A NEW ESTIMATION THEORY FOR SAMPLE SURVEYS, II

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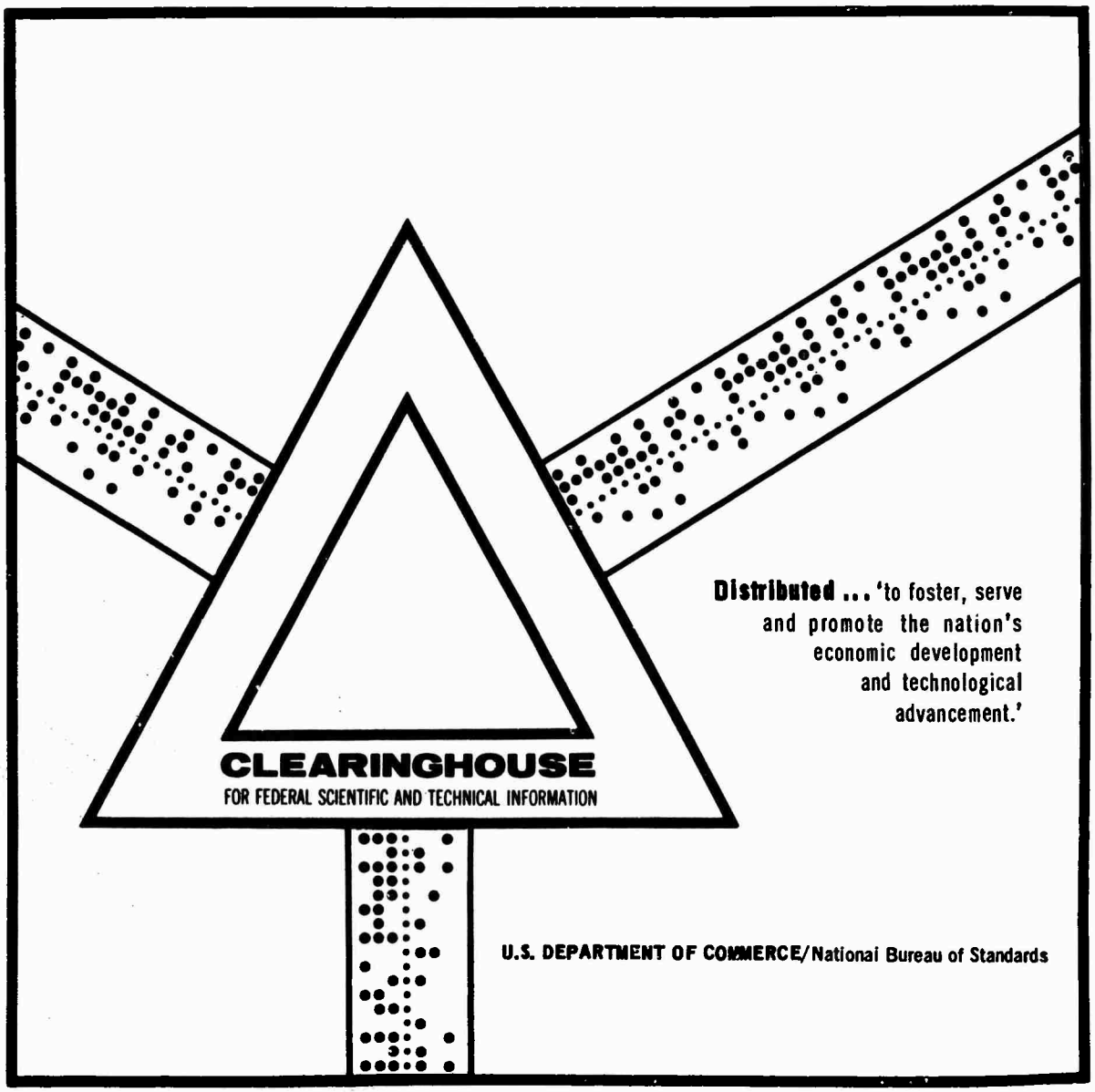
College Station, Texas

1969



Unclassified
Security Classification

AD 698484

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)
Texas A & M University

2a. REPORT SECURITY CLASSIFICATION
Unclassified

2b. GROUP
NA

3. REPORT TITLE
A New Estimation Theory for Sample Surveys, II

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)
Reprint

5. AUTHOR(S) (First name, middle initial, last name)
H. O. Hartley
J. N. K. Rao

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
1969	19	7

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
DAHC04 69 C 0058	NA
b. PROJECT NO. 20061102B14C	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) 8049.4-M
d.	

10. DISTRIBUTION STATEMENT
Distribution of this report is unlimited.

DDC
RELEASE
A

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
None	U.S. Army Research Office-Durham Box CM, Duke Station Durham, North Carolina 27706

13. ABSTRACT

(AD 516 466)

This paper is a sequel to an earlier one (Hartley and Rao, 1968) on the same topic. Our first paper was predominantly concerned with simple random sampling (with or without replacement) from a finite population. In the present paper we are concerned with examining the relation of our findings to the more complex sampling procedures such as unequal probability sampling as well as stratified and multistage sampling.

14. KEY WORDS
Estimating
Sampling
Statistical samples
Population (statistics)

DD FORM 1 NOV 68 1473 REPLACES DD FORM 1473, 1 JAN 64, WHICH IS OBSOLETE FOR ARMY USE.

Unclassified
Security Classification

ARO-D PROPOSAL

No. CRDARD-M-8049

"New Developments in Sample Survey Theory"

DAHC04 69 C 0058

TECHNICAL REPORT NO. 2

"A New Estimation Theory for Sample Surveys, II"

by H. O. Hartley and J.N.K. Rao

A New Estimation Theory for Sample Surveys, II

H. O. Hartley and J. N. K. Rao

Texas A & M University

1. Introduction.

This paper is a sequel to an earlier one (Hartley and Rao, 1968) on the same topic. Accordingly, it will be necessary to briefly recall the basic results of the earlier paper and relate that paper to the present one. Our first paper was predominantly concerned with simple random sampling (with or without replacement) from a finite population. In the present paper we are concerned with examining the relation of our findings to the more complex sampling procedures such as unequal probability sampling as well as stratified and multi-stage sampling.

The basic feature of our theory was a special parametrisation of a finite population of N units with k characteristics attached to each unit. Denote by $\underset{\sim}{y}_i$ the k-vector attached to the i-th unit. We assume that all elements of the $\underset{\sim}{y}_i$ are measured on discrete scales and that only a finite set of T measurement vectors $\underset{\sim}{y}_t$ (t = 1, 2,..,T) are possible for the $\underset{\sim}{y}_i$. Denote then by

$$N_t = \text{no. of units in the population having } \underset{\sim}{y}_t \tag{1}$$

satisfying the conditions

$$N_t \geq 0 \text{ and } \sum_{t=1}^{T} N_t = N. \tag{2}$$

Henceforth, sums and products for t are over 1, 2,..,T.

The parameters $N_1,..,N_T$ completely describe any finite population. The number T is usually large although sometimes occasions arise when T is small

or moderate and the estimation of the N_t is of intrinsic interest, as for example when the N_t represent a frequency distribution such as the number of households in the community comprising t persons. However, in most cases we shall be concerned with the estimation of a few simple parametric functions such as the population moments and not with the separate estimation of the excessively large number of parameters N_t.

Finite population sampling will normally consist of (a) the sample design, i.e., the procedure of drawing a sample of n distinct units (where n may be fixed or random) and with measuring the $\underset{\sim}{y}_t$ for these units, (b) the use of the measured $\underset{\sim}{y}_t$ to compute estimators of the population parameters.

In our previous paper we restricted (a) to simple random sampling and we confined the computation of estimators (b) to what we termed 'scale-load' estimators. These were defined as mathematical functions of the scale vectors $\underset{\sim}{y}_t$ and of their sample loads (frequencies) n_t = no. of units in the sample having $\underset{\sim}{y}_t$. Thus any identifying labels, i, that may be attached to the units may or may not be used for the implementation of the sample design; however, labels are not directly used in the computation of the estimators. Nevertheless, in situations where the labels, i, are observable characteristics of the units and are considered informative observables, the labels may be adjoined to the vectors $\underset{\sim}{y}_i$ as a (k + 1)-th component.

We were able to show that within the class of 'scale-load' estimators many of the estimators in current use possess interesting optimality properties in simple random sampling. Specifically the estimators are either UMV (unbiased minimum variance) or maximum likelihood estimators or both. Some of these results are briefly restated in Section 2. In the remaining sections of the present paper we are

concerned with the role these results play in the more complex sampling procedures. Briefly our findings are: (1) The above parametrization of finite populations will continue to yield useful likelihood formulations for sampling designs providing maximum likelihood and Bayesian estimation procedures. UMV property will be the exception rather than the rule. (2) We consider that identifying labels of primary units (or all but the last stage units) will often be available as well as informative. There are, however, situations in which higher stage units are not labelled as is the case, for example, for certain subsets of machine parts produced in bulk, the water supply of water works produced during certain time periods, etc. Certain situations where labels of higher stage units are not informative also exist, for example identifiable subsets of certain lists. Both 'scale-load' and label-dependent estimators are therefore required. As would be expected, there is usually no UMV estimator in the class of label-dependent estimators. (3) A particular problem arises when label dependence of estimators is used in conjunction with Bayesian concepts and separate prior distributions are allowed for the individually identifiable units. The resulting posterior distributions and hence Bayesian inferences do not depend on the survey design which in the frame work of Bayesian theory becomes a randomization procedure irrelevant in making posterior inferences. However, the absurd result that Bayesian theory leads to when applied to simple sampling or ultimate-stage unit sampling (Godambe, 1966) is perhaps our strongest point in favor of examining estimators that do not depend on the labels of the ultimate-stage units.

2. Simple random sampling.

If a simple random sample of fixed size n is drawn without replacement

from the population of N units, the likelihood of the n_t is given by

$$L(N_1,\ldots,N_T) = \Pi\binom{N_t}{n_t}\Big/\binom{N}{n} \tag{3}$$

where $n_t \geq 0$ and $\Sigma n_t = n$. We confine ourselves here to the case of a single character y attached to the units (i.e., k = 1). In our previous paper we have shown that any function of the n_t is an UMV estimator of its expectation. Specifically some of the more important parametric functions and their UMV estimators are given below:

Parametric function	UMV estimator
N_t/N	n_t/n
$\mu_r' = N^{-1}\Sigma N_t y_t^r$	$m_r' = n^{-1}\Sigma n_t y_t^r$
$\sigma^2 = \mu_2' - \mu_1'^2$	$\frac{n(N-1)}{N(n-1)}(m_2' - m_1'^2)$

(4)

Notice that the estimators do not depend on T or the non-observed y_t. When N/n is an integer, n_t/n and m_r' are also the maximum likelihood estimators (see the Appendix). When N/n is not integral, the maximization of (3) over the integral grid N_t can be achieved by the algorithm given in the Appendix; however, since UMV estimators exist, the maximum likelihood estimators may not have particular merit for small samples. The possibility of using maximum likelihood estimators of the N_t when T is small and the N_t are parameters of interest is being examined by a Monte Carlo study.

Turning now to Bayesian estimation, we have used in our previous paper the mathematically convenient prior distribution suggested by Hoadley (1968) and given by

$$\varphi(N_1,\ldots,N_T) \propto \Pi \frac{(N_t + \nu_t - 1)!}{N_t!(\nu_t - 1)!}, \quad \nu_t > 0. \tag{5}$$

The 'Bayes estimator' of μ_r' is the posterior expectation of μ_r' and is given by

$$E'(\mu_r') = (1 - \tfrac{n}{N})[wm_r' + (1 - w)M_r'] + \tfrac{n}{N} m_r' \qquad (6)$$

where

$$w = n/(n + \nu), \quad \nu = \Sigma\nu_t \qquad (7)$$

and

$$M_r' = \nu^{-1}\Sigma\nu_t y_t^r . \qquad (8)$$

It should be noted that the estimator (6) only requires the knowledge of M_r' (the prior mean of μ_r') and w, i.e., in the case of r = 1 the knowledge only of the prior mean M_1' and the relative weight w of the sample and prior information. Moreover, although the ν_t are akin to a prior sample frequencies, the posterior mean is not simply the mean of the pooled 'sample' $\nu_t + n_t$. It duly recognizes the fact that, as $n \to N$, the sample mean m_1' will tend to μ_1' and that the prior is ignored.

The expected loss which the decision maker faces by chosing the 'Bayes estimator' is given by the posterior variance

$$V'(\mu_r') = \frac{(N-n)(N+\nu)}{N^2(n+\nu+1)}\left[wm_{2r}' + (1-w)M_{2r}' - \left\{wm_r' + (1-w)M_r'\right\}^2\right] . \qquad (9)$$

The 'Bayes estimator' of σ^2 is given by

$$E'(\sigma^2) = \frac{(n+\nu-\nu/N)}{(1+\nu/N)} V'(\mu_1') + \tfrac{n}{N}\left[m_2' - m_1'^2 + (1-\tfrac{n}{N})(1-w)^2(m_1'-M_1')^2\right] . \qquad (10)$$

It should be noted that, if the prior information is <u>solely</u> based on a pilot sample, M_r' and ν would <u>roughly</u> represent the r-th sample moment based on the pilot sample and the pilot sample size respectively.

Turning to simple random sampling with replacement, suppose a random sample of fixed size m is drawn with equal probability and with replacement. Let

n denote the number of distinct units in the sample and n_t the number of distinct units having the value y_t in the sample. The total likelihood is given by

$$L(N_1,\ldots,N_T) = P(n) \frac{\prod\binom{N_t}{n_t}}{\binom{N}{n}} \qquad (11)$$

where the probability P(n) is a function only of m and N. For this sample design no UMV exists, but the maximum likelihood estimator of μ_r' is $m_r' = n^{-1}\Sigma n_t y_t^r$ provided N = cx least common multiple of 1, 2,...,m (c = integer). In particular, the maximum likelihood estimator of the population mean μ_1' is the sample mean based only on the distinct units in the sample and it is uniformly more efficient than the customary sample mean based on all the sample draws. With the prior distribution (5), the 'Bayes estimator' of μ_r', the posterior variance of μ_r' and the 'Bayes estimator' of σ^2 are respectively given by (6), (9) and (10), where n and the n_t are as defined above.

3. Estimation with concomitant variables.

In our earlier paper we have considered a situation customarily dealt with by ratio or regression method of estimation in which two variates y and x are attached to each of the units and the population mean $\bar{Y}$ of 'target variate' y is to be estimated utilizing the available information about x. Assuming that only the population $\bar{X}$ of x is known, we have shown that an approximation to the maximum likelihood estimator of $\bar{Y}$ is closely related to the customary regression estimator, provided the sample size n is moderately large. In this section we extend this result to multiple concomitant variables $x_1,\ldots,x_k$, assuming that only the population means $\bar{X}_1,\ldots,\bar{X}_k$ are known. We show that, for moderately large n, an approximation to the maximum likelihood estimator of $\bar{Y}$ is closely

related to the customary multiple regression estimator.

As before, we assume that a finite set of T distinct, known values y_t are feasible for y. Likewise, we assume that I_j distinct, known values x_{ji_j} are feasible for x_j $(j = 1,\ldots,k)$. Let $N_{i_1..i_k t}$ denote the number of units in the population which have $x_{1i_1},\ldots, x_{ki_k}$ and y_t attached to them. Let $n_{i_1..i_k t}$ be the number of units in the simple random sample of size n (drawn without replacement) which have $x_{1i_1},\ldots,x_{ki_k}$ and y_t attached to them.

We consider only the multinomial situation in which $N \to \infty$ and $N_{i_1..i_k t}/N \to P_{i_1..i_k t}$ while n is held fixed. The likelihood L is then given by the multinomial distribution with probabilities $P_{i_1..i_k t}$. The restrictions on the $P_{i_1..i_k t}$ are given by

$$\underset{\sim}{P} \geq 0,\ \underset{\sim}{P}'\underset{\sim}{i} = 1 \text{ and } \underset{\sim}{P}'\underset{\sim}{Z} = \underset{\sim}{\bar{X}} \tag{12}$$

where $\underset{\sim}{P}'$ is the nxl vector of the $P_{i_1..i_k t}$, $\underset{\sim}{i}$ is the lxn vector of 1's, $\underset{\sim}{\bar{X}}' = (\bar{X}_1,\ldots,\bar{X}_k)$ and $\underset{\sim}{Z} = (\underset{\sim}{x}_1^*|..|\underset{\sim}{x}_k^*)$ where $\underset{\sim}{P}'\underset{\sim}{x}_j^* = \bar{X}_j$ $(j = 1,\ldots,k)$. As in our previous paper, it can be shown that for moderate sample sizes n the global maximum of the multinomial likelihood can only be attained if $P_{i_1..i_k t} = 0$ for all those variate combinations for which $n_{i_1..i_k t} = 0$, and $P_{i_1..i_k t} > 0$ for the remainder. Confining then the maximization to the latter $P_{i_1..i_k t}$ only and introducing the Lagrangian multipliers λ and $\underset{\sim}{\mu}' = (\mu_1,\ldots,\mu_k)$, the maximization of log L subject to (12) is attained for $\underset{\sim}{P} = \underset{\sim}{\hat{P}}$ where

$$\hat{P}_{i_1..i_k t} = \frac{n_{i_1..i_k t}}{n}\left[1 + \frac{1}{n}\sum_1^k \mu_j(x_{ji_j} - \bar{X}_j)\right]^{-1}. \tag{13}$$

Expanding $\underset{\sim}{\hat{P}}'\underset{\sim}{i} = 1$ to first three terms we obtain

$$n(\underset{\sim}{\bar{x}} - \underset{\sim}{\bar{X}})'\underset{\sim}{\mu} = \underset{\sim}{\mu}'\underset{\sim}{X}^{*\prime}\underset{\sim}{X}^{*}\underset{\sim}{\mu} \tag{14}$$

where $\underset{\sim}{\bar{x}}' = (\bar{x}_1,\ldots,\bar{x}_k)$ is the vector of sample means and $\underset{\sim}{X}^{*\prime}\underset{\sim}{X}^{*} = \underset{\sim}{S}^{*} = (s^{*}_{jp})$ where

$$s^{*}_{jp} = n^{-1}\Sigma..\Sigma n_{i_1..i_k t}(x_{ji_j} - \bar{X}_j)(x_{pi_p} - \bar{X}_p).$$

It is readily seen that the solution of (14) is given by

$$\underset{\sim}{\mu} = n(\underset{\sim}{X}^{*\prime}\underset{\sim}{X}^{*})^{-1}(\underset{\sim}{\bar{x}} - \underset{\sim}{\bar{X}}). \tag{15}$$

Now using (15) and expanding (13) to the first two terms we get

$$\underset{\sim}{\hat{P}} = \frac{1}{n}\left[\underset{\sim}{n} + \underset{\sim}{X}^{+}(\underset{\sim}{X}^{*\prime}\underset{\sim}{X}^{*})^{-1}(\underset{\sim}{\bar{X}} - \underset{\sim}{\bar{x}})\right] \tag{16}$$

where $\underset{\sim}{n}$ is the 1xn vector of the $n_{i_1..i_k t}$ and X^{+} is given by $\underset{\sim}{y}'\underset{\sim}{X}^{+} = (s^{*}_{1y},..,s^{*}_{ky})$ where $\underset{\sim}{P}'\underset{\sim}{y} = \bar{Y}$ and

$$s^{*}_{jy} = n^{-1}\Sigma...\Sigma n_{i_1..i_k t}y_t(x_{ji_j} - \bar{X}_j),\ j = 1,..,k.$$

An improved approximation, along the lines of our previous paper, can be obtained by expanding (13) to the first three terms.

Using (16), an approximation to the maximum likelihood estimator of the population mean $\bar{Y} = \underset{\sim}{P}'\underset{\sim}{y}$ is given by

$$\hat{\bar{Y}} = \underset{\sim}{\hat{P}}'\underset{\sim}{y} \doteq \bar{y} + (\underset{\sim}{\bar{X}} - \underset{\sim}{\bar{x}})'\underset{\sim}{S}^{*-1}\underset{\sim}{s}^{*}_{.y} \tag{17}$$

where $\underset{\sim}{s}^{*}_{.y}$ is the k-vector of the s^{*}_{jy}. The customary multiple regression estimator is given by

$$\hat{\bar{Y}}_r = \bar{y} + (\underset{\sim}{\bar{X}} - \underset{\sim}{\bar{x}})'\underset{\sim}{S}^{-1}\underset{\sim}{s}_{.y} \tag{18}$$

where $\underset{\sim}{S} = (s_{jp})$, $\underset{\sim}{s}_{.y}$ is the k-vector of the s_{jy} and

$$s_{jp} = s^{*}_{jp} - (\bar{x}_j - \bar{X}_j)(\bar{x}_p - \bar{X}_p)$$

$$s_{jy} = s^{*}_{jy} - \bar{y}(\bar{x}_j - \bar{X}_j).$$

Although (17) differs slightly from (18), the above development clearly shows that, at least in large samples, the customary multiple regression estimator is essentially the maximum likelihood estimator.

4. Stratified simple random sampling without replacement.

4.1 UMV Estimator.

Suppose there are L strata in the population with N_i units in the i^{th} stratum ($i = 1,..,L$). Denote by N_{it} the number of units in the population belonging to the i^{th} stratum and having the measurement y_{it} ($t = 1,..,T_i$) so that $\sum_t N_{it} = N_i$ ($\sum\sum N_{it} = N$). A stratified simple random sample $(n_1,..,n_L)$ is drawn without replacement, ($\sum n_i = n$), and n_{it} denotes the number of units in the sample belonging to the i^{th} stratum and having the measurement y_{it}, ($\sum_t n_{it} = n_i$). Now the likelihood of the n_{it} is given by

$$L(N_{11},..,N_{LT_L}) = \prod_1^L \left[\frac{\binom{N_{i1}}{n_{i1}} \cdots \binom{N_{iT_i}}{n_{iT_i}}}{\binom{N_i}{n_i}} \right]. \tag{19}$$

Therefore, the n_{it} are complete sufficient for the N_{it} and, hence, the UMV estimator of the population mean $\bar{Y} = N^{-1}\sum\sum N_{it} y_{it}$ is the customary estimator

$$\hat{\bar{Y}} = N^{-1}\sum_{it}\sum \hat{N}_{it} y_t = N^{-1}\sum_i N_i \bar{y}_i \tag{20}$$

where $\hat{N}_{it} = (N_i/n_i)n_{it}$ is the UMV estimator of N_{it}. It also follows that the maximum likelihood estimators of the N_{it} and $\bar{Y}$ are the UMV estimators $\hat{N}_{it}$ and $\hat{\bar{Y}}$ respectively, when the N_i/n_i are integral. Notice that each stratum is described by its separate set of parameters, i.e., we have an additional subscript i to index the strata.

An interesting special case occurs when the stratification is according to the size of the units, say x_i. If we assume that x_i is constant within strata and use the allocation proportional to total size, i.e., $n_i = n(N_i x_i / \Sigma N_i x_i) = N_i P_i$ (say) where $\Sigma\Sigma_{it} P_i = n$, we get

$$\hat{\bar{Y}} = N^{-1} \Sigma\Sigma_{it} \hat{N}_{it} y_{it} = N^{-1} \Sigma\Sigma_{it} \frac{n_{it} y_{it}}{P_i} \tag{21}$$

which is a 'Horvitz-Thompson' type estimator.

4.2. Bayesian optimization of stratified sampling.

Ericson (1965) has presented a solution to the problem of optimum allocation when prior information in the form of a prior distribution is available. He has, however, assumed: (a) $N_i = \infty$, $i = 1,..,L$, (b) normality of the within stratum populations and (c) known within stratum population variances σ_i^2. Assuming that the within stratum population means μ_i have independent normal priors with means m_i and variances v'_{ii}, he has shown that the posterior variance of the population mean $\mu = \Sigma \pi_i \mu_i$ is given by

$$v'' = \Sigma_i \left[\pi_i^2 / \left(\frac{1}{v'_{ii}} + \frac{n_i}{\sigma_i^2} \right) \right] \tag{22}$$

where π_i is the known proportion of the population units falling within the i^{th} stratum. Ericson has given a computational algorithm to find $n_i \geq 0$ $(i = 1,..,L)$ such that (22) is minimized subject to the cost constraint

$$\Sigma c_i n_i = C \tag{23}$$

where C is the given budget.

Recently, Draper and Guttman (1968) have relaxed the assumption (c) and presented a sequential allocation scheme which appears simpler than Ericson's al-

gorithm. They have also considered the case of unknown proportions π_i. Using our present approach, one of us (J. N. K. Rao, 1968) has given a solution which is free from the restrictive assumptions (b) and (c). Extension to multiple priors and/or multiple characteristics by the use of convex programming was also considered. In this section we present a complete solution by relaxing the assumption (a) also.

We assume that prior information on the N_{it} is available in the form of (5) for each i and that the priors are independent. Therefore, the prior distribution of $N_{11},..,N_{LT_L}$ is

$$\varphi(N_{11},..,N_{LT_L}) \propto \prod_i \left[\prod_t \frac{(N_{it} + \nu_{it} - 1)!}{N_{it}!(\nu_{it} - 1)!}\right] \tag{24}$$

$$\nu_{it} > 0, \ \Sigma_t \nu_{it} = \nu_i \ .$$

Now, since $\bar{Y} = N^{-1}\Sigma_i N_i \bar{Y}_i$ where $\bar{Y}_i$ is the i^{th} stratum population mean, we get using (6) and (9) the posterior mean of $\bar{Y}$ as

$$E'(\bar{Y}) = N^{-1}\Sigma_i N_i \left[\left(1 - \frac{n_i}{N_i}\right) \Sigma_t \frac{n_{it} + \nu_{it}}{n_i + \nu_i} y_{it} + \frac{n_i}{N_i} \Sigma_t \frac{\nu_{it}}{\nu_i} y_{it}\right] \tag{25}$$

and the posterior variance of $\bar{Y}$ as

$$V'(\bar{Y}) = N^{-2}\Sigma_i N_i^2\left(1 - \frac{n_i}{N_i}\right)\left(1 + \frac{\nu_i}{N_i}\right)(n_i + \nu_i + 1)^{-1}.$$
$$\left[\Sigma_t \frac{n_{it} + \nu_{it}}{n_i + \nu_i} y_{it}^2 - \left(\Sigma_t \frac{n_{it} + \nu_{it}}{n_i + \nu_i} y_{it}\right)\right] \tag{26}$$

Since the posterior variance (26) depends on the to be observed sample values n_{it}, we take the expectation of (26) with respect to the marginal distribution of the n_{it}. It follows from Hoadley (1968) that the marginal distribution of the n_{it} is given by

$$f(n_{11},..,n_{LT_L}) = \prod_i \left[\frac{\prod_t \binom{n_{it} + \nu_{it} - 1}{n_{it}}}{\binom{n_i + \nu_i - 1}{n_i}} \right] \tag{27}$$

which is identical to that in the case of infinite populations with Dirichlet prior distributions. Therefore, using the results of J. N. K. Rao (1968) it follows from (26) that the expected posterior variance of $\bar{Y}$ is

$$V''(\bar{Y}) = \sum_i \frac{N_i^2}{N^2} \left(1 - \frac{n_i}{N_i}\right)\left(1 + \frac{\nu_i}{N_i}\right) \frac{A_i}{n_i + \nu_i} \tag{28}$$

where

$$\frac{A_i}{\nu_i} = (\nu_i + 1)^{-1} \sum_t \frac{\nu_{it}}{\nu_i} \left[y_{it} - \left(\sum_t \frac{\nu_{it}}{\nu_i} y_{it} \right) \right]^2. \tag{29}$$

It follows, using (9) and (10), that

$$\text{Prior variance of } \bar{Y}_i = \left(\frac{1}{\nu_i} + \frac{1}{N_i}\right) A_i \tag{30}$$

and

$$\text{Prior mean of } S_i^2 = A_i \tag{31}$$

where $N_i \sigma_i^2 = (N_i - 1)S_i^2$.

Now (28) is a separable convex function in the n_i and, therefore, the values n_i which minimize (28) subject to (23) and $0 \leq n_i \leq N_i$ $(i=1,..,L)$ can be obtained by convex programming* It is also possible to develop a sequential allocation procedure analogous to that of Draper and Guttman (1968).

It is important to note that the knowledge of the complete priors is not essential for the optimum allocation — only that of the prior mean of σ_i^2 and prior variance of μ_i is needed. If the priors are <u>solely</u> based on pilot samples within each stratum, then $[(\nu_i+1)/\nu_i]A_i$ and ν_i would <u>roughly</u> represent the pilot-sample variance and the pilot sample size respectively.

The extension of the above results to multiple priors and/or multiple

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* In our original version we ignored the restriction $n_i \leq N_i$ and Ericson has pointed it out.

characteristics follows along the lines of J. N. K. Rao (1968) and the optimum allocation is obtained by convex programming.

5. Single-stage unequal probability sampling.

In the preceding sections we have been mainly concerned with sampling procedures in which all the units had an equal chance of selection. The only exception is stratified sampling (Section 4.1) in which strata allocations n_i proportional to the products $N_i x_i$ gave all the N_i units in the i^{th} size stratum an equal inclusion probability of $P_i = n(N_i x_i / \Sigma N_i x_i)$ which was varied from ratum to stratum. While unequal probability sampling by 'size strata' may be satisfactory for many practical purposes, situations often arise in which we desire to vary the inclusion probability from unit to unit. However, this type of unequal probability sampling mainly arises in the selection of primary sampling units in multi-stage sampling which we discuss in Section 6. Here we confine ourselves to the (rare) situations where unequal probability sampling is used in 'single-stage' or 'ultimate-stage' sampling of units which are not necessarily identifiable in advance of sampling.

As an example of p.p.s. sampling of this kind, we may mention here the sampling of farm operators in Iowa counties proportional to the land acreages they operate. If a county map can be covered by a rectangle with dimensions Z by W and $(z_i, w_i), i = 1,..,r(r \geq m)$ denote uniform variables with $0 \leq z_i \leq Z$ and $0 \leq w_i \leq W$, co-ordinates (z_i, w_i) can be pinpointed on the map and the interviewer can be instructed to ascertain (in order of draw) the first m operators whose land acreages contain the pinpointed land marks. This results in p.p.s. sampling with replacement in which $p_i = x_i/X$ (x_i = land

acreage of i^{th} operator only known for sampled operators, X = total land acreage known in advance of sampling) where p_i denotes the probability of selection of i^{th} operator at a single draw. This well-known situation of 'multinomial sampling' is the only one discussed in this section. We show that it can be reparametrized in such a way that optimality properties can be formulated for certain estimators.

Let $r_i = y_i/p_i$ and denote by r_t $(t = 1,..,T)$ the set of T discrete scale points feasible for the r_i. Let the score m_i denote the number of times i^{th} unit is included in the sample $(i = 1,..,N; \Sigma m_i = m)$. We now classify the r_i into the T groups and denote by

$$p_{it} = \begin{cases} p_i & \text{if for the } i^{th} \text{ unit } r_i = r_t \\ 0 & \text{otherwise} \end{cases}$$

$$y_{it} = \begin{cases} y_i & \text{if for the } i^{th} \text{ unit } r_i = r_t \\ 0 & \text{otherwise} \end{cases}$$

$$m_{it} = \begin{cases} m_i & \text{if for the } i^{th} \text{ unit } r_i = r_t \\ 0 & \text{otherwise} \end{cases} .$$

The multinomial distribution of scoring m multinomial scores into N classes with probabilities p_i may then be written in the form

$$L(p_{11},..,p_{NT}) = \frac{m!}{\prod_{i,t} m_{it}!} \prod_{i,t} p_{it}^{m_{it}} \tag{32}$$

and may be reparametrised as follows:

$$\left.\begin{aligned} p_t &= \sum_i p_{it} \\ \gamma_{it} &= \begin{cases} p_{it}/p_t & \text{if } p_t > 0 \\ 1 & \text{if } p_t = 0 \end{cases} \end{aligned}\right\} \tag{33}$$

so that

$$\sum_i \gamma_{it} = 1 \quad \text{if } p_t > 0. \tag{34}$$

Writing $m_t = \sum_i m_{it}$, we may factorize (32) as

$$L(p_{11},\ldots,p_{NT}) = \left[\frac{m!}{\prod_t m_t!} \prod_t p_t^{m_t}\right]\left[\frac{\prod_t m_t!}{\prod_{i,t} m_{it}!} \prod_{i,t} \gamma_{it}^{m_{it}}\right] . \tag{35}$$

Equation (35) shows that the m_t are sufficient for the p_t since the latter are only involved in the marginal distribution of the m_t and not in the conditional distribution of the m_{it} given the m_t.

The maximum likelihood estimators of the p_t are given by the ratios m_t/m and, hence, the maximum likelihood estimator of the population total

$$Y = \sum_i y_i = \sum\sum_{it} y_{it} = \sum_t r_t \sum_i p_{it} = \sum_t r_t p_t \tag{36}$$

is given by

$$\hat{Y} = \sum_t r_t \frac{m_t}{m} = \frac{1}{m} \sum_t r_t \sum_i m_{it} = \frac{1}{m} \sum_i \frac{y_i m_i}{p_i} \tag{37}$$

which is the customary unbiased estimator of Y in p.p.s. sampling with replacement.

Finally it should be noted that (35) is the likelihood for the scores which do not necessarily represent counts of distincts units in the population. However, it is possible to obtain the likelihood of the number of distinct units in the sample with scale ratio r_t which we denote by n_t. The distribution of the m_t is given by

$$L(p_1,\ldots,p_T) = \frac{m!}{\prod_t m_t!} \prod_t p_t^{m_t} \tag{38}$$

and the conditional distribution of the n_t given the m_t can be obtained in terms of the γ_{it} from formula (4.3) of Kullback (1937). Finally the likelihood of the n_t can be obtained by summation of the product (i.e., the joint distribution) over $m_t = n_t$ to m subject to $\Sigma m_t = m$. We intend to examine this distribution in more detail elsewhere.

Although only one single method of unequal probability sampling is examined in this section and although the method examined is known not to be particularly efficient, the discussion clearly indicates the possibility of deriving concrete likelihoods for other unequal probability sampling methods with the help of our technique of parametrisation.

6. Two-stage sampling.

In order to simplify the discussion we confine ourselves to two-stage sampling in which the primaries are selected with equal or unequal probabilities. Consider then a population consisting of L primary units $i = 1,..,L$ of which ℓ will be sampled and denote by N_i the number of secondary units in the i^{th} primary. Denote by N_{it} the number of units in the i^{th} primary which have the scale value $y_{it}(t = 1,..,T_i)$ so that $\Sigma_t N_{it} = N_i$. Let $u_i = 1$ if the i^{th} primary is in the sample and zero otherwise. Denote by $P(u_1,..,u_L)$ the joint distribution of the u_i corresponding to the primary sampling procedure adopted and let n_i denote the number of secondary units to be drawn from the i^{th} primary if it is in the sample. The n_i are all specified apriori for $i = 1,..,L$. In this paper we only consider equal probability sampling of secondaries without replacement.

If we denote by n_{it} the number of secondaries having scale value y_{it} in the i^{th} sampled primary, then the joint likelihood of the u_i and the

n_{it} is given by

$$L(N_{11},..,N_{LT_L}) = P(u_1..u_L) \prod_i \left[\prod_t \binom{N_{it}}{u_i n_{it}} \Big/ \binom{N_i}{u_i n_i} \right] . \qquad (39)$$

6.1. Maximum likelihood estimation.

We confine ourselves here to the case of $N_i = \infty$, $i = 1,..,L$. The likelihood (39) reduces to

$$L(p_{11},..,p_{LT_L}) = P(u_1,..,u_L) \prod_i \left[\frac{(n_i u_i)!}{\prod_t u_i n_{it}!} \prod_t p_{it}^{u_i n_{it}} \right] . \qquad (40)$$

Maximisation of (40) subject to $\sum_t p_{it} = 1$ for $i = 1,..,L$ leads to

$$\hat{p}_{st} = n_{st}/n_s \quad \text{(primary s in the sample)} \qquad (41)$$

while any values of p_{jt} are permissible for j not in the sample. The maximum likelihood solution will, therefore, in general not be unique. Furthermore, we do not have complete sufficiency here and, hence, no UMV estimator exists. We have not considered here 'scale-load' estimators which do not depend on primary labels.

6.2. Bayesian estimation.

Since the complete likelihood is given by (39), the posterior distribution of the N_{it} is identical to that in the case of stratified sampling (section 4) noting that $n_i = 0$ is allowed for the latter. Therefore, the 'Bayes estimator' of $\bar{Y}$ is given by (25) and it may be recast as

$$E'(\bar{Y}) = N^{-1} \Sigma_1 N_i \left[\left(1 - \frac{n_i}{N_i}\right) \sum_t \frac{n_{it} + \nu_{it}}{n_i + \nu_i} y_{it} + \frac{n_i}{N_i} \sum_t \frac{\nu_{it}}{\nu_i} y_{it} \right]$$

$$+ N^{-1} \Sigma_2 N_i \sum_t \frac{\nu_{it}}{\nu_i} y_{it} \qquad (42)$$

where Σ_1 and Σ_2 respectively the summations over sampled and non-sampled primaries. It should be noted that we must have a prior distribution from each primary. If the prior distribution is solely based on pilot samples, this implies that the pilot sample must include at least one secondary unit from each primary.

The above analysis clearly shows that the sampling procedure adopted for selection of the primaries is entirely irrelevant as far as a full Bayesian analysis is concerned. However, if the likelihood based on a selected estimator is used for a (partial) Bayesian analysis based on insufficient statistics, then the posterior distribution and, hence, the 'Bayes estimator' would depend on the sampling procedure. These are the two alternatives available to the Bayesian analyst and, at this stage, we do not wish to take sides in this issue.

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