] The

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19/ The author wishes to express his great appreciation to Ilan Adler and Mrs. Patricia Peterson for developing the codes, and to Francisco Pereira for running them to obtain the computational results given in this section.

20/ The procedure used is labeled 1-2A-1 in [6].

main objective in designing the codes was reliability rather than efficiency or operational convenience.

The next step was to generate all of the parameters described in Section 2 for ten basic test problems, each involving ten investment proposals $(m = 10)$. This was done on the computer by randomly generating some integers from each of the intervals indicated in Table I.[21/] In the case of the μ_{jk} and ρ_{jk} $(k > j)$, some of them were randomly selected to have a value of zero according to a prespecified probability given in the second column of Table I, and only the rest of them were assigned randomly generated integers. In the case of the σ_j^2, another parameter s_j was generated instead, and then used to calculate σ_j^2 in different ways to obtain a variety of risk profiles, as will be described shortly. Each of the test problems includes five constraints of the form, $\sum_{k=1}^{m} a_{jk}\delta_k \leq b_j$ (which might, for example, represent budget constraints in the first five time periods), and these a_{jk} and b_j also were randomly generated from the intervals indicated in Table I.

21/ A square bracket at the end of an interval indicates that the end-point is included among the integers eligible for random selection, whereas a round parenthesis indicates that it is not included.

Table I

Description of the Randomly Generated Parameters for the Basic Test Problems

Type of Parameter	Prob. of Presetting to Zero	Interval Used for Random Generation
μ_j	0	[-10,90)
μ_{jk}	1/2	(-25,25)
s_j	0	[0,100)
$10\rho_{jk}$	1/2	(-10,10)
a_{1k}	0	[0,100)
a_{2k}	0	[-10,90)
a_{3k}	0	[-20,80)
a_{4k}	0	[-30,70)
a_{5k}	0	[-40,60)
b_j	0	[125,375)

In order to fully define the problems, it was next necessary to calculate the σ_j^2 in terms of the s_j and to set the values of the three utility function parameters described in Section 3. This was done in three different ways, as shown in Table II, in order to obtain sets of problems involving investment proposals with very little risk, a moderate degree of risk, and a very high level of risk associated with them, respectively. The indicated value of b_2 (which is $\frac{1}{2} + \frac{1}{2}e$ rounded to three significant digits) was so chosen in order to have $k = 1$ in the high risk aversion model for $U(p)$. This yields the same linear asymptote when $p \to \infty$ as for the basic model for $U(p)$, thereby providing a better comparison of other features of the two models.

Table II

Value of the Fixed Parameters for the Basic Test Problems

Parameter	Low Risk Problems	Moderate Risk Problems	High Risk Problems
σ_j^2	$\sigma_j^2 = s_j$	$\sigma_j^2 = s_j^2$	$\sigma_j^2 = s_j^3$
d	40	40	200
b_1	0.5	0.5	0.5
b_2	1.86	1.86	1.86

The resulting 30 problems were then used to test the effectiveness of the approximate procedure by comparing its solution with that obtained from the exact procedure, as well as with the exact solution for maximizing $\mu(\underline{\delta})$ if risk were ignored (which is obtained from the first phase of the exact procedure). This was done first with the basic model for $U(p)$, and then with the high risk aversion model for $U(p)$ when the distribution of $P(\underline{\delta})$ is assumed to be normal. The results are shown in Tables III, IV, and V for the three types of problems. For each problem and approach, the table gives (1) "E," the value of $E\{U(P(\underline{\delta}))\}$ for the solution obtained, (2) "μ," the value of $\mu(\underline{\delta})$ for this solution, (3) "σ," the value of $\sigma(\underline{\delta})$ for this solution, and (4) "Time," the time in seconds required to obtain the solution on an IBM 360/67 computer. For each model, the three columns in the tables refer respectively to (1) the feasible solution which maximizes $\mu(\underline{\delta})$ without considering risk, (2) the solution obtained by the approximate procedure, and (3) the optimal solution as obtained by the exact procedure.

Table III

Performance of the Approximate Procedure

with Low Risk Test Problems

Basic Problem Used		Basic Model			High Risk Aversion Model		
		max $\mu(\underline{\delta})$	Approx. Proc.	Exact Proc.	max $\mu(\underline{\delta})$	Approx. Proc.	Exact Proc.
#1	E	213.0	212.5	213.0	232.5	232.0	232.5
	μ	425.0	424.0	425.0	425.0	424.0	425.0
	σ	10.6	8.8	10.6	10.6	8.8	10.6
	Time	0.73	0.78	3.02	0.73	0.70	3.04
#2	E	264.2	237.9	264.2	284.0	257.5	284.0
	μ	528.0	475.0	528.0	528.0	475.0	528.0
	σ	22.9	15.7	22.9	22.9	15.7	22.9
	Time	0.59	0.64	1.20	0.59	0.61	1.47
#3	E	142.5	87.7	142.5	162.0	107.7	162.0
	μ	284.0	176.0	284.0	284.0	176.0	284.0
	σ	6.6	20.5	6.6	6.6	20.5	6.6
	Time	1.38	0.86	8.01	1.38	0.85	7.74
#4	E	152.9	152.9	152.9	172.5	172.5	172.5
	μ	305.0	305.0	305.0	305.0	305.0	305.0
	σ	12.9	12.9	12.9	12.9	12.9	12.9
	Time	1.10	0.64	3.08	1.10	0.66	3.69
#5	E	129.5	128.9	129.5	149.5	149.0	149.5
	μ	259.0	258.0	259.0	259.0	258.0	259.0
	σ	19.8	21.9	19.8	19.8	21.9	19.8
	Time	0.93	0.67	2.70	0.93	0.75	2.67
#6	E	117.0	112.4	117.1	136.9	132.4	137.0
	μ	234.0	225.0	234.0	234.0	225.0	234.0
	σ	17.8	21.0	16.9	17.8	21.0	16.9
	Time	1.63	0.75	7.65	1.63	0.77	7.64
#7	E	53.1	53.1	53.1	71.5	71.5	71.5
	μ	106.0	106.0	106.0	106.0	106.0	106.0
	σ	12.0	12.0	12.0	12.0	12.0	12.0
	Time	0.78	0.75	1.95	0.78	0.67	1.99
#8	E	167.6	167.6	167.6	187.5	187.5	187.5
	μ	335.0	335.0	335.0	335.0	335.0	335.0
	σ	20.9	20.9	20.9	20.9	20.9	20.9
	Time	0.85	0.68	4.30	0.85	0.66	4.17

#9	E	190.8	190.8	190.8	210.5	210.5	210.5
	μ	381.0	381.0	381.0	381.0	381.0	381.0
	σ	16.4	16.4	16.4	16.4	16.4	16.4
	Time	2.83	0.71	8.10	2.83	0.74	8.03
#10	E	128.5	121.7	128.5	148.0	141.5	148.0
	μ	256.0	243.0	256.0	256.0	243.0	256.0
	σ	8.7	15.9	8.7	8.7	15.9	8.7
	Time	0.78	0.71	4.24	0.78	0.71	4.21
Average	E	155.9	146.6	156.0	175.5	166.2	175.5
	μ	311.3	292.8	311.3	311.3	292.8	311.3
	σ	14.9	16.6	14.8	14.9	16.6	14.8
	Time	1.16	0.72	3.42	1.16	0.71	4.46

Table IV

Performance of the Approximate Procedure

with Moderate Risk Test Problems

Basic Probl. Used		Basic Model			High Risk Aversion Model		
		max $\mu(\underline{\delta})$	Approx. Proc.	Exact Proc.	max $\mu(\underline{\delta})$	Approx. Proc.	Exact Proc.
#1	E	207.9	208.0	208.0	232.5	232.5	232.5
	μ	425.0	424.0	424.0	425.0	425.0	425.0
	σ	80.2	75.4	75.4	80.2	80.2	80.2
	Time	0.73	0.73	2.71	0.73	0.76	3.61
#2	E	242.9	228.5	242.9	282.6	257.5	282.6
	μ	528.0	475.0	528.0	528.0	475.0	528.0
	σ	183.5	115.9	183.5	183.5	115.9	183.5
	Time	0.59	0.65	2.46	0.59	0.61	2.19
#3	E	139.5	111.9	139.5	162.0	-109.6	162.0
	μ	284.0	236.0	284.0	284.0	176.0	284.0
	σ	50.5	67.8	50.5	50.5	147.3	50.5
	Time	1.38	0.78	7.40	1.38	0.85	7.99
#4	E	142.0	144.3	144.3	172.3	172.3	172.3
	μ	305.0	289.0	289.0	305.0	305.0	305.0
	σ	99.6	24.8	24.8	99.6	99.6	99.6
	Time	1.10	0.66	3.27	1.10	0.66	3.15
#5	E	95.6	97.0	110.9	34.7	-724.6	134.3
	μ	259.0	216.0	229.0	259.0	258.0	229.0
	σ	162.2	85.7	53.1	162.2	180.9	53.1
	Time	0.93	0.68	2.88	0.93	0.67	3.75
#6	E	93.0	93.0	101.5	125.5	-522.4	133.4
	μ	234.0	234.0	229.0	234.0	225.0	229.0
	σ	130.2	130.2	95.5	130.2	170.8	95.5
	Time	1.63	0.76	6.25	1.63	0.79	8.05
#7	E	20.9	30.1	30.1	38.5	57.5	57.5
	μ	106.0	75.0	75.0	106.0	98.0	98.0
	σ	101.1	42.0	42.0	101.1	78.0	78.0
	Time	0.78	0.60	1.82	0.78	0.69	2.13
#8	E	140.7	140.7	140.7	166.2	166.2	166.2
	μ	335.0	335.0	335.0	335.0	335.0	335.0
	σ	164.3	164.3	164.3	164.3	164.3	164.3
	Time	0.85	0.69	4.72	0.85	0.71	5.01

#9	E μ σ Time	178.6 381.0 118.4 2.83	178.6 381.0 118.4 0.70	178.6 381.0 118.4 7.35	210.4 381.0 118.4 2.83	210.4 381.0 118.4 0.72	210.4 381.0 118.4 8.03
#10	E μ σ Time	124.6 256.0 54.9 0.78	112.9 241.0 76.0 0.69	124.6 256.0 54.9 4.25	147.9 256.0 54.9 0.78	134.8 243.0 126.2 0.72	147.9 256.0 54.9 4.36
Average	E μ σ Time	138.6 277.8 114.5 1.16	134.5 290.6 90.0 0.69	142.1 221.1 86.2 4.31	157.2 277.8 114.5 1.16	-12.5 292.1 128.2 0.72	169.9 307.0 97.8 4.84

Table V

Performance of the Approximate Procedure with High Risk Test Problems

Basic Problem Used		Basic Model			High Risk Aversion Model		
		max $\mu(\underline{\delta})$	Approx. Proc.	Exact Proc.	max $\mu(\underline{\delta})$	Approx. Proc.	Exact Proc.
#1	E	-101	40.0	56.9	-1384	-1082	180.3
	μ	425	95	281	425	424	281
	σ	630	53.8	268	630	617	268
	Time	0.73	0.76	3.65	0.73	0.73	6.60
#2	E	-1050	-7558	0	-9.8×10^{11}	-734.6	64.6
	μ	528	370	0	528	314	91.
	σ	1431	1017	0	1431	556	139
	Time	0.59	0.64	4.91	0.59	0.60	24.98
#3	E	-112	-212	77.9	-118	-343	166.6
	μ	284	236	219	284	236	276
	σ	465	482	149	465	482	289
	Time	1.38	0.82	6.92	1.38	0.78	6.00
#4	E	-541	-541	41.1	-55,571	-35,230	164.5
	μ	305	305	261	305	231	261
	σ	792	792	267	792	750	267
	Time	1.10	0.72	3.78	1.10	0.71	3.43
#5	E	-2468	48.2	48.2	-6.2×10^{10}	100.2	100.2
	μ	259	118	118	259	118	118
	σ	1411	68.9	68.9	1411	68.9	68.9
	Time	0.93	0.60	5.74	0.93	0.61	5.96
#6	E	-1207	27.3	27.3	-3,042,622	47.9	52.6
	μ	234	51	51	234	51	81
	σ	959	15.6	15.6	959	15.6	152
	Time	1.63	0.63	4.53	1.63	0.64	5.46
#7	E	-2261	-47.2	0	-554,102	-36.0	30.9
	μ	106	43	0	106	75	43
	σ	856	96.2	0	856	272	96.2
	Time	0.78	0.59	1.61	0.78	0.66	1.70
#8	E	-1622	48.6	48.6	-8.1×10^{10}	114.9	114.9
	μ	335	142	142	335	142	142
	σ	1332	103	103	1332	103	103
	Time	0.85	0.67	2.70	0.85	0.65	3.32

#9	E	-514	-473	51.2	-312,930	-239	160.6
	μ	381	122	146	381	250	269
	σ	892	442	103	892	472	289
	Time	2.83	0.64	8.91	2.83	0.67	9.45
#10	E	-60.5	95.7	95.7	57.9	57.9	154.7
	μ	256	188	188	256	256	188
	σ	381	24.0	24.0	381	381	24.0
	Time	0.78	0.64	3.33	0.78	0.70	3.35
Average	E	-993.6	-176.9	45.7	-1.123×10^{11}	-37344.4	119.0
	μ	311.3	167.0	140.6	311.3	209.7	175.0
	σ	914.9	309.5	99.9	914.9	371.8	159.6
	Time	1.16	0.60	4.26	1.16	0.68	7.51

An examination of these three tables reveals that the approximate procedure obtained the optimal solution in slightly less than half of the problems. For the low risk test problems, optimality was attained in four of the ten cases with both models for $U(p)$, and the objective function was within 10% of the optimal value in five of the remaining six problems. The optimal solution was obtained for five of the ten moderate risk problems. For the high risk test problems, the optimal solution for the basic model was obtained four times, and the optimal solution for the high risk aversion model was obtained in two cases, but the solution was far from optimal for most of the other problems of this kind. However, the approximate procedure was successful in surpassing the max $\mu(\underline{\delta})$ solution (the best solution if risk were ignored) in eight of the ten high risk test problems (with a tie in one of the remaining two cases).

The results described are all for problems having essentially the same set of values for the parameters of $U(p)$. Therefore, a number of different changes were made in these parameter values and incorporated into the moderate risk version (i.e., $\sigma_j^2 = {}_j^2$) of test problem 5 and the high risk version (i.e., $\sigma_j^2 = s_j^3$) of test problem 6. The objective was to gain some insight into both the effectiveness of the approximate procedure and the behavior of the solutions as the utility function changes. The results are shown in Tables VI and VII, where the first three columns indicate the parameter values used for each run. In order to provide a convenient standard of comparison, the first set of rows repeats the results with the original set of parameter values from the preceding tables.

Table VI

Effect of Changing the Utility Function
on Moderate Risk Test Problem 5

d	b_1	b_2		Basic Model max $\mu(\underline{\delta})$	Basic Model Approx. Proc.	Basic Model Exact Proc.	High Risk Aversion Model max $\mu(\underline{\delta})$	High Risk Aversion Model Approx. Proc.	High Risk Aversion Model Exact Proc.
40	0.5	1.86	E	95.6	97.0	110.9	34.7	-724.6	134.3
			μ	259.0	216.0	229.0	259.0	258.0	229.0
			σ	162.2	85.7	53.1	162.2	180.9	53.1
			Time	0.93	0.68	2.88	0.93	0.67	3.75
20	0.5	1.86	E	95.3	96.7	110.6	-4.6×10^{9}	-1.4×10^{13}	124.5
			μ	259.0	216.0	229.0	259.0	258.0	229.0
			σ	162.2	85.7	53.1	162.2	180.9	53.1
			Time	0.93	0.72	2.96	0.93	0.70	3.75
80	0.5	1.86	E	96.1	97.5	111.4	157.2	148.5	159.9
			μ	259.0	216.0	229.0	259.0	258.0	255.0
			σ	162.2	85.7	53.1	162.2	180.9	139.8
			Time	0.93	0.69	2.88	0.93	0.70	3.01
40	0.1	1.86	E	-10.1	19.0	26.2	51.5	9.1	74.9
				259.0	118.0	229.0	259.0	258.0	229.0
			σ	162.2	19.0	53.1	162.2	180.9	53.1
			Time	0.93	0.67	5.40	0.93	0.67	3.23
40	0.9	1.86	E	205.9	198.7	208.6	239.4	237.2	239.4
			μ	259.0	258.0	255.0	259.0	258.0	259.0
			σ	162.2	180.9	139.8	162.2	180.9	162.2
			Time	0.93	0.67	2.69	0.93	0.67	2.78
40	0.5	1.11	E	125.7	121.0	127.8	191.3	187.4	191.7
			μ	259.0	258.0	255.0	259.0	258.0	255.0
			σ	162.2	180.9	139.8	162.2	180.9	139.8
			Time	0.93	0.64	2.77	0.93	0.64	2.67
40	0.5	10	E	-114.0	50.8	84.2	-3.24×10^{23}	50.8	84.2
			μ	259.0	118.0	229.0	259.0	118	229
			σ	162.2	19.0	53.1	162.2	180.9	139.8
			Time	0.93	0.65	5.43	0.93	0.86	5.30

Table VII

Effect of Changing the Utility Function on High Risk Test Problem 6

d	b_1	b_2		Basic Model max $\mu(\underline{\delta})$	Basic Model Approx. Proc.	Basic Model Exact Proc.	High Risk Aversion Model max $\mu(\underline{\delta})$	High Risk Aversion Model Approx. Proc.	High Risk Aversion Model Exact Proc.
200	0.5	1.86	E	-1207	27.3	27.3	-3,042,622	47.9	52.6
			μ	234	51	51	234	51	81
			σ	959	15.6	15.6	959	15.6	152
			Time	1.63	0.63	4.53	1.63	0.64	5.46
100	0.5	1.86	E	-1213	26.0	26.0	-4.4×10^{20}	45.3	45.3
			μ	234	51	51	234	51	51
			σ	959	11.2	11.2	959	11.2	11.2
			Time	1.63	0.66	4.54	1.63	0.67	5.87
400	0.5	1.86	E	-1197	-880	29.6	-1655	65.5	84.3
			μ	234	105	51	234	81	151
			σ	959	545	11.2	959	149	326
			Time	1.63	0.65	4.55	1.63	0.66	5.04
200	0.1	1.86	E	-1709	30.6	30.6	-21,108	47.1	47.1
			μ	234	51	51	234	51	51
			σ	959	11.2	11.2	959	11.2	11.2
			Time	1.63	0.65	4.43	1.63	0.66	5.04
200	0.9	1.86	E	-1103	-1584	10.4	-2.0×10^{25}	-5.1×10^{7}	--
			μ	234	105	51	234	225	--
			σ	959	545	11.2	959	533	--
			Time	1.63	0.66	4.89	1.63	0.67	> 180
200	0.5	1.11	E	-841	48.3	48.3	-10.6	74.3	122.0
			μ	234	51	51	234	81	151
			σ	959	11.2	11.2	959	149	326
			Time	1.63	0.72	4.79	1.63	0.66	4.30
200	0.5	10	E	-9531	13.7	13.7	-2.0×10^{43}	-38.3	43.2
			μ	234	51	51	234	81	51
			σ	959	11.2	11.2	959	149	11.2
			Time	1.63	0.64	4.41	1.63	0.65	785

The next objective was to test the efficiency of the exact procedure with problems of different sizes. It was seen in the preceding tables that it is quite efficient with problems having five constraints and ten variables, requiring between one and ten seconds in all but one of the 84 cases.[22] However, the nature of this branch-and-bound procedure is that its running time should tend to grow quite rapidly with the size of the problem (especially as m, the number of variables increases). In order to estimate this rate of growth, and the resulting maximum number of variables (investment proposals) for which the procedure should be computationally feasible, some of the basic test problems were modified and/or combined to construct problems of other sizes. In each case, the values of the parameters shown in Table II were retained with the one exception that d was multiplied by $\frac{m}{10}$. All of the original values of the μ_j, μ_{jk}, s_j, and ρ_{jk} also were retained. When new values of μ_{jk} and ρ_{jk} had to be introduced because of combining problems, these new values were automatically set equal to zero. All variables kept their same a_{jk} coefficients in the constraints. In those cases were the number of constraints was reduced from five to one, this was done by eliminating all but the constraint involving the a_{3k} and b_3 parameters. When a basic test problem was used to contribute only half of its variables to the new problem, the first five variables were chosen, and the values of the b_i were divided by two. When combining two basic test problems or

[22] When the high risk aversion model was used with the $b_1 = 0.9$ problem in Table VII, the procedure did not terminate in three minutes of computer time. The reason is not evident.

portions thereof, their current values of b_i were added to obtain the b_i for the new problem.

Table VIII shows the results of the runs on an IBM-360/67 system with these new test problems. The first column indicates which basic test problem(s) was used to construct the problem for that run, and the next two columns give the resulting number of constraints and variables. The remaining three pairs of columns identify the version of the problem (corresponding to the three versions in Table II) and the type of utility function model (basic model or high risk aversion model) that were used. To provide a frame of reference, the average time for all of the basic test problems, as well as the individual times for the ones used to construct new problems here, are repeated from Tables III, IV, and V.

Recall from the preceding section that an exact expression for $E\{U(P(\underline{\delta}))\}$ is now known only when the high risk aversion model is being used and the distribution of $P(\underline{\delta})$ is normal. Otherwise, only an approximate expression, obtained from using a Taylor series expansion, is available. The exact procedure finds the feasible solution that maximizes whatever objective function is being used. Therefore, when the approximate expression is taken to be the objective function, there is no guarantee that the resulting solution will be the exact optimal solution for the real problem. Thus, it is important to verify that this approximation really is an adequate one, especially in terms of the resulting solution still being at least nearly optimal.

It is straightforward to study this question analytically for the case of the high risk aversion model. Define

Table VIII

Time (in sec.) Required by the Exact Procedure

for Test Problems of Different Sizes

Basic Problem(s) Used	No. of Con.	No. of Var.	Type of Problem: Low Risk: Basic M.	Low Risk: High R.A.M.	Moderate Risk: Basic M.	Moderate Risk: High R.A.M.	High Risk: Basic M.	High Risk: High R.A.M.
Ave., #1-10	5	10	3.42	4.46	4.31	4.84	4.26	4.82
$\frac{1}{2}$ of #1	5	5				0.41	0.33	
$\frac{1}{2}$ of #2	5	5				0.64	0.47	
$\frac{1}{2}$ of #5	5	5			0.40			0.30
$\frac{1}{2}$ of #8	5	5			0.39			0.37
#1	1	10				1.13	1.09	
#2	1	10				0.92	4.40	
#5	1	10			0.95			5.92
#8	1	10			1.08			3.23
#1	5	10	3.02	3.04	2.71	3.61	3.65	6.60
#2	5	10	1.20	1.47	2.46	2.19	1.44	2.91
#5	5	10	2.70	2.67	2.88	3.75	5.74	5.96
#8	5	10	4.30	4.17	4.72	5.01	2.70	3.32
#1 + $\frac{1}{2}$ of #2	5	15				72.83	87.80	
#5 + $\frac{1}{2}$ of #8	5	15			17.69			120.67
#1,#2	5	20				162.16	> 180	
#5,#8	5	20			> 180			> 180

$$c = \left(\frac{1-b_1}{a_1}\right)^2 \sigma^2(\underline{\delta})/2 \; .$$

Then set $p = \mu(\underline{\delta})$ and substitute the expressions for $U(p)$ and $U^{(2)}(p)$ given for this model into the approximate expression for $E\{U(P(\underline{\delta}))\}$. This yields

$$E\{U(P(\underline{\delta}))\} \approx a_1 + b_1\mu(\underline{\delta}) - a_1 e^{-\left(\frac{1-b_1}{a_1}\right)\mu(\underline{\delta})} [1+c] \; .$$

Now notice that the exact expression can be written as

$$E\{U(P(\underline{\delta}))\} = a_1 + b_1\mu(\underline{\delta}) - a_1 e^{-\left(\frac{1-b_1}{a_1}\right)\mu(\underline{\delta})} [e^c] \; ,$$

and that the standard expansion for e^c is

$$e^c = 1 + c + \frac{c^2}{2!} + \frac{c^3}{3!} + \cdots \; .$$

Thus, the approximate expression is equivalent here to expanding the e^c part of the exact expression and then ignoring terms above the first-order. The effect is to give somewhat less weight to the penalty for risk (as measured by $\sigma(\underline{\delta})$) than is warranted. The approximation is quite good if c is somewhat less than one, but is quite poor if c is somewhat larger than one. Thus, the critical factor is the relative size of $\frac{a_1}{1-b_1}$ and the $\sigma(\underline{\delta})$. Recall that $\frac{a_1}{1-b_1} = kd$, and that (kd,kd) is the point at which the asymptote $(a_1 + b_1 p)$ of the utility function intersects the 45° line through the origin. Thus, $\frac{a_1}{1-b_1}$ can be interpreted loosely as a point of diminishing returns for gains in present value. If this point is at least comparable in size to $\sigma(\underline{\delta})$ for the interesting values of $\underline{\delta}$, then the approximation should be quite adequate.

For the basic test problems and high risk aversion model, the value of kd is 40 for the low risk and moderate risk versions and 200 for the high risk version of these problems. The corresponding values of the $\sigma(\underline{\delta})$ for interesting $\underline{\delta}$ are such that the value of c would be a rather small fraction of one for the low risk problems, but would tend to be approximately one or even considerably larger for most of the other problems. Therefore, if the approximate expression for $E\{U(P(\underline{\delta}))\}$ were to be used instead of the exact one, it appears that the exact procedure should still give very good solutions for the low risk problems but might not perform well for the other problems. To test this experimentally, all 30 problems were rerun with the exact procedure using the approximate expression. The results are shown in Table IX. The first column identifies the problem, where LR is an abbreviation for Low Risk, MR for Moderate Risk, and HR for High Risk. The second column indicates the time in seconds required by the exact procedure run in this way, and the next two columns give the values of $\mu(\underline{\delta})$ and $\sigma(\underline{\delta})$ for the resulting solution which maximizes the approximate objective function. The fifth column recalculates the value of the objective function for this solution by using the exact expression. The last column then gives the ratio (expressed as a percentage) of this value to the corresponding value for the real optimal solution as given in Tables II, III, or IV. An examination of this column also reveals that the solution obtained actually was

Table IX

Effect of Using the Approximate Expression for E{U(P($\underline{\delta}$))}

with the Exact Procedure and High Risk Aversion Model

Problem	Time	μ	σ	E	$\frac{E}{E^*}$
LR, #1	3.02	425	10.6	232.5	100%
LR, #2	1.47	528	22.9	284.0	100%
LR, #3	7.97	284	6.6	162.0	100%
LR, #4	3.70	305	12.9	172.5	100%
LR, #5	2.66	259	19.8	149.5	100%
LR, #6	7.63	234	16.9	137.0	100%
LR, #7	1.94	106	12.0	71.5	100%
LR, #8	4.05	335	20.9	187.5	100%
LR, #9	8.02	381	16.4	210.5	100%
LR, #10	4.21	256	8.7	148.0	100%
LR, Ave.	4.47	311.3	14.8	175.5	100%
MR, #1	3.61	425	80.2	232.5	100%
MR, #2	2.16	528	183.5	282.6	100%
MR, #3	8.00	284	50.5	162.0	100%
MR, #4	3.08	305	99.6	172.5	100%
MR, #5	2.69	259	162.2	34.7	25.8%
MR, #6	7.70	234	130.2	125.5	94.2%
MR, #7	1.95	106	101.1	38.5	67.1%
MR, #8	4.06	335	164.3	166.2	100%
MR, #9	8.04	381	118.4	210.4	100%
MR, #10	4.25	256	54.9	147.9	100%
MR, Ave.	4.55	311.3	114.5	157.3	88.7%
HR, #1	2.34	396	459	106.2	58.9%
HR, #2	1.64	367	151	282.6	100%
HR, #3	5.11	276	289	166.6	100%
HR, #4	3.21	261	267	164.5	100%
HR, #5	5.95	118	68.9	100.2	100%
HR, #6	5.31	151	326	-1.9	-3.7%
HR, #7	1.69	43	96.2	30.9	100%
HR, #8	3.14	142	103	114.9	100%
HR, #9	7.59	323	420	81.3	50.6%
HR, #10	3.30	188	24.0	154.7	100%
HR, Ave.	3.93	226.5	220.4	120.0	80.6%

optimal for all ten low risk problems, and for seven out of ten of both the moderate risk and high risk problems. Studying the effect of using the approximate expression with the basic model for $U(p)$ is more difficult. However, an analysis of the remainder term that has been deleted from this expression suggests that it probably would tend to be positive, and should tend to increase as $\sigma(\underline{\delta})$ increases, so that the expression underestimates $E\{U(P(\underline{\delta}))\}$ most for large values of $\sigma(\underline{\delta})$. Therefore, it appears that, in contrast to the conclusion drawn for the high risk aversion model, the effect of the approximation with the basic model is to give somewhat more weight to the penalty for risk than is warranted. This conjecture is strongly supported by the computational results given in this section. In those cases where the basic model (using the approximate expression) and the corresponding high risk aversion model (using an exact expression) obtained different solutions on a given problem, it was always the basic model that yielded the solution with the smaller value of $\sigma(\underline{\delta})$, despite the less conservative utility function for this model.

6. Conclusions

The companion volume [9] comprehensively studies the problem of evaluating risky interrelated investments. One basic approach developed there has been selected for further study and elaboration here. This approach involves considering the probability distribution of present value for alternative combinations of investments, and then using powerful solution procedures to seek the feasible combination that maximizes the expected value of a utility function of present value. It takes into account various kinds of interrelationships between the proposed investments in some relatively convenient ways.

The utility function approach to considering risk is a relatively sophisticated technique that usually has been shunned in practice in favor of subjective evaluation of risk profiles by management. However, when a substantial number of proposed projects are being evaluated simultaneously (due to such interrelationships as competing for the same capital resources), so that the number of possible investment combinations is huge, it is then necessary to use some other device to identify the few best alternatives for careful evaluation by management. The utility function approach is a powerful and flexible way of doing this. Furthermore, a practical way of constructing a reasonable utility function by merely obtaining three meaningful judgment decisions from management has been developed in Section 3. Computational experience (see Tables VI and VII) also suggests that the optimal solution tends to be not too sensitive to the choice for each of these decisions. Therefore, the use of a utility function in this way appears to be a much more attractive and practical approach to the problem than might have been anticipated.

A useful area for future research would be a behavioral study of the process by which management would make the three judgment decisions described in Section 3 and of how the resulting answers would relate to management's true preferences on risk tradeoffs. This might lead to a refined procedure for obtaining the required information from management.

It is important to recognize that, when trying to identify the best combination of investments, it is the risk profile for the combination (primarily $\mu(\underline{\delta})$ and $\sigma(\underline{\delta})$) that is crucial rather than the risk profiles for the individual investments. Thus, even though a number of proposals appear individually to be very risky (i.e., σ_j is large relative to μ_j), there may still be an acceptably small degree of risk associated with approving all of these investments jointly. This diversification effect occurs because $\sigma(\underline{\delta})$ tends to grow less rapidly than $\mu(\underline{\delta})$ as more investments are approved,[23] and this effect is even more pronounced if some of the P_j are negatively correlated. Thus, it is dangerous to rely too much on subjective or compartmentalized decision-making here.

Computational experience suggests that the approximate solution is fairly effective in obtaining good solutions (and slightly more so when using the basic model for $U(p)$ rather than the high risk aversion model), since it actually yielded the optimal solution for almost half of the test problems. However, it does tend to give somewhat less weight to the element of risk than is warranted. As a result, it may yield a poor solution when considering a very risky set of proposed investments. Therefore, it is especially important in this situation to calculate the value of $E\{U(P(\underline{\delta}))\}$ for the solution $\underline{\delta}$ obtained in order to verify that this value actually is positive and reasonably large. It also would be advisable to rerun the procedure several times with more conservative values of the σ_j^2 and of the parameters of the utility function to see if less risky solutions yielding a higher expected utility are obtained.

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[23] For example, if N independent and identically distributed investments are approved, then $\sigma(\underline{\delta}) = \sqrt{N}\sigma_j$ whereas $\mu(\underline{\delta}) = N\mu_j$.

Since the approximate procedure is based primarily on using linear programming, it can handle a large number of constraints with ease, and is computationally feasible for problems involving an extremely large number of investment proposals. It also is quite well-suited for conducting an extensive sensitivity analysis (by referring to the final fractional solution rather than the final integer solution). Another advantage is that it can be readily adapted to the case where some of the investment decisions are of the "how much" rather than "yes-or-no" type.

The exact solution procedure does not enjoy any of these advantages of the approximate procedure. However, it does, of course, have the crucial advantage of guaranteeing an optimal solution (with respect to the objective function used) for any problem small enough for it to solve. It is quite efficient for problems involving approximately ten investment proposals or less. However, the computation time grows rapidly with the size of the problem, so it is almost certain to be computationally feasible only up to approximately fifteen proposals. It can also be tried with at least a small hope of success when there are say 20 to 25 proposals. It would seem that the presence of mutually exclusive sets of investment proposals also should increase the efficiency, although this has not been tested.

When it is assumed that $P(\underline{\delta})$ has a normal distribution and that $U(p)$ follows the high risk aversion model, then the corresponding exact expression for $E\{U(P(\underline{\delta}))\}$ should be used with the exact procedure. For other cases, an approximate expression based on a Taylor series expansion of $E\{U(P(\underline{\delta}))\}$ is available. As discussed at the end of the preceding section, this approximate expression performs quite well (still leading to the optimal solution in 24 of the 30 test problems run with the high risk aversion model), but it is sufficiently imprecise that its effect should not be ignored. With the high risk aversion model, it usually gives somewhat less consideration to risk than it should. Therefore, even when the distribution of $P(\underline{\delta})$ is far from normal, the expression that is exact for the normal distribution usually should be used instead in order to better approximate the appropriate risk tradeoffs. The only exceptions might be distributions that are skewed substantially to the right, since then the risk for a given $\mu(\underline{\delta})$ and $\sigma(\underline{\delta})$ tends to be somewhat less than with the normal distribution. When the basic model for $U(p)$ is being used, the approximate expression has the opposite effect, giving somewhat more weight to risk than is warranted. Since no exact expression is now available for the model, it is then especially important to conduct sensitivity analysis to check for alternative solutions that may give a better balance between risk and the expected return in present value.

The choice between the two alternative models for $U(p)$ should be based both on their inherent properties (as discussed in Section 3) and the behavior of the available solution procedures with them. It appears that the two models should be quite comparable in their evaluation of combinations of investments having only a moderate degree of risk, but the high risk aversion model should be much less tolerant of very risky combinations than the basic model. In effect, the high risk aversion model has a threshold level of tolerable risk beyond which it increases the penalty for risk very rapidly (due to the exponential form of the function). Thus, the optimal solutions for the two models (with respect to exact expressions for $E\{U(P(\underline{\delta}))\}$) should tend to be fairly comparable (and often identical), but somewhat riskier solutions may be still rated good by the basic model but very poor by the high risk aversion model. Because of the conservative effect of using the approximate expression for $E\{U(P(\underline{\delta}))\}$ with the basic model, the solution procedures tend to occasionally yield somewhat less risky solutions for this model than the high risk aversion model. This effect should be taken into account in choosing between the two models. However, it should also be noted that this tendency may be less pronounced (and perhaps even reversed) when the utility function parameters are such that k is somewhat less than one (which would be common) in the high risk aversion model rather than the value of one used for almost all of the computational experimentation. A final factor in making the choice is that the approximate solution procedure appears to be at least slightly more effective in obtaining good solutions with the basic model than with the high risk aversion model.

In conclusion, it should be reemphasized that the main practical purpose of this approach to analyzing risky interrelated investments is to identify and quantitatively describe the few best combinations of investments for careful evaluation by management. Thus, rather than using one set of parameter values, one utility function, and one solution procedure to find "the optimal solution," it usually is important to conduct a sensitivity analysis on the data to identify those solutions that are best under some plausible set of estimates.

This may well involve using more than one utility function model and solution procedure, much like in Tables VI and VII. This process will identify a small number of alternatives requiring further consideration, and will provide relevant quantitative measures of performance $(\mu(\underline{\delta}), \sigma(\underline{\delta})$, the resulting risk profile, and $E\{U(P(\underline{\delta}))\})$ for these alternatives under different assumptions. Management can then analyze this information, as well as other relevant factors not incorporated into the model, and make its decision on the best combination of investment proposals to approve.

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(See [9, pp. 101-110] for a comprehensive bibliography)

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APPENDIX

Listing of Test Problems

This appendix presents the ten basic test problems that were randomly generated as described by Table I of Section 4. The numbering of the problems coincides with that used in the various tables. For each problem, the diagonal elements of "MU-MATRIX" are the μ_j in order, and the off-diagonal elements are the μ_{jk}. The diagonal elements of the second matrix in each listing are the s_j, and the off-diagonal elements are the ρ_{jk}. The third matrix lists the a_{jk} and, on the far right, the b_j.

TEST PROBLEM NUMBER 1

MU-MATRIX IS

-6.	0.	22.	4.	0.	0.	-3.	0.	0.	0.
0.	54.	4.	23.	0.	0.	-20.	19.	0.	24.
22.	4.	41.	11.	0.	-4.	0.	0.	0.	0.
4.	23.	11.	18.	-23.	0.	-19.	-3.	0.	6.
0.	0.	0.	-23.	-10.	0.	-5.	23.	0.	6.
0.	0.	-4.	0.	0.	42.	13.	-24.	0.	0.
-3.	-20.	0.	-19.	-5.	13.	-2.	0.	13.	0.
0.	19.	0.	-3.	23.	-24.	0.	66.	9.	-4.
0.	0.	0.	0.	0.	0.	13.	9.	74.	-9.
0.	24.	0.	6.	6.	0.	0.	-4.	-9.	11.

DATA FOR FINDING COVARIANCE MATRIX ARE

22.0	0.0	0.0	0.0	0.0	0.5	0.0	0.0	0.0	-0.5
0.0	41.0	0.2	0.0	0.0	0.2	0.8	0.0	0.0	0.0
0.0	0.2	46.0	-0.1	0.0	0.0	-0.7	0.0	0.0	0.0
0.0	0.0	-0.1	4.0	0.0	0.0	0.0	-0.7	0.8	0.6
0.0	0.0	0.0	0.0	3.0	0.0	-0.3	0.0	0.0	0.0
0.5	0.2	0.0	0.0	0.0	81.0	-0.2	0.0	0.0	0.0
0.0	0.8	-0.7	0.0	-0.3	-0.2	57.0	-0.3	0.0	0.3
0.0	0.0	0.0	-0.7	0.0	0.0	-0.3	15.0	0.0	0.0
0.0	0.0	0.0	0.8	0.0	0.0	0.0	0.0	73.0	-0.3
-0.5	0.0	0.0	0.6	0.0	0.0	0.3	0.0	-0.3	6.0

CONSTRAINT MATRIX AND RIGHT HAND SIDE ARE

45.	29.	36.	85.	84.	26.	27.	38.	83.	25.	$\leqslant$	324.
25.	40.	78.	82.	29.	77.	56.	1.	61.	20.	$\leqslant$	276.
23.	54.	33.	37.	37.	11.	51.	24.	2.	12.	$\leqslant$	372.
-8.	-19.	39.	-10.	65.	66.	-6.	40.	-18.	3.	$\leqslant$	197.
-13.	36.	-16.	10.	46.	27.	-13.	11.	28.	8.	$\leqslant$	218.

TEST PROBLEM NUMBER 2

MU-MATRIX IS

86.	-19.	-2.	0.	0.	0.	-21.	8.	0.	0.
-19.	74.	19.	0.	0.	-17.	-23.	12.	0.	0.
-2.	19.	29.	-9.	-18.	0.	0.	13.	0.	0.
0.	0.	-9.	63.	0.	0.	5.	0.	-17.	-22.
0.	0.	-19.	0.	75.	0.	0.	0.	0.	0.
0.	-17.	0.	0.	0.	41.	0.	0.	0.	0.
-21.	-23.	0.	5.	0.	0.	32.	-3.	0.	-12.
8.	12.	13.	0.	0.	0.	-3.	81.	0.	9.
0.	0.	0.	-17.	0.	0.	0.	0.	59.	0.
0.	0.	0.	-2.	0.	0.	-12.	9.	0.	80.

DATA FOR FINDING COVARIANCE MATRIX ARE

60.0	0.7	0.8	0.0	0.4	0.0	0.0	-0.4	0.0	0.2
0.7	51.0	0.6	0.2	-0.5	0.0	0.0	0.0	0.0	0.0
0.8	0.6	68.0	0.5	-0.2	0.0	0.4	0.0	0.8	0.0
0.0	0.2	0.5	44.0	0.2	-0.6	0.0	0.0	0.8	0.2
0.4	-0.5	-0.2	0.2	49.0	-0.3	0.0	0.0	0.0	0.0
0.0	0.0	0.0	-0.6	-0.3	0.0	0.5	-0.4	0.0	0.0
0.0	0.0	0.4	0.0	0.0	0.5	45.0	0.0	-0.9	0.0
-0.4	0.0	0.0	0.0	0.0	-0.4	0.0	91.0	0.0	-0.8
0.0	0.0	0.8	0.8	0.0	0.0	-0.9	0.0	37.0	0.0
0.2	0.0	0.0	0.2	0.0	0.0	0.0	-0.8	0.0	97.0

CONSTRAINT MATRIX AND RIGHT HAND SIDE ARE

64.	18.	41.	19.	33.	1.	3.	5.	97.	43.	≤	346.
55.	58.	22.	86.	70.	40.	9.	22.	42.	63.	≤	341.
9.	0.	-15.	70.	-18.	75.	63.	48.	77.	45.	≤	349.
13.	-24.	40.	5.	-8.	69.	53.	1.	56.	-25.	≤	146.
15.	-11.	8.	40.	-14.	46.	-12.	0.	55.	-37.	≤	256.

TEST PROBLEM NUMBER 3

MU-MATRIX IS

-5.	0.	0.	0.	0.	7.	0.	9.	0.	0.
0.	11.	0.	0.	-8.	0.	23.	0.	0.	15.
0.	0.	66.	0.	0.	-15.	0.	0.	-8.	1.
0.	0.	0.	57.	0.	0.	11.	0.	0.	0.
0.	-8.	0.	0.	5.	1.	0.	0.	0.	0.
7.	0.	-15.	0.	1.	73.	0.	-13.	0.	23.
0.	23.	0.	11.	0.	0.	52.	0.	0.	4.
9.	0.	0.	0.	0.	-13.	0.	17.	-24.	0.
0.	0.	-8.	0.	0.	0.	0.	-24.	88.	24.
0.	15.	1.	0.	0.	23.	4.	0.	24.	36.

DATA FOR FINDING COVARIANCE MATRIX ARE

47.0	0.0	0.0	0.0	-0.4	0.6	0.0	0.0	0.8	0.0
0.0	29.0	-0.7	0.0	-0.6	0.0	0.1	-0.4	0.3	-0.4
0.0	-0.7	53.0	0.0	0.0	0.0	0.0	0.4	0.2	-0.7
0.0	0.0	0.0	52.0	0.2	-0.6	-0.7	0.0	-0.3	0.0
-0.4	-0.6	0.0	0.2	31.0	-0.1	-0.4	0.5	0.0	-0.4
0.6	0.0	0.0	-0.6	-0.1	21.0	0.5	0.0	0.8	-0.3
0.0	0.1	0.0	-0.7	-0.4	0.5	45.0	-0.2	-0.4	0.0
0.0	-0.4	0.4	0.0	0.5	0.0	-0.2	75.0	0.8	0.1
0.8	0.3	0.2	-0.3	0.0	0.8	-0.4	0.8	44.0	-0.6
0.0	-0.4	-0.7	0.0	-0.4	-0.3	0.0	0.1	-0.6	85.0

CONSTRAINT MATRIX AND RIGHT HAND SIDE ARE

18.	31.	26.	52.	54.	20.	87.	38.	83.	45.	$\leq$	304.
42.	13.	86.	49.	-9.	89.	36.	41.	6.	12.	$\leq$	278.
66.	58.	23.	40.	19.	75.	26.	-12.	38.	78.	$\leq$	171.
40.	1.	58.	64.	-26.	44.	67.	-28.	67.	8.	$\leq$	234.
35.	12.	-36.	-12.	-4.	-20.	41.	-8.	41.	1.	$\leq$	176.

TEST PROBLEM NUMBER 4

MU-MATRIX IS

79.	0.	18.	0.	6.	0.	0.	-13.	0.	0.
0.	58.	-23.	0.	0.	17.	0.	-15.	0.	-6.
18.	-23.	-10.	-24.	12.	2.	0.	0.	0.	0.
0.	0.	-24.	77.	-2.	0.	-23.	-20.	-18.	21.
6.	0.	12.	-2.	12.	0.	0.	0.	-14.	-1.
0.	17.	2.	0.	0.	35.	3.	0.	16.	13.
0.	0.	0.	-23.	0.	3.	61.	0.	12.	0.
-13.	-15.	0.	-20.	0.	0.	0.	5.	0.	0.
0.	0.	0.	-18.	-14.	16.	12.	0.	39.	0.
0.	-6.	0.	21.	-1.	13.	0.	0.	0.	45.

DATA FOR FINDING COVARIANCE MATRIX ARE

98.0	0.2	0.0	-0.6	-0.1	-0.6	0.0	-0.3	0.0	-0.9
0.2	0.0	-0.9	0.0	0.0	0.0	0.0	0.0	-0.8	0.0
0.0	-0.9	13.0	0.9	-0.3	-0.1	0.0	-0.6	0.0	0.0
-0.6	0.0	0.9	76.0	3.9	0.0	-0.4	0.8	0.0	0.0
-0.1	0.0	-0.3	3.9	0.0	0.0	-0.5	0.7	0.0	-0.2
-0.6	0.0	-0.1	0.0	0.0	47.0	0.0	0.6	-0.6	0.0
0.0	0.0	0.0	-0.4	-0.5	0.0	25.0	0.0	0.5	0.0
-0.3	0.0	-0.6	0.8	0.7	0.6	0.0	27.0	-0.3	0.0
0.0	-0.8	0.0	0.0	0.0	-0.6	0.5	-0.3	31.0	0.0
-0.9	0.0	0.0	0.0	-0.2	0.0	0.0	0.0	0.0	44.0

CONSTRAINT MATRIX AND RIGHT HAND SIDE ARE

18.	78.	48.	32.	61.	18.	30.	80.	83.	4.	$\leq$	142.
72.	3.	15.	4.	22.	74.	59.	48.	61.	50.	$\leq$	183.
-8.	4.	77.	73.	45.	26.	79.	3.	71.	7.	$\leq$	303.
-12.	6.	34.	12.	43.	-27.	4.	26.	52.	-10.	$\leq$	204.
-20.	16.	18.	24.	-35.	-21.	-25.	28.	-18.	38.	$\leq$	159.

TEST PROBLEM NUMBER 5

MU-MATRIX IS

78.	-8.	0.	-22.	6.	0.	-16.	-18.	0.	0.
-8.	56.	-5.	-20.	-24.	3.	3.	0.	2.	4.
0.	-5.	-10.	18.	-17.	-19.	0.	11.	0.	0.
-22.	-20.	18.	-10.	0.	11.	-5.	-5.	-6.	0.
6.	-24.	-17.	0.	-10.	12.	0.	0.	0.	-3.
0.	3.	-19.	11.	12.	87.	0.	-18.	0.	0.
-16.	3.	0.	-5.	0.	0.	70.	0.	0.	-3.
-18.	0.	11.	-5.	0.	-18.	0.	1.	0.	20.
0.	2.	0.	-6.	0.	0.	0.	0.	30.	22.
0.	4.	0.	0.	-3.	0.	-3.	20.	22.	31.

DATA FOR FINDING COVARIANCE MATRIX ARE

14.0	0.0	-0.8	-0.6	-0.3	0.0	0.0	-0.1	0.0	0.0
0.0	99.0	0.9	0.0	0.0	0.5	0.9	0.0	0.2	0.8
-0.8	0.9	69.0	0.0	0.0	0.0	-0.2	0.8	0.6	0.0
-0.6	0.0	0.0	22.0	0.0	0.9	0.0	0.0	-0.7	0.0
-0.3	0.0	0.0	0.0	8.0	0.2	0.0	0.0	-0.9	0.1
0.0	0.5	0.0	0.9	0.2	48.0	0.0	0.0	0.0	0.0
0.0	0.9	-0.2	0.0	0.0	0.0	13.0	-0.8	0.0	0.0
-0.1	0.0	0.8	0.0	0.0	0.0	-0.8	48.0	0.0	0.0
0.0	0.2	0.6	-0.7	-0.9	0.0	0.0	0.0	66.0	0.0
0.0	0.8	0.0	0.0	0.1	0.0	0.0	0.0	0.0	61.0

CONSTRAINT MATRIX AND RIGHT HAND SIDE ARE

7.	20.	65.	72.	14.	92.	93.	52.	69.	23.	$\leq$	331.
60.	21.	49.	0.	-1.	1.	26.	86.	86.	84.	$\leq$	160.
23.	5.	29.	-3.	12.	8.	67.	-2.	45.	57.	$\leq$	280.
49.	3.	11.	30.	49.	38.	8.	60.	5.	-8.	$\leq$	296.
13.	3.	38.	-27.	-2.	-21.	2.	0.	-16.	-37.	$\leq$	331.

TEST PROBLEM NUMBER 6

MU-MATRIX IS

74.	0.	-2.	-11.	4.	0.	0.	-14.	0.	0.
0.	20.	-21.	0.	0.	-24.	-13.	0.	-7.	13.
-2.	-21.	12.	0.	0.	0.	12.	-15.	2.	0.
-11.	0.	0.	52.	0.	18.	0.	0.	19.	0.
4.	0.	0.	0.	60.	-24.	0.	0.	-12.	-20.
0.	-24.	0.	18.	-24.	51.	0.	0.	0.	0.
0.	-13.	12.	0.	0.	0.	21.	-11.	0.	0.
-14.	0.	-15.	0.	0.	0.	-11.	30.	0.	0.
0.	-7.	2.	19.	-12.	0.	0.	0.	48.	0.
0.	13	0.	0.	-20.	0.	0.	0.	0.	75.

DATA FOR FINDING COVARIANCE MATRIX ARE

82.0	0.0	0.2	0.0	0.0	0.0	0.0	-0.6	0.0	0.0
0.0	95.0	-0.3	0.6	0.6	0.0	0.0	-0.9	0.0	-0.8
0.2	-0.3	36.0	-0.5	0.3	0.0	0.4	0.0	0.0	0.0
0.0	0.6	-0.5	53.0	0.0	-0.7	0.8	0.8	0.0	0.0
0.0	0.6	0.3	0.0	98.0	0.3	0.0	-0.5	0.0	0.3
0.0	0.0	0.0	-0.7	0.3	5.0	0.0	0.0	0.0	0.0
0.0	0.0	0.4	0.8	0.0	0.0	46.0	0.0	0.0	0.2
-0.6	-0.9	0.0	0.8	-0.5	0.0	0.0	28.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	52.0	0.0
0.0	-0.8	0.0	0.0	0.3	0.0	0.2	0.0	0.0	65.0

CONSTRAINT MATRIX AND RIGHT HAND SIDE ARE

66.	63.	5.	19.	45.	30.	50.	41.	50.	99.	≤	222.
28.	10.	13.	60.	29.	85.	26.	32.	38.	31.	≤	160.
-18.	68.	27.	52.	-3.	41.	3.	40.	13.	12.	≤	129.
20.	36.	42.	24.	2.	-19.	28.	32.	51.	-15.	≤	374.
6.	-37.	-23.	9.	20.	12.	-30.	28.	37.	-4.	≤	153.

TEST PROBLEM NUMBER 7

MU-MATRIX IS

37.	0.	0.	0.	-11.	0.	17.	20.	0.	0.
0.	19.	-18.	0.	-16.	4.	-11.	8.	0.	0.
0.	-18.	43.	0.	-10.	0.	10.	0.	0.	5.
0.	0.	0.	42.	0.	-17.	0.	0.	-7.	0.
-11.	-16.	-10.	0.	21.	6.	0.	0.	-1.	0.
0.	4.	0.	-17.	6.	11.	-16.	0.	0.	0.
17.	-11.	10.	0.	0.	-16.	35.	0.	0.	-19.
20.	8.	0.	0.	0.	0.	0.	65.	0.	0.
0.	0.	0.	-7.	-1.	0.	0.	0.	34.	-10.
0.	0.	5.	0.	0.	0.	-19.	0.	-10.	75.

DATA FOR FINDING COVARIANCE MATRIX ARE

70.0	0.0	0.0	-0.9	0.0	0.0	0.0	0.2	0.0	0.0
0.0	25.0	-0.6	0.0	0.0	0.0	-0.7	-0.2	-0.8	-0.9
0.0	-0.6	21.0	0.1	0.0	0.0	0.1	0.0	0.1	0.0
-0.9	0.0	0.1	96.0	0.0	0.0	0.7	-0.7	-0.7	0.0
0.0	0.0	0.3	0.0	84.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	39.0	0.0	0.0	0.4	0.4
0.0	-0.7	0.1	0.7	0.0	0.0	73.0	0.0	-0.2	-0.2
0.2	-0.2	0.0	-0.7	0.0	0.0	0.0	85.0	0.0	0.0
0.0	-0.8	0.1	-0.7	0.0	0.4	-0.2	0.0	51.0	-0.2
0.0	-0.9	0.0	0.0	0.0	0.4	-0.2	0.0	-0.2	42.0

CONSTRAINT MATRIX AND RIGHT HAND SIDE ARE

77.	58.	62.	94.	53.	36.	49.	92.	82.	95.	≤	127.
34.	57.	-1.	54.	-7.	-7.	26.	24.	45.	79.	≤	251.
29.	21.	64.	19.	27.	-12.	43.	27.	-16.	33.	≤	248.
52.	50.	19.	28.	-15.	60.	6.	-27.	59.	67.	≤	279.
-27.	56.	-39.	-2.	52.	9.	25.	-7.	-6.	-7.	≤	290.

TEST PROBLEM NUMBER 8

MU-MATRIX IS

0.	0.	0.	13.	-11.	20.	-8.	0.	-10.	-1.
0.	79.	0.	0.	2.	21.	0.	-17.	-3.	12.
0.	0.	38.	-5.	-20.	-5.	0.	-15.	0.	0.
13.	0.	-5.	72.	11.	0.	7.	0.	21.	20.
-11.	2.	-20.	11.	48.	0.	0.	0.	0.	-6.
20.	21.	-5.	0.	0.	-8.	-7.	0.	0.	0.
-8.	0.	0.	7.	0.	-7.	84.	0.	0.	0.
0.	-17.	-15.	0.	0.	0.	0.	32.	0.	10.
-10.	-3.	0.	21.	0.	0.	0.	0.	-10.	24.
-1.	12.	0.	20.	-6.	0.	0.	10.	24.	12.

DATA FOR FINDING COVARIANCE MATRIX ARE

91.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.1	0.0	0.0
0.0	69.0	0.0	0.0	0.0	-0.1	0.0	0.0	0.8	0.8
0.0	0.0	90.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	21.0	0.0	-0.4	0.1	-0.5	0.0	0.0
0.0	0.0	0.0	0.0	11.0	0.0	-0.8	-0.9	-0.5	-0.2
0.0	-0.1	0.0	-0.4	0.0	78.0	0.5	0.0	0.4	0.0
0.0	0.0	0.0	0.1	-0.8	0.5	67.0	0.0	-0.6	0.0
-0.1	0.0	0.0	-0.5	-0.9	0.0	0.0	94.0	0.7	0.0
0.0	0.8	0.0	0.0	-0.5	0.4	-0.6	0.7	62.0	0.0
0.0	0.8	0.0	0.0	-0.2	0.0	0.0	0.0	0.0	26.0

CONSTRAINT MATRIX AND RIGHT HAND SIDE ARE

10.	47.	62.	4.	55.	6.	15.	12.	2.	43.	≤	302.
50.	28.	74.	34.	46.	13.	53.	76.	-6.	72.	≤	147.
9.	22.	19.	71.	18.	23.	37.	0.	2.	50.	≤	345.
-7.	-1.	-22.	58.	46.	-23.	12.	-20.	8.	4.	≤	282.
-3.	-1.	52.	32.	41.	-12.	53.	43.	56.	-24.	≤	236.

TEST PROBLEM NUMBER 9

MU-MATRIX IS

21.	0.	22.	15.	19.	-4.	0.	3.	1.	-10.
0.	75.	19.	0.	0.	0.	0.	-1.	-21.	-19.
22.	19.	18.	0.	-9.	0.	0.	0.	0.	0.
15.	0.	0.	[illegible]	0.	0.	0.	0.	15.	[illegible]
19.	0.	-9.	0.	16.	21.	0.	0.	0.	[illegible]
-4.	0.	0.	0.	21.	78.	0.	0.	0.	0.
0.	0.	0.	0.	0.	0.	80.	-3.	0.	20.
3.	-1.	0.	0.	0.	0.	-3.	36.	0.	14.
1.	-21.	0.	15.	0.	0.	0.	0.	58.	0.
-10.	-19.	0.	0.	16.	0.	20.	14.	0.	82.

DATA FOR FINDING COVARIANCE MATRIX ARE

61.0	0.7	0.0	-0.9	0.0	0.0	-0.5	-0.8	-0.4	0.3
0.7	31.0	0.0	-0.5	-0.6	0.0	0.6	0.0	0.0	0.0
0.0	0.0	43.0	0.0	0.0	0.7	0.6	0.0	-0.5	-0.9
-0.9	-0.5	0.0	58.0	0.0	0.9	0.3	0.0	0.0	0.0
0.0	-0.6	0.0	0.0	85.0	-0.2	0.0	-0.1	0.0	-0.9
0.0	0.0	0.7	0.9	-0.2	89.0	0.6	0.3	-0.2	0.0
-0.5	0.6	0.6	0.3	0.0	0.6	68.0	0.0	0.0	0.5
-0.8	0.0	0.0	0.0	-0.1	0.3	0.0	0.0	-0.1	-0.4
-0.4	0.0	-0.5	0.0	0.0	-0.2	0.0	-0.1	87.0	0.3
0.3	0.0	-0.9	0.0	-0.9	0.0	0.5	-0.4	0.3	22.0

CONSTRAINT MATRIX AND RIGHT HAND SIDE ARE

79.	0.	98.	31.	82.	86.	96.	34.	2.	31.	≤	235.
28.	15.	27.	65.	63.	70.	22.	26.	35.	4.	≤	327.
55.	-15.	12.	44.	33.	36.	47.	9.	58.	78.	≤	194.
-22.	4.	7.	44.	26.	46.	6.	14.	-12.	34.	≤	252.
2.	51.	25.	27.	33.	-9.	18.	-39.	-18.	4.	≤	171.

TEST PROBLEM NUMBER 10

MU-MATRIX IS

50.	0.	0.	-17.	20.	0.	0.	0.	0.	0.
0.	78.	0.	18.	0.	-10.	0.	0.	19.	0.
0.	0.	77.	0.	-16.	0.	17.	0.	0.	23.
-17.	18.	0.	19.	0.	0.	-9.	15.	0.	-16.
20.	0.	-16.	0.	83.	0.	0.	0.	0.	6.
0.	-10.	0.	0.	0.	83.	16.	0.	0.	0.
0.	0.	17.	-9.	0.	16.	15.	-17.	0.	0.
0.	0.	0.	15.	0.	0.	-17.	-4.	-2.	-10.
0.	19.	0.	0.	0.	0.	0.	-2.	2.	0.
0.	0.	23.	-16.	6.	0.	0.	-10.	0.	3.

DATA FOR FINDING COVARIANCE MATRIX ARE

4.0	0.0	0.0	0.0	0.0	0.7	0.0	0.0	0.2	0.0
0.0	64.0	-0.5	-0.9	0.0	0.7	0.0	-0.6	0.0	0.0
0.0	-0.5	51.0	-0.2	-0.9	0.0	0.5	0.1	0.0	0.0
0.0	-0.9	-0.2	28.0	-0.9	-0.8	-0.7	0.8	-0.5	0.6
0.0	0.0	-0.9	-0.9	0.0	0.1	0.0	0.0	0.0	0.0
0.7	0.7	0.0	-0.8	0.1	52.0	0.0	0.5	0.4	0.0
0.0	0.0	0.5	-0.7	0.0	0.0	8.0	0.0	-0.3	0.0
0.0	-0.6	0.1	0.8	0.0	0.5	0.0	76.0	0.0	0.0
0.2	0.0	0.0	-0.5	0.0	0.4	-0.3	0.0	84.0	0.0
0.0	0.0	0.0	0.6	0.0	0.0	0.0	0.0	0.0	21.0

CONSTRAINT MATRIX AND RIGHT HAND SIDE ARE

50.	84.	96.	91.	95.	11.	5.	18.	58.	0.	$\leqslant$	179.
73.	52.	27.	46.	57.	32.	39.	12.	1.	53.	$\leqslant$	183.
53.	18.	10.	51.	-16.	11.	79.	40.	29.	69.	$\leqslant$	147.
35.	19.	-21.	-18.	-28.	57.	20.	-5.	59.	34.	$\leqslant$	352.
19.	50.	-22.	-30.	-17.	-36.	35.	10.	39.	-11.	$\leqslant$	192.

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13 ABSTRACT

When evaluating proposals for capital investment, it often is necessary to consider interrelationships of various kinds between the proposed projects. The amount of risk associated with the investments also is a very important factor. This paper studies the problem of how to simultaneously take both considerations into account in order to determine the best overall combination of projects to approach. In particular, it focuses on one basic approach developed in the author's monograph, The Evaluation of Risky Interrelated Investments (North-Holland, 1969), and extends it further in terms of formulation and interpretation needed for implementation. This approach uses present value (treated as a random variable) and expected utility of present value as criteria.

The paper begins by discussing the nature of various possible project interrelationships and how to take their effects into account in the analysis. It then develops a model for also considering risk, including an easily implemented method of constructing an appropriate utility function. An approximate linear programming approach and an exact integer nonlinear programming algorithm for finding the best combination of investments are reviewed. Finally, the results of extensive computational experimentation with these solution procedures are reported and evaluated.

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