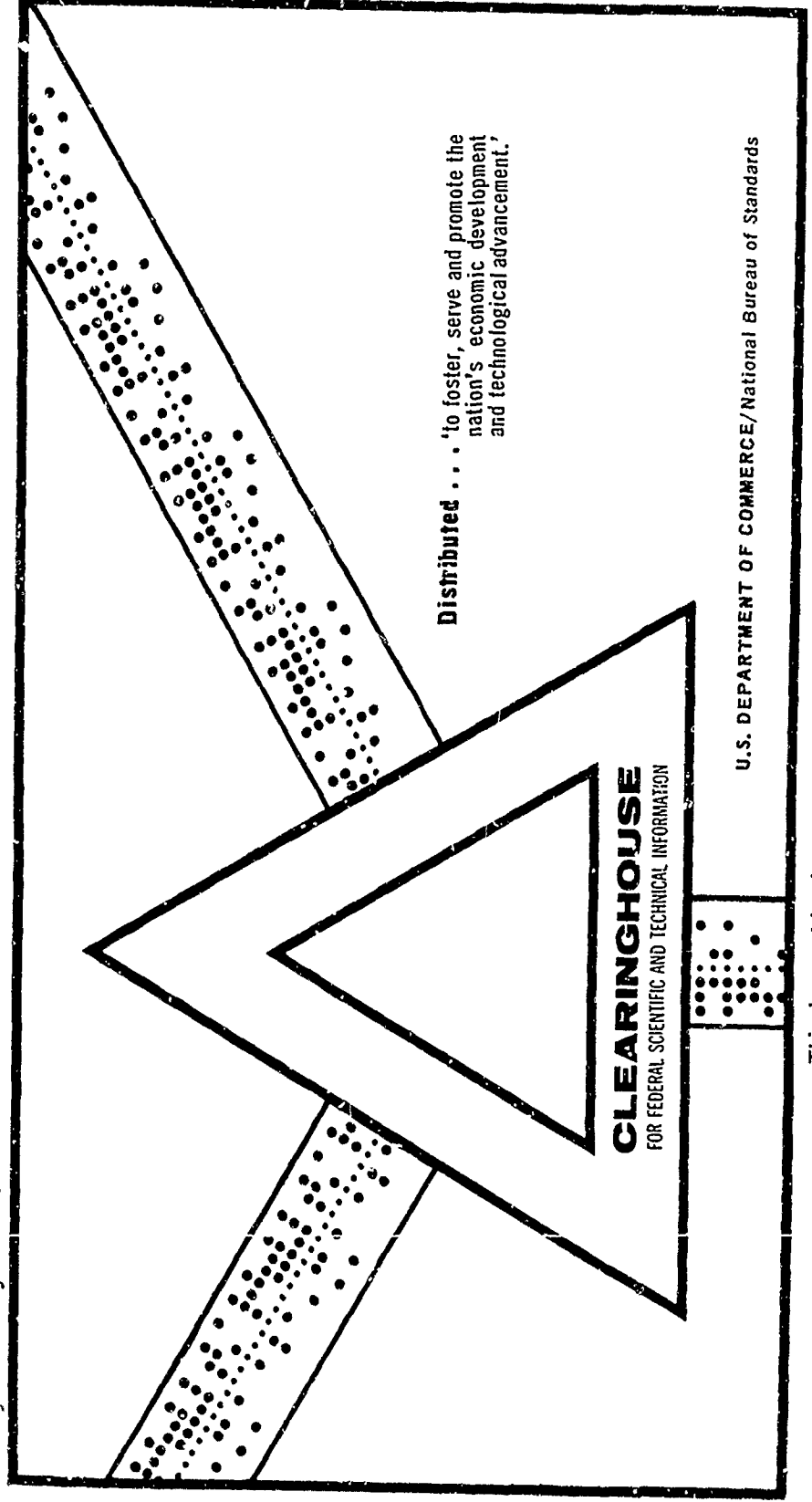AD 699 411

INVESTIGATION OF GROUND SHOCK EFFECTS IN NONLINEAR HYSTERETIC MEDIA: REPORT 3. A NOTE ON THE PLANE WAVES OF PRESSURE AND SHEAR IN A HALF-SPACE OF HYSTERETIC MATERIAL

Alva Matthews, et al

Weidlinger (Paul)
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INVESTIGATION OF GROUND SHOCK EFFECTS IN
NONLINEAR HYSTERETIC MEDIA

Report 3

A NOTE ON THE PLANE WAVES OF PRESSURE AND SHEAR

IN A HALF-SPACE OF HYSTERETIC MATERIAL

by

Alva Matthews with the cooperation of Hans H. Bleich

January 1969

FOREWORD

The current work on variable moduli models is part of Contract No. DACA-39-67-C-0048, "Investigation of Ground Shock Effects in Nonlinear Hysteretic Media," being conducted for the U. S. Army Engineer Waterways Experiment Station (WES) under DASA sponsorship.

The contract is being monitored by Dr. J. S. Zelasko, Impulse Loads Section, Soil Dynamics Branch, under the general supervision of Mr. A. A. Maxwell, Acting Chief, Soils Division, WES. COL Levi A. Brown, CE, is the contracting officer.

ABSTRACT

The problem of one dimensional plane wave propagation is considered in isotropic, nonlinear, hysteretic materials. Solutions for surface loads of step pressure and/or shear, in cases of loading or unloading are studied, and the presence of shocks and regions of continuous stress change are discussed.

The mathematical models used are a generalization of conventional elastic models. The moduli K are assumed to be functions of the stress invariant J_1 , and the moduli G are functions of both J_1 and the stress deviators. Different expressions for K and G are used during loading and unloading, leading to energy loss through hysteresis.

A NOTE ON THE PLANE WAVES OF PRESSURE AND SHEAR
IN A HALF-SPACE OF HYSTERETIC MATERIAL

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LIST OF SYMBOLS[*)]

G, G_{LD}, G_{UN}	Shear modulus, shear moduli in loading and unloading.
G_o, K_o	Elastic shear modulus, elastic bulk modulus.
g	Nondimensional shear modulus.
$g_{1...7}$	Coefficients used in variable shear moduli, Eqs. (24) and (25).
J_1, J_2	Stress invariants.
K, K_{LD}, K_{UN}	Bulk modulus, bulk moduli in loading and unloading.
k	Nondimensional bulk modulus.
$k_{o...4}$	Coefficients used in variable bulk moduli, Eqs. (22) and (23).
n	Fractional coefficient.
s_{ij}, s_x, s_y, s_z	Stress deviators.
t	Time.
U	Nondimensional dependent variable defined by Eq. (14).
$\dot{u}$, $\dot{v}$, $\dot{w}$	Velocity components in x, y, z direction, respectively.
x, y, z	Cartesian coordinates.
ε_{kk}	Volumetric strain.
e_{ij}	Deviatoric strains.
ρ	Mass density.
σ_x, σ_y, σ_z	Normal stress components.
τ	Shear stress.
$\cdot$	Differentiation with respect to time.
$'$	Differentiation with respect to U.

[*)]   Other symbols are defined as they appear in the text.

I   INTRODUCTION.

The following is a study of one dimensional wave propagation of pressure and/or shear for isotropic materials with nonlinear properties resulting in hysteretic effects. The one dimensional plane case is considered because it permits easy mathematical treatment leading to an understanding of the behavior in multi-dimensional situations. In addition, it can be used as a check on numerical schemes and gives limiting solutions for two dimensional situations at high Mach numbers. Solutions for the one dimensional problems for the step loads studied exist for all input combinations of pressure and shear, and are unique, giving at least a hint that the artificially constructed des-cription of the material does not lead to absurdities.

The mathematical models used are a generalization of con-ventional elastic models, the elastic moduli K being now functions of the first stress invariant J_1 , while the moduli G are functions of the stress invariants J_1 and J_2 . The functions K and G differ appropriately during loading and unloading, leading to energy loss through hysteresis. Following Ref. [1] the governing volumetric relations for initial loading are

$$\dot{J}_1 = 3K_{LD}(J_1) \, \dot{\epsilon}_{kk} \tag{1}$$

while for unloading and reloading

$$\dot{J}_1 = 3K_{UN}(J_1) \, \dot{\epsilon}_{kk} \tag{2}$$

where

$$K_{UN} \geq K_{LD} > 0 \qquad (3)$$

It is noted that Eq. (1) permits determination of J_1 as a function of ε_{kk}, so that the equation can be written in the alternative form $\dot{J}_1 = 3K_{LD}(\varepsilon_{kk}) \dot{\varepsilon}_{kk}$ without change of meaning. The alternative form is used for computational convenience in Ref. [1] because it results in simple expressions for the velocity of compressive shocks.

The deviatoric relations are, for loading and reloading[*],

$$\dot{s}_{ij} = 2G_{LD}(J_1, J_2)\dot{e}_{ij} \qquad (4)$$

and for unloading

$$\dot{s}_{ij} = 2G_{UN}(J_1, J_2)\dot{e}_{ij} \qquad (5)$$

where

$$G_{UN} \geq G_{LD} \geq 0 \qquad (6)$$

The choice of shear modulus is based on the sign of $\dot{J}_2$, G_{LD} being used for $\dot{J}_2 > 0$, while G_{UN} is used for $\dot{J}_2 \leq 0$.

Note that this paper uses tensile stresses and strains as positive while Ref. [1] uses compressive stresses and strains as positive.

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[*] Whether G_{LD} or G_{UN} is appropriate to use for the modulus of reloading is a question still undergoing discussion and investigation. This paper assumes tentatively that G_{LD} is appropriate.

II FORMULATION OF THE BASIC EQUATIONS.

To investigate the propagation of plane waves of either pressure and/or shear in the material described by Eqs. (1) to (6), consider the half-space shown in Fig. 1, subjected to a step load consisting of surface stresses $\bar{\sigma}_{xo}$ and/or $\bar{\tau}_{o}$. Let x, y, z be Cartesian coordinates, x in the direction of propagation of the disturbances, while y and the shear stress $\bar{\tau}$ are in the plane of the page. The symbols $\dot{u}$, $\dot{v}$, $\dot{w}$ represent the x, y, z components of the velocity, respectively. The premise of plane waves requires $\dot{w} = 0$, and that all derivatives with respect to y and z vanish.

Designating stress quantities prior to the introduction of nondimensional variables by $\bar{J}_1$, $\bar{s}_x$, etc., and considering the strains to be small, the constitutive relations have the same appearance as in conventional linear elasticity,

$$\frac{1}{9K} \dot{\bar{J}}_1 + \frac{1}{2G} \dot{\bar{s}}_x = \frac{\partial \dot{u}}{\partial x} \tag{7}$$

$$\frac{1}{9K} \dot{\bar{J}}_1 + \frac{1}{2G} \dot{\bar{s}}_y = 0 \tag{8}$$

$$\frac{1}{9K} \dot{\bar{J}}_1 + \frac{1}{2G} \dot{\bar{s}}_z = 0 \tag{9}$$

$$\frac{1}{2G} \dot{\bar{\tau}} = \frac{1}{2} \frac{\partial \dot{v}}{\partial x} \tag{10}$$

but K and G are functions of the stress invariants. Noting $\dot{\bar{s}}_x + \dot{\bar{s}}_y + \dot{\bar{s}}_z = 0$, Eqs. (7) to (9) combined give

$$\dot{\bar{J}}_1 = 3K \frac{\partial \dot{u}}{\partial x} \tag{11}$$

The equations of motion are

$$\frac{\partial \bar{\tau}}{\partial x} = \rho \frac{\partial \dot{v}}{\partial t} \tag{12}$$

$$\frac{1}{3} \frac{\partial \bar{J}_1}{\partial x} + \frac{\partial \bar{s}_x}{\partial x} = \rho \frac{\partial \dot{u}}{\partial t} \tag{13}$$

The five differential equations (7) and (10) to (13) in the five unknowns $\bar{s}_x$, $\bar{J}_1$, $\bar{\tau}$, $\dot{u}$ and $\dot{v}$ define the problem. The two equations, (8) and (9), serve solely for the determination of s_y and s_z, $s_y = s_z = -\frac{s_x}{2}$, a trivial relation which will not be carried along.

Following a procedure used in a previous investigation of plane waves in elasto-plastic materials, Ref. [2], dimensional considerations require that all stresses and velocities depend on a nondimensional combination of parameters and of the variables x and t,

$$U = \sqrt{\rho/G_o} \ \frac{x}{t} \tag{14}$$

Introduction of the variable U is conveniently combined with the definition of the nondimensional quantities

$$J_1 = \frac{\bar{J}_1}{G_o} \quad , \quad s_x = \frac{\bar{s}_x}{G_o} \quad , \quad \tau = \frac{\bar{\tau}}{G_o} \tag{15}$$

$$g = \frac{G(J_1, J_2)}{G_o} \quad , \quad k = \frac{K(J_1)}{G_o} \tag{16}$$

Introducing Eqs. (14) to (16) into Eqs. (7) and (10) to
(13) yields, after elimination of the velocities $\dot{u}$ and $\dot{v}$,

$$(U^2 - g) \tau' = 0 \qquad (17)$$

$$\frac{4g}{9k} J_1' - s_x' = 0 \qquad (18)$$

$$(\frac{1}{3} - \frac{U^2}{3k}) J_1' + s_x' = 0 \qquad (19)$$

where primes indicate differentiation with respect to U.
These three differential equations apply in regions of con-
tinuous solution, i.e., provided the derivatives are finite.
The differential equations are linear with respect to the
first derivatives J_1' , etc., and homogeneous. They thus
permit only the solution $J_1' = s_x' = \tau' = 0$, with J_1 = constant,
s_x = constant, τ = constant, unless the determinant of the
coefficients of Eqs. (17) to (19) vanishes. Noting that
Eq. (17) depends on τ, but not on the two other variables,
while Eqs. (18) and (19) do not contain τ, the respective
subdeterminants may be considered separately.

Equation (17) leads to the conclusion that τ must be
constant in any region, unless

$$U^2 - g = 0 \qquad (20)$$

Equations (18) and (19) indicate that s_x and J_1 will
be constants unless, in the region considered, the determinant
of the coefficients vanishes.

$$3U^2 - 3k - 4g = 0 \qquad (21)$$

While it will be seen in the applications that this does
not actually occur, the possibility of simultaneous variation
of Eqs. (20) and (21) must be allowed for.

The continuous solutions of the differential equations
(17) to (19) do not completely describe the situations which
may be encountered. There may be locations of discontinuities
(shocks) which must be obtained from other considerations. It
is known that materials with a hardening pressure-volume
relation permit (pressure) shocks. For the materials considered
in this study the velocity of such shocks may be obtained by
integrating the stress-strain relations in uniaxial strain.
In principle, shear shocks are also possible, but such shocks
do not occur for the dependency of G as a function of J_2 con-
sidered here, because the material in this respect is a
"softening one".

The next section will consider specific situations for
the material designated Type II Variable Modulus Material,
Ref. [3]. The pertinent point lies not in the details of the
solution, but in the fact that there is always a solution,
and that there is just one solution even if the possibility
of shocks is included.

III SOLUTIONS FOR SPECIAL CASES.

Typical suitable expressions for the moduli of soils have been obtained for uniaxial and triaxial test results of the type represented by Figs. 2 and 3, respectively. The expressions proposed are of the following form.

For initial compressive loading

$$k_{LD} = k_o + k_1 \varepsilon_{kk} + k_2 \varepsilon_{kk}^2 \qquad (22)$$

For unloading and reloading

$$k_{UN} = k_3 - k_4 J_1 > k_{LD} \qquad (23)$$

For initial deviatoric loading and reloading

$$g_{LD} = 1 - g_1 \left| \sqrt{J_2} \right| - g_2 J_1 - g_3 J_1^2 \qquad (24)$$

For deviatoric unloading

$$g_{UN} = g_7 + g_4 \left| \sqrt{J_2} \right| - g_5 J_1 - g_6 J_1^2 > g_{LD} \qquad (25)$$

All constants k_i and g_i are positive. Further, the above expressions apply for $J_1 < 0$, $\varepsilon_{kk} < 0$ and in stress ranges where all expressions, Eqs. (22)-(25) remain positive. For the plane case to be studied the value of J_2 becomes simply

$$J_2 = \frac{3}{4} s_x^2 + \tau^2 \qquad (26)$$

while J_1 remains

$$J_1 = \sigma_x + \sigma_y + \sigma_z \qquad (27)$$

A number of special problems will now be considered.

Case A - <u>Shear waves, initial loading</u>.

At the time $t = 0$, a stressless half-space, Fig. 1, is subjected to a load τ_o, while $\sigma_{xo} \equiv 0$. Noting that Eq. (14) gives $U = 0$ when $x = 0$, the boundary condition on the surface is

$$\tau(0) = \tau_o \qquad (28)$$

Further, for large values of $x \to \infty$, $U \to \infty$,

$$\tau(\infty) = 0 \qquad (29)$$

If a continuous solution exists, Eq. (20),

$$U^2 = g_{LD} \qquad (30)$$

must hold. Noting that $J_1 = 0$, $s_x = 0$, Eqs. (24) and (26) give

$$U^2 = 1 - g_1 |\tau| \qquad (31)$$

This simple relation defines the variation of τ as a function of U in any region where τ is not a constant. However, regions where τ is simply constant must also be included in the construction of a solution. It is important that Eq. (31) indicates that if $|\tau|$ is not a constant it will <u>increase</u> towards the surface where U is smaller. Equation (31) is therefore suitable for the particular case of loading. Provided $g_1 |\tau| < 1$ there will be a point, $U_1 = 1$, where the shear stress begins to increase, see Fig. 4a. Equation (31)

describes the solution until the point U_2 , defined by

$$U_2^2 = 1 - g_1 \left| \tau_o \right| \tag{32}$$

is reached. For $U < U_2$ the stress τ simply remains constant, so that the solution shown typically in Fig. 4a is obtained.

The relation Eq. (31) being monotonic, it is impossible to find an alternate continuous solution to satisfy the boundary condition[*)].

No solutions containing a discontinuity can be obtained because the weakening character of the stress-strain relationship in shear, Fig. 3, does not permit discontinuities in shear.

Case B - Shear waves, unloading.

Let the half-space be initially uniformly stressed by a shear stress $\tau = \tau_o$. At the instant $t = 0$ the surface stress is removed. The boundary conditions are, in this case,

$$\left. \begin{array}{l} \tau(0) = 0 \\[2mm] \tau(\infty) = \tau_o \end{array} \right\} \tag{33}$$

To investigate continuous solutions the reasoning of Case A applies again, but the expression g_{UN} of Eq. (25) must now be used, so that

$$U^2 = g_7 + g_4 \left| \tau \right| \tag{34}$$

[*)]   It is also not possible to construct alternative, inappropriate solutions of the type discussed in Case F.

g_4 and g_7 being positive. The relation indicates that $|\tau|$ increases with U, as required for an unloading case. Figure 4b shows the solution to be $\tau = \tau_0$ for values $U > U_1$, where

$$U_1^2 \geq g_7 + g_4 \tau_0 \qquad (35)$$

The value $\tau = 0$ is reached for $U_2 = \sqrt{g_7}$ and τ remains at this value for $U < U_2$.

No alternative[*] solution involving a shear shock is possible, because of the "weakening" stress-strain relation for unloading in shear shown in Fig. 3.

If a material has a stress-strain relation for unloading in which the curvature shown in Fig. 3 is reversed, a shear shock is possible and would give a solution to the problem. However, in this case the sign of g_4 in Eq. (25) would be negative and there would be no continuous solution possible. The uniqueness and existence requirement would still be satisfied.

Case C - <u>Pressure waves, unloading</u>.

Consider the half-space subject to a uniform state of stress defined by the given value $J_{1o} < 0$ and $s_{xo} < 0$, so that the normal surface stress is $\sigma_{xo} = s_{xo} + \frac{1}{3} J_{1o} < 0$. At the time $t = 0$ the surface load is reduced to a fraction, say $n\sigma_{xo}$. The boundary conditions in terms of the independent

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[*]   It is also not possible to construct alternative, inappropriate solutions of the type discussed in Case F.

variable U are

$$s_x(0) + \frac{1}{3} J_1(0) = n\sigma_{xo} \tag{36}$$

$$s_x(\infty) = s_{xo} < 0 \tag{37}$$

$$J_1(\infty) = J_{1o} < 0 \tag{38}$$

The determination of continuous solutions is more complicated than in Cases A and B, because the problem concerns now two unknown functions J_1 and s_x , while the earlier cases concerned only one unknown. As previously stated the differential equations (18), (19) permit no other solution than J_1 = constant, s_x = constant and thus σ_x = constant, unless the determinantal equation (21) is satisfied. If this is the case, Eqs. (18) and (19) are inherently equivalent and only one of them need be retained. It is convenient here to select Eq. (18)

$$\frac{4g}{9k} J_1' - s_x' = 0 \tag{39}$$

which relation is to be solved in conjunction with Eq. (21).

Before considering specific cases some general conclusions can be drawn from Eq. (39). The quantities g and k are inherently positive, so that J_1' and s_x' must necessarily have the same sign. In the problem of unloading considered here it is therefore inherent that negative values of J_1' and s_x' will represent unloading as long as $s_x < 0$, so that the unloading relations, Eqs. (23) and (25), for k and g are appropriate[*].

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[*]   If n is sufficiently small, s_x will change in sign and Eq. (24) will be appropriate. For simplicity, this case is not pursued here.

In order to find out if a continuous solution applies in the present situation, use is made of the fact, stated in the previous paragraph, that J_1' and s_x' have the same sign. In the case of unloading from negative (compressive) values

$$J_1' < 0 \qquad s_x' < 0 \qquad (40)$$

In addition, unloading implies that the derivatives of the absolute values satisfy

$$\left| J_1 \right|' > 0 \qquad \left| s_x \right|' > 0 \qquad (41)$$

The above inequalities can now be used to see if Eq. (21) permits a continuous solution. Taking the derivative of this equation with respect to U one finds the condition

$$\delta U = 3k' + 4g' > 0 \qquad (42)$$

The inequality follows from the fact that U is inherently positive. From Eq. (23)

$$\frac{d}{dU} (k_{UN}) = -k_4 J_1' > 0 \qquad (43)$$

Eq. (25) gives, for $\tau \equiv 0$, the relation $g_{UN} = g_7 + \frac{\sqrt{3}}{2} g_4 \left| s_x \right| - g_5 J_1 - g_6 J_1^2$, so that

$$\frac{d}{dU} (g_{UN}) = \frac{\sqrt{3}}{2} g_4 \left| s_x \right|' - g_5 J_1' - 2 g_6 J_1 J_1' \qquad (44)$$

While the first two terms are positive the last one is not.
Combining the last three equations the continuous solution
will apply if

$$2\sqrt{3}\, g_4 \left| s_x \right|' - J_1' \,(3k_4 + 4g_5 + 8g_6 J_1) > 0 \qquad (45)$$

The first term is inherently positive. The second term is
positive if the expression in parentheses is positive, but
as this depends on the relative magnitude of $8g_6 J_1 < 0$
compared to $3k_4 + 4g_5$ no general statement can be made.
The above relation investigates the situation on a purely
mathematical basis. The mathematical condition, Eq. (42),
however, expresses simply the fact that the unloading stress-
strain diagram in Fig. 2 has proper curvature as shown. If,
therefore, the value of the material constants give a curve
of this type the mathematical condition, Eq. (45), is in-
herently satisfied and need not be investigated. If the
stress-strain diagram is of this type it is clear, again on
physical grounds, that no (unloading) shock may exist and
the continuous solution of Eqs. (39) and (42) will be the
only solution[*].

To obtain specific solutions when the general relations,
Eqs. (23) and (25), are used, requires numerical solution of
the two Eqs. (39) and (42). However, the principle may be
demonstrated in the special case $g_5 = g_6 = 0$ where a closed
solution can be obtained. In this case g is a function of s_x
only, while k is a function of J_1 only so that Eq. ('9) can be

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[*] Continuous, but inappropriate solutions of the type dis-
cussed in Case F may occur.

integrated. With $b = -2\sqrt{3} \; g_4/9k_4$

$$g(s_x) = [Ck(J_1)]^b \tag{46}$$

where C is an open constant. This relation must apply at the head of the unloading wave $U = U_1$ where $s_x = s_{xo}$ and $J_1 = J_{1o}$, which permits the determination of the constant C

$$C = \frac{[g(s_{xo})]^{1/b}}{k(J_{1o})} \tag{47}$$

Substitution of this relation into the determinantal equation gives

$$3U^2 = 3k(J_1) + 4g(s_{xo}) \left[\frac{k(J_1)}{k(J_{1o})} \right]^b \tag{48}$$

Equation (46) may then be used to determine s_x

$$s_x = \frac{2}{\sqrt{3} \; g_4} \left\{ g(s_{xo}) \left[\frac{k(J_1)}{k(J_{1o})} \right]^b - g_7 \right\} \tag{49}$$

Knowledge of the relation between U and J_1 and s_x and J_1 permits a numerical determination of σ_x as a function of U in the region U_1 to U_2 where σ_x varies. The typical shape of this stress profile is shown in Fig. 6a.

Case D - Pressure Waves, Loading.

Consider the case of a stressless half-space when at the time $t = 0$ a normal surface pressure $\sigma_{xo} < 0$ is applied, while the shear stress vanishes, $\tau = 0$. In terms of the variable U

the boundary conditions are

$$\sigma_x(0) = \sigma_{xo} \tag{50}$$

$$\left.\begin{array}{l} J_1(\infty) = 0 \\[2mm] s_x(\infty) = 0 \end{array}\right\} \tag{51}$$

The procedure for the determination of continuous solutions, if any, is in principle the same as in Case C, except that the expressions g_{LD} and k_{LD} have to be used. In regions of continuous solution Eqs. (18), (19) and the determinantal equation

$$3U^2 = 4g + 3k \tag{52}$$

must hold.

The possibility of continuous solutions for the present case exists only if the Eq. (52) indicates that U decreases when $J_1^{'} < 0$, $s_x^{'} < 0$, while $J_1 < 0$, $s_x < 0$. Substitution of Eqs. (22) and (24) shows that the term k in Eq. (52) has exactly the opposite behavior, while the second and fourth terms of g in Eq. (24) change in the direction required for continuous solutions. For high stress levels the behavior of k controls, but no full statement can be made without introducing specific numbers.

As an alternative to a computational investigation on the basis of Eqs. (22) and (24) qualitative predictions can be made based on the character of the uniaxial stress-strain

diagram implied in these equations. The latter can be used
to find a uniaxial stress-strain diagram which is an approxi-
mation of the actual diagram, Fig. 2. This approximate
diagram, Fig. 5, ought to, and will usually have the same
character as Fig. 2, i.e., it will show a weakening of the
material at low stress ranges up to point C, a hardening
thereafter. The nature of the pressure waves for this
situation is well known, qualitatively. Numerical details
for specific values of the coefficients, Eqs. (22), (24)
could be obtained from Eqs. (18), (19) and (52).

a. If the value of $\left|\sigma_{xo}\right|$ is less than $\left|\sigma_c\right|$ in Fig. 5
the solution is continuous, changing between
points U_1 and U_2, .ig. 6b, while the stress is
constant $\sigma_x = \sigma_{xo}$ for $U \leq U_2$.

b. If the value of $\left|\sigma_{xo}\right|$, $\left|\sigma_A\right|$, is larger than $\left|\sigma_T\right|$ in
Fig. 5, the result will be a shock, Fig. 6c. The
velocity is defined by the slope of the line OA in
Fig. 5. The limiting value $\left|\sigma_T\right|$ is defined by the
statement that OT in Fig. 5 is tangent to the stress-
strain diagram at 0.

c. For intermediate stress levels $\left|\sigma_c\right| < \left|\sigma\right| < \left|\sigma_T\right|$
there will be a continuous precursor, followed by
a shock. The stress level up to which the solution
is continuous is defined by point D in Fig. 5, while
the shock velocity is defined by the slope of the
line BD. The result has the character of Fig. 6d.

Case E - Combined Pressure and Shear.

The situation when the applied surface load is a combination of pressure and shear can be discussed in a general way, and reduced to a combination of the solutions for pressure and shear alone.

In the case now considered where σ_x and τ must change, all three differential equations (17), (18) and (19) must be considered in regions where the solution changes continuously. However, the earlier conclusions still hold that τ can only change if the determinant, Eq. (20), vanishes, while J_1, σ_x and thus σ_x can only change if Eq. (21) is satisfied. Potential solutions can therefore be found in which σ_x changes only say from U_1 to U_2, while τ changes only for U_3 to U_4, the determination of the distribution remains as outlined in Cases A to D.

There is, however, also the additional potential possibility that Eqs. (20) and (21)

$$U^2 = g \tag{53}$$

$$U^2 = \frac{4}{3} g + k \tag{54}$$

are satisfied simultaneously in a region, U_5 to U_6. Noting that g and k are inherently positive, regardless whether the expressions g and k for loading, unloading or reloading are used, the two conditions can not be simultaneously satisfied.

There can thus be no region where all three quantities J_1 , s_x and τ change simultaneously. The pieces for construction of the solution are thus regions of uniform stress, regions where τ only changes, regions where J_1 , s_x and thus σ_x change, and compressive shocks.

The sequence of the combination of solutions is essentially the same for all situations, and only one case needs be discussed. Consider a half-space in which at $t < 0$, the stresses are $J_1 = J_{1o} < 0$, $s_x = s_{xo} < 0$, $\tau = \tau_o$ everywhere. At the instant $t = 0$ the surface loads are removed. The boundary conditions for $t > 0$ are then

$$
\left.
\begin{aligned}
\sigma_x(0) &= 0 \\
\tau(0) &= 0
\end{aligned}
\right\}
\quad (55)
$$

$$
\left.
\begin{aligned}
J_1(\infty) &= J_{1o} \\
s_x(\infty) &= s_{xo} \\
\tau(\infty) &= \tau_o
\end{aligned}
\right\}
\quad (56)
$$

Due to the fact that in the region of change of τ only J_2 changes while J_1 remains unaffected, the construction of a solution can be started with the region of compressive change U_1 , $U_2 > U_3$, U_4 . This solution is simply the solution to the problem of pressure relief, when the stress τ_o on the surface would be maintained, i.e., Case C treated before. Let the values of J_1 and s_x at point U_2 found in this manner be J_{12} and s_{x2} .

The changes in stress which will occur between U_3 and $U_4 < U_2$, are then found for the secondary problem

$$\sigma_x(0) = \tau(0) = 0 \tag{57}$$

$$\left.\begin{array}{l} J_1(\infty) = J_{12} \\[2mm] J_2(\infty) = \frac{3}{4} s_{x2}^2 + \tau_o^2 \\[2mm] \tau(\infty) = \tau_o \end{array}\right\} \tag{58}$$

The typical history of the stress is shown in Fig. 7.

It is not possible to start the solution by determination of the region of change in shear, and find the pressure change later. The reason is that the solution must be begun from a boundary where all quantities J_1 , s_x , τ are known. This is only true for $U = \infty$, Eqs. (56). At $U = 0$ only the value of $\sigma_x = \frac{1}{3} J_1 + s_x$ is known, not J_1 and s_x separately.

Other cases lead to appropriate combinations of the solutions shown in Figs. 4 and 6, which include pressure shocks.

The reduction of the problem of combined pressure and shear to a combination of the individual cases implies that uniqueness and existence found in Cases A-D, automatically apply here.

Case F - <u>Elimination of Multiple Solutions</u>.

In footnotes in Cases A, B and C mention was made of additional "inappropriate" continuous solutions. The following will give a typical example and show that such solutions, while mathematically correct, should be dismissed on physical grounds. This dismissal is based on the requirement that any solution containing a singularity, in this case a step discontinuity in the load at $x=t=0$, must be obtainable by a limiting process from a rapidly but monotonically changing surface load.

Consider a variation of Case B of unloading in shear, where the shear stress on the surface is not completely removed, but reduced from τ_o to $n\tau_o$, where $0 < n < 1$. The boundary conditions are then

$$\left.\begin{array}{l} \tau(0) \;=\; n\tau_o \\[2mm] \tau(\infty) \;=\; \tau_o \end{array}\right\} \tag{59}$$

These boundary conditions can be satisfied by using Eq. (34) for unloading from τ_o to $n\tau_o$, resulting in the monotonic stress history ABC shown in Fig. 8a. However, it is also possible to use Eq. (34) beyond point B reducing the shear stress to a lower level, $n_1\tau_o < n\tau_o$, point B_1 in Fig. 8b. The surface boundary condition can still be satisfied by obtaining from Eq. (31) a continuous solution where τ rises to the required value $n\tau_o$ as indicated in Fig. 8b, points C_1C.

Points CC_1 for a continuous solution with a rise in stress are in the required location, closer to the surface than point B_1 because $g_{LD} < g_{UN}$ applies for reloading.

In the alternative solution the time history of the stress at any point is not monotonic. Without going into details, the concept of characteristics requires that the solution to the problem of a rapid but gradual change of the surface stress from τ_o to $n\tau_o$ would result in a monotonic stress history. The solution shown in Fig. 8b is therefore to be dismissed. If equivalent nonmonotonic results can be constructed for other problems they are similarly inappropriate.

IV   <u>CONCLUSIONS</u>.

The investigation demonstrated that in the simple situ-
ations considered the use of the hysteretic stress-strain
relations proposed in Ref. [3] does not lead to difficulties,
i.e., solutions exist in all cases and are unique. In some
cases uniqueness is obtained only by using an additional
physical consideration outlined in Section III, Case F. It
leads to the conclusion that solutions for the problem stated
here are admissable only if the stress changes are monotonic.

In the case of elastic-plastic materials a similar study
of plane wave propagation was extended to the case of the
superseismic steady-state solution for a step load progressing
on a half-space. It is expected that the present work can
similarly be extended, thereby providing solutions suitable
to check purely numerical approaches.

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$$\overline{\sigma}_{xo} = G_o \, \sigma_{xo}$$

$$\tau_o \equiv G_o \, \tau_o$$

FIG. 1

$-\sigma_x$

$-\epsilon_x$

FIG. 2

$-s_x$

$-\widetilde{e}_x$
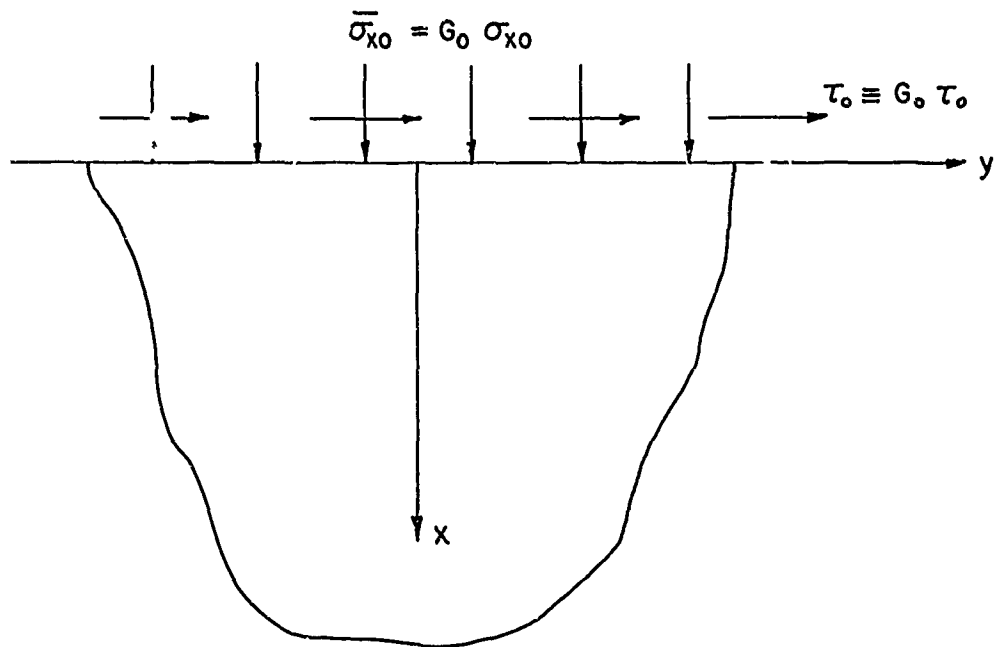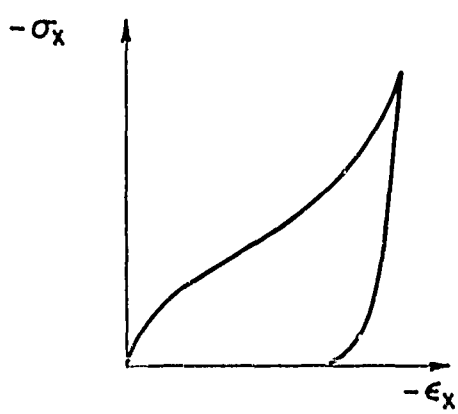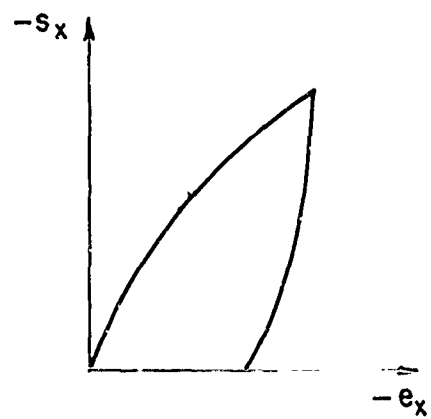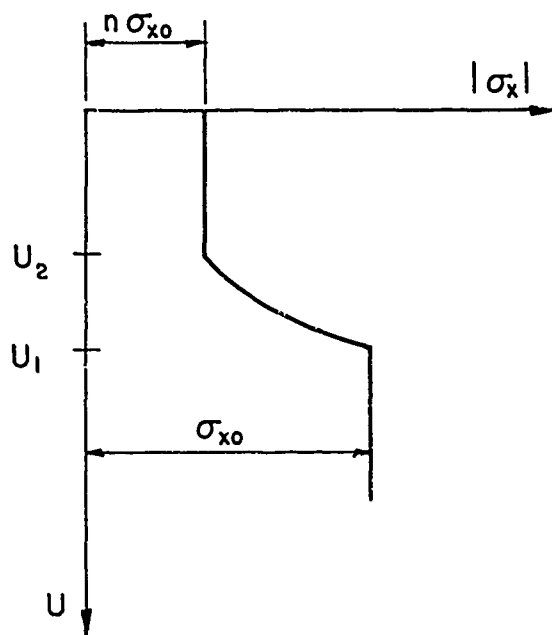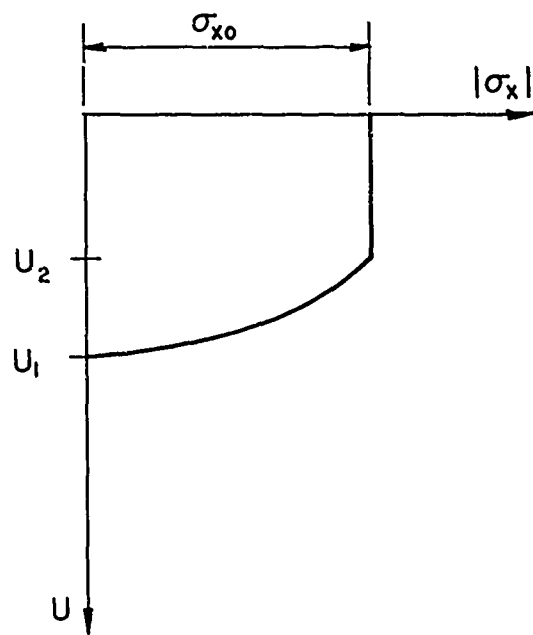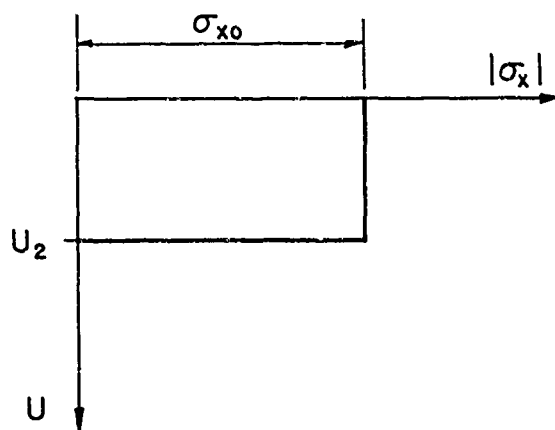
FIG. 3
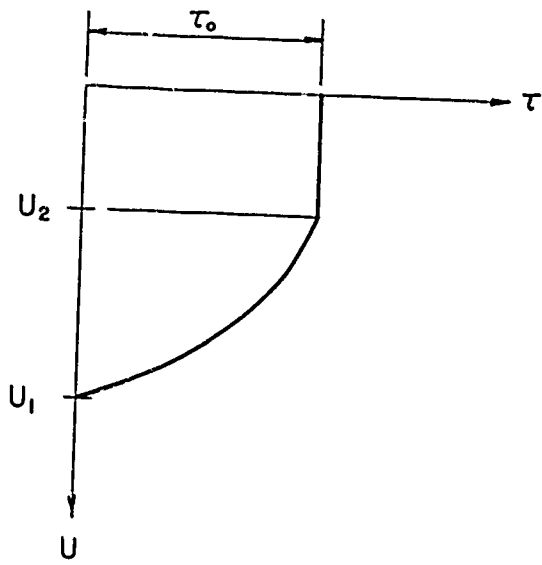
SHEAR LOADING

FIG. 4 a
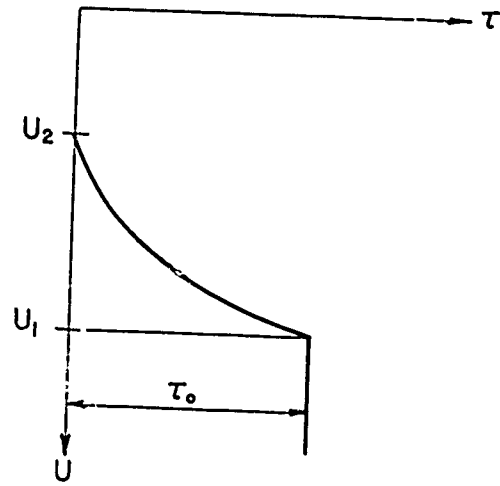
SHEAR UNLOADING

FIG. 4b
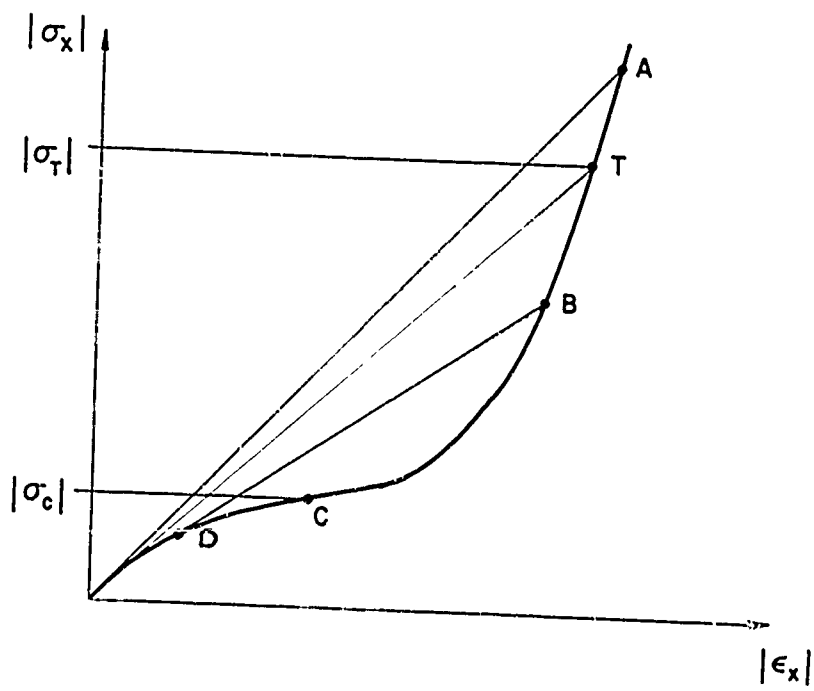
FIG. 5

FIG. 6a

FIG. 6b

FIG. 6c
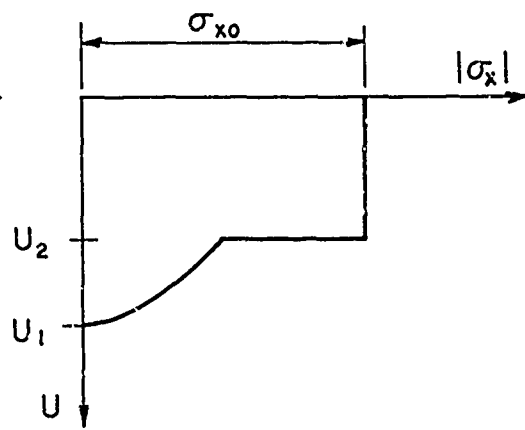
FIG. 6d

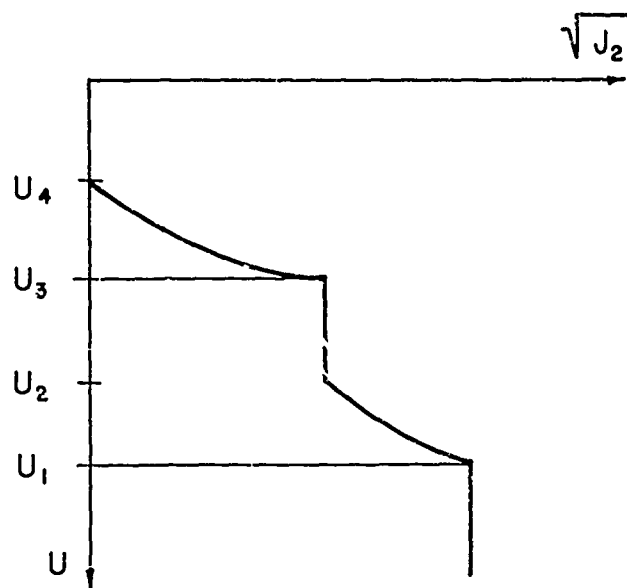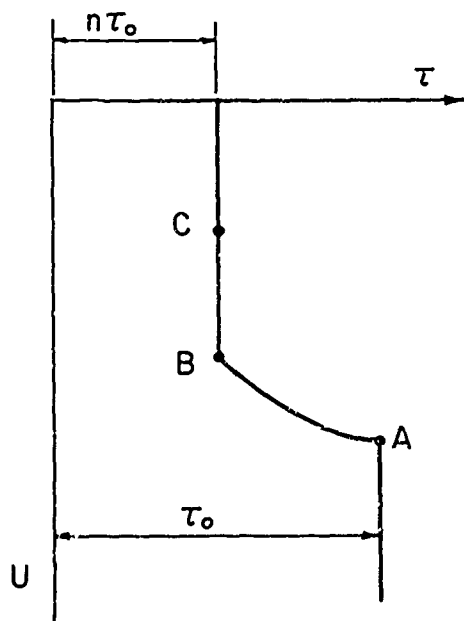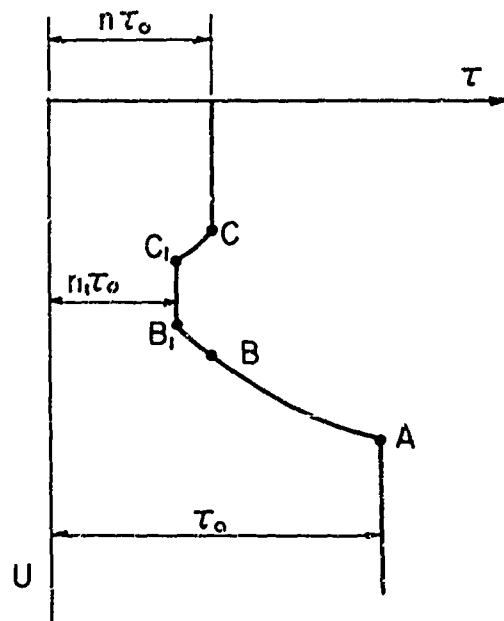FIG. 7

FIG. 8a

FIG. 8b

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13. ABSTRACT

The problem of one dimensional plane wave propagation is considered in isotropic, nonlinear, hysteretic materials. Solutions for surface loads of step pressure and/or shear, in cases of loading or unloading are studied, and the presence of shocks and regions of continuous stress change are discussed. The mathematical models used are a generalization of conventional elastic models. The moduli $-K$ are assumed to be functions of the stress invariant J_1, and the moduli G are functions of both J_1 and the stress deviators. Different expressions for K and G are used during loading and unloading, leading to energy loss through hysteresis.

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14. KEY WORDS	LINK A		LINK B		LINK C	
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Hysteretic materials						
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