

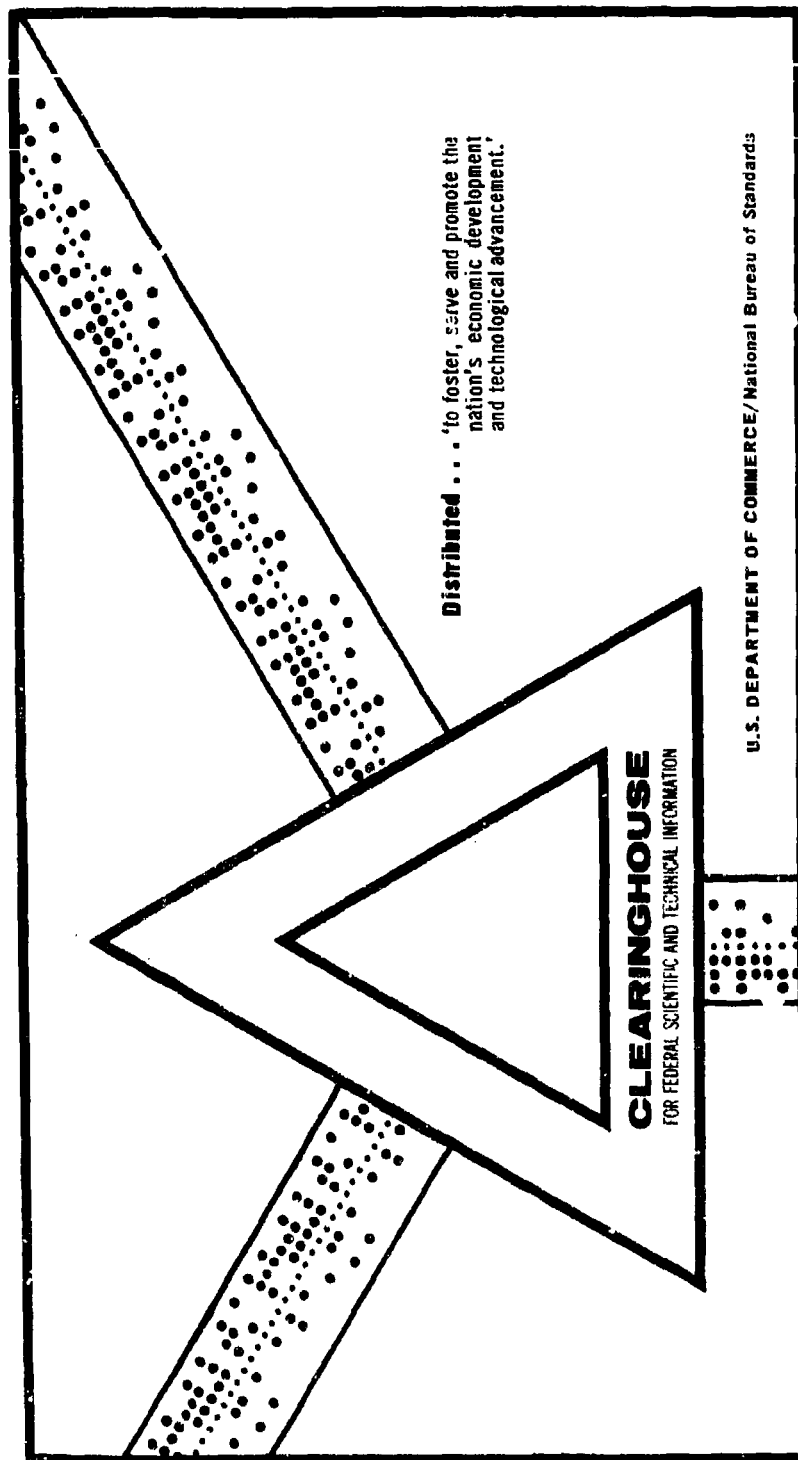
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A REVIEW OF SEARCH AND RECONNAISSANCE THEORY LITERATURE

Michael L. Moore

Michigan University
Ann Arbor, Michigan

January 1970



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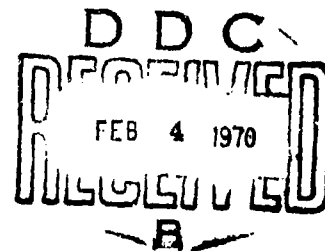
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TECHNICAL REPORT

**A REVIEW OF SEARCH AND RECONNAISSANCE
THEORY LITERATURE**

MICHAEL L. MOORE

JANUARY 1970



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THE UNIVERSITY OF MICHIGAN
ANN ARBOR, MICHIGAN

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SRL 2147 TR 70-1
Technical Report

A REVIEW OF SEARCH AND RECONNAISSANCE
THEORY LITERATURE

Michael L. Moore

SYSTEMS RESEARCH LABORATORY
Department of Industrial Engineering
The University of Michigan

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1.0 INTRODUCTION

Research being performed by the Systems Research Laboratory (SRL) under contract number N00014-67-A-0181-0012 with the Office of Naval Research is concerned with the development of more generalized mathematical structures of military processes. Emphasis has been directed to the modeling of combat processes and the development of associated allocation strategies. These efforts all assume perfect intelligence. As noted in the first progress report (SRL, 1969), intelligence could reasonably have a large effect on combat effectiveness predictions, especially when one considers its interaction with the allocation strategy.

It was thought that many of the existing search and reconnaissance theories would be useful for predicting the amount of intelligence-gathering capability possessed by a tactical unit. A thorough literature review in this area, however, indicated that existing theories are less than useful for this purpose. Most of the research efforts have been devoted to the development of strategies for the optimal allocation of search effort and little to the development of descriptive models of intelligence-gathering processes. The existing results do not consider important aspects such as intermittent target visibility, multiple targets, moving targets, and others. Accordingly, part of the research effort on this

contract is being devoted to the development of models of intelligence-gathering processes.

The purpose of this interim technical report is to present the results of the literature review both as a base for our research and to indicate fruitful areas of research for other investigators. Principal results in the field and the techniques used in attaining them are presented in an annotated bibliography. A comprehensive bibliography, organized under subject classifications, is included. Finally, some relevant areas for future research are described.

1.1 Definitions and Notations

This section contains some basic definitions and notations used throughout the paper. Additional notations and exceptions to those specified herein will be noted in the text.

Detection - The act of gathering information pertaining to the object being sought, the sifting out of what is important information and the relaying of that information in some efficient form to the decision maker.

Incremental Detection Model - Let q_i be the instantaneous probability of detection on the i^{th} scan of an area. Given n such scans, the probability of detection is

$$P(D) = 1 - \prod_{i=1}^n (1 - q_i).$$

Continuous Detection Model - The probability of detecting the target in the interval $(t, t + dt)$ is given by $\gamma(t) dt$. Given continuous observance over an interval $(0, \tau)$, the probability of detection is

$$P(\tau) = 1 - e^{-\int_0^{\tau} \gamma(t) dt}$$

Search Strategy - The decision made on the basis of information obtained from the detection process. A "search strategy" will be that set of rules which associates decisions with every conceivable result of the detection process.

Target - The object of the search, a military target, a mineral deposit, or any other object about which information is desired.

Notation:

- P_i = prior probability of the target being in the i^{th} subregion
- q_i = the conditional probability of detection for the i^{th} subregion
- α_i = $1 - q_i$
= the conditional overlook probability for the i^{th} subregion
- σ = $(\sigma_1, \dots, \sigma_j, \dots)$
= a search strategy (possibly infinite) where σ_j denotes which region is to be searched on the j^{th} trial
- c_i = cost of searching the i^{th} subregion
- t_i = the time spent searching the i^{th} subregion

1.2 Classification of Detectors and Targets

Models of search and reconnaissance processes treat detectors and targets with varied combinations of properties or assumptions regarding their behavior. This section presents a classification of analytic assumptions that may be used to describe the behavior of detectors and targets.

Detectors

1. Single Detector with a Single Scan
 - a) Binary detection (Incremental Detection Model)
 - b) Interval detection (Continuous Detection Model)
 - 1) Non-cumulative probability of detection
 - 2) Cumulative probability of detection
 - (a) partial loss of information
 - (b) no loss of information
2. Single Detector with Multiple Scan Capability
 - a) Binary detection
 - b) Interval detection
 - 1) Non-cumulative probability of detection
 - 2) Cumulative probability of detection
 - (a) partial loss of information
 - (b) no loss of information

3. Multiple Detectors with Single Scan Capability

a) Binary detection

- 1) Detectors act independently
- 2) Detectors act dependently

b) Interval detection

1) Independent action

- (a) non-cumulative probability of detection
- (b) cumulative probability of detection
 - (1) partial loss of information
 - (2) no loss of information

2) Dependent action

- (a) non-cumulative probability of detection
- (b) cumulative probability of detection
 - (1) partial loss of information
 - (2) no loss of information

4. Multiple Detectors with Multiple Scan Capability

a) Binary detection

- 1) Independent action
- 2) Dependent action

b) Interval detection

1) Independent action

- (a) non-cumulative probability of detection
- (b) cumulative probability of detection

- (1) partial loss of information
 - (2) no loss of information
- (2) Dependent action
 - (a) non-cumulative probability of detection
 - (b) cumulative probability of detection
 - 1) partial loss of information
 - 2) no loss of information

Targets

- 1. Single Target (which may be an entire group)
 - a) The target can exhibit binary visibility, i.e., it is either visible or not with specified probability.
 - b) The target may have only a single interval of visibility, the length of this interval having a known probability density function.
 - (1) The single visibility interval can begin at time $t = 0$.
 - (2) The single visibility interval can begin at some time $t \neq 0$.
 - c) The target can exhibit multiple periods of visibility.

2. Multiple Targets

- a) The members act independently with:
 - (1) binary visibility
 - (2) single interval visibility
 - (a) beginning at time $t = 0$,
 - (b) beginning at time $t \neq 0$.
- b) The members act in a dependent fashion with:
 - (1) binary visibility
 - (2) single interval visibility
 - (a) beginning at time $t = 0$,
 - (b) beginning at time $t \neq 0$.
 - (3) multiple periods of visibility.

The diagram¹ shown in Figure 1 presents, in flow chart format, the various attributes of search problems and analytic assumptions used in modeling them. Each paper discussed in this literature review can be characterized by a path through the diagram.

1.3 Organization of Review

The papers listed in the bibliography (Chapter 6) are presented in alphabetical order under the general headings

¹The diagram is a modification of one given by H. Beiman, "An Investigation of Sequential Search Algorithms," Operations Research, Inc., Silver Spring, Maryland, AD 657050.

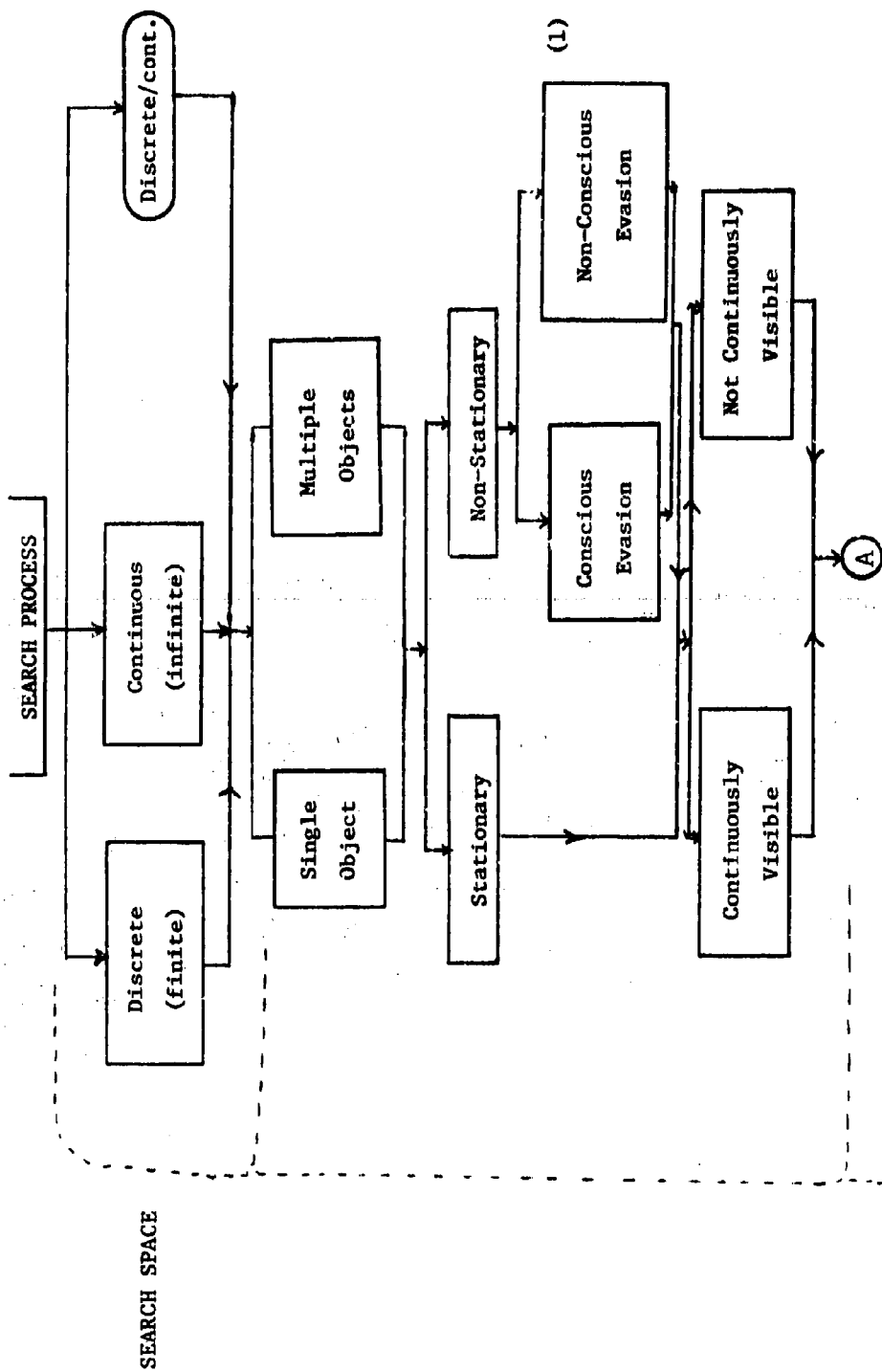
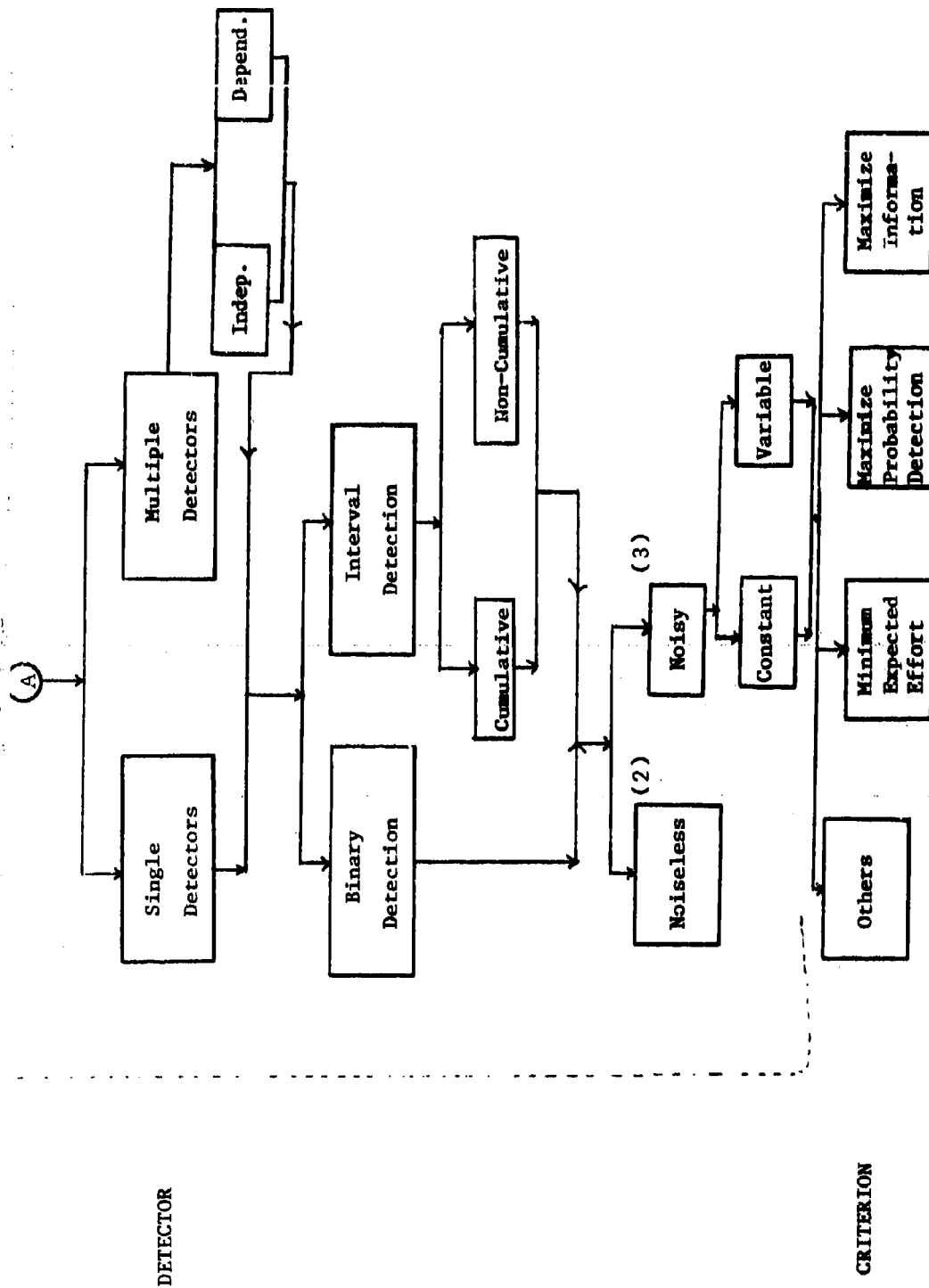


Figure 1 Diagram of Search Problem Attributes and Assumptions



(2) Perfect detector.

(3) Includes type I and II errors.

Figure 1 Diagram of Search Problem Attributes and Assumptions

noted below. Categories C, D, and E are discussed in Chapters 2, 3, and 4, respectively. Some areas for future research are described in Chapter 5.

(A) General Discussion

This category includes the pioneering work of Koopman, the applications of Morse and Kimball, and the bibliographies of Dobbie and Enslow.

(B) Measures of Performance

This category includes papers which consider various search objectives. Although it is usually assumed that the objective of search is the detection of the target, other objectives such as maximizing the information gain have been proposed and studied.

(C) Allocation of Effort in One-Sided Search

This category considers the problem of the distribution of effort required to find a target when the distribution of the target is known to the searcher. These subheadings are included under this category.

(1) Stationary Targets

The target is assumed stationary although some authors consider targets that suddenly appear and remain visible.

(2) Large Stationary Targets

The size and shape of the target may have some effect on the formulation and solution of the problem.

(3) Moving Targets

The target is moving without conscious evasion and the searcher knows the motion or distribution of motion.

(D) Two-Sided Search

This category, which considers the game theoretic aspects of search, investigates the search problem with a conscious evader. Included in this category are the search/evasion problems in which the searcher and evader can alter their motions differentially by choices of continuously varying parameters, e.g., the theory of differential games as formulated by Isaacs.

(E) Miscellaneous

This category includes papers containing important results in the development of search theory and methodology or application of search concepts to the operations of reconnaissance and surveillance.

2.0 ALLOCATION OF EFFORT IN ONE-SIDED SEARCH

The purpose of this chapter is to summarize the published results obtained to date in regard to the allocation of effort in the one-sided search for a stationary target. In this context, the distribution of the target is known to the searcher although it may not be present at the start of the search. The major results of investigations in this area are presented, as well as the interrelationships between them, if any. Since the entire field seems to have originated from the investigations of B. Koopman, these results will be the starting point of the review.

Koopman (1946) describes two types of detection processes; the "glimpse" or discrete mode, and the continuous mode. In the former, one has a single scan or glimpse probability of detection q_i which may be functionally dependent upon range, time, etc. Given n such looks, the probability of detection is determined as

$$P_n = 1 - \prod_{i=1}^n (1 - q_i) .$$

The continuous mode is characterized by the assumption that the probability of detection in a short time interval of length dt is given by $\gamma(t)dt$. Given continuous searching over a time interval of length t , the probability of detection is given by

$$P(t) = 1 - e^{-\int_0^t \gamma(t)dt} .$$

The optimal allocation of searching effort for a stationary target was derived by Koopman in the following fashion:

Let a stationary target be located in a known region A with known probability density function $p(x,y)$ continuous in the region A with the properties

$$\int_A p(x,y) dx dy = p_0 > 0,$$

$$\iint_A p(x,y) dx dy = 1.$$

Assume the searcher has certain constraints on the amount of effort, ϕ , that can be allocated to the search. Consider a search density function $\phi(x,y)$ defined on the region A with the properties that

$$\iint_A \phi(x,y) dx dy = \phi, \quad (1)$$

$$\phi(x,y) \geq 0 \text{ on } A. \quad (2)$$

Assume further that the searcher is operating in the continuous detection mode. Then the probability of detecting the target, $p[\phi]$, is given by

$$p[\phi] = \iint_A p(x,y) \left(1 - e^{-\phi(x,y)}\right) dx dy. \quad (3)$$

The conditional probability of detection, $1 - e^{-\phi(x,y)}$, is the result of the two-dimensional "law of random search." The fundamental problem is to determine from among all the functions satisfying equations 1 and 2 that which gives 3 its maximum value.

Koopman obtains the optimal solution as

$$\phi(x,y) = \log p(x,y) - \frac{1}{\hat{A}} \iint_{\hat{A}} \log p(x,y) dx dy + \phi/\hat{A},$$

for

$$(x,y) \in \hat{A},$$

and

$$\phi(x,y) = 0 \text{ for } (x,y) \in A - \hat{A},$$

where

$$\hat{A} = \left\{ (x,y) \mid p(x,y) \geq b, \log b - \frac{1}{\hat{A}} \iint_{\hat{A}} \log p(x,y) dx dy + \frac{\phi}{\hat{A}} = 0 \right\}$$

By considering $A = A_1 + A_2 + \dots + A_n$, one can obtain the solution to the n region search problem. Some generalizations suggested by Koopman, include the case of visibility varying from position to position, the case of weighting the probability of detection by a function dependent upon where the target is detected, and weighting the search density function by a cost function dependent upon the region being searched.

Charnes and Cooper (1958), develop an algorithm for the solution of a discrete version of Koopman's problem.

Let $\{P_j\}$, $j = 1, 2, \dots, N$ denote the probability that the target is in the j^{th} region. Then if $\{\phi_j\}$, $j = 1, 2, \dots, N$ denotes the normalized search density vector, Koopman's problem becomes

$$\begin{aligned} \text{Min} \quad & \sum_{j=1}^N P_j e^{-\beta \phi_j}, \\ \text{S.T.} \quad & \sum_{j=1}^N \phi_j = 1, \phi_j \geq 0, \\ & \text{and } P_j \geq 0, \sum_{j=1}^N P_j = 1, \end{aligned}$$

where β is a scale factor relating the allocations, ϕ_j , of search effort to the total amount of search effort available. The algorithm is obtained from the application of the Kuhn-Tucker conditions for optimality to the above convex programming problem.

The detection processes in Koopman's formulations were quite restrictive. de Guenin (1961) generalized these processes as follows: Let $p[\phi(x)]$ denote the probability of detecting the target with an effort $\phi(x)$ when the target is at x . The following assumptions are made with respect to $p[\phi(x)]$,

- (1) $p(0) = 0$
- (2) $p'(\phi) \geq 0$
- (3) $p'(\phi) =$ a decreasing function of ϕ
- (4) $p'(0) \geq 0, p'(\infty) = 0$.

From the above properties $p'(\phi)$ admits of an inverse function $\psi = f(p')$. The basic problem becomes

$$\text{Max } P = \int_{-\infty}^{\infty} g(x)p[\phi(x)]dx$$

$$\text{S.T. } \phi(x) \geq 0$$

$$\int_{-\infty}^{\infty} \phi(x) = \phi,$$

where $g(x)$ is the probability density function for target location. de Guenin derives the following necessary conditions for optimality under the above assumptions.

Theorem: A necessary condition for p to be optimum is that at any point x such that $\phi(x) > 0$,

$$g(x) p'_{\phi}[\phi(x)] = \text{constant}$$

$$g(x) / \left(d\phi/dp \right) = \text{constant},$$

where $d\phi/dp$ = the marginal effort to increase the detection probability. One might restate this result as follows:

Whenever the distribution of effort is optimum, the marginal effort required to increase the detection probability at any point is proportional to the probability density, $g(x)$, of the location of the object.

Koopman (1946) observed that the distribution which maximizes the detection probability with a given amount of effort has the interesting property that it is the sum of conditionally optimal distributions. That is, the optimal distribution of $E_1 + E_2$ is the sum of the optimal distribution of E_1 and the conditionally optimal distribution of E_2 , given that the target has not been found with the previous distribution of E_1 . Dobbie (1963) develops sufficient conditions for this additive property to hold, then shows that the solution to this class of problems can be attained by "optimizing conditionally in the small." Let $p(x, f(x))$, be the conditional probability that a target at x will be detected by the searching effort of intensity $f(x)$ at x . If the detection rate, $k(x)$, is independent of the searching effort $f(x)$, then Dobbie shows that

$$p(x, f(x)) = 1 - \exp[-k(x)f(x)].$$

Furthermore, it is also shown that if $\frac{\delta p(x, f)}{\delta f}$ is a positive monotonic non-increasing function of f for every x , then the distribution obtained by maximizing the probability of detection in-the-small will maximize the overall detection probability. It is also shown that the expected effort required to detect the target is given by

$$E = \int_0^{\infty} Q(E) dE = \int_0^{\infty} (1-p(E)) dE,$$

$p(E)$ is the probability of detecting the target with effort distributed according to a particular distribution function. From the above equation, one can see that the expected effort is minimized by always distributing the effort to maximize the probability of detection with the effort expended thus far. In contrast, the distribution that maximizes the probability of detection with a given amount of effort can be non-optimal for all values of effort less than the total, as long as the schedule attains the final distribution when all the effort has been applied.

Pollock (1960) introduces a discrete search model for two regions and determines the optimal sequential strategies for this model. A single searcher is given the a priori probability P that the target is in region 1. Conditional detection probabilities q_1 and q_2 are also given. It is assumed that each glimpse in either region takes one unit of time. As the search progresses, the a posteriori probabilities are obtained using Bayes' theorem. For example, suppose the searcher is unsuccessful in his look into region 1, the a posteriori probability that the target is in that region is given by

$$p' = \frac{(1-q_1)p}{1-q_1p}$$

Similarly, the a posteriori probability of the target being in region 1, given an unsuccessful look into region 2, is

$$p' = \frac{p}{1 - q_2(1-p)} .$$

Let the expected length of search using an arbitrary strategy be denoted by $E(p)$, where

$$E(p) = \left\{ \begin{array}{ll} 1 + (1 - q_1)p E\left(\frac{(1 - q_1)p}{1 - q_1p}\right) & : \text{Start in \#1} \\ 1 + (1 - (1 - p)q_2) E\left(\frac{p}{1 - (1 - p)q_2}\right) & : \text{Start in \#2} \end{array} \right\}$$

Pollock shows that the optimal sequence of looks is determined from the following conditions: Let p denote the "current" estimate of the probability of the target being in region 1, the selection of the next region to be searched is accomplished via the rule:

"For $p \geq \frac{q_2}{q_1 + q_2}$ look in box 1, otherwise look in box 2."

The optimal value of $E(p)$ under the above strategy is determined via a "bootstrap" technique of extending the region in which the optimal value of $E(p)$ is known.

In comparing the optimal values of the expected length of search between the discrete and continuous models, Pollock observed that for small values of q they are very close, indeed; however, as $q \rightarrow 1$ they become quite different. He also observed that the criteria of (1) maximizing the probability of detection by the end of a fixed time; and (2) minimizing the expected length of time until detection; lead to the same results for the allocation of effort.

Gilbert (1959) considers the continuous version of the two-box search problem including non-zero switching times. First, he notes that in general, search, under the assumptions made up to this point in our discussion, may be compared to a one-person game. Although not solvable as such, he concludes that all attention should be restricted to pure (deterministic) strategies, rather than mixed (probabilistic) strategies. He also notes that with $p = 1 - p = 1/2$, and $q_1(t) = q_2(t) = 1 - e^{-t}$, it is optimal to switch from one box to another whenever the box being searched has received a longer time of search than the other box. Then, by switching from box to box rapidly enough, one can get expected search times as close to two as desired. This leads to the definition of a "limit strategy," which will approach the true optimum strategy in the limit as switching becomes instantaneous. The limit strategy is defined as a pair of monotone non-decreasing functions $x(t)$ and $y(t)$ such that

$$x(t) + y(t) = t, \quad t \geq 0,$$

$x(t)$ and $y(t)$ are interpreted as the times which will be spent (using the optimal strategy) searching boxes A_1 and A_2 respectively when a total time t has been spent searching. The probability $Q(x(t), y(t))$ is defined as the probability that A_1 and A_2 can be searched for times x and y without detection, then

$$Q(x(t), y(t)) = p[1 - q_1(x)] + (1-p)[1 - q_2(y)],$$

$$q_1(x) = 1 - e^{-\alpha_1 x},$$

$$q_2(y) = 1 - e^{-\alpha_2 y}.$$

Since the distribution function for the time spent searching is $1-Q$, the optimal strategy is that $(x(t), y(t))$ which minimizes

$$-\int_0^\infty t dQ(x(t), y(t)) = \int_0^\infty Q(x(t), y(t)) dt.$$

The solution to the above problem yields the strategy:

(a) For $(1-p)\alpha_2 \geq p\alpha_1$, first look in box 2 for

$$\frac{1}{\alpha_2} \ln \frac{(1-p)\alpha_2}{p\alpha_1} \text{ units of time, then follow the}$$

limit strategy

$$\alpha_1 x = \alpha_2 y + \ln \left[\frac{\alpha_1 p}{\alpha_2 (1-p)} \right].$$

(b) For $(1-p)\alpha_2 \leq p\alpha_1$, first look in box 1 for

$$\frac{1}{\alpha_1} \ln \frac{\alpha_1 p}{\alpha_2 [1-p]} \text{ units of time, then follow}$$

the limit strategy.

Allowing for a non-zero switching time S , the strategies of interest become those which follow "staircase" paths in the (x,y) plane. If switches occur at the points (x_i, y_i) , $i = 1, 2, \dots$, then the expected search time of a strategy is

$$E = \int Q(x,y)(dx + dy) + S \sum_{i=1}^{\infty} Q(x_i, y_i)$$

where the integral is a line integral taken along the staircase path. Gilbert develops the following theorems pertaining to this case.

Theorem

Let C be a line segment between two switch points (x_i, y_i) and (x_{i+1}, y_{i+1}) of a minimizing strategy. If C is horizontal, there must be points on C at which $Q_x - Q_y \leq 0$. If C is vertical, there exist points of C at which $Q_x - Q_y \geq 0$.

Theorem

Let $p = 1-p = 1/2$, $q_1(t) = q_2(t) = q(t)$, and let the distribution function $1 - q(t)$ have mean T . Then bounds on the minimum expected time E_0 are given by

$$T + s/2 < E_0 < 2T + S/2 + \sqrt{2ST}.$$

The above theorems will yield the optimal solutions in special cases. For the case in which $q_1 = 1 - e^{-x}$, and $q_2 = 1 - e^{-y}$, Gilbert determines that the switch points are $(w, 0)$, $(w, 2w)$, $(3w, 2w)$ where w satisfies the equation

$$S + w_0 = \sinh w_0,$$

and the minimum value of E is

$$E_0 = 1 + \cosh^2(w_0/2).$$

However, these results will not yield solutions in more general situations. Kisi (1966) obtained the same result independently using somewhat more direct arguments.

Blachman (1959) considers the following variation of the search problem formulated by Koopman. The object is not present at the beginning of the search but has a distribution of arrival times, and the aim is not to maximize the probability of detection but to minimize the expected delay between arrival and detection. An object may appear in any one of n locations and will thereafter remain there, the probability of the i^{th} location being p_i , with $\sum p_i = 1$. The time of appearance of the object is distributed uniformly over a long interval of length T . A look in the i^{th} location takes a time t_i and, if the object is there, the look detects it with probability q_i .

The basic question is: In what order should the various locations be scanned during the time T to minimize the expected delay between the appearance of the object and its detection? The search pattern is characterized by the intervals T_{ij} between the beginning of the $(j-1)^{\text{st}}$ look in the i^{th} location and the beginning of the j^{th} . T_{i1} is defined as the interval between the start of the search and the first look in the i^{th} location. It is assumed that the target will not appear before the start of the search.

For a given search procedure, the expected delay between the arrival and the discovery of the object is

(4)

$$t = \sum_{i=1}^N p_i \left[t_i + \sum_{j=1}^{J_i} \frac{T_{ij}}{T} \left[\frac{T_{ij}}{2} + \sum_{k=1}^{J_i-1} \alpha_i^k T_{i(j+k)} \right] \right],$$

where J_i is the total number of looks in the i^{th} location and $\alpha_i = 1 - q_i$ is the probability of failing to detect the target. The procedure is to choose positive quantities T_{ij} that minimize the expected delay subject to

$$\sum_{j=1}^{J_i} T_{ij} \approx T \quad (i = 1, 2, \dots, n).$$

Minimizing the expected delay subject to the above constraint yields, treating J_i as fixed,

$$\begin{aligned} T_{ij} &= \frac{T}{J_i} \quad \left(2 \leq j \leq J_i - 1 \right) \\ &= \frac{T}{(1-\alpha_i)J_i} \quad i = 1, J_i \end{aligned}$$

under the assumption that T is great enough so that all J_i are large. The optimum expected delay is given by

$$t = \sum_{i=1}^N p_i \left[t_i + \frac{T}{J_i} \left(\frac{1}{q_i} - \frac{1}{2} \right) \right]. \quad (5)$$

To determine the optimum J_i , (5) must be minimized subject to

$$\sum_{i=1}^N J_i t_i = T.$$

The results of this minimization are:

$$(a) \quad J_i = \frac{T \sqrt{\frac{p_i}{t_i} \left(\frac{1}{q_i} - \frac{1}{2} \right)}}{\sum_{i=1}^N \sqrt{p_i t_i \left(\frac{1}{q_i} - \frac{1}{2} \right)}}$$

$$t_{\min} = \sum_{i=1}^n p_i t_i + \left[\sum_{i=1}^n \sqrt{p_i t_i \left(\frac{1}{q_i} - \frac{1}{2} \right)} \right]^2.$$

In general, it is not possible to arrange a search pattern that satisfies the above conditions for all locations, because the condition that "looks" in different locations must not overlap, has not been taken into account. Hence, one can conclude only that a search pattern which approximately satisfies the above condition is, at least, approximately optimum.

Blachman and Proschan (1959) consider the following general search problem. Objects arrive in accordance with a Poisson process, the rate of arrival being A . Having arrived, an object appears (and remains until detected) in box i with probability p_i . A single scan of box i costs c_i (possibly including the cost of false alarms), takes time t_i , and, if the object is present in box i at the beginning of the scan, will detect it with probability q_i . The resultant gain, $g_i(t)$, $i = 1, 2, \dots, n$, is a non-increasing function of, t , the delay between arrival and the beginning of the detecting look. Considering only cyclic search schedules, i.e., search schedules which repeat after D units of time, where D is arbitrary, the authors derive the optimum search procedure. The expected net gain per unit of time from f_i regularly spaced looks per unit of time allocated to box i , $i = 1, 2, \dots, n$, is given by

$$\begin{aligned}
 \Gamma &= \sum_{i=1}^N A p_i f_i q_i \left[\sum_{m=0}^{\infty} (1 - q_i)^m \int_{m/\bar{f}_i}^{(m+1)/f_i} g_i(t) dt \right] - f_i c_i \\
 &= \sum_{i=1}^N \Gamma_i
 \end{aligned}$$

The problem is to maximize the expected net gain subject to

$$\sum_{i=1}^N f_i t_i \leq 1, \quad f_i \geq 0, \quad (i = 1, 2, \dots, n).$$

The solution to the above problem is obtained as follows:

Define $f_i(r)$ for $r > 0$ as

$$f_i(r) = 0 \quad \text{if} \quad r_i(0) \leq r$$

$$f_i(r) = \infty \quad \text{if} \quad r_i(\infty) \geq r$$

$$f_i(r) = f_i \quad \text{such that} \quad r_i(f_i) = r, \quad \text{otherwise,}$$

where

$$r_i(f_i) = \frac{1}{t_i} \left(\frac{d\Gamma_i}{df_i} \right).$$

Also define

$$F(r) = \sum_{i=1}^N f_i(r)t_i$$

$$r^* = \min r, \\ \begin{cases} r > 0 \\ F(r) \leq 1 \end{cases}$$

and

$$f_i^* = f(r^*).$$

The major result of this paper is the following theorem:

Theorem:

$$\left\{ f_i^* \right\} \quad (i = 1, 2, \dots, n) \text{ maximizes } \Gamma \text{ among} \\ \left\{ f_i \right\} \quad i = 1, 2, \dots, n \text{ satisfying}$$

$$\sum_{i=1}^N f_i t_i = 1.$$

In the foregoing discussion, the optimal schedule was obtained by ignoring conflicts among boxes, however, the authors show that by taking n sufficiently large, and at the same time the p_i , t_i , c_i correspondingly small one can always produce a conflict-free schedule with the expected gain per unit of time as close as desired to

$$r^* = r(f_1^*, \dots, f_n^*).$$

Some additional results obtained in this paper include:

- (a) In considering how best to schedule scans in any one given box (ignoring all other boxes, for the moment), the optimum schedule calls for scans uniformly spaced in time.
- (b) The following theorem provides a sufficient condition for answering the question: Under what circumstances should a given box be searched?

Theorem:

If $\int_0^{\infty} t dg_i(t) = -\infty$, then $f_i^* > 0$.

- (c) By taking the gain function to be the negative of the delay between arrival and detection, i.e.,

$$g_i(t) = -(t + t_i), \quad i = 1, 2, \dots, n,$$

the optimal frequency, f_i^* , is shown to be proportional to

$$\sqrt{(p_i/t_i) \left(\frac{1}{q_i} - .5 \right)}.$$

Matula (1964) derives conditions for the existence of an ultimately periodic search program in the following context: An object is in one of a finite set I of possible locations, with a priori probability p_i , $\sum_{i=1} p_i = 1$. Associated with each location i is a cost for searching that location, c_i , and an overlook probability, α_i , if the object is in i and i is searched, it is not detected. The problem is to find a program $\sigma = (\sigma(1), \sigma(2), \dots)$, i.e., a sequence of locations to be searched such that the expected cost, $v(\sigma)$ of finding the object is minimal. A program is called ultimately periodic if $\sigma(j + \theta) = \sigma(j)$ for all $j > T$, where T denotes the length of the transient phase and θ the length of the period.

The major result of this paper is the conditions for the existence of an ultimately periodic optimal program as well as the minimal period and the minimal transient length. It is to be noted that the general dynamic programming solution gives an optimal program recursively, whereas the results of this paper have the advantage of yielding a closed form expression and require evaluation of only the first $T + \theta$ terms. In addition, a periodic optimal program yields for the expected cost a power series that is algebraically summable in closed form.

The results of the paper are summarized in the following assertions:

Lemma If σ is an ultimately periodic optimal program of transient length T and period $\theta = \sum_{i=T} n_i$ where n_i is the

number of searches of location i per period, then

$$\alpha_i^{n_i} = \alpha_j^{n_j} \quad \text{for } i, j \in I.$$

Corollary A necessary condition for the existence of an ultimately periodic optimal program is that the set of ratios

$$\left\{ \log \alpha_i / \log \alpha_j \right\} \quad (i, j) \in I$$

consist only of rational numbers.

Theorem: For the search problem where the ratios

$$\left\{ \log \alpha_i / \log \alpha_j \right\}$$

are rational numbers for $(i, j) \in I$, there exists a program σ^* such that

- (a) σ^* is ultimately periodic of period θ and transient length T , where

$$\theta = \min \left\{ \theta' \mid \theta' \text{ and } \theta' / \sum_{j \in I} (\log \alpha_i) / (\log \alpha_j) \text{ are integers} \right\}$$

$$T = \sum_{i \in I} \left[\min_{n=0,1,2,\dots} \left\{ n \mid \alpha_i^n q_i p_i / c_i \leq \min_{j \in I} \left\{ q_j p_j / \alpha_j c_j \right\} \right\} \right]$$

- (b) σ^* is optimal
- (c) θ is the minimal possible period
- (d) T is the minimal transient length.

Combining the previous results, Matula obtains:

Periodic Search Theorem:

A necessary and sufficient condition for the existence of an ultimately periodic optimal program is that the ratios

$$\left\{ \log \alpha_i / \log \alpha_j \right\} \quad (i, j) \in I,$$

all be rational.

It is interesting to note that the limiting frequency of search of a location for any optimal program depends only upon the overlook probabilities, not on the initial probability distribution or even the relative costs.

In the following example, due to Klein (1968), one can note the more detailed structure of the transition mechanism. Klein considers the following problem. An object moves about within a finite number of regions, one per time unit, according to known probabilistic laws. A single searcher, using a detection system whose effectiveness is a function of the amount of effort used and the region searched, checks one region at a time until the object is found, his budget effort is exhausted, or he decides that it is "uneconomical" to continue. The problem is to find an optimal sequential search policy, i.e.,

one which tells the searcher, at each point in time, whether to search, where to search, and how much effort to use. It is further assumed that the target's movements are independent of its location and that the searcher is "noisy" enabling the target to base his movements on knowledge of the searcher's location at the end of each period. The following assumptions are also made:

- (a) $L + 1$ regions are to be searched, $0, 1, \dots, L$.
- (b) The searcher starts in region 0 (the base) and the object is in any region. The budget, of size B , consists of a finite number of discrete units.

The two classes of states and their associated labels are defined as:

- (1) i_b^0 , region i has been searched, $0 \Rightarrow$ unsuccessfully, and b units of the budget remain for further use.
- (2) i_b^1 , same as above, except $1 \Rightarrow$ successful search.

The state space of the decision process is given by

$$S = \left\{ i_b^\alpha; i = 0, \dots, L; b = 0, 1, \dots, B-1; \alpha=0,1 \right\} \cup \left\{ 0_B^0 \right\},$$

where 0_B^0 is the initial state.

It is assumed that the target discovers the searcher's location at the end of each period. His evasion strategy, based on this information, is assumed to be randomized and represented in the form of a stochastic matrix

$$H = \left\{ h_{ij} \right\},$$

where i denotes the searcher's current location and j the target's next. Then, corresponding to each searcher position (i), the target moves to position j with probability h_{ij} . Note that this implies that the target's ability to move is independent of its location. This may not be true of the searcher's mobility. The effectiveness of the searcher's detection process depends upon the region searched and the amount of effort used, i.e.,

$v_j(e)$ = probability that a search of region j
using effort e will find a target if it
is in the region.

After each determination of the current state of the decision process, say i_b^α , the searcher chooses a decision, j_e , from a finite set $k(i_b^\alpha)$, i.e., the searcher chooses the next region to be examined (j) and the amount of effort to be used ($e = 1, 2, \dots, b$). It is assumed that the decision is made with probability $d(i_b^\alpha, j_e)$. The process is controlled by a randomized stationary decision rule (Derman (1962) has shown that attention may be restricted to this class of rules):

$$D = \left\{ d(i_b^\alpha, j_e) \right\}, \quad d(i_b^\alpha, j_e) \geq 0.$$

$$\sum_{j_e} d(i_b^\alpha, j_e) = 1.$$

The problem is to select an optimal rule D from the class of all randomized stationary rules. Next, the stopping states for the chain are defined as follows:

- (a) let A equal the set of all states in which the target is found

$$A = \left\{ i_b^1; i = 0, \dots, L; b = 0, 1, \dots, B-1 \right\},$$

- (b) let G equal the set of all states in which the budget is exhausted,

$$G = \left\{ i_0^0; i = 0, \dots, L \right\},$$

then $T = A \cup G$ is the complete set of stopping states for the chain. It is also assumed that the process starts in state i_0^0 , with probability 1. The transition probabilities for the controlled chain, $p(i_b^a, j_f^a)$ follow:

Let $r_{ij} = 1, 2, \dots$, be the travel effort needed to go from i to j ; then, for all integers b, f : $0 \leq f = b - e - r_{ij} < b \leq B$,

$$P(i_b^0, j_f^1) = h_{ij} v_j(e) d(i_b^0, j_e^0), i_b^0 \in S-T, j_f^1 \in A,$$

$$P(i_b^0, j_f^0) = \left[1 - h_{ij} v_j(e) \right] d(i_b^0, j_e^0), i_b^0 \in S-T, j_f^0 \in S-A.$$

The chain, as defined, is absorbing by virtue of the stopping states T . It is made cyclic by forcing its return to the starting state (0_B) whenever the set T is reached, i.e.,

$$P(i_b^\alpha, 0_B^\circ) = d(i_b^\alpha, 0_B^\circ) = 1 \text{ for } i_b^\alpha \in T.$$

This new chain consists of, at most, one ergodic class of states. The following cost structure is introduced: $c(i_b^\alpha, j_e)$ denotes the cost if the system is in state i_b^α , at the end of a period and decision j_e is made; that is

$$c(i_b^\alpha, j_e) = e + r_{ij}, \quad i_b^\alpha \in S-T$$

$$c(i_b^\alpha, j_e) = 0, \quad i_b^\alpha \in T.$$

The total expected cost is given by

$$Q(D) = E \left\{ \sum_{\tau=0}^{\tau(D)-1} c \right\}$$

where $\tau(D)$ is the random number of periods taken by the process to reach a stopping state using a specific rule D . Let

$\pi(i_b^\alpha) : i_b^\alpha \in S$ represent the (unique) steady state probabilities of the controlled chain (note that the π 's will be functions of the decision rule). The total expected cost can be written in the form

$$Q(D) = \left[\frac{1}{\pi(0_B^\circ)} - 1 \right] \sum_{i_b^\alpha} \sum_{j_e} \pi(i_b^\alpha) d(i_b^\alpha, j_e) c(i_b^\alpha, j_e),$$

where from Markov chain theory $1/\pi(O_B^0)$ is the mean recurrence time for state O_B and

$$E(\tau(D)) = \frac{1}{\pi(O_B^0)} - 1$$

is expected duration of the search. A successful search terminates in class A, hence the probability of a successful search using rule D is

$$P(D) = \frac{1}{\pi(O_B^0)} \sum_{i_B^a \in A} \pi(i_B^a).$$

Klein offers the following objective functions for consideration:

$$\begin{aligned} & \text{(a) } \min_{\{D\}} Q(D) \\ & \text{S.T. } P(D) \geq \theta \end{aligned}$$

$$\begin{aligned} & \text{(b) } \max_{\{D\}} P(D) \end{aligned}$$

$$\begin{aligned} & \text{(c) } \min_{\{D\}} E \{ \tau(D) \} \\ & \text{S.T. } P(D) \geq \theta \\ & \quad Q(D) \leq \Gamma \end{aligned}$$

$$\begin{aligned} & \text{(d) } \max_{\{D\}} P(D) \\ & \text{S.T. } E(\tau(D)) \geq A \\ & \quad Q(D) \leq \Gamma. \end{aligned}$$

Formulation (b) may be solved using dynamic programming, the other formulations can be transformed into linear programming problems by utilizing the techniques described in Derman (1962).

Pollock (1964) develops search strategies to minimize the expected cost of search which are sequential in the sense that a decision at any time is dependent upon what has been observed up to that time. The search process is represented in terms of a stochastic dynamic program including consideration of false alarm probabilities. The optimal search strategies as well as the associated minimum costs are given. The state variable, the probability that the target is present, is adjusted by Bayes' rule after every observation. It is shown that the optimal sequential strategy is similar to the Wald sequential probability ratio test. The target is assumed stationary, although if the target is not yet present in the region of interest, it has probability λ of arriving in each successive time interval.

Kadane (1968) studies the problem of choosing a strategy to maximize the probability of finding a stationary object when a budget ceiling is imposed. It is assumed that the probability of overlooking the object in the j^{th} search of box K , given that it is in box K and has not been found before the j^{th} search of box K , is a function, α_{jk} , of j and k alone. Therefore, the (unconditional) probability that the j^{th} search of box K is conducted and is successful is 0 if the strategy does not include a j^{th} search of box K , and is

$$P_k \prod_{j' < j} \alpha_{j'k} (1 - \alpha_{jk}) = P_{jk},$$

where P_k is the probability that the object is hidden in box K . Let E_{jk} be the event that the j^{th} search of box K is conducted and is successful and

P_{jk} = probability that the event E_{jk} occurs.

Let σ denote a search strategy, then the probability of finding the object using σ is $\sum_{\delta} P_{jk}$, P_{jk} is to be included in the summation if there is a j^{th} search of the K^{th} box in σ . The simplification and extension achieved in this paper are a consequence of the possibility of restricting the discussion to the unconditional probabilities of these mutually exclusive events E_{jk} . It is possible to compute the conditional probability that the j^{th} search of box K will be conducted and will be successful, as,

$$\frac{P_{jk}}{1 - \sum_{(r,s) \in \delta} P_{rs}}$$

where δ is the set of searches conducted up to this point.

Let the j^{th} search of box K cost c_{jk} . Then the largest cost one can occur using strategy σ is

$$\sum c_{jk}$$

where c_{jk} is included in the summation if there is a j^{th} search of box K in σ . In short, a strategy is sought to

$$\text{MAX} \quad \sum_{\sigma} P_{jk}$$

subject to

$$\sum_{\sigma} c_{jk} \leq C,$$

with the usual remarks concerning the extent of the summations.

The author extends the Neyman-Pearson Lemma to measures of arbitrary total measure. The theorem is stated as follows ($B = \sum c_{jk}$ over all positive P_{jk}):

Theorem:

Let $\{P_i\}$ and $\{c_i\}$ be arbitrary non-negative sequences such that $\sum P_i < \infty$. Let X be the class of sequences x_i , such that $0 \leq x_i \leq 1, \forall i$. If $0 < C < B$, then the maximum of

$$\sum x_i P_i$$

subject to

$$\sum x_i c_i \leq C$$

and $x_i \in X$ is attained, and it occurs when and only when

$$x_i = \begin{cases} 1 & \text{if } P_i > rc_i \\ 0 & \text{if } P_i < rc_i \end{cases} \quad (6)$$

for some $r, 0 < r < \infty$, and

$$\sum x_i c_i = C.$$

The set of r 's satisfying (6) is the same for each optimal x and is a single point or a closed interval. The author describes an integer programming algorithm (branch and bound variety) adapted to the problem of finding the object subject to a budget ceiling C when discreteness is insisted upon.¹ The implications of the previous theory towards the problem of minimizing the expected cost are summarized in the following results.

The author defines a set of searches to be locally optimal if the inclusion of (j', k') and exclusion of (j, k) implies

$$\frac{P_{j'k'}}{c_{j'k'}} \geq \frac{P_{jk}}{c_{jk}}$$

The following theorem is given:

Theorem: Let P_{jk}/c_{jk} be non-increasing in j for each k . Any locally optimal feasible strategy including all searches for which $P_{jk} \neq 0$ minimizes the expected cost of all unsuccessful searches plus half the cost of the last, successful search. Such a strategy

¹For $c_k=1$, $k=1, 2, \dots, N$, Chew (1967) gives basically the following optimal strategy: To maximize the probability of finding the target in a fixed number, N , of searches, choose those N searches (j, k) for which $P_k \alpha_k^{j-1} (1-\alpha_k)$ is largest.

exists if and only if

(a) In all boxes k for which $P_{jk} \neq 0$ for all j

$$b_k = b = \lim_{j \rightarrow \infty} \frac{P_{jk}}{c_{jk}}$$

where $b \geq 0$.

(b) If b is positive and $P_{jk}/c_{jk} = b$ for some (j,k) , then for every sufficiently large j , P_{jk}/c_{jk} is b or 0 in each box.

(c) $P_{jk}/c_{jk} \geq b$ for all (j,k) such that $P_{jk} \neq 0$.

Black (1965) presents a graphical argument for the optimal sequential search procedure for the following problem: A stationary target is in one of $\hat{n}$ regions. It is in region i with prior probability P_i , a look in region i costs c_i , and the target can be overlooked with probability α_i .

Let

$P(k)$ = probability that the target is found on or before k^{th} look,

$c(k)$ = total cost of the first k looks, and c the random total cost. Then the expectation of c is

$$E(c) = \sum_{k=1}^{\infty} (c(k) - c(k-1)) (1 - P(k-1)) \quad (7)$$

which is obtained from

$$E(c) = \lim_{N \rightarrow \infty} \sum_{k=1}^N c(k) (P(k) - P(k-1)) + c(N)(1-P(N)).$$

Black then plots $P(k)$ versus $C(k)$ as in Figure 2.

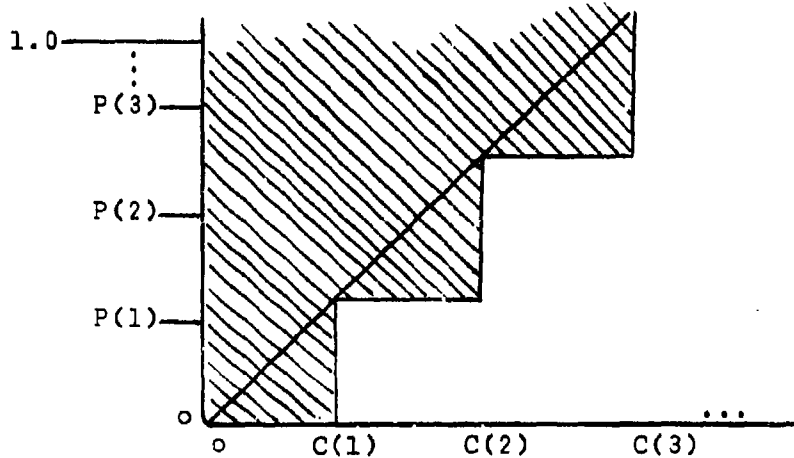


Figure 2
Probability of Detection in K Looks

As shown in equation (7) the expected cost of a search using this policy is equal to the shaded area. It is noted that all policies with finite expected cost have the same triangles in their probability-cost plot, with only their order changed.

The heights of the triangles are given by

$$P_i(1-\alpha_i) \alpha_i^{n-1}$$

and the base by c_i . Clearly, the policy that places the triangles in order of decreasing steepness is optimal, if it is feasible.

Consider all the numbers

$$\frac{P_i(1-\alpha_i) \alpha_i^{n-1}}{C_i}$$

arranged in a two-dimensional array. Note that the

$$\frac{P_i(1-\alpha_i) \alpha_i^{n-1}}{C_i}$$

are monotone decreasing in n .

It is observed that the application of Bayes' rule shows that the policy with minimum expected cost is identical with that generated by the rule:

"Always look in the region for which the posterior probability (given the failure of earlier looks) of finding the object divided by the cost is maximum."

Since the logarithm is monotone increasing in its argument, one can construct the optimal policy by arranging the numbers:

$$\log \left[\frac{P_i(1-\alpha_i)}{C_i} \right] + (n-1) \log \alpha_i$$

in decreasing order. Viewing these numbers as points along a line, the points corresponding to any particular region will be equally spaced. If $\log \alpha_i$ are commensurate, the optimal policy is eventually periodic.

Renyi (1965) considers the following search problem: let S_N be a finite set having $n \geq 2$ distinguishable elements. Suppose one wishes to find an unknown point X of the set S_N . It is further assumed that one cannot observe X directly; however, one may choose some functions $f_1, f_2, \dots, f_k$ from a given set F of functions defined on S_N , and observe the values $f_1(x), f_2(x), \dots, f_k(x)$. It is assumed that F contains M functions, $M \leq n$. A strategy of search is a method for the successive choice of $f_1, f_2, \dots, f_k$, which leads to the determination of X . The usual definitions of pure and mixed strategies are applied to the choice of the function $f_1, f_2, \dots, f_k$. The author attains some general theorems concerning the duration of a search using random search methods, and it is shown that, in general, these random search methods are almost as good as the best pure strategy, and are usually much simpler.

Miehle (1954) discusses numerical techniques for determining the optimal distribution of effort under constraints. In particular, one has various types of effort to expend on corresponding tasks applied towards a desired result. The effect is represented by $E(x_1, x_2, \dots, x_k)$. In particular, Miehle studies the case in which the effects are additive, i.e.,

$$E(x_1, x_2, \dots, x_k) = f_1(x_1) + f_2(x_2) + \dots + f_n(x_n).$$

The objective is, of course,

$$\text{Max} \quad E(x_1, x_2, \dots, x_k)$$

$$\text{S.T.} \quad \sum x_i \leq C, \\ x_i \geq 0.$$

The numerical solution technique consists of searching an array, the columns of which represent the efforts $x_1, \dots, x_k$ and the rows the allowable allocation to each effort type ranging from 0 to C, for the maximum value of $E(x_1, x_2, \dots, x_k)$.

Staroverov (1963) considers the following search problem. A point is located in the k^{th} cell with probability P_k , $\sum_k P_k = 1$. One cell is inspected per unit of time; if the point lies in the cell being inspected, it is discovered with probability $q > 0$. The results of such investigations are considered independent. Let σ_t denote the number of the cell being investigated at time t , if the point was not discovered up to the time $t-1$. Let $\sigma = (\sigma_1, \dots, \sigma_t, \dots)$ denote the search strategy and T_σ the time required for discovering the point. In this paper, a procedure of searching, σ^* , is determined so that

$$E(T_{\sigma^*}) = \inf_{\sigma} E(T_\sigma).$$

Arkin (1964a) extends the results of Staroverov and considers simultaneous search of a number of cells. Explicit formulae are given for the optimal strategy of search and for the corresponding distribution and mean value of its duration.

In another paper, Arkin (1964b) considers the problem of obtaining uniformly optimal strategies in the context of the stationary search problem. The a priori distribution of a particle in $\mathbb{R}^n$ is given by the density function $f(x)$. The search strategy is defined by the function

$$\sigma = \sigma(x, t), \quad \sigma \geq 0,$$

$$\int_{\mathbb{R}^n} \sigma(x, t) dx = 1.$$

Let $P_\sigma(T)$ denote the probability of finding the particle using strategy σ during time T . A strategy σ^* is uniformly optimal if

$$P_{\sigma^*}(T) = \sup_{\sigma} P_{\sigma}(T) \quad \text{for any } T > 0.$$

In a very general case, the author proves the existence of the strategy σ^* and is able to find its explicit form.

Chew (1967) considers the following variation on the stationary search problem. Let the a priori distribution of the object's location be denoted by $\{P_k\}$, $k = 1, 2, \dots, n$, where

$$\sum_k P_k = 1 - q < 1.$$

Since in this case the search has a positive probability of never terminating, one must couple a stopping rule S with any search procedure σ . A loss function is defined by imposing a penalty cost ($c > 0$) on the searcher for stopping before the object is found. A procedure (σ, S) which minimizes the expected cost to the searcher (i.e. which yields Bayes' risk) is derived.

MacQueen and Miller (1960) deal with the problem of whether or not a search activity should be started and, if started, whether or not it should be continued. Their model gives rise to a general functional equation for which existence and uniqueness conditions are given.

Gluss (1961) considers a model in which there are N neighboring cells in one of which there is an object that it is required to find. The a priori probabilities of the object being in cells $1, \dots, N$ are $P_1, \dots, P_N$ respectively, and the costs of examination of these cells are $c_1, \dots, c_N$. The search policy is considered to be optimal when the statistical expectation of the total cost of search is minimized. It is assumed that costs comprise a travel cost dependent upon the distance from the last cell examined, in addition to a fixed examination cost. It is assumed initially that the searcher is next to cell 1, $c_i = i + c$, where c is constant, and from then onwards (assuming that the j^{th} cell has just been examined) $c_i = |i-j| + c$. An optimal search strategy is found in the case where the P_i 's are all equal, and an approximately optimal search strategy is found in the case where P_i is proportional to i . The latter case has application to defense situations where complete searches occur at successive intervals of time, and hence the enemy objects are thinned out the nearer they come to the defense base.

Pollock (1969) considers a target moving in a Markovian fashion between two regions. The objective functions for the standard problems of the minimization of the expected time until detection and maximization of the probability of detection under a constraint on search effort are derived. For certain special forms of the transition matrix, decision rules are derived for the minimum expected time problem. Upper and lower bounds are also derived for the minimum expected time problem.

3.3 TWO-SIDED SEARCH

Neuts (1963) develops, among other things, stationary minimax strategies for a multistage search game. A stationary strategy for the hunter is an n -tuple

$$y = (y_1, \dots, y_n);$$

$$y_i \geq 0, \quad \sum_{j=1}^n y_j = 1,$$

which denotes a probability distribution, chosen once and for all, and by which the region to be examined at each stage is selected. A mixed strategy for the stationary target is an n -tuple $x = (x_1, \dots, x_n)$, with $x_i \geq 0$; ($i = 1, 2, \dots, n$) and $\sum_{j=1}^n x_j = 1$. x_i denotes the probability of the target being in the i th box. If the searcher uses the stationary strategy y and the target the mixed strategy x , then the expected return to the target at each stage of the game is given by

$$A(x, y) = \sum_{k=1}^n y_k (c_k - a q_k x_k),$$

where,

- c_i = cost to the searcher for a look in region i
- q_i = probability of finding the target given the correct region is searched
- a = reward to searcher for detecting the target.

The probability $p(x,y)$ that the object will be found during a given search equals:

$$p(x,y) = \sum_{k=1}^n q_k x_k y_k.$$

The discounted expected return to the target during the entire search is given by

$$\begin{aligned} F(x,y) &= \sum_{r=0}^{\infty} \delta^r [1 - p(x,y)]^r A(x,y) \\ &= \frac{A(x,y)}{1 - \delta[1 - p(x,y)]}. \end{aligned}$$

Denoting by $x^0 = (x_1^0, \dots, x_n^0)$, $y^0 = (y_1^0, \dots, y_n^0)$ and V_δ respectively a pair of minimax strategies and the value of the game with payoff $F(x,y)$, one must have

$$F(x^0, y) \geq V_\delta \quad \text{for all } y,$$

$$F(x, y^0) \leq V_\delta \quad \text{for all } x.$$

Neuts obtains as the solution to the above formulation:

$$\begin{aligned} V_\delta &= \left[\sum_{k=1}^n \frac{c_k}{q_k} - a \right] \left[\delta + (1-\delta) \sum_{k=1}^n \frac{1}{q_k} \right]^{-1} \\ x_j^0 &= \frac{1}{q_j} \left[\frac{c_j - (1-\delta) V_\delta}{a + V_\delta \delta} \right] \quad j = 1, 2, \dots, n \end{aligned}$$

$$y_i^0 = \frac{1}{q_i} \left[\sum_{k=1}^n \frac{1}{q_k} \right]^{-1} \quad i = 1, 2, \dots, n.$$

Note the independence of y^0 (searcher's strategy) of all parameters except the detection probabilities q_k .

The same remark holds for the expected duration of the game, i.e.,

$$\begin{aligned} \tau &= \sum_{k=1}^{\infty} k \left[1 - p(x^0, y^0) \right]^{k-1} \\ &= \left(\sum_{k=1}^n \frac{1}{q_k} \right)^2 \end{aligned}$$

It should be noted that stationary minimax strategies correspond to the following cases:

- (a) a memoryless searcher
- (b) the target is allowed to move after each region is searched.

Let $x = (x_1, \dots, x_n)$ denote an arbitrary mixed strategy for the target. Suppose one is interested in determining the optimum sequential response for the searcher against x and for the minimum expected loss. Bellman's principle of optimality implies that the following functional equation must be satisfied

$$f_{\delta}(x) = \min_{1 \leq i \leq n} \left\{ c_i - a q_i x_i + \delta(1 - q_i x_i) f_{\delta}(T_i x) \right\}$$

with

$$c_i > 0, \quad 0 < q_i \leq 1, \quad 0 < \delta \leq 1, \quad a \geq 0 \text{ and}$$

$$T_i x = \zeta = (\zeta_1, \dots, \zeta_n)$$

defined by

$$\zeta_i = x_i \left[\frac{1 - q_i}{1 - q_i x_i} \right]$$

$$\zeta_j = x_j \left[\frac{1}{1 - q_i x_i} \right] \quad j \neq i.$$

The n -tuple ζ is the a posteriori distribution derived from x , given that one unsuccessful search of box i was made. For $0 < \zeta < 1$, Bellman (1957) settles the questions of existence, uniqueness, and continuity of the solution of the above functional equation. Neuts obtains the following results on this equation for $\delta = 1$. Let $f_{n+1}(x)$ be defined by

$$f_{n+1}(x) = \min_{1 \leq i \leq n} \left[c_i - a q_i x_i + \delta [1 - q_i x_i] f_n(x) \right]$$

Theorem:

The sequence $f_n(x)$, $n = 0, 1, \dots$, for $\delta = 1$ is monotone decreasing in n for all $x \in X$. A sufficient condition for this to be true for all x and $0 < \delta < 1$ is that

$$\sum_{k=1}^n \frac{c_k}{q_k} \leq a.$$

Theorem:

There exists a bounded concave solution $f(x)$ to the functional equation for $\delta = 1$.

Charnes and Schroder (1967) develop models and methods to find optimal tactics in an idealization of antisubmarine warfare, viewed as a game of pursuit between the hunter-killer force and a possible submarine. The status of the pursuit at every move t ($t = 1, 2, \dots$) is taken to be one of a finite number of possible states. A state summarizes the tactical information available to both players for decision making. A finite collection of tactical plans (decisions) is associated with each state. When the players move they each choose a plan and thereby jointly determine an intermediate payoff from the hunted to the hunter and a transition probability distribution over the states. The objective is to find an optimal strategy for each player. A strategy is a decision (possibly randomized) for each state and move, an optimal strategy is one of a minimax pair for the total expected payoff. These concepts are presented in terms of a terminating stochastic game (TSG) which may be defined as a game played in a sequence of moves. At each move, the

game is said to be in one of a finite number of states $i = 1, 2, \dots, n$. If the game is in state i ($i = 1, 2, \dots, n$) and the hunter chooses alternative K , while the hunted chooses alternative L , then the payoff from hunted to hunter is a_i^{KL} ($K = 1, 2, \dots, M_i$; $L = 1, 2, \dots, N_i$).

The choice of alternatives K and L also determines the transition probabilities:

$$P_{ij}^{KL} \geq 0 \quad (i, j) = 1, 2, \dots, n$$

$$K = 1, 2, \dots, M_i$$

$$L = 1, 2, \dots, N_i$$

$$(i) \quad \sum_{j=1}^n P_{ij}^{KL} < 1, \quad \text{all } K, L, i$$

$$(ii) \quad |a_i^{KL}| < M, \quad \text{all } K, L, i.$$

Under the above assumptions, the game terminates with probability 1 and the accumulated payoffs received by either player are bounded. A behavior strategy for either player is an n -tuple of probability distributions $x = (x_1, \dots, x_n)$ where $x_i = (x_i^1, \dots, x_i^{M_i})$.

If the hunter uses a behavior strategy, he chooses the mixed strategy x_i whenever the game is in state i regardless of what move it is or the manner of arrival at state i . By choosing

a starting state i we obtain an infinite (the number of moves may not be bounded) game G_i ($i = 1, 2, \dots, n$). A terminating stochastic game is defined as a collection $G = (G_1, \dots, G_n)$. Let $\hat{w}_i$ denote the value of G_i ; the minimax of its total expected payoffs. The value of G may be defined to be the vector $\hat{w} = (\hat{w}_1, \dots, \hat{w}_n)$. Consider a two-person zero-sum game with payoff matrix $A_i(\alpha)$ where $A_i(\alpha)$, $i = 1, \dots, n$ is the $M_i \times N_i$ matrix whose K - L th element is

$$a_i^{KL} + \sum_{j=1}^n p_{ij}^{KL} \alpha_j,$$

and $\alpha = (\alpha_1, \dots, \alpha_n)$ is an n -vector of real numbers.

Let $\text{VAL}(B)$ denote the minimax value of the two-person zero-sum game with payoff matrix B and let $X(B)$ and $Y(B)$ denote the sets of optimal mixed strategies for the respective players. The following theorems characterize the optimal solutions to the terminating stochastic game.

Theorem 1:

The value of the terminating stochastic game G is the unique solution $\hat{w}$ of the nonlinear system of equations

$$\hat{w}_i = \text{VAL} [A_i(\hat{w})] \quad i = 1, 2, \dots, n.$$

Theorem 2:

The behavior strategies $\hat{x}, \hat{y}$ where $\hat{x} = x_i[A_i(\hat{w})]$, $\hat{y} \in y_i[A_i(\hat{w})]$ ($i=1,2,\dots, n$) are optimal for the first and second players, respectively, in every game G_i belong to G .

Charnes and Schroder then show that the nonlinear problem can be replaced by a sequence of linear programming problems. Stopping criteria are developed which insure the desired approximation to $\hat{w}$. The preceding results are then applied to a problem in antisubmarine warfare. It is shown that the objective function of the minimization of the expected duration of the search can be expressed in terms of a terminating stochastic game. In the event the hunter knows or is willing to assume certain behavior on the part of the submarine, the game becomes a one-person game. In this case, the determination of the hunters optimal strategy is reduced to solving a discounted Markovian decision process of the type studied by Howard (1960). Finally, the authors study a finite terminating stochastic game which terminates in n moves or a terminal state, whichever occurs first. It is shown that in this case the optimal strategies depend upon the move and are not behavior strategies.

Norris (1962) considers the two-sided extension of a one-sided search problem. The search is conducted against a conscious evader who is able to observe the searcher's actions and capitalize on any errors he makes. The evasion device of moving between looks is treated. The game is zero-sum and incorporates a fairly general reward structure which can include discounting. The reward coefficients associated with this structure, as well as the location of the boxes and

their detection probabilities, are known to both players. Good strategies are developed for the players when the game involves two boxes. In the case of an infinite moving cost, designated by G^∞ , exact solutions may be obtained when the escape probabilities, α_1 and α_2 , (the complements of the detection probabilities) satisfy the relationship

$$\alpha_1^{n_1} = \alpha_2^{n_2}$$

for a pair of intergers n_1 and n_2 . This relationship is the necessary condition for an ultimately periodic optimal program derived by Matula (1964).

In the case of a finite moving cost, designated by G , the evaders position (2 Box Case) as the search progresses is described by a probability vector. If the probability that he is in one box becomes sufficiently high, he should move from this box with a certain probability. This causes the probability vector describing his position to be transformed to the nearest boundary of the no-move region. The searcher's good strategy can be generated by a finite Markov process. In some states of the process the next look is made deterministically. In others called mixed states, the next look is made according to a probability distribution. As moving costs increase, deterministic looks are made more frequently, and the situations in which a move is admissable occur less frequently. In the case of infinite (prohibitive) moving costs, the searcher makes a random selection from two

infinite search sequences. Once this choice has been made, the search process is completely deterministic.

In the N-box formulation of the finite moving cost game, the good search strategy cannot be generated by a finite Markov process. A limited memory approach to finding an approximation to the good search strategy is suggested for future research. In the game designated G^0 , no such cost is incurred by the evader when he moves. As a result, the searcher cannot gain any inference concerning the evader's position from his past sequence of unsuccessful looks, and each look should be made according to the same probability distribution (this is the stationary minimax case discussed by Neuts (1963)). When the N-box form of G^0 was considered, it is noted that the good search strategy may be useful when the evader arrives sometime after the start of the game or leaves. Finally, it is noted that the results for the N-box case in which the position of evader is specified by a probability vector known to the searcher may be useful in studying some one-sided search problems.

Johnson (1964) considers the following search problem: Blue chooses a region i ($i = 1, 2, \dots, n$) in which to hide. Red selects one of n regions to search; if unsuccessful, he is told whether he is too high or too low, and repeats until he determines the correct region. Detection occurs with probability one, given the selection of the region chosen by Blue.

Although such a scenario is unrealistic in a military context, it may be quite the opposite in an information retrieval context. Theorems concerning the necessary conditions for optimality are presented. Optimal strategies are obtained (trial and error) for $n \leq 11$. For larger problems one has recourse to linear programming techniques on a digital computer. If $\{P_j\}$ ($j = 1, 2, \dots, n$) is a vector containing the probabilities with which Blue selects the j^{th} region, then it is shown that $P_1 \geq P_2$. Let $S_i = \{S_{ij}\}$ denote the i^{th} strategy for Red, i.e., S_{ij} equals the number of look when region j is searched under strategy i . The following theorems pertain to Red's optimal strategies.

Theorem:

Assume at given stage that Red, playing S_i , has located Blue within the region $k \leq j \leq M$, and that S_i calls for the next look at a , left of Blue's frequency distribution on this interval, and if a is too small, next playing at b to the right of a . Then a necessary condition for the optimality of S_i against $\{P_j\}$ is that

$$\sum_{k \leq j \leq a} P_j \geq \sum_{b \leq j \leq m} P_j.$$

Theorem:

At each stage Red should make his guess inside the middle third of Blue's probability distribution on the current interval of uncertainty.

Giammo (1963) considers the following problem: Consider two opposing mobile battle forces that are able to change position only at fixed time intervals, not necessarily equal. Each force knows the area in which the other is operating and is assumed to be efficiently searching this area for the enemy's position. Labeling the forces Blue and Red, Giammo defines R to be the total area of Blue's operating region and assumes that Red can search a region of area $r dt$ in a time interval dt , where r is some constant. B and b are defined in a similar fashion with reference to Red's operating region and Blue's rate of search. It is assumed that the Blue force moves periodically every t_M time units with the first move occurring at random with a uniform probability density in the time interval $0 \leq t \leq t_M$. Each move is considered to be instantaneous and to terminate with equally likely probability at any point in its own operating region. It should be noted that each time Red (Blue) moves, a new stage of the search starts which is independent of the preceding stages.

The objective of this paper is to develop expressions for the probability that Red will detect Blue without Blue's having previously detected Red, $P_{r,b}$ and the probability that Blue will detect Red without Red's having previously detected Blue, $P_{b,r}$.

Define:

$P_b(t)$ = the probability that Blue has discovered Red before time t ,

$P_r(t)$ = the probability that Red has discovered Blue before time t .

In these definitions, it is assumed that the searchers are independent, i.e., that the discovery of Red by Blue does not interfere with continuation of Red's search and visa-versa. Given that $P_b(t)$ and $P_r(t)$ represent the integrals of corresponding probability density functions, one can write:

$$P_b(\tau) = \int_0^{\tau} p_b(t) dt,$$

$$P_r(\tau) = \int_0^{\tau} p_r(t) dt.$$

Certainly, one can obtain the probability that Blue will discover Red before time τ without Red's having discovered Blue as

$$P_{b,r}(\tau) = \int_0^{\tau} [1 - P_r(t)] p_b(t) dt,$$

and for Red

$$P_{r,b}(\tau) = \int_0^T [1 - P_b(t)] p_r(t) dt.$$

The desired parameters are:

$$P_{r,b} = P_{r,b}(\infty),$$

and

$$P_{b,r} = P_{b,r}(\infty).$$

Integration by parts yields

$$P_{r,b} = \int_0^{\infty} P_r(t) p_b(t) dt,$$

and

$$P_{b,r} = \int_0^{\infty} P_b(t) p_r(t) dt.$$

Giammo then derives exact as well as approximate expressions for $P_{r,b}$ and $P_{b,r}$ under the above assumptions concerning the motion and search structure of the problem.

Koopman (1963) presents some of his original work (Koopman, 1957) in terms of a zero-sum game. He considers the problem of detecting an enemy unit located at a point x in some region R with a limited amount of search effort Φ . One is interested in determining a distribution of random search intensity $\phi(x)$,

with the provision that

$$\int_R \phi(x) dx = \Phi, \quad \phi(x) \geq 0.$$

According to the law of random search (Koopman (1957)), the probability of detecting the target at x is

$$1 - e^{-\phi(x)},$$

and therefore the probability of detecting the target when its probability of being at x has density $p(x)$ is

$$P = \int_R p(x) [1 - e^{-\phi(x)}] dx.$$

If the searcher assumes that his distribution of effort $\phi(x)$ is known to the target, and the target can then choose his position (or position density $p(x)$) to minimize the probability of detection P , then the searcher can select $\phi(x)$ to achieve maximum P . Conversely, the target may not know $\phi(x)$ and may assume that the searcher knows $p(x)$ and selects $\phi(x)$ to maximize P . In both cases, for $\phi(x) = \Phi/R$, one has

$$\text{maximum } P = \text{minimax } P = 1 - e^{-\Phi/R}.$$

In the heterogeneous case in which the "visibility", $g(x)$, depends upon position, the probability of detection becomes

$$P = \int_R p(x) [1 - e^{-g(x)\phi(x)}] dx,$$

and the constraint on search effort is weighted by position, i.e.,

$$\int_R h(x) \phi(x) dx = \Phi, \quad h(x) \phi(x) \geq 0.$$

Koopman obtains the following result for this case: The target's strategy $p(x)$ is given by $p(x) = a h(x)/g(x)$, and the search density function $\phi(x) = b/g(x)$. The constants a and b in the above expressions are determined from

$$\frac{1}{a} = \int_R \frac{h(x)}{g(x)} dx, \quad \text{and} \quad \frac{1}{b} = \frac{1}{a\Phi}.$$

The value of the game is still $1 - e^{-\Phi/R}$.

The case of a moving target is also considered. The target has to move along a path C from a point x^0 on a given curve K_0 to a point x^1 on the given curve K_1 , C passing through a field R (bounded by the given curves) in which the search is being conducted. The searcher can choose any $\phi(x)$ subject to

$$\int \phi(x) dx = \Phi, \quad \phi(x) \geq 0;$$

and the target can select his curve C , which he follows at a constant speed. It is shown that the expression for the probability of detection is given by

$$P = P(c, \phi) = 1 - \exp \left[- \int_C \phi(x)g(x)ds \right]$$

where ds is the arc length, and the integration denotes a line integral along the path C . Since $P(c, \phi)$ increases or decreases with $\int_C \phi(x)g(x)ds$, the problem of minimax can be stated in terms of this line integral.

Beltrami (1961) studies a random patrol on a straight line and gives a rigorous mathematical discussion leading to the paradox that the requirement of uniform coverage in a random patrol where the searcher has fixed speed imposes the condition of a non-random back and forth patrol. The following scenario is considered: A search craft S patrols a linear barrier in some back and forth manner. Using detection gear it has an effective search radius ρ (definite range law) which is assumed small in comparison with the barrier length. The penetrator P , approaches to within some distance of the barrier and appraises the patrol pattern of S . If the patrol is regular, then an intelligent tactic on the part of P is to coincide its barrier crossing with the moment in which S will be moving away or is at the extreme distance from the cross-over point. A random patrol for S is chosen in

order to completely eliminate any advantage to P; it being essential that the probability that a given point is covered in a move by S is as nearly constant as possible. It is shown that this policy will assure that the maximum penetration threat of P is minimized.

Dresher (1961) considers two formulations of a reconnaissance problem. In the first model, it is assumed that the attacker and defender have two strategies each. Blue, the attacker, wishes to seize a defended enemy position. It is assumed that he has two courses of action:

- (a) Attack with the entire force,
- (b) attack with part of his force, leaving the remainder as reserves and a rear guard.

Let the payoff matrix A be given by

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

where, for example, a_{21} represents the value to Blue if he attacks with part of his force and Red defends with his entire force.

It is further assumed that the attacker can send out a detachment to reconnoiter in an attempt to discover the plans of the defender. In order to defend himself against such possible action, the defender may take counter measures. The new game now has 16 strategies for the attacker and 4 for the

defender. The matrix for the new game can however, by testing for dominance, be reduced to a 4×4 matrix. A particular reconnaissance game is solved by way of illustration.

Dresher's second example deals with the value of reconnaissance information in the context of a bombing attack. It is assumed that there is an uncertainty concerning the worth of a target. Such uncertainty may arise from unknown or partially known results of earlier strikes on the same target. If the exact worth of the target is discovered through reconnaissance, then it is possible to dispatch the most efficient size attacking force against it. In order for a reconnaissance to be successful, at least one reconnaissance aircraft must fly to the target and return. The following notation is introduced:

- B = Military worth of one bomber.
- R = Military worth of one reconnaissance aircraft.
- T = Military worth of the target.
- $\phi(t)$ = Probability that the value of the target does not exceed t . This probability distribution is known prior to reconnaissance.
- r = Number of reconnaissance aircraft sent out prior to the mission.
- b = Number of bombers dispatched to the target during the mission.
- p = One-way survival probability of bomber and reconnaissance aircraft between base and target.

αT = Probable worth of the target after being hit by one bomber.

$\alpha^2 T$ = Probable worth of the target after being hit by two bombers.

The object of the attacker is to maximize the net outcome of the mission, the difference between the target damage and the aircraft losses.

The payoff, depending upon r and b , is given by

$$M(r, b) = \int \left[t(1 - \alpha^{pb}) - (1 - p^2) Bb - (1 - p^2) Rr \right] d\phi(t).$$

The optimal solutions are given by

$$r^* = 1 + \frac{1}{p} \ln \frac{AP}{R}$$

and

$$b^* = \begin{cases} \frac{\ln \left(\frac{T}{D} \right)}{-p \ln \alpha} & \text{If reconnaissance reports } T. \\ \frac{\ln \left(\frac{\phi_1}{D} \right)}{-p \ln \alpha} & \text{If reconnaissance does not report.} \end{cases}$$

where

$$P = -\ln(1 - p^2)$$

$$D = -\frac{(1 - p^2)B}{p \ln \alpha},$$

$$\phi_1 = \int t d\phi(t),$$

$$A = D \int \ln \frac{\phi_1}{t} d\phi(t).$$

For these optimal values the payoff is given by

$$M(r^*, b^*) = \phi_1 - D - D \ln \frac{\phi_1}{D} + \frac{1}{p} [AP - (1-p^2)R] - (1-p^2)Rr^*.$$

Issacs (1965) discusses extensions of his theory of differential games to games with incomplete information, e.g., search games. It is shown that when the hidden objects are numerous and immobile, the time to find them (payoff function) is nearly independent of the searcher's strategy as long as no effort is wasted re-searching territory already scouted and the overlook probability is zero. In the case of search games with mobile hiders, Isaacs conjectures that the details of the randomization are unimportant, but certain basic parameters, such as the hider's speed, are not. He argues that in either case there appear to be strong grounds for an approximate theory.

4.0 MISCELLANEOUS TOPICS

Dobbie (1964) considers the following surveillance problem. A region of the ocean is to be kept under surveillance to determine the probable number of enemy submarines in the region and their locations. It is desired to estimate additional measures of effectiveness of the surveillance operation, such as the expected fraction of submarines in the region being tracked at a given time. He is also interested in determining how the above measures depend upon the capabilities of the various components of the detection and tracking forces. The following assumptions are made:

- (a) Submarines enter the region at a known rate. It is also assumed that their time on station is a random variable with known distribution.
- (b) Two modes of detection are considered:
 - (1) Detection at barrier line, the detection process described by a single probability of detection,
 - (2) Area search detection, the detection capabilities are described by two search rates, one applies to submarines not previously detected, the other to previously detected submarines.

- (c) Contact between the tracker and the submarine can be broken and reacquisition occur, both events are described by their respective rates. It is also assumed that a contact is passed from a detection unit to track unit with probability one in zero time units.

In order to characterize the surveillance system, Dobbie describes the following state space:

1. Submarine is being tracked,
2. Submarine not being tracked, contact has been lost,
3. Submarine not detected.

Using renewal-type arguments, Dobbie derives expressions for

- (a) The expected number of submarines in the i^{th} state at time t , the expected number of submarines in the region at t ,
- (b) The probability that a submarine in the region is in state i at time t .

The author then relaxes the assumption that contacts are passed from detection units to tracking units in zero time with probability one. In addition, the following assumptions are also made:

- (a) Given detection by a barrier unit or by an area sensor, the detecting unit will attempt to maintain contact until a tracking unit arrives in the vicinity.
- (b) Targets can be reacquired either by area search or by special search. If contact is regained by special search, it is assumed that tracking will be accomplished by the detecting unit until transfer is made to a similar unit and during this time, the rate of losing contact is λ .

In this case the expanded state space includes:

- (1) Targets (submarines) tracked by a mobile unit in the vicinity of the target;
- (2) Target previously tracked, contact recently lost, local search being made to regain tracking contact;
- (3) Target previously tracked, new detection recently made by area search, tracking units enroute to area or searching in an effort to obtain tracking contact;
- (4) Target previously tracked, search to regain contact discontinued, no new detection;
- (5) Target not previously tracked, recently detected by area search, tracking units enroute;
- (6) Target detected by the barrier as it enters the region, tracking units enroute or searching to obtain tracking contact.
- (7) Target not previously tracked and no previous detection.

As before, Dobbie develops expressions for the probability that a submarine is in state i at time t , given that it was in state 6 or 7 at time $t = 0$ and stays in the region during $(0, t)$; $i = 1, 2, \dots, 7$.

Koopman (1946) developed the fundamental theory of target detection for two limited cases. In the one case, the detection equipment is assumed to sweep or scan at regular intervals, with the "glimpses" of the target long enough apart so that the probability of detection on one glimpse is independent of the probability on the preceding glimpses. In the other case the detector is assumed to be continuous in its action, and it is assumed that there is a probability γdt of detecting the target in any interval of time dt . Kimball (1963) observes that actual equipment in use has detection properties which lie between these limits. He shows that, in spite of this, actual detection equipment can be considered as equivalent to a certain continuously operating detector whose properties are derivable from those of the actual equipment. In addition, he also considers the problem of holding the target. Assuming the detection process to be a one-step Markov process, Kimball notes that it can be described by the matrix

$$\begin{pmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{pmatrix} ,$$

where, e.g., g_{00} is the probability that there is no detection on a given scan if there was no detection on the previous scan. New parameters r and g are defined as

$$r = g_{01} + g_{10} ,$$

$$g = \frac{g_{01}}{g_{01} + g_{10}} ,$$

where g is the unconditional probability of detection on an arbitrary trial and r is a measure of the lack of correlation between trials. It is shown that if the scanning frequency is f , the frequency of transitions in either direction (from the detected to the undetected, or visa versa) is

$$w = frg(1 - g) .$$

Kimball defines two detectors as equivalent if their g and w parameters are the same. In particular, any detector is in this sense equivalent to a continuous detector with the following properties. If the detector is in the "undetected" state, the probability that it begins to detect in any interval, dt , is gdt , and if the detector is in the "detecting"

state, it has a probability of becoming "undetected" equal to βdt . The proper values of β and γ are

$$\beta = \frac{w}{g},$$

$$\gamma = \frac{w}{1 - g}.$$

Kimball forms the following model of tracking: The entire system, detector plus operator, can be in any one of four states:

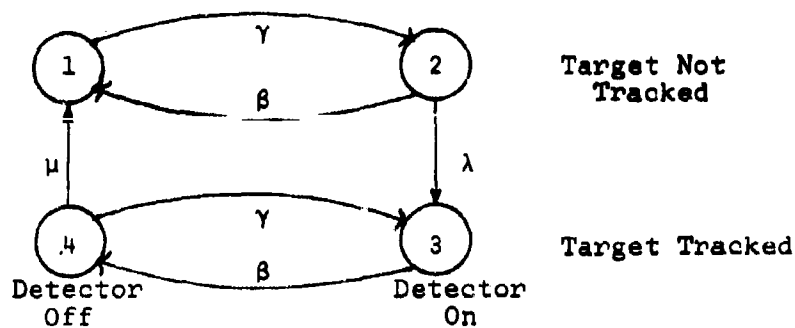
- (1) Detector off, target not tracked;
- (2) Detector on, target not tracked;
- (3) Detector on, target tracked;
- (4) Detector off, target tracked.

It is assumed that the behavior of the detector and the operator can be modeled in a continuous fashion. Let

λdt = probability of a transition in dt from state 2 to state 3, and

μdt = probability of a transition in dt from state 4 to state 1.

The state diagram is given by



The steady state probabilities of being in the four states are derived in terms of γ , β , μ , and λ , as well as the frequencies with which both the tracking and detection phases start and stop.

The problem of the target visibility changing over time has been formulated by Bonder (1969) and Disney (1969).

Bonder considered the situation in which the target and the searcher (detector) may not be continuously visible during the period of time in which the searcher is examining the subregion containing the target. The searcher has a detection capability only when the target is visible. The author considered the following situations:

- (a) The target may be visible to the searchers for the entire search interval with some known probability p ,
- (b) The target may be visible at the start of the search period, the length of the visible period being a random variable with known probability density function, and not reappear,

- (c) A single period of visibility may be exhibited starting at some random time during the search interval and lasting a random amount of time.

In each of these cases, the probability density functions for the time until the first detection, the time spent searching the area until a fixed number of detections occur, and the time spent searching the total area are derived.

Disney characterized the visibility process in which the target alternates between visible and invisible states as an alternating renewal process. The transition matrix for this process is

$$\begin{pmatrix} 0 & f_1(t) \\ f_2(t) & 0 \end{pmatrix},$$

where $f_1(t)$ is the probability density function for the time in the visible state and $f_2(t)$ the probability density function for the time in the invisible state.

Employing some renewal theory arguments, the author obtained, among other things,

- (a) $\pi_1(t)$, the density function for the probability that the target is visible at time t ,
- (b) for a fixed time interval of length τ_d , the distribution of
 - (1) the number of times the target is visible,
 - (2) the total time of visibility.

Analysis of interactions between the visibility and detection processes represents an important extension of the scope of knowledge in search theory related to the results concerning the stationary target. Physically, the structure of their interactions can be considered as a model in which the search environment acts to aid the target, e.g., the terrain, foliage, etc., common to the subregion in which the target is operating, or, in the ASW context, the existence of thermal barriers, and other local phenomena which tend to increase (and decrease) the level of concealment of the target over time. In the situation in which a single interval of visibility exists, the probability distribution of the length of the visible period may be interpreted as the time required for the hunted to become aware of the hunter's presence. Multiple periods of visibility may reflect the situation in which the enemy periodically activates some form of sensing equipment which makes him vulnerable to detection by the searcher.

Danskin (1962a) makes a study of the optimum distribution of aerial reconnaissance effort against land targets in the presence of decoys. The model considered is one in which the reconnoitering forces allocate effort among various regions, their objective being the location of the targets, assuming the side being reconnoitered is passive.

The information function of communication theory is chosen as the measure of effectiveness. That is, the information of a reconnaissance is defined to be the change in the uncertainty of the region resulting from that reconnaissance. For each of the (K_0) regions, one has an information function $I_K(x)$, where x is the level of reconnaissance. The allocation problem is stated as: Given X units of reconnaissance effort to distribute among the K_0 regions, how shall this be done so as to maximize the information?

One wishes to maximize

$$\sum_{K=1}^{K_0} I_K(x_K)$$

subject to

$$\sum_{K=1}^{K_0} x_K = X, \quad x_K \geq 0.$$

The solution to the problem depends entirely on the form of the functions $I_K(x)$. Under the most realistic assumptions concerning the detection probabilities associated with aerial reconnaissance, the author is unable to determine the behavior of the second derivative of $I_K(x)$ and thus the form of the objective function. In Part II of the two-part paper, Danskin (1962b) considers the two-sided reconnaissance

problem, in which the side being reconnoitered seeks to minimize the information (maximize the confusion) obtained by the reconnoiterer, while maintaining at least a certain minimum acceptable threat with a fixed budget. This problem formulated as a zero-sum, two-person game, is solved for a special case (fixed equipment) and it is shown that there exists a solution in mixed strategies for the general use.

Smallwood (1965) considers a model for the placement of n detection stations for optimum coverage of an arbitrary area. The stations are assumed to be identical and to have a probability of detection that is a function only of the distance between the station and the event to be detected. Furthermore, stations are assumed to operate independently of each other. It is also assumed that the enemy has complete knowledge of the station locations and effectiveness and is interested only in eluding detection by the detection stations. The situation is reduced to the minimax problem of placing the stations so that the maximum probability of not detecting an enemy event is minimized. Necessary conditions for the optimal locations are given, and a hill climbing interactive technique based on these conditions is described in some detail. The technique is applied to the problem of the location of detection stations within the United States and the Soviet Union.

Pollock (1969) points out that there has been a tendency to model the three phases of a general surveillance operation (search, detection, and ensuing action) separately, the output parameters of one such model are often used as the inputs to another. He considers some of the interfaces between these phases and presents some examples of the relation between search, detection, and decision theories involving false alarms, continuous surveillance, localization, and the selection of appropriate measures of effectiveness.

W. Edwards (1962) notes that the development of a dynamic decision theory will be central to the expansion of research on human decision problems. A taxonomy of decision problems is presented, most require a dynamic theory in which the decision-maker is assumed to make a sequence of decisions, basing decision $n + 1$ on what he learned from decision n and its consequences. The relevance of the mathematical developments in dynamic programming and Bayesian statistics to dynamic decision theory is examined.

Along these lines, Rapoport (1966) considers a dynamic programming model of a controller, i.e., a dynamic decision-maker, who can actively manipulate the environment by his decisions. An experiment is described in which subjects were given dynamic decision-making tasks, the results fit

well the analytic solution obtained from the dynamic programming model.

Smallwood (1966) notes that in many practical situations the discount factor for future rewards and costs is not known precisely. The dependence of the optimum policy on the discount factor is often noted in the modeling of these problems. He discusses the dependence of the optimum policy on the discount factor for the class of finite-state, time-invariant, Markov models. A procedure is developed for finding the value of the discount factor for which the decision-maker is indifferent between two policies. The procedure is extended to a discussion of how one can find the complete description of the optimum policy regions over any range of the discount factor.

5.0 AREAS FOR FUTURE RESEARCH

Table 1 is an attempt at summarizing the current "state of the art" in search and reconnaissance theory (at least in subjects relevant to the goals of this report). The numbered entries refer to the papers in the bibliography given in Chapter 6.0. No attempt was made to enumerate *all* the papers in a given category, but only to indicate that the area had been treated in the literature.

Examination of Table 1 clearly reveals areas in which little or no research activity has been devoted and which are considered important topics for future research. These are briefly noted below along with some areas suggested by Pollock (1969) and Dobbie (1963).

1. *Interaction Between Detection and Visibility Processes*

The table suggests that the visibility problem as defined by Bonder (1969) and Disney (1969) has not been treated. As noted in the text, the visibility process has been modeled as follows:

- (a) The target may be visible to the searcher for the entire search interval with some known probability p .
- (b) The target may be visible at the start of the search period, the length of the visible period being a random variable with known probability density function, and not reappear.

Table 1
Summary of Research Efforts

	One-Sided Search						Two-Sided Search*		
	Single Target			Multiple Target			Single Target		
	Stationary	Non-Stationary		Stationary	Non-Stationary		Stationary	Non-Stationary	
	Vis. No. Vis.	Vis.	No. Vis.	Vis. No. Vis.	Vis.	No. Vis.	Vis. No. Vis.	Vis.	No. Vis.
<i>Detection</i>									
Single-Scan									
a) Binary	(37)		(56)			(14)		(58) (57)	(59)
b) Interval									
1) Non-Cumulative	(20) (33)		(49)				(57) **	(57)	(57)
2) Cumulative									(55)
<i>Multiple-Scan</i>									
a) Binary	(67)						(67)		(65)
1) Independent									
2) Dependent									
b) Interval									
1) Independent									
a) Non-Cumulative	(67)		(90)				(67) (90)		
b) Cumulative									
2) Dependent									
a) Non-Cumulative	(68)						(68)		
b) Cumulative									

*The multiple-target category was eliminated because the literature contains no work in this area.
 **Visibility is considered as a function of position and not as a function of time per Bonder (1969) and Disney (1969).

- (c) A single period of visibility may be exhibited starting at some random time during the search interval and lasting a random amount of time.
- (d) The target may exhibit alternating periods of visibility and invisibility, the durations of each being random variables.

These forms may interact with all modes of detection and targets, thus giving rise to many research possibilities. This area is currently being studied extensively in the SRL under this ONR contract.

2. *Non-Stationary Targets*

- (a) Target motion independent of position and known precisely.
- (b) Target motion independent of position and drawn at random from a population known to the searcher.
- (c) Target motion dependent upon position and known precisely to the searcher.
- (d) Target motion drawn randomly from a known population which is a function of target position.
- (e) Target motion chosen in advance by the evader from a probability distribution known to the searcher.
- (f) Evader chooses motions, subject to limitations known partially to the searcher, throughout the search as he obtains information on the past activities and location of searcher.

3. *Structure and Capabilities of Operational Detectors*

Although the description of assumptions regarding detectors given in Section 1.2 differentiated between detectors that had single- and multiple-scan capabilities, this difference is not reflected in the diagram. With the exception of Kimball's paper (1963), all research papers considered in this literature review take as given the capabilities of the detector and do not distinguish single-scan versus multiple-scan effects. Research is needed in this area to understand the behavior of operationally useful devices, e.g., the effect of multiple scans, independence between successive looks, etc.

4. Optimization criteria will, in general, depend upon the objective of the operation. If additional action is to be taken after detection, then neither the maximization of the probability of detection nor the minimization of the expected search time may be optimal for the combined operation. Research should be devoted to the structuring of the total activity, which includes search, detection, track, and ensuing action, before selecting the optimization criteria. For example, search activity can readily be interfaced with the combat activity which results from mutual detection.

5. The output of many of the optimization search studies has been the *fixed* amount of time to search in a box. One might instead consider the likely possibility that the actual search time will be a random variable and examine its effect on the optimal policy. It is not unlikely that searchers may have various modes of operation, each of which has a characteristic distribution of search time as well as associated Type I and II errors.

6. The likelihood that searchers will not (or cannot) follow optimal search procedures suggests research be devoted to the problem of converting theoretical results into practical rules of application.

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13. ABSTRACT			
Research being performed by the Systems Research Laboratory (SRL) under contract number N00014-67-A-0181-0012 with the Office of Naval Research is concerned with the development of more generalized mathematical structures of military processes. Emphasis has been directed to the modeling of combat processes and the development of associated allocation strategies. It was thought that many of the existing search and reconnaissance theories would be useful for predicting the amount of intelligence-gathering capability possessed by a tactical unit. A thorough literature review in this area, however, indicated that existing theories are less than useful for this purpose. Most of the research efforts have been devoted to the development of strategies for the optimal allocation of search effort and little to the development of descriptive models of intelligence-gathering processes. The purpose of this interim technical report is to present the results of the literature review, both as a base for our research and to indicate fruitful areas of research for other investigators. Principal results in the field and the techniques used in attaining them are presented in an annotated bibliography. A comprehensive bibliography, organized under subject classifications, is included. Finally, some relevant areas for future research are described.			

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