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THEORY OF ANISOTROPIC THICK PLATES

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THEORY OF ANISOTROPIC THICK PLATES

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The goal of these remarks is to show how the classical theory of thick plates is generalized to the case of a plate possessing anisotropy of a particular type, that is, a plate manufactured from transversally isotropic material and consequently possessing five independent elastic constants.

1. General information. In the classical statement, the problem of equilibrium of an isotropic thick plate is reduced to the determination of six component stresses σ_x, σ_y, ..., τ_{xy} and three projections of displacement – u, v, w – which strictly satisfy all equations of elasticity theory and also the boundary conditions on the flat surfaces (faces). The conditions on the lateral surface, i.e., on the edge of the plate, are satisfied approximately, "on the average," as in the theory of the generalized plane stress state or in the theory of bending of thick plates.

If the plate is deformed only by a load distributed along the edge, two basic cases of equilibrium are distinguished: a) the plane stress state, and b) bending. Each case reduces to the determination in the region of the plate of a single biharmonic function of the variables x, y (a function of stresses or, correspondingly, of the deflection of the middle plane), satisfying the boundary conditions on the contour; the corresponding solutions are called homogeneous ([1], page 200). If the flat surfaces are also loaded the solution is obtained by superimposing the homogeneous solutions and the solution for an infinite elastic layer bounded by two parallel planes over which given forces are distributed. The homogeneous solutions for an isotropic plate are determined by simple formulas which connect stresses and displacements with the biharmonic function. It is easy to show that in the case of a transversally isotropic plate the homogeneous solutions also have a very simple structure and also connect stresses and displacements with the biharmonic function.

Let there be a plate which is cut out of transversally isotropic material in such a way that in it the plane of isotropy is parallel to the middle surface. We will take the latter as plane xy and designate the thickness of the plate as h. The equations of the generalized Hooke law are written as follows ([2], page 28):

$$
\begin{aligned}
&\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) - \frac{\nu_1}{E_1}\sigma_z, &\quad& \gamma_{yz} = \frac{1}{G_1}\tau_{yz}\\
&\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) - \frac{\nu_1}{E_1}\sigma_z, && \gamma_{xz} = \frac{1}{G_1}\tau_{x}\\
&\varepsilon_z = -\frac{\nu_2}{E}(\sigma_x + \sigma_y) + \frac{1}{E_1}\sigma_z, && \gamma_{xy} = \frac{1}{G}\tau_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}
\end{aligned}
\qquad (1.1)
$$

Here E and E_1 are the Young moduli for tension-compression in directions parallel and perpendicular to the middle plane; ν is the Poisson coefficient, characterizing the contraction in the plane of isotropy during tension in the same plane; ν_1 is the same parameter for stretching in the transverse direction z; ν_2 is the Poisson coefficient which characterizes contraction in the direction which is normal to the middle surface during tension in planes which are parallel to the middle surface; $G = E/2(1 + \nu)$; and G_1 is the shear modulus for planes which are parallel and normal to the middle surface. In all there are five different constants, since $E\nu_1 = E_1\nu_2$.

Ten unknown functions must satisfy the basic system of equations of equilibrium for a transversally isotropic body; we will obtain this system by adding to (1.1) three equations of equilibrium of a continuous medium:

$$\frac{d\sigma_x}{dx} + \frac{d\tau_{xy}}{dy} + \frac{d\tau_{xz}}{dz} + X = 0 \quad (xyz) \tag{1.2}$$

Here X, Y, and Z are projections of volume forces.

We shall introduce homogeneous solutions for a transversally isotropic plate without derivation; the solutions differ from those for an isotropic plate only in the coefficients which depend on elastic constants, and they can be derived by an analogous method ([1], pages 200-222; [3], Ch. 22). We shall also present formulas for a plate which is bent by an evenly distributed load.

2. <u>Plane stress state</u>. Forces are acting along the edge of a plate; these forces are symmetrical with respect to the middle plane and, consequently, lead to forces acting in this plane; there are no volume forces.

The component stresses and displacements are determined according to the following formulas:

$$u = \bar{u} - \frac{\nu_2}{2E}\left(\frac{h^2}{12} - z^2\right)\frac{\partial}{\partial x}\nabla^2 F, \qquad w = -\frac{\nu_2}{E} z\nabla^2 F \tag{2.1}$$

$$v = \bar{v} - \frac{\nu_2}{2E}\left(\frac{h^2}{12} - z^2\right)\frac{\partial}{\partial y}\nabla^2 F$$

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} + a\left(\frac{h^2}{12} - z^2\right)\frac{\partial^2}{\partial y^2}\nabla^2 F \tag{2.2}$$

$$\sigma_y = \frac{\partial^2 F}{\partial x^2} + a\left(\frac{h^2}{12} - z^2\right)\frac{\partial^2}{\partial x^2}\nabla^2 F$$

$$\tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} - a\left(\frac{h^2}{12} - z^2\right)\frac{\partial^2}{\partial x \partial y}\nabla^2 F \qquad \left(a = \frac{\nu_2}{2(1+\nu)}\right) \tag{2.3}$$

$$\tau_{yz} = \tau_{xz} = \sigma_z = 0$$

Here F is a function of Airy stresses, satisfying the equation

$$\nabla^2\nabla^2 F = 0 \qquad \left(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{2.4}$$

The first terms of the expressions – u, v, σ_x, σ_y, and τ_{xy} – will be the average over the thickness of the values of displacements and stresses and are connected by the equations of the plane problem

$$\frac{\partial \bar{u}}{\partial x} = \frac{1}{E}\left(\frac{\partial^2 F}{\partial y^2} - \nu\frac{\partial^2 F}{\partial x^2}\right), \qquad \frac{\partial \bar{v}}{\partial y} = \frac{1}{E}\left(\frac{\partial^2 F}{\partial x^2} - \nu\frac{\partial^2 F}{\partial y^2}\right)$$
$$\frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} = -\frac{2(1+\nu)}{E}\frac{\partial^2 F}{\partial x \partial y} \tag{2.5}$$

The second terms, depending on z, are correction members which take into account the change in the displacements and stresses over the thickness. In order to obtain the distribution of stresses in a thick plate it is necessary first to solve the plane problem, i.e., to find F, $\bar{u}$, and $\bar{v}$.

In the case of an isotropic material, $\nu_2 = \nu$; $\alpha = \nu/2(1 + \nu)$.

3. Bending. If the plate is deformed by a bending load distributed only along the edge, displacements and stresses are expressed through a biharmonic function as follows:

$$u = -z\frac{\partial w_1}{\partial x}, \qquad v = -z\frac{\partial w_1}{\partial y}, \qquad w = w_0 + z^2\frac{\nu_2}{2(1-\nu)}\nabla^2 w_0 \tag{3.1}$$

$$\sigma_x = -z\frac{E}{1-\nu^2}\left(\frac{\partial^2 w_1}{\partial x^2} + \nu\frac{\partial^2 w_1}{\partial y^2}\right), \qquad \tau_{xy} = -z\frac{E}{1+\nu}\frac{\partial^2 w_1}{\partial x \partial y}$$
$$\sigma_y = -z\frac{E}{1-\nu^2}\left(\nu\frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^2 w_1}{\partial y^2}\right), \qquad \tau_{xz} = -\frac{E}{2(1-\nu^2)}\left(\frac{h^2}{4} - z^2\right)\frac{\partial}{\partial x}\nabla^2 w_0 \tag{3.2}$$
$$\sigma_z = 0 \qquad \tau_{yz} = -\frac{E}{2(1-\nu^2)}\left(\frac{h^2}{4} - z^2\right)\frac{\partial}{\partial y}\nabla^2 w_0$$

Here

$$w_1 = w_0 + \frac{E}{2G_1(1-\nu^2)}\left(\frac{h^2}{4} - \frac{\beta z^2}{3}\right)\nabla^2 w_0, \qquad \beta = 1 - \frac{G_1\nu_2}{2G} \tag{3.3}$$

Stresses σ_x and σ_y lead to bending moments M_x and M_y; τ_{xy} leads to the torsion moment H_{xy}; τ_{xz}, τ_{yz} lead to shearing forces. The latter are expressed through w_0 just as in the theory of thin plates, while for the moments we obtain expressions which are constructed in the same way as in the theory of thin plates, although in them the role of w_0 is played by the quantity w_2:

$$M_x = -D\left(\frac{\partial^2 w_2}{\partial x^2} + \nu\frac{\partial^2 w_2}{\partial y^2}\right), \qquad M_y = -D\left(\nu\frac{\partial^2 w_2}{\partial x^2} + \frac{\partial^2 w_2}{\partial y^2}\right)$$
$$H_{xy} = -D(1-\nu)\frac{\partial^2 w_2}{\partial x \partial y}, \qquad D = \frac{Eh^3}{12(1-\nu^2)} \tag{3.4}$$

In these formulas D is the rigidity of an isotropic plate with the elastic constants E, ν:

$$w_2 = w_0 + \frac{3D\beta_1}{2G_1 h}\nabla^2 w_0, \qquad \beta_1 = 0.8 + 0.1\frac{G_1\nu_2}{G} \tag{3.5}$$

The structure of the formulas of stresses, moments, and shearing forces is the same as that for an isotropic thin plate ([4], page 246).

In the case of an isotropic plate,

$$\beta = 0.5(2-\nu), \qquad \beta_1 = 0.1(8+\nu) \tag{3.6}$$

and we obtain known formulas ([1], pages 205-208).

If the load also acts on the faces $z = \mp h/2$, in the simplest cases of its distribution the solution for the layer, which must be added to the homogeneous solution (3.1)-(3.2), can sometimes be found by the elemental method. We shall show, without derivation, the solution for a plate which is bent by load q distributed evenly over the upper boundary $z = -h/2$, and by its own weight (the middle plane is assumed to be horizontal).

In this case w_0 satisfies the inhomogeneous equation

$$D\nabla^2\nabla^2 w_0 = q + \gamma h \quad (\gamma \text{ is the specific weight of the material}) \tag{3.7}$$

The formulas for displacements, stresses, and moments are obtained from (3.1), (3.2), and (3.4) by adding to their right sides, respectively, the expressions

$$
\begin{gathered}
u' = \frac{q\nu_2}{2E}x, \qquad v' = \frac{q\nu_2}{2E}y \\
w' = -\frac{q}{2E_1}z + \frac{qh(1-\nu-2\nu_1\nu_2)}{2E_1(1-\nu)}\frac{z^2}{h^2} + \\
+\frac{(q+\gamma h)h}{2E_1(1-\nu)}\left\{\frac{1}{2}\left(\frac{3E\nu_1}{G_1}+1-\nu-2\nu_1\nu_2\right)\frac{z^2}{h^2}-\left[\frac{E\nu_1}{G_1}+1-\nu-(3+\nu)\nu_1\nu_2\right]\frac{z^4}{h^4}\right\}
\end{gathered}
\tag{3.8}
$$

$$
\begin{gathered}
\sigma_x' = \sigma_y' = \frac{q\nu_2}{2(1-\nu)}\left(\frac{3z}{h}-\frac{4z^3}{h^3}\right)+\frac{\gamma h\nu_2}{2(1-\nu)}\left(\frac{z}{h}-\frac{4z^3}{h^3}\right) \\
\tau_{xy}' = \tau_{yz}' = \tau_{xz}' = 0 \\
M_x' = M_y' = \frac{h^2\nu_2}{10(1-\nu)}\left(q+\frac{\gamma h}{6}\right), \qquad H_{xy}' = 0
\end{gathered}
\tag{3.9}
$$

The normal stress on areas which are parallel to the middle plane is different from zero:

$$
\sigma_z = -\frac{q}{2}\left(1-\frac{3z}{h}+\frac{4z^3}{h^3}\right)+\frac{\gamma h}{2}\left(\frac{z}{h}-\frac{4z^3}{h^3}\right)
\tag{3.10}
$$

4. Bending of a round plate. We shall consider in greater detail the case of a round plate with radius a, supported along the entire edge and bent by uniformly distributed load q; the weight of the plate is ignored.

We will obtain the expressions for displacements u_r, u_θ and stresses σ_r, σ_θ, and $\tau_{r\theta}$ in cylindrical coordinates from the corresponding expressions of the theory of thin plates; we will replace w_0 in the latter by the quantity w_1, which depends on z, and we will add to the right sides the functions

$$
\begin{gathered}
u_r' = \frac{q\nu_2}{2E}r, \qquad u_\theta' = 0 \\
\sigma_r' = \sigma_\theta' = \frac{q\nu_2}{2(1-\nu)}\left(\frac{3z}{h}-\frac{4z^3}{h^3}\right), \qquad \tau_{r\theta}' = 0
\end{gathered}
\tag{4.1}
$$

In order to find the moments M_r, M_θ, and $H_{r\theta}$, it is necessary to take the formulas of the theory of thin plates, to replace w_0 in them by the expression w_2, and to add the moments corresponding to σ_r' and σ_θ', i.e.,

$$M_r' = M_\theta' = \frac{qh^2\nu_2}{10(1-\nu)} . \tag{4.2}$$

It is clear that w_0 is a function only of dispersion r:

$$w_0 = \frac{qr^4}{64D} + A + Br^2 . \tag{4.3}$$

(terms which give a singularity in the center are discarded).

The boundary conditions have the form

$$w_0 = 0, \quad M_r = 0 \quad \text{when } r = a. \tag{4.4}$$

We shall introduce the final formulas for displacement w at any point and for the stresses:

$$\begin{gathered} w = \frac{q}{64D}(a^2 - r^2)\left[\frac{5+\nu}{1+\nu}a^2 - r^2 + \frac{2h^2}{5(1+\nu)}n\right] - \\ - z^2 \frac{q\nu_2}{16(1-\nu)D}\left(\frac{3+\nu}{1+\nu}a^2 - 2r^2 + \frac{h^2}{5}n\right) + w' \end{gathered} \tag{4.5}$$

$$\begin{gathered} \sigma_r = \frac{3qz}{4h^3}\left[(3+\nu)(a^2 - r^2) + m\left(\frac{z^2}{3} - \frac{h^2}{20}\right)\right] \\ \sigma_\theta = \frac{3qz}{4h^3}\left[(3+\nu)a^2 - (3\nu+1)r^2 + m\left(\frac{z^2}{3} - \frac{h^2}{20}\right)\right] \\ \sigma_z = -\frac{q}{2}\left(1 - \frac{3z}{h} + \frac{4z^3}{h^3}\right), \quad \tau_{rz} = -\frac{3qr}{h^3}\left(\frac{h^2}{4} - z^2\right), \quad \tau_{rz} = \tau_{\theta z} = 0 . \end{gathered} \tag{4.6}$$

Here the following designations are used:

$$m = \frac{4}{1-\nu}\left[\frac{E}{G_1} - \nu_2(3+\nu)\right], \qquad n = \frac{4}{1-\nu}\left[\frac{E}{G_1} - 0.25\nu_2(7-\nu)\right]. \tag{4.7}$$

The deflection in the center equals

$$f = \frac{qa^4}{64D}\frac{5+\nu}{1+\nu}\left[1 + \frac{2n}{5(5+\nu)}\left(\frac{h}{a}\right)^2\right]. \tag{4.8}$$

The largest normal stress is obtained on the axis of the plate at the flat surfaces, i.e., at the points r = 0, z = $\mp h/2$:

$$\sigma_{max} = \frac{3qa^2}{8h^2}(3+\nu)\left[1+\frac{m}{30(3+\nu)}\left(\frac{h}{a}\right)^2\right]. \tag{4.9}$$

For an isotropic plate,

$$m = 4(2+\nu), \qquad n = \frac{8+\nu+\nu^2}{1-\nu}. \tag{4.10}$$

In formulas (4.8) and (4.9) the terms which are proportional to $(h/a)^2$ represent corrections to the results of the theory of thin plates. Calculations show that for isotropic plates these corrections are usually small in comparison with the first terms. Using the designations f_0 and σ^0_{max} for deflection and the greatest stress, calculated according to the theory of thin plates, we find the following values for an isotropic plate in which $\nu = 1/3$ and $h/a = 0.2$:

$$f = 1.038f_0, \qquad \sigma_{max} = 1.004\sigma^0_{max}.$$

The deflection obtained on the basis of the theory of thick plates differs from f_0 by less than 4%, while the stress differs from σ^0_{max} by a total of 0.04%. However, similar conclusions about the insignificant value of corrections may turn out to be inaccurate for a transversally isotropic plate in which the shear modulus G_1 is, generally speaking, in no way connected with the other constants. We will present results of calculations for two transversally isotropic plates in which $\nu = \nu_2 = 1/3$ and G_1 is small in comparison with E.

1) $G_1 = 0.1E$, $f = 1.17f_0$, $\sigma_{max} = 1.021\sigma^0_{max}$, i.e., the difference in deflections f and f_0 comprises 17%, while the difference between σ_{max} and σ^0_{max} is 2.1%.

2) $G_1 = 0.01E$, $f = 2.79f_0$, $\sigma_{max} = 1.237\sigma^0_{max}$.

In this case the correction term for deflection is 1.79 times greater than f_0, i.e., the obtained deflection is almost double that calculated by the theory of thin plates, while the correction for stress comprises about 24% of σ^{o}_{max}. From this it is clear that the formulas for deflection of the approximate theory of thin plates can give a significant error for transversally isotropic plates in which the resistance to shear in the planes of the transverse section is small.

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ABSTRACT

(U) The material of the plate is assumed to be transversely isotropic, with the planes of isotropy parallel to the middle surface, and the material therefore possesses five independent elastic constants. Equations are given without detailed derivation for the displacements and stresses corresponding to a state of plane stress, and for the displacements, stresses and moments in a bent plate, both for the general case, and for the particular case of a plate bent by a uniformly distributed pressure in addition to its own weight. For a simply supported circular plate subjected to a uniformly distrubuted pressure the equations for displacements, stresses and moments are given, neglecting the weight of the plate. For the particular cases considered, the errors of the thin plate approximation are negligible if the material is isotropic, but are appreciable if the material is anisotropic, the errors becoming larger the smaller the ratio. There are 4 Soviet references.