SOME LOWER BOUNDS OF RELIABILITY

by

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1. Introduction and Summary

Let us consider a complex system consisting of n components in series so that the system fails if one component fails. Let X_i represent the time to failure of the i^{th} component. Then the reliability of the system at time t is given by

$$R(t) = P(X_1 > t, \ldots, X_n > t) \quad . \tag{1.1}$$

We will consider the problem of finding a lower bound on $R(t)$. Various authors have considered this problem from time to time; in particular one may refer to Barlow, Proschan and Hunter [3] and Barlow and Marshall [2]. The approach usually taken is to assume that the form of the joint distribution of $X_1, \ldots, X_n$ is given and then find a lower bound on $R(t)$. We will take a different approach. We will assume that an easily calculable and useful function $R^*(t)$ is given. We will find a condition such that $R^*(t)$ is, in fact, a lower bound on $R(t)$ for all $t \geq 0$. We want these conditions to be meaningful in the sense that they should have simple physical interpretations.

In the following discussion we will take

$$R^*(t) = \prod_{i=1}^{n} P(X_i > t) \quad . \tag{1.2}$$

We note that if $X_1,\ldots,X_n$ are independent then $R(t) = R^*(t)$, for all $t \geq 0$. Hence if $R^*(t)$ is a lower bound on $R(t)$ then it is a "sharp" lower bound in the sense that $R^*(t)$ may actually be attained by $R(t)$ for all $t \geq 0$.

We will consider a slightly more general problem as follows:

Problem A: To find meaningful sufficient conditions such that

$$P(X_1 > x_1,\ldots,X_n > x_n) \geq \prod_{i=1}^{n} P(X_i > x_i) \; , \quad \text{for all } x_1,\ldots,x_n \quad . \tag{1.3}$$

In reliability applications non-negative random variables are considered, in which case inequality (1.3) will be meaningful for $(x_1,\ldots,x_n) \geq 0$.

One answer to the problem has already been given by Esary, Proschan and Walkup [6]. They have introduced the concept of associated random variables.

Definition: We say that random variables $\underline{X} = (X_1,\ldots,X_n)$ are associated if

$$\mathrm{cov}[f(\underline{X}),g(\underline{X})] \geq 0$$

for all non-decreasing functions f and g for which $E[f(\underline{X})]$, $E[g(\underline{X})]$, and $E[f(\underline{X})g(\underline{X})]$ exists.

They have shown that if $X_1,\ldots,X_n$ are associated, then

$$\left.\begin{aligned} &(1.4)\quad P(X_1 > x_1,\ldots,X_n > x_n) \geq \prod_{i=1}^{n} P(X_i > x_i) \quad , \\ &(1.5)\quad P(X_1 \leq x_1,\ldots,X_n \leq x_n) \geq \prod_{i=1}^{n} P(X_i \leq x_i) \end{aligned}\right\} \text{ for all } x_1,\ldots,x_n \; .$$

Unfortunately association is not an easy property to verify. It is not an intuitively meaningful property either, except for binary random variables with $n = 2$ where association is equivalent to non-negative covariance. Hence we consider the following problem.

<u>Problem B</u>: To find meaningful sufficient conditions such that

$$(1.6) \qquad (X_1,\ldots,X_n) \text{ are associated} \quad .$$

It is clear that any solution to problem B is a solution to problem A, but the reverse may not be true.

Lehmann [10] has considered problem A for two variables. For $n = 2$, inequality (1.3) is equivalent to

$$(1.7) \qquad P(X_1 \leq x_1,\ X_2 \leq x_2) \geq P(X_1 \leq x_1)P(X_2 \leq x_2) \quad .$$

Lehmann has considered the problem in this form. Before we summarize his results we will define the concept of monotone likelihood ratio. The following definition is due to Lehmann [9].

Definition: The real-parameter family of densities $f_\theta(x)$ is said to have monotone likelihood ratio in $T(x)$ if there exists a real-valued function $T(x)$ such that for any $\theta' < \theta$ the distributions P_θ and $P_{\theta'}$ are distinct, and the ratio $f_\theta(x)|f_{\theta'}(x)$ is a non-decreasing function of $T(x)$.

If $x > x'$ and if we replace $T(x)$ by x, then the definition is equivalent to

$$(1.8)\quad f_\theta(x)f_{\theta'}(x') \geq f_\theta(x')f_{\theta'}(x) \quad \text{for all } x > x' \text{ and all } \theta > \theta' ,$$

such that the densities are defined. In fact, there is no need to distinguish between x the variable and θ the parameter. We can write inequality (1.8) as

$$(1.9)\quad f(\theta,x)f(\theta',x') \geq f(\theta,x')f(\theta',x) \quad \text{for all } x > x' \text{ and all } \theta > \theta' .$$

We can generalize further and define a generalized monotone likelihood ratio (abbreviated g.m.l.r.) for a multivariate density $f(u_1,\ldots,u_n)$.

Definition: Let $u_i > u_i'$, $u_j > u_j'$, $i \neq j$. $f(u_1,\ldots,u_n)$ is said to have g.m.l.r. in u_i, u_j iff

$$(1.10)\quad \begin{aligned} &f(u_1,\ldots,u_i,\ldots,u_j,\ldots,u_n)f(u_1,\ldots,u_i',\ldots,u_j',\ldots,u_n) \\ &-f(u_1,\ldots,u_i',\ldots,u_j,\ldots,u_n)f(u_1,\ldots,u_i,\ldots,u_j',\ldots,u_n) \geq 0 \end{aligned}$$

for all u_i, u_i', u_j, u_j' as above and for the other $(n-2)$ u's fixed at any value $u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_{j-1},u_{j+1},\ldots,u_n$ with the proviso that the densities must be properly defined at these values.

One consequence of this definition is that if f has g.m.l.r. in u_i, u_j, $i \neq j$ then the ratio

$$\frac{f(u_1,\ldots,u_i,\ldots,u_j,\ldots,u_n)}{f(u_1,\ldots,u_i,\ldots,u_j',\ldots,u_n)} \tag{1.11}$$

is non-decreasing in u_i for the other u's fixed provided the ratio is defined. We note that the concept of g.m.l.r. is symmetric in u_i, u_j.

We note that the ratio in (1.11) may become ∞ or $\frac{0}{0}$, depending upon the behaviour of f. To avoid such complications we assume in the proof that each of the random variables $X_1,\ldots,X_n$ has the same range of definition and the density f is non-zero over this range except perhaps at the end points. For simplicity let us assume that the range of $X_1,\ldots,X_n$ is the n dimensional Euclidean space. In reliability applications one deals with non-negative random variables. In this case we assume that if $X_1,\ldots,X_n$ are non-negative then the density f is non-zero over the positive orthant.

We will assume further that there exists a dominating measure μ with respect to which the density $f(u_1,\ldots,u_n)$ of $X_1,\ldots,X_n$ exists. We will use this density whenever required.

To summarize Lehmann's results let us consider the following conditions:

(1.12) $P(X_2 \le x_2 | X_1 \le x_1) \ge P(X_2 \le x_2)$ for all x_1, x_2 such that the conditional cumulative distribution functions are defined.

(1.13) $P(X_2 \le x_2 | X_1 \le x_1') \ge P(X_2 \le x_2 | X_1 \le x_1)$ for all $x_1' < x_1$ and all x_2 such that both sides are defined.

(1.14) $P(X_2 \le x_2 | X_1 = x_1)$ is non-increasing in x_1 for each fixed x_2.

(1.15) For all x_1, x_1' such that $x_1 > x_1'$, the ratio of the conditional densities

$$\frac{f(x_2 | X_1 = x_1)}{f(x_2 | X_1 = x_1')}$$

is non-decreasing in x_2 over all values such that the ratio is defined. This condition means that $f(x_2 | X_1 = x_1)$ has g.m.l.r. in x_1, x_2.

Lehmann [9 ,10] has shown that

$$(1.16) \qquad \left\{ \begin{array}{l} (1.15) \text{ implies } (1.14) \ , \\ (1.14) \text{ implies } (1.13) \\ (1.13) \text{ implies } (1.12) \\ (1.12) \text{ is equivalent to } (1.7) \ . \end{array} \right.$$

We will try to generalize these conditions for an arbitrary number n of components.

We have already mentioned the importance of associated random variables in our discussion. Esary et al., [6] have proved the following results:

(1.17) Any subset of associated random variables is associated.

(1.18) The set consisting of a single random variable is associated.

(1.19) Non-decreasing functions of associated random variables are associated.

(1.20) Independent random variables are associated.

As mentioned earlier associativity is not an easy condition to check. Moreover, the idea is not intuitive. Esary, et al., [6] have shown the following:

(1.21) Let $\text{cov}[\gamma(\underline{X}), \delta(\underline{X})] \geq 0$ for all binary non-decreasing functions γ, δ. Then $\underline{X} = (X_1, \ldots, X_n)$ is associated.

(1.22) If $P[X_1 > x_1 | X_2 = x_2]$ is non-decreasing in x_2 for all x_1, then X_1, X_2 are associated.

We note that for $n = 2$, statement (1.22) is a solution of problem B. We note that statement (1.14) is equivalent to statement (1.22) and (1.14) is a solution to problem A for $n = 2$. Hence these two statements establishes a bridge between the results of the Lehmann [10] and Esary et al., [6].

We will generalize the conditions for associativity to an arbitrary number n of variables and try to give a meaningful interpretation to these conditions.

We will consider some special cases, specially, the cases of multivariate exponential distribution as introduced by Harris [7] and Marshal and Olkin [11], and the case of multivariate normal distribution. Numerical examples will be given from time to time to illustrate our results.

We will apply analogous techniques to the case of n components in parallel.

Definition: A system with more than one component is said to be a parallel system if it fails only when all the components fail.

Let there be n components in the system. Let $X_1, \ldots, X_n$ denote the times to failure of components 1 to n. The reliability of the system at time t is given by

$$R(t) = 1 - P(X_1 \leq t, \ldots, X_n \leq t) \quad . \tag{1.23}$$

We want a lower bound to this reliability. Following our approach in the case of a series system we take

(1.24) $$R^*(t) = 1 - \prod_{i=1}^{n} P(X_i \leq t)$$

as a possible lower bound and want to find condition such that $R^*(t)$ is in fact a lower bound on $R(t)$ for all t. We note that if $X_1,\ldots,X_n$ are independent then $R^*(t) = R(t)$ for all t. Hence if $R^*(t)$ is a lower bound it is a "sharp" lower bound in the sense that $R^*(t)$ will actually be attained under the condition of independence.

As before, we will consider a slightly more general problem as follows:

Problem C: To find meaningful sufficient conditions such that

(1.25) $$1 - P(X_1 \leq x_1,\ldots,X_n \leq x_n) \geq 1 - \prod_{i=1}^{n} P(X_i \leq x_i)$$

for all $x_1,\ldots,x_n$. This is equivalent to finding sufficient conditions for

(1.26) $$P(X_1 \leq x_1,\ldots,X_n \leq x_n) \leq \prod_{i=1}^{n} P(X_i \leq x_i)$$

In reliability application we consider non-negative random variables so that the above is meaningful only for $(x_1,\ldots,x_n) \geq 0$.

Our analysis will show that, although some results analogous to those in the series case hold, not all results in the series case have analogues in the parallel case.

2. A Set of Sufficient Conditions for $\prod_{i=1}^{n} P(X_i > x_i)$ to be a Lower Bound on $P(X_1 > x_1,\ldots,X_n > x_n)$

In this section we will consider problem A as mentioned in the introduction. To recapitulate, let $X_1,\ldots,X_n$ be n random variables, not necessarily independent. We want to find meaningful sufficient conditions such that

$$(1.3)\quad P(X_1 > x_1,\ldots,X_n > x_n) \geq \prod_{i=1}^{n} P(X_i > x_i)\ , \quad \text{for all} \quad (x_1,\ldots,x_n)\ .$$

We establish the notation that the left hand side of inequality (1.3) can be written as $P(X_i > x_i,\ i = 1,2,\ldots,n)$.

As mentioned in the introduction, our main results in this section will generalize the conditions in statements (1.12) to (1.15) of Lehmann given in [9 ,10].

In particular we will prove the following main theorems:

Theorem 2.1: If, $P(X_j > x_j,\ j = 1,\ldots,i-1 | X_i > x_i)$ $\geq P(X_j > x_j,\ j = 1,\ldots,i-1)$, $i = 2,\ldots,n$ and for all $x_1,\ldots,x_n$, then inequality (1.3) holds.

Theorem 2.2: If, $P(X_j > x_j,\ j = 1,\ldots,i-1 | X_i > x_i)$ is non-decreasing in x_i, $i = 2,\ldots,n$ for all other x's fixed then the hypothesis of Theorem 2.1 is satisfied, and hence inequality (1.3) holds.

Theorem 2.3: If, $P(X_j > x_j,\ j = 1,\ldots,i-1 | X_i = u_i)$ is non-decreasing in u_i for fixed $x_1,\ldots,x_{i-1}$, $i = 2,\ldots,n$ then the hypothesis of Theorem 2.2 is satisfied, and hence inequality (1.3) holds.

Theorem 2.4: If, $f(u_1,\ldots,u_n)$ has g.m.l.r. in every pair u_i, u_j, $i \neq j$, $i,j = 1,\ldots,n$ then the hypothesis of Theorem 2.3 is satisfied, and hence inequality (1.3) holds.

We hope that in practical applications even if Theorems 2.1, 2.2 and 2.3 are difficult to verify, Theorem 2.4 may be easily verifiable and hence we will know whether $\prod_{i=1}^{n} P(X_i > x_i)$ is a lower bound for $P(X_i > x_i,\ i = 1,\ldots,n)$. We note that if inequality (1.3) holds for all $x_1,\ldots,x_n$ then letting any subset of $x_1,\ldots,x_n$ tend to $-\infty$, inequality (1.3) will hold for the remaining variables.

From time to time we will give examples to show that the conditions of Theorems 2.1-2.4 are sufficient and not necessary conditions. We will also show that the first three conditions may be "asymmetrical". We will try to interpret these conditions and will show that these interpretations are physically meaningful.

Lastly we will consider the case of multivariate exponential distribution as introduced by Marshall and Olkin [11] and Harris [7]. We will show that for this distribution Theorem 2.3 can be applied but Theorem 2.4 may be inapplicable.

2.1 Proof of Theorem 2.1

Theorem 2.1 is really a generalization of condition (1.12) to n-variables. Apart from proving the theorem we will also show by examples that the theorem gives a set of sufficient conditions for $n \geq 3$ and not a set of necessary conditions and that the conditions are "asymmetrical" in the sense to be defined later.

We now prove the theorem:

Theorem 2.1: If,

$$P(X_j > x_j,\ j = 1,\ldots,i-1 | X_i > x_i) \geq P(X_j > x_j,\ j = 1,\ldots,i-1) \tag{2.1}$$

$$i = 2,\ldots,n \quad \text{and for all} \quad x_1,\ldots,x_n ,$$

then inequality (1.3) holds.

Proof: Take $i = 2$. Then inequality (2.1) implies

$$P(X_1 > x_1,\ X_2 > x_2) \geq P(X_1 > x_1)\ P(X_2 > x_2) \quad .$$

Suppose,

$$P(X_j > x_j,\ j = 1,\ldots,i-1) \geq \prod_{j=1}^{i-1} P(X_j > x_j) \tag{2.2}$$

Then, by inequalities (2.1) and (2.2),

$$P(X_j > x_j,\ j = 1,\ldots,i) \geq P(X_j > x_j,\ j = 1,\ldots,i-1)P(X_i > x_i)$$
$$\geq \prod_{j=1}^{i} P(X_j > x_j) \ .$$

Hence by induction inequality (1.3) holds.

Inequality (2.1) is asymmetrical in the following sense. Let $n = 3$, then (2.1) reduces to

$$(2.3) \quad \begin{cases} P(X_1 > x_1 | X_2 > x_2) \geq P(X_1 > x_1) \\ \\ P(X_1 > x_1, X_2 > x_2 | X_3 > x_3) \geq P(X_1 > x_1,\ X_2 > x_2) \ , \\ \qquad\qquad\qquad\qquad\qquad \text{for all } (x_1, x_2, x_3) \end{cases}$$

Even if these inequalities hold, they do not imply that, for example,

$$(2.4) \quad \begin{cases} P(X_3 > x_3 | X_2 > x_2) \geq P(X_3 > x_3) \\ \\ P(X_3 > x_3, X_2 > x_2 | X_1 > x_1) \geq P(X_3 > x_3,\ X_2 > x_2) \ , \\ \qquad\qquad\qquad\qquad\qquad \text{for all } (x_1, x_2, x_3) \end{cases}$$

The following is a counterexample.

<u>Example 2.1</u>: Let $n = 3$ and consider the following data for random variables taking only values 0 and 1.

X_1	X_2	X_3	Probability
0	0	0	.50
0	1	1	.35
1	0	1	.05
1	1	0	.05
1	1	1	.05

The marginal distributions are given by

X_1	X_2	Prob.	X_1	X_3	Prob.	X_2	X_3	Prob.
0	0	.50	0	0	.50	0	0	.50
0	1	.35	0	1	.35	0	1	.05
1	0	.05	1	0	.05	1	0	.05
1	1	.10	1	1	.10	1	1	.40

X_1	Prob.	X_2	Prob.	X_3	Prob.
0	.85	0	.55	0	.55
1	.15	1	.45	1	.45

In the example we will put $x_i = 0^-$ or 1^-, $i = 1,2,3$ so that $P(X_1 > a^-, X_2 > b^-, X_3 > c^-) = P(X_1 \geq a, X_2 \geq b, X_3 \geq c)$. Keeping this in mind the following figures show that inequalities (2.3) hold.

x_1	x_2	$P(X_1 \geq x_1, X_2 \geq x_2)$	$P(X_1 \geq x_1)P(X_2 \geq x_2)$
0	0	1	1
0	1	.45	.45
1	0	.15	.15
1	1	.10	.0675

x_1	x_2	x_3	$P(X_i \geq x_i, i=1,2,3)$	$P(X_1 \geq x_1, X_2 \geq x_2)P(X_3 \geq x_3)$
0	0	0	1	1
1	0	0	.15	.15
0	1	0	.45	.45
0	0	1	.45	.45
0	1	1	.40	.2025
1	0	1	.10	.0675
1	1	0	.10	.10
1	1	1	.05	.045

But $P(X_2 \geq 1, X_3 \geq 1 | X_1 \geq 1) - P(X_2 \geq 1, X_3 \geq 1)$

$$= \frac{.05}{.15} - .40 = \frac{.05-.06}{.15} < 0 \quad .$$

Hence inequalities (2.4) do not hold.

For a system with n components, there are $n!$ ways of numbering them $1,2,\ldots,n$. Theorem 2.1 says that if inequalities (2.1) hold for any one system of numbering, then inequality (1.3) holds. In that case it is quite possible that inequalities (2.1) will not hold for some other system of numbering. Inequalities (2.1) give a set of sufficient conditions for ineqaulity (1.3) to hold. These conditions are also necessary for $n = 2$. But for higher values of n, the conditions are no longer necessary as the following counterexample shows.

Example 2.2: Let us consider the following data for $n = 3$. The random variables take the value 0 or 1 and the comments made under Example 2.1 apply.

X_1	X_2	X_3	Probability
0	0	0	.50
0	1	1	.15
1	0	1	.15
1	1	0	.15
1	1	1	.05

The marginal distributions are given by

X_1	X_2	Probability
0	0	.50
0	1	.15
1	0	.15
1	1	.20

Exactly the same probabilities hold for X_1, X_3 and X_2, X_3.

X_1	Probability
0	.65
1	.35

Exactly the same probabilities hold for X_2, X_3.

The following figures show that inequality (1.3) holds.

x_1	x_2	x_3	$P(X_i \geq x_i,\ i=1,2,3)$	$\prod_{i=1}^{3} P(X_i \geq x_i)$
0	0	0	1	1
1	0	0	.35	.35
0	1	0	.35	.35
0	0	1	.35	.35
0	1	1	.20	.1225
1	0	1	.20	.1225
1	1	0	.20	.1225
1	1	1	.05	.042875

But $P(X_1 \geq 1,\ X_2 \geq 1 | X_3 \geq 1) - P(X_1 \geq 1,\ X_2 \geq 1)$

$$= \frac{.05}{.35} - .20 = \frac{.05-.07}{.35} < 0 \quad .$$

From symmetry, $P(X_1 \geq 1,\ X_3 \geq 1 | X_2 \geq 1) - P(X_1 \geq 1,\ X_3 \geq 1) < 0$ and $P(X_2 \geq 1,\ X_3 \geq 1 | X_1 \geq 1) - P(X_2 \geq 1,\ X_3 \geq 1) < 0$. Hence inequalities (2.1) do not hold.

2.2 Proof of Theorem 2.2

Theorem 2.2 is a generalization of condition (1.13) to n-variables. We will prove the theorem and as in Section 2.1 we will give counter-examples to show that the conditions of the theorem are "asymmetrical" and sufficient without being necessary.

We now prove the theorem.

Theorem 2.2: If,

$$(2.5)\quad \begin{cases} P(X_j > x_j,\ j = 1,\dots,i\text{-}1 \mid X_i > x_i) \\ \quad \geq P(X_j > x_j,\ j = 1,\dots,i\text{-}1 \mid X_i > x_i') \ , \quad i = 2,\dots,n \\ \text{for all } \ x_1,\ x_i' < x_i \ , \quad i = 2,\dots,n \end{cases}$$

then inequalities (2.1) hold and by Theorem 2.1, inequality (1.3) holds.

Proof: By hypothesis, for $i = 2,3,\dots,n$,

$$P(X_1 > x_1,\dots,X_{i-1} > x_{i-1} \mid X_i > x_i) \geq \frac{P(X_1 > x_1,\dots,X_{i-1} > x_{i-1}, X_i > x_i')}{P(X_i > x_i')} \ .$$

Let $x_i' \to -\infty$. (For non-negative random variables we let $x_i' \to 0^-$).
Then $P(X_1 > x_1,\dots,X_{i-1} > x_{i-1} \mid X_i > x_i) \geq P(X_1 > x_1,\dots,X_{i-1} > x_{i-1})$.
Hence, inequalities (2.1) hold and the theorem is proved.

Again, inequalities (2.5) are asymmetrical in the following sense. Let $n = 2$. Then (2.5) reduces to

$$P(X_1 > x_1 | X_2 > x_2) \geq P(X_1 > x_1 | X_2 > x_2')$$

for $x_2' < x_2$ and all x_1. However, this does not imply that

$$P(X_2 > x_2 | X_1 > x_1) \geq P(X_2 > x_2 | X_1 > x_1')$$

for $x_1' < x_1$ and all x_2. The following is a counterexample.

Example 2.3: The following is a bivariate distribution:

X_2 \ X_1	0	1
0	.03	.10
1	.07	.30
2	.10	.40

By actual calculations, we note

$$P(X_1 \geq 0,\ X_2 \geq 0) = 1 = P(X_1 \geq 0)P(X_2 \geq 0)$$

$$P(X_1 \geq 0,\ X_2 \geq 1) = P(X_2 \geq 1) = P(X_1 \geq 0)P(X_2 \geq 1)$$

$$P(X_1 \geq 1,\ X_2 \geq 0) = P(X_1 \geq 1) = P(X_1 \geq 1)P(X_2 \geq 0)$$

$$P(X_1 \geq 0,\ X_2 \geq 2) = P(X_2 \geq 2) = P(X_1 \geq 0)P(X_2 \geq 2)$$

$$P(X_1 \geq 1,\ X_2 \geq 1) = .70 > P(X_1 \geq 1)P(X_2 \geq 1) = .80 \times .87 = .696$$

$$P(X_1 \geq 1,\ X_2 \geq 2) = .40 = P(X_1 \geq 1)P(X_2 \geq 2) = .80 \times .50 = .40$$

Also,

$$P(X_2 \geq 0 | X_1 \geq 0) = 1, \quad P(X_2 \geq 0 | X_1 \geq 1) = 1$$

$$P(X_2 \geq 1 | X_1 \geq 0) = .87, \quad P(X_2 \geq 1 | X_1 \geq 1) = .875$$

$$P(X_2 \geq 2 | X_1 \geq 0) = .50, \quad P(X_2 \geq 2 | X_1 \geq 1) = .50$$

But

$$P(X_1 \geq 0 | X_2 \geq 0) = 1, \; P(X_1 \geq 0 | X_2 \geq 1) = 1, \; P(X_1 \geq 0 | X_2 \geq 2) = 1$$

$$P(X_1 \geq 1 | X_2 \geq 0) = .80, \; P(X_1 \geq 1 | X_2 \geq 1) = \frac{.70}{.87} = .8046, \; P(X_1 \geq 1 | X_2 \geq 2) = .80.$$

Hence, inequalities (2.5) hold for $P(X_2 \geq x_2 | X_1 \geq x_1)$ but not for $P(X_1 \geq 1 | X_2 \geq 1)$ and $P(X_1 \geq 1 | X_2 \geq 2)$.

As in Theorem 2.1 for inequality (1.3) to hold it is enough that inequalities (2.5) hold for at least one set of the $n!$ numbering systems possible. Again, if inequalities (2.5) hold for one system it may not hold for another.

Inequalities (2.5) give only a set of sufficient conditions for inequality (2.1) to hold. These are by no means necessary, as can be seen from the following example:

Example 2.4: Consider the following bivariate distribution:

X_2 \ X_1	0	1	2
0	.30	0	.10
1	0	0	.09
2	.12	.09	.30

By actual calculations, we find that

$$P(X_1 \geq 0) = 1,\ P(X_1 \geq 1) = .58,\ P(X_1 \geq 2) = .49$$

$$P(X_2 \geq 0) = 1,\ P(X_2 \geq 1) = .60,\ P(X_2 \geq 2) = .51$$

$$P(X_1 \geq 0,\ X_2 \geq 0) = 1 = P(X_1 \geq 0)\ P(X_2 \geq 0)$$

$$P(X_1 \geq 0,\ X_2 \geq 1) = P(X_2 \geq 1) = .60 = P(X_1 \geq 0)\ P(X_2 \geq 1)$$

$$P(X_1 \geq 0,\ X_2 \geq 2) = P(X_2 \geq 2) = .51 = P(X_1 \geq 0)\ P(X_2 \geq 2)$$

$$P(X_1 \geq 1,\ X_2 \geq 0) = P(X_1 \geq 1) = .58 = P(X_1 \geq 1)\ P(X_2 \geq 0)$$

$$P(X_1 \geq 1,\ X_2 \geq 1) = .48 > .348 = P(X_1 \geq 1)\ P(X_2 \geq 1)$$

$$P(X_1 \geq 1,\ X_2 \geq 2) = .39 > .2958 = P(X_1 \geq 1)\ P(X_2 \geq 2)$$

$$P(X_1 \geq 2,\ X_2 \geq 0) = P(X_1 \geq 2) = .49 = P(X_1 \geq 2)\ P(X_2 \geq 0)$$

$$P(X_1 \geq 2,\ X_2 \geq 1) = .39 > .294 = P(X_1 \geq 2)\ P(X_2 \geq 1)$$

$$P(X_1 \geq 2,\ X_2 \geq 2) = .30 > .2499 = P(X_1 \geq 2)\ P(X_2 \geq 2)$$

But

$$P(X_1 \geq 2 | X_2 \geq 2) - P(X_1 \geq 2 | X_2 \geq 1) = \frac{.30}{.51} - \frac{.39}{.60} = \frac{.18-.1989}{.51 \times .60} < 0\ ,$$

$$P(X_2 \geq 2 | X_1 \geq 2) - P(X_2 \geq 2 | X_1 \geq 1) = \frac{.30}{.49} - \frac{.39}{.58} = \frac{.174-.1911}{.49 \times .58} < 0\ .$$

2.3 Proof of Theorem 2.3

Theorem 2.3 is a generalization of condition (1.14) to n-variables. As before we will prove the validity of the theorem and give counterexamples to show that the conditions of the theorem are "asymmetrical" and are sufficient without being necessary.

Let $F_i(u)$ be the cumulative distribution function of X_i, i.e.,

$$F_i(u) = P(X_i \leq u) \quad .$$

Then we have the following theorem.

Theorem 2.3: Let

$$h_i(u_i) = P(X_1 > x_1,\ldots,X_{i-1} > x_{i-1} | X_i = u_i) \ , \quad i = 2,\ldots,n$$

$$= \int_{u_1=x_1}^{\infty} \int_{u_2=x_2}^{\infty} \cdots \int_{u_{i-1}=x_{i-1}}^{\infty} dF(u_1,\ldots,u_{i-1}|u_i) \quad .$$

If,

$$(2.6) \quad \begin{cases} h_i(u_i) \text{ is non-decreasing in } u_i \text{ for fixed } x_1,\ldots,x_{i-1} \text{ and} \\ i = 2,\ldots,n \ , \end{cases}$$

then inequalities (2.5) hold and by Theorem 2.2, inequality (1.3) holds.

Proof: By definition,

$$P(X_1 > x_1,\ldots,X_i > x_i) = \int_{x_i}^{\infty} h_i(u)dF_i(u) \ , \quad i = 2,\ldots,n \ .$$

Let $x_i' < x_i$. Then

$$P(X_1 > x_1,\ldots,X_{i-1} > x_{i-1}|X_i > x_i) - P(X_1 > x_1,\ldots,X_{i-1} > x_{i-1}|X_i > x_i')$$

$$= \frac{\int_{x_i}^{\infty} h_i(u)dF_i(u)}{P(X_i > x_i)} - \frac{\int_{x_i'}^{\infty} h_i(u)dF_i(u)}{P(X_i > x_i')}$$

$$= \frac{1}{P(X_i > x_i)P(X_i > x_i')} [P(X_i > x_i')\int_{x_i}^{\infty} h_i(u)dF_i(u)$$

$$- P(X_i > x_i)\int_{x_i'}^{\infty} h_i(u)dF_i(u)]$$

$$= \frac{1}{P(X_i > x_i)P(X_i > x_i')} [\{P(x_i \geq X_i > x_i')+P(X_i > x_i)\}\int_{x_i}^{\infty} h_i(u)dF_i(u)$$

$$- P(X_i > x_i)\{\int_{x_i'}^{x_i} h_i(u)dF_i(u) + \int_{x_i}^{\infty} h_i(u)dF_i(u)\}]$$

$$= \frac{1}{P(X_i > x_i)P(X_i > x_i')} [P(x_i \geq X_i > x_i')\int_{x_i}^{\infty} h_i(u)dF_i(u)$$

$$- P(X_i > x_i)\int_{x_i'}^{x_i} h_i(u)dF_i(u)]$$

$$= \frac{1}{P(X_i > x_i)P(X_i > x_i')} \left[\int_{x_i'}^{x_i} dF_i(v) \int_{x_i}^{\infty} h_i(u)dF_i(u)\right.$$

$$\left. - \int_{x_i}^{\infty} dF_i(v) \int_{x_i'}^{x_i} h_i(u)dF_i(u)\right]$$

$$= \frac{1}{P(X_i > x_i)P(X_i > x_i')} \left[\int_{v=x_i'}^{x_i} \int_{u=x_i}^{\infty} \{h_i(u)-h_i(v)\}dF_i(u)dF_i(v)\right]$$

Now $u \epsilon (x_i, \infty)$

$$v \epsilon (x_i', x_i] \quad .$$

Hence, $u > v$.

By hypothesis, $h_i(u) \geq h_i(v)$. Hence,

$$\int_{v=x_i'}^{x_i} \int_{u=x_i}^{\infty} \{h_i(u)-h_i(v)\}dF_i(u)dF_i(v) \geq 0 \quad ,$$

and inequalities (2.5) hold. Now by Theorem 2.2, inequality (1.3) holds, and Theorem 2.3 is proved.

As in Theorem 2.1, conditions (2.6) are asymmetrical. Let $n = 2$. The following counterexample shows that although $P(X_2 > x_2 | X_1 = u)$ is nondecreasing in u for any fixed x_2, $P(X_1 > x_1 | X_2 = u)$ is not.

Example 2.5: We will use the data of Example 2.3.

$$P(X_1 \geq 0 | X_2 = 0) = 1, \; P(X_1 \geq 0 | X_2 = 1) = 1, \; P(X_1 \geq 0 | X_2 = 2) = 1 \; .$$

$$P(X_1 \geq 1 | X_2 = 0) = \frac{.10}{.13} = .7692, \; P(X_1 \geq 1 | X_2 = 1) = \frac{.30}{.37} = .8108 \quad ,$$

$$P(X_1 \geq 1 | X_2 = 2) = \frac{.40}{.50} = .80 \quad .$$

Hence, (2.6) does not hold for $P(X_1 \geq x_1 | X_2 = u)$. But

$$P(X_2 \geq 0 | X_1 = 0) = 1, \; P(X_2 \geq 0 | X_1 = 1) = 1$$

$$P(X_2 \geq 1 | X_1 = 0) = \frac{.17}{.20} = .85, \; P(X_2 \geq 1 | X_1 = 1) = \frac{.70}{.80} = .875 \; ,$$

$$P(X_2 \geq 2 | X_1 = 0) = \frac{.10}{.20} = .50, \; P(X_2 \geq 2 | X_1 = 1) = \frac{.40}{.80} = .50 \quad .$$

Hence, (2.6) holds for $P(X_2 \geq x_2 | X_1 = u)$.

As before, for inequality (1.3) to hold it is enough that (2.6) holds for at least one set of the $n!$ numbering systems possible. Again, if (2.6) holds for one system it may not hold for another.

Theorem 2.3, of course, gives a set of sufficient conditions for Theorem 2.2 to hold. The following example shows that the conditions are not necessary.

Example 2.6:

X_2 \ X_1	0	1	2
0	.40	.04	.04
1	.04	0	0
2	.04	.04	.40

$$P(X_1 \geq 0|X_2 \geq 1) - P(X_1 \geq 0|X_2 \geq 0) = \frac{.52}{.52} - \frac{1}{1} = 0$$

$$P(X_1 \geq 0|X_2 \geq 2) - P(X_1 \geq 0|X_2 \geq 0) = \frac{.48}{.48} - \frac{1}{1} = 0$$

$$P(X_1 \geq 0|X_2 \geq 2) - P(X_1 \geq 0|X_2 \geq 1) = \frac{.48}{.48} - \frac{.52}{.52} = 0$$

$$P(X_1 \geq 1|X_2 \geq 1) - P(X_1 \geq 1|X_2 \geq 0) = \frac{.44}{.52} - \frac{.52}{1} = \frac{.44-.2704}{.52} > 0$$

$$P(X_1 \geq 1|X_2 \geq 2) - P(X_1 \geq 1|X_2 \geq 0) = \frac{.44}{.48} - \frac{.52}{1} = \frac{.44-.2496}{.48} > 0$$

$$P(X_1 \geq 1|X_2 \geq 2) - P(X_1 \geq 1|X_2 \geq 1) = \frac{.44}{.48} - \frac{.44}{.52} > 0$$

$$P(X_1 \geq 2|X_2 \geq 1) - P(X_1 \geq 2|X_2 \geq 0) = \frac{.40}{.52} - \frac{.44}{1} = \frac{.40-.2288}{.52} > 0$$

$$P(X_1 \geq 2|X_2 \geq 2) - P(X_1 \geq 2|X_2 \geq 0) = \frac{.40}{.48} - \frac{.44}{1} = \frac{.40-.2112}{.48} > 0$$

$$P(X_1 \geq 2|X_2 \geq 2) - P(X_1 \geq 2|X_2 \geq 1) = \frac{.40}{.48} - \frac{.40}{.52} > 0 \quad .$$

But

$$P(X_1 \geq 1|X_2 = 1) - P(X_1 \geq 1|X_2 = 0) = \frac{0}{.04} - \frac{.08}{.48} < 0 \quad .$$

2.4 Physical Interpretations of the Hypotheses of Theorem 2.1, 2.2 and 2.3 and the Example of Multivariate Exponential Distribution

Let us try to interpret the inequalities (2.1), (2.5) and conditions (2.6). Let $n = 3$. Then (2.1) reduce to

$$P(X_1 > x_1, X_2 > x_2 | X_3 > x_3) \geq P(X_1 > x_1, X_2 > x_2) \quad ,$$

$$P(X_1 > x_1 | X_2 > x_2) \geq P(X_1 > x_1) \quad .$$

Suppose we have components 1 and 2 in series. We note the joint probability that component i survives beyond a given time x_i, $i = 1,2$. If we add a third component to the system and condition on the survival of the third component beyond x_3 then the conditional probability that the other two components survive beyond times x_1, x_2 tends to increase beyond the probability of the same event when no component 3 were present. Similar interpretation holds for component 1 and a series system with components 1 and 2. Hence each new component has a beneficial effect on the system in the sense that if the new component survives then the other components tend to survive longer than the time to which they would have survived if no new components were added. If the components were independent, the survival of one component would have no effect on the survival of the other components.

Let us now consider the general case. Let us number the components $1,2,\ldots,n$ in any way. Theorem 2.1 requires that for at least one of these $n!$ numbering, for $i = 2,\ldots,n$, the addition of component

i and its survival, tends to increase the conditional probability of survival of all the other previous components beyond the value which would have been obtained if component i were not added. It is sufficient for the theorem if this condition holds for all other components and not only for the previous components. Hence, for the particular numbering system there is a beneficial effect of the survival of the i^{th} component $(i = 2,\ldots,n)$ on the survival of all other components. Note, the same may not be true for a different numbering system.

Let us now consider inequalities (2.5). Let $n = 3$. Then (2.5) reduce to

$P(X_1 > x_1 | X_2 > x_2)$ non-decreasing in x_2 for fixed x_1.

$P(X_1 > x_1, X_2 > x_2 | X_3 > x_3)$ non-decreasing in x_3 for fixed (x_1, x_2).

Let us consider the first condition. It says that the longer the second component survives, the conditional probability that the first component survives longer than a given time also tends to increase. In other words, long survival of the second component tends to improve the chance of survival of the first component. Similarly, from the second condition a long survival of the third component tends to improve the joint probability of survival of the other two components. In a way the component we are conditioning on is working for the benefit of the other components. This should be contrasted with a system with independent components. In that case, the conditioning on one component has no effect on the chance of survival of the other components.

Let us now consider the general case. As before let us number the components $1,2,\ldots,n$ in any way. Theorem 2.2 requires that for at least one of these $n!$ numbering systems, for $i = 1,\ldots,n$, the longer component i survives the higher is the probability that all the other previous components survive past any given set of time periods. In fact, it is sufficient to say that the chance of survival of the other (not necessarily the previous) components beyond any given set of time periods is going up with the continued survival of the i^{th} component for $i = 1,\ldots,n$. Hence, for the particular numbering system there is a beneficial effect of the survival of any component on all the previous components. Note that the same may not be true for a different numbering.

The hypothesis of Theorem 2.3 gives a set of sufficient conditions for Theorem 2.2 to hold. It says that for Theorem 2.1 to hold it is sufficient that a numbering system exist for which, for $i = 1,2,\ldots,n$, if the time to failure of the i^{th} component goes up, the joing survival probability of all previous components beyond any set of time periods must improve. As before, we can drop the word "previous" from this statement. Hence, Theorem 2.3 is essentially a restatement of Theorem 2.2 in terms of time to failure rather than in terms of survival beyond given points of time. Again, if conditions (2.6) hold for one set of numbering, they need not hold for another.

Example 2.7: Let us consider the case of multivariate exponential distribution introduced by Marshall and Olkin [11] and Harris [7]. For simplicity in algebra we will consider the cases of trivariate exponential and bivariate exponential.

Let X_1, X_2, X_3 be three random variables having trivariate exponential distribution. Then,

$$P(X_1 > x_1, X_2 > x_2, X_3 > x_3)$$

$$= \exp[-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_3 - \lambda_4 \max(x_1,x_2) - \lambda_5 \max(x_2,x_3) - \lambda_6 \max(x_1,x_3) - \lambda_7 \max(x_1,x_2,x_3)]$$

$$(\lambda_1,\dots,\lambda_7) > 0 , \quad (x_1,x_2,x_3) \geq 0 .$$

The trivariate exponential distribution has a discontinuous density function with 13 different forms including singular components. Hence we will only consider inequalities (2.5) for this case.

By actual calculations

$$P(X_1 > x_1 | X_2 > x_2) - P(X_1 > x_1 | X_2 > x_2')$$

$$= \exp[-(\lambda_1+\lambda_6)x_1 - (\lambda_4+\lambda_7) \max(x_1-x_2,0)] - \exp[-(\lambda_1+\lambda_6)x_1 - (\lambda_4+\lambda_7) \max(x_1-x_2',0)]$$

Let $x_2 > x_2'$. Then $x_1-x_2 < x_1-x_2'$.

Hence, $\max(x_1-x_2,0) \leq \max(x_1-x_2',0)$.

This implies that $P(X_1 > x_1|X_2 > x_2) - P(X_1 > x_1|X_2 > x_2') \geq 0$. Also,

$$P(X_1 > x_1, X_2 > x_2|X_3 > x_3) - P(X_1 > x_1, X_2 > x_2|X_3 > x_3')$$

$$= \exp[-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_4 \max(x_1, x_2)][\exp\{-\lambda_5 \max(x_2 - x_3, 0)$$

$$-\lambda_6 \max(x_1 - x_3, 0) - \lambda_7 \max(x_1 - x_3, x_2 - x_3, 0)\} - \exp\{-\lambda_5 \max(x_2\ x_3', 0)$$

$$-\lambda_6 \max(x_1 - x_3', 0) - \lambda_7 \max(x_1 - x_3', x_2 - x_3', 0)\}] \quad .$$

If $x_3 > x_3'$, we can easily see, as in the previous case, that the difference is nonnegative.

These two results verifies (2.5). By Theorem 2.2 inequalities (2.1) and (1.3) hold. Similar calculations will show that for multivariate exponential inequalities (2.5), (2.1), and (1.3) hold. In fact, in this case, it is easy to verify inequality (1.3) directly. We also note that inequalities (2.5), (2.1) and (1.3) hold for all possible numbering systems.

To verify condition (2.6) we will consider the bivariate exponential distribution of two variables X_1 and X_2. Let us write the distribution as,

$$P(X_1 > x_1, X_2 > x_2) = \exp[-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 \max(x_1, x_2)] \quad ,$$

$$(\lambda_1, \lambda_2) > 0 \ , \qquad (x_1, x_2) \geq 0 \quad .$$

The density has three different forms including a singular component. Let us denote them by $f_i(\cdot,\cdot)$, $i = 1,2,3$.

a) Let $x_1 > x_2$. Then

$$f_1(x_1, x_2) = \lambda_2(\lambda_1 + \lambda_3)e^{-(\lambda_1+\lambda_3)x_1 - \lambda_2 x_2} \quad .$$

b) Let $x_1 < x_2$. Then

$$f_2(x_1,x_2) = \lambda_1(\lambda_2+\lambda_3)e^{-\lambda_1 x_1-(\lambda_2+\lambda_3)x_2} .$$

c) Let $x_1 = x_2 = x$. Then

$$f_3(x,x) = \lambda_3 e^{-(\lambda_1+\lambda_2+\lambda_3)x} .$$

Let us consider $P(X_1 > x_1 | X_2 = u_2) = \int_{x_1}^{\infty} f(u_1|u_2)du_1$.

Let $u_2 < x_1$. Then

$$P(X_1 > x_1|X_2 = u_2) = \frac{\lambda_2}{\lambda_2+\lambda_3} e^{-\lambda_3(x_1-u_2)-\lambda_1 x_1} ,$$

increasing in u_2 for fixed $x_1 \geq 0$.

Let $u_2 \geq x_1$. Then

$$P(X_1 > x_1, X_2=u_2) = \int_{x_1}^{u_2^-} f_2(u_1,u_2)du_1 + \int_{u_2^+}^{\infty} f_1(u_1,u_2)du_1 + f_3(u_2,u_2)$$

$$= (\lambda_2+\lambda_3)e^{-\lambda_1 x_1-(\lambda_2+\lambda_3)u_2}$$

Hence, $P(X_1 > x_1|X_2=u_2) = e^{-\lambda_1 x_1}$, constant in u_2 for fixed x_1.

Let $u_2 \geq x_1$, $u_2' < x_1$. Then $u_2 > u_2'$ and

$$P(X_1 > x_1|X_2=u_2) - P(X_1 > x_1|X_2=u_2') = e^{-\lambda_1 x_1}\left[1 - \frac{\lambda_2}{\lambda_2+\lambda_3} e^{-\lambda_3(x_1-u_2')}\right] > 0 .$$

Hence, condition (2.6) holds and Theorem 2.3 applies.

2.5 Generalized Monotone Likelihood Ratio and the Proof of Theorem 2.4

In the case of multivariate exponential one would probably verify inequality (1.3) directly or at most verify the conditions involving probability only. One can similarly consider multivariate normal distribution and determine under what conditions the different theorems hold. But the multivariate normal density is not too easy to integrate. In fact, exact integration is not possible. In general, we may have a situation where the density function cannot be integrated easily and one would like to determine whether inequality (1.3) holds without performing any integration. This suggests that we should look for sufficient conditions involving density only. Such conditions are available and can be stated in terms of generalized monotone likelihood ratio abbreviated g.m.l.r. We have already defined the concept in the introduction. To recapitulate,

Let $u_i > u_i'$, $u_j > u_j'$, $i \neq j$, $i,j = 1,\ldots,n$ · $f(u_1,\ldots,u_n)$ is said to have g.m.l.r. in u_i,u_j iff

$$
\begin{aligned}
&f(u_1,\ldots,u_i,\ldots,u_j,\ldots,u_n)f(u_1,\ldots,u_i',\ldots,u_j',\ldots,u_n) \\
(1.10)\qquad & \\
&-f(u_1,\ldots,u_i',\ldots,u_j,\ldots,u_n)f(u_1,\ldots,u_i,\ldots,u_j',\ldots,u_n) \geq 0 \quad ,
\end{aligned}
$$

for all u_i, u_i', u_j, u_j' as above and for the other $(n-2)$ u's fixed at any value $u_1,\ldots,u_{i-1}$, $u_{i+1},\ldots,u_{j-1}$, $u_{j+1},\ldots,u_n$. This definition has a proviso and a consequence, both of which have been stated in the introduction.

In terms of g.m.l.r. we have the following

Theorem 2.4: If,

$$(2.7)\quad \begin{cases} f(u_1,\ldots,u_n) \text{ has g.m.l.r. in every pair } u_i,\ u_j, \\ \\ i \neq j,\ i,j = 1,\ldots,n\ , \end{cases}$$

then (2.6) holds and by Theorem 2.3, inequality (1.3) holds.

We will need a few results to prove this theorem. To prove these results we will use the following result of Lehmann.

Theorem 2.5: Let $p(x|y)$ be a family of densities, indexed by y, on the real line with g.m.l.r. in x,y. Then

(i) If $\psi(\cdot)$ is a non-decreasing function of x, then $E\ \psi(X|y)$, (expectation with respect to X only) is a non-decreasing function of y.

(ii) For any $y > y'$,

$$P(X > x|y) \geq P(X > x|y')\ , \quad \text{for all } x\ .$$

Proof: See Lehmann [9 , lemma 2, p. 74].

Let $g_j(u_k) = P(X_j > x_j|u_{j+1},\ldots,u_k,\ldots,u_i)\ ,\ 1 \leq j < k \leq i \leq n\ .$

Lemma 2.6: If,

$$(2.8)\quad \begin{cases} f(u_j | u_{j+1},\ldots,u_k,\ldots,u_i) \text{ has g.m.l.r. in } u_j,\ u_k \text{ for} \\ \\ \text{other } u\text{'s fixed,}\quad 1 \le j < k \le i \le n\ , \end{cases}$$

then $g_j(u_k)$ is a non-decreasing function of u_k for other variables fixed.

Proof: When all variables other than u_j, u_k are fixed, we can look upon $f(u_j | u_{j+1},\ldots,u_k,\ldots,u_i)$ as the density of X_j indexed by $X_k = u_k$. Let $u_k > u_k'$. By part (ii) of Theorem 2.5,

$$P(X_j > x_j | u_{j+1},\ldots,u_k,\ldots,u_i) \ge P(X_j > x_j | u_{j+1},\ldots,u_k',\ldots,u_i)\ ,$$

$$1 \le j < k \le i \le n\ ,$$

which proves the lemma.

Let

$$r(x_1,\ldots,x_j;u_{j+1},u_k) = P(X_1 > x_1,\ldots,X_j > x_j | u_{j+1},\ldots,u_k,\ldots,u_i)\ .$$

Then

$$r(x_1,\ldots,x_{j+1};u_{j+2},u_k) = P(X_1 > x_1,\ldots,X_{j+1} > x_{j+1} | u_{j+2},\ldots,u_k,\ldots,u_i)\ .$$

Lemma 2.7: If,

(2.9)
(a) $f(u_{j+1}|u_{j+2},\dots,u_k,\dots,u_i)$ has g.m.l.r. in u_{j+1} and u_k for the other u's fixed, and

(b) $r(x_1,\dots,x_j;u_{j+1},u_k)$ is non-decreasing in u_{j+1}, u_k for other variables fixed,

then $r(x_1,\dots,x_{j+1};u_{j+2},u_k)$ is non-decreasing in u_k for other variables fixed.

Proof: Let $u_k > u_k'$. Then

$$r(x_1,\dots,x_{j+1};u_{j+2},u_k) - r(x_1,\dots,x_{j+1};u_{j+2},u_k')$$

$$= \int_{u_{j+1}=x_{j+1}}^{\infty} [P(X_1>x_1,\dots,X_j>x_j|u_{j+1},\dots,u_k,\dots,u_i)f(u_{j+1}|u_{j+2},\dots,u_k,\dots,u_i)$$

$$- P(X_1 > x_1,\dots,X_j > x_j|u_{j+1},\dots,u_k',\dots,u_i)f(u_{j+1}|u_{j+2},\dots,u_k',\dots,u_i)]d\mu$$

$$= \int_{u_{j+1}=x_{j+1}}^{\infty} [r(x_1,\dots,x_j;u_{j+1},u_k)f(u_{j+1}|u_{j+2},\dots,u_k,\dots,u_i)$$

$$- r(x_1,\dots,x_j;u_{j+1},u_k')f(u_{j+1}|u_{j+2},\dots,u_k',\dots,u_i)]d\mu$$

$$\geq \int_{u_{j+1}=x_{j+1}}^{\infty} [r(x_1,\dots,x_j;u_{j+1},u_k')f(u_{j+1}|u_{j+2},\dots,u_k,\dots,u_i)$$

$$- r(x_1,\dots,x_j;u_{j+1},u_k')f(u_{j+1}|u_{j+2},\dots,u_k',\dots,u_i)]d\mu \text{ , by (2.9b)}$$

$$= \int_{u_{j+1}=x_{j+1}}^{\infty} r(x_1,\dots,x_j;u_{j+1},u_k')[f(u_{j+1}|u_{j+2},\dots,u_k,\dots,u_i)$$

$$- f(u_{j+1}|u_{j+2},\dots,u_k',\dots,u_i)]d\mu \quad .$$

For all other variables fixed, let us define

$$\psi(u_{j+1}) = 0 \quad \text{for} \quad u_{j+1} < x_{j+1}$$

$$= r(x_1,\ldots,x_j;u_{j+1},u_k') \quad \text{for} \quad u_{j+1} \geq x_{j+1}$$

Hence $\psi(\cdot)$ is a non-decreasing function of u_{j+1}. Using (2.9a) and part (i) of Theorem 2.5, we get the lemma.

The following theorem is a generalization of (1.15) and gives a set of sufficient conditions for (2.6) to hold. These conditions are given in terms of monotone likelihood ratio.

<u>Theorem 2.8</u>: If

$$(2.10) \quad \begin{cases} f(u_j|u_{j+1},\ldots,u_k,\ldots,u_i) \text{ has g.m.l.r. in } u_j \text{ and} \\ u_k \text{ for other variables fixed, } k = j+1, i; j = 1,\ldots,i-1; \\ \qquad i = 2,\ldots,n \end{cases},$$

then $h_i(u_i) \equiv P(X_1 > x_1,\ldots,X_{i-1} > x_{i-1}|X_i = u)$ s non-decreasing in u for fixed $(x_1,\ldots,x_{i-1})$, $i = 2,\ldots,n$. This means that (2.6) holds and by Theorem 2.3, inequality (1.3) holds.

<u>Proof</u>: We will prove the result by induction. Let $j = 1$. Then by (2.10),

$$\frac{f(u_1|u_2,\ldots,u_k,\ldots,u_i)}{f(u_1|u_2,\ldots,u_k',\ldots,u_i)}$$

is non-decreasing in u_1 for $u_k > u_k'$ and the other u's fixed. By Lemma 2.6, $g_1(u_k) = P(X_1 > x_1 | u_2,\ldots,u_k,\ldots,u_i)$ is non-decreasing in u_k for other variables fixed.

Let $r(x_1,\ldots,x_j;u_{j+1},u_k) \equiv P(X_1 > x_1,\ldots,X_j > x_j | u_{j+1},\ldots,u_k,\ldots,u_i)$ be non-decreasing in u_k for other variables fixed, $k = j+1,i$. Then by (2.10) and Lemma 2.7,

$$r(x_1,\ldots,x_{j+1};u_{j+2},u_k)$$

is non-decreasing in u_k for the other variables fixed. Hence, by induction on j,

$$P(X_1 > x_1,\ldots,X_j > x_j | u_{j+1},\ldots,u_k,\ldots,u_i)$$

is non-decreasing in u_k for other variables fixed for $k = j+1,i; j = 1,\ldots,i-1; i = 2,\ldots,n$. Putting $j = i-1$, $k = i$, the result follows.

The hypothesis of Theorem 2.8 requires conditioning on more than one variable and computation of marginal distributions. It is possible to avoid multiple conditioning as is shown by the lemma below.

<u>Lemma 2.9</u>: $f(u_j | u_{j+1},\ldots,u_k,\ldots,u_i)$ has g.m.l.r. in u_j,u_k for other u's fixed <u>iff</u> $f(u_j,\ldots,u_{k-1},u_{k+1},\ldots,u_i | u_k)$ has g.m.l.r. in u_j,u_k for other u's fixed, $k = j+1,\ldots,i$; $j = 1,\ldots,i-1$; $i = 2,\ldots,n$.

<u>Proof</u>: Let $u_k > u_k'$. Now

$$\frac{f(u_j|u_{j+1},\ldots,u_k,\ldots,u_i)}{f(u_j|u_{j+1},\ldots,u_k',\ldots,u_i)} = \frac{f(u_j,u_{j+1},\ldots,u_k,\ldots,u_i)}{f(u_j,u_{j+1},\ldots,u_k',\ldots,u_i)} \frac{f(u_{j+1},\ldots,u_k',\ldots,u_i)}{f(u_{j+1},\ldots,u_k,\ldots,u_i)}$$

$$= \frac{f(u_k)f(u_j,\ldots,u_{k-1},u_{k+1},\ldots,u_i|u_k)}{f(u_k')f(u_j,\ldots,u_{k-1},u_{k+1},\ldots,u_i|u_k')} \frac{f(u_{j+1},\ldots,u_k',\ldots,u_i)}{f(u_{j+1},\ldots,u_k,\ldots,u_i)}$$

$$= \frac{f(u_j,\ldots,u_{k-1},u_{k+1},\ldots,u_i|u_k)}{f(u_j,\ldots,u_{k-1},u_{k+1},\ldots,u_i|u_k')} \frac{f(u_{j+1},\ldots,u_{k-1},u_{k+1},\ldots,u_i|u_k')}{f(u_{j+1},\ldots,u_{k-1},u_{k+1},\ldots,u_i|u_k)}$$

Since $u_{j+1},\ldots,u_k,u_k',\ldots,u_i$ are fixed, the second ratio in the last expression is fixed. Hence, if u_j increases, then

$$\frac{f(u_j|u_{j+1},\ldots,u_k,\ldots,u_i)}{f(u_j|u_{j+1},\ldots,u_k',\ldots,u_i)} \quad \text{and} \quad \frac{f(u_j,\ldots,u_{k-1},u_{k+1},\ldots,u_i|u_k)}{f(u_j,\ldots,u_{k-1},u_{k+1},\ldots,u_i|u_k')}$$

increases (or decreases) together. Hence the lemma.

Corollary 2.10: If

(2.11) $f(u_j,\ldots,u_{k-1},u_{k+1},\ldots,u_i|u_k)$ has g.m.l.r. in u_j,u_k for other u's fixed, $k = j+1,\ldots,i$; $j = 1,\ldots,i-1$; $i = 2,\ldots,n$,

then (2.10) holds and hence condition (2.6) holds..

Proof: Follows from Lemma 2.9 and Theorem 2.8.

We can completely avoid considering conditional densities as is shown by the lemma below.

Lemma 2.11: $f(u_j,\ldots,u_{k-1},u_{k+1},\ldots,u_i|u_k)$ has g.m.l.r. in u_j,u_k and other u's fixed iff $f(u_j,\ldots,u_k,\ldots,u_i)$ has m.l.r. in u_j,u_k and the other u's fixed, $k = j+1,\ldots,i$; $j = 1,\ldots,i-1$; $i = 2,\ldots,n$.

Proof: Obvious from the proof of Lemma 2.9.

From Lemma 2.11, Corollary 2.10, and Theorem 2.8, it is sufficient to consider $f(u_j,\ldots,u_k,\ldots,u_i)$ for $k = j+1,\ldots,i$; $j = 1,\ldots,i-1$; $i = 2,\ldots,n$, for inequality (1.3) to hold.

We can completely avoid considering marginal densities, as is shown by the lemmas below.

Lemma 2.12: Let $r \leq n$, $(i_1,i_2,\ldots,i_r) \subset (1,2,\ldots,n)$, $P = (u_{i_2},\ldots,u_{i_{j-1}})$ and $Q = (u_{i_{j+1}},\ldots,u_{i_{r-1}})$, $j = 1,2,\ldots,r$ with the proviso that $P = \Phi$ when $j = 1$ and $Q = \Phi$ when $j = r$. If $f(u_{i_1},P,u_{i_j},Q,u_{i_r})$ has g.m.l.r. in u_{i_s},u_{i_t} for $s,t = 1,\ldots,r$; $s \neq t$ and other u's fixed, then $f(u_{i_1},P,Q,u_{i_r})$ has g.m.l.r. u_{i_s},u_{i_t} for $s,t = 1,\ldots,r$; $s \neq t$; $s \neq j$; $t \neq j$, and other u's fixed.

Proof: It is enough to prove this lemma for $s = 1$, $t = r$, and $1 \neq j \neq r$. We have to show that for every $u_{i_1} > u'_{i_1}$, $u_{i_r} > u'_{i_r}$.

$$(2.12)\quad f(u_{i_1},P,Q,u_{i_r})f(u'_{i_1},P,Q,u'_{i_r}) - f(u'_{i_1},P,Q,u_{i_r})f(u_{i_1},P,Q,u'_{i_r}) \geq 0 .$$

We will follow essentially the technique of Khursheed Alam [1] and Lehmann [8]. (There was an error in Lehmann's original version. The error has been corrected by Khursheed Alam.)

$$(2.13)\quad f(u_{i_1},P,Q,u_{i_r}) = \int_{u_{i_j}=-\infty}^{\infty} f(u_{i_1},P,u_{i_j},Q,u_{i_r})d\mu$$

$$= \int_{u_{i_j}=-\infty}^{\infty} \frac{f(u_{i_1},P,u_{i_j},Q,u_{i_r})}{f(u'_{i_1},P,u_{i_j},Q,u_{i_r})} \frac{f(u'_{i_1},P,u_{i_j},Q,u_{i_r})}{f(u'_{i_1},P,Q,u_{i_r})} f(u'_{i_1},P,Q,u_{i_r})d\mu$$

$$= f(u'_{i_1},P,Q,u_{i_r}) \int_{u_{i_j}=-\infty}^{\infty} \frac{f(u_{i_1},P,u_{i_j},Q,u_{i_r})}{f(u'_{i_1},P,u_{i_j},Q,u_{i_r})} f(u_{i_j}|u'_{i_1},P,Q,u_{i_r})d\mu$$

Similarly,

$$(2.14)\quad f(u_{i_1},P,Q,u'_{i_r}) = f(u'_{i_1},P,Q,u'_{i_r}) \int_{u_{i_j}=-\infty}^{\infty} \frac{f(u_{i_1},P,u_{i_j},Q,u'_{i_r})}{f(u'_{i_1},P,u_{i_j},Q,u'_{i_r})} \times$$

$$f(u_{i_j}|u'_{i_1},P,Q,u'_{i_r})d\mu \quad .$$

Since by hypothesis

$$\frac{f(u_{i_1},P,u_{i_j},Q,u'_{i_r})}{f(u'_{i_1},P,u_{i_j},Q,u'_{i_r})} \text{ is non-decreasing in } u'_{i_r} \quad \text{for} \quad u_{i_1} > u'_{i_1}$$

and other u's fixed, we have for $u_{i_r} > u'_{i_r}$,

$$\frac{f(u_{i_1},P,u_{i_j},Q,u'_{i_r})}{f(u'_{i_1},P,u_{i_j},Q,u'_{i_r})} \leq \frac{f(u_{i_1},P,u_{i_j},Q,u_{i_r})}{f(u'_{i_1},P,u_{i_j},Q,u_{i_r})} .$$

Hence, from (2.14),

$$f(u_{i_1},P,Q,u'_{i_r}) \leq f(u'_{i_1},P,Q,u'_{i_r}) \int_{u_{i_j}=-\infty}^{\infty} \frac{f(u_{i_1},P,u_{i_j},Q,u_{i_r})}{f(u'_{i_1},P,u_{i_j},Q,u_{i_r})} \times$$

$$f(u_{i_j}|u'_{i_1},P,Q,u'_{i_r})d\mu .$$

Hence, the left-hand side of (2.12) is <u>greater than or equal to</u>

$$(2.15)\ f(u'_{i_1},P,Q,u_{i_r})f(u'_{i_1},P,Q,u'_{i_r}) \int_{u_{i_j}=-\infty}^{\infty} \frac{f(u_{i_1},P,u_{i_j},Q,u_{i_r})}{f(u'_{i_1},P,u_{i_j},Q,u_{i_r})} \times$$

$$[f(u_{i_j}|u'_{i_1},P,Q,u_{i_r}) - f(u_{i_j}|u'_{i_1},P,Q,u'_{i_r})]d\mu .$$

By hypothesis,

$$\frac{f(u_{i_j}|u'_{i_1},P,Q,u_{i_r})}{f(u_{i_j}|u'_{i_1},P,Q,u'_{i_r})} = \frac{f(u'_{i_1},P,u_{i_j},Q,u_{i_r})}{f(u'_{i_1},P,u_{i_j},Q,u'_{i_r})} \cdot \frac{f(u'_{i_1},P,Q,u'_{i_r})}{f(u'_{i_1},P,Q,u_{i_r})}$$ is non-decreasing

in u_{i_j} for $u_{i_r} > u'_{i_r}$ and other u's fixed. Hence,

$f(u_{i_j} | u'_{i_1}, P, Q, u_{i_r})$ has g.m.l.r. in u_{i_1} and u_{i_r}. Again by hypothesis,

$$\psi(u_{i_j}) = \frac{f(u_{i_1}, P, u_{i_j}, Q, u_{i_r})}{f(u'_{i_1}, P, u_{i_j}, Q, u_{i_r})}$$

is non-decreasing in u_{i_j} for $u_{i_1} > u'_{i_1}$ and other u's fixed.

Hence by Theorem 2.5, part (i), the integral in (2.15) is nonnegative and the lemma is proved. See example 3.1 for reverse implication.

Lemma 2.13: Let A and B be subsets of $(u_1,\ldots,u_n)$ such that $A \cap B = \Phi$, $i \notin A \cup B$, $j \notin A \cup B$, $A \cup B \subset (u_1,\ldots,u_n)$. Let $f(u_1,u_2,\ldots,u_n)$ have g.m.l.r. in every pair u_i,u_j when the other u's are held fixed; $i,j = 1,\ldots,n$; $i \neq j$. Then $f(u_i,A,B,u_j)$ have g.m.l.r. in u_i,u_j for $i,j = 1,\ldots,n$, $i \neq j$ when the other u's are held fixed.

Proof: The proof is by induction. We integrate out one variable at a time from the set $(u_1,\ldots,u_n) - (u_i,A,B,u_j)$. Lemma 2.12 proves the inductive steps and also sets out the induction.

We will now prove Theorem 2.4. To recapitulate, the statement of Theorem 2.4 is as follows.

Theorem 2.4: If

$$(2.7)\quad \begin{cases} f(u_1,\ldots,u_n) \text{ has g.m.l.r. in every pair } u_i,u_j,\ i \neq j\ , \\ i,j = 1,\ldots,n\ , \end{cases}$$

then (2.6) holds and by Theorem 2.3, inequality (1.3) holds.

Proof: Follows from Lemmas 2.13, 2.11, and Corollary 2.10.

2.6 Interpretation of the Conditions Involving Generalized Monotone Likelihood Ratio and the Example of Bivariate Exponential Distribution

Let us now interpret the meaning of the conditions involving g.m.l.r.. Let A and B be a partition of the set $(u_1,\ldots,u_n)$. Let $(u_1,\ldots,u_n) \geq (u_1',\ldots,u_n')$. Let A', B' be the corresponding partition of $(u_1',\ldots,u_n')$. It is obvious that for the hypothesis of Theorem 2.6 to hold,

$$\frac{f(A|B)}{f(A|B')} - \frac{f(A'|B)}{f(A'|B')} \geq 0 \tag{2.16}$$

for all possible partitions A, B, A', B' such that none of these sets is empty.

What is the physical interpretation of inequality (2.16)? Let us consider the case $n = 2$. Let $A = \{u_1\}$, $B = \{u_2\}$. Then inequality (2.16) reduces to

$$\frac{f(u_1|u_2)}{f(u_1|u_2')} \geq \frac{f(u_1'|u_2)}{f(u_1'|u_2')} \,, \quad \text{for every} \quad u_1 > u_1' \quad \text{and} \quad u_2 > u_2' \;.$$

Suppose $f(u_1'|u_2') = p$, $f(u_1|u_2') = kp$,

$$f(u_1'|u_2) = q \;.$$

Then $f(u_1|u_2) \geq kq$.

For a smaller value u_2' of X_2, $f(X_1|X_2)$ has increased by a factor k as X_1 increased from u_1' to u_1. For a higher value u_2 of X_2, $f(X_1|X_2)$ will increase by a factor larger than k as X_1 increased from u_1' to u_1.

Let X_1, X_2 be discrete random variables. Then $f(\cdot|\cdot)$ represent actual probabilities of survival. Given two times to failure of the second component, the conditional probability of survival of the first component, up to a given time, increases with increase in X_1, at a faster rate when X_2 is large than when X_2 is small.

If X_1, X_2 represent continuous random variables, we can no longer talk about probabilities; we have to talk about likelihoods. The generalized monotone likelihood ratio property in this case implies that the likelihood of observing a large value of X_1 increases at a faster rate when we have observed a large value of X_2 than when we have observed a small value of X_2.

Similar interpretations exist for an arbitrary number n of components. We have to consider X_1 and X_2 as random vectors rather than random variables.

These interpretations become clearer if we consider the following facts. Let $n = 2$. Suppose we are given a random sample of size 2 of X_1. Let the sample values be $u_1 > u_1'$. Similarly, let $u_2 > u_2'$ be a random sample of size 2 from X_2. We are also given that the density $f(u_1,u_2)$ has g.m.l.r. in u_1,u_2 i.e.,

$$f(u_1,u_2)f(u_1',u_2') \geq f(u_1,u_2')f(u_1',u_2) \quad .$$

We want to match (u_1, u_1') with (u_2, u_2') such that the likelihood is maximized in some sense.

There are two sets of possible matchings

$$\begin{array}{ccc} u_1 \longleftrightarrow u_2 & & u_1 \longleftrightarrow u_2' \\ & \text{or,} & \\ u_1' \longleftrightarrow u_2' & & u_1' \longleftrightarrow u_2 \end{array} .$$

The two sets of likelihoods are $f(u_1,u_2)f(u_1',u_2')$ and $f(u_1,u_2')f(u_1',u_2)$. Due to the g.m.l.r. property the first set of matchings tends to have a larger likelihood than the second set. This means that larger values of X_2 implies larger values of X_1 and conversely. The extension to n variables $X_1,\ldots,X_n$ is immediate. Note, if we are considering discrete-valued random variables then the likelihoods can be replaced by actual probabilities and the interpretation becomes very intuitive.

The physical interpretations of the hypotheses of Theorems 2.1, 2.2, 2.3 and 2.4 point to the fact that each component (or each set of components) has beneficial effects on the other components in the sense that if the given set survives for a long time, then there is a greater chance (or likelihood) that the other components will survive for a long time also.

Theorem 2.4 gives a set of sufficient conditions for inequality (1.3) to hold. These conditions are by no means necessary as is shown by the following example.

Example 2.8: Let X_1, X_2 have bivariate exponential distribution as given in Example 2.7. We have shown that (1.3) holds for this distribution. Let us check if $f(u_1,u_2)$ has g.m.l.r. in u_1,u_2.

Let $u_1 > u_1' > u_2 > u_2'$. Then using the values given in Example 2.7,

$$f(u_1,u_2)f(u_1',u_2') - f(u_1,u_2')f(u_1',u_2)$$

$$= f_1(u_1,u_2)f_1(u_1',u_2') - f_1(u_1,u_2')f_1(u_1',u_2) = 0$$

Let $u_1 > u_2 > u_1' > u_2'$.

$$f(u_1,u_2)f(u_1',u_2') - f(u_1,u_2')f(u_1',u_2)$$

$$= f_1(u_1,u_2)f_1(u_1',u_2') - f_1(u_1,u_2')f_2(u_1',u_2)$$

$$= \lambda_2^2(\lambda_1+\lambda_3)^2 \exp\{-(\lambda_1+\lambda_3)u_1-(\lambda_2+\lambda_3)u_2-\lambda_1u_1'-\lambda_2u_2'\}$$

$$[e^{\lambda_3(u_2-u_1')} - \frac{\lambda_1(\lambda_2+\lambda_3)}{\lambda_2(\lambda_1+\lambda_3)}]$$

This difference can be negative if $\lambda_3(u_2-u_1') < \log \frac{\lambda_1(\lambda_2+\lambda_3)}{\lambda_2(\lambda_1+\lambda_3)}$. For example, if $\lambda_1 = 2$, $\lambda_2 = \lambda_3 = 1$, $u_2 - u_1' < \log (\frac{4}{3})$. Hence bivariate exponential density need not have g.m.l.r.

To find if the density ever has g.m.l.r. we proceed as follows. We will consider the ratio $\frac{f(u_1,u_2)}{f(u_1,u_2')}$ as a function of u_1 for fixed $u_2 > u_2'$ for the following five cases.

a) $u_2 > u_2' > u_1$.

$$\frac{f(u_1,u_2)}{f(u_1,u_2')} = \frac{f_2(u_1,u_2)}{f_2(u_1,u_2')} = e^{(\lambda_2+\lambda_3)(u_2'-u_2)}, \quad \text{constant in } u_1.$$

b) $u_2 > u_2' = u_1$.

$$\frac{f(u_1,u_2)}{f(u_1,u_2')} = \frac{f_2(u_1,u_2)}{f_3(u_1,u_1)} = \frac{\lambda_1(\lambda_2+\lambda_3)}{\lambda_3} e^{(\lambda_2+\lambda_3)(u_1-u_2)}, \text{ increasing}$$

in u_1 if $\lambda_2 + \lambda_3 > 0$.

c) $u_2 > u_1 > u_2'$.

$$\frac{f(u_1,u_2)}{f(u_1,u_2')} = \frac{f_2(u_1,u_2)}{f_1(u_1,u_2')} = \frac{\lambda_1(\lambda_2+\lambda_3)}{\lambda_2(\lambda_1+\lambda_3)} e^{\lambda_3(u_1-u_2)+\lambda_2(u_2'-u_2)},$$

increasing in u_1 if $\lambda_3 > 0$.

d) $u_1 = u_2 > u_2'$.

$$\frac{f(u_1,u_2)}{f(u_1,u_2')} = \frac{f_3(u_1,u_1)}{f_1(u_1,u_2')} = \frac{\lambda_3}{\lambda_2(\lambda_1+\lambda_3)} e^{\lambda_2(u_2'-u_1)}, \text{ decreasing in}$$

u_1 if $\lambda_2 > 0$.

e) $u_1 > u_2 > u_2'$.

$$\frac{f(u_1,u_2)}{f(u_1,u_2')} = \frac{f_1(u_1,u_2)}{f_1(u_1,u_2')} = e^{-\lambda_2(u_2'-u_2)}, \text{ constant in } u_1.$$

We note that as u_1 increases from 0, it passes through the five cases e) to e) in that order. From d), the density do not have g.m.l.r. if $\lambda_2 > 0$. If we consider the ratio $f(u_1,u_2)|f(u_1',u_2)$ $u_1 > u_1'$ then we will find that the density does not have g.m.l.r. if $\lambda_1 > 0$.

If $\lambda_2 = 0$, it is more convenient to consider the difference

$$f(u_1,u_2)f(u_1',u_2') - f(u_1,u_2')f(u_1',u_2)$$

for $u_1 > u_1'$, $u_2 > u_2'$, rather than the ratio as done previously. First we note that

$$f_1(u_1,u_2) = 0 \qquad \text{for } u_1 > u_2$$

$$f_2(u_1,u_2) = \lambda_1\lambda_3 e^{-\lambda_1 u_1 - \lambda_3 u_2} \qquad \text{for } u_1 < u_2$$

$$f_3(u,u) = \lambda_3 e^{-(\lambda_1+\lambda_3)u} \qquad \text{for } u_1 = u_2 = u$$

According to the relative positions of u_1, u_2, u_1', u_2' there are 13 cases to consider. In 7 of these cases the difference is zero, because at least one f in each term is zero. For 4 cases, $u_1 = u_2 > u_2' = u_1'$, $u_1 = u_2 > u_2' > u_1'$, $u_2 > u_1 > u_1' = u_2'$ and $u_2 > u_1 > u_2' > u_1'$ the difference is non-negative the second term being zero. Let us consider the two remaining cases.

$$u_2 > u_2' = u_1 > u_1' \ .$$

The difference $= f_2(u_1,u_2)f_2(u_1',u_1) - f_3(u_1,u_1)f_2(u_1',u_2)$

$= \lambda_1\lambda_3^2 e^{-\lambda_1(u_1'+u_1)-\lambda_3(u_1+u_2)}[\lambda_1 - 1]$. If $\lambda_1 < 1$ then this expression is always negative. If $\lambda_1 \geq 1$ this expression is always non-negative. In the one remaining case the difference is zero.

Similarly, we can take $\lambda_1 = 0$ and find that the density do not have g.m.l.r. if $\lambda_2 < 1$. If $\lambda_1 = \lambda_2 = 0$, then we do not have a bivariate distribution. Hence we conclude that a proper bivariate exponential do not have g.m.l.r. under any circumstances, if $(\lambda_1, \lambda_2, \lambda_3) > 0$.

We will consider the case of multivariate normal distribution in the next section.

3. Associated Random Variables

We have already defined associated random variables in the introduction. To recapitulate,

Definition: We say that random variables $\underline{X} = (X_1,\ldots,X_n)$ are associated if

$$\mathrm{cov}[f(\underline{X}),g(\underline{X})] \geq 0$$

for all non-decreasing functions f and g for which $E[f(\underline{X})]$, $E[g(\underline{X})]$ and $E[f(\underline{X})g(\underline{X})]$ exists. In our problem, their importance stems from inequalities (1.4) and (1.5). By (1.4), associativity of $X_1,\ldots,X_n$ is a solution to our problem A. By (1.5) we obtain an upper bound to the reliability of a parallel system. To recapitulate,

Definition: A set of components is said to be a parallel system if it fails only when all the components fail.

Hence, if X_i is the time to failure of the i^{th} component in a parallel system of n components, then the reliability at time t, $R(t)$, of the system is given by

$$R(t) = 1 - P(X_1 \leq t,\ldots,X_n \leq t) \quad . \tag{3.1}$$

If $X_1,\ldots,X_n$ are associated, then by (3.1) and (1.5)

$$R(t) = 1 - P(X_1 \leq t,\ldots,X_n \leq t) \leq 1 - \prod_{i=1}^{n} P(X_i \leq t) \quad .$$

This gives an upper bound to the reliability of a parallel system. Perhaps this upper bound is not as useful as a lower bound. However, note that this upper bound is sharp in the sense that it is actually attained if the components are independent.

The definition of associated random variables, or its equivalent statement (1.21) is rather difficult to verify. Hence, as stated in Section 1 we consider problem B. To recapitulate,

<u>Problem B</u>: To find meaningful sufficient conditions such that

(1.6) $\underline{X} = (X_1,\ldots,X_n)$ are associated .

In this section we will give some sufficient conditions for associativity. These conditions will be expressed in terms of g.m.l.r. whose physical interpretation has already been given in Section 2. In particular, we will show that if the joint density $f(u_1,\ldots,u_n)$ has g.m.l.r. in every pair u_i, u_j, $i \neq j$, $i,j = 1,\ldots,n$ then $(X_1,\ldots,X_n)$ are associated.

We will apply the above result to a multivariate normal density and deduce some sufficient conditions for associativity in this case.

3.1 Generalized Monotone Likelihood Ratio and Associativity

In this subsection we will give some sufficient conditions for associativity of $(X_1,\ldots,X_n)$. These conditions will be expressed in terms of g.m.l.r.

Our main theorem is as follows:

Theorem 3.1: If,

$$(3.2)\quad \begin{cases} f(u_1,\ldots,u_n) \text{ has g.m.l.r. in every pair } u_i,u_j,\ i \neq j\ , \\ \\ i,j = 1,\ldots,n\ , \end{cases}$$

then $(X_1,\ldots,X_n)$ are associated.

We note that the hypothesis of Theorem 3.1 is the same as the hypothesis of Theorem 2.4. In fact by inequality (1.4), 3.1 implies Theorem 2.4. We will prove Theorem 3.1 by taking a slightly different approach than that taken in proving Theorem 2.4. We will require a few results before we prove the above theorem.

Lemma 3.2: Let $X_1,\ldots,X_n$ be a set of random variables. Then X_j, given $X_i = u_i$, $i = 1,2,\ldots,n$; $i \neq j$, is associated.

Proof: This is essentially result (1.18) as given by Esary, et al., in [6]. It is sufficient to take $j = 1$. Let $\alpha(x_1,u_2,\ldots,u_n)$, $\beta(x_1,u_2,\ldots,u_n)$ be non-decreasing binary functions of x_1 when $u_2,\ldots,u_n$ are fixed. Then either $\alpha \leq \beta$ or $\beta \leq \alpha$. Let $\alpha \leq \beta$. Note $\alpha^2 = \alpha$ and $\beta^2 = \beta$. Also, $\alpha \leq \beta$ implies $\alpha^2 \leq \alpha\beta$ (since

$\alpha \geq 0$, $\beta \geq 0$). Let μ be a dominating measure with respect to which the density $f(x_1,\ldots,x_n)$ of $X_1,\ldots,X_n$ exists. Then, assuming that the relevant moments exist,

$$\mathrm{cov}_{X_1|X_i=u_i;i=2,\ldots,n}[\alpha(X_1,\ldots,X_n),\ \beta(X_1,\ldots,X_n)]$$

$$= \int_{x_1=-\infty}^{\infty} \alpha(x_1,u_2,\ldots,u_n)\ \beta(x_1,u_2,\ldots,u_n)\ f(x_1|u_2,\ldots,u_n)d\mu$$

$$- \int_{x_1=-\infty}^{\infty} \alpha(x_1,u_2,\ldots,u_n)\ f(x_1|u_2,\ldots,u_n)d\mu$$

$$\int_{x_1=-\infty}^{\infty} \beta(x_1,u_2,\ldots,u_n)\ f(x_1|u_2,\ldots,u_n)d\mu$$

$$\geq \int_{x_1=-\infty}^{\infty} \alpha^2(x_1,u_2,\ldots,u_n)\ f(x_1|u_2,\ldots,u_n)d\mu$$

$$- \int_{x_1=-\infty}^{\infty} \alpha(x_1,u_2,\ldots,u_n)\ f(x_1|u_2,\ldots,u_n)d\mu$$

$$\int_{x_1=-\infty}^{\infty} \beta(x_1,u_2,\ldots,u_n)\ f(x_1|u_2,\ldots,u_n)d\mu$$

$$= [\int_{x_1=-\infty}^{\infty} \alpha(x_1,u_2,\ldots,u_n)\ f(x_1|u_2,\ldots,u_n)d\mu]$$

$$[1 - \int_{x_1=-\infty}^{\infty} \beta(x_1,u_2,\ldots,u_n)\ f(x_1|u_2,\ldots,u_n)d\mu]$$

≥ 0 since α and β are binary functions; similarly, when $\beta \leq \alpha$.

By (1.21) the result follows.

From now on, unless otherwise stated, we will assume that all the moments that we need exist.

Let $\underline{X} = (X_1,\dots,X_n)$. Then for any two real-valued functions α and β of $\underline{X}$, $\mathrm{cov}[\alpha(\underline{X}),\beta(\underline{X})]$ can be easily expressed in terms of conditional moments as follows:

$$\mathrm{cov}_{\underline{X}}[\alpha(\underline{X}),\beta(\underline{X})] = E_{X_2,\dots,X_n}\mathrm{cov}_{X_1|X_2=u_2,\dots,X_n=u_n}[\alpha(\underline{X}),\beta(\underline{X})]$$

$$+ \sum_{r=2}^{n-1} E_{X_{r+1},\dots,X_n}\mathrm{cov}_{X_r|X_{r+1}=u_{r+1},\dots,X_n=u_n}$$

$$(3.3)\qquad [E_{X_1,\dots,X_{r-1}|X_r=u_r,\dots,X_n=u_n}\{\alpha(\underline{X})\},$$

$$E_{X_1,\dots,X_{r-1}|X_r=u_r,\dots,X_n=u_n}\{\beta(\underline{X})\}]$$

$$+ \mathrm{cov}_{X_n}[E_{X_1,\dots,X_{n-1}|X_n=u_n}\{\alpha(\underline{X})\}, E_{X_1,\dots,X_{n-1}|X_n=u_n}\{\beta(\underline{X})\}]$$

where $E_{X|Y=y}$ means expectation with respect to the conditional distribution of X, given $Y = y$, and similarly for the covariance.

Lemma 3.3: If,

(3.4)
- (a) $\alpha(\underline{X})$ is a non-decreasing binary function of $\underline{X}$ and
- (b) $f(u_j|u_1,\dots,u_{j-1},u_{j+1},\dots,u_k,\dots,u_n)$ has g.m.l.r. in u_j,u_k for other u's fixed, $j,k = 1,\dots,n$; $k \neq j$;

then, $h_{j\alpha}(u_k) \equiv E_{X_j|X_1=u_2,\ldots,X_{j-1}=u_{j-1},X_{j+1}=u_{j+1},\ldots,X_k=u_k,\ldots,X_n=u_n}[\alpha(\underline{X})]$

is a non-decreasing function of u_k, $j,k = 1,\ldots,n;\ k \neq j$.

Proof: Let $u_k > u_k'$. Since $\alpha(\underline{X})$ is a non-decreasing binary function of $\underline{X}$, there exists an x_j^* (depending on $u_1,\ldots,u_{j-1}$, $u_{j+1},\ldots,u_n$) such that

$$\text{(3.5)} \quad \alpha(X_j;u_1,\ldots,u_{j-1},u_{j+1},\ldots,u_k,\ldots,u_n) = 0 \quad \text{for} \quad X_j \leq x_j^* \ ,$$
$$= 1 \quad \text{for} \quad X_j > x_j^* \ .$$

Now,

$$\text{(3.6)} \quad E_{X_j|X_i=u_i,i\neq j,i=1,\ldots,n}[\alpha(\underline{X})] = P[\alpha(u_1,\ldots,X_j,\ldots,u_n) = 1 | X_i = u_i \ , \ i \neq j,\ i = 1,\ldots,n] \ .$$

Since α is a non-decreasing binary function,

$$\alpha(u_1,\ldots,X_j,\ldots,u_k,\ldots,u_n) \geq \alpha(u_1,\ldots,X_j,\ldots,u_k',\ldots,u_n) \ ,$$

since $u_k > u_k'$. Hence, $\alpha(u_1,\ldots,X_j,\ldots,u_k',\ldots,u_n) = 1$ implies

$$\alpha(u_1,\ldots,X_j,\ldots,u_k,\ldots,u_n) = 1 \ .$$

Therefore,

$$P[\alpha(u_1,\ldots,X_j,\ldots,u_k,\ldots,u_n) = 1] \geq P[\alpha(u_1,\ldots,X_j,\ldots,u_k',\ldots,u_n) = 1] \ .$$

Hence, from equation (3.6),

$$(3.7)\quad E_{X_j|X_i=u_i,i\neq j,i=1,\ldots,n}[\alpha(\underline{X})] \geq P[\alpha(u_1,\ldots,X_j,\ldots,u_k',\ldots,u_n) = 1|$$

$$X_i = u_i,\ i\neq j,\ i = 1,\ldots,n]$$

$$= P[X_j > x_j^*(u_1,\ldots,u_{j-1},u_{j+1},\ldots,u_k',\ldots,u_n)|X_i=u_i,\ i\neq j,\ i=1,\ldots,n]\ ,$$

by equations (3.5).

From (3.4b) and Lemma 2.6 we know that $P(X_j > x_j|X_i = u_i,\ i\neq j,\ i = 1,\ldots,n)$ is non-decreasing in u_k for the other variables fixed. Hence, from inequation (3.7),

$$E_{X_j|X_i = u_i,\ i\neq j,\ i = 1,\ldots,n}[\alpha(\underline{X})]$$

$$\geq P[X_j > x_j^*(u_1,\ldots,u_{j-1},u_{j+1},\ldots,u_k',\ldots,u_n)|X_i=u_i,X_k=u_k',i\neq j,i\neq k\ ,$$

$$i = 1,\ldots,n]$$

$$= P[\alpha(u_1,\ldots,X_j,\ldots,u_k',\ldots,u_n) = 1|X_i = u_i,X_k = u_k',i\neq j,\ i\neq k\ ,$$

$$i = 1,\ldots,n]\ ,$$

by equation (3.5)

$$= E_{X_j|X_i = u_i,X_k = u_k',\ i\neq j,\ i\neq k,\ i = 1,\ldots,n}[\alpha(\underline{X})]\ .$$

Hence, $h_{j\alpha}(u_k)$ is a non-decreasing function of u_k, and the lemma is proved.

Let us define

$$(3.8)\quad h^*_{r-1,\alpha}(u_k) \equiv E_{X_1,\ldots,X_{r-1}|X_i = u_i,\ i = r,\ldots,n}[\alpha(\underline{X})],\ k = r,\ldots,n.$$

The following lemma shows that under certain conditions $h^*_{r,\alpha}(u_k)$ is a non-decreasing function of u_k.

<u>Lemma 3.4</u>: If

(3.9)

(a) $h^*_{r-2,\alpha}(u_k)$ is a non-decreasing function of u_k for other u's fixed, $k = r-1,k$, and

(b) $f(u_{r-1}|u_r,\ldots,u_k,\ldots,u_n)$ has g.m.l.r. in u_{r-1},u_k, for the other u's fixed,

then $h^*_{r-1,\alpha}(u_k)$ is a non-decreasing function of u_k.

<u>Proof</u>: Let $u_k > u_k'$. Then

$$(3.10)\quad h^*_{r-1,\alpha}(u_k) - h^*_{r-1,\alpha}(u_k')$$

$$= \int_{u_{r-1}=-\infty}^{\infty} [h^*_{r-2,\alpha}(u_k)f(u_{r-1}|u_r,\ldots,u_k,\ldots,u_n)$$

$$- h^*_{r-2,\alpha}(u_k')f(u_{r-1}|u_r,\ldots,u_k',\ldots,u_n)]d\mu \quad ,$$

using equation (3.8) and the definition of conditional expectation;

$$\geq \int_{u_{r-1}=-\infty}^{\infty} h^*_{r-2,\alpha}(u_k')[f(u_{r-1}|u_r,\ldots,u_k,\ldots,u_n) -$$

$$- f(u_{r-1}|u_r,\ldots,u_k',\ldots,u_n)]d\mu \quad ,$$

using (3.9a) and the fact that $u_k > u_k'$;

$$= \int_{u_{r-1}=-\infty}^{\infty} E_{X_1,\ldots,X_{r-2}|X_i=u_i, X_k=u_k', i\neq k, i=r-1,\ldots,n}[\alpha(\underline{X})] \times$$

$$[f(u_{r-1}|u_r,\ldots,u_k,\ldots,u_n) - f(u_{r-1}|u_r,\ldots,u_k',\ldots,u_n)]d\mu \quad .$$

Using (3.9b) and part (i) of Theorem 2.5 on $h^*_{r-2,\alpha}(u_k')$ we get that the above expression is non-negative. This proves the lemma.

We next prove the following theorem.

Theorem 3.5: Let,

$$\text{(3.11)} \quad \begin{array}{l} f(u_r|u_{r+1},\ldots,u_k,\ldots,u_n) \text{ have g.m.l.r. in } u_r,u_k, \\ k = r+1,\ldots,n;\ r = 1,\ldots,n-1 \ \ (\text{ignore } k = r+2 \text{ when } r = n-1). \end{array}$$

Then $\underline{X} = (X_1,\ldots,X_n)$ are associated, and by (1.17) every subset of $(X_1,\ldots,X_n)$ is associated.

Proof: Let $\alpha(\underline{X})$, $\beta(\underline{X})$ be two non-decreasing binary functions of $\underline{X}$. By (1.21) we have to show that

$$\operatorname{cov}[\alpha(\underline{X}),\beta(\underline{X})] \geq 0 \quad .$$

We will use the expansion of the covariance given in equation (3.3). Given $X_2 = u_2,\ldots,X_n = u_n$, $\alpha(\cdot)$ and $\beta(\cdot)$ are non-decreasing binary functions of a single random variable X_1. By Lemma 3.2, $X_1|X_i = u_i$, $i = 2,\ldots,n$, is associated. Hence,

$$\operatorname{cov}_{X_1|X_i=u_i,i=2,\ldots,n}[\alpha(\underline{X}),\beta(\underline{X})] \geq 0 \quad \text{for all} \quad u_2,\ldots,u_n \quad .$$

Hence,

$$E_{X_2,\ldots,X_n}\operatorname{cov}_{X_1|X_i=u_i,i=2,\ldots,n}[\alpha(\underline{X}),\beta(\underline{X})] \geq 0 \quad .$$

Let us now consider the second term on the right-hand side of equation (3.3), viz.,

$$E_{X_3,\ldots,X_n}\operatorname{cov}_{X_2|X_3=u_3,\ldots,X_n=u_n}[E_{X_1|X_2=u_2,\ldots,X_n=u_n}\{\alpha(\underline{X})\} \,,$$

$$E_{X_1|X_2=u_2,\ldots,X_n=u_n}\{\beta(\underline{X})\}] \quad .$$

By hypothesis the conditions of Lemma 3.3 are satisfied for $j = 1$, $k = 2$. Hence, $E_{X_1|X_2=u_2,\ldots,X_n=u_n}\{\alpha(\underline{X})\}$ and $E_{X_1|X_2=u_2,\ldots,X_n=u_n}\{\beta(\underline{X})\}$ are non-decreasing functions of a single random variable, X_2, when $X_3 = u_3,\ldots,X_n = u_n$ are fixed. By Lemma 3.2, $X_2|X_3 = u_3,\ldots,X_n = u_n$ is associated. By the definition of associated random variables,

$$U \equiv \operatorname{cov}_{X_2|X_3=u_3,\ldots,X_n=u_n}[E_{X_1|X_2=u_2,\ldots,X_n=u_n}\{\alpha(\underline{X})\}\,,$$

$$E_{X_1|X_2=u_2,\ldots,X_n=u_n}\{\beta(\underline{X})\}] \geq 0 \;.$$

Hence, $E_{X_3,\ldots,X_n}[U] \geq 0$. This shows that the second term on the right-hand side of equation (3.3) is non-negative.

We will now show by induction on r that

$$E_{X_1,\ldots,X_{r-1}|X_r = u_r,\ldots,X_n = u_n}[\alpha(\underline{X})]$$

is non-decreasing in u_r for $u_{r+1},\ldots,u_n$ fixed. The above result starts the induction for $r = 2$. Lemma 3.4 proves the inductive step from r to $r+1$. Hence, the assertion. Similarly,

$$E_{X_1,\ldots,X_{r-1}|X_r = u_r,\ldots,X_n = u_n}[\alpha(\underline{X})]$$

is non-decreasing in u_k for other u's fixed. Replacing α by β, we get similar results in β. This starts the next inductive step.

Using Lemma 3.2, $X_r|X_{r+1} = u_{r+1},\ldots,X_n = u_n$ is associated. Using the result just proved and the definition of associated random variables,

$$\operatorname{cov}_{X_r|X_{r+1}=u_{r+1},\ldots,X_n=u_n}[E_{X_1,\ldots,X_{r-1}|X_r=u_r,\ldots,X_n=u_n}\{\alpha(\underline{X})\}\,,$$

$$E_{X_1,\ldots,X_{r-1}|X_r=u_r,\ldots,X_n=u_n}\{\beta(\underline{X})\}] \geq 0 \quad \text{for all fixed } u_{r+1},\ldots,u_n \;.$$

Hence, $E_{X_{r+1},\ldots,X_n} \text{cov}_{X_r | X_{r+1}=u_{r+1},\ldots,X_n=u_n}$

$$[E_{X_1,\ldots,X_{r-1}|X_r=u_r,\ldots,X_n=u_n}\{\alpha(\underline{X})\},\ E_{X_1,\ldots,X_{r-1}|X_r=u_r,\ldots,X_n=u_n}\{\beta(\underline{X})\}]$$

is non-decreasing for $r = 2,\ldots,n-1$.

The inductive proof given above and Lemma 3.4 imply that $E_{X_1,\ldots,X_{n-1}|X_n=u_n}\{[\alpha(\underline{X})]\}$ and $E_{X_1,\ldots,X_{n-1}|X_n=u_n}\{[\beta(\underline{X})]\}$ are non-decreasing in u_n. Hence, as before,

$$\text{cov}_{X_n}[E_{X_1,\ldots,X_{n-1}|X_n=u_n}\{\alpha(\underline{X})\}\ ,\ E_{X_1,\ldots,X_{n-1}|X_n=u_n}\{\beta(\underline{X})\}] \geq 0 \quad .$$

We have shown that each term on the right-hand side of equation (3.3) is non-negative. Hence, the left-hand side is also non-negative, i.e.,

$$\text{cov}[\alpha(\underline{X}),\beta(\underline{X})] \geq 0 \quad ,$$

and the theorem is proved.

We now come to the proof of Theorem 3.1. To recapitulate the statement of the theorem is as follows:

Theorem 3.1: If,

$$(3.2) \quad \begin{cases} f(u_1,\ldots,u_n) \text{ has g.m.l.r. in every pair } u_i,u_j,\ i \neq j\ , \\ i,j = 1,\ldots,n\ , \end{cases}$$

then $(X_1, \ldots, X_n)$ are associated.

Proof: Follows from Lemma 2.13, 2.11, 2.9 and Theorem 3.5.

We note at this point that all the results of Section 3 gives sufficient conditions and it is not known whether any of them are necessary conditions also.

3.2 Multivariate Normal Distribution

Let $X_1,\ldots,X_n$ have multivariate normal distribution with mean $\mu_1,\ldots,\mu_n$ and covariance matrix V. Assume V is positive definite (i.e., no singularity). Then $\det V \geq 0$ and V^{-1} exists. Let $V^{-1} = R$. Let $\det V \equiv |V|$, and

$$x-\mu = \begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \\ \cdots\cdots \\ x_n-\mu_n \end{pmatrix}, \quad R = (r_{ij}) .$$

We note that $r_{ij} = r_{ji}$ for all $i,j = 1,\ldots,n$. The density function of $X_1,\ldots,X_n$ is given by

$$f(x) = C(|V|)\exp[-\tfrac{1}{2}(x-\mu)^T R(x-\mu)] . \quad -\infty \leq x_i \leq \infty$$

$$-\infty < \mu_i < \infty , \; i = 1,\ldots,n .$$

where $C(|V|)$ is a function of $|V|$

Let ρ_{ij} = correlation between X_i and X_j. We note that $\rho_{ij} = \rho_{ji}$ for all $i,j = 1,\ldots,n$.

Lemma 3.6: The multivariate normal density f has g.m.l.r. in every pair x_i,x_j iff $r_{ij} \leq 0$ for all $i \neq j$, $i,j = 1,\ldots,n$.

Proof: Let us consider x_1,x_2. Let $x_1 > x_1'$, $x_2 > x_2'$ and $x_3,\ldots,x_n$ be fixed

$$f(x) = C(|V|)\exp[-\tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i-\mu_i)(x_j-\mu_j)r_{ij}] .$$

Now $\sum_{i=1}^{n} \sum_{j=1}^{n} (x_i-\mu_i)(x_j-\mu_j)r_{ij}$

$$= (x_1-\mu_1)^2 r_{11} + 2(x_1-\mu_1)(x_2-\mu_2)r_{12} + (x_2-\mu_2)^2 r_{22}$$

$$+ 2\sum_{i=3}^{n} (x_1-\mu_1)(x_i-\mu_i)r_{i1} + 2\sum_{i=3}^{n} (x_2-\mu_2)(x_i-\mu_i)r_{i2}$$

$$+ \sum_{i=3}^{n} \sum_{j=3}^{n} (x_i-\mu_i)(x_j-\mu_j)r_{ij}$$

$$f(x_1,x_2,x_3,\ldots,x_n)f(x_1',x_2',x_3,\ldots,x_n)-f(x_1,x_2',x_3,\ldots,x_n)f(x_1',x_2,x_3,\ldots,x_n)$$

$$= C^2(|V|)\exp[-\tfrac{1}{2}\{(x_1-\mu_1)^2 r_{11} + (x_2-\mu_2)^2 r_{22} + 2\sum_{i=3}^{n}(x_1-\mu_1)(x_i-\mu_i)r_{i1}$$

$$+ 2\sum_{i=3}^{n} (x_2-\mu_2)(x_i-\mu_i)r_{i2} + \sum_{i=3}^{n}\sum_{j=3}^{n} (x_i-\mu_i)(x_j-\mu_j)r_{ij}$$

$$+ (x_1'-\mu_1)^2 r_{11} + (x_2'-\mu_2)^2 r_{22} + 2\sum_{i=3}^{n} (x_1'-\mu_1)(x_i-\mu_i)r_{i1}$$

$$+ 2\sum_{i=3}^{n} (x_2'-\mu_2)(x_i-\mu_i)r_{i2} + \sum_{i=3}^{n}\sum_{j=3}^{n} (x_i'-\mu_i)(x_j'-\mu_j)r_{ij}\}]$$

$$[e^{-r_{12}\{(x_1-\mu_1)(x_2-\mu_2)+(x_1'-\mu_1)(x_2'-\mu_2)\}} - e^{-r_{12}\{(x_1-\mu_1)(x_2'-\mu_2)+(x_1'-\mu_1)(x_2-\mu_2)\}}]$$

By hypothesis,

$$0 < (x_1-x_1')(x_2-x_2') = (x_1-\mu_1)(x_2-\mu_2)+(x_1'-\mu_1)(x_2'-\mu_2)-(x_1-\mu_1)(x_2'-\mu_2)-(x_1'-\mu_1)(x_2-\mu_2)$$

Hence f has g.m.l.r. in x_1, x_2 iff $-r_{12} \geq 0$ i.e., iff $r_{12} \leq 0$. Similarly, considering every pair we see that f has g.m.l.r. in every pair x_i, x_j iff $r_{ij} \leq 0$, for all $i \neq j$, $i,j = 1,\ldots,n$.

Since R is positive definite, if f has g.m.l.r. in every pair x_i, x_j every column of R has exactly one positive element viz., the diagonal element.

Corollary 3.7: If the multivariate normal density f has g.m.l.r. in every pair x_i, x_j, $i \neq j$, $i,j = 1,\ldots,n$, then $\rho_{ij} \geq 0$ for all $i,j = 1,\ldots,n$.

Proof: If $i = j$, then $\rho_{ij} = 1 > 0$. Let $i \neq j$. Let f have g.m.l.r. in every pair x_i, x_j. Let us consider x_1, x_2. By repeated application of Lemma 2.12, the marginal bivariate density $f(x_1, x_2)$ has g.m.l.r. in x_1, x_2. We can easily see, as in the proof of Lemma 3.6, that $f(x_1, x_2)$ has g.m.l.r. in x_1, x_2 iff $\rho_{12} \geq 0$. Hence considering every pair x_i, x_j we get the corollary.

The concept of Leontieff matrix has been used in mathematical programming by various authors including Dantzig[4 , 5]. This matrix is defined as follows.

Definition: An $m \times n$ matrix A is called a Leontieff matrix if

(i) each column of A has exactly one positive element, and

(ii) there exists an $n \times 1$ non-negative non-zero vector X such that AX is a strictly positive vector.

In the present discussion the importance of Leontieff matrix stems from the following well-known theorem.

Theorem 3.8: The following are equivalent,

1. A is a square Leontieff matrix

2. A is non-singular, (i) in the definition holds and $A^{-1} \geq 0$, $A^{-1} \neq 0$.

Theorem 3.9: The multivariate normal density f has g.m.l.r. in every pair x_i, x_j $i \neq j$, $i,j = 1,\dots,n$ iff R is a Leontieff matrix.

Proof: Let f have g.m.l.r. in every pair x_i, x_j $i \neq j$, $i,j = 1,\dots,n$. Then by Corollary 3.7, $\rho_{ij} \geq 0$, for all $i,j = 1,\dots,n$. Hence $V = R^{-1} \geq 0$ and $R^{-1} \neq 0$, since diagonal elements of V are positive. Also by Lemma 3.6, $r_{ij} \leq 0$, for all $i \neq j$, $i,j = 1,\dots,n$. Since R is positive definite all diagonal elements of R are positive. Hence part 2 of Theorem 3.8 holds and R is a square Leontieff matrix.

Let R be a square Leontieff matrix. Then since $r_{ii} > 0$, for all $i = 1,\dots,n$, by Theorem 3.8, $r_{ij} \leq 0$, for all $i \neq j$, $i,j = 1,\dots,n$. By Lemma 3.6, f has g.m.l.r. in every pair x_i, x_j. Hence the theorem is proved.

We have shown that $\rho_{ij} \geq 0$, $i,j = 1,\dots,n$ is a necessary condition for f to have g.m.l.r. in every pair x_i, x_j, $i \neq j$, $i,j = 1,\dots,n$. Unfortunately it is not a sufficient condition as the following counterexample shows.

Example 3.1: Let $\mu = 0$, $V = \begin{pmatrix} 1 & .8 & .9 \\ .8 & 1 & .9 \\ .9 & .9 & 1 \end{pmatrix}$. Since V is symmetric and every principal minor is positive, V is positive definite. Hence V is a valid co-variance matrix. Note for this V, $\rho_{ij} > 0$ for all $i,j = 1,2,3$. But

$$R = V^{-1} = \frac{1}{|V|}\begin{pmatrix} .19 & .01 & -.18 \\ .01 & .19 & -.18 \\ -.18 & -.18 & .36 \end{pmatrix}, \quad |V| = .036$$

Hence by Lemma 3.6, f does not have g.m.l.r. in x_1, x_2 for x_3 fixed. But the marginal density $f(x_1, x_2)$ has g.m.l.r. in x_1, x_2.

Note, this example shows that reverse implication in Lemma 2.12 need not hold.

4. A Set of Sufficient Conditions for $1 - \prod_{i=1}^{n} P(X_i \leq x_i)$ to be a Lower Bound on $1 - P(X_1 \leq x_1,\ldots,X_n \leq x_n)$ and the Case of Binary Random Variables

We have already defined a parallel system and stated our problem in the introduction. To recapitulate,

Definition: A system consisting of more than one component, is said to be a parallel system if it fails only when all the components fail.

Let $X_1,\ldots,X_n$ be the times to failure of the n components comprising the system. The reliability of the system at time t is given by

$$R(t) = 1 - P(X_1 \leq t,\ldots,X_n \leq t) \quad .$$

We take

$$R^*(t) = 1 - \prod_{i=1}^{n} P(X_i \leq t)$$

as a possible lower bound on $R(t)$. We consider the following problem.

Problem C: To find meaningful sufficient conditions such that

$$(1.25) \qquad 1 - P(X_1 \leq x_1,\ldots,X_n \leq x_n) \geq 1 - \prod_{i=1}^{n} P(X_i \leq x_i)$$

for all $x_1,\ldots,x_n$

(1.25) is equivalent to

$$(1.26) \qquad P(X_1 \leq x_1, \ldots, X_n \leq x_n) \leq \prod_{i=1}^{n} P(X_i \leq x_i) \quad .$$

We note that under such sufficient conditions $R^*(t)$ is a sharp lower bound on $R(t)$ in the sense stated in the introduction.

We will prove the following analogues of Theorem 2.1, 2.2 and 2.3.

<u>Theorem 4.1</u>: If, $P(X_j \leq x_j,\ j = 1,\ldots,i-1 | X_i \leq x_i) \leq P(X_j \leq x_j,\ j = 1,\ldots,i-1)$, $i = 2,\ldots,n$ and for all $x_1,\ldots,x_n$, then inequality (1.26) holds.

<u>Theorem 4.2</u>: If, $P(X_j \leq x_j,\ j = 1,\ldots,i-1 | X_i \leq x_i) \leq P(X_j \leq x_j,\ j = 1,\ldots,i-1 | X_i \leq x_i')$, $i = 2,\ldots,n$, for all x_1, $x_i < x_i'$, $i = 2,\ldots,n$, then the hypothesis of Theorem 4.1 is satisfied and hence inequality (1.26) holds.

<u>Theorem 4.3</u>: Let $L_i(u_i) = P(X_j \leq x_j,\ j = 1,\ldots,i-1 | X_i = u_i)$, $i = 2,\ldots,n$. If $L_i(u_i) \geq L_i(u_i')$ for $u_i > u_i'$ and $x_1,\ldots,x_{i-1}$ fixed for $i = 2,\ldots,n$, then the hypothesis of Theorem 4.2 is satisfied and hence inequality (1.26) holds.

Unfortunately, no analogue of Theorem 2.8 could be found for the general case. Hence we have considered the special case of binary random variables.

<u>Definition</u>: A random variable X is said to be binary if X takes only values 0 and 1.

In this case, the result is based on the concept of decreasing generalized monotone likelihood ratio.

Definition: Let $u_i > u_i'$, $u_j > u_j'$, $i \neq j$. The function $f(u_1,\dots,u_n)$ is said to have decreasing generalized monotone likelihood ratio (abbreviated d.g.m.l.r.) iff

$$(4.1)\quad \begin{aligned} &f(u_1,\dots,u_i,\dots,u_j,\dots,u_n)f(u_1,\dots,u_i',\dots,u_j',\dots,u_n) \\ &-f(u_1,\dots,u_i',\dots,u_j,\dots,u_n)f(u_1,\dots,u_i,\dots,u_j',\dots,u_n) \le 0 \end{aligned}$$

for all u_i, u_i', u_j, u_j' as above and for the other $(n-2)$ u's fixed at any value $u_1,\dots,u_{i-1}$, $u_{i+1},\dots,u_{j-1}$, $u_{j+1},\dots,u_n$.

This definition implies that

$$(4.2)\quad \frac{f(u_1,\dots,u_i,\dots,u_j,\dots,u_n)}{f(u_1,\dots,u_i,\dots,u_j',\dots,u_n)} \le \frac{f(u_1,\dots,u_i',\dots,u_j,\dots,u_n)}{f(u_1,\dots,u_i',\dots,u_j',\dots,u_n)} ,$$

provided the ratios are well-defined. This means that the left hand side of the inequality is a non-increasing function of u_i for other u's fixed. It will be noted that d.g.m.l.r. is the same as g.m.l.r. with the final inequality sign reversed.

In terms of d.g.m.l.r. we have the following analogue to Theorem 2.8 in the case of binary random variables.

Theorem 4.4: Let $(i_1,\dots,i_k) \subset (1,2,\dots,j)$ such that $k = 1,\dots,j-1$; $1 \le i_1 < i_2 < \cdots < i_k < j$, $j = 2,\dots,n$. If,

$f(u_{i_1} | u_{i_2} = 0, \ldots, u_{i_k} = 0, u_j)$ has d.g.m.l.r. in u_{i_1}, u_{i_j} then the hypothesis of Theorem 4.3 is satisfied and hence inequality (1.26) holds.

It might be thought that analogous to Lemmas 2.12 and 2.13, the d.g.m.l.r. property of the parent density is transmitted to marginal densities. Unfortunately this is not the case and we will give a counterexample to show this.

In the case of general random variables the following theorem holds. This is probably not as useful as Lemmas 2.12 and 2.13 or Theorem 2.8.

Let there exist a dominating measure μ with respect to which the joint density $f(u_1, \ldots, u_n)$ of $X_1, \ldots, X_n$ exists. Then

(4.3)
$$P(X_1 \le x_1, \ldots, X_j \le x_j,\ u_{j+1} | X_i = u_i) = \int_{u_1=-\infty}^{x_1} \cdots \int_{u_j=-\infty}^{x_j} f(u_1, u_2, \ldots, u_j,\ u_{j+1} | u_i) d\mu$$

<u>Theorem 4.5</u>: If, for $i = 2, \ldots, n$

a) $f(u_1 | u_i)$ has d.g.m.l.r. in u_1, u_i, and

b) $P(X_1 \le x_1, \ldots, X_j \le x_j,\ u_{j+1} | X_i = u_i)$ has d.g.m.l.r. in u_{j+1}, u_i for $j = 1, 2, \ldots, i-2$,

then $P(X_1 \le x_1, \ldots, X_{i-1} \le x_{i-1} | X_i = u_i)$ is non-decreasing in u_i for $i = 2, \ldots, n$ and by Theorem 4.3, inequality (1.26) holds.

Lastly, we have given physical interpretation of some of the conditions under which the theorems hold.

4.1 Proof of the Theorems

Theorem 4.1 is analogous to Theorem 2.1.

Theorem 4.1: If,

$$(4.4)\quad \begin{cases} P(X_j \le x_j,\ j = 1,\dots,i-1 \mid X_i \le x_i) \le P(X_j \le x_j,\ j = 1,\dots,i-1) \\ \\ i = 2,\dots,n \text{ and for all } x_1,\dots,x_n , \end{cases}$$

then inequality (1.26) holds.

Proof: The proof is analogous to the proof of Theorem 2.1. Let $i = 2$. Then from inequalities (4.4)

$$P(X_1 \le x_1 \mid X_2 \le x_2) \le P(X_1 \le x_1) \ .$$

Hence, $P(X_1 \le x_1,\ X_2 \le x_2) \le P(X_1 \le x_1)P(X_2 \le x_2)$.

Suppose,

$$(4.5)\qquad P(X_j \le x_j,\ j = 1,\dots,i-1) \le \prod_{j=1}^{i-1} P(X_j \le x_j) \ .$$

Then from inequalities (4.4) and (4.5),

$$P(X_j \le x_j,\ j = 1,\dots,i) \le P(X_j \le x_j,\ j = 1,\dots,i-1)P(X_i \le x_i)$$

$$\le \prod_{j=1}^{i} P(X_j \le x_j) \ .$$

The theorem is now proved by induction.

Analogous to Theorem 2.2 we will now show that if the conditional probability $P(X_j \leq x_j,\ j = 1,\dots,i-1 | X_i \leq x_i)$ is non-decreasing in x_i then the inequalities (4.4) hold. In particular, we prove the following theorem.

Theorem 4.2: If,

$$(4.6)\ \begin{cases} P(X_j \leq x_j,\ j = 1,\dots,i-1 | X_i \leq x_i) \leq P(X_j \leq x_j,\ j = 1,\dots,i-1 | X_i \leq x_i') , \\ \\ i = 2,\dots,n, \quad \text{for all} \quad x_1,\ x_i < x_i',\ i = 2,\dots,n \ , \end{cases}$$

then inequalities (4.4) hold and by Theorem 4.1, the inequality (1.26) holds.

Proof: The proof is analogous to that of Theorem 2.2.

By hypothesis for $i = 2,3,\dots,n$,

$$P(X_j \leq x_j,\ j = 1,\dots,i-1 | X_i \leq x_i) \leq \frac{P(X_j \leq x_j,\ j = 1,\dots,i-1,\ X_i \leq x_i')}{P(X_i \leq x_i')} .$$

Let $x_i' \to \infty$. Then,

$$P(X_j \leq x_j,\ j = 1,\dots,i-1 | X_i \leq x_i) \leq P(X_j \leq x_j,\ j = 1,\dots,i-1) \ .$$

This proves the theorem.

Instead of conditioning on the event $(X_i \leq x_i)$, we can condition on the event $(X_i = x_i)$ as in Theorem 2.3.

Let $L_i(u_i) = P(X_j \leq x_j,\ j = 1,\ldots,i-1 | X_i = u_i)$, $i = 2,\ldots,n$.

Theorem 4.3: If,

$$(4.7) \begin{cases} L_i(u_i) \geq L_i(u_i') & \text{for } u_i > u_i' \text{ and } x_1,\ldots,x_{i-1} \text{ fixed} \\ \text{for } i = 2,\ldots,n , & \end{cases}$$

then inequalities (4.6) hold and by Theorem 4.2, inequality (1.26) holds.

Proof: The proof is by induction and is analogous to the proof of Theorem 2.3.

Let $F_i(u) = P(X_i \leq u)$ be the cumulative distribution function of X_i. Then

$$P(X_j \leq x_j,\ j = 1,\ldots,i) = \int_{-\infty}^{x_i} L_i(u)dF_i(u) , \quad i = 2,\ldots,n .$$

Let $x_i \leq x_i'$. Then,

$$P(X_j \leq x_j,\ j = 1,\ldots,i-1 | X_i \leq x_i) - P(X_j \leq x_j,\ j = 1,\ldots,i-1 | X_i \leq x_i')$$

$$= \frac{\int_{u_i=-\infty}^{x_i} L_i(u)dF_i(u)}{P(X_i \leq x_i)} - \frac{\int_{u=-\infty}^{x_i'} L_i(u)dF_i(u)}{P(X_i \leq x_i')}$$

$$= \frac{1}{P(X_i \leq x_i)P(X_i \leq x_i')} \int_{v=x_i}^{x_i'} \int_{u=-\infty}^{x_i} [L_i(u)-L_i(v)]dF_i(u)dF_i(v) .$$

$$\text{But,} \qquad v \in (x_i, x_i']$$

$$u \in (-\infty, x_i] \quad .$$

Hence, by hypothesis $L_i(u) \leq L_i(v)$, which proves the theorem.

As in Section 2 we would like to find sufficient conditions in terms of conditional densities such that inequalities (4.7) hold. Unfortunately, the only set of conditions that we could find appeared to be vacuous, since no example could be produced to satisfy these conditions. For $n = 3$, these conditions are expressed in terms of g.m.l.r. and d.g.m.l.r. as follows.

Let μ be a dominating measure with respect to which the joint density $f(u_1,\ldots,u_n)$ of $X_1,\ldots,X_n$ exists. If,

$$(4.8) \quad \begin{cases} \text{a) } f(u_1|u_2,u_3) \text{ has g.m.l.r. in } (u_1,u_2) & , \\ \text{b) } f(u_1|u_2,u_3) \text{ has d.g.m.l.r. in } (u_1,u_3) & , \\ \text{c) } f(u_1|u_2) \text{ has d.g.m.l.r. in } (u_1,u_2) & , \\ \text{d) } f(u_2|u_3) \text{ has d.g.m.l.r. in } (u_2,u_3) & , \end{cases}$$

then inequalities (4.7) hold for $n = 3$.

This theorem can be generalized to any arbitrary n. Unfortunately, no example could be devised which satisfies (4.8) (a) and (b) simultaneously along with (c) and (d). So it is quite possible that the set of conditions given by (4.8) is vacuous.

Even if (4.8) were not vacuous, as in Theorem 2.4 we would like to find conditions on $f(u_1,u_2,u_3)$ such that (4.8) holds. As we have said before the d.g.m.l.r. property of $f(u_1,u_2,u_3)$ may not be transmitted to its marginal densities as the following counterexample shows:

Example 4.1: Consider the following probabilities.

X_1	X_2	X_3	Probability
0	0	0	.05
1	0	0	.15
0	1	0	.10
0	0	1	.10
0	1	1	.10
1	0	1	.10
1	1	0	.30
1	1	1	.10

X_1	X_2	Probability
0	0	.15
1	0	.25
0	1	.20
1	1	.40

X_1	X_3	Probability
0	0	.15
1	0	.45
0	1	.20
1	1	.20

X_2	X_3	Probability
0	0	.20
1	0	.40
0	1	.20
1	1	.20

By actual calculations we find that $P(X_1 = x_1, X_2 = x_2, X_3 = x_3)$ has d.g.m.l.r. in every pair. However

$$P(X_1=0,X_2=0)P(X_1=1,X_2=1) - P(X_1=0,X_2=1)P(X_1=1,X_2=0)$$

$$= .15 \times .40 - .20 \times .25 = .06 - .05 > 0 \quad .$$

Hence $f(u_1,u_2)$ does not have d.g.m.l.r. in (u_1,u_2). We also get,

$$P(X_1 \le 0, X_2 \le 0, X_3 \le 1) - P(X_1 \le 0)P(X_2 \le 0)P(X_3 \le 1)$$

$$= .15 - .35 \times .40 > 0 \quad .$$

Hence inequality (1.26) does not hold.

In view of these difficulties we will now consider the special case of binary random variables. For this case we have the following analogue of Theorem 2.8.

Theorem 4.4: Let $(i_1, i_2, \ldots, i_k) \subset (1, 2, \ldots, j)$ such that $k = 1, \ldots, j-1$; $1 \le i_1 < i_2 < \cdots < i_k < j$, $j = 2, \ldots, n$. If $f(u_{i_1} | u_{i_2} = 0, \ldots, u_{i_k} = 0, u_j)$ has d.g.m.l.r. in u_{i_1}, u_j then $P(X_1 \le x_1, \ldots, X_{j-1} \le x_{j-1} | X_j = u_j)$ is non-decreasing in u_j, $j = 2, \ldots, n$. Hence by Theorem 4.3, inequality (1.26) holds.

Proof: First let us note that the d.g.m.l.r. property implies,

$$P(X_{i_1}=0|X_{i_2}=0,\ldots,X_{i_k}=0,X_j=0)P(X_{i_1}=1|X_{i_2}=0,\ldots,X_{i_k}=0,X_j=1)$$

$$\le P(X_{i_1}=1|X_{i_2}=0,\ldots,X_{i_k}=0,X_j=0)P(X_{i_1}=0|X_{i_2}=0,\ldots,X_{i_k}=0,X_j=1)$$

But since,

$$P(X_{i_1}=0|X_{i_2},\ldots,X_{i_k},X_j=0) + P(X_{i_1}=1|X_{i_2},\ldots,X_{i_k},X_j=0) = 1$$

and

$$P(X_{i_1}=0|X_{i_2},\ldots,X_{i_k},X_j=1) + P(X_{i_1}=1|X_{i_2},\ldots,X_{i_k},X_j=1) = 1 ,$$

we have,

$$(4.9) \quad P(X_{i_1}=0|X_{i_2},\ldots,X_{i_k},X_j=1) \geq P(X_{i_1}=0|X_{i_2},\ldots,X_{i_k},X_j=0) \quad .$$

Since $x_i = 0$ or 1 for all i, we need only consider the subsets of $(x_1,\ldots,x_{j-1})$ in $P(X_k \leq x_k,\ k = 1,\ldots,j-1|X_j = u_j)$ which are 0. Suppose $x_k = 0$, $1 \leq k \leq j-1$ and $x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_{j-1}$ are 1. We have to show that

$$(4.10) \qquad P(X_k=0|X_j=1) - P(X_k=0|X_j=0) \geq 0 \quad .$$

Let $i_1 = k$, $(u_{i_2},\ldots,u_{i_k}) = \emptyset$. Then by hypothesis $f(u_k|u_j)$ has d.g.m.l.r. in u_k,u_j. Hence by inequality (4.9), inequality (4.10) holds.

Now suppose out of $(x_1,\ldots,x_{j-1})$, $x_{i_1},\ldots,x_{i_k}$ are 0 and the rest are 1. Then we have to show that

$$(4.11) \quad P(X_{i_1}=0,X_{i_2}=0,\ldots,X_{i_k}=0|X_j=1) - P(X_{i_1}=0,X_{i_2}=0,\ldots,X_{i_k}=0|X_j=0) \geq 0 \ .$$

The left hand side of (4.11) can be written as

$$= \prod_{m=1}^{k} P(X_{i_m}=0|X_{i_{m+1}}=0,\ldots,X_{i_k}=0,X_j=1) - \prod_{m=1}^{k} P(X_{i_m}=0|X_{i_{m+1}}=0,\ldots,X_{i_k}=0,X_j=0) \ .$$

Since $f(u_{i_m}|u_{i_{m+1}}=0,\ldots,u_{i_k}=0,u_j)$ has d.g.m.l.r. in u_{i_m},u_j by inequality (4.9),

$$P(X_{i_m}=0 \mid X_{i_{m+1}}=0,\ldots,X_{i_k}=0,X_j=1) \geq P(X_{i_m}=0 \mid X_{i_{m+1}}=0,\ldots,X_{i_k}=0,X_j=0) ,$$

$$\text{for} \quad m = 1,\ldots,k \quad .$$

Hence inequality (4.11) holds and the theorem is proved.

The following example shows that the conditions under which the theorem holds are not vacuous.

Example 4.2: Consider the following probabilities.

X_1	X_2	X_3	Probability
0	0	0	.04
1	0	0	.20
0	1	0	.20
0	0	1	.20
0	1	1	.20
1	0	1	.06
1	1	0	.06
1	1	1	.04

X_1	X_2	Probability
0	0	.24
0	1	.40
1	0	.26
1	1	.10

X_1	X_3	Probability
0	0	.24
0	1	.40
1	0	.26
1	1	.10

X_2	X_3	Probability
0	0	.24
0	1	.26
1	0	.26
1	1	.24

By actual calculations we easily see that $f(u_i|u_j)$ has d.g.m.l.r. in (u_i,u_j), $i < j$; $i,j = 1,2,3$. Also, $f(u_1|u_2=0,u_3)$ has d.g.m.l.r. in u_1,u_3. But,

$$P(X_1=0|X_2=1) - P(X_1=0|X_2=0) = \frac{.40}{.50} - \frac{.24}{50} > 0 \quad ,$$

$$P(X_1=0|X_3=1) - P(X_1=0|X_3=0) = \frac{.40}{.50} - \frac{.24}{.50} > 0 \quad ,$$

$$P(X_2=0|X_3=1) - P(X_2=0|X_3=0) = \frac{.26}{.50} - \frac{.24}{.50} > 0 \quad ,$$

$$P(X_1=0,X_2=0|X_3=1) - P(X_1=0,X_2=0|X_3=0) = \frac{.20}{.50} - \frac{.04}{.50} > 0 \quad .$$

Again,

x_1	x_2	x_3	$P(X_1 \le x_1, X_2 \le x_2, X_3 \le x_3)$	$\prod_{i=1}^{3} P(X_i \le x_i)$
0	0	0	.04	.16
1	0	0	.24	.25
0	1	0	.24	.32
0	0	1	.24	.32
0	1	1	.64	.64
1	0	1	.50	.50
1	1	0	.50	.50
1	1	1	1	1

We have already explained our difficulties in finding any result analogous to Theorem 2.4 even in the case of binary random variables. We will not pursue this quest any further. However, in the general case the following theorem is a slight improvement on Theorem 4.3.

Theorem 4.5:

(4.12) If, for $i = 2,\ldots,n$

a) $f(u_1|u_i)$ has d.g.m.l.r. in (u_1,u_i), and

b) $P(X_1 \le x_1,\ldots,X_j \le x_j,\ u_{j+1}|X_i=u_i)$ has d.g.m.l.r. in u_{j+1},u_i for $j = 1,2,\ldots,i-2$,

then $P(X_1 \le x_1,\ldots,X_{i-1} \le x_{i-1}|X_i=u_i)$ is non-decreasing in u_i

for $i = 2,\ldots,n$ and by Theorem 4.3, inequality (1.26) holds.

The proof of this theorem will require the following theorem which is analogous to Theorem 2.5.

Theorem 4.6: Let $p(x|y)$ be a family of densities indexed by y on the real line with d.g.m.l.r. in x,y. Then,

(i) If $\psi(\cdot)$ is a non-decreasing (non-increasing) function of x, then $E\psi(X|y)$, (expectation with respect to X only) is a non-increasing (non-decreasing) function of y.

(ii) For any $y > y'$

$$P(X \leq x|y) \geq P(X \leq x|y') \ , \quad \text{for all} \quad x \quad .$$

Proof: The proof is analogous to that of Theorem 2.5. We will prove part (i) "without parenthesis". The proof for with parenthesis will be analogous.

Let $y > y'$. Then

$$E\psi(X|y) - E\psi(X|y') = \int_{-\infty}^{\infty} \psi(x)[p(x|y) - p(x|y')]d\mu$$

$$= \int_{-\infty}^{\infty} \psi(x)p(x|y')[\frac{p(x|y)}{p(x|y')} - 1]d\mu$$

Since $p(x|y)$ has d.g.m.l.r. in (x,y) and

$$(4.13) \qquad \int_{-\infty}^{\infty} [p(x|y) - p(x|y')]d\mu = 0 \quad ,$$

there exists an x^* such that

$$(4.14) \qquad \frac{p(x|y)}{p(x|y')} \begin{cases} \geq 1, & \text{for } x < x^* \\ < 1, & \text{for } x \geq x^* . \end{cases}$$

$$E\psi(X|y) - E\psi(X|y') = \int_{-\infty}^{x^*} \psi(x)p(x|y')[\frac{p(x|y)}{p(x|y')} - 1]d\mu$$

$$+ \int_{x^*}^{\infty} \psi(x)p(x|y')[\frac{p(x|y)}{p(x|y')} - 1]d\mu$$

$$\leq \sup_{x < x^*} \psi(x) \int_{-\infty}^{x^*} p(x|y')[\frac{p(x|y)}{p(x|y')} - 1]d\mu$$

$$+ \inf_{x \geq x^*} \psi(x) \int_{x^*}^{\infty} p(x|y')[\frac{p(x|y)}{p(x|y')} - 1]d\mu$$

$$= [\sup_{x < x^*} \psi(x) - \inf_{x \geq x^*} \psi(x)] \int_{-\infty}^{x^*} p(x|y')[\frac{p(x|y)}{p(x|y')} - 1]d\mu \text{ , using (4.13)}$$

Since $\psi(x)$ is non-decreasing in x,

$$\sup_{x < x^*} \psi(x) - \inf_{x \geq x^*} \psi(x) \leq 0$$

and by definition of x^* in (4.14)

$$p(x|y')[\frac{p(x|y)}{p(x|y')} - 1] \geq 0 \quad \text{for} \quad x < x^*$$

Hence $E\psi(X|y) - E\psi(X|y') \leq 0$, proving part (i).

$$\text{Let } \psi(X) = 1 \quad \text{for} \quad X \leq x$$

$$= 0 \quad \text{for} \quad X > x \ .$$

The $\psi(X)$ is a non-increasing function of X. Hence by part (i) "with parenthesis",

$$E\psi(X|y) - E\psi(X|y') \geq 0$$

$$\text{i.e., } P(X \leq x|y) \geq P(X \leq x|y')$$

proving part (ii).

Proof of Theorem 4.5: By part (ii) of Theorem 4.6,

$$f(u_1|u_i) \text{ d.g.m.l.r. in } u_1,u_i \text{ implies } P(X_1 \leq x_1|X_i=u_i)$$

is non-decreasing in u_i.

Since $P(X_1 \leq x_1,u_2|u_i)$ has d.g.m.l.r. in (u_2,u_i), and $P(X_1 \leq x_1|X_i = u_i)$ is non-decreasing in u_i, in a manner similar to the proof of Theorem 4.6, part (i), we get that

$$P(X_1 \leq x_1,X_2 \leq x_2|X_i = u_i) - P(X_1 \leq x_1,X_2 \leq x_2|X_i = u_i')$$

$$= \int_{-\infty}^{x_2} [P(X_1 \leq x_1,u_2|u_i) - P(X_1 \leq x_1,u_2|u_i)]d\mu \geq 0 \ .$$

Proceeding in this manner we prove that

(a) $P(X_k \leq x_k,\ k = 1,\ldots,j \mid X_i = u_i)$ non-decreasing in u_i ,

(b) $P(X_k \leq x_k,\ k = 1,\ldots,j, u_{j+1} \mid X_i = u_i)$ d.g.m.l.r. in u_{j+1}, u_i

implies that $P(X_k \leq x_k,\ k = 1,\ldots,j+1 \mid X_i = u_i)$ is non-decreasing in u_i. Hence by induction the theorem follows.

4.2 Interpretations

We will now interpret some of the conditions under which our theorems hold. Let us consider the hypothesis of Theorem 4.1. For n = 3 the conditions are,

$$P(X_1 \le x_1, X_2 \le x_2 | X_3 \le x_3) \le P(X_1 \le x_1, X_2 \le x_2) ,$$

$$P(X_1 \le x_1 | X_2 \le x_2) \le P(X_1 \le x_1) .$$

Suppose we have components 1 and 2 in parallel. We note the joint probability that component i does not survive beyond a given time x_i, $i = 1,2$. If we add a third component to the system and condition on the failure of the third component before x_3, then the conditional probability that the other two components do not survive beyond times x_1, x_2 tends to decrease below the probability of the same event when no component 3 were present. Hence addition of component 3 tends to reduce the chance of failure. Similar interpretations can be given in the case of a system with n components. In general if the components are such that their addition tends to reduce the chance of failure of the system then the hypothesis of Theorem 4.1 holds.

For $n = 3$, the conditions of Theorem 4.2 reduce to

$$P(X_1 \le x_1, X_2 \le x_2 | X_3 \le x_3) \text{ non-decreasing in } x_3 \text{ for fixed } (x_1, x_2),$$

$$P(X_1 \le x_1 | X_2 \le x_2) \text{ non-decreasing in } x_2 \text{ for fixed } x_1 .$$

Consider a system with components 1 and 2 in parallel. If x_2 increases, then the second component tends to survive longer and the conditional probability of failure of the first component before x_1 tends to increase. But since we have a parallel system the survival of one component ensures the survival of the system. Hence if the second component survives longer then the system survives longer even if the first component fails quickly. Similar result holds for a system with three components. In general, the hypothesis of Theorem 4.2 seems to say that if we progressively add components to the system it does not matter if the previous components fail quickly, the system will survive if the new component survives for a long time.

The hypothesis of Theorem 4.3 says the same thing except that we now condition on the actual time to failure of the new component rather than on the event of non-survival beyond a given time.

It seems that, as in the case of series system, in parallel system each component must act beneficially for the other components. Either it decreases the chance of failure of the other components or it must survive for a long time if other components fail quickly.

Since d.g.m.l.r. is the same as g.m.l.r. with the direction of the final inequality reversed, the interpretation of d.g.m.l.r. property is analogous to g.m.l.r. property with the inequalities reversed. Hence, d.g.m.l.r. implies that large values of one variable tend to be associated with small values of the other variable and vice versa. Since the variables we are considering are the times to failure, we get, that the quick failure of one component is associated with long survival of another component. In general, this should be fine because the system survives as long as one component survives. Unfortunately, this does not always work as Example 4.1 and our inability to get any theorem analogous to Theorem 2.4 shows.

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13. ABSTRACT

Consider a series system with n components. Let X_i be the time to failure of the i^{th} component. Let $R(t)$ be the reliability of the system at time t and $R^*(t)$ be the reliability when the components are independent. It is desired to find physically meaningful conditions such that $R^*(t)$ is a lower bound of $R(t)$ for all t. In particular, the following problem has been considered.

Problem A: To find meaningful sufficient conditions such that

$$P(X_1 > x_1, \dots, X_n > x_n) \geq \prod_{i=1}^{n} P(X_i > x_i) \quad \text{for all } x_1, \dots, x_n.$$

It is well-known that if $X_1, \dots, X_n$ are associated then the inequality under Problem A holds. This leads to the following problem.

Problem B: To find meaningful sufficient conditions such that $X_1, \dots, X_n$ are associated.

Let $f(u_1, \dots, u_n)$ be the joint density of $X_1, \dots, X_n$. The main result in the report says that, "if $f(u_1, \dots, u_n)$ has generalized monotone likelihood ratio in every pair u_i, u_j, $i \neq j$, then (a) the inequality under Problem A holds, and (b) $X_1, \dots, X_n$ are associated".

For a parallel system with n components the following problem is considered.

Problem C: To find meaningful sufficient conditions such that

$$1 - P(X_1 \leq x_1, \dots, X_n \leq x_n) \geq 1 - \prod_{i=1}^{n} P(X_i \leq x_i) \quad \text{for all } x_1, \dots, x_n.$$

Unfortunately these conditions are not as elegant as those in the case of a series system.

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