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The Haloing Effect of the Third Integral

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THE HALOING EFFECT OF THE THIRD INTEGRAL

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SUMMARY

Whether Contopoulos' galactic system is separable (unlikely) or not (likely), the fact is that there exists a vicinity of the equilibrium in which numerical integration of high accuracy cannot separate the system from its image through a Birkhoff's normalization of high order. To all practical purposes, Stellar Dynamics is then justified in pretending that the model is, in that region, structured by a so-called third integral.

1. A PRAGMATIC APPROACH TO THE THIRD INTEGRAL

The usefulness of the so-called third integral in stellar dynamics has been recently debated (Goudas 1969a, Barbanis and Contopoulos 1969, Goudas 1969b). We like to think that we have several facts to contribute to the debate. But before we do so, we wish to express very clearly that our intention is not to enter into an argument with the authors we just mentioned, but simply to report some facts and to indicate how, in our opinion, they seem to support a pragmatic approach to the third integral.

Essentially we have to deal with conservative Hamiltonian systems with two degrees of freedom. Repeatedly, in the past, mathematicians have warned us that the structure of their phase space is generally far from trivial. Nevertheless, the physicist inclines to look there for the kind of phase portraits he is most acquainted with, namely those of *separable* dynamical systems. We understand the concept of separability in a general way: in principle two distinct integrals in involution make it possible to generate a transformation to a set of phase variables in which the angular coordinates become ignorable. Evidently the meaningful problem is not to categorically decide whether or not a given dynamical system possesses two independent integrals in involution in a *generic* way, i.e. not only for the particular system under consideration, but also for all--or only almost all--systems close by, provided of course an adequate topology has been defined on a set of specified Hamiltonian functions. Questions of that kind have received clear cut answers in a few particular instances under well-defined topological and function-theoretic conditions; some are negative (Siegel 1954, Moser 1968), others are in the positive (Rüssmann 1967). Therefrom a physicist is justified in concluding that this rigorous approach is not the proper one. For non-integrable dynamical systems are with us to be investigated, and if straightforward statements of existence and convergence are felt to prove wrong or inadequate, they should not be used to bar further numerical investigations, quite the contrary; they should urge to assemble more evidences and help direct the mathematician toward constructive approaches.

We like to look at the problem from this angle. Three algorithms essentially have been proposed to analyze the phase space around an equilibrium: cross sections by numerical integration (Hénon and Heiles 1964), construction of the adelphic integral (Whittaker 1917) and normalization by recurrent canonical transformations (Birkhoff 1927). Each of these algorithms may be compared to a lens focused on the equilibrium itself. It magnifies some details of the phase portrait, it obscures others. Islands of stability come out enhanced, although their finer structures are blurred; zones of instability usually fall off the field of view unless a special effort is made to untangle some of their complexities (Danby 1968). In sum the lens has built in itself a definite power of resolution. Where a high commensurability ratio causes a chain of thin, elongated islands, the algorithm deforms the picture into a continuing flow of quasi-periodic orbits, distorting the isolated natural families of periodic orbits into tori of ordinary families of periodic orbits, and smoothing out the manifolds of asymptotic orbits. In domains where two resonances overlap, however wild the portrait becomes, elimination even at a low order of all periodic terms but the resonant ones helps in discerning mechanisms likely responsible for the instability of the system and its ergodicity (Walker and Ford 1969, Ford and Lunsford 1969).

Distorted and fuzzy as they often come out of a numerical experimentation, these phase portraits are far from being the ultimate truth about the phase space. But authors have given so much of their time, effort and ingenuity to develop them that they can be forgiven for not strongly cautioning the reader against identifying them with the reality. Thus instead of concluding that, within a certain accuracy and provided that the interval of integration does not exceed a certain upper bound, the elliptical restricted problem behaves roughly *as if* it were separable, one announces that "two integrals of motion have been found" (Contopoulos 1967). Which statement, of course, will prompt a supercillious colleague to produce evidence to the contrary. At this point there is great risk that the debate will sink into a sterile exchange. A comparison may help to convey our point. The sine function does not belong to the vector space of polynomials in one letter; yet, according to the Weierstrass' theorem, it can be uniformly approximated in that space over a finite interval: The former of these two statements is a triviality; but the latter one is eminently useful. In the same way, we all accept that a dynamical system generally is not a separable one. Yet, in the recent years, we have learned to give to the formal approximations of a non-separable system by separable ones more credit than elementary mathematics is willing to grant (see for instance Moser 1968, pp. 14 and 15).

2. THE NUMERICAL LENS THAT IS A BIRKHOFF'S NORMALIZATION

To us the search for a "third integral" in a conservative Hamiltonian system having an equilibrium proceeds essentially in two steps: first we use the given Hamiltonian system to generate separable systems related to it, then we check to what extent in the phase space and within what threshold of accuracy we can trust the physical information brought forth by the separable model.

Birkhoff's normalization rather than Whittaker's adelphic integral seems to provide a better approach to these problems. For not only does Birkhoff's normalization yield all the geometric information contained in the invariant sections computed from the adelphic integral, but it also exhibits the dynamical features of the problem. This is especially relevant, considering that Siegel's objection to the existence of an adelphic integral is of a dynamical nature. For, should the system admit an adelphic integral, then there would appear around the elliptic equilibrium tori filled with periodic orbits--i.e. *ordinary* families of periodic orbits in the sense of Whittaker (1917). But Siegel proves that systems presenting this feature form a rare set in a certain set of dynamical systems provided with a usual topology. Hence in order to assess the credibility of an adelphic integral in the neighborhood of an elliptic equilibrium, we recommend to analyze the behavior of the actual system on the tori of ordinary families of periodic orbits exhibited by the normalized "approximations". Birkhoff's normalization locates immediately where these tori could be. Can the same be done as easily by means of only the explicit expression of the adelphic integral? Quite recently Contopoulos (1968) suggested that it is feasible.

We have treated the Hamiltonian function of a galactic field proposed by Contopoulos (1960), namely

$$\mathcal{H} = \tfrac{1}{2}(X^2 + Y^2) + \tfrac{1}{2}(Ax^2 + By^2) - \epsilon xy^2 - \frac{1}{3}\epsilon' x^3 \qquad (1)$$

where

$$A = 0.076 \qquad \epsilon = 0.206$$
$$B = 0.55 \qquad \epsilon' = 0.052$$

This system presents four configurations of equilibrium E_1, E_2, E_3 and E_4. For increasing values of the energy, we have drawn in Fig. 1 the evolution of the curves of zero velocity: the shaded areas are prohibited, since in them the kinetic energy would be negative. Only for energies below the level corresponding to the equilibrium configurations E_3 and E_4, and only in the closed admissible domain around E_1, could the Hamiltonian (1) be suggestive of the galactic field in our solar neighborhood.

We introduce the frequencies

$$\omega_\ell = A^{\frac{1}{2}} \quad \text{and} \quad \omega_s = B^{\frac{1}{2}}. \qquad (2)$$

Since $A < B$, we have $\omega_\ell < \omega_s$, and we agree to refer to ω_ℓ

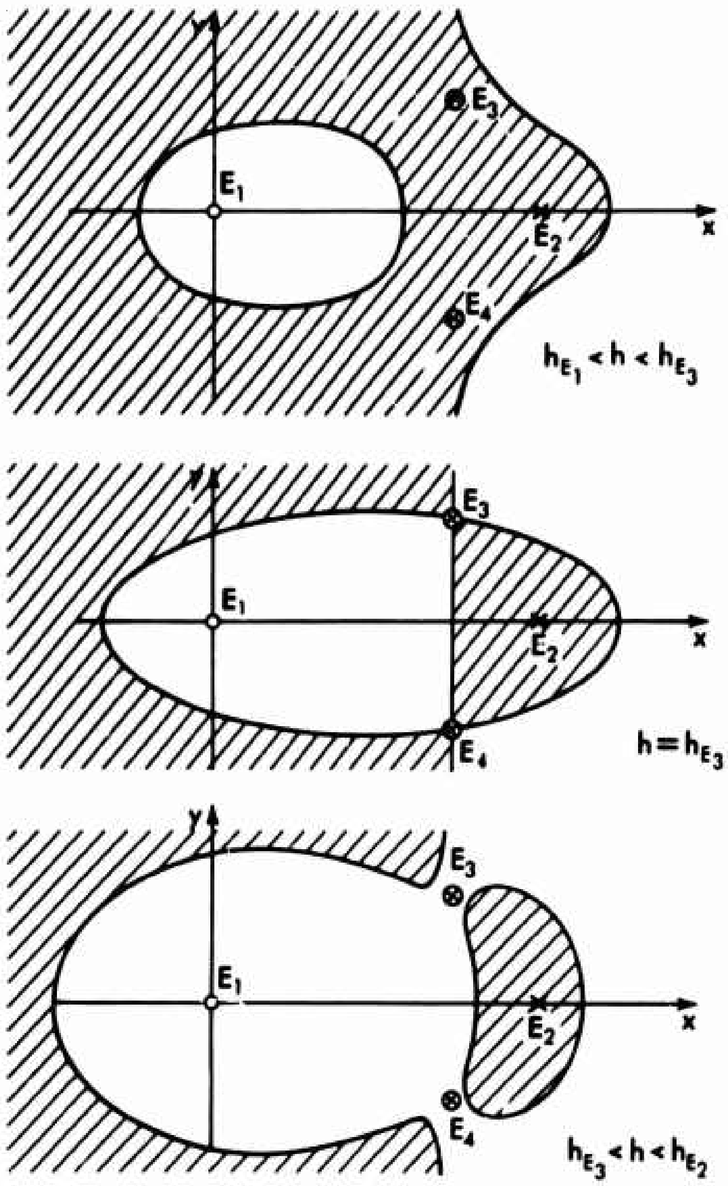

Fig. 1. Evolution of the curves of zero velocity.

(resp. ω_s) as the long period (resp. the short period) frequency.

In order to initialize Birkhoff's normalization, we propose the completely canonical transformation

$$x = (2L/\omega_\ell)^{\frac{1}{2}}\sin \ell, \qquad y = (2S/\omega_s)^{\frac{1}{2}}\sin s$$
$$X = (2L\omega_\ell)^{\frac{1}{2}}\cos \ell, \qquad Y = (2S\omega_s)^{\frac{1}{2}}\cos s \tag{3}$$

from the original Cartesian state variables (x,y,X,Y) to the angle coordinates (ℓ,s) and the action momenta (L,S). The Hamiltonian now becomes the function

$$\mathcal{H} \equiv \mathcal{H}_0 + \mathcal{H}_1 \tag{4}$$

where

$$\mathcal{H}_0 = \omega_\ell L + \omega_s S, \tag{5}$$

and

$$\mathcal{H}_1 \equiv \mathcal{H}_1(\ell,s,S,L) \tag{6}$$

is a homogeneous polynomial of degree 3 in $L^{\frac{1}{2}}$ and $S^{\frac{1}{2}}$, the coefficients being trigonometric lines in s and ℓ with real coefficients; $\mathcal{H}_1$ presents the usual d'Alembert characteristic.

Birkhoff's normalization consists in constructing a completely canonical transformation from the angle-action state variables (ℓ,s,L,S) to a new set of angle-action state variables (ℓ',s',L',S') having the property that the transformed of (4) decomposes into the sum

$$\mathcal{H} = \mathcal{N} + \mathcal{R} \tag{7}$$

where $\mathcal{N}$ is polynomial of an assigned degree N in L' and S', whereas the residual $\mathcal{R}$ is a series in $L'^{\frac{1}{2}}$ and $S'^{\frac{1}{2}}$ beginning with terms of degree $2N + 1$, the coefficients being trigonometric sums in ℓ' and s'. The canonical normalization can be constructed explicitly by means of Lie transforms (Deprit *et al* 1969). We have carried it automatically by computer up to degree $N = 10$, thus neglecting a residual of at least order 21 in $L'^{\frac{1}{2}}$ and $S'^{\frac{1}{2}}$. The main advantage of a normalization by Lie transforms is that it makes the transformation of any function of the old angles and actions depend only on the generating function $W(\ell', s', L', S')$. The conversion into the new angles and actions consists then in recursively constructing a triangle made of binomial combinations of Poisson brackets; the construction can easily be carried automatically by computer. We have had access to an IBM 360-44, and the codes consisted in calling subroutines out of a package called MAO (Rom 1969).

3. THE TEST OF A THIRD INTEGRAL BY LIE TRANSFORMS

Considering the ease with which conversion to the normalized state variables can be carried out, we undertook to check various expressions of the third integral recently published. Once the Hamiltonian (4) has been given the form (7), if the residual $\mathcal{R}$ is neglected, the principal part $\mathcal{N}$ defines a separable dynamic system: the actions L' and S' are thus integrals. Therefore the various expressions proposed in the literature as third integrals, when converted into the normalizing variables, must turn out to be functions of the actions L' and S' exclusively. As a matter of fact, we found that Hori's (1968) formula for the third integral is precisely the action S', at least as far as degree $N = 2$, beyond which Hori did not carry out the developments.

On the other side, the expression given in Goudas (1969, formulas (4) and (5) on pages 8 and 9), when converted into Birkhoff's normalizing variables, still contains trigonometric terms in ℓ' and s': actually it is only the contribution of the fifth order (formula (5)) which is defective. To exclude the possibility of an error on our part, we undertook a more elementary check: we calculated the Poisson bracket of Goudas' expression with the Hamiltonian, and we found that it still contains terms of degree 5 in the Cartesian coordinates and momenta. We made an attempt at detecting possible misprints in Goudas' formula (5). The author gives explicitly the partial derivatives of his third integral; actually, in their contributions of degree 4, three of these derivatives, namely Formulas (6), (7) and (9), show inconsistencies in signs and coefficients among themselves and with the differentiated function. Apparently the third integral in Goudas (1968) is a transcription of Formulas (9) and (10) obtained earlier (Goudas and Barbanis 1962). But, as we have checked by taking its Poisson bracket with the integral of energy, the expression given there--slightly at variance with Formula (5) in Goudas (1968)--is not formally an integral. Eventually we arrived at what seems the source of both expressions, namely Formula (5) in Barbanis (1962). Now this is formally a third integral up to the fifth degree in the Cartesian coordinates and momenta. Converted into Birkhoff's normalizing variables, Barbanis' third integral becomes

$$\Phi = 0.275680L' - 0.195060L'^2 - 0.675410L'S' - 0.544584S'^2,$$

a function exclusively of the two formal integrals L' and S' produced by the normalization.

4. SEPARABLE PICTURES PRODUCED BY A NORMALIZATION

The truncated system described by the normalized portion $\mathcal{N}$ is separable. The structure of its phase space is simple. The energy manifolds

$$\mathcal{N}(L',S') = h$$

for small values of the energy constant h surround the equilibrium E, at which $L' = S' = 0$. In the diagram (L',S') they are represented by polynomial curves (the solid curves in Fig. 2). The solutions of the normalized system fall into three classes:

a) the natural family $\mathcal{L}_s$ of singular periodic solutions for which $L' = 0$; these are the short period orbits emanating from E_1 in association with the frequency ω_s, as per Liapunov's theorem.

b) the natural family $\mathcal{L}_\ell$ of singular periodic solutions for which $S' = 0$; their existence is likewise insured by Liapunov's theorem.

c) invariant tori of quasi-periodic orbits, each of them characterized by the quantity

$$\sigma(L',S') = \frac{\partial\mathcal{N}}{\partial L'}/\frac{\partial\mathcal{N}}{\partial S'}$$

which we call the rotation number of the torus. The tori degenerate into families of ordinary periodic orbits whenever the rotation number σ takes rational values.

The natural families $\mathcal{L}_s$ and $\mathcal{L}_\ell$ survive in the full dynamical system from its normalizing truncation. The perturbation $\mathcal{R}$ introduces

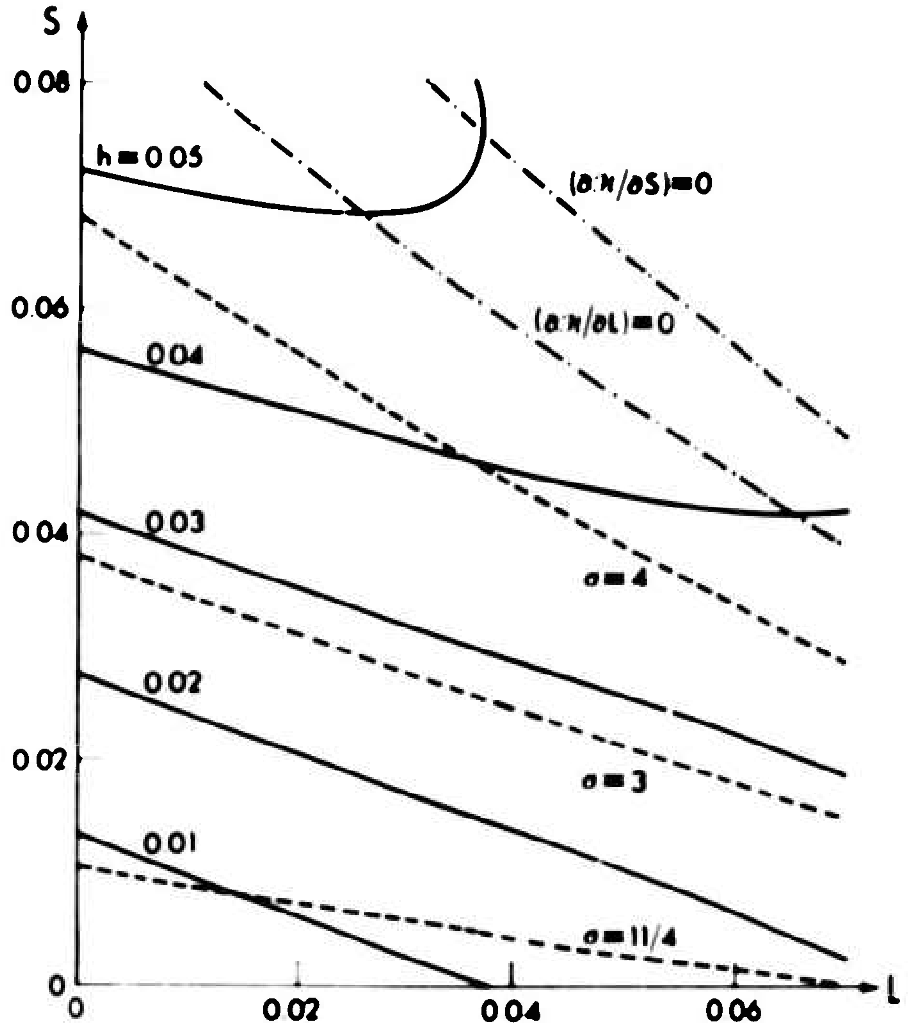

Fig. 2. Energy manifolds and tori of ordinary periodic orbits for the normalized system in the plane of actions L' and S'.

of course variations in the initial conditions, the period and the index of stability, the size of which may be interpreted as a measure of the adequacy of the normalization to the non-integrable problem. As the energy increases from its absolute minimum at E_1, the agreement deteriorates progressively. This expected evolution can be followed in Table I where we have listed for a few orbits of $\mathcal{L}_s$ the energy constant h, the period T, the initial conditions x_0, $\dot{y}_0$ at a crossing normal to the axis Ox (N.B. These orbits are symmetric with respect to that axis), the sum Tr of the non-trivial multipliers; we also mention the variations Δx_0, $\Delta \dot{y}_0$ on the initial conditions, ΔT on the period and ΔTr on the index stability caused by the perturbation $\mathcal{R}$ on the corresponding short period orbit in the normalized system, as yielded by the equations of the normalizing transformation for $L' = 0$.

Table I

Singular Periodic Orbits of Short Period

	1	2	3
h	0.007364765	0.267511315	0.355657405
T	8.593381628	8.893873967	9.933926945
x_0	0.039415822	0.130354242	0.249396558
$\dot{y}_0$	0.120886637	0.205639589	0.258731789
Tr	-1.315353	-1.118749	-0.755201
Δx_0	$<10^{-12}$	1.3×10^{-6}	3.5×10^{-4}
$\Delta \dot{y}_0$	$<10^{-12}$	-2.1×10^{-7}	-4.2×10^{-5}
ΔT	$<10^{-12}$	5.6×10^{-6}	1.6×10^{-3}
ΔTr	$<10^{-10}$	9.3×10^{-6}	2.8×10^{-3}

To our knowledge, the orbits of the long period family $\mathcal{L}_\ell$ have not appeared yet in the literature of the galactic potential. For the sake of completeness, we have followed this family. In the immediate neighborhood of E_1, we used the normalizing transformation to produce approximate initial conditions, which we improved by successive isoenergetic corrections. In the region where normalization proves inadequate, we continued the family analytically by numerical integration (Deprit and Henrard 1967). Actually the family $\mathcal{L}_\ell$ is totally degenerate: it constitutes the class of (periodic) oscillations for the conservative system

$$\mathcal{K} = \frac{1}{2} X^2 + \frac{1}{2} Ax^2 - \frac{1}{3} \varepsilon' x^3,$$

which is a kind of anharmonic oscillator. It is thus easily shown that $\mathcal{L}_\ell$ terminates with an orbit doubly asymptotic to the equilibrium E_2. We list in Table II a few orbits of this family, and we present in Fig. 3 the period curve (period T versus energy h) which summarizes its evolution.

Table II

Singular Periodic Orbits of Long Period

	1	2	3	4
h	0.002736854	0.008081604	0.013228295	0.018228245
T	23.130312798	23.927982302	24.962432597	26.459314310
x_0	0.034162526	0.110154765	0.200465837	0.200465837
$\dot{x}_0$	0.073392040	0.123642037	0.153888559	0.183525717
Tr	-0.661723	-0.494549	-0.291875	-0.039733
$\Delta\dot{x}_0$	$<10^{-12}$	-1.3×10^{-8}	-1.8×10^{-6}	*
ΔT	$<10^{-12}$	4.5×10^{-7}	10^{-4}	*
ΔTr	$<10^{-10}$	1.2×10^{-7}	2.7×10^{-5}	*

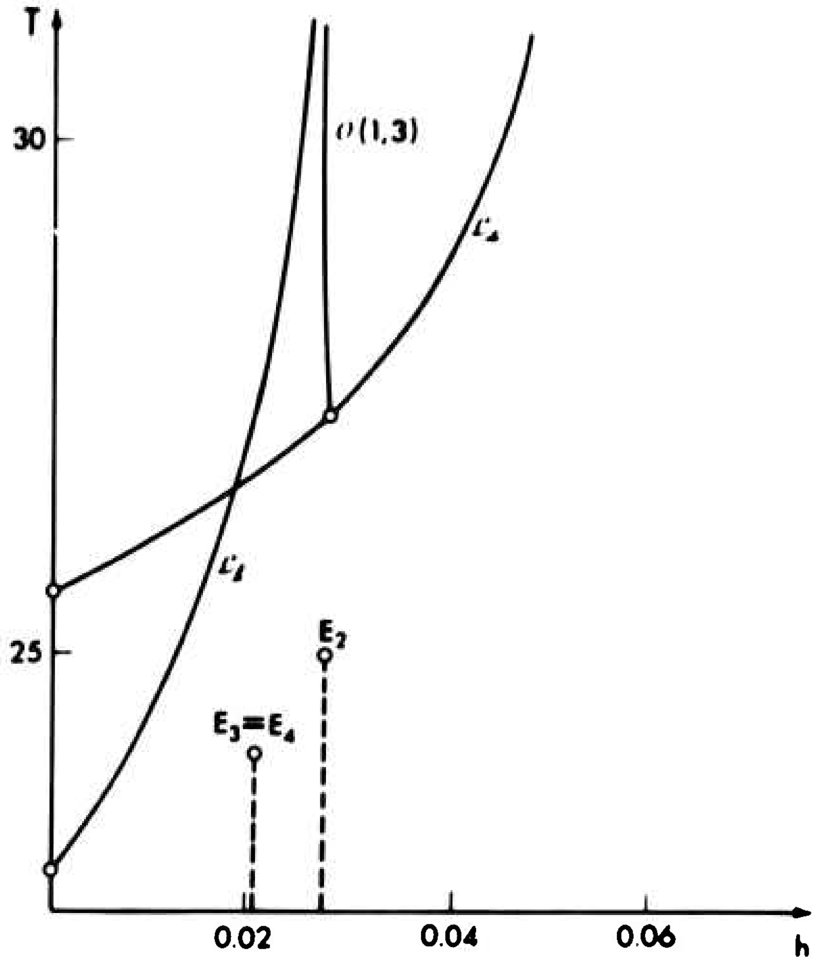

Fig. 3. Period curves for the family $\mathcal{L}_l$ and the family $O(3/1)$. For the family $\mathcal{L}_s$, the periods have been multiplied by 3 so as to show how $O(3/1)$ branches out of $\mathcal{L}_s$.

5. ORDINARY FAMILIES OF PERIODIC ORBITS AS TEST CASES OF SEPARABILITY

In the complete system as well as in the normalized one, the orbits of $\mathcal{L}_s$ and $\mathcal{L}_\ell$ are singular; in particular their characteristic exponents are not all four equal to zero. Consequently, whether in the normalized approximation or in the complete system, any integral must have its gradient collinear with that of the energy at every point of any orbit in the families $\mathcal{L}_s$ and $\mathcal{L}_\ell$. This basic theorem of Poincaré may serve as a check. Either a state function is known to be an integral; then checking the collinearity of its gradient with that of the energy is a way of assessing the accuracy of the mechanical quadrature; conversely, once it is reasonably certain that the algorithm of numerical integration is accurate, the test of collinearity may serve to decide whether or not a given state function is an integral on the cones defined by the families $\mathcal{L}_s$ and $\mathcal{L}_\ell$. In this respect, noting that the orbits of Table I in Goudas (1968) belong to the family $\mathcal{L}_s$, we conclude from the tests of collinearity in Tables III-VI that the expression given by Eq. (4) and (5) in the same paper is not an integral, either because of the errors we found in Eq. (5) or, assuming that Goudas' programs use the correct expression from Barbanis (1962), because Contopoulos' third integral expanded to the fourth order only is inadequate beyond the immediate neighborhood of E_1.

The numerical test, as long as it is restricted to singular periodic orbits, does not yield by itself information on the functional relation between the energy integral and the so-called third integral. Of course, if all the periodic orbits of a system turn out to be singular and, moreover, if the set of periodic orbits is everywhere dense in the set of solutions, then the "third integral" would be in a pathological dependence with respect to the energy, as the points at which its gradient is collinear with that of energy would be everywhere dense. Consequently one way of advancing the question of the third integral consists in examining whether the galactic system admits periodic orbits other than singular. In fact to rule out a priori their existence amounts to decree from the onset of the investigation that the galactic potential is not separable. Indeed the normalizing approximation $\mathcal{N}$ offers the example of a system having an everywhere dense family of ordinary periodic orbits; at any point of these orbits, the gradients of each action integral L' or S' are not parallel to that of energy.

From this standpoint we offer what we think is a more significant test of the third integral. Take an ordinary family of periodic orbits for $\mathcal{N}$; check whether it survives in the full system or breaks down into a finite number of natural families of periodic orbits. At the intervals of energy where the breakdown occurs, we know then that $\mathcal{N}$ is not a valid approximation of $\mathcal{H}$. And, of course, in the intervals of energy where the ordinary family survives, we could only conclude that the method is unable to separate the system from its approximative portrait as given by the truncation $\mathcal{N}$. Assuming that the normalization has been carried far enough, and that the calculations have performed very accurately over a sufficiently long interval of time, such a conclusion should be good news to a physicist. It would mean that, to all practical purposes, it would take observations accurate beyond the present state of the art and extending over an exceedingly long period of time to recognize that, in a not too unreasonably large vicinity of the equilibrium, the galactic system $\mathcal{H}$ may not be separable after all.

Like any numerical verification, the test of separability we suggest to draw from a Birkhoff's normalization is merely pragmatic. If an ordinary family of periodic orbits for the truncated system fails to survive in the complete system $\mathcal{H}$, we are unable to interpret the failure as either an *error in the model*--which would force us to categorically claim that indeed the complete system is not integrable in the sense of Arnol'd--, or to a mere *error in the truncation*--which then should rather prompt us to push the normalization to an even higher order. Tests of separability based on other numerical experimentations are equally void of compelling rigor. This is obviously the case for the tests which consisted in producing orbits that do not cover densily the available coordinate space within a curve of zero velocity on a given manifold of energy (Contopoulos 1963). And it is also the case for Goudas' test on collinear gradients, as we shall presently show.

6. THE ORDINARY FAMILY AT THE RESONANCE 1/3

In Table II of Goudas (1968), we are presented with a few symmetric orbits that originate from the resonance $\sigma = 1/3$ in the normalized system $\mathcal{N}$. The intrinsic elements and initial conditions given there should be substantially improved. Let us take for instance the

Orbit No. 1. Starting from the initial conditions, we integrated the trajectory, until we reached the value assigned by Goudas to its period, at which instant we computed the stability index Tr (see the first column in Table III). Then, fixing the energy h, we applied to this periodic orbit the usual isoenergetic corrections it badly needed (see the second column in Table III). Finally we kept the period T fixed and we applied isoperiodic corrections (see the third column in Table III). The most interesting result is that this orbit has all four characteristic exponents equal to zero. It is thus an instance of a periodic orbit which, within the accuracy of a double precision integration, keeps its *ordinary* character in the transition from the truncated normalization to the full system.

Similarly we question the values produced by Goudas for the adelphic integral along this periodic orbit. For Professor Contopoulos kindly communicated to us the expression of this integral which he is capable of generating automatically by computer (Contopoulos 1966). We truncated this expression first after degree 3, then after degree 4, and in each case we evaluated the truncated adelphic integral along the orbit. The results are presented in columns 3 and 4 of Table IV; they differ significantly from those of column 1 which are the numbers published by Goudas.

This partial checking confirms what we already found in other situations, namely that numerical experimentation fails to be conclusive less it is carried out with very good accuracy.

Table III. Periodic Orbit in the Ordinary Family $O(1/3)$

	Goudas	Isoenergetic Improvement	Isoperiodic Improvement
Initial abscissa	0.298704	0.300318	0.298618
Initial velocity	0.221645	0.221513	0.221656
Period	27.223990	27.225836	27.223990
Energy	0.0274918	0.0274918	0.0274917
Stability index	2.000004	1.9999999	1.9999999

Table IV. The Adelphic Integral along the Periodic Orbit of Table III

t	Goudas	3rd order	4th order
0	-0.472	-0.436	-0.454
1	-0.488	-0.450	-0.454
2	-0.503	-0.463	-0.453
3	-0.500	-0.451	-0.452
4	-0.489	-0.448	-0.453
5	-0.485	-0.456	-0.453
6	-0.494	-0.456	-0.453
7	-0.494	-0.456	-0.453
8	-0.491	-0.453	-0.452
9	-0.490	-0.451	-0.452
10	-0.491	-0.452	-0.452
11	-0.491	-0.452	-0.452
12	-0.490	-0.452	-0.452
13	-0.491	-0.452	-0.452
14	-0.490	-0.452	-0.452
15	-0.490	-0.452	-0.452

In the instance in case, we just discover that a double precision integration is unable to decide whether the orbit is singular or belongs to a one-parameter family of periodic orbits generated from one another by the group of adelphic transformations corresponding to the "third integral". Incidentally Goudas' inconclusive test of constancy for that integral suggests that his program might have used the erroneous formula of his paper. Meanwhile the improvement we observe from column 3 to column 4 in Table IV indicates that a normalization of sufficiently high order may account adequately for what a double precision integration would tell us about the galactic potential at the energy $h = 0.0274918$. Our point, however, is not that the galactic potential should be declared to be there a separable system, but that it is impossible to distinguish it from its separable image as given by $\mathcal{N}$; numerical integration is not sharp enough to resolve the fine details of the phase portrait beyond its blurred appearance through that magnifying glass which is the normalization.

It thus turns out to be an interesting proposition to follow the ordinary family $\mathcal{O}(1/3)$ and detect at what level of energy its breakdown into a two-lane bridge of natural periodic orbits becomes noticeable. For, once the breakdown is patent, normalization loses its practical value, the galactic potential can no longer be identified there to the separable system it describes, and functions of L' and S' are no longer practical substitutes of a "third integral". Unless, of course, as Contopoulos proposes, one refocuses the normalizing procedure from the equilibrium to the regions where the commensurability islands for $\sigma = 1/3$ are expected to appear.

Tables V and VI list a few of the orbits in the stable and unstable lanes of the family $\mathcal{O}(1/3)$ after it has broken into a natural bridge. The initial conditions were computed in this way: in the plane of the normalized actions (see Fig. 2), at equally spaced values of L', we determine the values of S' along the commensurability curve $\sigma(L',S') = 1/3$, and the corresponding values of the energy constant $h = \mathcal{N}(L',S')$. Then we insert these values of L' and S' in the d'Alembert series in L',S',ℓ',s' for the coordinates and the velocities. The initial conditions thus found correspond in the normalized system to a one-parameter multiplicity of ordinary periodic orbits.

Table V. Stable Lane of the Family $\mathcal{O}(1/3)$

	1	2	3	4
L'	0.005	0.015	0.030	0.050
S'	0.036450736	0.032881282	0.027646400	0.020973229
h	0.027437895037	0.027276426504	0.027118912258	0.027113776654
T	27.336564126068	27.729592648265	28.263773295676	28.285681461062
Tr	1.999998435	1.999956689	1.999754826	1.999742799
x_0	0.377830642608	0.535671190923	0.669748586640	0.674307902683
$\dot{y}_0$	0.214233918684	0.195124734402	0.174819453540	0.174068634945
Δ	10^{-4}	4×10^{-4}	10^{-3}	7×10^{-3}

Table VI. Unstable Lane of the Family $\mathcal{O}(1/3)$

	1	2	3	4
L'	0.005	0.015	0.030	0.050
S'	0.036450736	0.032881282	0.027646400	0.020973229
h	0.027437895037	0.027276426504	0.027118912258	0.027113776654
T	27.333016757818	27.691445063627	28.110665427880	28.126016099351
Tr	2.000001535	2.000040523	2.000213749	2.000223591
x_0	0.179226717123	0.182810120059	0.185599062618	0.185686038544
y_0	-0.021473993709	-0.020482050750	-0.019327587853	-0.019285550500
$\dot{x}_0$	-0.043705279166	-0.075247032911	-0.098332841891	-0.099037981818
$\dot{y}_0$	0.224731752511	0.215321874529	0.204927666159	0.204559141921

When such initial conditions yield a periodic orbit whose stability index Tr is not close to 2, isoenergetic corrections will be applied to them. The quantity Δ in Table V is an indication of the corrections brought on the initial conditions provided by the truncated normalizing transformations.

Although at the energy levels where the orbits are located, $\mathcal{N}$ is no longer an adequate picture of $\mathcal{H}$, the normalization has yielded an important feature of the phase space: the stable lane of $\mathcal{O}(1/3)$ is like the dorsal spine of a family of invariant quasi-periodic orbits, while the orbits in the unstable lane are epicentres of instability zones. We may reasonably anticipate that tori and instability zones are themselves contained in a halo of invariant quasi periodic orbits. Although we have not investigated further the situation, we suggest that the normalization might still be a satisfactory approximation of the galactic potential on the halo of enveloping quasi-periodic tori. For the present situation is similar to the two-lane bridge of long period orbits discovered in the Trojan problem (Deprit and Rabe 1969).

Let us add that the family $\mathcal{O}(1/3)$ originates from an orbit of $\mathcal{L}_s$ traveled 3 times; the other termination is not known to us.

Orbits 2-6 of Goudas' Table II belong to the stable lane of $\mathcal{O}(1/3)$. They are singular. Consequently they fall outside the domain where either Barbanis' function to the fifth degree or Birkhoff's normalization even to the twentieth order could be applied meaningfully. Therefore it does not make sense to test here for the collinearity in gradients of the energy and the third integral: We know a priori that, at any point of such a natural periodic orbit, gradients of independent integrals are collinear.

7. THE ORDINARY FAMILY AT THE RESONANCE 4/11

So far we dealt with families of periodic orbits that were ordinary for the normalized system but broke down into natural families for the complete problem. Are all ordinary families in that category? We cannot answer categorically. But we know of ordinary families which a double precision integration cannot resolve into bridges of singular orbits. Consider for instance the commensurability ratio $\sigma = 4/11$. At the energy $h = 0.008$, it gives rise to an infinity of periodic orbits. The initial conditions generated by the normalization equations have served to start an integration of the complete problem. The calculation of each orbit was stopped when the estimated period $T = 94.710170931337$ had been reached. In the last column of Table VII, we list under Δ the maximum of the quantities $|x(T)-x_0|$, $|y(T)-y_0|$, $|\dot{x}(T)-\dot{x}_0|$, $|\dot{y}(T)-\dot{y}_0|$. Obviously these initial conditions (reproduced in Table VII) require no significant corrections. Yet for the symmetric orbits 1 (Fig. 4) and 7 (Fig. 5) as

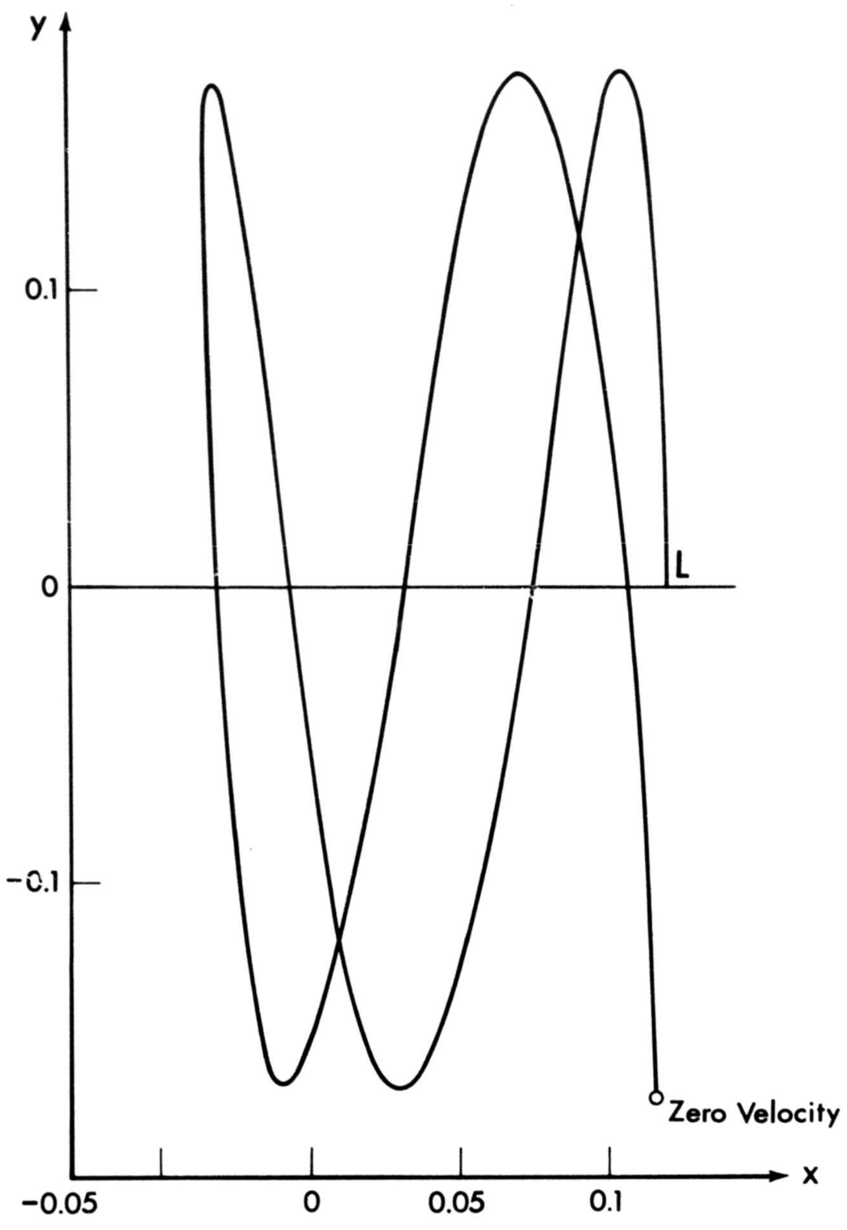

Fig. 4. Orbit No. 1 in the family $\mathcal{O}(4/11)$. Only a quarter of the orbit is plotted; the full orbit consists of the curve here drawn and of its symmetric with respect to the x-axis, the complete arc being traveled twice.

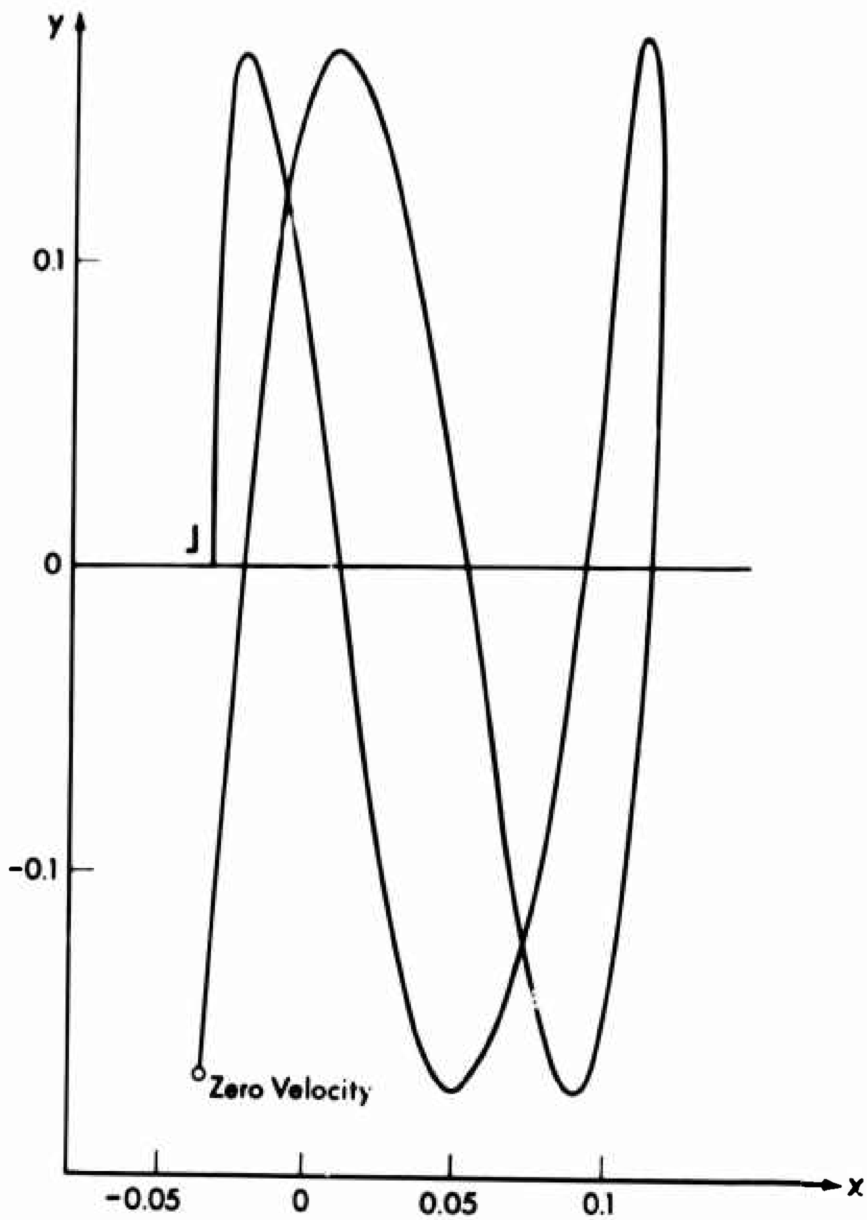

Fig. 5. Orbit No. 7 in the family $O(4/11)$ (symmetric with respect to the x-axis).

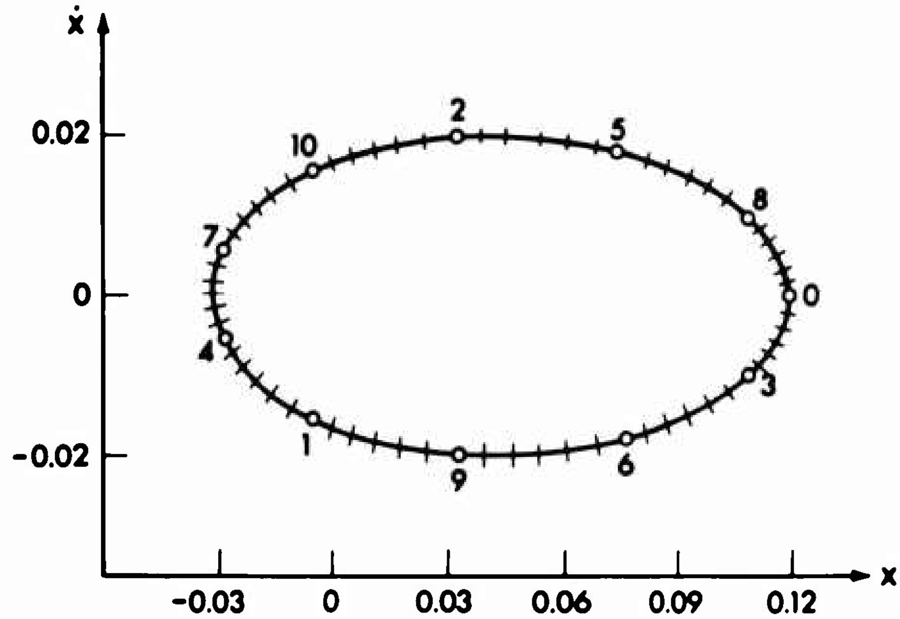

Fig. 6. Cross section in the ordinary family $O(4/11)$ at the energy level $h = 0.008$. (The circles indicate the successive intersections of one of the symmetric orbits in the family; tick marks refer to other orbits in the same torus).

well as for orbit 2, we attempted to improve on the periodicity; the corrections entered in Table VIII brought no change to the stability index which remained set at 2. The cross section by the plane $y_0 = 0$, as projected on the plane $(x,\dot{x})$ of the flow in the direction $y > 0$ (see Fig. 6) presents the usual characteristic of a Lissajoux family of periodic orbits for a separable system, each orbit in the family deriving from one another by a shift of the phase.

We want to emphasize that this example does not contribute to deciding whether, at that energy level and so close to the equilibrium, the galactic system is integrable or not. All it shows is that, within the resolution power of a double precision integration, the galactic system cannot be distinguished from the separable system represented by its Birkhoff's normalization of the twentieth order.

As the energy increases away from the equilibrium E_1, we start noticing the dissolution of the ordinary tori of periodic orbits into pairs of isolated natural periodic orbits, one with stable characteristic exponents, the other with unstable ones. In these regions, it then has become possible to dissociate the galactic potential from the (separable) normalized model (see Table IX).

Table VII. Periodic Orbits in the Family $\mathcal{O}(4/11)$

($h = 0.008$, $T = 94.710170931337$, $Tr = 2$, $y_0 = 0$)

No.	x_0	$\dot{x}_0$	$\dot{y}_0$	Δ
1	0.118781953476964	0	0.122416513260330	2×10^{-10}
2	0.104845001238049	-0.011380578262750	0.122780323885444	3×10^{-10}
3	0.118459202988883	-0.001809303978216	0.122424981650740	3×10^{-10}
4	0.117493500640730	-0.003604356267765	0.122450307545493	3×10^{-10}
5	0.113668831031758	-0.007095093426141	0.122550429205126	3×10^{-10}
6	0.110840221679624	-0.008762872516124	0.122624290701551	4×10^{-10}
7	-0.032191479174304	0	0.126174820734764	2×10^{-10}

Table VIII. Corrections on Some Orbits of $\mathcal{O}(4/11)$ at $h = 0.008$

	1	2	7
Δx_0	1.05×10^{-8}	-2.68×10^{-6}	1.02×10^{-8}
Δy_0	--	-2.43×10^{-8}	--
$\Delta \dot{x}_0$	--	1.01×10^{-7}	--
ΔT	1.84×10^{-8}	1.84×10^{-8}	1.84×10^{-8}
ΔTr	6×10^{-12}	6×10^{-12}	6×10^{-12}

Table IX. Natural Periodic Orbits Resulting from the Dissolution of the Ordinary Torus $\mathcal{O}(4/11)$ at the Energy Levels h = 0.01 and 0.014

h	0.01	0.01	0.014	0.014
T	96.324102914682	96.324102826247	100.195807187502	100.195802185201
Tr	1.999999974	2.000000026	1.999992668	2.000007332
x_0	0.400291011112	-0.278803306683	0.677011519908	-0.479422738872
y_0	0	0	0	0
$\dot{x}_0$	0.486×10^{-9}	0	0	0
$\dot{y}_0$	0.100228800665	0.115503833180	0.062634141987	0.081924667338

8. CONCLUSIONS

The topic of the third integral in stellar dynamics is closely akin to the Theory of Trojan Planets: it bears essentially on the possibility of representing over a finite interval of time the motion around an equilibrium by means of truncated series in the powers of action momenta, their coefficients being quasi-periodic functions in angle coordinates.

Extensive investigations in the Trojan case have shown that solutions of this kind are gratifying to a large extent. We indicated here that they would meet with the same success in the case of stellar dynamics. Provided of course that the investigator proceeds with great care, and refrains from unanswerable conclusions.

In keeping with the traditions of celestial mechanics, the overall approach should be a pragmatic one (Poincaré 1893, pp. 1-2).

Mathematicians take it for very plausible that almost all Hamiltonian systems have no regular integral other than the energy (Abraham 1967, pp. 112 and 170). But they do not count on truncated formal expansions and numerical integrations to irrefutably establish their contention.

Astronomers, on the other hand, cannot help but apply to their models, whether they are integrable or not, formalisms or numerical procedures which substitute to the full system a separable approximation.

The two approaches are justified: the first in theoretical researches, the second in numerical applications. Both should be kept open, and they should be maintained independently of one another until the time comes where they would have established constructive cooperation.

On the whole astronomers, even if they do not know in full rigor the limitations of their formalisms, rarely overstep them. The approximations they consider as sufficient keep them well within the boundaries of credibility. Anyway a certain intuition is there to guide the application of a formalism to a physical situation, and as soon as that affinity of a physicist with his world ceases being helpful, control from observations would not fail from eventually uncovering the violations of accepted standards of interpretation.

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