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The Turbulent Boundary Layer With Mass Transfer and Pressure Gradient

by

Stephen C. Lubard and Fernando L. Fernandez

70 JAN 05

Prepared for
SPACE AND MISSILE SYSTEMS ORGANIZATION
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San Bernardino Operations

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FOREWORD

This report by The Aerospace Corporation, San Bernardino Operations has bee prepared under Contract No. F04701-69-C-0066 as TR-0066(S5816-43)-1. The Air Force program monitor is Capt. J. Hull, USAF (SMYA). The dates of research for this report include the period January 1969 through September 1969. This report was submitted by the author in September 1969.

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This technical report has been reviewed and is approved.

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UNCLASSIFIED ABSTRACT

THE TURBULENT BOUNDARY LAYER WITH MASS TRANSFER AND PRESSURE GRADIENT, by Stephen C. Lubard and Fernando L. Fernandez

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An analysis of the incompressible, turbulent boundary layer, including the combined effects of mass transfer and pressure gradient is presented in this paper. An integral method employing the integral mechanical energy equation forms the basis of the analysis. Stevenson's velocity profiles are used to obtain the functional dependence of the integral properties and also obtain a skin friction law. A definition of an equilibrium flow with mass transfer and pressure gradient is given in order to evaluate the dissipation integral (C_D) which appears in the integral mechanical energy equation. This definition requires a pressure gradient parameter similar to Clauser's β_T with a modification to include the effect of mass transfer to be held constant. An expression for C_D in the case of equilibrium turbulent flows is then obtained which depends directly on this new pressure gradient parameter (β_T^*). In order to treat the general case of non-equilibrium flows, this expression for C_D is uncoupled from β_T^*, through the use of a single empirical curve fit of the existing <u>no mass transfer</u> equilibrium flow data relating β_T to the Clauser shape parameter.

In addition to uncoupled C_D from the pressure gradient parameter, several specified variations in mass transfer rate are assumed in order to obtain an expression for C_D which is not a function of the mass transfer rate derivative. The numerical results are found to be weakly dependent on which of these variations is used. Comparisons of the numerical results with a wide variety of experimental data, including cases where the blowing rate and pressure are varying simultaneously, shows good agreement. In addition, several problems with discontinuities in blowing or suction are solved and also seen to be in good agreement with the data. (Unclassified Report)

CONTENTS

TABLES

FIGURES

FIGURES (Continued)

LIST OF SYMBOLS

A	Constant in Eq(60)
B	$^{2\lambda}/C_f$, Laminar blowing similarity parameter
C	Constant in Coles' and Stevenson's Velocity Profile Law, taken as 5.0
C_D	Dissipation Integral Eq (7)
C_f	Skin Friction Coefficient $\frac{2\tau_w}{\rho U^2}$
F	Defect Function defined by Eq (9)
F_1	Defect Function in Mickley and Davis' defect law
F_2	Defect Function in Stevenson's defect law (same as F)
f	u/U in Appendix B
f	$\left[C_f/_2\right]^{1/2}$
f^*	$\left[\frac{C_f}{2} + \lambda\right]^{1/2}$
G	Clauser's equilibrium shape parameter Eq (50)
G^*	Equilibrium Shape Parameter with Mass Transfer Eq (53)
g	Functual relation Eq (47)
H	Shape Factor $\equiv \delta^*/\theta$
$\overline{H}$	1/H
I_n	See appendix A
i	Eq (48) either -1, 0, or +1
J	θ^*/δ^*
K	Constant in Velocity Laws taken as 0.41
P	Function of f^*, Π, ϵ appearing in Eq (32)
p	External Pressure
Q	Function of f^*, Π, ϵ appearing in Eq (32)

R	Function of f^*, Π, ϵ appearing in Eq (32)
R_1	Defined by Eq (23)
R_2	Defined by Eq (24)
R_3	Defined by Eq (25)
$R_{e\theta}$	$U\theta/\nu$ Momentum thickness Reynold's Number
U	Velocity at the edge of the boundary layer
U^+	Generalized velocity in Stevenson's Velocity Eq (12)
u	Mean streamwise velocity in the boundary layer
u'	Fluctuation streamwise velocity in the boundary layer
u_τ	Friction velocity at the wall $U\sqrt{\frac{1}{2}C_f}$
u_τ^*	McQuaid's friction velocity $= Uf^* = U\left(^{C}f/_{2} + \lambda\right)^{1/2}$
u^*	Mickley and Davis' friction velocity Eqs (21, 22)
v	Mean normal velocity in the boundary layer
v'	Fluctuation normal velocity in the boundary layer
W	Coles' Wake function
x	Streamwise coordinate
y	Normal coordinate
β	Faulkner-Skan parameter in Appendix B
β_T	Clauser Pressure Gradient Parameter no mass transfer Eq (40)
β_T^*	Pressure Gradient Parameter with mass transfer Eq (38)
δ	Boundary layer thickness
δ^*	Displacement thickness
ϵ	λ/f^* new mass transfer parameter
η	Faulkner Skan normal coordinate

ν	Kinematic viscosity
θ	Momentum thickness
θ^*	Mechanical energy defect
λ	v_w/U non dimensional mass transfer rate
Π	Coles' shape parameter Eq (9)
τ	Shear Stress

Subscripts

e	edge of the boundary layer
w	wall ($y = 0$)
o	initial conditions
1	just upstream of the discontinuity in λ
2	just downstream of the discontinuity in λ

Superscript

$(\ )'$	differentiation with respect to η in Appendix B

SECTION I

INTRODUCTION

The common occurance of turbulent flow in nature and the need to predict skin friction and flow separation in the case of adverse pressure gradient justifies a continued high level of research effort in turbulent boundary layers. In addition to the prediction of turbulent boundary layer development with pressure gradient, the surface may be porous and mass may be injected or sucked through the surface. The injection of mass has been used as a highly effective method of cooling the surface [1, 2] while the use of suction has been employed to prevent boundary layer separation [3]. Since in general a turbulent boundary layer may have both pressure gradient and mass transfer at the surface, it is necessary to be able to predict its development with both of these effects occurring simultaneously.

Several techniques have been developed for calculating a turbulent boundary layer with an arbitrary pressure gradient, but zero mass transfer. These techniques generally fall into either the eddy viscosity methods [4, 5] or the integral methods [6, 7]. An extensive review of the current state of the art in turbulent boundary layers with pressure gradient (and zero mass transfer) has been published in Ref 8. In Ref 8, it was concluded that the eddy viscosity methods do not appear to offer an improvement in the calculation of the integral properties and skin friction coefficient over the results obtained using an integral method such as Alber's [9]. The simplicity and short calculation times of the integral approach makes this method extremely attractive if only the integral properties are desired.

In the case of combined mass transfer and pressure gradient, no satisfactory integral approach has previously been developed. Several techniques have been used in the case of mass transfer but zero pressure gradient, for example, Refs 10, 11, and 12.

It should be noted that although the eddy viscosity methods can be used directly to solve problems with injection (or suction) and pressure gradient, no analysis has been made to indicate the accuracy of the results. In addition, additional assumptions as to the dependence of the eddy viscosity on mass transfer will probably be necessary [13].

In this paper, an integral method of solution of the combined pressure gradient and mass transfer turbulent boundary layer is developed. The approach taken is similar to that used by Alber [9] in solving the no mass transfer pressure gradient case. Alber uses the mechanical energy equation and obtains an exact expression for the dissipation integral, (C_D) for the case of equilibrium flows [14]. The general case of non-similar flows with arbitrary pressure gradients is treated by uncoupling the dissipation integral from the equilibrium pressure gradient parameter β_T using a single empirical curve fit of existing equilibrium flow data, which relates β_T to the Clauser Shape Parameter, G. (See Figure 4.)

From laminar similar flow results and a dimensional argument analogous to the one used by Clauser [14], it is argued that a generalization of the concept of equilibrium flow to include mass transfer requires the parameter

$$\beta_T^* \equiv \frac{\delta^* \dfrac{dp}{dx}}{\tau_w + \rho v_w U}$$

be held constant: An expression for the dissipation integral, C_D which is directly dependent on this new pressure gradient parameter is then obtained. The same empirical curve fit used by Alber to uncouple C_D from the Clauser parameter β_T is then used directly simply by replacing β_T with β_T^*.

The idea of uncoupling the dissipation integral from the external pressure gradient is extremely important in solving relaxing flow problems. These are problems where, for example, the pressure gradient is changed from a non-zero to a zero value in a discontinuous fashion. If the dissipation integral were coupled to the external pressure gradient, it would immediately go to its zero pressure gradient value. This would be incorrect, since the flow actually relaxes over a finite length. In addition to a jump in pressure gradient, it is also possible to have a jump in mass transfer rate. This type of problem also requires a finite length for relaxation. Problems of this type will be solved in this paper.

An additional complication which is absent in the no mass transfer case has to be considered for the present problem. This is due to the appearance in the expression for C_D of the derivative of the mass transfer rate, $\frac{d\lambda}{dx}$. This dependence on $\frac{d\lambda}{dx}$ is discussed and a method of handling it is presented and used in the analysis.

In the following sections, the integral method is developed in detail for the two dimensional case. Numerical results are then obtained for a wide variety of problems and the results compared with data. These problems include cases with constant, low and high blowing and suction rates at constant pressure, varying blowing rates at constant pressure and cases where the blowing rate and pressure varied simultaneously. Good agreement with the data is obtained in these examples and in fact all the examples which have been considered. As an additional extreme test of the analysis, the solution of problems with a discontinuity in blowing or suction is considered. Several examples are solved and compared with the experimental data. Good agreement is obtained.

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SECTION II

ANALYSIS

A. BASIC EQUATIONS

The starting point of any mean flow analysis of the turbulent boundary layer is the time averaged boundary layer equations. These are written below for a two dimensional, incompressible flow field [8]:

CONTINUITY: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

MOMENTUM: $$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{dp_e}{dx} + \frac{\partial \tau}{\partial y} - \rho \frac{\partial}{\partial x} \cancelto{0}{(\overline{u'^2} + \overline{v'^2})} \tag{2}$$

SHEAR STRESS: $$\tau = \mu \frac{\partial u}{\partial y} - \rho \overline{u'v'} \tag{3}$$

EXTERNAL FLOW: $$\frac{1}{\rho} \frac{dp_e}{dx} = -U \frac{dU}{dx} \tag{4}$$

The last term in the momentum equation is normally considered small and is neglected compared to the other quantities.

Integrating the momentum equation and the equation obtained by multiplying the momentum equation by u from y = 0 to δ, we obtain, after rearranging and substituting the continuity equation, two integral equations which may be written as

$$\overline{H} \frac{d\delta^*}{dx} + \delta^* \frac{d\overline{H}}{dx} = C_f/2 + \lambda - \frac{\delta^*}{U} \frac{dU}{dx} (2\overline{H} + 1) \tag{5}$$

$$J \frac{d\delta^*}{dx} + \delta^* \frac{dJ}{dx} = C_D + \lambda - 3J \frac{\delta^*}{U} \frac{dU}{dx} \tag{6}$$

where $\overline{H} \equiv \theta/\delta^*$ and $J \equiv \theta^*/\delta^*$.

C_D is the dissipation integral defined by

$$C_D = \frac{2}{\rho U^3} \int_0^\delta \tau \frac{\partial u}{\partial y}\, dy \tag{7}$$

Eqs (5) and (6) represent two equations for the five unknowns ($\overline{H}$, J, δ^*, C_f, C_D). It is clear here as in any of the integral methods, that additional assumptions must be employed. These assumptions can be divided into three areas. These are:

1) The Integral properties $\overline{H}$, J or some velocity defect law

2) A Skin Friction Law

3) The Dissipation Integral (C_D)

The remainder of the analysis will be devoted to considering these three areas.

B. DEFECT LAWS AND THE INTEGRAL PROPERTIES

The functional dependence of the integral properties, $\overline{H}$ and J, may be written once a defect law is obtained. This defect law is normally obtained after studying a large body of experimental data.

For the case of zero mass transfer, Coles [15] after studying hundreds of experimental turbulent velocity profiles for adverse and favorable pressure gradients, was able to obtain an excellent fit to the data using his Law of the Wake [15]. This defect law may be written as

$$\frac{U-u}{u_\tau} = F(y/\delta, \Pi) \tag{8}$$

$$F(y/\delta, \Pi) = -\frac{1}{K} \ln (y/\delta) + \frac{\Pi}{K} \left[W(1) - W(y/\delta) \right] \tag{9}$$

In this equation, K is a constant, where Coles [16] recommends the value of 0.41, W (y/δ) is Coles' Wake Function and is tabulated in Ref (15) and Π is a shape parameter and is at most a function of x. In the case of equilibrium layers, i.e., β_T = constant, Clauser showed that the velocity defect is independent of x which implies Π = constant.

Several defect forms have been proposed for layers with mass transfer based on constant or slowly varying pressure data.

Mickley and Smith [17] suggests a relation of the form

$$\frac{U-u}{u^*} = F_1 (y/\delta) \tag{10}$$

where u* is "the friction velocity based on the shear stress at the inner edge of the outer flow" and is a function of the mass transfer rate.

Stevenson [18] uses an argument similar to that used by Millikan [19] in the impermeable case to obtain a defect law of the form

$$U_e^+ - U^+ = F_2 (y/\delta) \tag{11}$$

where U^+ is the generalized velocity defined by

$$U^+ = \frac{2u_\tau}{v_w} \left(1 + \frac{v_w u}{u_\tau^2 U}\right) \tag{12}$$

Taking the limit as $v_w \rightarrow 0$ of the left hand side of Eq (11), we obtain

$$(U_e^+ - U^+) = \frac{U-u}{u_\tau} \tag{13}$$

which implies that, since F_2 is independent of v_w, that

$$F_2 (y/\delta) \equiv F(y/\delta, \Pi_{FP}) \tag{14}$$

Here, Π_{FP} is the flat plate (or $\frac{dp}{dx} = 0$) value of Π and has been found experimentally to be approximately 0.55. Figure 1 shows a comparison of McQuaid's [20] blowing data plotted using Stevenson's defect representation.

A third proposed defect formulation is given by McQuaid [20] as

$$\frac{U-u}{u_\tau^*} = F\,(y/\delta,\ \Pi_{FP}) \tag{15}$$

where

$$u_\tau^* = U\left(\frac{c_f}{2} + \lambda\right)^{1/2} \tag{16}$$

This proposed law is the simplest; however, it is seen to have large errors at the higher blowing rates (Figure 2).

Both Mickley and Smith's and Stevenson's representation seem to do an adequate job even at the higher blowing rates. However, because of the need for an empirically determined u^* in Eq (10) and the fact that Stevenson's representation is felt to have a firmer theoretical basis [18], it was decided to use Stevenson's Law in the analysis. It should be noted that all of the blowing (or suction) data used in confirming Stevenson's Law is for constant pressure ($\frac{dp}{dx} = \beta_T = 0$) or slowly varying pressure. It is therefore assumed that Stevenson's Law is valid for $\frac{dp}{dx} \neq 0$ simply by using Eqs (11) and (14) with Π_{FP} replaced by Π thus giving

$$U_e^+ - U^+ = F\,(y/\delta,\ \Pi) \tag{17}$$

Instead of using Eq (17) directly, it is found convenient to introduce two new parameters defined by

$$\frac{u_\tau^*}{U} = f^* = (C_f/2 + \lambda)^{1/2} \tag{18}$$

and

$$\epsilon = \lambda/f^* \tag{19}$$

Stevenson's Law may then be rewritten after some rearranging as

$$\frac{U-u}{u_\tau^*} = F\,(y/\delta,\,\Pi)\,\left[1 - \frac{\epsilon}{4}\,F\,(y/\delta,\,\Pi)\right] \tag{20}$$

It should be noted that at least for the case of blowing that ϵ is always a small quantity since for blowing $\epsilon<\sqrt{\lambda}$. The maximum value of λ which is considered in this analysis is 0.01 which implies $\epsilon<0.1$. Since ϵ is always small, Eq (20) indicates that the mass transfer form of the defect law may be obtained from the no mass transfer form by adding a small correction term ($-\epsilon/4$ F). Writing Stevenson's defect law in this way has several advantages. First, it shows clearly how all the defect laws are related. We see, for example, for $\lambda\to o$, Eq (20) approaches McQuaid's proposed formulation. Mickley and Smith's proposed representation is obtained from Eq (20) if their friction velocity u^* is assumed equal to

$$u^* = u_\tau^*\left[1 - \frac{\epsilon}{4}\,F\,(y/\delta,\,\Pi)\right] \tag{21}$$

or from the definition of u_τ^*, this is equivalent to

$$\frac{u^*}{u_\tau} = \left[1 - \frac{\epsilon}{4}\,F\,(y/\delta,\,\Pi)\right]\left[1 + 2\lambda/C_f\right]^{1/2} \tag{22}$$

Evaluating F at $y/\delta = 0.1$ (approximately equal to 8) as suggested by Mickley and Smith and using McQuaid's data, Eq (22) is calculated in Figure 3. Also shown is the curve given by Mickley and Smith [21]. The points are seen to fall close to the proposed curve.

A second advantage to writing Stevenson's defect law in the form of Eq (19) is that the dependence of δ^*, θ and θ^* on f^*, Π, and ϵ is easily obtained. The results may be written as, (neglecting the sublayer contribution to the integral properties)

$$\delta^* = \delta R_1(f^*,\Pi,\epsilon) \tag{23}$$

$$\theta = \delta R_2(f^*,\Pi,\epsilon) \tag{24}$$

$$\theta^* = \delta R_3 (f^*, \Pi, \epsilon) \tag{25}$$

$$\overline{H} = R_2/R_1 \tag{26}$$

$$J = R_3/R_1 \tag{27}$$

The functional form of R_1, R_2 and R_3 is given in Appendix A.

Before leaving this section on the defect laws with mass transfer, two important factors should be noted. First, since almost all of the blowing data is for constant pressure, we have had to assume that Stevenson's law is valid for $\frac{dp}{dx} \neq 0$ simply by assuming that Eq (17) is valid. The question which remains and which will be considered in the section on the dissipation integral is what is the physical parameter similar to Clauser's Equilibrium pressure gradient parameter β_T which if held constant implies Π = constant.

A second factor which is related to the first is whether a prescribed variation in λ is necessary for equilibrium to exist and, if so, what is this prescribed variation. It should be noted in this regard that Stevenson's defect law depends only on the local value of λ and not on $\frac{d\lambda}{dx}$. In fact, data for $\lambda \sim x^{-.2}$ in addition to constant λ data has been fitted to Stevenson's form [21]. This question of a prescribed variation in mass transfer rate will also be considered further in the section on the dissipation integral.

C. SKIN FRICTION LAW

An additional equation which can be used to close the system of integral equations is some form of skin friction law. This is simply some empirically observed dependence of C_f on the other parameters of the problem. This relation may be explicit or implicit. One way of obtaining a skin friction law which was used by Coles [15] is to obtain a velocity profile which is valid for the entire layer (except possibly the laminar sub-layer) and use the boundary condition that $u \rightarrow U$ as $y \rightarrow \delta$.

A velocity profile which has been found to fit the mass transfer data reasonably well is obtained by combining Stevenson's Law of Wall for transpiration [22] with his defect law. The velocity profile is then written as [22]

$$\frac{2f}{\lambda}\left[\left(1 + \frac{\lambda}{f^2}\frac{u}{U}\right)^{1/2} - 1\right] = C + \frac{1}{K}\ln\left(\frac{Ufy}{\nu}\right) + \frac{\Pi}{K} W(y/\delta) \tag{28}$$

Using the condition that $u \rightarrow U$ as $y \rightarrow \delta$, we obtain from Eq (28)

$$\frac{2f}{\lambda}\left[\left(1 + \frac{\lambda}{f^2}\right) - 1\right] = C + \frac{1}{K}\ln\left(\frac{Uf\delta}{\nu}\right) + \frac{\Pi}{K} W(1) \tag{29}$$

Subtracting Eq (28) from Eq (29) gives Stevenson's defect law Eq (17) which is consistent.

Using the definitions of f^* and ϵ and substituting in Eq (29), we obtain

$$\frac{2}{\epsilon}\left[1 - (1 - \epsilon/f^*)^{1/2}\right] = C + \frac{1}{K}\ln\left[\frac{U\delta f^*}{\nu}(1 - \epsilon/f^*)^{1/2}\right] + \frac{2\Pi}{K} \tag{30}$$

The δ appearing in Eq (30) can be replaced in terms of δ^* using Eq (23) giving finally an implicit skin friction law which may be written as

$$\frac{2}{\epsilon}\left[1 - (1 - \epsilon/f^*)^{1/2}\right] = C + \frac{1}{K}\ln\left[\frac{U\delta^*}{R_1\nu}f^*(1 - \epsilon/f^*)^{1/2}\right] + \frac{2\Pi}{K} \tag{31}$$

It is found convenient to use Eq (31) in differentiated form. Carrying out this differentiation, we obtain

$$\frac{d\delta^*}{dx} + \delta^* Q\frac{df^*}{dx} + \delta^* P\frac{d\Pi}{dx} + \delta^* R\frac{d\epsilon}{dx} = -\frac{\delta^*}{U}\frac{dU}{dx} \tag{32}$$

where Q, P, and R are functions of f^*, Π and ϵ.

We may now write down the four differential equations necessary to describe the development of the four quantities, δ^*, f^*, Π, and ϵ. These are

$$\overline{H}\,\frac{d\delta^*}{dx} + \delta^*\left(\frac{\partial\overline{H}}{\partial f^*}\,\frac{df^*}{dx} + \frac{\partial\overline{H}}{\partial\Pi}\,\frac{d\Pi}{dx} + \frac{\partial\overline{H}}{\partial\epsilon}\,\frac{d\epsilon}{dx}\right) = \frac{C_f}{2} + \lambda - \frac{\delta^*}{U}\,\frac{dU}{dx}(2\overline{H} + 1) \quad (33)$$

$$J\,\frac{d\delta^*}{dx} + \delta^*\left(\frac{\partial J}{\partial f^*}\,\frac{df^*}{dx} + \frac{\partial J}{\partial\Pi}\,\frac{d\Pi}{dx} + \frac{\partial J}{\partial\epsilon}\,\frac{d\epsilon}{dx}\right) = C_D + \lambda - 3J\,\frac{\delta^*}{U}\,\frac{dU}{dx} \quad (34)$$

$$\frac{d\delta^*}{dx} + \delta^*\left(Q\,\frac{df^*}{dx} + P\,\frac{d\Pi}{dx} + R\,\frac{d\epsilon}{dx}\right) = -\frac{\delta^*}{U}\,\frac{dU}{dx} \quad (35)$$

$$\epsilon\,\frac{df^*}{dx} + f^*\,\frac{d\epsilon}{dx} = \frac{d\lambda}{dx} \quad (36)$$

Eqs (33) and (34) are the integral momentum and moment of momentum equations, Eq (35) is the differentiated skin friction law and Eq (37) is obtained by differentiating the definition of ϵ, Eq (19).

If an expression for the dissipation integral as a function of f^*, Π, and ϵ were known, the above system could be integrated. The method of obtaining this expression is detailed in the following section.

D. EQUILIBRIUM FLOW AND THE DISSIPATION INTEGRAL

Alber [9], in solving the no mass transfer case, obtains an exact expression for the dissipation integral in the case of equilibrium flow (β_T or Π = constant). Before this method can be used for the combined pressure gradient and mass transfer case, it is necessary to establish a definition of an equilibrium flow with mass transfer. Going back to Stevenson's defect law generalized to include pressure gradient, i.e.,

$$U_e^+ - U^+ = F\,(y/\delta, \Pi) \quad (17)$$

it is necessary to obtain a physical parameter related to Clauser's β_T which, if held constant, implies Π is a constant or $U_e^+ - U^+ \neq f(x)$.

Following an argument similar to that used by Clauser, we are looking for a parameter which represents a balance between the external driving force $\delta' \frac{dp}{dx}$ and the wall dissipative effect. In the zero mass transfer case, the only wall effect is represented by the shear stress, τ_w. However, we now have to include the force necessary to accelerate the mass injected (ρv_w) or removed to some representative velocity u'. Therefore, β_T generalizes with mass transfer to

$$\beta_T^* = \frac{\delta' \frac{dp}{dx}}{\tau_w + \rho v_w u'} \tag{37}$$

Clauser was able to show experimentally that δ' should be δ^* for no mass transfer. Since u' is some representative velocity in the defect region, it is postulated that $u' \approx U$. Therefore, we assume

$$\beta_T^* = \frac{\delta^* \frac{dp}{dx}}{\tau_w + \rho v_w U} \tag{38}$$

Obviously β_T^* has the proper limits as the mass transfer approaches zero. In addition, for $\frac{dp}{dx} = 0$, $\beta_T^* = 0$ and we have an equilibrium flow which is the result shown by Stevenson in his original defect law. It should be noted that Eq (38) is only a postulate and requires experimental confirmation.

As a further check on Eq (38), consider Eq (20) for small values of mass transfer. This implies ϵ small and we can write Eq (20) as

$$\frac{U-u}{Uf^*} \approx F\,(y/\delta, \Pi) \tag{39}$$

This is exactly the same expression as the no mass transfer case with f replaced by f^*. If β_T is written in terms of f, we obtain

$$\beta_T = \frac{\delta^* \frac{dp}{dx}}{\rho U^2 f^2} = \frac{-\delta^*}{f^2 U} \frac{dU}{dx} \tag{40}$$

Replacing f by f^* as indicated by Eq (39) gives exactly β_T^* since

$$\beta_T^* = \frac{\delta^* \frac{dp}{dx}}{\rho U^2 f^{*2}} = \frac{-\delta^*}{f^{*2} U} \frac{dU}{dx} \tag{41}$$

As further confirmation that β_T^* defines an equilibrium flow with mass transfer, a discussion of the laminar similar flow case is given in Appendix B. It is shown there that the laminar case also indicates the β_T^* is the correct parameter.

It is now possible to calculate an exact expression for the dissipation integral in the case of equilibrium flows. Setting Π = constant or $\frac{d\Pi}{dx} = 0$ and using the definition of β_T^* Eqs (33) - (36) are reduced to the form

$$\overline{H} \frac{d\delta^*}{dx} + \delta^* \left(\frac{\partial \overline{H}}{\partial f^*} \frac{df^*}{dx} + \frac{\partial \overline{H}}{\partial \epsilon} \frac{d\epsilon}{dx} \right) = f^{*2} \left[1 + \beta_T^* \left(2\overline{H} + 1\right) \right] \tag{42}$$

$$J \frac{d\delta^*}{dx} + \delta^* \left(\frac{\partial J}{\partial f^*} \frac{df^*}{dx} + \frac{\partial J}{\partial \epsilon} \frac{d}{dx} \right) = C_D + \epsilon f^* + 3Jf^{*2} \beta_T^* \tag{43}$$

$$\frac{d\delta^*}{dx} + \delta^* \left(Q \frac{df^*}{dx} + R \frac{d\epsilon}{dx} \right) = f^{*2} \beta_T^* \tag{44}$$

$$\epsilon \frac{df^*}{dx} + f^* \frac{d\epsilon}{dx} = \frac{d\lambda}{dx} \tag{45}$$

These equations could be solved for C_D giving

$$C_D = C_D \left(f^*, \Pi, \epsilon, \beta_T^*, \frac{d\lambda}{dx} \right) \tag{46}$$

Two points should be made about this expression for the dissipation integral. First, it is seen that the value of C_D is directly tied to the local value of the pressure gradient through β_T^*. This fact will be considered later in the analysis. The second point is the appearance of $d\lambda/dx$ in Eq (46). The expression for C_D has been obtained by assuming Π = constant or that the flow is locally in equilibrium. Using the property of equilibrium flows given by Walz [8], which states that in the defect region τ is only a function of the local velocity u, this implies that C_D is only a function of u. Since u is not a function of $d\lambda/dx$, C_D should not be dependent on $d\lambda/dx$. This difficulty can be eliminated if a similar or equilibrium mass transfer variation is known.

Several variations of mass transfer rate have been suggested as being "similar". These include the following:

1) $\frac{2\lambda}{Cf}$ = constant: This is known as being the correct parameter in the laminar case and has been suggested by several authors for the turbulent boundary layer [21].

2) λ = constant: This has been suggested by McQuaid (Ref 20) on the basis of his higher blowing rate data.

3) ϵ = constant: This is suggested by considering Eq (20), which is the rearranged Stevenson's law. We see that, if Π and ϵ are held constant, then

$$\frac{U-u}{Uf^*} \neq g\,(x) \tag{47}$$

In fact, it is possible to use Eq (47) as an alternate complete definition of an equilibrium flow with mass transfer.

Considering the third variation, ϵ = constant, at the higher blowing rates this implies that $\epsilon \approx \sqrt{\lambda}$ = constant which is the same as McQuaid's proposal.

All the suggestions for a similar flow with mass transfer may be used to obtain a relation between $\frac{d\epsilon}{dx}$ and $\frac{df^*}{dx}$. This relation may be written for all the variations considered as

$$\frac{d\epsilon}{dx} = i\frac{\epsilon}{f^*}\frac{df^*}{dx} \tag{48}$$

where

$$\frac{2\lambda}{C_f} = \text{Constant} \rightarrow i = +1$$

$$\lambda = \text{Constant} \rightarrow i = -1$$

$$\epsilon = \text{Constant} \rightarrow i = 0$$

Using Eq (48) in Eqs (42) and (45) and then solving for C_D, we obtain

$$C_D = -\epsilon f^* - 2J\beta_T^* f^{*2} + \frac{f^{*2}\left[1 + \beta_T^*(\overline{H}+1)\right]\left[\frac{\partial J}{\partial f^*} - JQ + \frac{i\epsilon}{f^*}\left(\frac{\partial J}{\partial \epsilon} - JR\right)\right]}{\left[\frac{\partial \overline{H}}{\partial f^*} - \overline{H}Q + \frac{i\epsilon}{f^*}\left(\frac{\partial \overline{H}}{\partial \epsilon} - \overline{H}R\right)\right]} \tag{49}$$

As noted previously, C_D is directly tied to the external pressure gradient through β_T. This is an unacceptable situation if it is desired to solve a relaxing flow problem. Alber [9] was able to "unhook" the dissipation integral from the pressure gradient in the no mass transfer case by using the Clauser shape parameter, G defined by

$$G \equiv \frac{\int_0^1 \left(\frac{U-u}{u_\tau}\right)^2 d\,(y/\delta)}{\int_0^1 \left(\frac{U-u}{u_\tau}\right) d\,(y/\delta)} \tag{50}$$

Alber [23] obtained from the equilibrium flow data a curve fit of G as a function of β_T. The curve fit and the data are shown in Figure 4.

Using Cole's Law of the Wake, G is simply a function of Π given by

$$G \equiv \frac{\int_0^1 F^2\,(y/\delta, \Pi)\; d\,(y/\delta)}{\int_0^1 F\,(y/\delta, \Pi)\; d\,(y/\delta)} = G\,(\Pi) = \frac{I_2\,(\Pi)}{I_1\,(\Pi)} \tag{51}$$

Therefore from the curve fit, we can solve for

$$\beta_T = \beta_T(G) = \beta_T(\Pi) \tag{52}$$

and thus completely "unhook" C_D from the pressure gradient.

The above idea can be generalized to include mass transfer by using the β_T^* previously defined. We also define a new shape parameter, G^* as

$$G^* \equiv \frac{\int_0^1 (U_e^+ - U^+)^2 \, d(y/\delta)}{\int_0^1 (U_e^+ - U^+) \, (d(y/\delta)} = \frac{I_2(\Pi)}{I_1(\Pi)} \tag{53}$$

We see from Eq (53) that G^* is only a function of Π and in fact is the same function of Π as in the no mass transfer case.

It is now assumed that the same empirical curve of G vs β_T used by Alber (Figure 4) in the no mass transfer case can be used with mass transfer simply by replacing β_T with β_T^* and G with G^*. Therefore, from Alber's curve fit, [23] we write

$$\beta_T^* = 0.026875 \left[G^{*2} + 2.8\, G^* - 65.39 \right] \tag{54}$$

This is substituted in Eq (49) for β_T^* thus completing the "unhooking".

Except for a consideration of the initial conditions which are discussed below, this completes the analysis.

E. INITIAL CONDITIONS

Initial conditions are necessary in order to numerically integrate the above system of differential equations. Therefore, values of δ^*, f^*, Π and ϵ at some x_o are needed to begin the integration. Generally, experimental data will provide the integral properties, δ_o^* and $\overline{H}_o$ and on occasion, the skin friction coefficient, C_{f_o}. If the skin friction coefficient is given, then f_o^* and ϵ_o are simply obtained from the boundary conditions and C_{f_o} as

$$f_o^* = \left(1/2\, C_{f_o} + \lambda_o\right)^{1/2} \tag{55}$$

and

$$\epsilon_o = \lambda_o / f_o^*$$

The initial value of Coles' shape parameter (Π) is obtained using $\overline{H}_o$ and Eq (26) and solving for Π_o using an iterative scheme.

If the skin friction coefficient is not given, a value of C_{f_o} is guessed which implies f_o^* and ϵ_o. A value of Π_o is obtained for the guessed C_{f_o} from Eq (26) for $\overline{H}_o$. A new value of f_o^* is then obtained from the skin friction law, Eq (30).

This completes the analysis of the incompressible turbulent boundary layer with mass transfer. Given boundary conditions for U and λ as a function of x and the initial conditions, the system of equations is integrated numerically using a Runga-Kutta integration routine and a CDC 6600 computer. Typical integration times on any one problem are on the order of seconds.

Before considering the numerical examples and a comparison of the numerical results with data, it is felt desirable to list the assumptions and empirical constants used in the analysis:

ASSUMPTIONS

1) β_T and ϵ = constant defines an equilibrium flow with mass transfer
2) G^* vs β_T^* is the same as the no mass transfer G vs β_T curve

EMPIRICISM

1) K = 0.41 and C = 5.0 (no mass transfer values)[17]
2) G vs β_T curve (no mass transfer curve fit)

It is seen that nowhere has experimental data with mass transfer been used to obtain empirical constants for the analysis.

SECTION III

NUMERICAL EXAMPLES

In order to check the analysis and assumptions presented in the previous section, the numerical results of the analysis are compared with experimental data for a wide variety of problems. The experimental data used in the comparison is that obtained by McQuaid [20] and Simpson [24]. This data includes cases with 1) low and high constant blowing and suction rates at constant pressure, 2) variable blowing rates at constant pressure and, 3) cases where both the pressure and the blowing rate are varied simultaneously. In addition, several problems with a discontinuity in blowing or suction are considered and compared with the experimental data.

Table 1 gives a list of the examples considered together with the boundary conditions for each case.

In using McQuaid's data, the skin friction coefficient used in all examples is that which is listed by McQuaid as obtained by fitting Stevenson's law of the wall to the data. McQuaid uses the constants $K = 0.435$ and $C = 4.0$ in performing this fit. In Simpson's data, the value of $C_f/2$ which is used, is that which is listed as the "best estimate". In no case has any attempt been made to determine the error in this quantity, although the error is expected to be significant at the lower values ($C_f \leq 0.5 \times 10^{-3}$). The error in C_f given by Simpson is included in the figures. In addition, no attempt has been made to determine if and to what extent three dimensional affects are present in the data.

In evaluating the dissipation integral, Eq (49) with $i = 0$ (ϵ = constant) was used. In addition, for several examples $i = 1$ ($2\lambda/C_f$ = constant) and $i = -1$ (λ = constant) was used in evaluating C_D. The results for these different values of i are tabulated in Tables 2 and 3 for two problems, together with the $i = 0$ results, and the data. It should be noted that a value of $K = 0.41$ has been used in all of the examples considered.

A. CONSTANT PRESSURE - CONSTANT MASS TRANSFER

Figures 5 through 10 show the results for the cases where the pressure and the mass transfer rate are constant. For the lower blowing and suction rates, the agreement with the data is excellent; however, for the higher rates the values of C_f, θ and H show some disagreement. This disagreement for H and θ is less than 10 percent. This is felt to be good for these higher blowing rates since there is some question as to the adequacy of Stevenson's defect law at the higher rates. (See Figure 1.) The disagreement for the skin friction coefficients at the higher blowing rates is generally within the accuracy of the data. In the case of the higher suction rate ($\lambda = -0.00240$) data of Simpson, the agreement is again within 10 percent, which is again considered quite good.

B. CONSTANT PRESSURE - VARIABLE MASS TRANSFER RATE

Two examples with constant pressure and continuously varying blowing rates are considered and compared with the data of Simpson.

1) $\lambda = 0.0148\ (12x)^{-1/2}$; $U = 42.8$ fps

The results for this example are shown in Figure 11. The agreement with the data is excellent over the entire range of the experiment.

2) $\lambda = 6.0 \cdot 10^{-4}\ (x)$; $U = 43.0$

This is a much more severe example than the previous case because of the relatively high rate of mass transfer increase. Figure 12 shows the comparison between the data and the analysis. Again excellent agreement is obtained even for the skin friction at the lower values.

C. VARIABLE PRESSURE - VARIABLE BLOWING RATE

McQuaid performed three experiments where he varied the pressure and the blowing rate simultaneously. The boundary conditions of U and λ and a comparison of the analysis with the data is shown in Figures 13 through 18. In all three cases, the agreement with the data is good (within 10 percent).

1) McQuaid Pressure Gradient I

This case involves a mild adverse pressure gradient and a mild increase in blowing rates, Figure 13. The maximum value of blowing rate is fairly low. Figure 14 shows the excellent agreement between the analysis and the data over the entire range. The agreement is within 5 percent.

2) McQuaid Pressure Distribution II

This case is a mild favorable pressure gradient and a decrease in λ. Figure 15 shows the boundary conditions for this example. It is seen that the blowing rate remains quite high over the entire range. Figure 16 gives a comparison between the analysis and data. The integral properties are within a few percent, while the agreement in skin friction is not quite as good. However, because of the low values of skin friction, this difference is probably within the error of the data.

3) McQuaid Pressure Distribution III

This case is the most severe pressure gradient considered. The boundary conditions are shown in Figure 17 where it is seen the velocity increases a factor of two while the blowing rate increases a factor of three. The results for this case are shown in Figure 18 and are considered quite good considering the boundary conditions.

The results that have been presented are all for the case where C_D is evaluated with $i = 0$ ($\epsilon = 0$). In Tables 2 and 3, the results for the skin friction coefficient and shape factor, H are shown assuming $i = 1$ ($2\lambda/C_f$ = constant) and $i = -1$ (λ = constant) for two cases. These cases are:

1) McQuaid Pressure Gradient III

2) Simpson Suction $\lambda = -0.00240$

Also shown in these tables are the $i = 0$ results and the data. It is seen that the results are very insensitive to which variation in blowing rate is assumed.

D. DISCONTINUITIES IN BLOWING OR SUCTION

An extremely difficult problem for an integral method and in fact any method is one where the boundary conditions are changed discontinuously. Both McQuaid and Simpson obtained experimental data on several problems where the blowing or suction rates were changed discontinuously. The analysis of problems with steps in mass transfer rate is given below followed by several numerical examples and a comparison with the data.

We assume the Integral Eqs (33) - (36) are valid on both sides of the step. Therefore, given initial conditions upstream of the step, we can integrate the equations up to the discontinuity. It is then necessary to obtain the values of δ^*, f^*, Π and ϵ downstream of the discontinuity so that the solution can be continued.

The conditions just downstream of the discontinuity in mass transfer are obtained by assuming the continuity of mass and momentum across the discontinuity and also that the skin friction law Eq (31) is valid on either side of the discontinuity.

The conservation of mass implies $(\delta - \delta^*)$ is continuous across the step. Therefore, using Eq (23),

$$\delta_1^* \left[\frac{1}{R_1 (f_1^*, \Pi_1, \epsilon_1)} - 1\right] = \delta_2^* \left[\frac{1}{R_1 (f_2^*, \Pi_2, \epsilon_2)} - 1\right] \tag{56}$$

The conservation of momentum implies θ is continuous or

$$\theta_1 = \theta_2 = \delta_2^* R_2 (f_2^*, \Pi_2, \epsilon_2) \Big/ R_1 (f_2^*, \Pi_2, \epsilon_2) \tag{57}$$

A third relation is obtained from the integral form of the skin friction law (Eq 31). Writing the skin friction law on each side of the discontinuity and subtracting gives

$$\ell n \left[\frac{f_1^* \delta_1^* \sqrt{1 - \epsilon_1/f_1^*}}{R_1 (f_1^*, \Pi_1, \epsilon_1)}\right] - \ell n \left[\frac{f_2^* \delta_2^* \sqrt{1 - \epsilon_2/f_2^*}}{R_1 (f_2^*, \Pi_2, \epsilon_2)}\right] = -2 (\Pi_1 - \Pi_2)$$
$$+ \frac{2K}{\epsilon_1} \left[1 - \sqrt{1 - \epsilon_1/f_1^*}\right] - \frac{2K}{\epsilon_2} \left[1 - \sqrt{1 - \epsilon_2/f_2^*}\right] \tag{58}$$

We obtain a fourth condition by using the definition of ϵ which may be written downstream of the step as

$$\lambda_2 = f_2^* \epsilon_2 \tag{59}$$

Eqs (55) - (58) are four algebraic equations for the four quantities, δ_2^*, f_2^*, Π_2 and ϵ_2, downstream of the discontinuity. Given the conditions just upstream of the step, and the λ downstream, these equations can be solved.

Three examples were solved using this technique for obtaining the initial conditions downstream of the step. The results for these examples are shown compared to the data in Figures 19 and 21. In all these examples, the data was only used to obtain initial conditions a distance upstream of the step. The solution was then calculated up to the jump. New initial conditions were calculated using Eqs (55) - (59) and the solution continued downstream from this point.

The first example, Figure 19, is from McQuaid's data and represents a case where the blowing is discontinuously stopped at x = 17.40 inches. The agreement with the data is reasonably good. The theoretical predictions for skin friction seem to lie above the data. However, taking flat plate data for C_f as a function of Re_θ, we find at x = 30 inches, $Re_\theta = 6150$, and the flat plate skin friction coefficent for this value of Re_θ is approximately 2.70×10^{-3} compared with the analysis value of 2.75×10^{-3} and an experimental value of 2.5×10^{-3}. It is felt that the effects of the initially injected boundary layer should have almost disappeared by x = 30 inches.

The second example, Figure 20, is from Simpson's data and represents a case where the blowing is begun discontinuously at x = 3 ft. The calculation is started just upstream of the step. The agreement for θ and H is excellent (within 5 percent). The calculated skin friction coefficient does not agree with the data as well but considering the large error given by Simpson for this quantity, the disagreement is not very large.

The third example (Figure 21) is also from Simpson and is a case where suction is discontinued at x 3 ft. These calculations also show good agreement with the data.

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SECTION IV

DISCUSSIONS AND CONCLUSIONS

An integral method of calculating the development of the incompressible turbulent boundary layer with both mass transfer and pressure gradient has been developed in this paper. No new empirical constants which depend on mass transfer data have been utilized in the analysis. A large number of examples have been solved and the results compared with experimental data. In all cases, good agreement with the data has been obtained. No effort has been made to compare this integral method with the eddy viscosity methods. This is due to the fact that no analysis of the combined pressure gradient and mass transfer problems using an eddy viscosity approach has been found in the literature. However, it is felt the inherent simplicity and rapid computing times of the present integral approach make it competitive if not superior to the present eddy viscosity methods.

It is desirable at this time to discuss some of the assumptions and concepts used in the analysis. One of the basic assumptions used in this analysis is the validity of Stevenson's Defect Law and its extension to flows with pressure gradient. (Eq (17)). Figure 1 shows a comparison of this defect law with data where we see excellent agreement for the lower blowing rates. However, at $\lambda = 0.008$ there is a definite disagreement. This disagreement also exists for the higher blowing rate data in Ref (25). It is, therefore, felt that Stevenson's defect form may be inadequate at the higher blowing rates, which may explain the disagreement between the data and calculations at these higher values. One possibility of improving on Stevenson's defect form is suggested by Eq (20) which is Stevenson's form in terms of ϵ and f^*. We could consider Eq (20) as an expansion in ϵ and thus write

$$\frac{U-u}{u_\tau^*} = F\,(y/\delta,\,\Pi)\left[1 - \frac{\epsilon}{4}\,F\,(y/\delta,\,\Pi) + A\,\epsilon^2 F^2\,(y/\delta,\,\Pi) + \ldots\right] \qquad (60)$$

where A is a constant which would be obtained from the higher blowing and/or suction data. Besides modifying the defect law, Stevenson's law of the wall should certainly be looked at further for the higher blowing rates and possibly also be modified.

An additional basic assumption in this analysis is the definition of β_T^* as the equilibrium pressure gradient parameter to use in the case of mass transfer. This assumption must be tested experimentally before its validity can be definitely established. Together with this pressure gradient parameter, the determination of the required mass transfer parameter for the turbulent flow case requires additional effort. The three parameters indicated in this report ($2\lambda/C_f$, λ and ϵ) are certainly not the only possibilities and additional experimental work seems necessary to resolve this question.

In addition to the above questions which arise due to the mass transfer, there is a significant error in the definition of the zero mass transfer G vs β_T curve which is used in the analysis. (Figure 4.) It should be noted in this regard that the third pressure gradient problem falls on the extrapolated portion of this curve. It is felt that a better definition of this curve would certainly increase the accuracy of the analysis.

An additional point, which will only be mentioned here, is the concept of boundary layer "memory". This idea has been advanced by several authors [8] to explain the failure of previous methods to predict boundary layer behavior in equilibrium and relaxing flows. This concept is in disagreement with Alber's [9] results and also the results in this report. The method presented here includes no upstream history effects and yet good agreement with the data is obtained even for problems with steps in blowing or suction.

In conclusion, an integral method of solving the turbulent boundary layer with both pressure gradient and mass transfer has been developed. The method is simple and calculation times for any one example are extremely short. The results of the analysis have been compared with experimental data for a wide variety of examples with good agreement obtained in all cases. In addition, the method has been shown capable of handling problems where there is a discontinuity in either blowing or suction.

REFERENCES

1. Rubesin, N.W., "An Analytical Estimation of the Effect of Transpiration Cooling on the Heat Transfer and Skin Friction Characteristics of a Compressible Turbulent Boundary Layer", NACA TN 3341, (December, 1954)

2. Rubesin, M.W. and Pappas, C.E., "An analysis of the Turbulent Boundary Layer Characteristics on a Flat Plate with Distributed Light Gas Injection", NACA TN 4149, (February, 1958)

3. Pankhurst, R.C., "Recent British Work on Methods of Boundary Layer Control". Proc. Symp. at Nat. Phys. Lab. (1955)

4. Mellor, G.L., "Turbulent Boundary Layers with Arbitrary Pressure Gradients and Divergent or Convergent Cross Flows", Gas Dynamics Laboratory, Princeton University, Rept. 775, March 1966.

5. Smith, A.M.O., Jaffe, N.A. and Lind, R.C., "Study of a General Method of Solution to the Incompressible Turbulent Boundary Layer Equations", Douglas Aircraft Co., Report LB52949, November, 1965.

6. Von Doenhoff, A.E., and Tetervin, N., "Determination of the General Relations for the Behavior of Turbulent Boundary Layers", NACA Rep. No. 772 (1943).

7. Truckenbrodt, E., "A Method of Quadrature for Calculation of the Laminar and Turbulent Boundary Layers", NACA TM 2178, 1953.

8. Coles, D., Hirst, E., "Proceedings AFOSR-IFP-Stanford 1968 Conference on Turbulent Boundary Layer Prediction," Vol. 1 and Vol. 2, August 1968.

9. Alber, I.E., "Application of an Exact Expression for the Equilibrium Dissipation Integral to the Calculation of Turbulent Non-Equilibrium Flows", presented at the AFOSR-IFP Stanford Conference on Computation of Turbulent Boundary Layers, August 1968.

10. Smith, K.A., "The Transpired Turbulent Boundary Layer", ScD thesis, M.I.T. 1962.

11. Mickley, H.S. and Davis, R.S., "Momentum Transfer for Flow Over a Flat Plate with Blowing," NACA, TN 4017 (1957).

12. Thompson, B.G.J., "The Calculation of Shape Factor Development in Incompressible Turbulent Boundary Layers with or Without Transpiration", Proc. AGARD Symposium, Naples 1965, AGARDograph 97, pp 159-190.

13. Kleinstein, G., "Generalized Law of the Wall and Eddy Viscosity Model for Wall Boundary Layers", AIAA J. Vol. 5, No. 8, May 1967, pp 1402-1407.

14. Clauser, F.M., "Turbulent Boundary Layer in Adverse Pressure Gradients", J. Aero. Sciences, Vol. 21, pp 91-108 (1954)

15. Coles, D., "The Law of the Wake in the Turbulent Boundary Layer", Journal of Fluid Mechanics, Vol. 1, Part 2, July 1956.

16. Coles, D., "The Young Person's Guide to the Data", a survey lecture prepared for the 1968 AFOSR-IFP Stanford Conference on Computation of Turbulent Boundary Layers, Stanford, Calif. August 1968.

17. Mickley, H.S. and Smith, K.A., "Velocity Defect Law for a Transpired Turbulent Boundary Layer", AIAA J., Vol. 1, p. 1685.

18. Stevenson, T.N., "A Modified Velocity Defect Law for Turbulent Boundary Layers with Injection", Cranfield College of Aeronautics, Report 170 (1963)

19. Von Karman, T., "Theorie des Rubungswiderstands", Hydromechanische Probleme des Scheffsantriebs, Hamburg, 1932.

20. McQuaid, J. "Incompressible Turbulent Boundary Layers with Distributed Injection," Ph.D. thesis, Cambridge University (1966)

21. Mickley, H.S. and Smith, K.A. and Fraser, M.D., "Velocity Defect Laws for Transpired Turbulent Boundary Layers," AIAA J. Vol. 3, March 1965, pp 787-788.

22. Stevenson, T.N., "A Law of the Wall for Turbulent Boundary Layers With Suction or Injection", Cranfield College of Aeronautics, Report 166 (1963).

23. Alber, I.E., and Coats, D.E., "Analytic Investigation of Equilibrium and Non-Equilibrium Compressible Turbulent Boundary Layers", Dynamic Science TR-A69-104, May 1969.

24. Simpson, R. L. ; Kays, W. M. ; Moffat, R. J. ; "The Turbulent Boundary Layer on a Porous Plate: An Experimental Study of the Fluid Mechanics with Injection and Suction", Stanford University, Dept. of Mech. Engin., Report No. MMT 2 (1967).

25. Fernandez, F. L. and Zukoski, I., "Turbulent Boundary Layer with Large Distributed Surface Injection", AIAA J. Vol. 7 Sept. (1969)

26. Economos, C. "The Compressible Turbulent Boundary Layer With Mass Transfer", Polytechnic Institute of Brooklyn, Ph. D. Thesis (June 1968)

27. Hartree, D. R., "On an Equation Occurring in Flakner and Skan's approximate Treatment of the Equations of the Boundary Layer", Proc. Cambr. Phil., Soc., 33, Part II, 223 (1937)

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Table 1

EXAMPLES CONSIDERED AND THEIR BOUNDARY CONDITIONS

PROBLEM	BOUNDARY CONDITIONS	FIGURE NO. (results)
1	$\lambda = 0.001;$ $\frac{dU}{dx} = 0$	5
2	$\lambda = -0.001;$	6
3	$\lambda = 0.0033;$	7
4	$\lambda = -0.0024;$	8
5	$\lambda = 0.008;$	9
6	$\lambda = 0.0095;$	10
7	$\lambda = 5.68 \times 10^{-2} x^{-1/2};$	11
8	$\lambda = 6.0 \times 10^{-4} x;$	12
9	λ increases; $\frac{dU}{dx} < 0$ (Fig. 13)	14
10	λ decreases; $\frac{dU}{dx} > 0$ (Fig. 15)	16
11	λ increases; $\frac{dU}{dx} > 0$ (Fig. 17)	18
12	BLOWING DISCONTINUOUSLY STOPPED	19
13	BLOWING DISCONTINUOUSLY STARTED	20
14	SUCTION DISCONTINUOUSLY STOPPED	21

Table 2

COMPARISON OF THE RESULTS USING DIFFERENT VALUES OF i
IN THE EVALUATION OF C_D FOR McQUAID PRESSURE GRADIENT III

X IN	$C_f \times 10^3$				$H = \delta^*/\theta$			
	DATA	i = 0	i = +1	i = -1	DATA	i = 0	i = +1	i = -1
11.2	2.10	2.10	2.10	2.10	1.525	1.525	1.525	1.525
14.2	2.00	1.83	1.83	1.86	1.502	1.537	1.537	1.538
17.0	1.90	1.78	1.78	1.81	1.472	1.511	1.511	1.512
21.8	1.70	1.56	1.56	1.60	1.427	1.486	1.486	1.487
24.8	1.40	1.21	1.21	1.25	1.411	1.505	1.504	1.504
28.2	1.00	0.79	0.85	0.89	1.433	1.550	1.550	1.550
30.8	0.75	0.54	0.54	0.58	1.441	1.603	1.603	1.603
34.2	0.33	0.25	0.25	0.29	1.589	1.750	1.750	1.750

Table 3

COMPARISON OF THE RESULTS USING DIFFERENT VALUES OF i IN THE EVALUATION OF C_D FOR SIMPSON SUCTION ($\lambda = -0.0024$)

X FT	$C_f \times 10^3$				$H = \delta^*/\theta$			
	DATA	i = 0	i = +1	i = -1	DATA	i = 0	i = +1	i = -1
1.5	6.88	6.88	6.88	6.88	1.37	1.37	1.37	1.37
3.5	6.22	6.51	6.51	6.52	1.31	1.33	1.33	1.33
5.5	5.90	6.29	6.29	6.30	1.26	1.30	1.30	1.30
7.5	5.66	6.15	6.15	6.16	1.25	1.29	1.29	1.29

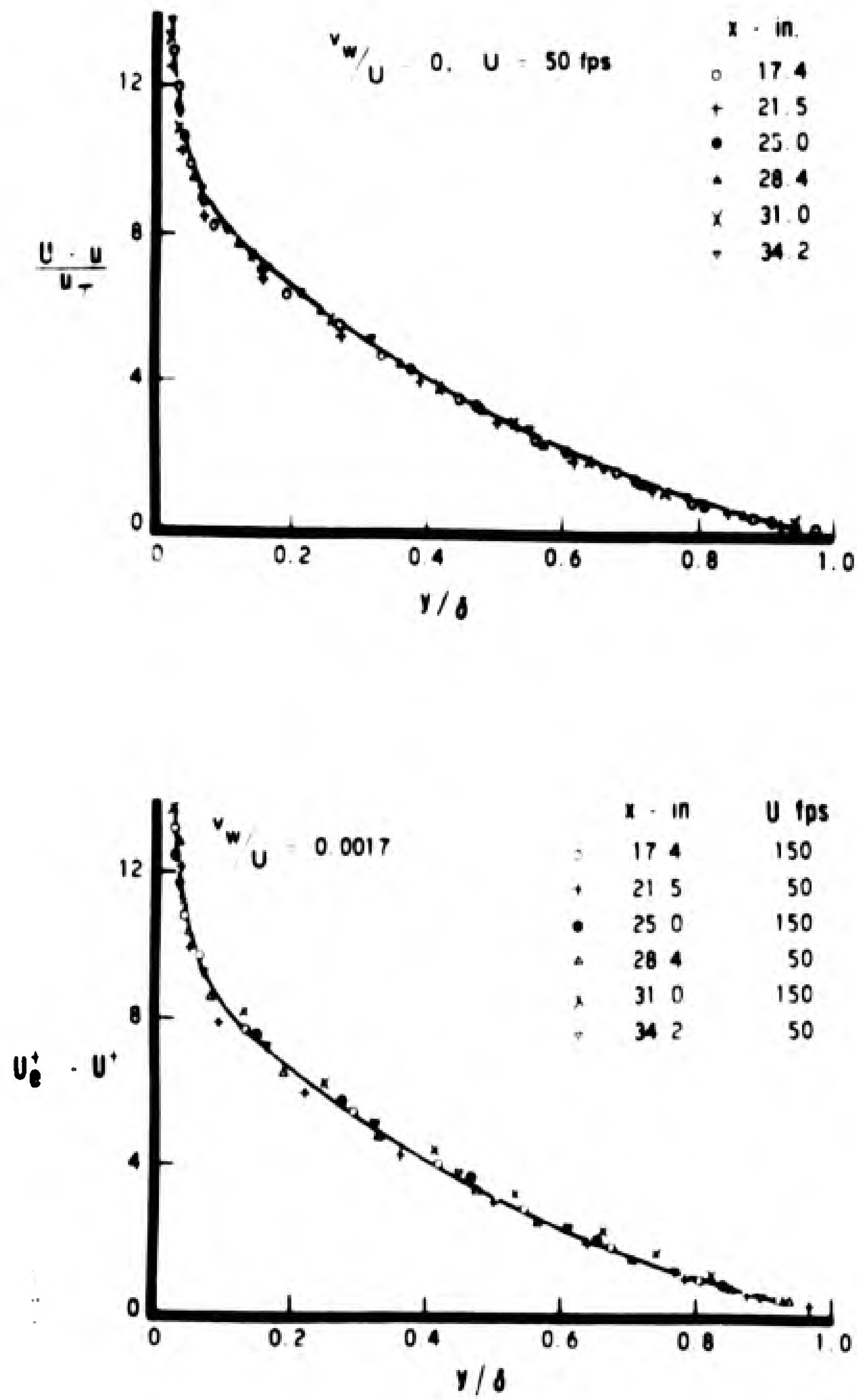
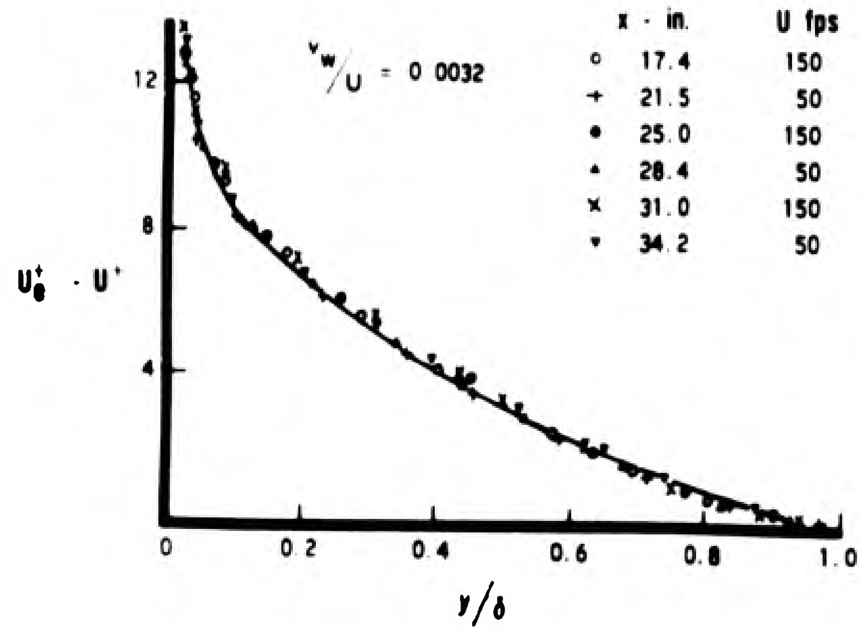


Figure 1. Comparison of Stevenson's Defect Law with McQuaid's Experimental Data (from Reference 20)



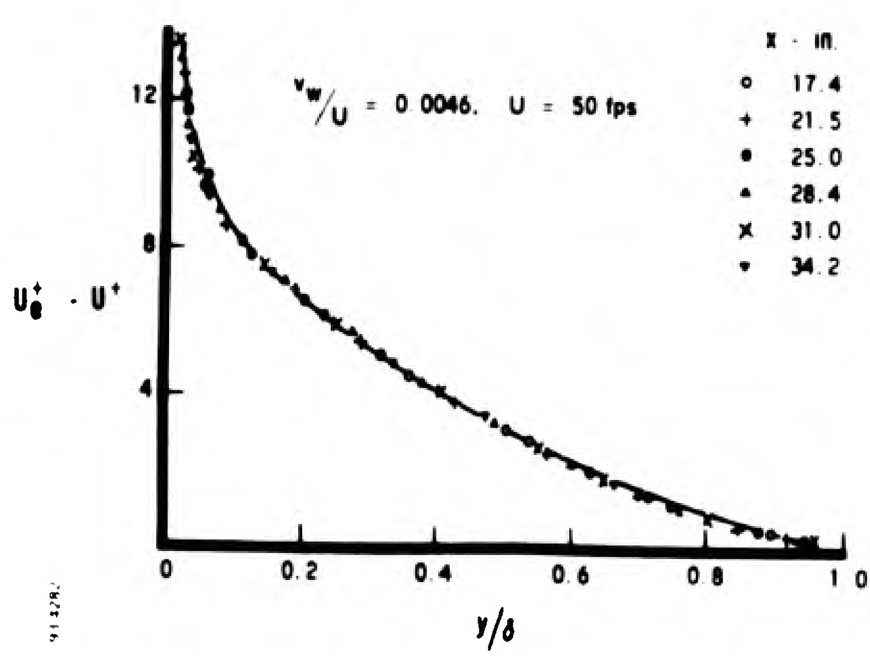


Figure 1. (continued)

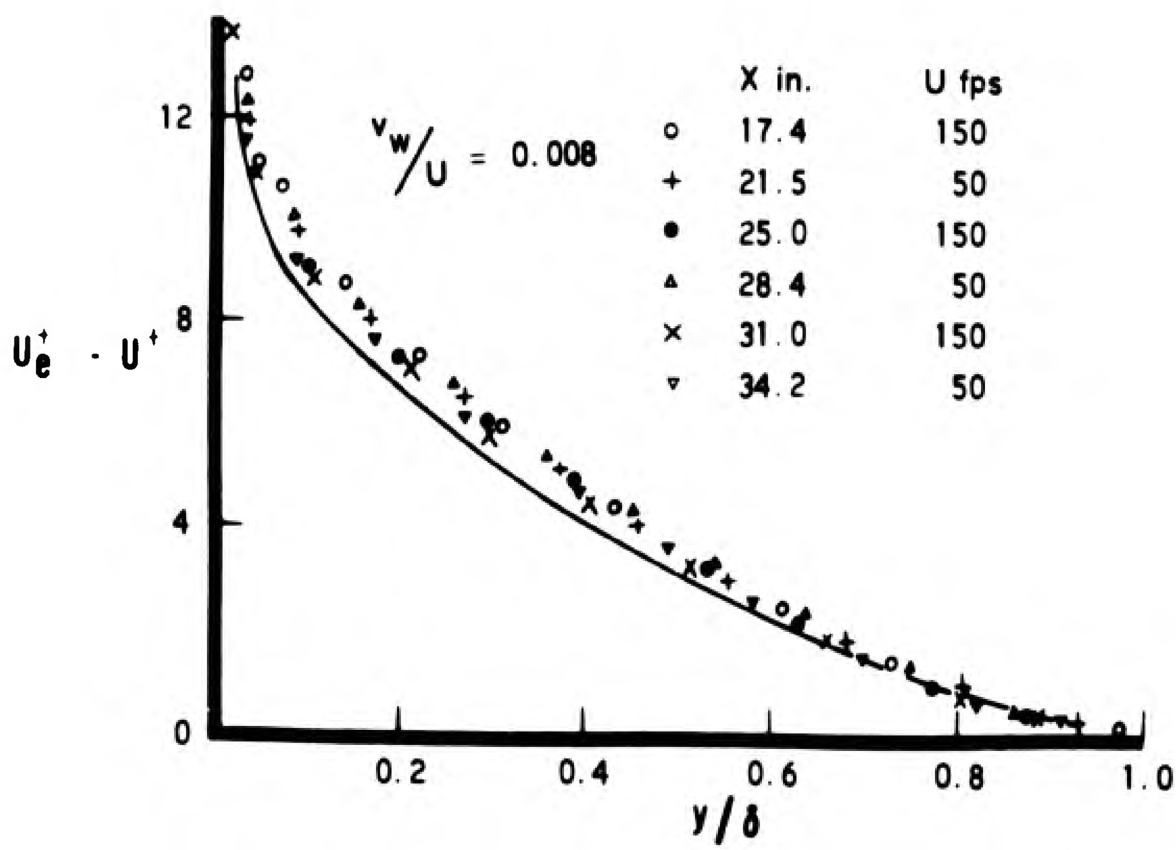


Figure 1. (continued)

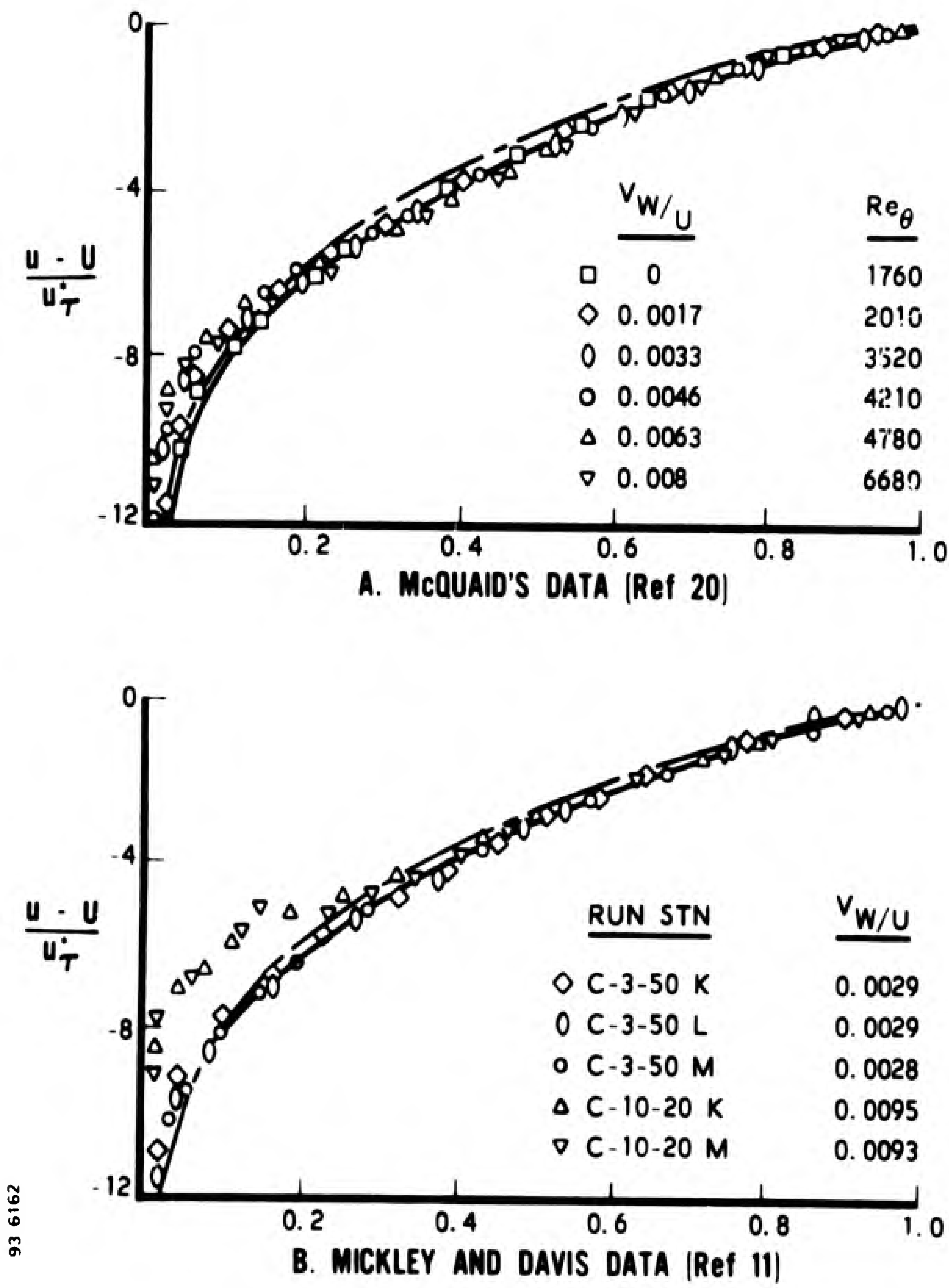


Figure 2. Comparison of McQuaid's Defect Law with Experiment

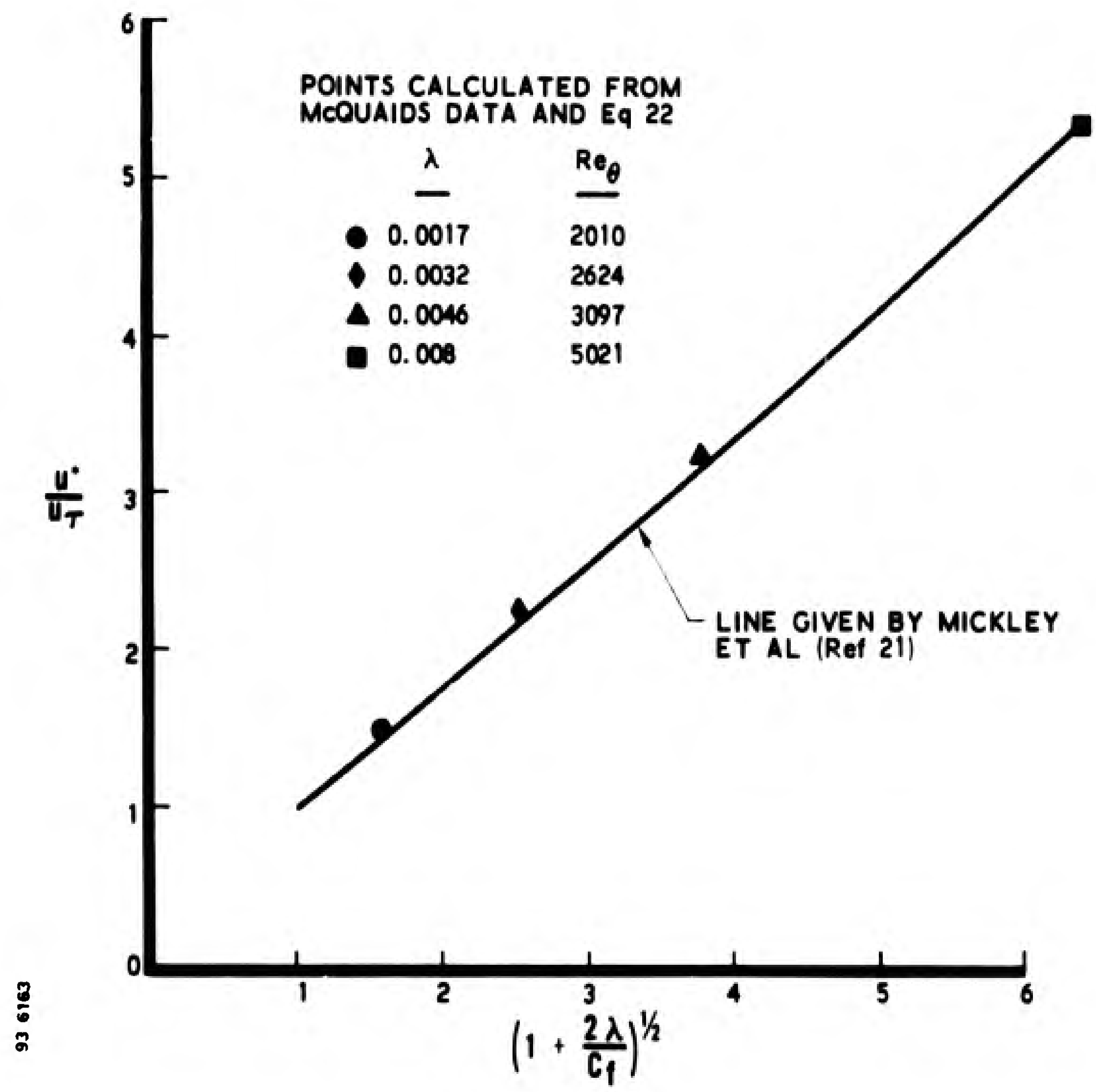


Figure 3. Mickley and Davis' Friction Velocity Obtained from Stevenson's Law

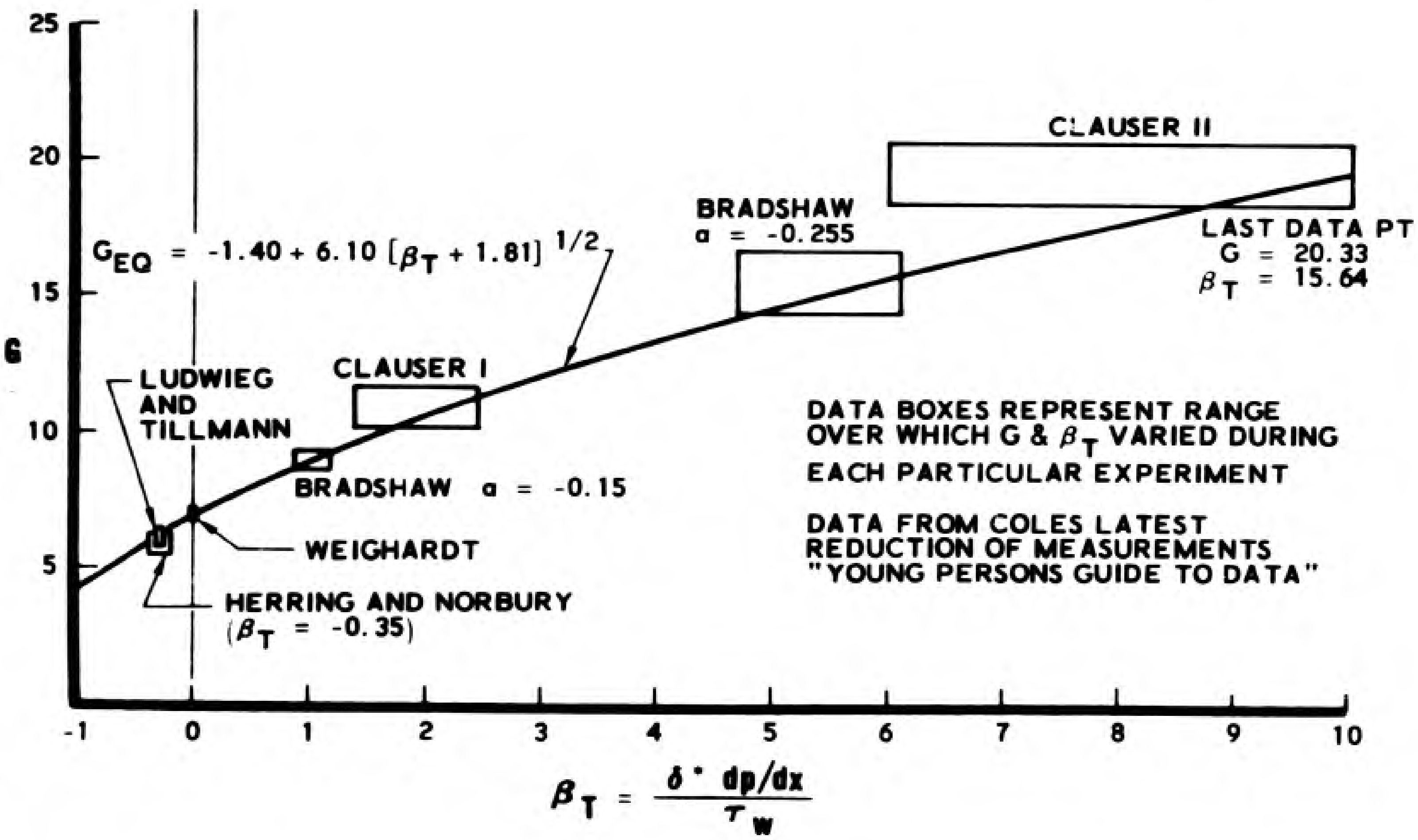


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Figure 4. Alber's G vs β_T Curve for No Mass Transfer

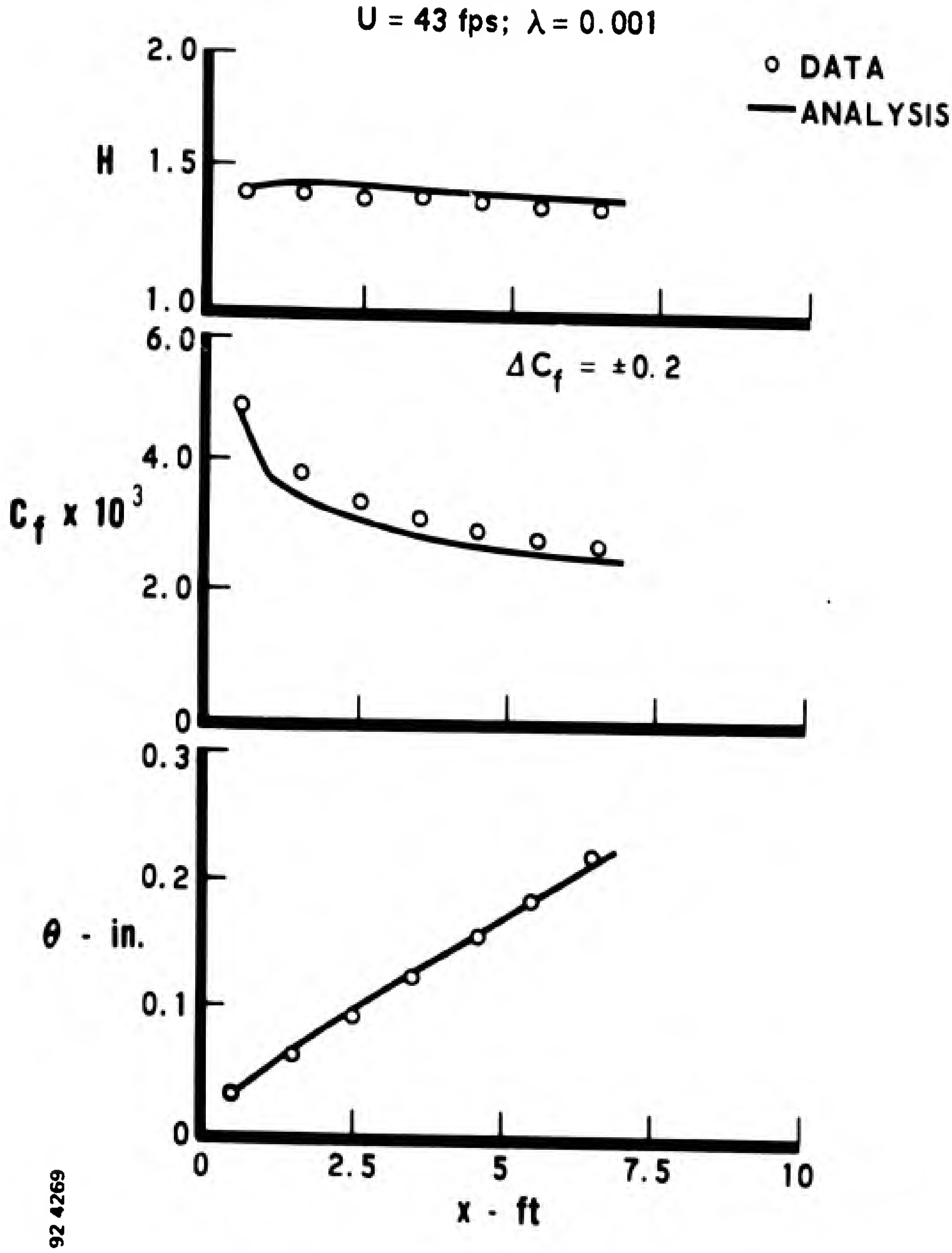

Figure 5. Comparison of the Analysis with Simpson's Blowing Data (U = constant; λ = +0.001)

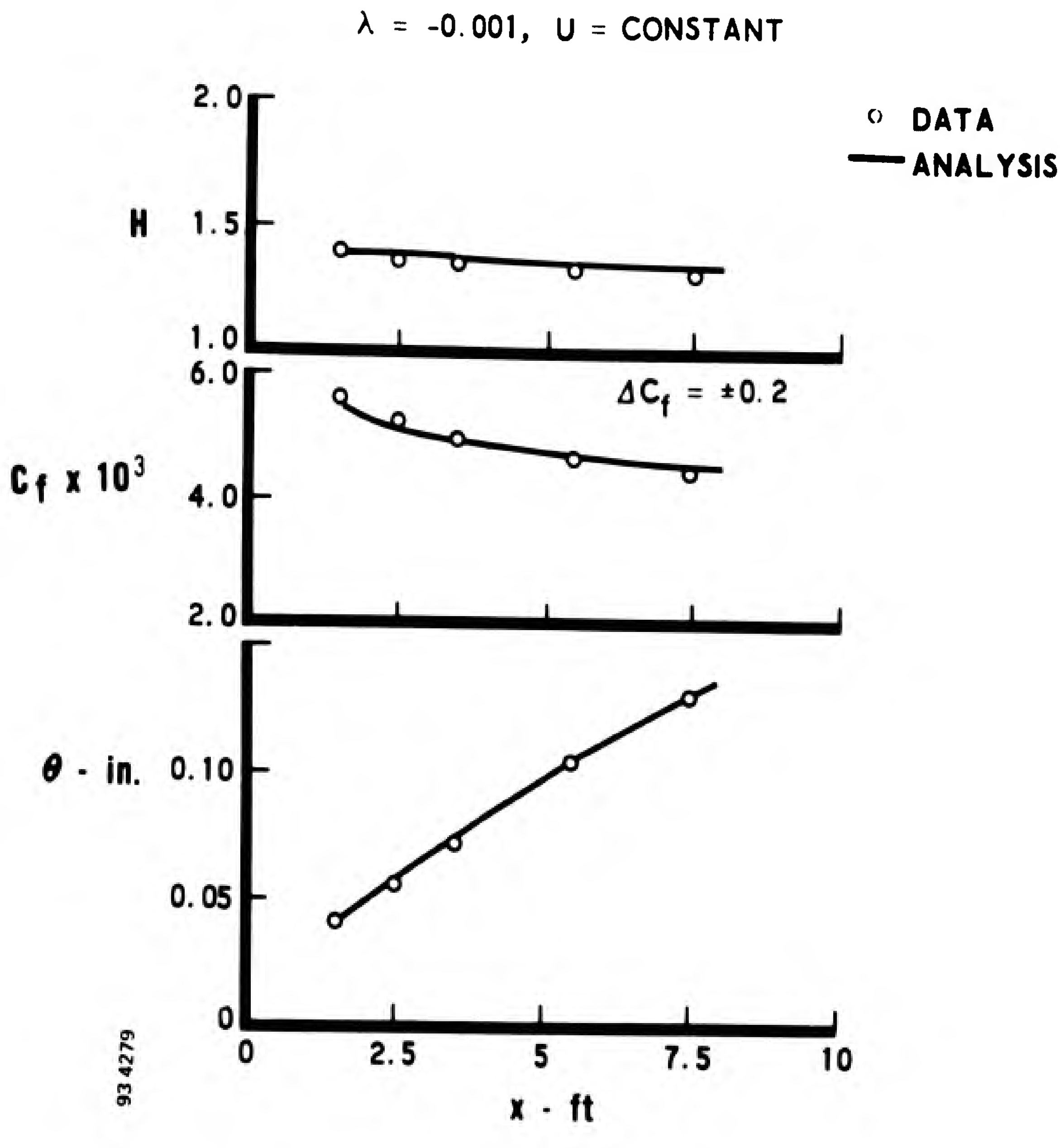

Figure 6. Comparison of the Analysis with Simpson's Suction Data (U = constant; λ = - 0.001)

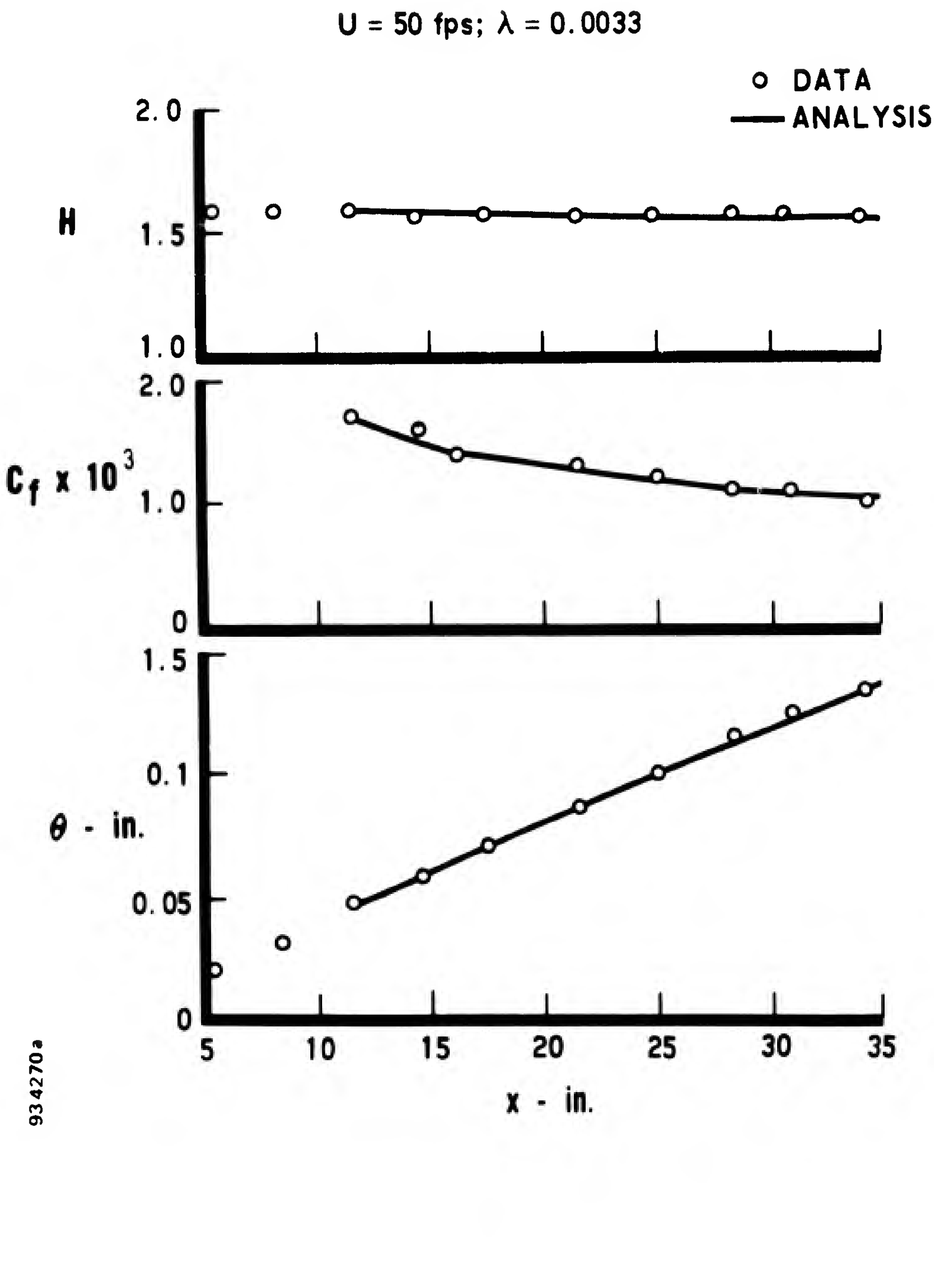

Figure 7. Comparison of the Analysis with McQuaid's Blowing Data (U constant: $\lambda = +0.0033$)

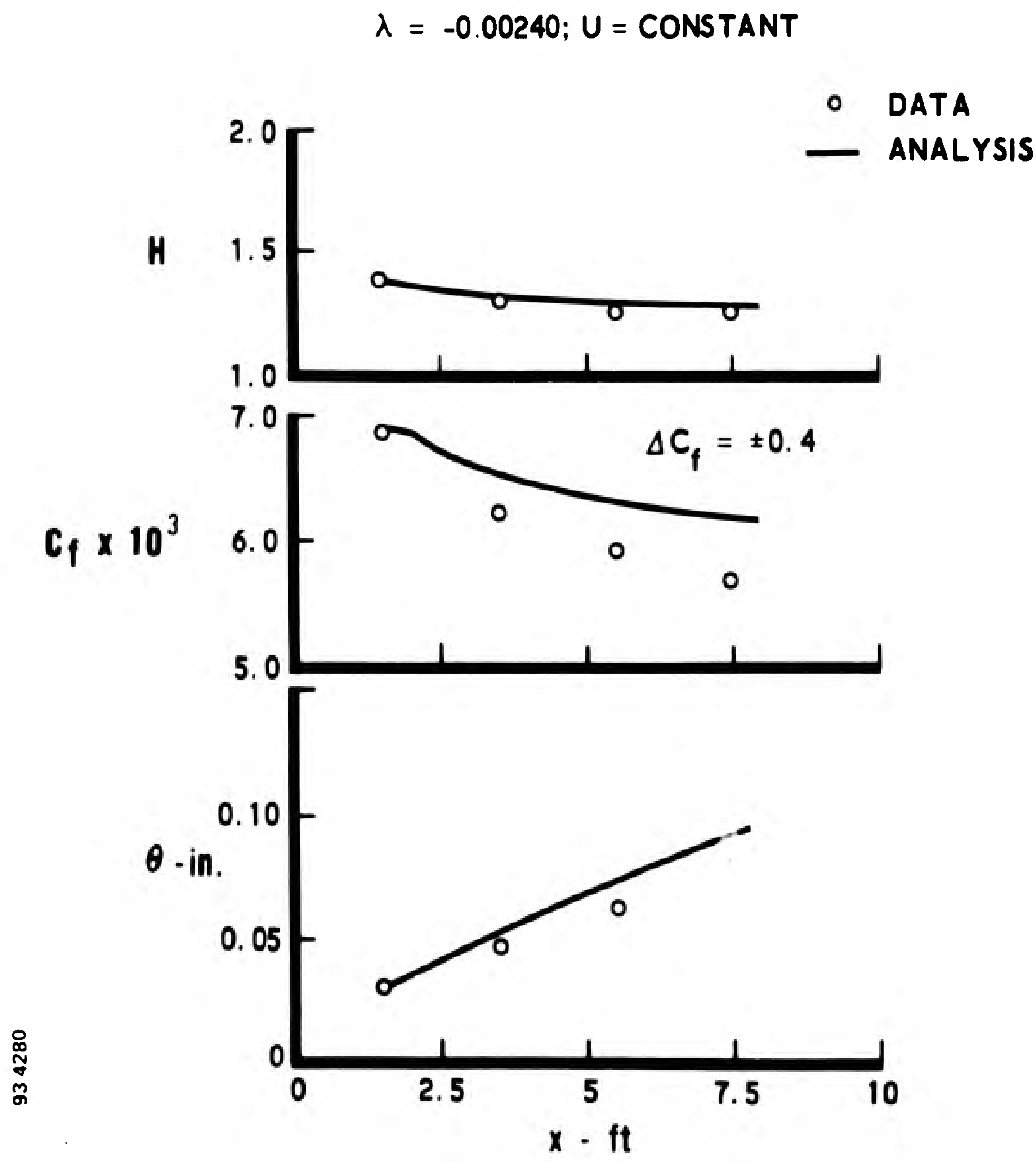

Figure 8. Comparison of the Analysis with Simpson's Higher Suction Rate Data (U = constant; λ = - 0.00240)

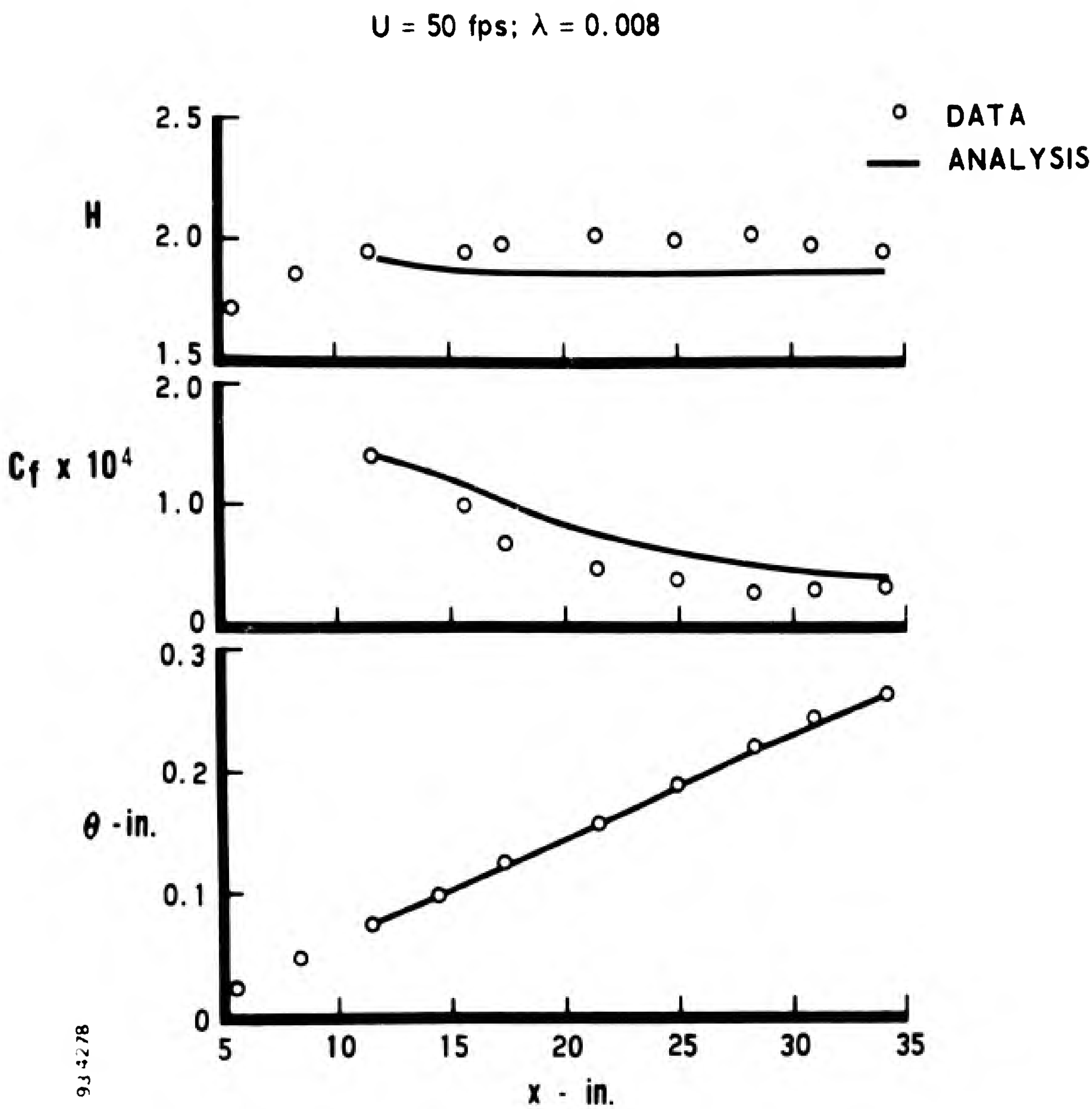

Figure 9. Comparison of the Analysis with McQuaid's Higher Blowing Data (U = constant; $\lambda = 0.008$)

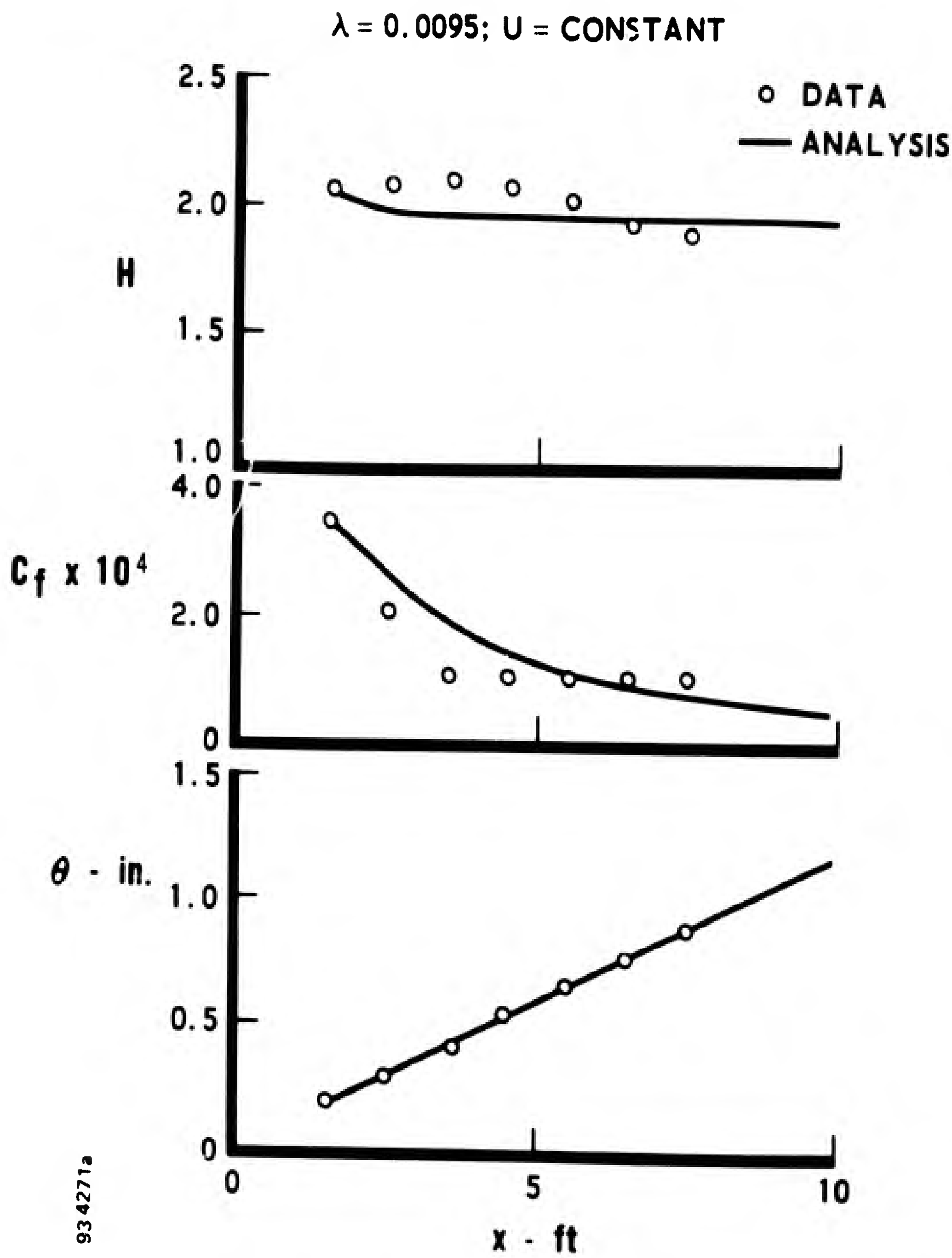

Figure 10. Comparison of the Analysis with Simpson's Higher Blowing Data (U = constant; λ = + 0.0095)

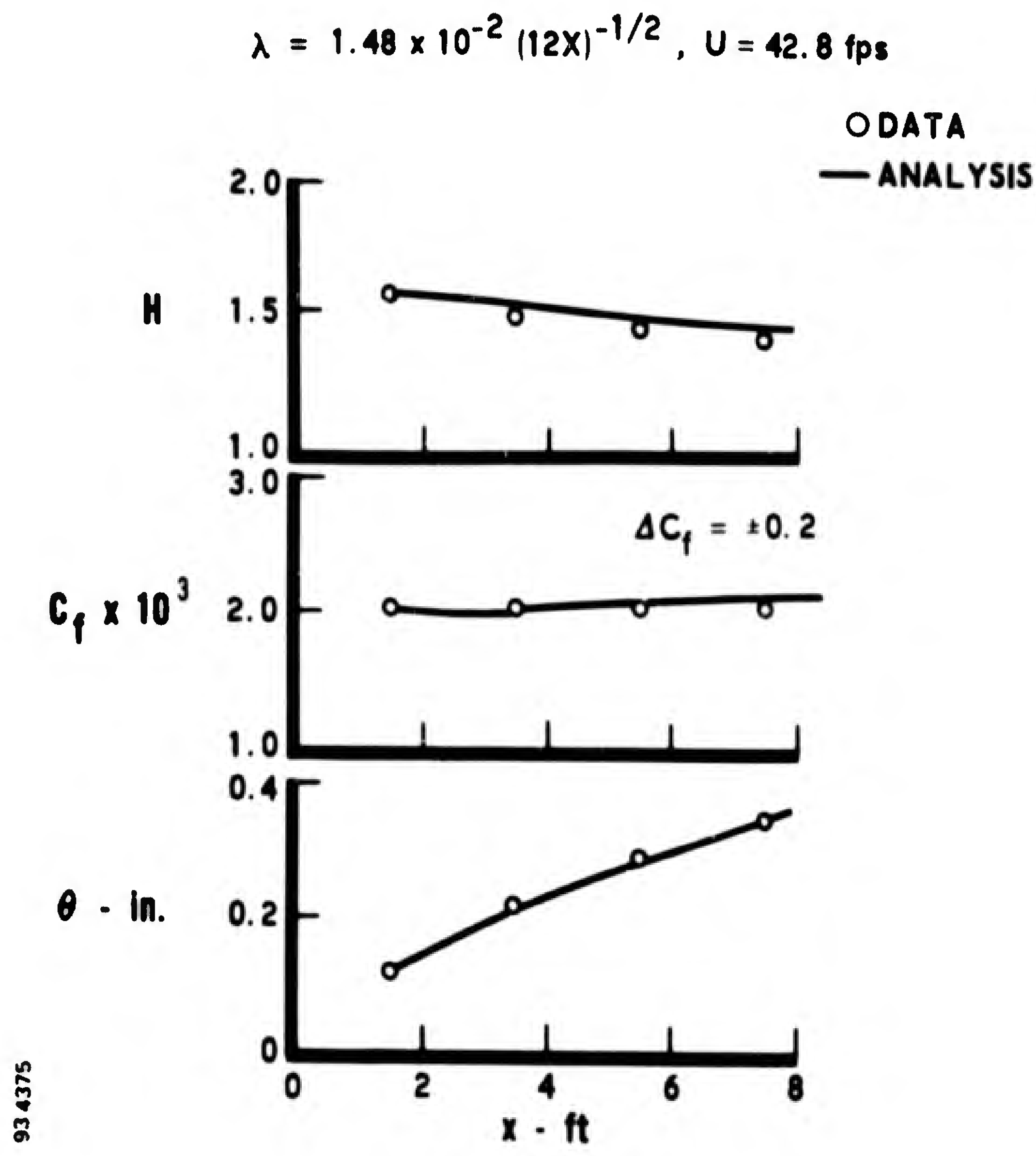

Figure 11. Comparison of the Analysis with Simpson's Variable Blowing Data ($\lambda \sim x^{-1/2}$)

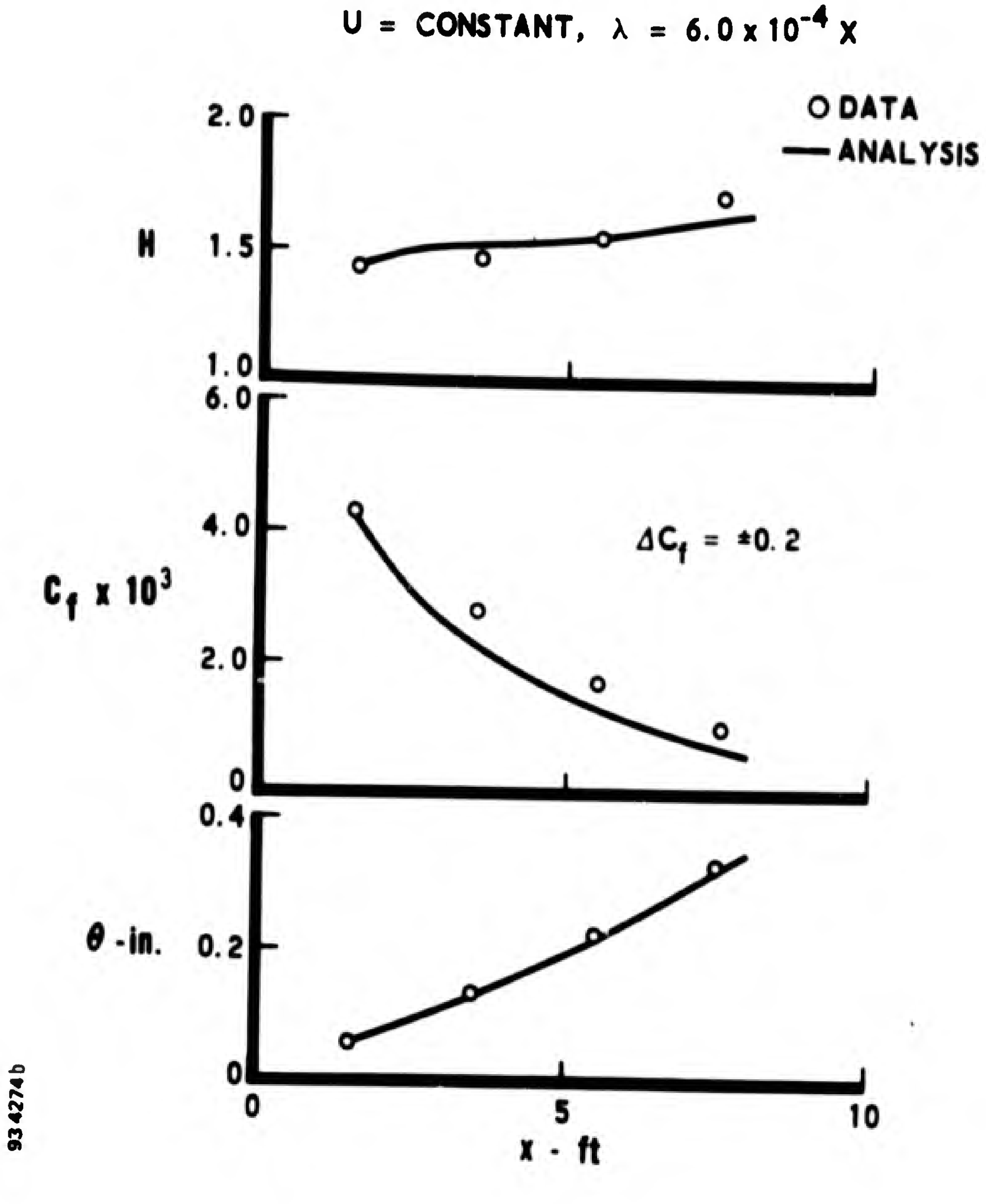

Figure 12. Comparison of the Analysis with Simpson's Variable Blowing Data ($\lambda \sim x$)

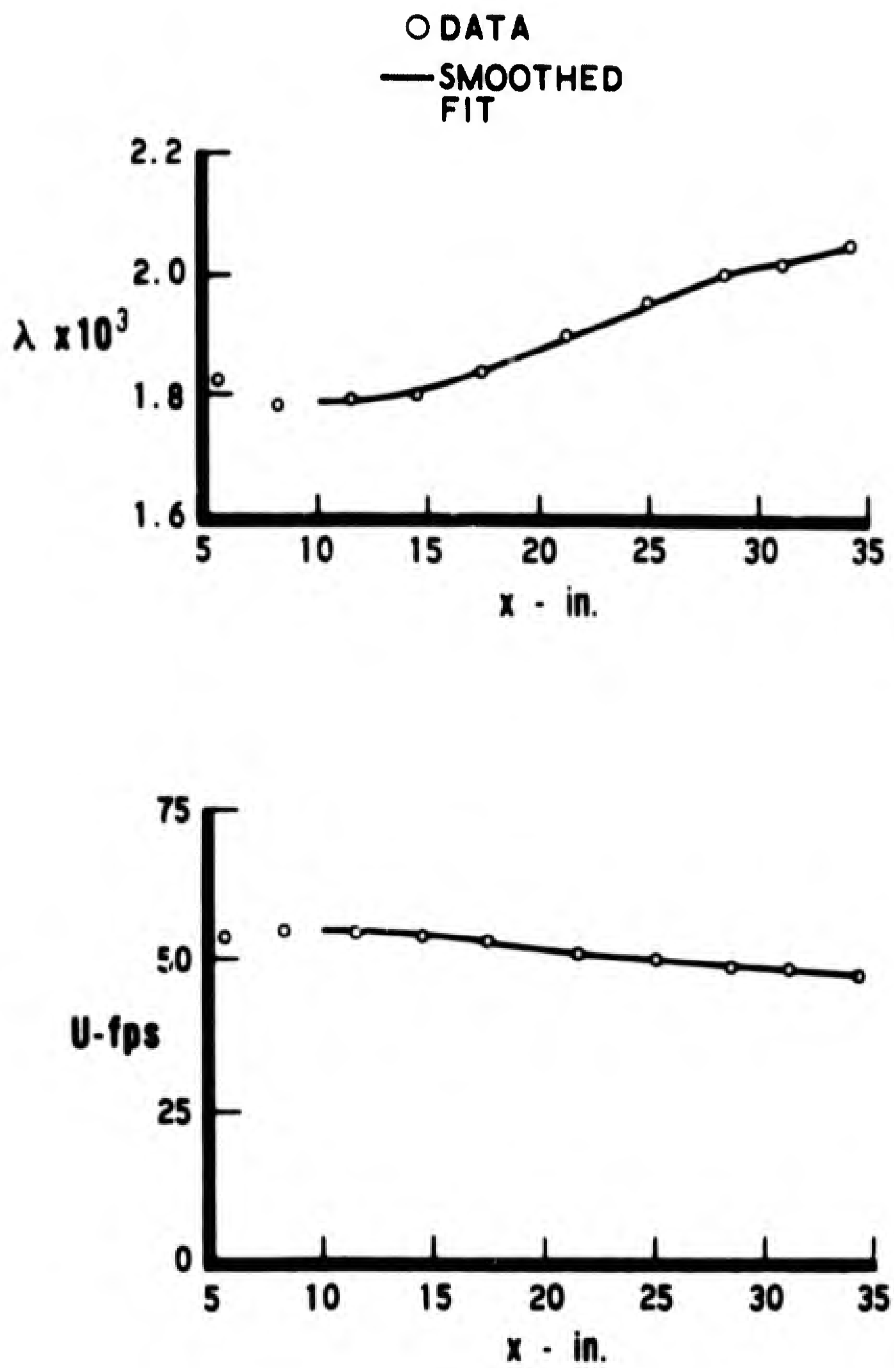

Figure 13. Boundary Conditions for McQuaid Pressure Distribution I

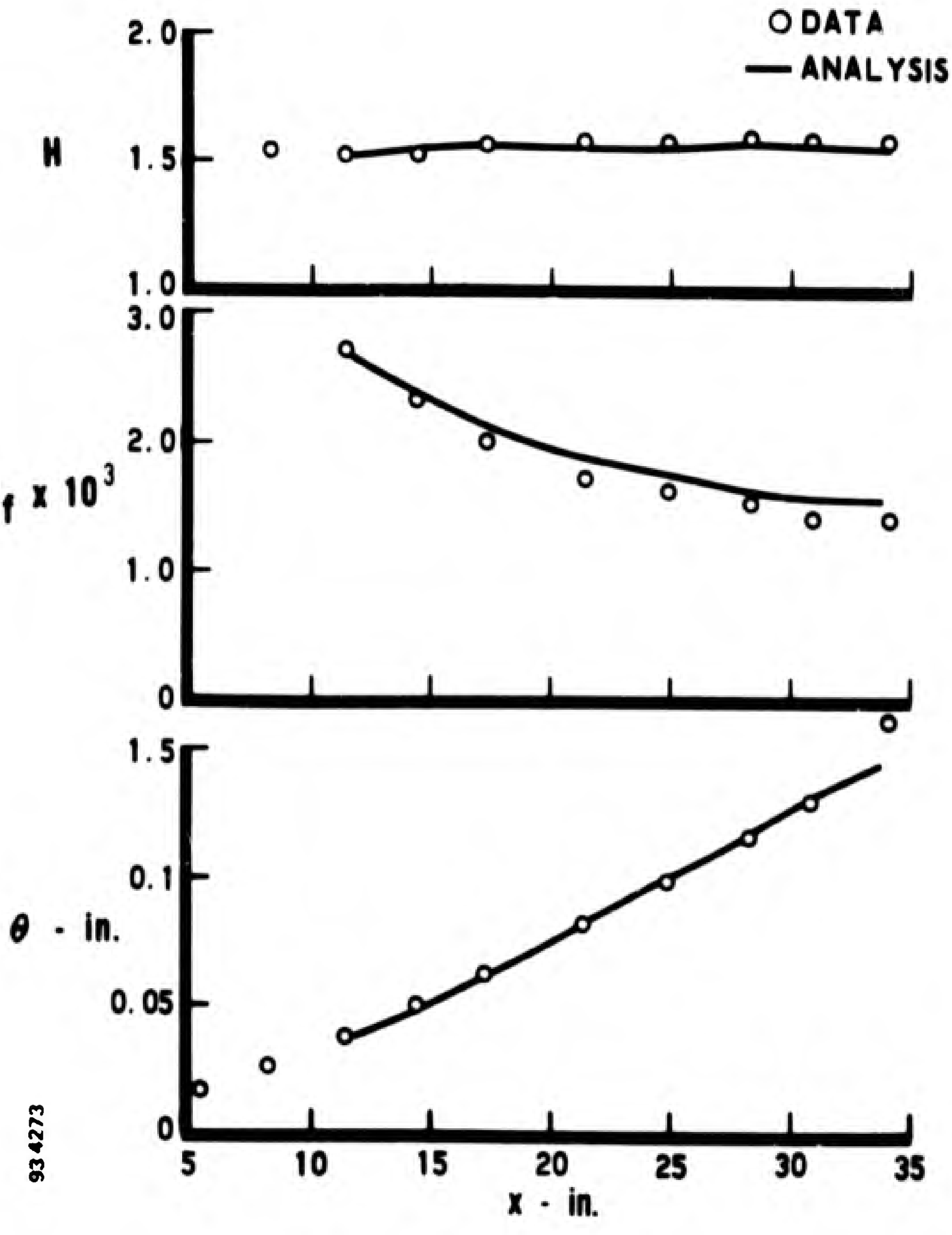

Figure 14. Comparison of the Analysis with McQuaid Pressure Distribution I Data

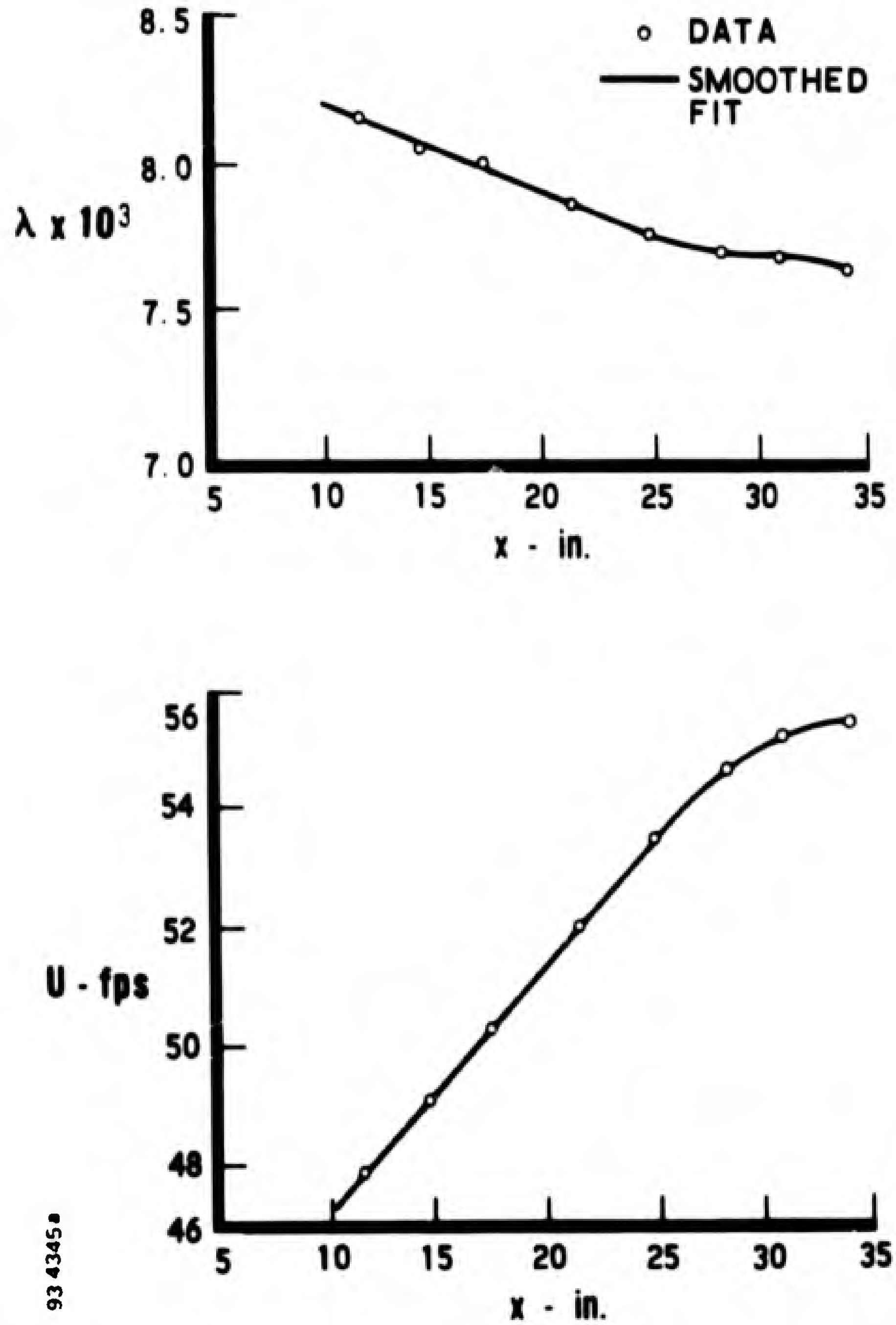

Figure 15. Boundary Conditions for McQuaid Pressure Distribution II

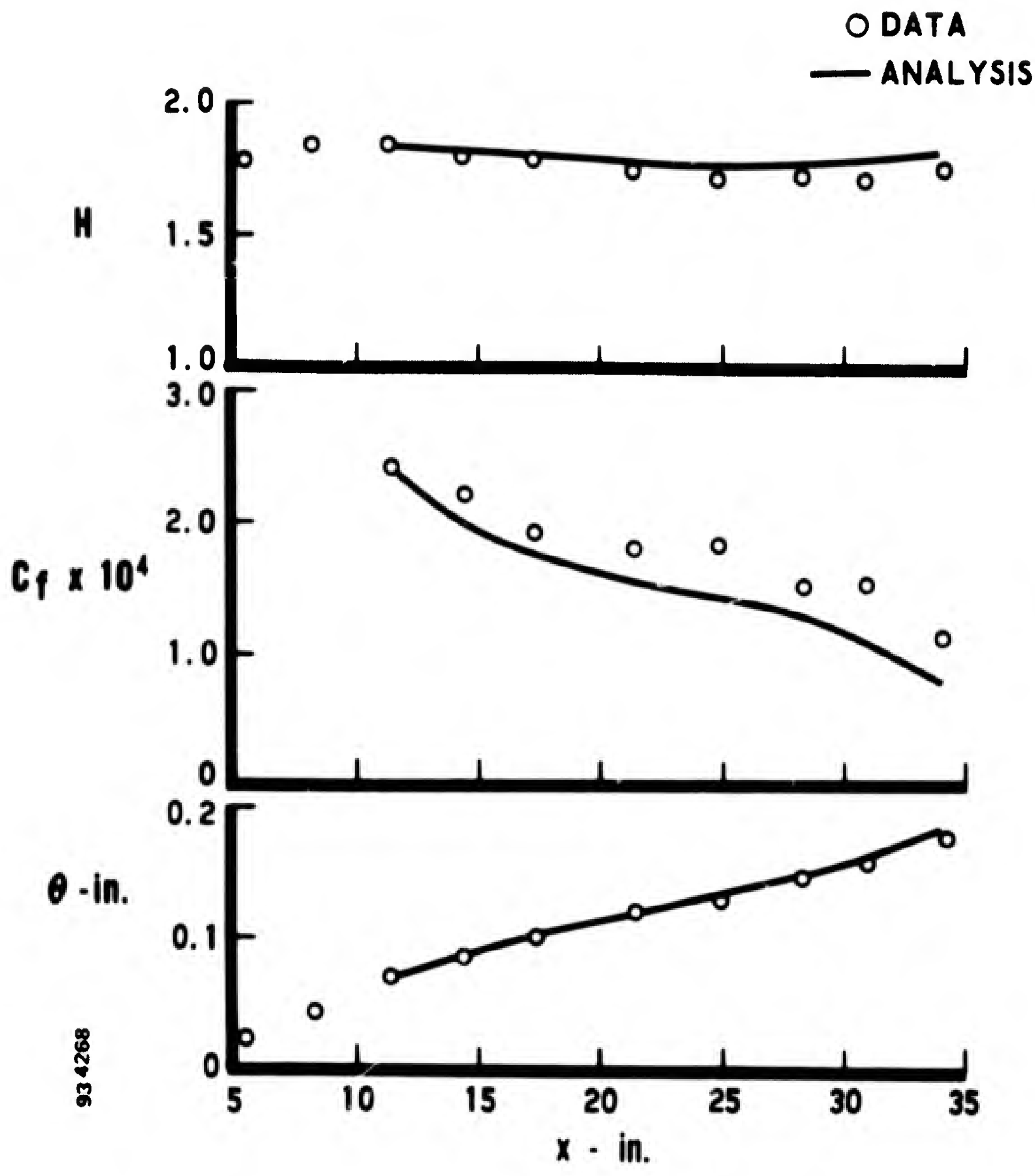

Figure 16. Comparison of the Analysis with McQuaid Pressure Distribution II Data

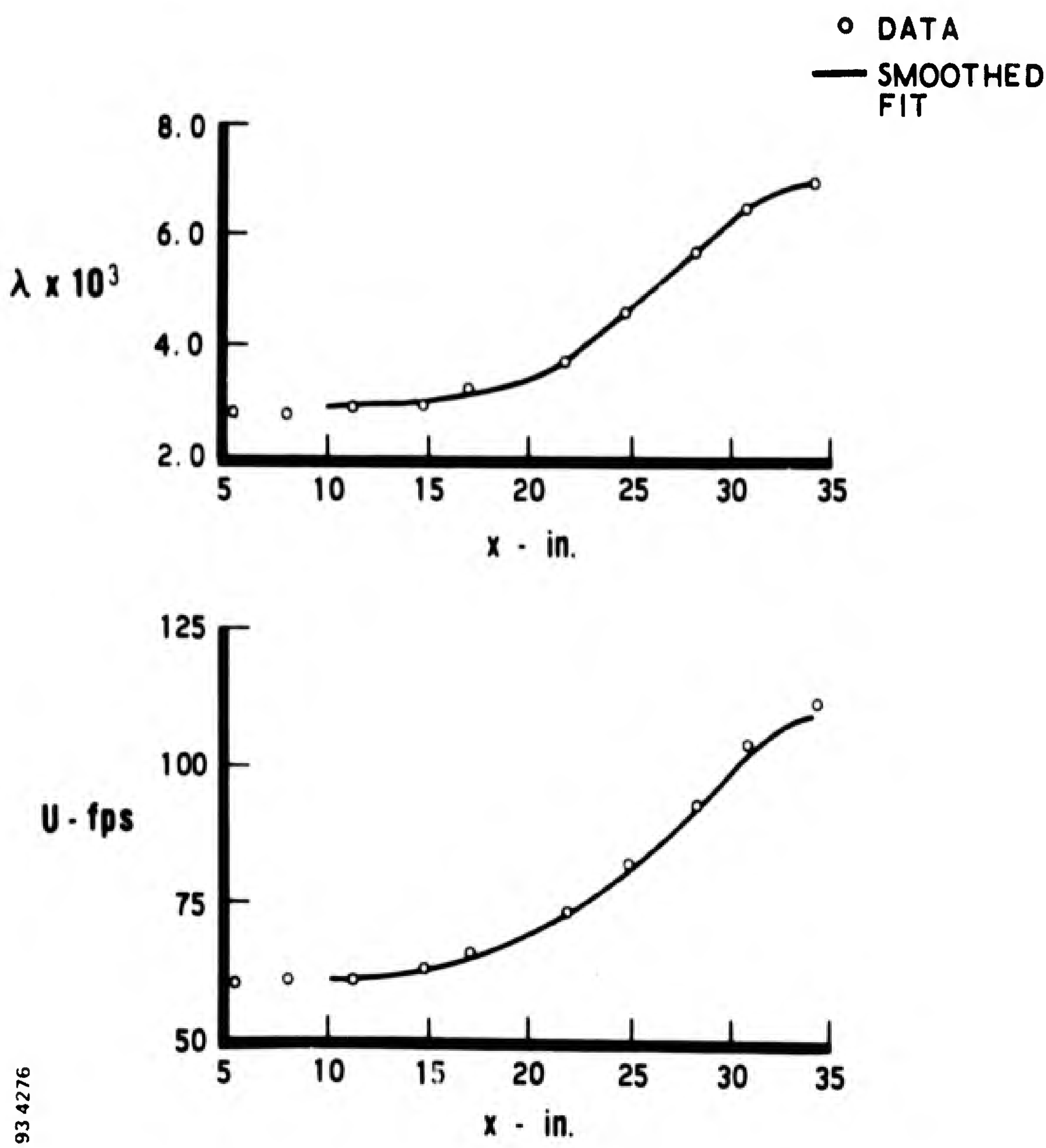

Figure 17. Boundary Conditions for McQuaid Pressure III

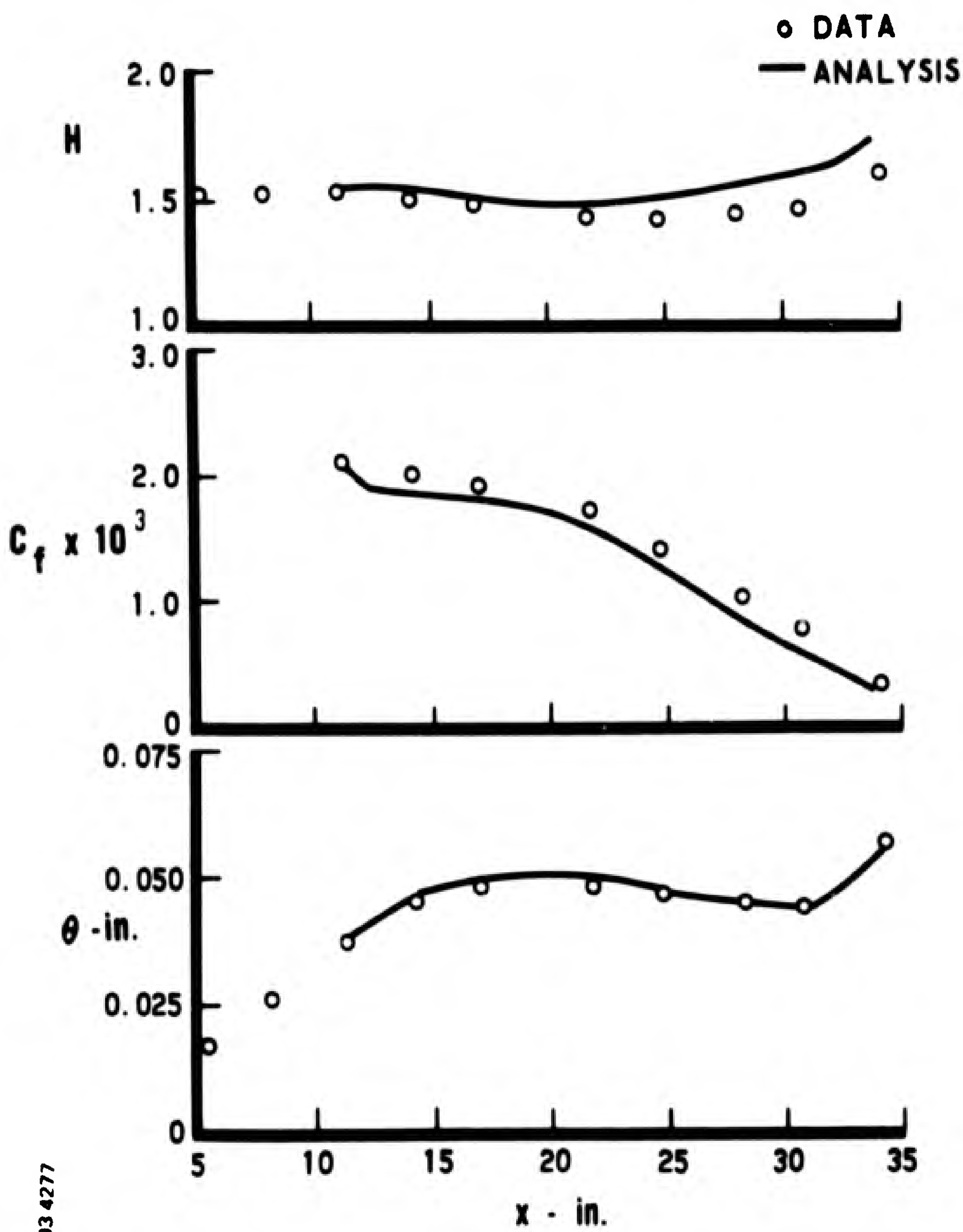

Figure 18. Comparison of the Analysis with McQuaid Pressure Distribution III Data

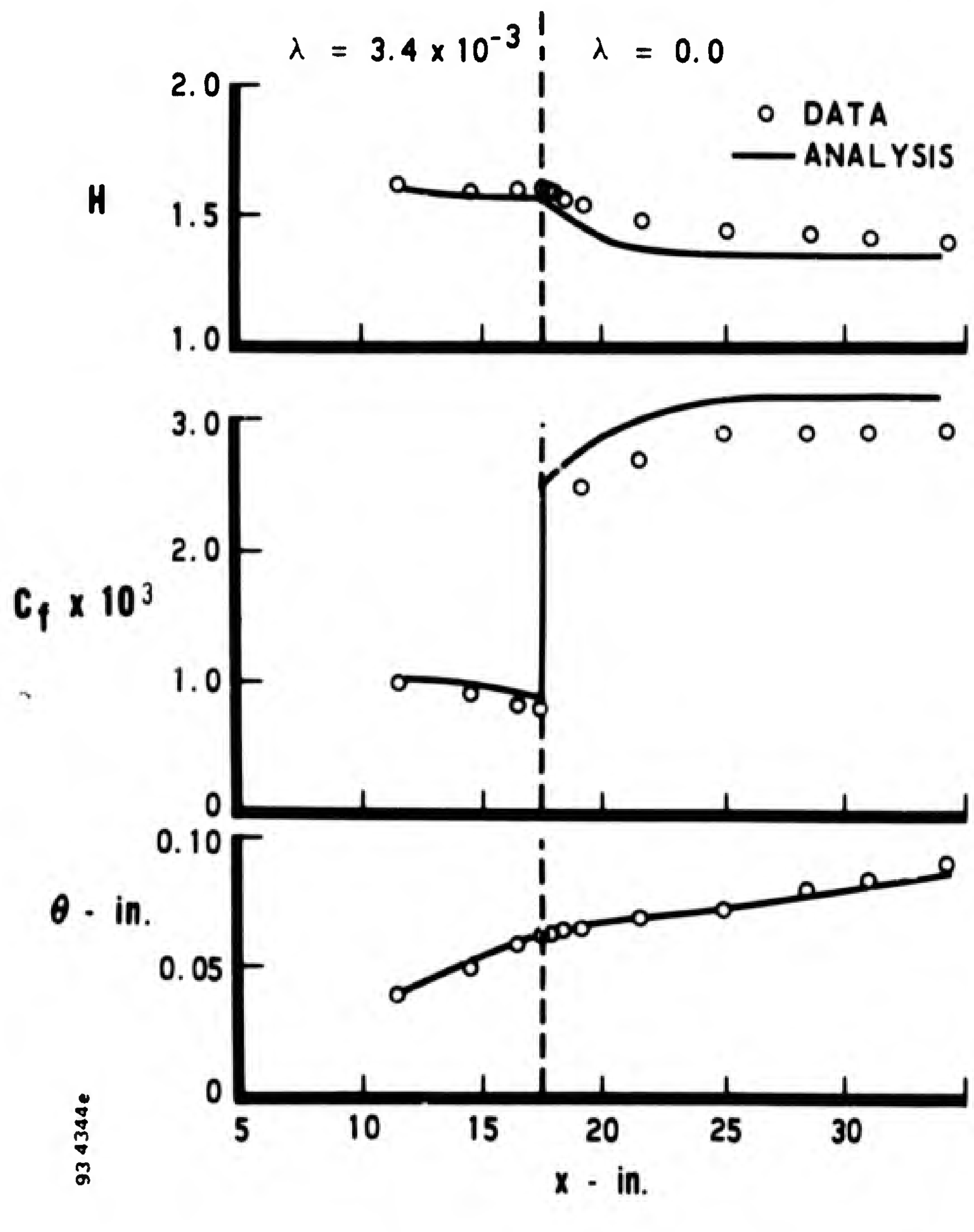

Figure 19. Comparison of the Analysis with McQuaid's Discontinuous Blowing Data

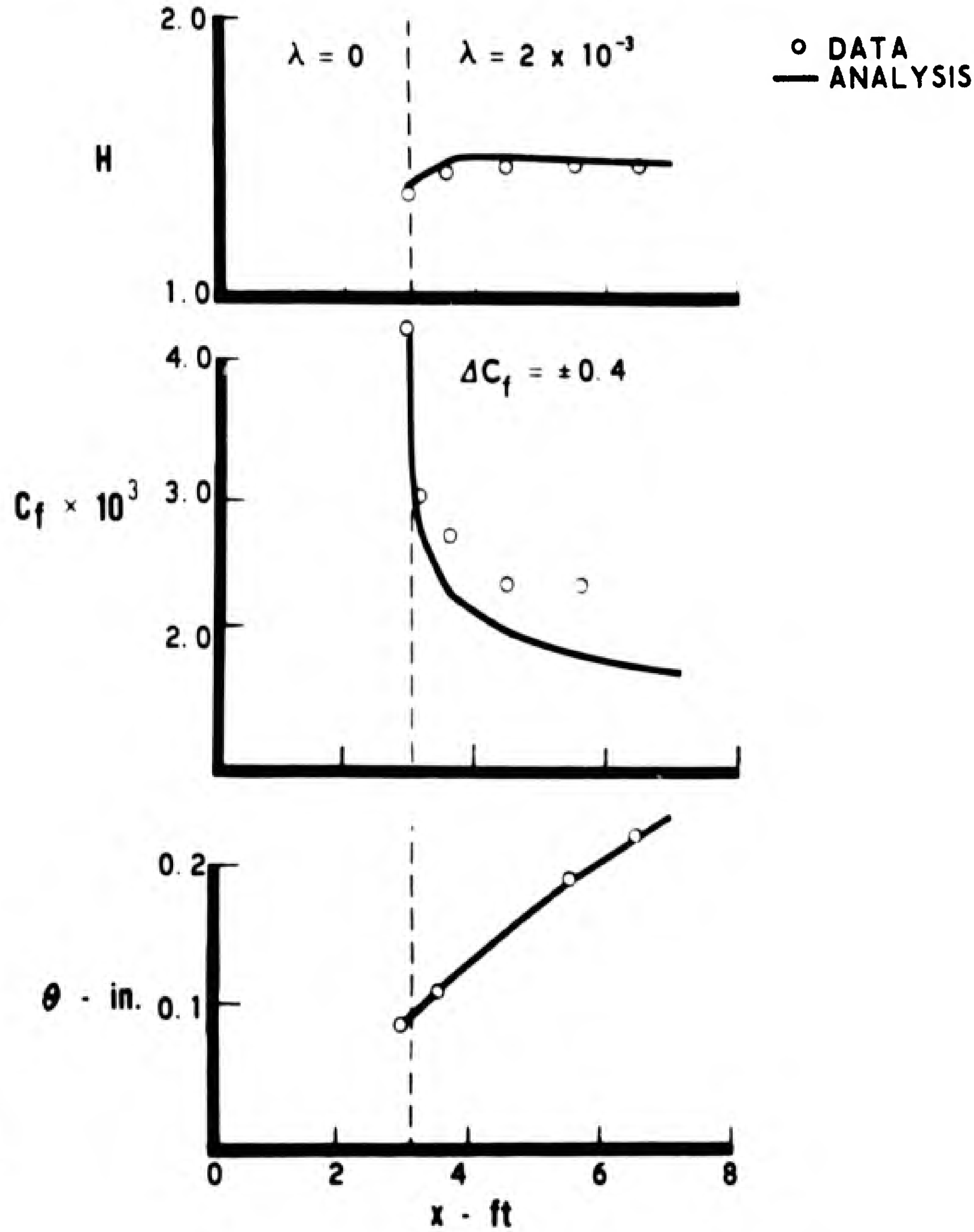

Figure 20. Comparison of the Analysis with Simpson's Discontinuous Blowing Data

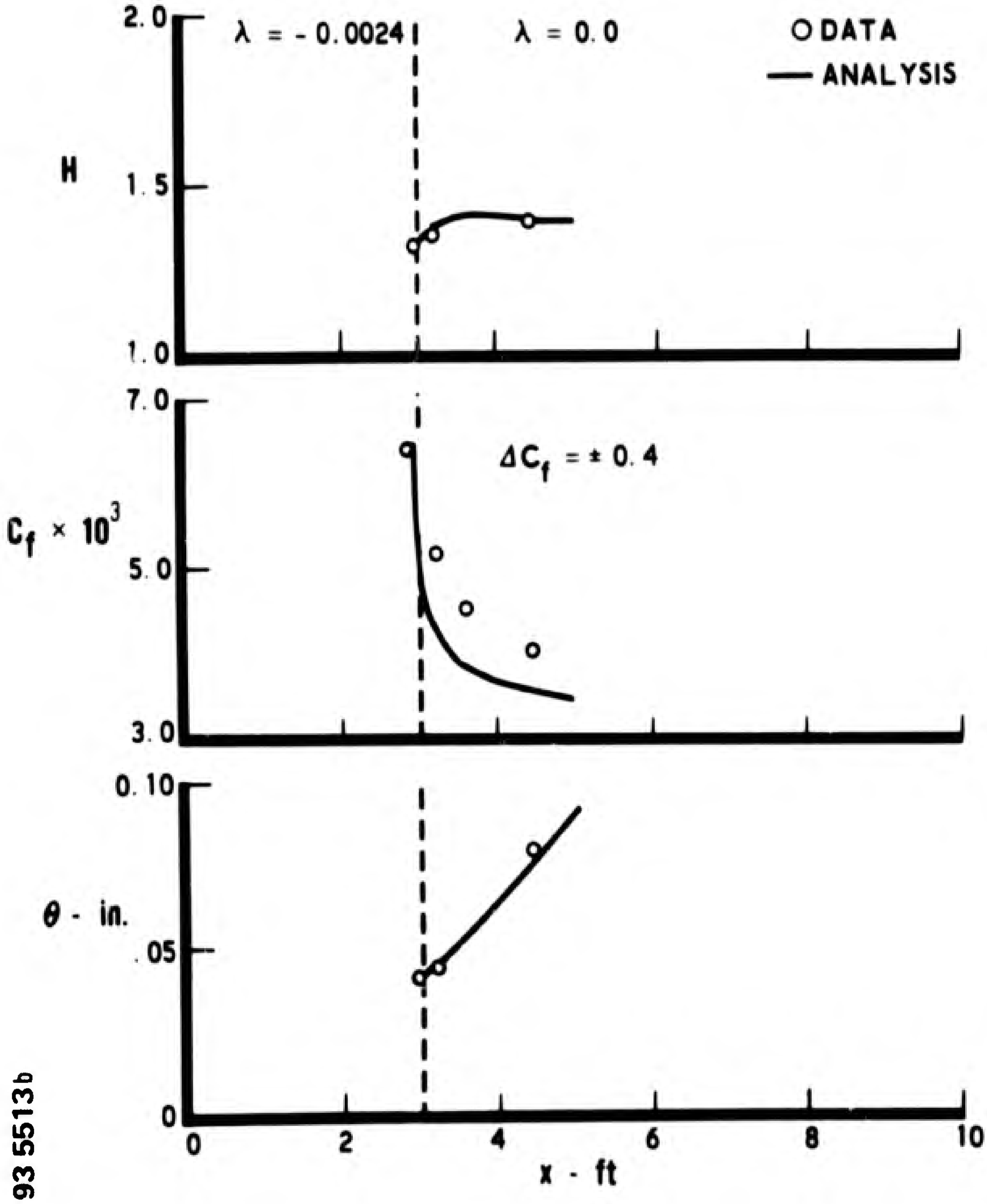

Figure 21. Comparison of the Analysis with Simpson's Discontinuous Suction Data

Ω

APPENDIX A

Solving Eq (20) for u/U gives

$$\frac{u}{U} = f^* \left[1 - F(y/\delta,\ \Pi) \left\{ 1 - \frac{\epsilon}{4}\ F(y/\delta,\ \Pi) \right\} \right] \tag{A-1}$$

Therefore from the definition of δ^*, θ and θ^*, we obtain

$$R_1 (f^*, \Pi, \epsilon) = f^* \left[I_1 - \frac{\epsilon}{4} I_2 \right]$$

$$R_2 (f^*, \Pi, \epsilon) = f^* \left[I_1 - I_2 (f^* + \epsilon/4) + (\epsilon f^*/2)\, I_3 - \frac{\epsilon^2 f^*}{16} I_4 \right]$$

$$R_3 (f^*, \Pi, \epsilon) = f^* \Big[2 I_1 - I_2 (\epsilon/2 + 3f^*) + I_3 (f^{*2} + \frac{3}{2} \epsilon f^*)$$

$$+ I_4 (- \frac{3}{4} \epsilon f^* - \frac{3}{16} f^* \epsilon^2) + I_5 (\frac{3}{16} \epsilon^2 f^{*2}) - I_6 (\epsilon^3 f^{*2}/64) \Big]$$

where

$$I_n (\Pi) \equiv \int_0^1 F^n (y/\delta,\ \Pi)\, d(y/\delta)\ ; \qquad n = 1 - 6$$

Using F $(y/\delta,\ \Pi)$ given in Ref (26), we obtain for I_n the following:

$$I_1 = (1 + \Pi) / K$$

$$I_2 = (2.0 + 3.17 \Pi + 1.49 \Pi^2) / K^2$$

$$I_3 = (6.0 + 11.0 \Pi + 8.30 \Pi^2 + 2.46 \Pi^3) / K^3$$

$$I_4 = (24 + 46.6 \Pi + 41.7 \Pi^2 + 20.1 \Pi^3 + 4.27 \Pi^4) / K^4$$

$$I_5 = (120. + 238 \Pi + 227 \Pi^2 + 133 \Pi^3 + 46.6 \Pi^4 + 7.66 \Pi^5) / K^5$$

$$I_6 = (720 + 1440 \Pi + 1413 \Pi^2 + 893 \Pi^3 + 384 \Pi^4 + 105 \Pi^5 + 14.0 \Pi^6) / K^6$$

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APPENDIX B

Consider the Falkner Skan [28] equation:

$$f''' + ff'' + \beta(1 - f'^2) = 0 \qquad \text{(B-1)}$$

with the boundary conditions

$$f(0) = -f_w$$

$$f'(0) = 0$$

$$f'(\infty) = 1$$

In general these solutions and the integral properties represent a two parameter family of curves. Therefore, H can be written as:

$$H = \frac{\int_0^{\infty} (1 - f')\, d\eta}{\int_0^{\infty} (1 - f')\, f' d\eta} = H(f_w, \beta) \qquad \text{(B-2)}$$

The solutions of these equations can be considered as experimental data to see if a "defect" plot can be made and an integral property similar to Clausers G can be obtained in terms of a modified pressure gradient parameter.

First considering the zero pressure gradient case ($\beta = 0 = \frac{dp}{dx}$), Figure B-1 shows the velocity profiles as a function of η for various blowing rates, f_w. Replotting these curves in a defect form, we obtain a reasonable collapsing of the data. This result is shown in Figure B-2 where we have used the quantity $(f''(o) + f_w)^{1/2}$ as the denominator in the defect. This is analogous to f^* which has been introduced for the turbulent case.

Returning to the integral properties, we consider using β_T^* as a single independent parameter where as indicated previously

$$\beta_T^* \equiv \frac{\delta^* \frac{dp}{dx}}{\tau_w + \rho v_w U} \tag{B-3}$$

For a Falkner Skan flow, it can easily be shown that

$$\beta_T^* = \frac{-\beta}{\left[f''(o) + f_w\right]} \int_o^\infty (1 - f')\, d\eta \tag{B-4}$$

Furthermore, in a manner analogous to Clauser, define for the Falkner Skan flows

$$G^* \equiv \frac{1 - \frac{1}{H}}{\left[f''(o) + f_w\right]^{1/2}} \tag{B-5}$$

Figure B-3 shows G^* vs β_T^* for the Falkner Skan solutions. It is seen that with this definition, the blowing, suction and no mass transfer data fall on the same curve. Hence, with G^* and β_T^* defined as in Eqs (B-4) and (B-5), the no mass transfer curve can be used.

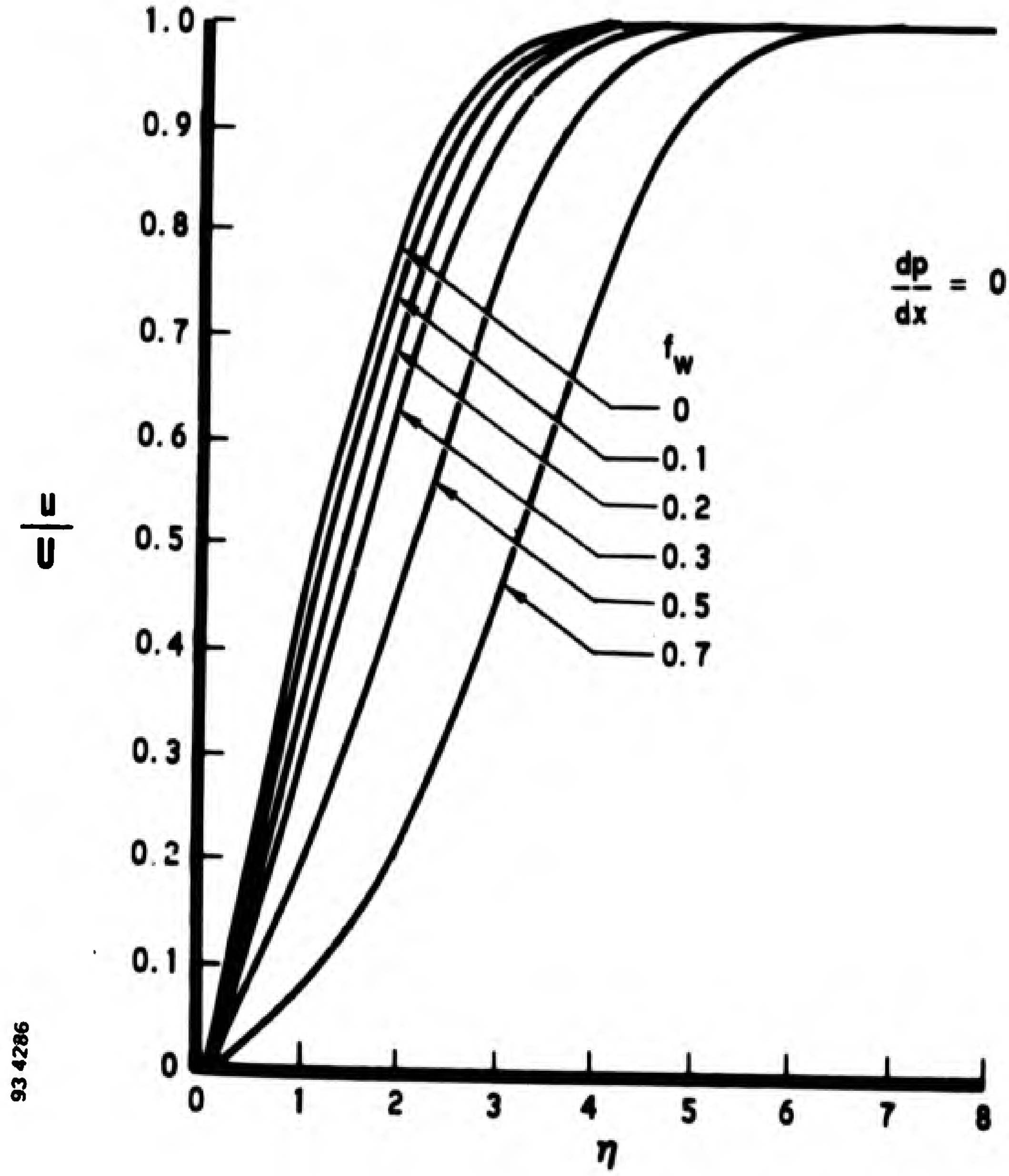

Figure B-1. Laminar Similar Velocity Profiles with Blowing

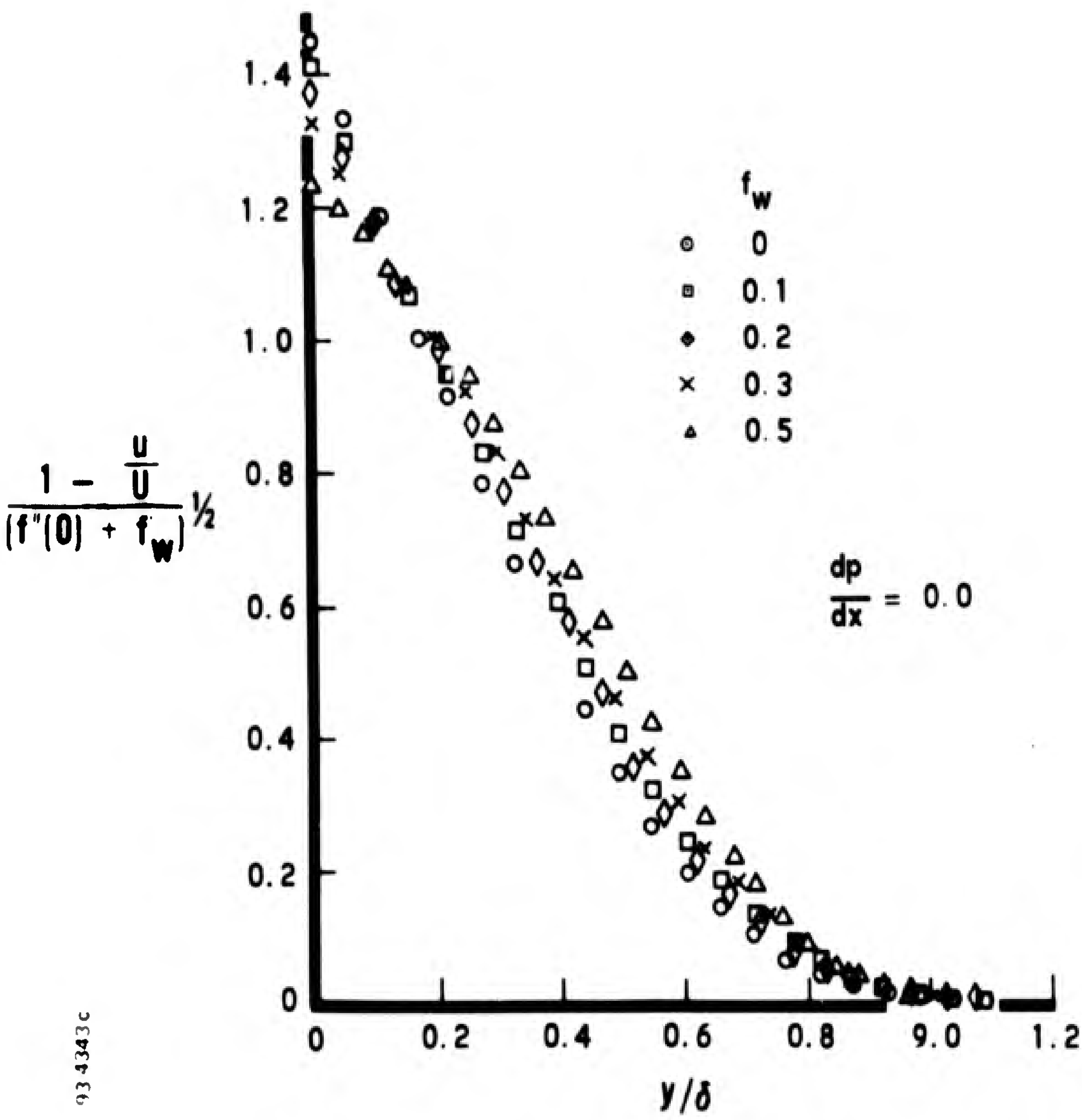

Figure B-2. Laminar Similar Profiles in Defect Coordinates

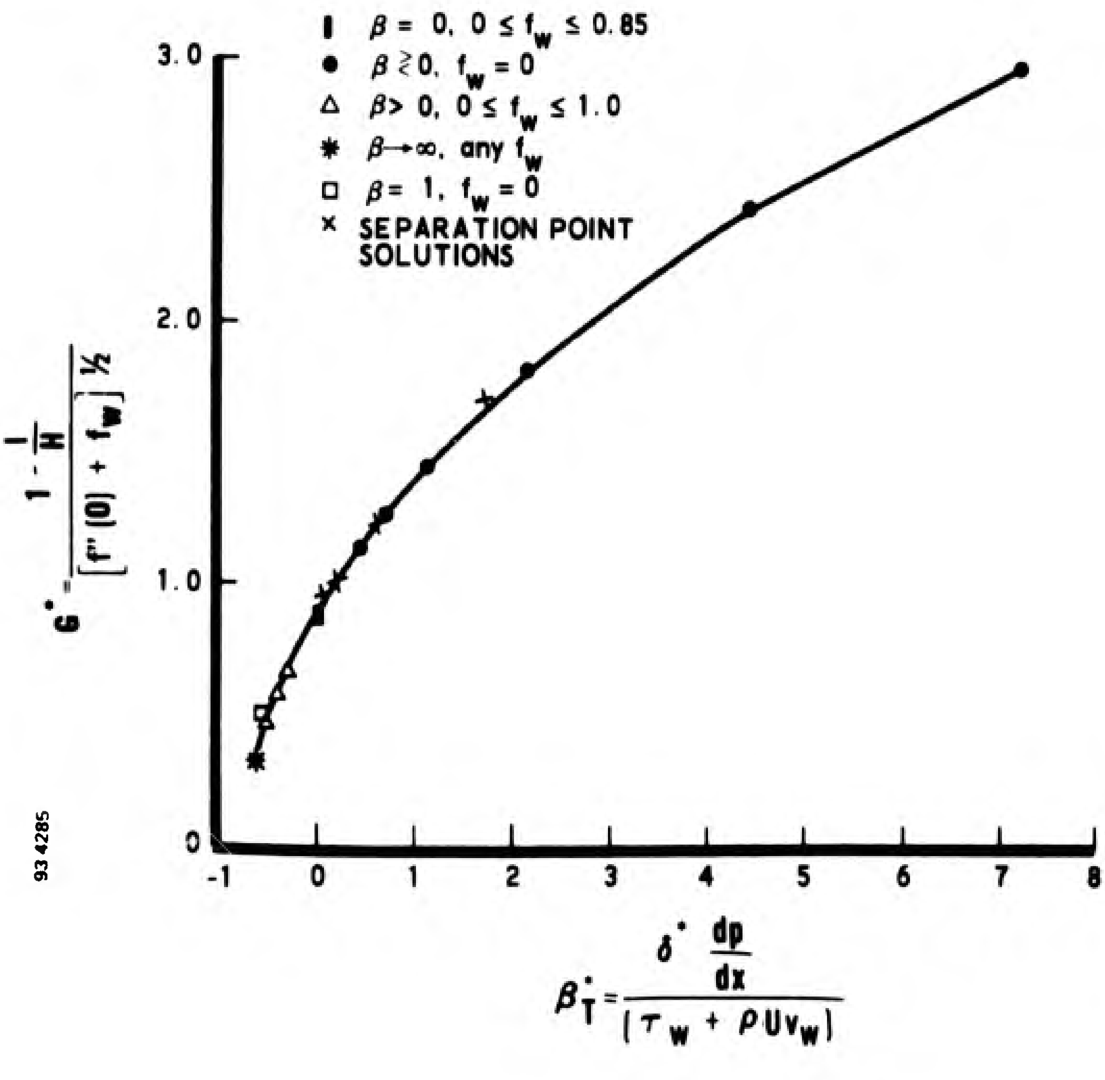

Figure B-3. Laminar Shape Factor vs New Similarity Parameter

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13. ABSTRACT An analysis of the incompressible, turbulent boundary layer, including the combined effects of mass transfer and pressure gradient is presented in this paper. An integral method employing the integral mechanical energy equation forms the basis of the analysis. Stevenson's velocity profiles are used to obtain the functional dependence of the integral properties and also obtain a skin friction law. A definition of an equilibrium flow with mass transfer and pressure gradient is given in order to evaluate the dissipation integral (C_D) which appears in the integral mechanical energy equation. This definition requires a pressure gradient parameter similar to Clauser's β_T with a modification to include the effect of mass transfer to be held constant. An expression for C_D in the case of equilibrium turbulent flows is then obtained which depends directly on this new pressure gradient parameter (β_T^*). In order to treat the general case of non-equilibrium flows, this expression for C_D is uncoupled from β_T^* through the use of a single empirical curve fit of the existing no mass transfer equilibrium flow data relating β_T to the Clauser shape parameter.

In addition to uncoupled C_D from the pressure gradient parameter, several specified variations in mass transfer rater are assumed in order to obtain an expression for C_D which is not a function of the mass transfer rate derivative. The numerical results are found to be weakly dependent on which of these variations is used. Comparisons of the numerical results with a wide variety of experimental data, including cases where the blowing rate and pressure are varying simultaneously, shows good agreement. In addition, several problems with discontinuities in blowing or suction are solved and also seen to be in good agreement with the data. (Unclassified Report).

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