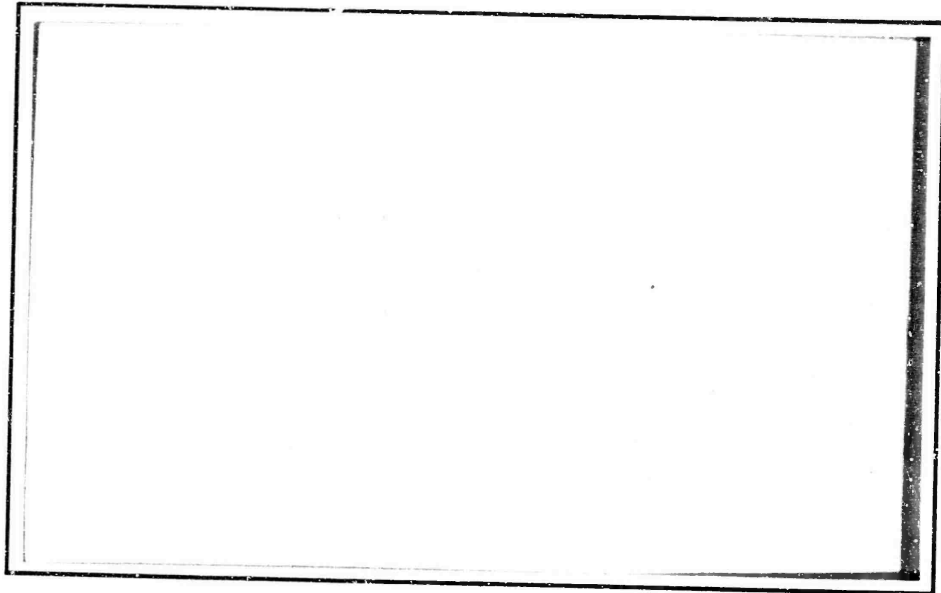


AD704620

This research was supported by the
Propulsion Division, AFOSR,
SREP
under Contract/Grant *AF-AFOSR-844-67*



Reproduced by the
CLEAR HOUSE
for Federal Scientific & Technical
Information Springfield Va. 22151

Department of Physics
University of Miami, Coral Gables, Florida
33124

This document is approved
for publication and sale; its
distribution is unlimited

RECEIVED
APR 22 1970
A

ORO - 3895 - 11 MIAPH - TP - 70.3

Functional Dependence of Lagrange Multipliers*

Lawrence Carl Hawkins

Department of Physics
University of Miami
Coral Gables, Florida 33124

AFOSR 70-1016 TR

* Supported by the United States Atomic Energy

Commission Contract No AT - (40-1) - 3895

ABSTRACT

Given three variational problems consisting of minimizing a given function(a) subject to one constraint function(a), the three equivalent Lagrange-multiplier problems are deduced and the proper functional dependence of the particular Lagrange multipliers is derived. The results prove the consistency of the use and subsequent physical interpretation of the constant Lagrange multipliers in the plasma, global-stability, variational calculations with integral constraints of Chandrasekhar, of Woltzer, and of Wells and Norwood.

I. INTRODUCTION

One of the more subtle questions in the use of Lagrange-multiplier techniques¹ is the physical interpretation² and explicit determination³ of particular multipliers without ad hoc assumptions⁴. Even more basic than these queries is the question of functional dependence; viz., when must Lagrange multipliers in a given variational problem be constant and when can they be spatially dependent? A perusal of the standard literature¹ shows cases of both constant and spatially-dependent⁴ Lagrange multipliers, but seemingly no sharp, necessary and sufficient conditions of distinction between these two cases.

In the current studies of Wells and Norwood⁵ on global stability of closed plasma configurations, the functional dependence of certain of the multipliers is crucial.

When one treads on the essentially unfamiliar ground of constraint conditions on minimum-entropy production rates, such as typified by the recent work of Robertson⁶, any guiding conditions on the Lagrange multipliers would seem to help considerably.

In section II three given variational problems consisting of minimizing a function(a) subject to one constraint function(a) are reset in the Lagrange-multiplier formalism.

In section III the functional dependence of the Lagrange-multiplier of problem 1 is first analyzed by the constraint-elimination method and then shown to be consistent with a similar analysis in the Lagrange-multiplier formalism.

Section IV and V contain similar analyses for problems 2 and 3.

II. THREE VARIATIONAL PROBLEMS

For simplicity⁷, we restrict the discussion below to the scalar function(al)s f ; F , J

$$f = f(\underline{x}) \quad , \quad (1)$$

$$F[x^1] = \int_{t_0}^{t_1} \tilde{F}(t, x^1, \dot{x}^1) dt \quad , \quad (2a)$$

$$J[\underline{x}] = \int_{t_0}^{t_1} \tilde{J}(t, \underline{x}, \dot{\underline{x}}) dt \quad , \quad (2b)$$

and one constraint function(al) g ; G

$$g(\underline{x}) \equiv k_1 \quad , \quad (3)$$

$$G[x^1] \equiv \int_{t_0}^{t_1} \tilde{G}(t, x^1, \dot{x}^1) dt = k_2 \quad , \quad (4)$$

where $x \equiv x^i \hat{e}_i$; ($i=1,2$), $\dot{x} \equiv d(\underline{x})/dt$, $\nabla \equiv \hat{e}^i \partial_i$.

The following three variational problems are formulated:

$$1'.) \text{ STN. } f(\underline{x}) \ni g(\underline{x}) = k_1 \quad , \quad (4)$$

$$2'.) \text{ STN. } J[\underline{x}] \ni g(\underline{x}) = k_1 \quad , \quad (5)$$

$$3'.) \text{ STN. } F[x^1] \ni G[x^1] = k_2 \quad , \quad (6)$$

using the notation (STN.) for stationarity of the given function(al).

It is easy to show that problems 1'-3' can be replaced in the Lagrange-multiplier formalism by

$$1.) \text{ STN. } \{H_1(\underline{x}, \lambda_1) \equiv f + \lambda_1(g - k_1)\} \quad , \quad (7)$$

$$2.) \text{ STN. } \int_{t_0}^{t_1} dt \{ \tilde{J} + \lambda_2(\underline{x}) dg \} \quad , \quad (8)$$

$$3.) \text{ STN. } \{H_3[x^1, \lambda_3] \equiv F + \lambda_3(G - k_2)\} \quad , \quad (9)$$

respectively, where now the unprimed problems are made stationary as a function of the given variables (assumed independent), subject to no constraints.

At this point all of the above noted sources¹⁻⁴ work out particular examples of problems 1-3 in which λ_1 and λ_3 are constant but occasionally⁴ permit λ_2 to be nonconstant.

The justification for making λ_3 constant usually¹ is left to the unsatisfactory question: How else can one perform the variation without using the constancy of λ_3 to commute with the integral? This can be easily repudiated by postulating that λ_3 is really $\lambda_3 \equiv \langle \tilde{\lambda}_3 \rangle$ and that λ_3 could be taken inside to $\tilde{\lambda}_3$.

Even more difficult to intuit is the constancy of λ_1 in the light of problem 2.

The main purpose of this note is to show that the Lagrange multiplier formalism explicitly requires λ_1 and λ_3 to be constants but permits spatial variation of λ_2 . Note that this result justifies the integral-constraint technique and the consistency of the density as a constant Lagrange multiplier in the incompressible-plasma model of Wells⁵. This interpretation of the Lagrange multiplier subsequently⁵ forced a modification of one of the constraint integrals for the compressible plasma model.

III. PROBLEMS 1' AND 1

It should of course be realized that any constrained variational problem can, in principle, be solved by explicit elimination of the constraint function(al) without recourse to the Lagrange-multiplier technique. In this context, the Lagrange-multiplier formalism is a mathematical technique which is equivalent to, but sometimes much more convenient than, the alternate constraint-elimination method. However, this sometimes ease of calculation is paid for with the added duty of analyzing and physically interpreting the multipliers. These problems were of course discussed briefly in section I.

It is easy to show that the constraint-elimination solution method applied to problem 1' reduces the necessary⁸ conditions of solution to

$$\nabla f \times \nabla g = 0 \quad , \quad (10a)$$

$$g = k_1 \quad , \quad (10b)$$

where the cross-product in Eq.(10a) is in the x^3 direction.

Equation (10a) implies that

$$\nabla f = -\tilde{\lambda}_1(x) \nabla g \quad , \quad (11)$$

where $\tilde{\lambda}_1$ is an arbitrary scalar function.

To determine $\tilde{\lambda}_1$ we examine three cases:

CASE 1. $f_1 = f_1(g)$, (f_1 a total function of g). (12)

By calculating the gradient of Eq. (12a), we obtain

$$\nabla f_1 = (df_1/dg) \nabla g \quad . \quad (13)$$

Since by assumption we have f_1 as a total function of g ,

$$df_1/dg = \phi(g), \quad (14)$$

$$= -\tilde{\lambda}_1 \text{ (constant along } g = k_1). \quad (15)$$

This case of course gives the trivial result that f is stationary at every point along $g = k_1$, but was included for completeness.

CASE 2: $f_2 \neq f_2(g)$ (f_2 not a total function of g). (12b)

A simple sketch of either f_2 evaluated along $g_2 = k_1$ or lines of $f_2 \equiv \text{constant}$ relative to $g \equiv k_1$ shows that Eq. (11) can be satisfied at only a discrete number of points. By evaluating $\tilde{\lambda}_1$ at these points, we obtain a set of constants which can be considered an equivalent set of constant Lagrange multipliers.

Hence for both cases the constraint-elimination method explicitly reduces the Lagrange multipliers to constants.

CASE 3: case 1 for part of domain of $g = k_1$ and case 2 for remainder. (12c)

Results of case 1 and case 2 follow directly, requiring again the constancy of the Lagrange multiplier $\tilde{\lambda}_1$.

Now as a consistency check, we analyze problem 1 initially assuming $\lambda_1 \equiv \lambda_1(x)$.

The necessary⁸ conditions for the H_1 of problem 1 to be stationary are

$$\underline{\nabla} f + \lambda_1 \underline{\nabla} g = -g \underline{\nabla} \lambda_1 \quad , \quad (16a)$$

$$g = k_1 \quad . \quad (16b)$$

Computing [Eq. (16a)] $\times \underline{\nabla} g$ gives

$$\underline{\nabla} f \times \underline{\nabla} g = -g \underline{\nabla} \lambda_1 \times \underline{\nabla} g \quad , \quad (16c)$$

which implies $\nabla g \sim \nabla \lambda_1$ to be consistent with Eq. (10a). However the calculation of $(\nabla g) \cdot$ [Eq. (16a)] shows that the only functional dependence of λ_1 permitted for consistency with the constraint-elimination solution of problem 1 is that of $\lambda_1 \equiv \text{constant}$.

Both methods are therefore consistent and require $\lambda_1 \equiv \text{constant}$.

IV. PROBLEM 3

Problem 3 is treated next as it also implies a constant λ_3 .

Note first the usual necessary ^{1,10} condition for the stationarity of a functional:

$$\delta F \equiv \epsilon_1 \left(\left. \frac{d(F[x_1 + \epsilon_1 n_1])}{d\epsilon_1} \right|_{\epsilon_1=0} \right) = 0, \quad (17)$$

where the δ operator defines the first variation of the functional F in the usual notation.

We now evaluate the functionals $F[x^1]$ and $G[x^1]$ at the point

$$x^1 \equiv x^1 + \epsilon_1 n_1 + \epsilon_2 n_2:$$

$$F[x^1] \equiv F[\epsilon_1, \epsilon_2] \quad , \quad (18)$$

$$\geq F[0,0] \quad , \quad (19)$$

where

$$F[0,0] = F[x_1] \quad , \quad (20)$$

and

$$G[x^1] \equiv G[\epsilon_1, \epsilon_2] \quad , \quad (21)$$

$$\equiv G[0,0] \quad , \quad (22)$$

$$\equiv G[x_1] = k_2 \quad , \quad (23)$$

where Eqs. (19) and (23) follow from the definitions of functional stationarity and functional constraint, respectively.

Substituting Eqs. (18) and (21) into Eq. (9) and making stationary the result with respect to $\epsilon_1, \epsilon_2, \lambda_3$ by using the δ variation from Eq. (17), we can reduce problem 3 to the form:

$$\left\{ \frac{\partial F}{\partial \epsilon_1} + \lambda_3 \frac{\partial G}{\partial \epsilon_1} \right\} \Big|_{\epsilon_2=0} = 0 \quad , \quad (24a)$$

$$\left\{ \frac{\partial F}{\partial \epsilon_2} + \lambda_3 \frac{\partial G}{\partial \epsilon_2} \right\} \Big|_{\epsilon_2=0} = 0 \quad , \quad (24b)$$

$$\{G = k_2\} \Big|_{\epsilon_1, \epsilon_2=0} \quad . \quad (24c)$$

The equivalence of Eq. (24) and the Eqs. (15) which define problem 1 proves by inspection the iff conditions for constancy of the λ_3 Lagrange multiplier. This constancy of λ_3 permits its commutation with the integration and the usual reduction to the Euler-Lagrange¹ operator $[\]_x \equiv d/dt (d/d\dot{x}) - d/dx$ acting on the function $\tilde{F} + \lambda_3 \tilde{G}$. Also note the consistency restriction on G of $[G]_x \neq 0$. A physical interpretation of this restriction seems lacking in the literature.

V. PROBLEM 2

Using the Euler-Lagrange operator $[\]_{x^i}$ and the variations $\tilde{x}^i = x^i + \epsilon^i \eta^i$, where $i = 1, 2$ and with no summation over repeated indices, we can easily reduce Eq. (8) of problem 2 to

$$[\tilde{J}]_{x^i} + \lambda_2(x) \frac{\partial g}{\partial x^i} = 0 \quad , \quad (25a)$$

$$g = k_1 \quad . \quad (25b)$$

Inspection of Eq. (25) shows that the possible nonconstancy of λ_2 results from noting either

- 1.) the lack of constraints analogous to those of problem 1 which require λ_2 to be constant.

or

- 2.) the integral of g over the interval is equivalent to an infinite number of point constraints, each of which has a constant Lagrange multiplier but which in the limit of the Riemann sum produces a $\lambda_2(x)$.

ACKNOWLEDGEMENTS

The author wishes to express his appreciation to Professor Harry S. Robertson for a critical reading of this communication and to Dr. James C. Nearing both for several significant criticisms of an earlier version of this work and for reference to Morse's book, Reference 10.

REFERENCES

- * Supported by the United States Atomic Energy Commission Contract No. AT-(40-1)-3895
- ¹ Standard references include: H. Goldstein, Classical Mechanics (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1959), Chapter 2; J.W. Dettman, Mathematical Methods in Physics and Engineering (McGraw-Hill Book Company, Inc., New York, 1962), Chapter 2; L.P. Smith, Mathematical Methods for Scientists and Engineers (Dover Publications, Inc., New York, 1953), Chapter 15; G. Arfken, Mathematical Methods for Physicists (Academic Press, Inc., New York, 1966), p. 637; K.R. Simon, Mechanics (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1960), second edition, Chapter 9.
- ² See the excellent discussion of Lagrange multipliers as forces of constraint, K.R. Simon, Reference 1, chapter 9; compare the discussion of the geometric basis of Lagrange multipliers, J.R. Gaskill, Jr. and M. Arenstein, Amer. J. Phys. 37, 93 (1969).
- ³ Note the nontrivial calculations needed to prove even the positive-definiteness of the viscosity coefficients of fluid mechanics: L.D. Landau and E.M. Lifshitz, Fluid Mechanics (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1959), p. 54; note the extensive vector calculations needed to determine and eliminate the Lagrange multipliers from even simple variational calculations: T.S. Lundgren, Phys. Fluids. 6, 393 (1963).

- ⁴ E.g., notice the ad hoc determination of the Lagrange multipliers in the variational calculation of P. Rosen, *Phys. Fluids*, 1, 251 (1953).
- ⁵ D.R. Wells, *J. Plasma Physics* 4, (1970)(in press); D.R. Wells and J. Norwood, Jr., *J. Plasma Physics*, 3, 21 (1969), c.f., S. Chandrasekhar, *Proc. Natn. Acad. Sci. (U.S.A.)* 42, 273 (1956); L. Woltzer, *Astrophys. J.* 130, 400 (1959).
- ⁶ H.S. Robertson, *Phys. Rev.* 188, 233 (1969).
- ⁷ In addition, numerous points of mathematical rigor and finery, such as using different symbols to distinguish between a function and its functional value and between the zero vector and the number zero, etc., have been omitted whenever context permits.
- ⁸ Sufficiency conditions here, of course, entail considerations of the sign and form of the second derivatives; i.e., the Hessian.
- ⁹ See. e.g., S. Chandrasekhar, *Proc. Natn. Acad. Sci. (U.S.A.)* 42, 5 (1956), Eqs. (17) and (18).

¹⁰ As is well known, the problem of sufficiency conditions is one of the more difficult subjects in the calculus of variations and is beyond the scope of this note. Moreover while Weierstrasz' Theorem insures the existence of the maximum-minimum problem of functions, no such comparably general results exist for the maximum-minimum problem of functionals in the calculus of variations. For a rigorous treatment and extensive bibliography of the results that existed through 1934, see M. Morse, The Calculus of Variations in the Large (American Mathematics Society, New York, 1934), Chapters I-IV. For a more modern treatment see C.B. Morrey, Jr., Multiple Integrals in the Calculus of Variations (Springer-Verlag Inc., New York, 1966).

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing exception must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) University of Miami Coral Gables, Florida 33124		2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED	
		2b. GROUP	
3. REPORT TITLE FUNCTIONAL DEPENDENCE OF LAGRANGE MULTIPLIERS			
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) Scientific Interim			
5. AUTHOR(S) (First name, middle initial, last name) Lawrence C Hawkins			
6. REPORT DATE March 1970		7a. TOTAL NO. OF PAGES 15	7b. NO. OF REFS 10
8a. CONTRACT OR GRANT NO. AF AFOSR-844-67		9a. ORIGINATOR'S REPORT NUMBER(S) MIAPH-TP-70.3	
b. PROJECT NO. 9752-01		9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) AFOSR 70-1016 TR	
c. 61102F			
d. 681308			
10. DISTRIBUTION STATEMENT 1. This document has been approved for public release and sale; its distribution is unlimited.			
11. SUPPLEMENTARY NOTES <i>Tech, other</i>		12. SPONSORING MILITARY ACTIVITY AF Office of Scientific Research (SREP) 1400 Wilson Boulevard Arlington, Virginia	
13. ABSTRACT Given three variational problems consisting of minimizing a given function(al) subject to one constraint function(al), the three equivalent Lagrange-multiplier problems are deduced and the <u>proper</u> functional dependence of the particular Lagrange multipliers is derived. The results <u>prove</u> the consistency of the use and subsequent physical interpretation of the <u>constant</u> Lagrange multipliers in the plasma, global-stability, variational calculations with integral constraints of Chandrasekhar, of Woltzer, and of Wells and Norwood.			