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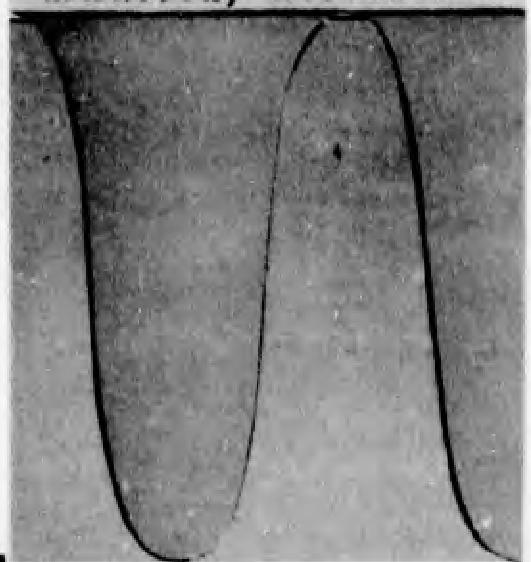

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STOCHASTIC INTEGRALS FOR NIGH-MARTINGALES

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ABSTRACT

The processes studied in this report generalize those known as martingales, the generalization being dependent on certain subsidiary functions φ, ρ, which in practice, could be, for instance, powers u^α, u^β between 0 and 1. The object of the paper is to define for such a process $x(t,\omega)$ the stochastic integral of a second process $y(t,\omega)$, where y is subject to conjugate conditions involving subsidiary functions ψ, σ such that $\varphi(u)\psi(u)/u^2$ and $\rho(u)\sigma(u)/u^2$ are integrable at $u = 0$. It is possible, for instance, for almost all the curves described by one of the processes for constant ω to be space-filling.

STOCHASTIC INTEGRALS FOR NIGH-MARTINGALES[*)]

L. C. Young

§1. Introduction. This is a sequel to two M. R. C. reports, the first of which consisted of three notes on derivatives and integrals, while the second concerned an extension of the notion of integral of a deterministic integrand with respect to a stochastic process, this process being, in some sense "close" to those with orthogonal increments.[**)] We now wish to define the integral of one stochastic process with respect to another, by a similar method, and for this purpose we consider processes which are "close" to being martingales, and which we call[***)] nigh-martingales. The need for such extensions of the class of martingales, or the class of processes with orthogonal increments, is illustrated by the fact that the corresponding classes of functions $x(t, \omega)$ include not a single non-constant function of the form $x(t)$, independent of ω; our extensions include every $x(t)$ which satisfies, for instance, a suitable fractional Lipschitz condition. Clearly, if a stochastic theory is to tackle, sensibly, stochastic generalizations of non-trivial deterministic problems, it must not itself reduce to triviality in the deterministic case !

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**) Mathematics Research Center Technical Reports nos. 677 and 932. References in the text to the "previous" report are intended to indicate no. 932. See also literature there given.

***) What a pity we cannot call them nightingales! We will do so privately, but just spell them as in the text, to avoid a public outcry.

As in the previous report, we regard a stochastic process as given by a complex-valued function $x(t, \omega)$, where t lies, for simplicity, in the unit time-interval T, except that we may have occasion to allow the values of $x(t, \omega)$ to range in a slightly more general Euclidean space, as this raises no new difficulties whatever. On the other hand, it will be convenient here to impose rather special conditions on the space of ω, and on the corresponding probability, or unit measure, $d\omega$; their purpose is to avoid side-issues, and also to avoid too many probabilistic terms: indeed, the functions of t, given by $x(t, \omega)$, could very well be those defining a family of generalized streamlines, or of light-rays, or optimal trajectories, in problems apparently far removed from random phenomena, but in which, for some reason, the uniqueness of solutions with given initial data has broken down. With an eye on such interpretations, we regard each value of ω as just a label, attached to a function of t, and this label is to be retained at all times, and is not to be tampered with. Thus the function $x(t, \omega)$ defines a family of such functions of t, and the members are distinguished by their labels ω. Moreover, on the space of these labels we are given a unit measure $d\omega$.

In practice such labels have to be capable of an easy classification, and for this reason they are made up of many parts. This means that each label can be regarded also as a pair of slightly simpler labels, or more generally, as the ordered set of a finite number of still simpler ones, so that the space of ω has a Cartesian structure. We shall suppose this structure made up as follows, in relation to the time-scale.

Given any time-interval Δ, we denote by ω_Δ a set of labels which distinguishes the restrictions of our functions of t to the closed interval Δ. Then given any subdivision $\mathfrak{J}$ of T into intervals Δ, we identify the space of ω with the Cartesian product of the spaces of partial labels ω_Δ. Thus the space of ω has a Cartesian decomposition corresponding to each subdivision $\mathfrak{J}$ of T. The decomposition of the unit measure $d\omega$ involves "conditional" unit measures $d\omega_\Delta$, each of which can depend on a prior label ω_{Δ^-}, where Δ^- is the set of t not exceeding the initial time of Δ. More precisely, if $\Delta^0, \Delta^1, \ldots, \Delta^N$ are the $\Delta \in \mathfrak{J}$ in the order of increasing t, the measure of a set $\tilde{\Omega}$ of labels $\omega = (\omega_{\Delta^0}, \omega_{\Delta^1}, \ldots, \omega_{\Delta^N})$ is to be given by

$$\int_{\tilde{\Omega}} d\omega = \int d\omega_{\Delta^0} \int d\omega_{\Delta^1} \cdots \int d\omega_{\Delta^N} \tilde{\Omega}(\omega_{\Delta^0}, \omega_{\Delta^1}, \ldots, \omega_{\Delta^N}),$$

where $\tilde{\Omega}(\omega)$ is unity in E, zero elsewhere, and where each $d\omega_{\Delta^i}$ is a unit measure which may depend on the previous labels ω_{Δ^j} $(j < i)$, but not on the later ones $(j > i)$.

The above assumptions on the space of ω and on the various measures $d\omega_\Delta$ amount to much more than one likes to make, but here they are largely a matter of notation, our object being simplicity of presentation rather than generality. Similarly we shall, as in the previous report, disregard complicated side-issues by suppressing a monotone increasing change of variable $\tau = \tau(t)$ on the t-axis. To aid intuition, given $\Delta \subset T$, we term ω_Δ and ω_{Δ^-} the present and the past of the label ω, relative to the interval Δ. We extend the notation to the case where Δ reduces to a point t, so that ω_{t^-} denotes the past of ω relative to the point t, while ω_t has cancelled;* we note that ω_{t^-} coincides with ω_{Δ^-} for each Δ whose initial point is t. Similarly, if Δ is an interval in T for which t is the final point, we term future of ω relative to Δ, or relative to t, and denote by ω_{Δ^+} or ω_{t^+}, the part of the label ω relative to the interval $\Delta^+ = (t, 1)$. Finally we agree that, in connection with an interval $\Delta \subset T$, or a point $t \epsilon T$, the terms for "almost every past" mean except for a set of ω_{Δ^+}, or ω_{t^+}, of $d\omega_{\Delta^+}$, or $d\omega_{t^+}$, measure zero.

We shall consider in the sequel only processes $x(t, \omega)$ which are, at each t, independent of the future label ω_{t^+}, that is, they depend only on t and on the past label ω_{t^-}. Such a stochastic process will be said to be labelled by its past. We shall suppose it square-integrable in dt and in $d\omega$, and therefore in $d\omega_\Delta$ for almost every past relative to Δ. We term it a nigh-martingale, if, further, for certain continuous increasing non-negative functions χ_1, χ_2 of a variable $u \geq 0$, there exists a monotone increasing function $\tau(t)$ $t \epsilon T$, such that, for every interval $\Delta \subset T$ and for almost every corresponding past, we have

$$(1.1) \qquad \begin{cases} \left| \int \Delta x d\omega_\Delta \right| \leq \chi_1(\Delta \tau), \\ \int |\Delta x|^2 d\omega_\Delta \leq \chi_2(\Delta \tau). \end{cases}$$

Here $\Delta x, \Delta \tau$ denote as usual the difference in t of the corresponding functions; moreover, if we write, as we will in the sequel, φ for $(\chi_1)^{1/2}(\chi_2)^{1/4}$, and $\rho(u)$ for $(u\chi_2(u))^{1/2}$, we shall further restrict χ_1, χ_2 by the following condition: there must exist functions $\psi(u)$, $\sigma(u)$,

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*We shall not need to distinguish by labels the restrictions of functions of t to single points t.

such that each of the pairs φ, ψ and ρ, σ constitutes a reversible pair of estimate functions according to the terminology of section 2 of the previous report. It will be convenient to fix two such functions ψ, σ, and we do so from now on. Further, for simplicity, we suppress the change of variable $\tau = \tau(t)$ and suppose that $\tau(t) = t$. Thus (1.1) becomes

$$(1.2) \qquad \begin{cases} \left| \int \Delta x \, d\omega_\Delta \right| \leq \chi_1(|\Delta|) , \\ \int |\Delta x|^2 d\omega_\Delta \leq \chi_2(|\Delta|) , \end{cases}$$

where $|\Delta|$ is the length of Δ. Finally, to avoid certain side-issues to which we shall return, we suppose $\chi_2(0) = 0$, although this is not really material. When these various hypotheses are fulfilled, we term the process $x(t, \omega)$ a canonical nigh-martingale.

In the main essentials, such processes are to be regarded as special cases of those of section 2 of the previous report, where we could very well have extended our hypotheses slightly, by an unsymmetrical form of the cross-correlation inequality there used, and by a corresponding interpretation of φ, which was mentioned in the introduction to that report. The additional hypotheses, that we now make, will enable us to use the same basic methods, to integrate with respect to our process $x(t, \omega)$, not only deterministic functions $y(t)$, considered in the previous report, but a wider class of functions $y(t, \omega)$, for almost all constant ω.

In the sequel, it will be convenient to denote, for any square-integrable function $f(\omega)$, by $n(f)$ the L^2-norm of f, i.e. the expression $(\int |f|^2 d\omega)^{1/2}$. We shall define, for almost every ω, the stochastic integral

$$(1.3) \qquad \int_T y(t,\omega)\,d_t x(t,\omega)\ ,$$

under the following hypotheses:

$$(1.4)\quad \begin{cases} \text{(i)}\ \ y(t,\omega)\ \text{is square-integrable in}\ dt, d\omega, dtd\omega\ ; \\ \text{(ii)}\ \ y(t,\omega)\ \text{is labelled by its past (i.e. depends only on}\ t, \omega_{t^-})\ ; \\ \text{(iii)} \qquad \int \eta(\Delta y)dt \leq \psi(h)\ , \\ \text{(iv)} \qquad \int \eta^2(\Delta y)dt \leq \sigma^2(h)\ , \end{cases}$$

where Δ is now (temporarily only) the "sliding" interval $(t-h, t)$.

We observe that classes of functions subject to conditions of the type (1.4) generalize in a natural way those considered in the previous report, and are similar to the integrated Lipschitz classes considered by Hardy and Littlewood and others in connection with fractional integration. They are thus fairly extensive classes.

§2. <u>The stochastic integral and the basic inequality</u>. We intend to define (1.3) as a limit of a corresponding elementary integral, in which $y(t,\omega)$ is replaced by a certain rather better-behaved function $\hat{y}(t,\omega)$. We therefore need to say something first about this elementary integral, and here the discussion deviates somewhat from that of the previous report, although we again consider square-integrable functions of ω as points, or vectors, situated in the Hilbert space of such functions, with the usual norm η. From this point of view, the function $x(t,\omega)$ again becomes a vector-valued function $\mathfrak{x}(t)$, which thus depends only on t ; similarly, $\hat{y}(t,\omega)$ becomes a vector-valued function $\hat{\mathfrak{y}}(t)$, whereas in the previous report y, $\hat{y}$ were scalar-valued functions of t alone. This means that we have a new symmetry between x and $\hat{y}$, and it becomes necessary to introduce the elementary product of two vectors f, g situated in our Hilbert space. This product $f.g$ will be simply the function of ω, i.e. by setting $(fg)(\omega) = f(\omega)g(\omega)$. However, this elementary product of two vectors in our Hilbert space L^2 is not itself a vector in L^2, but only a vector in the larger Banach space L^1. We shall write $\| \|_{L^1}$, $\| \|_{L^2}$ for the norms in L^1 and L^2, so that $\| \|_{L^2}$ is the same as n

With the help of this elementary product, we can now form the usual elementary Riemann-Stieltjes sums

$$S = \sum_{i>0} \hat{\psi}(\tau_i)\{X(t_i) - X(t_{i-1})\}$$

where $t_0 = 0$, $t_{i-1} \le \tau_i \le t_i$, and where the points t_i constitute a finite set, whose final member is the end 1 of T. The sums S are then vectors in L^1, and we define the elementary integral

$$I = \int_T \hat{y}(t,\omega)d_t(t,\omega) = \int_T \hat{\psi}(t)dX(t),$$

as the L^1-limit of the sums S, when the greatest of the lengths $t_i - t_{i-1}$ is made to tend to 0.

According to the classical theorem of Stieltjes, I exists, whenever one of the functions $X(t)$, $\hat{\psi}(t)$ is continuous and the other, of bounded variation. Here continuity and bounded variation of the individual functions have no direct reference to our elementary product, and are therefore understood in the L^2 metric. Continuity in this sense of $X(t)$ is ensured by the second relation (1.2), since $X_2(0) = 0$; bounded variation of $\hat{y}(t)$ will be an automatic consequence of good behavior of $\hat{y}(t,\omega)$ in the only cases in which the elementary definition of stochastic integral will be used. There will be two such cases.

The first and simplest case is that in which $\hat{y}(t)$ reduces to a step-function. We shall then allow ourselves the slight inaccuracy of also speaking of $\hat{y}(t,\omega)$ as a step-function. Moreover, in this case, we shall invariably employ other symbols in place of $\hat{y}$. We shall only be interested in a retarded step-function, and by this we mean a function $z(t,\omega)$ such that there exists a subdivision $\mathfrak{J}$ of T, for which, in the interior of each $\Delta \epsilon \mathfrak{J}$, the value of $z(t,\omega)$ depends only on Δ and on the past $\omega_\Delta -$, relative to Δ; we shall denote this value by $z(\Delta, \omega_\Delta -)$, and we shall allow ourselves the slight inaccuracy of denoting the step-function itself by $z(\Delta, \omega_\Delta -)$ $\Delta \epsilon \mathfrak{J}$, although this notation ignores the values at the ends of the $\Delta \epsilon \mathfrak{J}$, and rather corresponds to what we termed a (step)-function in the previous report. We term $z(t,\omega)$ a (step h)-function, if $\mathfrak{J}$ is a subdivision of T into equal parts of length h. With the above notation, our elementary definition leads, for a retarded step-function $z(t,\omega)$, to the formula

$$\int z(t,\omega) d_t x(t,\omega) = \sum_{\Delta \epsilon \mathfrak{J}} z(\Delta, \omega_\Delta -) \Delta x .$$

The other case to which we apply the elementary definition, is that in which

$$(2.1) \qquad \hat{y}(t, \omega) = \frac{1}{k} \int_{t-k}^{t} y(\tau, \omega) d\tau,$$

where k is small and positive, and where $y(t, \omega)$ is a function subject to the hypotheses (1.4) of the preceding section, slightly strengthened as in the previous report, by replacing T, where relevant by an enlarged interval T^+; this is because, when $t \in T$ and k is small, $t-k \in T^+$. Henceforth the symbol $\hat{y}(t, \omega)$, in this section, will only refer to a function of the form (2.1). It remains to verify that, in this case, the corresponding vector valued function $\hat{\mathfrak{y}}(t)$ is of bounded variation, i.e. that, for some finite constant $\mathfrak{K}$, depending only on the function $\hat{\mathfrak{y}}(t)$, we have, for every subdivision $\mathfrak{J}$ of T,

$$\sum_{\Delta \in \mathfrak{J}} \mathfrak{h}(\Delta \hat{\mathfrak{y}}) \leq \mathfrak{K}.$$

In verifying any such inequalities for $\mathfrak{h}$, it is convenient to bear in mind that $\mathfrak{h}$ is a homogeneous convex function, so that, for $p \geq 1$, it satisfies Jensen's inequality; the latter states that, if F is a centre of gravity $\int F(\alpha) d\alpha$ of vectors $F(\alpha)$, situated in our Hilbert space, where $d\alpha$ is a unit measure, then

$$\mathfrak{h}^p(F) \leq \int d\alpha \, \mathfrak{h}^p(\mathfrak{F}(\alpha)).$$

In fact here, if $\mathfrak{F}(\alpha)$ is the function $f(\alpha,\omega)$, the left-hand side is

$$\left(\int d\omega \left|\int d\alpha\, f\right|^2\right)^{p/2} \le \left(\int d\omega \int d\alpha |f|^2\right)^{p/2} =$$

$$= \left(\int d\alpha \int d\omega |f|^2\right)^{p/2} \le \int d\alpha \left(\int d\omega |f|^2\right)^{p/2} ,$$

which is the right-hand side. We apply this inequality with $p = 1$, so that, by homogeneity, $d\alpha$ can be replaced by any non-negative measure, and we use the fact that

$$\Delta \hat{\psi} = \frac{1}{k} \int_\Delta (\psi - \mathfrak{R}_k \psi) dt ,$$

where $\mathfrak{R}_k$ is the retardation $t \to t-k$. Thus

$$n(\Delta \hat{\psi}) \le \frac{1}{k} \int_\Delta n(\psi - \mathfrak{R}_k \psi)\, dt,$$

and therefore

$$\sum_{\Delta \epsilon \mathfrak{J}} n(\Delta \hat{\psi}) \le \frac{1}{k} \int_T n(\psi - \mathfrak{R}_k \psi) dt \le \frac{2}{k} \int_{T^+} n(\psi) dt$$

so that this last expression can be taken for $\mathcal{K}$.

We shall need also the same calculation when $\hat{\psi}$ is replaced by

$$(2.2) \qquad \hat{\psi}(t) - \mathfrak{R}_h \hat{\psi}(t) .$$

We see that the total variation of the vector-valued function (2.2) cannot exceed

$$\frac{2}{k}\int_{T+} n(\psi - R_h\psi)dt \le \frac{2}{k}\left(\int_{T+} n^2(\psi - R_h\psi)dt\right)^{1/2},$$

which, according to (1.4) (iv), is at most $2\sigma(h)/k$. We note also that, for every t, the norm of the vector (2.2) is at most $\sqrt{h}\,\sigma(k)/k$; we see this as follows: the norm in question is

$$\frac{1}{k}\, n\left(\int_{t-h}^{t} d\tau(\psi - R_k\psi)(\tau)\right) \le \frac{1}{k}\int_{t-h}^{t} d\tau\; n(\psi - R_k\psi)(\tau)$$

$$\le \frac{1}{k}\, h^{1/2}\left(\int_{t-h}^{t} d\tau\; n^2(\psi - R_k\psi)(\tau)\right)^{1/2} \le \frac{1}{k} h^{1/2}\sigma(k).$$

This shows that, for fixed k, both the maximum norm, and the total variation, of the vector-valued function (2.2) will tend to 0 with h. It follows that the elementary integral

$$\int_T \hat{\psi}(t)\,dX(t)$$

is not only the limit of the sums S defined above, but also that of the retarded sums

$$S_- = \sum_{t>0} \hat{\psi}(\tau_i - h)\{X(t_i) - X(t_{i-1})\},$$

provided that, in addition to our previous conditions, $h \to 0$. In fact, the L^1 norm of the difference $S - S_-$, is, by a classical partial summation argument, at most the product of the oscillation of X by the sum $(2\sigma(h)/k) + (\sqrt{h}\,\sigma(k)/k)$. This is an important remark, crucial later on.

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We now come to the basic lemma, which takes the place of lemma (2.3) of the previous report. We shall denote in it by $I(z)$ the elementary integral, with respect to the process $x(t,\omega)$, of a (step h)-function $z(t,\omega)$. The lemma will show incidentally that, contrary to expectations, the vector $I(z)$ does lie in the Hilbert space L^2.

(2.3) <u>Lemma</u>. <u>Let</u> $z(t,\omega)$ <u>be a (step</u> h)-<u>function</u>. <u>Then</u>

$$\eta^2(I(z)) \le h^{-2}\rho^2(h)\int_T \eta^2(z)dt + (h^{-1}\varphi(h)\int_T \eta(z)dt)^2.$$

<u>Proof</u>. Let $\mathfrak{I}$ be the subdivision of T into equal parts of length h. Then we can write $\eta^2(I(z))$ in the form

$$\sum_{\Delta\in\mathfrak{I},\Delta^*\in\mathfrak{I}} \int d\omega z(\Delta,\omega_{\Delta-})\bar{z}(\Delta^*,\omega_{\Delta^*-})\Delta x\,\Delta^*\bar{x} = \sum_{\Delta,\Delta^*} b(\Delta,\Delta^*).$$

Here, we first estimate separately the quantity b for $\Delta = \Delta^*$. It is an integral in $d\omega$ of an expression independent of the future relative to Δ, and we rewrite it as a repeated integral in $d\omega_\Delta d\omega_{\Delta-}$. If we denote by $\eta(z \mid \Delta)$ the norm of z restricted to Δ, we thus find that

$$b(\Delta,\Delta) = \int d\omega_{\Delta-}\{z\bar{z}\int d\omega_\Delta \Delta x\Delta\bar{x}\} = \int d\omega_{\Delta-}|z|^2\int d\omega_\Delta|\Delta x|^2$$

$$\le \chi_2(h)\int d\omega_{\Delta-}|z|^2 = \chi_2(h)\eta^2(z|\Delta) = h^{-1}\chi_2(h)\int_\Delta \eta^2(z)dt.$$

We can proceed similarly when Δ^* is prior to Δ, remembering that only the factor Δx then depends on ω_Δ. We find firstly that

$$b(\Delta,\Delta^*) = \int d\omega_{\Delta-}\{z(\Delta,\omega_{\Delta-})\bar{z}(\Delta^*,\omega_{\Delta^*-})\Delta^*\bar{x}\int d\omega_\Delta \Delta x\}$$

$$\leq \chi_1(h)\int d\omega_{\Delta-}\ |z(\Delta,\omega_{\Delta-})\bar{z}(\Delta^*,\omega_{\Delta^*-})\Delta^*\bar{x}\ |,$$

and secondly, by writing Δ' for the difference of the intervals $\Delta-$, Δ^*- and expressing the integral in $d\omega_{\Delta-}$ as a repeated integral in $d\omega_{\Delta'}$, $d\omega_{\Delta^*-}$, that the quantity last displayed above becomes

$$\chi_1(h)\int d\omega_{\Delta^*-}\{|\bar{z}(\Delta^*,\omega_{\Delta^*-})|\int d\omega_{\Delta'}\ |z(\Delta,\omega_{\Delta-})\Delta^*\bar{x}|\}$$

$$\leq \chi_1(h)\int d\omega_{\Delta^*-}\{|\bar{z}(\Delta^*,\omega_{\Delta^*-})|(\int d\omega_{\Delta'}|z(\Delta,\omega_{\Delta-})|^2)^{1/2}(\chi_2(h))^{1/2}\}$$

$$\leq \chi_1(h)(\chi_2(h))^{1/2}(\int d\omega_{\Delta^*-}|\bar{z}(\Delta^*,\omega_{\Delta^*-})|^2)^{1/2}(\int d\omega_{\Delta^*-}\int d\omega_{\Delta'}|z(\Delta,\omega_{\Delta-})|^2)^{1/2}$$

$$= \chi_1(h)(\chi_2(h))^{1/2}n(\bar{z}|\Delta^*)n(z|\Delta) = h^{-2}\chi_1(h)(\chi_2(h))^{1/2}\int_{\Delta^*}n(\bar{z})dt\int_{\Delta}n(z)dt.$$

By symmetry a similar inequality holds when Δ is prior to Δ^*, and, from these inequalities for the quantities b, the assertion of our lemma now follows at once by addition, in view of the relations connecting χ_1, χ_2 with φ and ρ.

We continue to follow the developments of section 2 of the previous report, with the changes called for by our present hypotheses. If $\mathfrak{T}$ is the subdivision of T into equal parts of length h, we shall denote by $z(t,\omega)$ the (step h)-function whose value for $t \in \Delta$, where $\Delta \in \mathfrak{J}$, retarded mean

$$\frac{1}{h} \int_{\Delta} y(t-h,\omega)dt.$$

Similarly we define $\hat{z}(t,\omega)$ in terms of the retarded mean formed with $\hat{y}$ in place of y. We shall term the functions z, $\hat{z}$ the (step h)-approximations to y, $\hat{y}$; they are evidently retarded step-functions, and we may denote them correspondingly also in the forms $z(\Delta,\omega_{\Delta-})$, $\hat{z}(\Delta,\omega_{\Delta-})$ where $\Delta \in \mathfrak{J}$. Further, we write z^* in place of z when h is replaced by h^*; and later we shall write y_ν, $\hat{y}_\nu$ for z, $\hat{z}$, when $h = 2^{-\nu}$, at which time the symbols z, $\hat{z}$, z^* will be free to take up other meanings, if we so desire.

Our next lemma is the analogue of lemma (2.4) of the previous report; we write in it, for $p = 1, 2$,

$$\psi_p(h) = \sup_{0 \leq c \leq h} \int_{T+} \mathfrak{n}^p(\mathfrak{y} - \mathfrak{R}_c \mathfrak{y})dt,$$

moreover the capitals Z, Z^*, $\hat{Z}$ denote the vector-valued step-functions, defined, as functions of t, by z, z^*, $\hat{z}$.

(2.4) Lemma. With the notation just explained, where we further suppose $h^* = h/N$ for some positive integer N, we have

(i) $\int_T n^p(\hat{\psi} - \mathfrak{R}_h\hat{\psi})dt \leq \psi_p(h)$,

(ii) $\int_T n^p(Z-\hat{Z})dt \leq \int_{T+} n^p(\psi - \hat{\psi})dt \leq \psi_p(k)$,

(iii) $\int_T n^p(Z-Z^*)dt \leq 2\psi_p(2h)$.

<u>Proof</u>. To prove (i), we observe that, by Jensen's inequality,

$$n^p\left(\frac{1}{k}\int_0^k \{\psi(t-\tau) - \psi(t-h-\tau)\}d\tau\right) \leq$$

$$\frac{1}{k}\int_0^k n^p\{\psi(t-\tau) - \psi(t-h-\tau)\}d\tau,$$

and therefore, by integrating in t and interchanging the order of the integrations on the right, that the left-hand side of (i) is at most $k^{-1}\int_0^k d\tau\ \psi_p(h)$. Similarly in regard to the second half of (ii), we have at a point t

$$n^p\left(\frac{1}{k}\int_0^k \{\psi(t) - \psi(t-\tau)\}d\tau\right) \leq \frac{1}{k}\int_0^k n^p\{\psi(t) - \psi(t-\tau)\}d\tau,$$

and the desired result follows just as in (i). The same argument, starting with the inequality

$$n^p\left(\frac{1}{h}\int_0^h d\tau\ \mathfrak{R}_h(\psi - \hat{\psi})(t-\tau)\right) \leq \frac{1}{h}\int_0^h d\tau\ n^p(\mathfrak{R}_h(\psi - \hat{\psi})(t-\tau))$$

leads to the first half of (ii).

It only remains to establish (iii). We denote by $\mathfrak{I}^*$ the subdivision of T into equal parts of length h^*, and by Δ^* any one of these parts. We denote further by Δ the interval of the subdivision into equal parts of length h, such that $\Delta \supset \Delta^*$, and by E the set of integers s, where $-N < s < N$, such that the translation of Δ^* by sh^* remains in Δ; the set E clearly has N members, and for $t \in \Delta^*$ we have

$$Z = \frac{1}{h} \int_{\Delta} \Re_h \psi \, dt = \frac{1}{Nh^*} \sum_{s \in E} \int_{\Delta^*} \Re_{h+sh^*} \psi \, dt .$$

Hence

$$Z - Z^* = \frac{1}{N} \sum_{s \in E} \frac{1}{h^*} \int_{\Delta^*} (\Re_{h+sh^*} - \Re_{h^*}) \psi \, dt ,$$

and by Jensen's inequality this implies

$$n^p(Z - Z^*) \leq \frac{1}{N} \sum_{s \in E} \frac{1}{h^*} \int_{\Delta^*} dt \; n^p(\Re_{h+sh^*} - \Re_{h^*}) \psi$$

$$\leq \frac{1}{N} \sum_{|s|<N} \frac{1}{h^*} \int_{\Delta^*} dt \; n^p(\Re_{h+sh^*} - \Re_{h^*}) \psi .$$

By multiplying by h^*, we get a corresponding inequality for the integral over Δ^*, and by summing with respect to Δ^* we find that the left-hand side of (iii) is at most

$$\frac{1}{N} \sum_{|s|<N} \int_T dt \; n^p(\Re_{h+sh^*} - \Re_{h^*}) \psi \leq \frac{1}{N} \sum_{|s|<N} \psi_p(2h),$$

which implies our assertion.

As a consequence of the preceding two lemmas, which correspond to lemmas (2.3) and (2.4) of the previous report, we can now proceed almost exactly as in section 2 of that report. We write h_ν for $2^{-\nu}$, we denote by y_ν, $\hat{y}_\nu$ the step-functions z, $\hat{z}$ of lemma (2.4) for $h = h_\nu$, and by I_ν, $\hat{I}_\nu$ the elementary integrals, with respect to the process of y_ν, $\hat{y}_\nu$, considered as functions of t. Applying lemma (2.4) (iii) with $N = 2$, so that z^* becomes $y_{\nu+1}$, we find, by taking the difference $y_\nu - y_{\nu+1}$ for z in lemma (2.3), that

$$h^2(I_\nu - I_{\nu+1}) \leq (h_\nu^{-1}\rho(h_\nu) \cdot 4\sigma(h_\nu))^2 + (h_\nu^{-1}\varphi(h_\nu) \cdot 4\psi(h_\nu))^2,$$

in view of the fact that, for $p = 1, 2$,

$$(2\psi_p(2h))^{1/p} \leq 2^{1+1/p}\psi_p(h)^{1/p} \leq 4\psi_p(h)^{1/p},$$

which is at most $4\sigma(h)$ for $p = 2$, and $4\psi(h)$ for $p = 1$. It follows that the series

$$\sum_\nu h(I_\nu - I_{\nu+1})$$

is majorised, term by term, by the sum of the series

$$\sum_\nu 4h_\nu^{-1}\rho(h_\nu)\sigma(h_\nu) + \sum_\nu 4h_\nu^{-1}\varphi(h_\nu)\psi(h_\nu).$$

Hence the L^2-limit I of I_ν exists and

$$h(I - \hat{I}_\nu)$$

is at most the sum of the remainders of the two series. In the same way the L^2-limit $\hat{I}$ of $\hat{I}_\nu$ exists and

$$n(\hat{I} - \hat{I}_\nu)$$

is at most the sum of these remainders, since by (2.4) (i) the function $\hat{y}$ satisfies inequalities of the type (1.4) (iii), (iv), with the same ψ, σ as the function y. Since $\hat{I}$ is then also the L^1-limit of $\hat{I}_\nu$, by Schwarz's inequality, and since this L^1-limit can be expressed as that of an average of retarded sums S_- considered earlier in this section, we see that $\hat{I}$ coincides with the elementary integral

$$\int \hat{y}(t,\omega)d_t x(t,\omega).$$

Finally, by applying (ii) of lemma (2.4) with $h = h_\nu$ so chosen that $h_\nu \leq k < h_{\nu-1}$, and by writing

$$n(I - \hat{I}) \leq n(I-I_\nu) + n(\hat{I}-\hat{I}_\nu) + n(I_\nu - \hat{I}_\nu)$$

we see, exactly as in the corresponding part of the previous report, that I is the L^2-limit of $\hat{I}$ as $k \to 0$, and that

$$n(I - \hat{I})$$

cannot exceed some constant multiple of the sum of the integrals

$$\int_0^k \frac{\rho(u)}{u}\, d\sigma(u) + \int_0^k \frac{\varphi(u)}{u}\, d\psi(u).$$

We define this L^2-limit to be our stochastic integral (1.3).

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13. ABSTRACT

This is a continuation of MRC reports 677, 932. It defines a stochastic integral in which the integrand and the process with respect to which we integrate are stochastic, the process being supposed in a certain sense "close" to being a martingale.

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