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General Relativity Two-Way Doppler Shift

Prepared by GILBERT IALONGO
Space Physics Laboratory

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Laboratory Operations
THE AEROSPACE CORPORATION

Prepared for SPACE AND MISSILE SYSTEMS ORGANIZATION
AIR FORCE SYSTEMS COMMAND
LOS ANGELES AIR FORCE STATION
Los Angeles, California

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FOREWORD

This report is published by The Aerospace Corporation, El Segundo, California, under Air Force Contract No. F04701-69-C-0066.

This report, which documents research carried out from April 1969 through June 1969, was submitted on 19 February 1970 to Colonel G. E. Burrell, SMON, for review and approval.

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Publication of this report does not constitute Air Force approval of the report's findings or conclusions. It is published only for the exchange and stimulation of ideas.

G. E. Burrell
Colonel, USAF
Director for Development
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ABSTRACT

On the basis of the metric in a gravitational field, a two-way Doppler shift is obtained for an arrangement in which a signal sent from a ground-based transmitter is received by a satellite, retransmitted, and picked up by a receiver. The geometry used is quite general, and the formula obtained is correct to order $(v/c)^3$ included. When both transmitter and receiver are on the earth, much of the general relativistic effect is seen to drop out. Various limits of the Doppler shift formula are considered.

CONTENTS

I. INTRODUCTION

This report derives a relativistic Doppler shift formula under the most general condition possible. It is assumed that both transmitter and receiver are moving in an arbitrary direction with rates not necessarily constant. The derivation is done with the use of general relativity and includes the corrections due to the gravitational field on the motion of light waves. The situation envisioned is one in which a transmitter sends a monochromatic light signal to a receiver. The receiver picks up the signal, multiplies it by a fixed ratio, and sends it back to a second receiver. This is a two-way Doppler arrangement. The second receiver may or may not be at the same location as the transmitter; at any rate, the geometry is left to be completely general.

The formula derived here can be used to calculate the range rate of a satellite, in which case the transmitter is a ground-stationed antenna, the receiver the orbiting satellite, and the second receiver a tracking antenna. The high degree of precision of the formula obtained in this report may be necessary for the correct tracking of satellites orbiting at large altitudes, in particular at synchronous altitudes.

The one-way Doppler shift is derived first; the derivation is subsequently extended to the two-way arrangement.

II. DERIVATION OF ONE-WAY DOPPLER SHIFT

A. METRIC IN A GRAVITATIONAL FIELD AND PATH OF A PHOTON

The general form of the space-time relativistic metric is

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu \qquad (1)$$

where the quantity ds is called the proper time.

In the case of a gravitational field at a point $\vec{r}$ outside a sphere of mass M whose center is at r = 0, the metric is given by the Schwartschild space-time. Its form is

$$ds^2 = (1 - \frac{2m}{r})dt^2 - \frac{1}{c^2}\left[\frac{dr^2}{1 - (2m/r)} + r^2 d\theta^2 + r^2 \sin^2\theta\, d^2\phi\right] \qquad (2)$$

where the constant m is defined as

$$m = G\, M_e/c^2. \qquad (3)$$

In Eq. (3), G is the gravitational constant, M_e the mass of the earth, and c the velocity of light. The geodesics of the electromagnetic radiation is found by differentiation with respect to some non-zero parameter μ. (For light $ds^2 = 0$.) One has[1]

$$\frac{d}{d\mu}\left(g_{\lambda\lambda}\frac{dx^\lambda}{d\mu}\right) - \frac{1}{2}\sum_{\nu=1}^{4}\frac{\partial g_{\nu\nu}}{\partial x^\lambda}\left(\frac{dx^\nu}{d\mu}\right)^2 = 0 \qquad \lambda = 1, 2, 3, 4 \qquad (4)$$

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[1] G. McVittie, General Relativity and Cosmology, Univ. of Illinois Press, Urbana (1965), p. 80

The following relation also holds

$$\sum_{\nu=1}^{4}{}' g_{\nu\nu}\left(\frac{dx^{\nu}}{d\mu}\right)^2 = 0 \qquad (5)$$

Only four of the relations defined by Eqs. (4) and (5) are independent. Taking the equations that are obtained from setting $\lambda = 1, 3, 4$ (i.e., $x^{\nu} = t, \theta, \phi$) and using Eq. (5), we obtain for the motion of electromagnetic waves

$$\frac{d}{d\mu}\left[\left(1 - \frac{2m}{r}\right)\frac{dt}{d\mu}\right] = 0 \qquad (6)$$

$$\frac{d}{d\mu}\left[r^2 \sin^2\theta \frac{d\phi}{d\mu}\right] = 0 \qquad (7)$$

$$\frac{d}{d\mu}\left(r^2 \frac{d\theta}{d\mu}\right) - r^2 \sin^2\theta \cos\theta\left(\frac{d\theta}{d\mu}\right)^2 = 0 \qquad (8)$$

$$\left(1 - \frac{2m}{r}\right)\left(\frac{dt}{d\mu}\right)^2 - \frac{1}{c^2}\left[\left(1 - \frac{2m}{r}\right)^{-1}\left(\frac{dr}{d\mu}\right)^2 + r^2\left(\frac{d\theta}{d\mu}\right)^2 + r^2 \sin^2\theta\left(\frac{d\phi}{d\mu}\right)^2\right] = 0 \qquad (9)$$

We are interested in motion in a plane passing through the origin $r = 0$ that traces an orbit about this point, and we obtain

$$\theta = \pi/2 \quad , \quad d\theta/d\mu = 0. \qquad (10)$$

This boundary condition is obviously a solution of Eq. (8). The other equations reduce to

$$\left(1 - \frac{2m}{r}\right)\frac{dt}{d\mu} = \beta \quad , \quad \text{a constant} \qquad (11)$$

$$\frac{d}{d\mu}\left(r^2 \frac{d\phi}{d\mu}\right) = 0 \qquad \text{or} \qquad r^2 \frac{d\phi}{d\mu} = h \quad , \quad \text{a constant} \tag{12}$$

and

$$\left(1 - \frac{2m}{r}\right)\left(\frac{dt}{d\mu}\right)^2 = \frac{1}{c^2}\left[\left(1 - \frac{2m}{r}\right)^{-1}\left(\frac{dr}{d\mu}\right)^2 + r^2\left(\frac{d\phi}{d\mu}\right)^2\right] \tag{13}$$

Eqs. (11) and (12) yield

$$\frac{r^2}{1 - (2m/r)} \frac{d\phi}{dt} \equiv \frac{h}{\beta} = h^*. \tag{14}$$

Setting $\mu = t$ and substituting Eq. (14) into Eq. (13) gives

$$\frac{dr}{dt} = \left(1 - \frac{2m}{r}\right) \frac{c}{r}\left[r^2 - \frac{h^{*2}}{c^2}\left(1 - \frac{2m}{r}\right)\right]^{1/2} \tag{15}$$

This equation represents the radial motion of photons. The quantity h^* is a constant along any path; however, the value of h^* is different for different paths. For radial propagation it is clear that $h^* = 0$, since $d\phi/dt = 0$.

B. ONE-WAY DOPPLER SHIFT CONFIGURATION

A transmitting station T is fixed on the moving earth. The space-time coordinates describing the transmission of an electromagnetic wave are given by the pair of events $(r_t, \pi/2, \phi_t, t_t)$ and $(r_t, \pi/2, \phi_t + \Delta\phi_t, t_t + \Delta t_t)$. The change in the ϕ coordinate $\Delta\phi_t$ is due to the rotation of the earth, and Δt_t is the time elapsed during the emission of the wave. The coordinates of the reception of this wave on the satellite S define two events $(r_s, \pi/2, \phi_s, t_s)$ and $(r_s + \Delta r_s, \pi/2, \phi_s + \Delta\phi_s, t_s + \Delta t_s)$. The position of the satellite changes by an amount

Δr_s and $\Delta\phi_s$ in the time Δt_s, which measures the change of the coordinate time during the arrival of the wave. By comparing the time intervals Δt_t and Δt_s, we can calculate the range rate of the satellite.

C. INVARIANT DEFINITION OF FREQUENCY AND FREQUENCY SHIFT

The frequency f_t of the wave emitted by the ground transmitter can be defined in an invariant way by

$$f_t = \frac{1}{\Delta s_t} \quad , \tag{16}$$

where Δs_t is the invariant proper time interval. The frequency $f_s = f_t + \Delta f_t$ of the wave emitted by the ground and received by the satellite can be similarly defined in an invariant fashion by

$$f_s = f_t + \Delta f_t = \frac{1}{\Delta s_s} \quad . \tag{17}$$

The two proper times Δs_t and Δs_s are defined in Section D. Using Eqs. (16) and (17), we can write

$$\frac{f_s}{f_t} = \frac{f_t + \Delta f_t}{f_t} = 1 + \frac{\Delta f_t}{f_t} = \frac{\Delta s_t}{\Delta s_s} \quad . \tag{18}$$

The frequency shift Δf_t will be shown to be related to the range rate of the satellite.

D. PROPER TIMES

In the geometry we are considering, the proper time ds is defined by

$$ds^2 = (1 - \frac{2m}{r})\, dt^2 - \frac{1}{c^2}\left[\left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2\, d\phi^2\right] \quad . \tag{19}$$

We can write this as

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 - \frac{1}{c^2}\left[\left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\phi^2\right] \frac{[1 - (2m/r)]dt^2}{[1 - (2m/r)]dt^2} \quad (20)$$

or

$$ds^2 = \left(1 - \frac{2m}{r}\right)\left[1 - \frac{1}{c^2} \frac{\{[1 - (2m/r)]^{-1} dr^2 + r^2 d\phi^2\}}{[1 - (2m/r)]dt^2}\right] dt^2 \quad . \quad (21)$$

If we define a velocity relativistically as[2]

$$q = \frac{1}{(1 - 2m/r)^{1/2}} \left\{[1 - (2m/r)]^{-1}\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\phi}{dt}\right)^2\right\} \quad , \quad (22)$$

we obtain

$$ds^2 = \left(1 - \frac{2m}{r}\right)\left(1 - \frac{q^2}{c^2}\right) dt^2 \quad . \quad (23)$$

Using this form for the definition of the proper time, we can write for the definition of the proper times Δs_t and Δs_s

$$\Delta s_t = \left(1 - \frac{2m}{r_t}\right)^{1/2}\left(1 - \frac{q_t^2}{c^2}\right)^{1/2} \Delta t_t \quad (24)$$

[2]We are keeping $d\theta/dt = 0$ in this discussion.

and

$$\Delta s_s = \left(1 - \frac{2m}{r_s}\right)^{1/2} \left(1 - \frac{q_s^2}{c^2}\right)^{1/2} \Delta t_s \quad . \tag{25}$$

E. COMPARISON OF THE COORDINATE TIME INTERVALS Δt_t AND Δt_s

To be able to compute the ratio f_r/f_t of Eq. (18), we need to develop an expression for $\Delta t_t/\Delta t_s$. This task can be simplified by reference to Figure 1, which relates the coordinates describing the emission and reception of the electromagnetic waves.

The start and the end of a wave are signaled by photons emitted at $\vec{r}_t$ and $\vec{r}_t + \Delta\vec{r}_t$, respectively. These photons travel radially to $\vec{r}_s$ and $\vec{r}_s + \Delta\vec{r}_s$, respectively. For radial propagation we have

$$\frac{dr}{dt} = c(1 - 2m/r) \quad . \tag{26}$$

Then for the first photon we have

$$t_s - t_t = \frac{1}{c}\int_{r_t}^{r_s} \frac{dr}{1 - 2m/r} \quad . \tag{27}$$

For the photon signaling the end of the wave we can write

$$(t_s + \Delta t_s) - (t_t + \Delta t_t) = \frac{1}{c}\int_{r_t+\Delta r_t}^{r_s+\Delta r_s} \frac{dr}{1 - 2m/r} \quad . \tag{28}$$

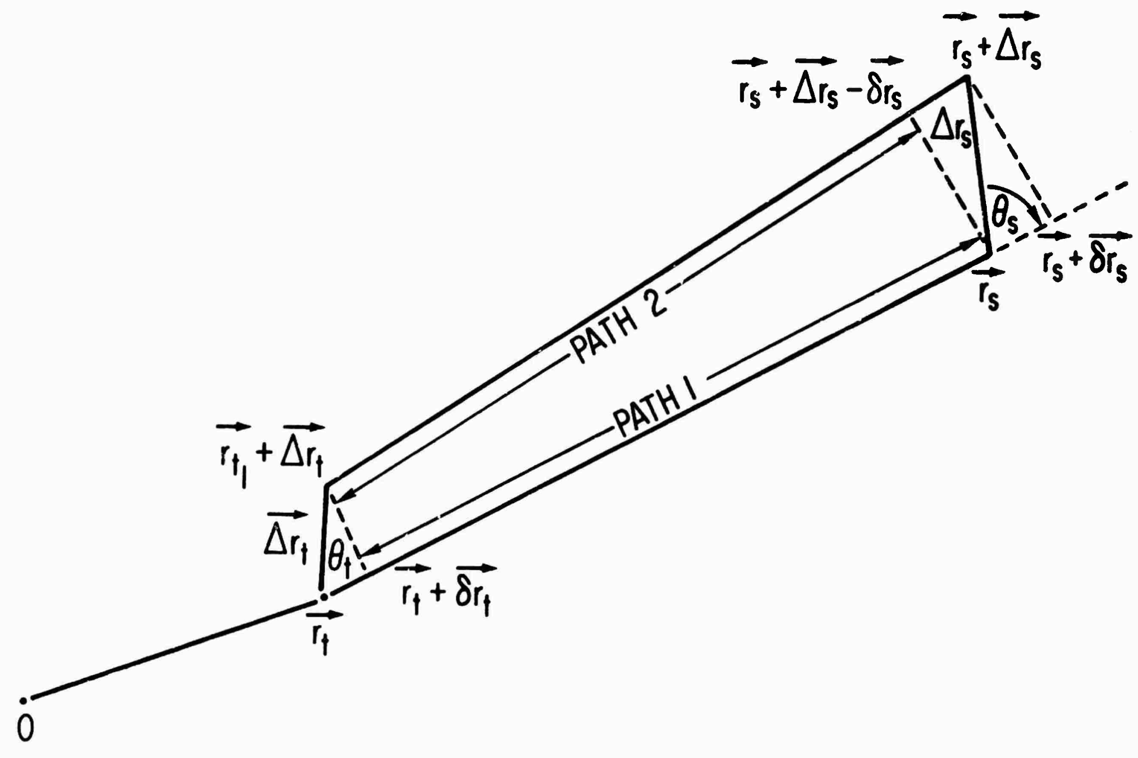

Figure 1. Relations Between the Coordinates Describing the Emission and Reception of the Signals

Subtracting Eq. (27) from Eq. (28), we obtain

$$\Delta t_s - \Delta t_t = \frac{1}{c}\left[\int_{r_t+\Delta r_t}^{r_s+\Delta r_s} \frac{dr}{1 - 2m/r} - \int_{r_t}^{r_s} \frac{dr}{1 - 2m/r}\right] . \quad (29)$$

With reference to Figure 1 we can rewrite Eq. (29) in the form

$$\Delta t_s - \Delta t_t = \frac{1}{c}\left[\int_{\text{path 1}} \frac{dr}{1 - 2m/r} + \int_{|r_s+\Delta r_s - \delta r_s|}^{|\vec{r}_s+\Delta\vec{r}_s|} \frac{dr}{1 - 2m/r} - \int_{|\vec{r}_t|}^{|\vec{r}_t+\delta\vec{r}_t|} \frac{dr}{1 - 2m/r} - \int_{\text{path 2}} \frac{dr}{1 - 2m/r}\right] . \quad (30)$$

In the evaluation of the integrals in Eq. (30), it is useful to bear in mind that Δr_t and Δr_s are the distances travelled by transmitter and satellite during the time of transmission and reception of one cycle of the signal. Using $f_t \simeq 1.8 \times 10^9$ cps we get $\Delta r_t \simeq 2.6 \times 10^{-7}$ m and $\Delta r_s \simeq 1.7 \times 10^{-6}$ m for satellites at synchronous altitudes. This means that paths 1 and 2 are parallel to a very high degree of accuracy. Furthermore, the greatest error we can make in saying that the lengths of paths 1 and 2 are the same can be found by letting $\Delta r_t = 0$ and $\Delta r_s \simeq 2 \times 10^{-6}$ m and assuming that $\Delta\vec{r}_s$ is normal to path 2. With r_1 and r_2 designating the lengths of paths 1 and 2, respectively, we have

$$r_1 = \sqrt{r_2^2 + \Delta r^2} \simeq r_2 + \frac{1}{2}\frac{\Delta r^2}{r_2} \quad \text{if} \quad \Delta r \ll r_2 . \quad (31)$$

With the satellite at synchronous altitude we obtain $r_1 - r_2 \lesssim 10^{-19}$ m. The remaining integrals in Eq. (30) are of the order of Δr_t and Δr_s; thus, setting the integrals over paths 1 and 2 equal amounts to rejecting quantities of order $0(\Delta r^3)$, which is perfectly justified. We now have

$$\Delta t_s - \Delta t_t = \frac{1}{c}\left(\int_{\delta r_s} \frac{dr}{1 - 2m/r} - \int_{\delta r_t} \frac{dr}{1 - 2m/r}\right) \quad . \tag{32}$$

This gives

$$\Delta t_s - \Delta t_t = \frac{1}{c}\left(\frac{\delta r_s}{1 - 2m/r_s} - \frac{\delta r_t}{1 - 2m/r_t}\right) \quad . \tag{33}$$

With reference to Figure 1 we can write

$$\delta r_s = \Delta r_s \cos \theta_s = \Delta \vec{r}_s \cdot \frac{\vec{r}_s - \vec{r}_t}{|\vec{r}_s - \vec{r}_t|} \tag{34}$$

and

$$\delta r_t = \Delta r_t \cos \theta_t = \Delta \vec{r}_t \cdot \frac{\vec{r}_s - \vec{r}_t}{|\vec{r}_s - \vec{r}_t|} \tag{35}$$

Substituting Eqs. (34) and (35) into Eq. (33), we obtain

$$\Delta t_s - \Delta t_t = \frac{1}{c}\left(\vec{A}\cdot\Delta\vec{r}_s - \vec{B}\cdot\Delta\vec{r}_t\right) \quad , \tag{36}$$

where we have let

$$\vec{A} \equiv \frac{1}{1 - 2m/r_s} \frac{\vec{r}_s - \vec{r}_t}{|\vec{r}_s - \vec{r}_t|} \tag{37}$$

and

$$\vec{B} \equiv \frac{1}{1 - 2m/r_t} \frac{\vec{r}_s - \vec{r}_t}{|\vec{r}_s - \vec{r}_t|} \quad . \tag{38}$$

Eq. (36) can also be written as

$$\Delta t_s - \Delta t_t = \frac{1}{c}\left(\vec{A}\cdot\frac{\Delta\vec{r}_s}{\Delta t_s} - \vec{B}\cdot\frac{\Delta\vec{r}_t}{\Delta t_s}\frac{\Delta t_t}{\Delta t_t}\right)\Delta t_s \quad , \tag{39}$$

or

$$\Delta t_s - \Delta t_t = \frac{1}{c}\left(\vec{A}\cdot\vec{v}_s - \vec{B}\cdot\vec{v}_t\frac{\Delta t_t}{\Delta t_s}\right)\Delta t_s \quad . \tag{40}$$

Dividing by Δt_s gives

$$1 - \frac{\Delta t_t}{\Delta t_s} = \frac{1}{c}\left(\vec{A}\cdot\vec{v}_s - \vec{B}\cdot\vec{v}_t\frac{\Delta t_t}{\Delta t_s}\right) \quad . \tag{41}$$

Rearranging terms we obtain

$$\frac{\Delta t_t}{\Delta t_s} - \frac{\vec{B}\cdot v_t}{c}\cdot\frac{\Delta t_t}{\Delta t_s} = 1 - \frac{\vec{A}\cdot v_s}{c} \tag{42}$$

or

$$\frac{\Delta t_t}{\Delta t_s} = \frac{1 - (\vec{A} \cdot \vec{v}_s / c)}{1 - (\vec{B} \cdot v_t / c)} \tag{43}$$

Writing $\vec{A}$ and $\vec{B}$ explicitly, we have

$$\frac{\Delta t_t}{\Delta t_s} = 1 - \frac{\vec{v}_s}{c} \cdot \frac{\vec{r}_s - \vec{r}_t}{|\vec{r}_s - \vec{r}_t|} \left(1 - \frac{2m}{r_s}\right)^{-1} \Bigg/ \; 1 - \frac{\vec{v}_t}{c} \cdot \frac{\vec{r}_s - \vec{r}_t}{|\vec{r}_s - \vec{r}_t|} \left(1 - \frac{2m}{r_t}\right)^{-1} , \tag{44}$$

which is the relation between Δt_t and Δt_s we sought.

F. ONE-WAY DOPPLER SHIFT FORMULA

We can now gather Eqs. (19), (24), (25), and (44) to obtain

$$\frac{f_s}{f_t} = \left(1 - \frac{2m}{r_t}\right)^{1/2} \left(1 - \frac{q_t^2}{c^2}\right)^{1/2} \Bigg/ \left(1 - \frac{2m}{r_s}\right)^{1/2} \left(1 - \frac{q_s^2}{c^2}\right)^{1/2}$$

$$\times \; 1 - \frac{\vec{v}_s}{c} \cdot \frac{\vec{r}_s - \vec{r}_t}{|\vec{r}_s - \vec{r}_t|} \left(1 - \frac{2m}{r_s}\right)^{-1} \Bigg/ \; 1 - \frac{\vec{v}_t}{c} \cdot \frac{\vec{r}_s - \vec{r}_t}{|\vec{r}_s - \vec{r}_t|} \left(1 - \frac{2m}{r_t}\right)^{-1} \tag{45}$$

This expression is relativistically correct to the order of the terms neglected in the calculation of Eq. (44). The terms neglected are of order $\frac{v_s}{c} (\Delta r)^2 \simeq (\frac{v}{c})^4$. To the same accuracy we can replace q_t with v_t and q_s with v_s. Thus we have

$$\frac{f_s}{f_t} = \left(1 - \frac{2m}{r_t}\right)^{1/2}\left(1 - \frac{v_t^2}{c^2}\right)^{1/2} \Bigg/ \left(1 - \frac{2m}{r_s}\right)^{1/2}\left(1 - \frac{v_s^2}{c^2}\right)^{1/2}$$

$$\times\ 1 - \frac{\vec{v}_s}{c}\cdot\frac{\vec{r}_s - \vec{r}_t}{|\vec{r}_s - \vec{r}_t|}\left(1 + \frac{2m}{r_s}\right) \Bigg/ 1 - \frac{\vec{v}_t}{c}\cdot\frac{\vec{r}_s - \vec{r}_t}{|\vec{r}_s - \vec{r}_t|}\left(1 + \frac{2m}{r_t}\right) , \tag{46}$$

which is valid relativistically to $0\left[(v/c)^3\right]$.

III. TWO-WAY DOPPLER SHIFT FORMULA

To obtain the two-way Doppler formula, we need a relation between the frequency transmitted by the satellite f_{ts} and the frequency received by the ground-based receiver f_r. If the satellite transmitter sends out a signal with a frequency k times the received frequency f_s, we have $f_{ts} = kf_s$, so that

$$\frac{f_r}{f_t} = \frac{f_r}{f_{ts}} \frac{f_{ts}}{f_s} \frac{f_s}{f_t} = k \frac{f_r}{f_{ts}} \frac{f_s}{f_t} \quad . \tag{47}$$

No new derivation is needed for the calculation of f_r/f_{ts}. We need only observe that the analysis carried out before still applies if we replace the subscript t with s and the subscript s with r in the righthand side of Eq. (46). We now have

$$\frac{f_r}{f_{ts}} = \left(1 - \frac{2m}{r_s}\right)^{1/2} \left(1 - \frac{v_s^2}{c^2}\right)^{1/2} \Bigg/ \left(1 - \frac{2m}{r_r}\right)^{1/2} \left(1 - \frac{v_r^2}{c^2}\right)^{1/2}$$
$$\times \; 1 - \frac{\vec{v}_r}{c} \cdot \frac{\vec{r}_r - \vec{r}_s}{|\vec{r}_r - \vec{r}_s|} \left(1 + \frac{2m}{r_r}\right) \Bigg/ 1 - \frac{\vec{v}_s}{c} \cdot \frac{\vec{r}_r - \vec{r}_s}{|\vec{r}_r - \vec{r}_s|} \left(1 + \frac{2m}{r_s}\right) \tag{48}$$

The final result is now obtained by substitution of Eqs. (46) and (48) into Eq. (47):

$$\frac{f_r}{f_t} = k \left(1 - \frac{2m}{r_t}\right)^{1/2} \left(1 - \frac{v_t^2}{c^2}\right)^{1/2} \Bigg/ \left(1 - \frac{2m}{r_r}\right)^{1/2} \left(1 - \frac{v_r^2}{c^2}\right)^{1/2}$$

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$$\times\ 1 - \frac{\vec{v}_s}{c} \cdot \frac{\vec{r}_s - \vec{r}_t}{|\vec{r}_s - \vec{r}_t|}\left(1 + \frac{2m}{r_s}\right) \Big/ \ 1 - \frac{\vec{v}_t}{c} \cdot \frac{\vec{r}_s - \vec{r}_t}{|\vec{r}_s - \vec{r}_t|}\left(1 + \frac{2m}{r_t}\right)$$

$$\times\ 1 - \frac{\vec{v}_r}{c} \cdot \frac{\vec{r}_r - \vec{r}_s}{|\vec{r}_r - \vec{r}_s|}\left(1 + \frac{2m}{r_r}\right) \Big/ \ 1 - \frac{\vec{v}_s}{c} \cdot \frac{\vec{r}_r - \vec{r}_s}{|\vec{r}_r - \vec{r}_s|}\left(1 + \frac{2m}{r_s}\right) . \qquad (49)$$

This formula is relativistically correct to $0\left[(v/c^3\right]$.

We will assume that the transmitter and receiving stations are ground based and located at the same geographical coordinates. Then we can set

$$r_r = r_t \text{ and } v_r = v_t \qquad (50)$$

but

$$\vec{r}_r \neq \vec{r}_t \text{ and } \vec{v}_r \neq \vec{v}_t \quad . \qquad (51)$$

The vectors defining the position and velocity of the receiver and transmitter are not the same, because during the time interval necessary for a photon to travel from the earth to the satellite and back, the earth has rotated and the direction of the vectors $\vec{r}$ and $\vec{v}$ has changed. The extent to which Eq. (49) is influenced by this effect will be calculated later.

Using Eq. (50) we can simplify Eq. (49) to yield

$$\frac{f_r}{f_t} = k\left[1 - \frac{\vec{v}_s}{c} \cdot \hat{R}_t \left(1 + \frac{2m}{r_s}\right) \Big/ \ 1 - \frac{\vec{v}_t}{c} \cdot \hat{R}_t \left(1 + \frac{2m}{r_t}\right)\right]$$

$$\times\ 1 - \frac{\vec{v}_r}{c} \cdot \hat{R}_s \left(1 + \frac{2m}{r_t}\right) \Big/ \ 1 - \frac{\vec{v}_s}{c} \cdot \hat{R}_s \left(1 + \frac{2m}{r_s}\right) \qquad (52)$$

where we have defined

$$\hat{R}_t \equiv \frac{\vec{r}_s - \vec{r}_t}{|\vec{r}_s - \vec{r}_t|} \quad \text{and} \quad \hat{R}_s \equiv \frac{\vec{r}_r - \vec{r}_s}{|\vec{r}_r - \vec{r}_s|} \quad . \tag{53}$$

It should be noted that in the two-way Doppler effect, in which transmitter and receiver are both on the earth, the general relativity effects of the order of 2m/r have dropped out and they enter only as (2m/r)(v/c), which is one order of magnitude smaller than 2m/r.

It is useful to know the order of magnitude of the quantities in the right-hand side of Eq. (52). With the assumption that the satellite is at geosynchronous altitude, we have

$$\frac{v_s}{c} \simeq 10^{-5} \quad , \quad \frac{v_t}{c} = \frac{v_r}{c} \simeq 1.5 \times 10^{-6}$$

$$\frac{2m}{r_s} \simeq 2 \times 10^{-10} \quad , \quad \frac{2m}{r_t} \simeq 1.4 \times 10^{-9} \tag{54}$$

Therefore, if we are to expand Eq. (52) to terms of order $(v/c)^3$, we must retain the terms of the form (2m/r)(v/c).

We noted previously that Eq. (52) is valid only to terms of order of $(v/c)^3$ included. Treating the scalar products in Eq. (52) as small in comparison to 1, expanding in powers of v/c, and retaining only quantities of the order of $(v/c)^3$ gives the following expression.

$$\frac{f_r}{f_t} = k\Big\{1 + \frac{1}{c}\left(\vec{v}_s \cdot \hat{R}_s + \vec{v}_t \cdot \hat{R}_t - \vec{v}_r \cdot \hat{R}_s - \vec{v}_s \cdot \hat{R}_t\right) +$$

$$+ \frac{1}{c^2}\Big[\left(\vec{v}_s \cdot \hat{R}_s\right)^2 + \left(\vec{v}_t \cdot \hat{R}_t\right)^2 + \left(\vec{v}_t \cdot \hat{R}_t\right)\left(\vec{v}_s \cdot \hat{R}_s\right) - \left(\vec{v}_r \cdot \hat{R}_s\right)\left(\vec{v}_s \cdot \hat{R}_s\right) - \left(\vec{v}_t \cdot \hat{R}_t\right)\left(\vec{v}_r \cdot \hat{R}_s\right)$$

$$-\left(\vec{v}_s \cdot \hat{R}_t\right)\left(\vec{v}_s \cdot \hat{R}_s\right) - \left(\vec{v}_s \cdot \hat{R}_t\right)\left(\vec{v}_t \cdot \hat{R}_t\right) + \left(\vec{v}_s \cdot \hat{R}_t\right)\left(\vec{v}_r \cdot \hat{R}_s\right)\Big] + \qquad \text{(Con't)}$$

$$+\frac{1}{c^3}\Big[\left(\vec{v}_s\cdot\hat{R}_s\right)^3+\left(\vec{v}_t\cdot\hat{R}_t\right)^3+$$

$$+\left(\vec{v}_t\cdot\hat{R}_t\right)^2\left(\vec{v}_s\cdot\hat{R}_s\right)-\left(\vec{v}_t\cdot\hat{R}_t\right)^2\left(\vec{v}_r\cdot\hat{R}_s\right)-\left(\vec{v}_t\cdot\hat{R}_t\right)^2\left(\vec{v}_s\cdot\hat{R}_t\right)+$$

$$+\left(\vec{v}_s\cdot\hat{R}_s\right)^2\left(\vec{v}_t\cdot\hat{R}_t\right)-\left(\vec{v}_s\cdot\hat{R}_s\right)^2\left(\vec{v}_s\cdot\hat{R}_t\right)-\left(\vec{v}_s\cdot\hat{R}_s\right)^2\left(\vec{v}_r\cdot\hat{R}_s\right)+$$

$$+\left(\vec{v}_s\cdot\hat{R}_t\right)\left(\vec{v}_t\cdot\hat{R}_t\right)\left(\vec{v}_r\cdot\hat{R}_s\right)-\left(\vec{v}_s\cdot\hat{R}_t\right)\left(\vec{v}_t\cdot\hat{R}_t\right)\left(\vec{v}_s\cdot\hat{R}_s\right)+\left(\vec{v}_s\cdot\hat{R}_s\right)\left(\vec{v}_r\cdot\hat{R}_s\right)\left(\vec{v}_s\cdot\hat{R}_t\right)$$

$$-\left(\vec{v}_s\cdot\hat{R}_s\right)\left(\vec{v}_r\cdot\hat{R}_s\right)\left(\vec{v}_t\cdot\hat{R}_t\right)$$

$$+\frac{2m}{r_s}\left[\left(\vec{v}_s\cdot\hat{R}_s\right)-\left(\vec{v}_s\cdot\hat{R}_t\right)\right]c^2+\frac{2m}{r_t}\left[\left(\vec{v}_t\cdot\hat{R}_t\right)-\left(\vec{v}_r\cdot\hat{R}_s\right)\right]c^2\Big]\Big\} \quad . \qquad (55)$$

The limit of this equation when we set $\vec{r}_r = \vec{r}_t$ and $\vec{v}_r = \vec{v}_t$ is examined in the the Appendix.

We can obtain a better understanding of the contribution of each term on the righthand side of Eq. (55) by looking at the relationships between the quantities $\vec{r}_t$, $\vec{r}_r$, $\vec{v}_t$, and $\vec{v}_r$. We begin by noting that for a point on the earth, moving with angular velocity $\vec{\omega}$,

$$\vec{v} = \vec{\omega} \times \vec{r} \qquad (56)$$

and

$$\vec{r}(t+\Delta t) \simeq \vec{r}(t) + \vec{v}\Delta t = \vec{r}(t) + \vec{\omega} \times \vec{r}(t)\Delta t \quad . \qquad (57)$$

Therefore we can write

$$\vec{v}_t = \vec{\omega} \times \vec{r}_t \qquad (58)$$

$$\vec{v}_r = \vec{\omega} \times \vec{r}_r \qquad (59)$$

$$\vec{r}_r = \vec{r}_t + \vec{\omega} \times \vec{r}_t \Delta t = \vec{r}_t + \omega r_t \Delta t \, \hat{v}_t \quad . \tag{60}$$

It is now possible to obtain a relationship between the unit vectors $\hat{R}_s$ and $\hat{R}_r$. By definition of $\hat{R}_s$ and using Eq. (60), we have

$$\hat{R}_s = \frac{\vec{r}_r - \vec{r}_s}{|\vec{r}_r - \vec{r}_s|} = \frac{\vec{r}_t + \omega r_t \Delta t \hat{v}_t - \vec{r}_s}{|\vec{r}_t + \omega r_t \Delta t \hat{v}_t - \vec{r}_s|} \quad . \tag{61}$$

The relative importance of the second term on the righthand side of Eq. (61) is $|\omega r_t \Delta t \, \hat{v}_t| / |\vec{r}_t| = \omega \Delta t \simeq 3 \times 10^{-5}$, of the order of v/c. The value of Δt used to estimate $\omega \Delta t$ is the time necessary for a light wave to go from the earth to the satellite at synchronous altitude and back. ω is the angular velocity of the earth. Since $\omega r_t \, \Delta t$ is small compared to r_t and r_s, a fruitful way of writing $\hat{R}_s$ is

$$\hat{R}_s = \frac{\vec{r}_r - \vec{r}_s}{|\vec{r}_r - \vec{r}_s|} = \frac{\vec{r}_t - \vec{r}_s}{|\vec{r}_t - \vec{r}_s + \omega r_t \Delta t \hat{v}_t|} + \frac{\omega r_t \Delta t \hat{v}_t}{|\vec{r}_t - \vec{r}_s + \omega r_t \Delta t \hat{v}_t|} \quad . \tag{62}$$

The absolute value in the denominator can be expanded as follows:

$$\begin{aligned} |\vec{r}_t - \vec{r}_s + \omega r_t \Delta t \hat{v}_t| &= \sqrt{(\vec{r}_t - \vec{r}_s + \omega r_t \Delta t \hat{v}_t)^2} \\ &= \left[(\vec{r}_t - \vec{r}_s)^2 + 2(\vec{r}_t - \vec{r}_s)\cdot \hat{v}_t \, \omega r_t \Delta t + \omega^2 r_t^2 \Delta t^2 \right]^{1/2} \\ &\simeq |r_t - r_s| \left(1 + 2 \frac{\vec{r}_t - \vec{r}_s}{|\vec{r}_t - \vec{r}_s|^2} \cdot \hat{v}_t \, \omega r_t \Delta t \right)^{1/2} + O\left[(\omega r_t \Delta t)^2 \right] \end{aligned} \tag{63}$$

or

$$|\vec{r}_t - \vec{r}_s + \omega r_t \Delta t \hat{v}_t| \simeq |\vec{r}_t + \vec{r}_s| + \frac{\vec{r}_t - \vec{r}_s}{|\vec{r}_t - \vec{r}_s|} \cdot \hat{v}_t \, \omega r_t \Delta t \tag{64}$$

Substituting Eq. (12) into Eq. (11), we obtain

$$\hat{R}_s = \frac{\vec{r}_t - \vec{r}_s}{|\vec{r}_t - \vec{r}_s|} \left[1 - \frac{\vec{r}_t - \vec{r}_s}{|\vec{r}_t - \vec{r}_s|^2} \cdot \hat{v}_t \, \omega r_t \Delta t \right] + \frac{\omega r_t \Delta t}{|\vec{r}_t - \vec{r}_s|} \hat{v}_t \quad . \tag{65}$$

Using the definition of $\hat{R}_t$, we can write Eq. (65) as

$$\hat{R}_s = - \hat{R}_t - \frac{\omega r_t \Delta t}{R_t} (\hat{R}_t \cdot \hat{v}_t) \hat{R}_t + \frac{\omega r_t \Delta t}{R_t} \hat{v}_t \tag{66}$$

$\hat{v}_t$ can be written in terms of its components parallel and perpendicular to $\hat{R}_t$, as follows:

$$\hat{v}_t = \cos\theta \, \hat{R}_t + \sin\theta \, \hat{R}_{\perp, t} \quad . \tag{67}$$

$\hat{R}_{\perp, t}$ is in the direction perpendicular to R_t. Using Eq. (67) to replace $\hat{v}_t$ in Eq. (66), we obtain

$$\hat{R}_s = -\hat{R}_t + \frac{\omega r_t \Delta t}{R_t} \sin\theta \, \hat{R}_{\perp, t} \quad . \tag{68}$$

Since $\omega r_t \Delta t / R_t \simeq \omega \Delta t/6 \simeq 5.10^{-6}$ is of the order of v/c, it follows that $\hat{R}_s$ equals $-\hat{R}_t$ to within corrections of the order of v/c. Referring back to Eq. (55), the result just found says that if we let $\vec{r}_t = \vec{r}_r$ and $\vec{v}_t = \vec{v}_r$ in any term of order $(v/c)^n$ in Eq. (55), the error made is of the order of $(v/c)^{n+1}$. Therefore, while it is necessary to retain the differentiation of $\vec{r}_t$ and $\vec{r}_r$ and of $\vec{v}_t$ and $\vec{v}_r$ in the terms of order (v/c) and $(v/c)^2$, we can let $\vec{r}_t = \vec{r}_r$ and $\vec{v}_t = \vec{v}_r$ in the terms of order $(v/c)^3$. The error made in so doing is of order $(v/c)^4$ and can be safely neglected.

IV. CONCLUSIONS

The Doppler formula obtained in this report is a general relativistic two-way Doppler shift for a signal emitted by a ground transmitter and received by a ground receiver after being reflected by an orbiting satellite. The analysis is carried out under most general conditions, but the approximation made in the computation of the integrals needed to evaluate the ratio $\Delta t_t/\Delta t_s$ makes the result valid to quantities of order $(v/c)^3$ when the satellite is at synchronous altitude. The formula obtained is sufficiently accurate to evaluate satellite range rate to $(v/c)^3 c \simeq 10^{-5}$ cm/sec.

APPENDIX

In this Appendix we want to analyze the limiting value that Eq. (55) assumes when we set $\vec{r}_r = \vec{r}_t$ and $\vec{v}_r = \vec{v}_t$. From the definitions of $\hat{R}_t$ and $\hat{R}_s$, it is clear that setting $\vec{r}_r = \vec{r}_t$ implies $\hat{R}_t = -\hat{R}_s \equiv \hat{R}$. $\hat{R}$ is a unit vector along the direction joining the earth-based stations and the satellite.

A little algebra shows that the terms in c^{-1} become $-2(\vec{v}_s - \vec{v}_t)\cdot\hat{R}$, the terms in c^{-2} become $2[(v_s - v_t)\cdot\hat{R}]^2$, and the terms in c^{-3} reduce to $-2[(v_s - v_t)\cdot\hat{R}]^3 + 2(v_t\cdot\hat{R})^2 (v_s\cdot\hat{R}) -2(v_s\cdot\hat{R}) - (v_s\cdot\hat{R})(4m/r_s)c^2 + (v_t\cdot\hat{R})(4m/r_t)c^2$. Using these results in Eq. (55) gives

$$\frac{f_r}{f_t} = k\left\{1 - \frac{2(\vec{v}_f - \vec{v}_t)\cdot\hat{R}}{c} + \frac{2[(\vec{v}_s - \vec{v}_t)\cdot\hat{R}]^2}{c^2} - \frac{2[(\vec{v}_s - \vec{v}_t)\cdot\hat{R}]^3}{c^3} + \right.$$

$$\left. + \frac{1}{c^3}\left[2(\vec{v}_t\cdot\hat{R})^2(\vec{v}_s\cdot\hat{R}) -2(\vec{v}_s\cdot\hat{R})^2(\vec{v}_t\cdot\hat{R}) -(\vec{v}_s\cdot\hat{R})\frac{4m}{r_s}c^2 + (\vec{v}_t\cdot\hat{R})\frac{4m}{r_t}c^2\right]\right\} \tag{A-1}$$

Defining $\vec{v}_s\cdot\hat{R} = u_s$ and $\vec{v}_t\cdot\hat{R} = u_t$, we can rewrite Eq. (A-1) in the following form

$$\frac{f_r}{f_t} = k\left[1 - \frac{2(u_s - u_t)}{c} + \frac{2(u_s - u_t)^2}{c^2} - \frac{2(u_s - u_t)^3}{c^3} + \frac{2u_t^2 u_s}{c^3} - \frac{2u_s^2 u_t}{c^3} - \right.$$

$$\left. - \frac{4m}{r_s}\frac{u_s}{c} + \frac{4m}{r_t}\frac{u_t}{c}\right] \quad . \tag{A-2}$$

We now define the range rate $\dot{r}$ to be the rate of the separation between the transmitter frame and satellite frame, as seen by an observer on the transmitter frame. Because of the law of addition of velocity in relativity, it should be clear that $\dot{r} \neq u_s - u_t$; the correct relation is, in fact,

$$\dot{r} = \frac{u_s - u_t}{1 - (u_s u_t/c^2)} \quad . \tag{A-3}$$

We can rewrite this expression more conveniently for our purpose as

$$u_s - u_t = \left(1 - \frac{u_s u_t}{c^2}\right)\dot{r} \quad , \tag{A-4}$$

or

$$u_s - u_t = \dot{r} - \frac{u_s u_t}{c^2}\dot{r} \quad . \tag{A-5}$$

To orders of $(u/c)^3$ we have

$$u_s - u_t = \dot{r} - \frac{u_s u_t}{c^2}(u_s - u_t) + O\left[(u/c)^4\right] \tag{A-6}$$

or

$$u_s - u_t = \dot{r} - \frac{u_s^2 u_t}{c^2} + \frac{u_t^2 u_s}{c^2} + O\left[(u/c)^4\right] \tag{A-7}$$

Substituting Eq. (A-7) into Eq. (A-2) and keeping only terms of the order of $(u/c)^3$, we obtain

$$\frac{f_r}{f_t} \simeq k\left[1 - 2\frac{\dot{r}}{c} + \frac{2u_s^2 u_t}{c^2} - \frac{2u_t^2 u_s}{c^2} + 2\left(\frac{\dot{r}}{c}\right)^2 - 2\left(\frac{\dot{r}}{c}\right)^3 + \right.$$
$$\left. + \frac{2u_t^2 u_s}{c^2} - \frac{2u_s^2 u_t}{c^2} - \frac{4m}{r_s}\frac{u_s}{c} + \frac{4m}{r_t}\frac{u_t}{c}\right]$$

or

$$\frac{f_r}{f_t} \simeq k\left[1 - 2\frac{\dot{r}}{c} + 2\left(\frac{\dot{r}}{c}\right)^2 - 2\left(\frac{\dot{r}}{c}\right)^3 - \frac{4m}{r_s}\frac{u_s}{c} + \frac{4m}{r_t}\frac{u_t}{c}\right] \quad . \tag{A-9}$$

The standard formula derived for a two-way doppler shift, which is valid for near-earth geometries and does not include general relativity effects, is

$$\left(\frac{f_r}{f_t}\right)' = k\frac{1 - (\dot{r}/c)}{1 + (\dot{r}/c)} \quad , \tag{A-10}$$

where the notation ( )' is used to differentiate from Eq. (A-9).

By expanding Eq. (A-10) to orders of $(\dot{r}/c)^3$, we obtain

$$\left(\frac{f_r}{f_t}\right)' = u\left(1 - \frac{\dot{r}}{c}\right)\left[1 - \frac{\dot{r}}{c} + \left(\frac{\dot{r}}{c}\right)^2 - \left(\frac{\dot{r}}{c}\right)^3 + \cdots\right] = 1 - 2\frac{\dot{r}}{c} + 2\left(\frac{\dot{r}}{c}\right)^2 - 2\left(\frac{\dot{r}}{c}\right)^3 \cdots \tag{A-11}$$

Comparison of Eqs. (A-9) and (A-11) shows that they agree to order $(\dot{r}/c)^3$, the only difference being the terms $\frac{4m}{r_t}\frac{u_t}{c}$ and $-\frac{4m}{r_s}\frac{u_s}{c}$, which depend exclusively on general relativity effects.

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On the basis of the metric in a gravitational field, a two-way Doppler shift is obtained for an arrangement in which a signal sent from a ground-based transmitter is received by a satellite, retransmitted, and picked up by a receiver. The geometry used is quite general, and the formula obtained is correct to order $(v/c)^3$ included. When both transmitter and receiver are on the earth, much of the general relativistic effect is seen to drop out. Various limits of the Doppler shift formula are considered.

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II. DERIVATION OF ONE-WAY DOPPLER SHIFT

A. METRIC IN A GRAVITATIONAL FIELD AND PATH OF A PHOTON

The general form of the space-time relativistic metric is

$$ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu \qquad (1)$$

where the quantity ds is called the proper time.

In the case of a gravitational field at a point $\vec{r}$ outside a sphere of mass M whose center is at r = 0, the metric is given by the Schwartschild space-time. Its form is

$$ds^2 = (1 - \frac{2m}{r})dt^2 - \frac{1}{c^2}\left[\frac{dr^2}{1 - (2m/r)} + r^2 d\theta^2 + r^2 \sin^2\theta \, d^2\phi\right] \qquad (2)$$

where the constant m is defined as

$$m = G \, M_e/c^2. \qquad (3)$$

In Eq. (3), G is the gravitational constant, M_e the mass of the earth, and c the velocity of light. The geodesics of the electromagnetic radiation is found by differentiation with respect to some non-zero parameter μ. (For light $ds^2 = 0$.) Following McVittie and Schusterman[2] one obtains the following expression.

$$\frac{d}{d\mu}\left(g_{\lambda\lambda} \frac{dx^\lambda}{d\mu}\right) - \frac{1}{2} \sum_{\nu=1}^{4} \frac{\partial g_{\nu\nu}}{\partial x^\lambda} \left(\frac{dx^\nu}{d\mu}\right)^2 = 0 \qquad \lambda = 1, 2, 3, 4 \qquad (4)$$

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[1] G. McVittie, General Relativity and Cosmology, Univ. of Illinois Press, Urbana (1965), p. 80

[2] L. Schusterman, Relativistic Red Shift and Its Application to the Range Rate Measurements of Satellites, LMSC/584748, Lockheed Missiles and Space Co., Sunnyvale, Calif. (11 May 1967).

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The following relation also holds

$$\sum_{\nu=1}^{4} g_{\nu\nu}\left(\frac{dx^{\nu}}{d\mu}\right)^{2} = 0 \qquad (5)$$

Only four of the relations defined by Eqs. (4) and (5) are independent. Taking the equations that are obtained from setting $\lambda = 1, 3, 4$ (i. e., $x^{\nu} = t, \theta, \phi$) and using Eq. (5), we obtain for the motion of electromagnetic waves

$$\frac{d}{d\mu}\left[\left(1 - \frac{2m}{r}\right)\frac{dt}{d\mu}\right] = 0 \qquad (6)$$

$$\frac{d}{d\mu}\left[r^{2} \sin^{2}\theta \frac{d\phi}{d\mu}\right] = 0 \qquad (7)$$

$$\frac{d}{d\mu}\left(r^{2} \frac{d\theta}{d\mu}\right) - r^{2} \sin^{2}\theta \cos\theta\left(\frac{d\theta}{d\mu}\right)^{2} = 0 \qquad (8)$$

$$\left(1 - \frac{2m}{r}\right)\left(\frac{dt}{d\mu}\right)^{2} - \frac{1}{c^{2}}\left[\left(1 - \frac{2m}{r}\right)^{-1}\left(\frac{dr}{d\mu}\right)^{2} + r^{2}\left(\frac{d\theta}{d\mu}\right)^{2} + r^{2}\sin^{2}\theta\left(\frac{d\phi}{d\mu}\right)^{2}\right] = 0 \qquad (9)$$

We are interested in motion in a plane passing through the origin $r = 0$ that traces an orbit about this point, and we obtain

$$\theta = \pi/2 \qquad , \qquad d\theta/d\mu = 0. \qquad (10)$$

This boundary condition is obviously a solution of Eq. (8). The other equations reduce to

$$\left(1 - \frac{2m}{r}\right)\frac{dt}{d\mu} = \beta \qquad , \qquad \text{a constant} \qquad (11)$$