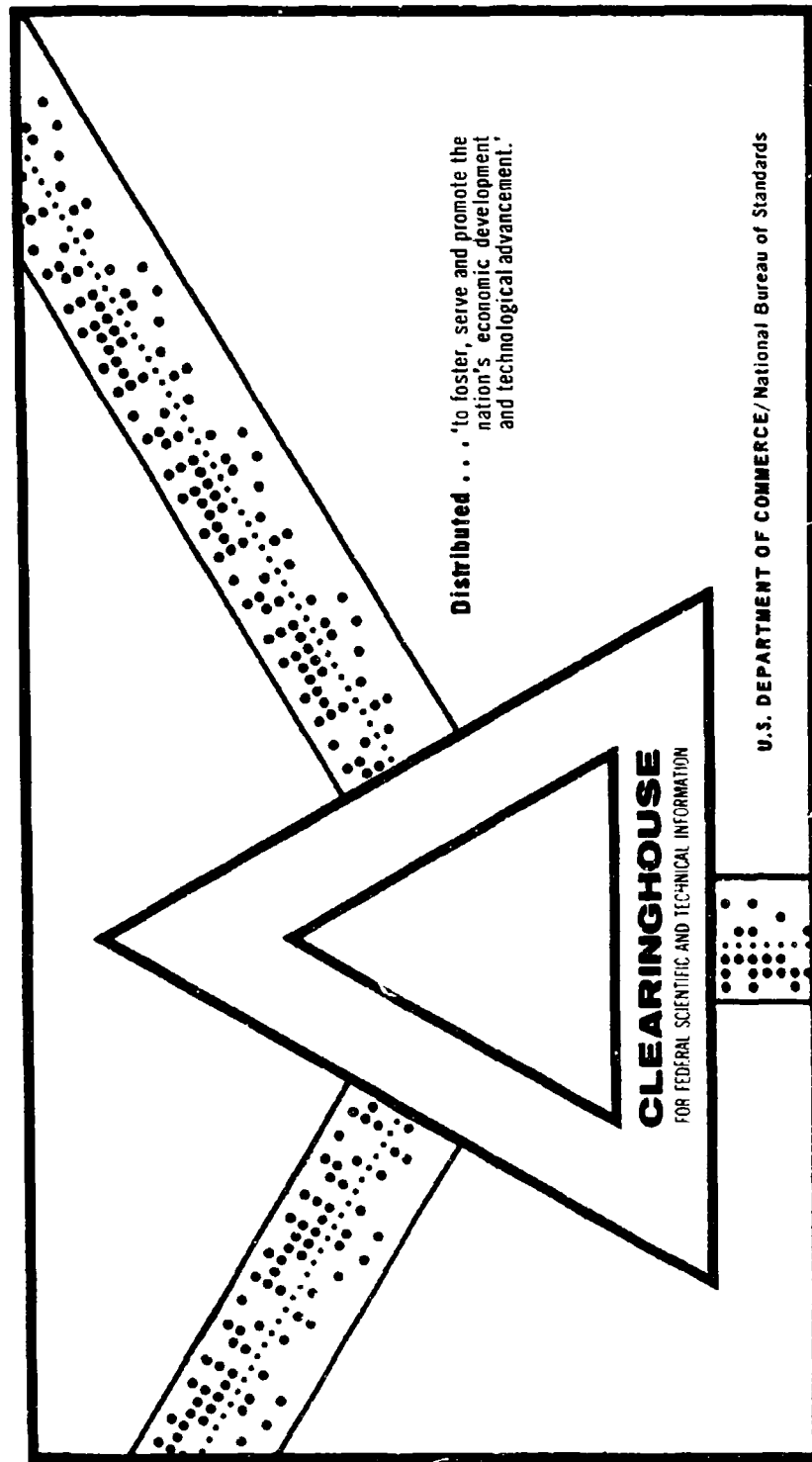


TECHNIQUES FOR DIRECTIONAL DATA

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4 November 1969



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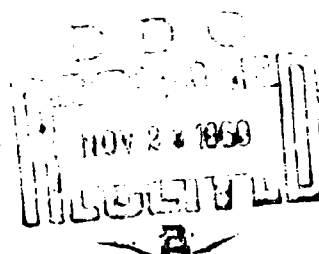
TECHNICAL REPORT NO. 150

November 4, 1969

PREPARED UNDER CONTRACT Nonr-225(52)
NR-342-022
OFFICE OF NAVAL RESEARCH

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DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
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TECHNIQUES FOR DIRECTIONAL DATA

by

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1. INTRODUCTION

1.1 In recent years techniques have been developed for dealing with statistical data where the observations are directions, and where the directions are assumed to be more or less concentrated around a single mode. In three dimensions, the distribution used to describe such directional data is Fisher's (1953) distribution, and in two dimensions it is the von Mises distribution. In this paper we extend the techniques for these distributions to deal with axial data, i.e. data consisting of vectors whose direction can be in either sense, and also for use with directed data from populations with two modes, in opposite directions. The techniques make use of tables already prepared for the Fisher and von Mises distributions. Examples of directed data are, in three dimensions, directions of magnetization of rocks, or, in two dimensions, directions of bird flights or of prevailing winds; examples of axial data are normals to planes of cleavage of rocks, or inclinations of the long axis of pebbles in till deposits.

The procedures will be given first for three dimensions, since later on it is easy to adapt them for two dimensions; the rest of the introduction deals with notation to be used. The Fisher distribution is described in section 2; its bimodal extension, assuming axial data or directed data with equal modal strengths, is given in section 3, with

examples; tests of hypotheses for three-dimensional data are in section 4. For two dimensions, the von Mises distribution and its bimodal extension are treated in section 5.

The adaptation of all the procedures for directed data but with unequal modes is given in section 6. Section 7 demonstrates the importance of knowing the type of data, and deals briefly with related topics. Examples are included throughout the paper.

1.2 Notation. Observations denoting directions are recorded by unit vectors; in three dimensions a typical vector is $\underline{OP}$, starting at the center O of a sphere of radius 1 and ending at P , a point on the surface of the sphere. Axial data could be best recorded by drawing the entire diameter POQ say, though in three dimensions this is then difficult, in practice, to show on the usual diagrams. Techniques for axial data must not depend on whether $\underline{QP}$ or $\underline{OQ}$ is used to represent an observation; for all data, thus the vector used is called $\underline{QP}$. Let P be located by the usual spherical polar coordinates θ, ϕ ; we shall regard these also as coordinates of the line OP . For simplicity, let $\theta = 0^\circ$ be thought of as pointing to the "north pole" of the sphere, so that $\theta = 90^\circ$ is the equator and $\theta = 180^\circ$ is the "south pole".

1.3 Other Notation. In the northern hemisphere, θ is then the colatitude of P , and $\theta = 90^\circ - \lambda$, where λ is northern latitude; in the southern hemisphere, $\theta = 90^\circ + \lambda$, where λ is southern latitude;

ϕ is the longitude measured from 0 to 360° eastwards from $\phi = 0^\circ$. ϕ is called also the orientation; P is sometimes measured by orientation ϕ and dip angle δ below the equator; δ corresponds to southern latitude λ , and $\theta = 90^\circ + \lambda$. In practical work, θ (or δ) and ϕ are often given in degrees, as has so far been assumed; in theoretical discussion, we shall assume θ, ϕ are in radians. This will not affect the practical techniques, which use the components of the given vectors. For these, we must introduce a set of rectangular coordinates; a natural set has the z-axis along $\theta = 0$, the x-axis along $\theta = \pi/2, \phi = 0$, and the y-axis along $\theta = \pi/2, \phi = \pi/2$. The components of a vector with coordinates θ, ϕ are then

$$x = \sin \theta \cos \phi, \quad y = \sin \theta \sin \phi, \quad z = \cos \theta.$$

For a given sample of size N , let $\underline{Q}P_i$ ($i=1,2,\dots,N$) be the i -th unit vector, and let x_i, y_i, z_i be its components; define $X = \sum x_i, Y = \sum y_i, Z = \sum z_i$. These are the components of the vector sum or resultant $\underline{R}$ of the $\underline{Q}P_i$; if $\underline{R}$ has length R , then $R^2 = X^2 + Y^2 + Z^2$. When dealing with several samples, the subscript r will give the value for the r -th sample, e.g. N_r is the sample size of the r -th sample, with resultant $\underline{R}_r$; N will denote the total sample size $N_1 + N_2 + \dots + N_s$, where s is the number of samples.

2. THE FISHER DISTRIBUTION FOR DIRECTED VECTORS.

2.1 Suppose the vectors $\underline{Q}P_i$ ($i=1,2,\dots,N$) represent directed data, with an arrowhead at P_i .

The Fisher distribution describes a probability density at the point θ_0, ϕ_0 which is proportional to $\exp(\kappa \cos \theta_0)$: more precisely, the density of θ is

$$(1) \quad f(\theta) = \frac{\kappa \sin \theta}{2 \sinh \kappa} \exp(\kappa \cos \theta), \quad 0 \leq \theta \leq \pi,$$

and ϕ is independently uniformly distributed between 0 and 2π . The distribution is symmetrical around A (along $\theta = 0$, pointing to the north pole), and with a single mode at A ; κ is a parameter ($\kappa > 0$), which describes the concentration of the distribution. When κ is large the distribution is highly concentrated around A , and when $\kappa = 0$ the vectors (i.e. the points P) are uniformly distributed over the surface of the sphere. This distribution was introduced by Fisher (1953) to describe vectors denoting remanent magnetization of rocks. Statistical procedures were given by Fisher, and by Watson (1956), Watson and Williams (1956), and Watson and Irving (1956); this work has been developed, and the necessary tables produced, by Stephens (1962b, 1967, 1969a). The techniques have begun to be used in applied work, particularly in a geological context; see, for example, Andrews and Shimizu (1966), and, for a wider discussion, with a long list of references, Watson (1968). It will be convenient now to summarize these procedures; in the next section they will be adapted for use with axial or bimodal three-dimensional data.

2.2 Estimation of modal vector and concentration parameter for the Fisher distribution. In the distribution (1) above, the modal

direction $\underline{A}$ was assumed along $\theta = 0$; in practice it will usually not be known, and, with κ , must be estimated. The maximum likelihood equations for these estimators are based on the statistics $\underline{R}$ and X , from a sample of N unit vectors, already described in section 1.

(a) The estimate of $\underline{A}$ is the direction of $\underline{R}$, i.e., a unit vector $\underline{a}$ estimating $\underline{A}$ is $\underline{a} = \underline{R}/R$.

(b) To estimate κ , solve

$$(2) \quad \coth \kappa - 1/\kappa = R/N;$$

if $\underline{A}$ is known, replace R by X in this equation. A table for solving (2) is in Stephens (1967).

3. BIMODAL EXTENSION OF THE FISHER DISTRIBUTION

3.1 The natural extension of the Fisher distribution to cover bimodal data is obtained by superimposing two Fisher distributions with opposite modal vectors. If the line along $\theta = 0$ and $\theta = \pi$, now called the modal axis A , represents the direction of the two modes, the density of θ is then

$$(3) \quad f(\theta) = \frac{\kappa \sin \theta}{2 \sinh \kappa} (a \exp(\kappa \cos \theta) + (1-a) \exp(-\kappa \cos \theta)),$$

$$0 \leq \theta \leq \pi$$

and ϕ has a uniform distribution between 0 and 2π as before.

The relative strength of the two modes is measured by the parameter

a , which lies between 0 and 1. When $a = 0.5$, (3) becomes

$$(4) \quad f(\theta) = \frac{\kappa \sin \theta}{2 \sinh \kappa} (\cosh (\kappa \cos \theta)), \quad 0 \leq \theta \leq \pi .$$

Distribution (3) will be used to describe directed data with unequal modes; distribution (4) will be used in analyzing axial data, if both ends are recorded. In general, the modal axis will not be known, and must be estimated. When A is known, pointing, say, to the North pole, we could choose to record axial data by one point only, in the northern hemisphere; the density of θ is then

$$(5) \quad f(\theta) = \frac{\kappa \sin \theta}{\sinh \kappa} (\cosh (\kappa \cos \theta)), \quad 0 \leq \theta \leq \pi/2 .$$

In practice, a given experimenter, say collecting directions of magnetization of rocks, will probably show a natural preference for recording the data in one sense more often than the other, so that a given sample of axial data, as recorded, may look like directed data with two unequal modes or even only one mode; this can be very deceptive. It is therefore important to know what type of data is in a given sample in order to decide the analysis to be undertaken; one should not rely only on the appearance of the sample. We illustrate this point in section 7.

We now discuss two estimation problems, for distribution (4); how to estimate the direction of the modal axis A , when this is not known (so that it does not lie along $\theta = 0$); and how to estimate κ . The techniques to be given will not depend on which end of a diameter is chosen to represent an observation with no preferred sense; thus for

this type of data one follows the procedures using the observations exactly as given. This is also the case for data representing directed vectors from a population with two opposite modes of equal amplitude. Modes of unequal amplitude ($a \neq 0.5$) are discussed in section 6.

3.2 Estimation of the Modal Axis A. Suppose a plane M is chosen through O , characterized by its normal $\underline{n}$; one end only of each recorded axial diameter is then chosen to give a directed vector, such that all the directed vectors lie on one side of M . The resultant $\underline{R}$ is then calculated, its value clearly depends on the choice of M , i.e. of $\underline{n}$. When $\underline{n}$ is along the modal diameter A , so that the plane M is at right angles to A , the expected length R of $\underline{R}$ is a maximum; and when A lies in plane M , so that $\underline{n}$ is at right angles to A , the expected length R will be a minimum. This suggests that to estimate A , we must find the plane M which gives a maximum R , and this is done iteratively by the following method, which applies to both axial and directed data.

(a) Directed data is recorded by the end P_1 of the vector $\underline{OP}_1$; for an axial observation choose either end of the diameter to be initially P_1 . In the steps which follow, the direction of a sample vector will sometimes be reversed, and P_1 always refers to the end of the vector which is currently used to assign it a direction $\underline{OP}_1$.

(b) If a good estimate $\underline{y}$, length V , of the modal vector exists, let the components of the unit vector $\underline{y} = \underline{y}/V$ be l, m, n .

(c) Suppose the sample vector $\underline{OP}_1$ has components x_1, y_1, z_1 ; for each sample vector, calculate $a = \underline{y} \cdot \underline{OP}_1$, i.e. $a = lx_1 + my_1 + nz_1$; if a is negative, change the signs of x_1, y_1 and z_1 . This reverses the original direction of $\underline{OP}_1$ and ensures that it now makes an angle less than $\pi/2$ with $\underline{y}$.

(d) For the final set of sample vectors, calculate the resultant (vector sum) $\underline{R}$, length R .

(e) Let $\underline{x} = \underline{R}/R$ be the new $\underline{y}$, components l, m, n , and repeat from step (c). When two successive values of $\underline{x}$ are identical, stop the procedure; let the final unit vector $\underline{x}$ be called $\underline{x}_0$; the line along which $\underline{x}_0$ lies is the estimate of the modal axis $\underline{A}$.

(f) If no good estimate $\underline{y}$ exists in (b) above, start as follows. Let (l, m, n) be $(1, 0, 0)$ and proceed with steps (c) and (d). Repeat with $(l, m, n) = (0, 1, 0)$ and $(l, m, n) = (0, 0, 1)$. Of the three values of $\underline{R}$ so obtained, choose that with the largest length R as the initial $\underline{y}$; take $\underline{y} = \underline{R}/R$ and continue from step (b) above.

3.3 Estimation of κ . 1. Modal axis known. If the direction of the modal axis $\underline{A}$ is known, and lies along $\theta = 0$, it is easy to derive the maximum likelihood estimating equation for κ . It is

$$(6) \quad \cosh \kappa - \frac{1}{\kappa} = \frac{1}{N} \sum_{i=1}^N \cos \theta_i \tanh (\kappa \cos \theta_i),$$

where $\cos \theta_i$ is the angle between $\underline{OP}_i$ and the modal axis $\underline{A}$, chosen to point in either direction.

2. Modal axis not known. When $\underline{A}$ is not known, we measure θ from the estimate of $\underline{A}$, along $\underline{x}_0$ calculated in the previous section.

This is analogous to replacing X by R when using equation (2) for the Fisher distribution.

For each sample vector, only $\cos \theta_1$ is actually required, and if the components of $\underline{r}_0$ are l_0, m_0, n_0 , and the final components of $\underline{OP}_1$ are x_1, y_1, z_1 , $\cos \theta_1$ is given by $\cos \theta_1 = l_0 x_1 + m_0 y_1 + n_0 z_1$. Every value of $\cos \theta_1$ will be positive. Equation (6) is then solved iteratively by the following steps. Let $\underline{R}^0$ be the final resultant, length R^0 , of the sample vectors, i.e. $\underline{r}_0 = \underline{R}^0/R^0$, and let $M(\kappa)$ be the right hand side of (6) for any κ .

(a) The quantity $(\sum \cos \theta_1)/N$ is R^0/N , say Y_0 ; solve $\cosh \kappa - 1/\kappa = Y_0$ to give an initial estimate κ_1 for κ . This may be done using e.g. Table 3 in Stephens (1967); if $Y_0 \geq 8$, κ_1 is $1/(1-Y_0)$.

(b) Solve $\cosh \kappa - 1/\kappa = M(\kappa_1)$; call the solution κ_2 .

(c) Solve $\cosh \kappa - 1/\kappa = M(\kappa_2)$; call the solution κ_3 , etc., and repeat this procedure; the sequence for κ converges, and the limit is $\hat{\kappa}$, the estimate of κ .

The above procedure is proved convergent as follows. First suppose κ^* is the solution of (6). Since $\tanh x < 1$, $M(\kappa_1) < Y_0$; and since $\cosh \kappa - 1/\kappa$ is monotonic in κ , $\kappa_2 < \kappa_1$. But $\tanh x$ is monotonic in x , so $M(\kappa_2) < M(\kappa_1)$; so by repetition of the argument, $\kappa_3 < \kappa_2$. Similarly $\kappa_4 < \kappa_3$, etc. Also $Y_0 > M(\kappa^*)$, so $\kappa_1 > \kappa^*$; then $M(\kappa_1) > M(\kappa^*)$, and so $\kappa_2 > \kappa^*$; similarly $\kappa_i > \kappa^*$ for all i . Thus the sequence of solutions κ_i is decreasing, bounded below by κ^* , and so converges. But $\cosh \kappa_n - 1/\kappa_n = M(\kappa_{n-1})$, and if we take the limit, as $n \rightarrow \infty$, on both sides, we have the limiting value of $\kappa = \kappa^*$.

As will be seen in the examples which now follow, the technique converges very rapidly for K -values of 5 or more.

3.4 Examples. The data are from measurements of inclination of till deposits, kindly made available by Dr. C. King of the Department of Geography, University of Nottingham, England; there are 4 samples each of 2 observations, measured to the nearest 5 degrees. The effect of the precision of measurement is not considered in this paper. Table 1 gives the data and the steps in the estimation procedures, for Sample 1. The table is divided into several parts:

(a) The 25 values of θ , ϕ are listed first, in degrees. The degree symbol is omitted.

(b) Estimation of R^0 . Unit vectors along the x , y , and z axes are taken as starting values of y , assuming no initial approximation known for the modal axis; the resultant length is given, together with its coordinates θ , ϕ , and in the last column are listed those vectors which must be reversed from the original given direction to lie at an angle less than 90 with the current y .

(c) The largest R obtained from (b) is 20.20, with vector No. 1 reversed; this R , reduced to unit length, becomes the next y with direction cosines $l = -0.329$, $m = 0.9211$, $n = -0.208$. Now vector No. 1 is returned to its original sense (thus all the vectors, as given, are within 90 of the current y) and the new $R = 20.54$; on using this to make the new y , we get no change in R , so that this is the final resultant, R^0 ; its direction cosines are -0.418 , 0.889 , -0.188 , and its coordinates are $\theta = 100.8$, $\phi = 115.2$. Since it is the R^0 for sample 1, it is designated R_1^0 .

(d) Estimation of κ . The first estimate of κ , derived from $R_1^0 = 20.54$, is $\kappa_1 = 5.61$. The successive estimates converge rapidly to $\hat{\kappa} = 5.56$.

Table 2 gives the final results for Samples 2, 3 and 4, for Samples 1 and 2 taken together as one sample of 50 observations, and for Samples 3 and 4 taken together. X, Y, Z , are the components of R^0 ; θ, ϕ are its coordinates in degrees. Thus $X = R^0 \sin \theta \cos \phi$, $Y = R^0 \sin \theta \sin \phi$, $Z = R^0 \cos \theta$. The techniques converge very rapidly for these values of κ . Even for Sample 3, only 3 iterations are needed to obtain $\hat{\kappa} = 3.92$.

4. TESTS OF HYPOTHESES.

4.1 We now consider tests for bimodal data. The tests to be proposed are devised to make use of methods and tables already prepared for the Fisher distributions; these may be briefly summarized as follows.

For the Fisher distribution, one-sample tests of hypotheses concerning A or κ are based on R and on X (Stephens, 1962, 1967). For the important test that s different samples have the same modal vector A , a test is based on the conditional distribution of $R_1 + R_2 + \dots + R_s$, given R ; R is the length of R , the overall resultant of all the s samples. The tables for this test are in Stephens (1969).

4.2 Possible test statistics for the Bimodal Distribution: R^0 and S .

For the bimodal distribution, when A is not known, it would be natural to base tests on the distribution of R^0 ; when A is known, possible test statistics could be the component of R^0 on A , or the

sum C of the components on A of all the given vectors, each one pointing so that its component is positive. (Note that these two statistics are not necessarily the same.) Unfortunately, even the distribution of R^0 is not known; and we consider an alternative statistic. Suppose, for a given sample of size N , we have found the estimates $\hat{A}$ and $\hat{\kappa}$ by the above methods; we then ask for the resultant vector S which would have given the same estimates, of modal vector A and of κ , on the assumption it is the resultant of a Fisher-distributed sample of the same size. Clearly S will lie along R^0 , and its length is easily found from the calculations leading to $\hat{\kappa}$ for the bimodal sample. Consider, for example, Sample 1. The final $\hat{\kappa}$ is 5.56, obtained by solving $\coth \kappa - 1/\kappa = 0.8200$; comparison with equation (2) for the Fisher distribution shows that 0.8200 must be the value of S/N , so that $S = 0.8200 \times 25 = 20.50$. We call S the adjusted resultant of the bimodal sample; note that it might point along either direction of the estimated modal axis, according to the final direction taken by R^0 . In effect we have imagined constructing a Fisher sample from the original data, with same N , κ , and modal vector (based on our estimates), and S would then be its resultant. The values of S are included in Table 2. Since we have tests and tables available for the Fisher distribution, based on the Fisher resultant R , we could now, as an approximate procedure for the bimodal distribution, use the same tests, with the adjusted resultant S replacing R .

4.3 One sample tests. This procedure will first be illustrated with one-sample tests on Sample 1.

Test for the modal axis A Suppose the null hypothesis is H_0 : A , for Sample 1, is along $\phi = 90$, i.e. the y-axis. The appropriate Fisher distribution test is based on the conditional distribution of R given the component C or the hypothesized modal vector; if R is too large, H_0 is rejected. Here, the value of S is 20.50; it lies along the direction of R^0 , and its component C is then 18.22. The test, for $\alpha = 0.05$, uses Figure 2 of Stephens (1962), or approximations which accompany it for the case when the component, called X in the Figure, is beyond the range given. In this case, the approximation in section 3.3 applies; the critical value of S , say S_0 , will be calculated from $\frac{S_0 - C}{N - S_0} = \frac{F_{2,48}(\alpha)}{24}$; $F_{2,48}(\alpha)$ is the usual F-distribution, here with 2 and 48 degrees of freedom, at the level α . For $\alpha = 0.05$, we get $S_0 = 19.016$. Since S is greater than S_0 , H_0 is rejected at the 5% level.

Use of R^0 . For reasonably large κ , R^0 very nearly equals S , as here, and the test could be made using R^0 ; the component C is now 18.26 (the Y component of R^0 , from Table 2) and the critical value for R^0 would then be 19.051. H_0 would again be rejected. In the next test, for κ , replacement of S by R^0 also does not change the result of the test.

Test for κ . Consider a 5% test of H_0 : the true κ of Sample 1 is 4.5. The test for the Fisher distribution is given in Stephens (1969); it uses the value of R/N , and a table of the critical values is given. For $N = 25$, $\kappa = 4.5$, the upper 5% critical value is 0.854. If for

Sample 1, we use R^0 as test statistic, the value of $R^0/N = 0.822$; this is not significant, so H_0 is not rejected, for a one-tail (or two-tail) test. Similarly, $C/N = 0.820$ and use of S would give the same result.

4.4 Tests for several samples. Notation.

Let the r -th sample have size N_r , and let R_r^0 be the resultant as calculated above, and S_r the adjusted resultant, with length R_r^0 , S_r as before, $N = N_1 + N_2 + \dots + N_s$, where s is the number of samples. We can combine the resultants R_r^0 or S_r , or the samples themselves, in several ways:

(a) Firstly, suppose the resultants of the individual samples j, k, l, m , say, are so aligned that they give the maximum length to the vector sum $R_j^0 + R_k^0 + R_l^0 + R_m^0$; we shall describe them as well-aligned. (Recall that any calculated R_i^0 is ambiguous in direction and may be reversed as desired.) For well-aligned vectors, this vector sum will be called R_{jklm}^0 ; its length is R_{jklm}^0 .

(b) Secondly, samples j, k, l, m , may all be pooled into one sample, and the procedure above followed to calculate R^0 for the overall sample; we shall use the notation R_{jklm}^0 for the resultant of pooled samples. It will not necessarily be the case that R_{jklm}^0 will be equal to R_{jklm}^0 ; a vector may point one way in sample j , say, in calculating R_j^0 , and be reversed in calculating the pooled sample resultant.

This can easily be checked by reference to the individual components; the final X component of the pooled sample will equal the sum of the individual X components if the vector sum equals the pooled resultant, and similarly for the Y and Z components. As an illustration, consider Table 2. Samples 3 and 4; even when R_4^0 is reversed, to make the X values of 3 and 4 both positive, their sum is not the X value of Samples 3 and 4 pooled together; we can see that the X of R_{34}^0 is 28.99 (16.74 + 12.25), while that of P_{34}^0 is 28.16. In this case there occurs one vector in Sample 3 which changes direction in the overall pooled sample; it is at approximately 89.3 degrees to R_3^0 and at 103.6 degrees to R_{34}^0 . In such circumstances P_{jk}^0 will always be greater than R_{jk}^0 ; R_{34}^0 has length $R_{34}^0 = 38.20$ and P_{34}^0 has length $P_{34}^0 = 38.62$. For more than two samples the disparity may be greater. In a similar way we can define $S_{jklm} = S_j + S_k + S_l + S_m$ when the S vectors are well-aligned; its length is S_{jklm} . It will not necessarily be the same as the value of S for the pooled samples j, k, l, and m; we shall call this Q_{jklm} , length Q_{jklm} .

4.5 Test Statistics. Tests for several samples might be based on R or S statistics, and on vector sums or pooled values. We shall suggest using S-statistics, since the appropriate tables to be used are based on Fisher distributions, and pooled resultant S values (Q_{jklm} rather than S_{jklm}), since these are easily obtained from the computational procedure described above, when the samples are pooled into one. Borderline decisions will in any case be treated

with reserve owing to the approximate nature of the tests. We now illustrate several multi-sample test procedures.

Test for a Common Modal Axis. A test for H_0 : that s samples with the same unknown κ , have a common modal axis, will be based on the arithmetic sum $S_1 + S_2 + \dots + S_s$ given $Q_{12\dots s}$. Suppose we test this hypothesis, at the 5% level for samples 1 and 2. $S_1 = 20.50$, $S_2 = 20.78$, and $Q_{12} = 40.47$. The test follows the procedure in Stephens (1969). The steps are as follows:

- (a) Calculate $W = Q_{12}/N = 0.810$.
- (b) Calculate $Z = (S_1 + S_2)/N = 0.826$.
- (c) Use section 2 of Stephens (1969) to find the critical value z for $N = 50$, $W = 0.810$, and $\alpha = 0.05$; using the F-approximation there given, we find $z = 0.821$.
- (d) Since Z exceeds z , reject H_0 at the 5% level. Z is in fact just significant at the 2.5% level (critical value $z = 0.824$).

The vector sum $S_{12} = S_1 + S_2$ has length $S_{12} = 40.49$; obviously replacement of Q_{12} by S_{12} gives the same result for the test. If one uses the R statistics, the pooled resultant $P_{12} = R_{12}$. The steps (a) and (b) give $W = 0.811$ and $Z = 0.827$. Again Z is just significant at the 2.5% level; the critical values go up very slightly with the slight increase in W , but so also does Z . The example illustrates again that for reasonably large κ , the S and R statistics will give the same results. This is especially so in the tests of a conditional nature: those for the modal axis for one sample, and those for a common modal axis for several samples.

5. THE VON MISES DISTRIBUTION AND BIMODAL EXTENSION.

5.1 The distribution analogous to Fisher's, for use in two dimensions, is the von Mises distribution; if θ is the polar coordinate of P, or equivalently of OP, the density is

$$f(\theta) = \frac{1}{2\pi I_0(\kappa)} \exp(\kappa \cos \theta) \quad -\pi < \theta \leq \pi$$

with modal direction A along $\theta = 0$ and κ measuring concentration as before. Treatment of this distribution is in Gumbel, Greenwood and Durand (1953) and Watson and Williams (1956); tests are in Stephens (1962a, 1969b); Batschelet (1965) gives a good account of the statistical procedures and a bibliography of applications, and May (1967) shows how the procedures may be applied to practical situations. Again estimation and testing procedures are based on the sample resultant R and on X , its component along A , known or hypothesized. The direction of R estimates A when this is not known, κ is estimated from $I_1(\kappa)/I_0(\kappa) = R/N$; when A is known, X replaces R . $I_0(\kappa)$, $I_1(\kappa)$ are the usual imaginary Bessel functions of order zero and one.

5.2 The Bimodal Extension. The extension of the von Mises distribution for bimodal data gives density

$$(7) \quad f(\theta) = \frac{1}{2\pi I_0(\kappa)} (a \exp(\kappa \cos \theta) + (1-a) \exp(-\kappa \cos \theta)), \quad -\pi < \theta \leq \pi;$$

when $a = 0.5$, this becomes

$$(8) \quad f(\theta) = \frac{1}{2\pi I_0(\kappa)} \cosh(\kappa \cos \theta), \quad -\pi < \theta \leq \pi;$$

with a modal diameter A along $\theta = 0$ or π . If A lies along $\theta = \theta_0$ or $\theta_0 + \pi$, $\cos \theta$ is replaced by $\cos(\theta - \theta_0)$ in (7) or (8). For distribution (8), used for axial data or directed data with opposite modes of equal amplitude, the technique for estimating A from a sample of size N is similar to that in section 3 for three dimensions. A suitable set of rectangular coordinates puts the x-axis along $\theta = 0$ (now no longer known to be the modal diameter A), so that $x = \cos \theta$, $y = \sin \theta$.

5.3 Estimation of R . Then, with suitable initial unit vector y , components $1, m$, A is estimated iteratively by following steps (c) to (e) of section 3.2 with the obvious change to two dimensions. If no good estimate of A exists, we follow (f), starting now with $(1, m) = (1, 0)$ and then $(0, 1)$.

5.4 Estimation of κ . The equation for estimating κ is now

$$(9) \quad I_1(\kappa)/I_0(\kappa) = \frac{1}{N} \sum_{i=1}^N \cos \theta_i \tanh(\kappa \cos \theta_i)$$

and this is solved iteratively exactly as in section 3.3, steps (a) to (c); the initial right hand side is R^0/N , where R^0 is the length of the vector estimating A . The successive κ values must be found by interpolation in a table giving values of $I_1(\kappa)/I_0(\kappa)$ for given κ ; a table is in Gumbel, Greenwood and Durand (1953) or Batschelet (1965).

5.5 Examples. Some axial data, supplied by Dr. A. Rees of the Department of Oceanography, University of Southampton, are in Table 3, Samples 5 and 6. The data represents axis of maximum susceptibility in magnetization of rocks, and comes from the Franciscan rocks in Diablo Range in California, (Rees and Hamilton, 1965). Table 3 gives the estimates of the parameters (θ , the polar coordinate of β , gives the estimated inclination of A), for the samples taken separately and also pooled into one sample. The modified statistic $\bar{S}$ is calculated as for three dimensions; the final right-hand side of (9) gives S/N , and $\bar{S}$ lies along R^0 . Tests are conducted as for three dimensions; those for the modal vector and for κ are in Stephens (1962a, 1962b); multi-sample tests are in Stephens (1969c). We will illustrate only a two-sample test with Samples 5 and 6.

Test for a Common Modal Axis. For a test of H_0 : the two samples have a common modal axis we have $Z = (S_1 + S_2)/N = 0.973$; $W = Q_{12}/N = 0.959$. The critical value z of Z is given in Stephens (1969c); this is a slight improvement on an F-test introduced by Watson and Williams (1956), and gives $z = 0.986$ at the 5% level. Thus we reject the null hypothesis H_0 at this level.

6. OPPOSITE MODES OF UNEQUAL AMPLITUDE.

6.1 Data, with direction, from populations with modes which are opposite but of unequal amplitude, will be treated using distributions (3) or (7). The vectors now have a definite sense. The estimation

of $\hat{A}$ follows the steps as before, in sections 3 and 5; then a may be estimated by $\hat{a}$, the proportion of vectors which is not reversed when estimating $\hat{A}$. The equations for estimating κ become

$$(10) \quad \coth \kappa - \frac{1}{\kappa} = \frac{1}{N} \sum_{i=1}^N \cos \theta_i \frac{a \exp(K \cos \theta_i) - (1-a) \exp(-K \cos \theta_i)}{a \exp(K \cos \theta_i) + (1-a) \exp(-K \cos \theta_i)}$$

for three dimensions; for two dimensions the left hand side of (10) is replaced by $I_1(\kappa)/I_0(\kappa)$. In the right hand side one inserts $\hat{a}$, and follows the same iteration procedure as in section 3; the sequence for κ converges to the estimate $\hat{\kappa}$.

Example. We illustrate this technique with some interesting data given several years ago by Dr. E. Gould of the Johns Hopkins School of Hygiene. The data represent directions taken by turtles after treatment; it is thought that the turtles have a preferred direction, but some are confusing forwards with backwards. Thus the distribution is (7); the actual values and the analysis are in Table 3. May (1967) has varied the parameters to attempt to find a best fit; his best fit values are $\vartheta = 61.5$, $\kappa = 3.167$, and $a = 0.803$. The results in Table 3 are in excellent agreement.

Tests and Confidence Intervals. These would follow the procedures already described, using again the right hand side of (10) to give an adjusted resultant $\hat{S}$. In this example it would have length 61.83, and, as before, lies along $\theta = 63.08$, the direction of $\hat{R}$. Suppose one wished to find a 10% confidence interval for the modal axis of (7), using $\hat{S}$;

Figure 1 of Stephens (1962a) is used, or the approximations given when the data is beyond the range of the figure; here, the approximation in section 3.4 applies, and gives a band for θ with a half-width equal to 7.4 degrees. The confidence interval is $55.7 < \theta < 70.5$.

7. FURTHER REMARKS.

7.1 Importance of knowledge of type of data. We illustrate this point, mentioned in section 3.1, with another two-dimensional sample of Rees and Hamilton (1965); the data, 10 values of θ , is from their site San Jose 9. The values are 2, 13, 14, 141, 152, 156, 166, 356, 357, 358. A rough glance might suggest a von Mises sample; if so analyzed, the modal direction would be along $\theta = 41.13$; the resultant is 3.18; κ is 0.672, indicating, of course, wide dispersion. In fact the data is axial; if diameters are drawn through the data points, we see one end of each diameter makes a set concentrated between 260 and 20; if the analysis of this paper is used, the modal axis is $\beta = 351.6$, R^0 is 9.59, $S = 9.59$, $\kappa = 12.10$. There is a big difference in the estimates of modal direction. A knowledgeable experimenter may of course present the data so that either analysis would give the same result for modal direction; this has happened in our data in Sample 1, where at the end we see that R^0 is found with none of the original vectors, as given, needing to be reversed. Thus straight Fisher-distribution techniques would have given the same resultant. But it would have been easy to reverse a selection of the given vectors

to produce an entirely different Fisher resultant. On the sphere, especially, it is not easy to see, especially with widespread data, which end should be chosen so that a Fisher analysis gives the same resultant as the techniques of section 2; the point of these techniques is to render such a choice unnecessary. Note that in any case, even with the same resultant, one would obtain a different κ for Fisher-distribution and for bi-modal analysis.

7.2 Alternate distributions. Another distribution for bimodal or axial data has been proposed by Watson (1966); the density for θ is proportional to $\exp(\lambda \cos^2 \theta)$ (if the modal axis $\hat{A}$ is along $\theta = 0$), and ϕ is uniform as before. Use of this distribution, with estimation as described in Watson (1966) gives the following modal axis estimates for Samples 1 to 4 in Table 1; Sample 1, $\theta = 102.6$, $\phi = 113.6$; Sample 2, $\theta = 78.2$, $\phi = 111.7$; Sample 3, $\theta = 98.2$, $\phi = 148.4$; Sample 4, $\theta = 106.2$, $\phi = 127.9$. The results are in good agreement with those given in Table 1. Testing procedures are not yet as well developed for this distribution as for the Fisher distribution.

In two dimensions, the corresponding density is equivalent to a density proportional to $\exp(\lambda \cos 2\theta)$; thus the doubled angles have a von Mises distribution, and analysis proceeds by doubling the angles given, estimating the modal vector, and halving its angle. Thus for the doubled-angle vectors, for Sample 5, $R = 7.40$, along $\theta = 183.6$, so that the modal axis estimate is $\hat{A}$, along $\theta = 91.8$; for Sample 6, R is 8.76, along $\theta = 222.5$, giving modal axis $\hat{A}$ along $\theta = 111.25$.

Direct application of von Mises techniques rejects the hypothesis, at the 5% level, that the samples have a common modal vector; details are in Stephens (1969c). These results compare well with those in Table 3.

Neither of the above distributions in two or three dimensions, is strictly applicable to directed data with unequal modes, except that the same techniques, to estimate the modal axis, can still be employed. If this is done with the turtle data of Sample 7, i.e. the angles are doubled and the direction of the resultant then found and halved, we have the estimate of $\hat{A}$ along $\theta = 62.57$.

7.3 Goodness-of-fit. One might wish to test the data to see if they are well-fitted by the distributions considered. This is an important subject and for the present only a few comments are offered here. For both sphere and circle, a distinction must be made between axial and bimodal directed data. When the axis $\hat{A}$ has been estimated, it should be taken as origin for θ and all coordinates transformed. Axial data should be so recorded that all vectors are within 90 degrees of $\hat{A}$, i.e. their θ -values are less than $\frac{\pi}{2}$ radians. For the sphere, the test is then made for a fit to distribution (5). For bimodal directed data, (4) is used, or, with unequal modes, (3). The tests can be made separately for θ and for ϕ , using say the χ^2 test. A rough measure of goodness of fit can be found by use of the U^2 and V goodness-of-fit statistics, but no precise tests can be made, as their distributions depend on the fact that parameters have been estimated. Similar remarks apply to the bimodal distributions on the circle. When

the coordinates have been transformed to make $\theta = 0$ lie along $\hat{A}$, distributions (7) or (8) are used for bimodal directed data. For axial data, distribution (8) is used, with twice the $f(\theta)$ shown and range $-\pi/2 < \theta < \pi/2$.

7.4 Acknowledgments. This work was begun at the University of Nottingham, and revised at Stanford University; the author is grateful particularly to Professors C. Granger and H. Solomon, for the opportunities to visit these Universities, and to Dr. C. King for her interest in the problems discussed. The research was partly supported at McGill by the National Research Council of Canada, and at Stanford by the U.S. Office of Naval Research, Contract Nonr 225(52); the author expresses his thanks for this support.

TABLE 1

Three-dimensional axial data: estimation procedures for Sample 1

Sample 1: Coordinates θ, ϕ of 25 vectors: 1 is the vector number.

1	θ	ϕ	1	θ	ϕ	1	θ	ϕ	1	θ	ϕ
1	80	190	7	110	95	13	115	130	19	70	70
2	110	70	8	110	120	14	100	140	20	80	90
3	115	70	9	110	120	15	115	140	21	65	100
4	130	80	10	125	120	16	125	140	22	75	110
5	100	90	11	100	125	17	120	150	23	75	110
6	105	90	12	105	125	18	110	115	24	80	115
									25	50	150

Estimation of modal axis A; final estimate underlined.

y direction cosines			R	coordinates of R		Vector numbers of vectors reversed
l	m	n		θ	ϕ	
1	0	0	12.21	84.3	332.3	1 ; 7 to 18 ; 21 to 25.
0	1	0	20.20	102.0	109.7	1
0	0	1	11.68	41.5	296.9	2 to 18.
-0.329	0.921	-0.208	20.54	100.8	115.2	none
<u>-0.418</u>	<u>0.889</u>	<u>-0.188</u>	<u>20.54</u>	<u>100.8</u>	<u>115.2</u>	none

Estimation of κ .

$$\begin{aligned}
 R^0 &= 20.54 ; Y_0 = R^0/N = 0.8218 & \kappa_1 &= 5.61 \\
 \text{new R.H.S.} & M(\kappa_1) = 0.8201 & \kappa_2 &= 5.56 \\
 \text{new R.H.S.} & M(\kappa_2) = 0.8200 & \kappa_3 &= 5.56 ; \hat{\kappa} = 5.56
 \end{aligned}$$

true final resultant $R_1^0 = 20.54$; adjusted Fisher resultant $S_1 = 20.50$

TABLE 2

Three-dimensional axial data: results for 4 samples, and pooled samples.

Sample No.	Resultant					Size	Adjusted	$\hat{\kappa}$
	X	Y	Z	θ	ϕ	R^0	S	
1	- 8.59	18.26	-3.86	100.8	115.2	20.54	20.50	5.56
2	- 7.29	19.05	4.10	78.6	110.9	20.81	20.79	5.95
3	16.74	- 8.11	2.95	99.0	154.2	18.84	18.64	3.92
4	-12.25	15.09	-6.03	107.2	129.1	20.35	20.26	5.28
1 & 2	-15.88	37.31	0.24	89.7	113.1	40.55	40.47	5.25
3 & 4	28.16	-24.98	8.63	77.1	318.4	38.62	38.15	4.21

TABLE 3

Two-dimensional axial and directional data:

Sample 5: N = 8	Sample 6: N = 10	Sample 7: N = 76				
θ	θ	θ	θ	θ	θ	θ
247	90 124	8 30	48 58	65 83	95 138	237 285
267	93 323	9 34	48 58	68 88	96 153	238 319
268	104	13 38	48 61	70 88	98 153	243 343
271	105	13 38	48 63	73 88	100 155	244 350
272	109	14 40	50 64	78 90	103 204	250
280	111	18 44	53 64	78 92	106 215	251
282	116	22 45	56 64	78 92	110 223	257
286	121	27 47	57 65	83 93	118 226	268

Results:

Sample No.	Resultant			Size	Adjusted	$\hat{K}$
	X	Y	θ	R^0		
5	-0.23	7.84	91.7	7.85	7.85	25.87
5	-3.55	9.00	111.52	9.68	9.68	15.38
5 & 6	-3.78	16.84	102.65	17.26	17.26	12.19
7	28.39	55.91	63.08	62.71 $a=0.803$	61.83	3.05

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DOCUMENT CONTROL DATA - R&D		
(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)		
1. ORIGINATING ACTIVITY (Corporate author) Department of Statistics Stanford University Stanford, California		2a. REPORT SECURITY CLASSIFICATION
		2b. GROUP
3. REPORT TITLE TECHNIQUES FOR DIRECTIONAL DATA		
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) TECHNICAL REPORT		
5. AUTHOR(S) (Last name, first name, initial) STEPHENS, M.A.		
6. REPORT DATE NOVEMBER 4, 1969	7a. TOTAL NO. OF PAGES 29	7b. NO. OF REFS 17
8a. CONTRACT OR GRANT NO. NOMR-225(52) b. PROJECT NO. NR-342-022 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) Technical Report No. 150 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	
10. AVAILABILITY/LIMITATION NOTICES DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED		
11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Logistics and Math. Sciences Branch Office of Naval Research Washington, D.C.	
13. ABSTRACT Statistical data in the form of directions may be recorded as two- or three-dimensional unit vectors from an origin O to points P on the circumference of a circle or the surface of a sphere. In recent years techniques have been developed to deal with such data, when the vectors are <u>directed</u> , i.e., have a preferred sense. This paper shows how the techniques may be adopted for use with <u>axial</u> data, i.e., vectors with no preferred sense, and also for <u>bimodal directed</u> data. Several practical illustrations are given.		

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14. KEY WORDS	LINK A		LINK B		LINK C	
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Directions, Fisher distribution, von Mises distribution Spherical data Circular data						

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