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REPORT NO. RE-TR-70-3

ERRORS IN RANGE ESTIMATION BY TRIANGULATION

by

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January 1970

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Redstone Arsenal, Alabama

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22 January 1970 Report No. RE-TR-70-3

ERRORS IN RANGE ESTIMATION BY TRIANGULATION

by

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Abstract

This report discusses the estimation of range to target when range information is unavailable; triangulation methods are used which require two angle measuring radars. Two separate means of estimation are presented: the differential analysis approach and the incremental analysis.

Contents

Illustrations

1. Introduction

In an environment in which radar range information is not available, an estimate of the range to the target can be found by using triangulation methods. These require two angle measuring radars separated by a known distance along a base line. The measured angles from the base line to the target and the measured distance between the radars can be used to obtain an estimate of the range to the target.

The range estimate found will in general not be exact because of errors in the measured quantities. The range error for given measurement errors is estimated by two separate means. A differential analysis leads to a range error estimate valid when the measurement errors are small. Assuming that the measurement errors are independent Gaussian random variables and using the differential analysis lead to a range error variance. If the errors are not small or the range is very large, a more suitable range error estimate is obtained from an incremental analysis. This analysis also leads to validity bounds for the differential analysis results.

2. Range Computation

The error-free triangulation problem is shown in Figure 1, where:

d = measured distance between radar sites

θ_1, θ_2 = measured angles to target

R = range to be estimated

r, ω = reference polar coordinates.

In terms of the measured quantities, the range is given by

$$R = d\,(\sin\theta_2 \operatorname{ctn}\theta_1 + \cos\theta_2)^{-1} . \tag{1}$$

3. Differential Error Analysis

For small perturbations in the measured quantities, the perturbation, ρ, in the range estimate is given by

$$\rho = \frac{\partial R}{\partial d}\delta + \frac{\partial R}{\partial \theta_1}\alpha + \frac{\partial R}{\partial \theta_2}\beta , \tag{2}$$

where

$$\delta = \text{the perturbation of } d$$

$$\alpha = \text{the perturbation of } \theta_1$$

$$\beta = \text{the perturbation of } \theta_2 .$$

The partial derivatives of R are

$$\frac{\partial R}{\partial d} = (\sin\theta \,\mathrm{ctn}\,\theta_1 + \cos\theta_2)^{-1} = \frac{R}{d} \tag{3}$$

$$\frac{\partial R}{\partial \theta_1} = d\,(\sin\theta_2 \csc^2\theta_1 \;\; \sin\theta_2 \,\mathrm{ctn}\,\theta_1 + \cos\theta_2)^{-2} = \frac{R^2 \sin\theta_2}{d \sin^2\theta_1} \tag{4}$$

$$\frac{\partial R}{\partial \theta_2} = d\,(\sin\theta_2 - \cos\theta_2 \,\mathrm{ctn}\,\theta_1)\;\;(\sin\theta_2 \,\mathrm{ctn}\,\theta_1 + \cos\theta_2)^{-2} = \frac{R^2}{d}\,(\sin\theta_2 - \cos\theta_2 \,\mathrm{ctn}\,\theta_1) \tag{5}$$

The error coefficients C_d, C_1, and C_2 are defined as:

$$C_d = \frac{\partial R}{\partial d} \tag{6}$$

$$C_1 = \frac{\partial R}{\partial \theta_1}$$

$$C_2 = \frac{\partial R}{\partial \theta_2}$$

The behavior of these error coefficients determines the sensitivity of the range calculation to errors in d, θ_1, and θ_2. Because the angle error coefficients, C_1 and C_2, grow as R^2 as compared to R for C_d, they will dominate the range error as R becomes large. For ranges such that $\frac{R}{d} \geq 5$ and for angles off bore-sight of $|\omega| \leq 30$ degrees, the coefficients C_1 and C_2 are approximately equal and simplify to

$$C_1 \simeq C_2 \simeq \frac{R^2}{d} \tag{7}$$

In order to show the behavior of C_1 and C_2 as a function of range and angle off boresight, it is convenient to plot C_1/R and C_2/R on a polar plot with a normalized reference coordinate $\frac{r}{d}$ replacing r. In Figures 2 and 3 constant contours of C_1/R and C_2/R are plotted by using the normalized coordinate.

4. Gaussian Random Errors

Equation (2) gives the range error as a linear function of the measurement errors δ, α, and β. Therefore, if δ, α, and β are zero mean, independent, Gaussian random variables with respective variances of σ_δ^2, σ_α^2, and σ_β^2, then the range error, ρ, will be a zero mean, random variable with a variance given by

$$\sigma_\rho^2 = C_d^2\sigma_\delta^2 + C_1^2\sigma_\alpha^2 + C_2^2\sigma_\rho^2 \; . \tag{8}$$

A fractional error variance is defined by dividing σ_ρ^2 by R^2 to get

$$e_\rho^2 = \sigma_\rho^2/R^2 \tag{9}$$

A simple approximate expression for e_ρ can be found when the angle error terms dominate the expression in equation (9). By using equation (7) and assuming $\sigma_\alpha = \sigma_\beta = \sqrt{2}\,\sigma$,

then

$$e_\rho \simeq \frac{R2\sigma}{d} \; . \tag{10}$$

When $\sigma_\alpha = \sigma_\beta = \sqrt{2}\,\sigma$, the fractional error on boresight, where $\theta_1 = \theta_2 = \theta$, reduces to

$$e_\rho^2 = \frac{\sigma_\delta^2}{d^2} + 2\sigma^2\,\frac{R^2}{d^2}\left\{\frac{1 + (\sin^2\theta - \cos^2\theta)}{\sin^2\theta}\right\} \tag{11}$$

It is easy to show that equation (11) has a minimum at $\theta = \pi/4$. When $\theta = \pi/4$ the distance r along boresight is half the separation distance and the measured range is $\frac{d}{\sqrt{2}}$. The fractional error variance reduces to

$$e_{\rho}^{2} = \left(\frac{\sigma_{\delta}^{2}}{d^{2}} + 2\sigma^{2}\right). \tag{12}$$

5. Incremental Error Analysis

In the special case of measuring ranges to targets on boresight, $\theta_1 = \theta_2$, and equation (1) simplifies to

$$R = \frac{d}{2 \cos \theta_1} \quad \text{(on boresight)} . \tag{13}$$

The error estimates in the preceding section predict finite range error (more accurately — finite error variances). But for every large target range the angle θ_1 is very close to $\pi/2$ so that even very small positive errors can give a measured angle of $\pi/2$ such that the range estimate and range error become infinite. Therefore, the estimates [equation (8) or (10)] must be useful only on a limited interval. Estimates of this interval will be obtained by using the following incremental error analysis.

In Figure 4 the effects of incrementing the measured quantities on the range estimate are shown. If measurement errors are bounded by the increments shown, then the range estimate will be bounded between R_{max} and R_{min} The "average" error in equation (14) will be used as the range error estimate for this analysis

$$\sigma_R = \frac{R_{max} - R_{min}}{2} . \tag{14}$$

As before, it is convenient to define the fractional error, e_R, by dividing σ_R by by the true range, R, to get

$$e_R = \frac{\sigma_R}{R} = \frac{R_{max} - R_{min}}{2R} . \tag{15}$$

In Figures 5 and 6 contours of constant fractional error are plotted for two cases. For Figure 5 the angle increments were $\alpha = \beta = 0.01$ radian and the separation increment was 1 percent of the separation with $\delta = 0.01d$. For Figure 6 $\alpha = \beta = 0.001$ radian and $\delta = 0.01$ d. Figure 5 illustrates the case where the range error becomes infinite.

6. Differential and Incremental Errors Compared

The fractional error, e_ρ, from the differential analysis and the fractional error, e_R, from the incremental analysis give similar results in many cases. Figure 7 shows e_ρ and e_R along boresight as a function of the normalized range, r/d. The errors are plotted for the two cases in which the variances and increments were as follows:

	Case I	Case II
variances		
σ_α	$= \sqrt{2} \times 10^{-2}$ rad	$\sqrt{2} \times 10^{-3}$ rad
σ_β	$= \sqrt{2} \times 10^{-2}$ rad	$\sqrt{2} \times 10^{-3}$ rad
$\frac{\delta}{d}$	$= \sqrt{2} \times 10^{-2}$	$\sqrt{2} \times 10^{-2}$
increments		
α	$= 1 \times 10^{-2}$ rad	1×10^{-3} rad
β	$= 1 \times 10^{-2}$ rad	1×10^{-3} rad
$\frac{\delta}{d}$	$= 1 \times 10^{-2}$	1×10^{-2}

The straight line portion of the e_ρ curve corresponds to the region in which the simplification in equation (10) is valid. A similar simplification for e_R is found by letting $\delta = 0$ and $\alpha = \beta = \sigma$. Then along boresight equation (15) becomes

$$e_R \simeq \frac{R}{d}\, 2 \sin\sigma \, \frac{(1 + \cos 2\theta)}{(\cos 2\sigma + \cos 2\ )} . \tag{16}$$

If $\theta + \sigma < \pi/2$ and σ is very small, then

$$\frac{1 + \cos 2\theta}{\cos 2\sigma + \cos \theta} \simeq 1, \tag{17}$$

and

$$\sin \sigma \simeq \sigma \; ; \tag{18}$$

substituting equations (17) and (18) into equation (16) results in

$$e_R \simeq \frac{R}{d} 2\sigma \; . \tag{19}$$

Thus as was illustrated in Figure 7, e_R, and e_ρ give similar results for large R/d as long as $\theta + \sigma < \pi/2$.

7. Error Estimate Limitations

The differential analysis would predict that equation (10) and hence equation (19) are valid for arbitrarily large R/d. But as R/d becomes very large, the exact value of e_R from equation (16) diverges from the estimate of equation (19). The value of R/d at which this occurs will give an upper limit on R/d for which equation (19) is valid. This limit is also the limit for the differential analysis and is found as follows.

From Figure 8, equation (16) can be written as

$$e_R \simeq \frac{R}{d} 2 \sin\sigma \; \frac{(\sin \eta \cos \sigma + \cos \eta \sin \sigma)^2}{\sin \eta \, (\sin \eta \cos 2\sigma + \cos \eta \sin 2\sigma)} \; . \tag{20}$$

If it is assumed that the σ is small and that $\frac{R}{d}$ is large such that η is small, then by using the approximation $\sin x \simeq x$, equation (20) becomes

$$e_R \simeq \frac{R}{d} 2\sigma \; \frac{(\eta + \sigma)^2}{\eta(\eta + 2\sigma)} \; . \tag{21}$$

Equation (21) indicates that equation (19) is valid as long as $\sigma < \eta$. Equation (21) begins to diverge from equation (19) as σ becomes equal or greater than η. The $\frac{R}{d}$ corresponding to $\eta = \sigma$ is

$$\left(\frac{R}{d}\right) = \frac{1}{2} \; \frac{\sin\left(\frac{\pi}{2} - 2\sigma\right)}{\cos\left(\frac{\pi}{2} - 2\sigma\right)} \tag{22}$$

$$\simeq \frac{1}{4\sigma} \quad \left(\text{upper limit on } \frac{R}{d}\right) \; .$$

For Case I in Figure 7, e_R diverged from e_ρ at about R/d = 25. This is equal to the upper limit which would have been predicted by equation (22) by using $\sigma = 0.01$. Thus, as long as R/d does not exceed the limit given by equation (22), the differential error estimates of equation (8) or (10) should be valid.

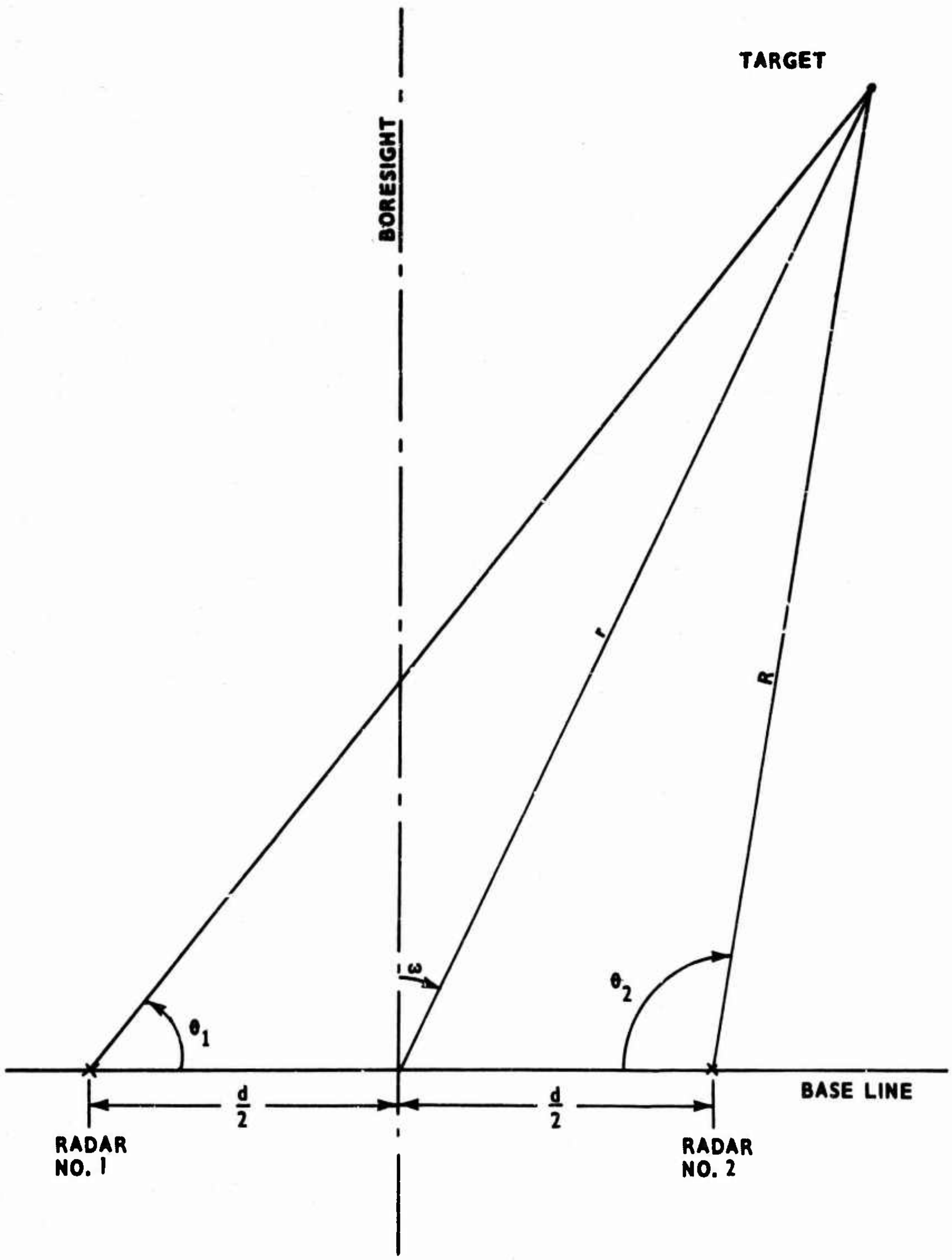

FIGURE 1. BASIC GEOMETRY

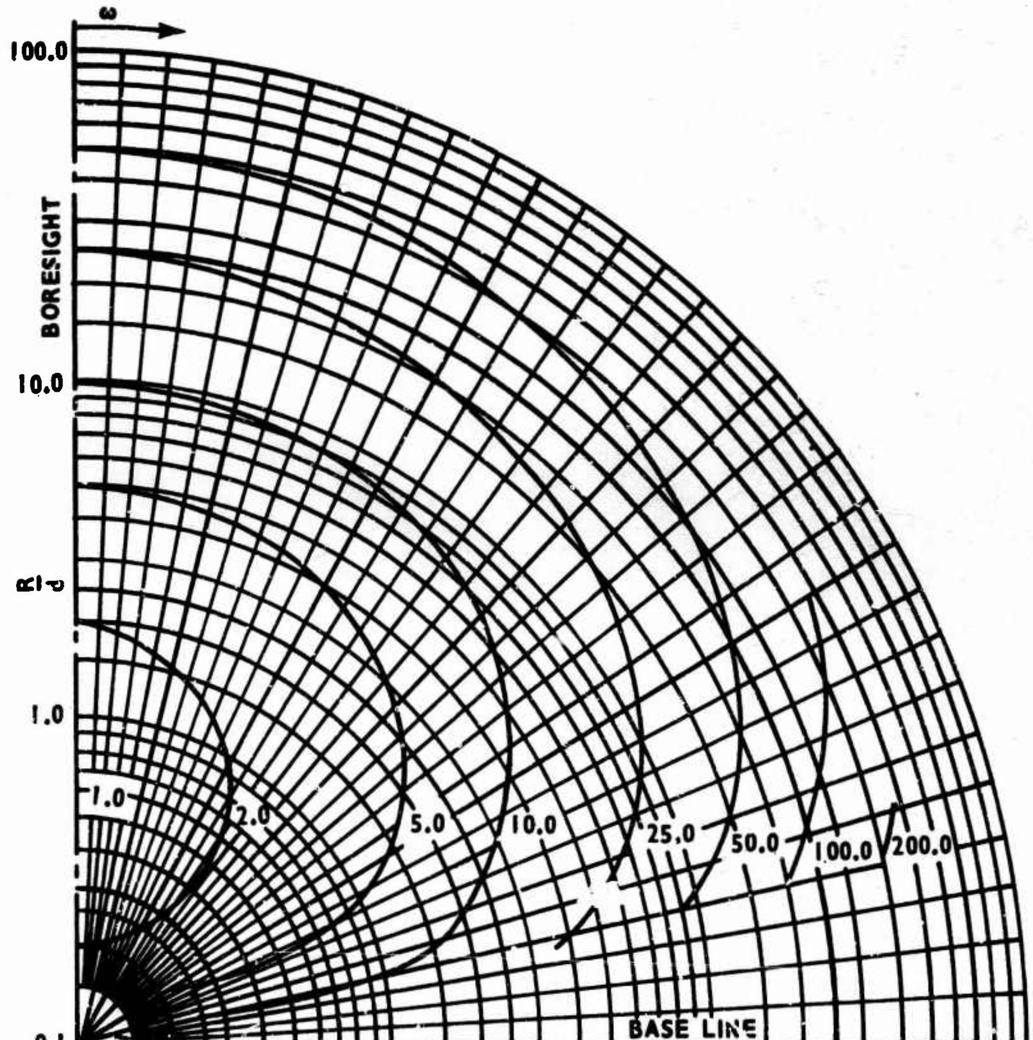

FIGURE 2. NORMALIZED COEFFICIENT $C_1/R = \frac{\partial R}{\partial \theta_1}/R$

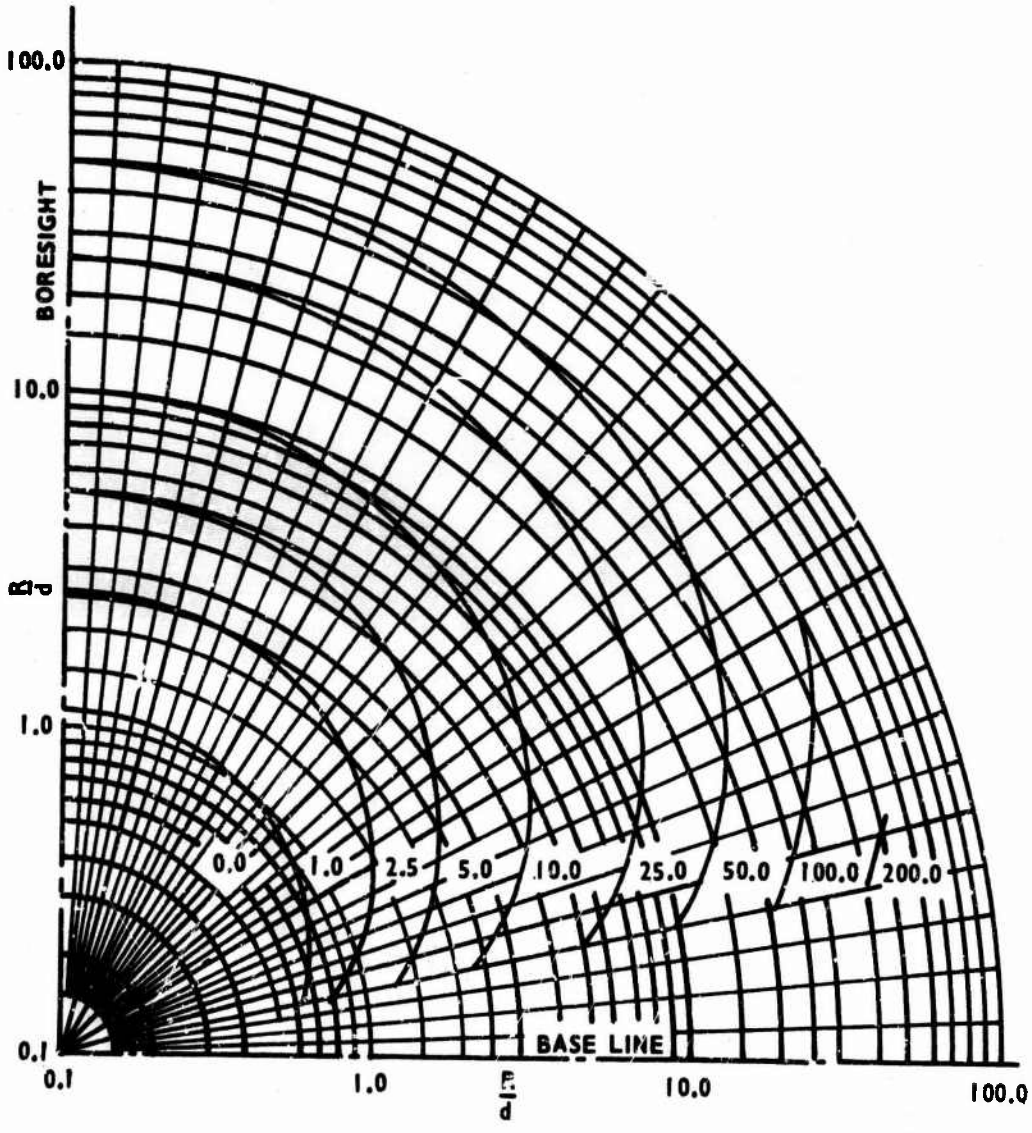

FIGURE 3. NORMALIZED COEFFICIENT $C_2/R = \frac{\partial R}{\partial \theta_2}/R$

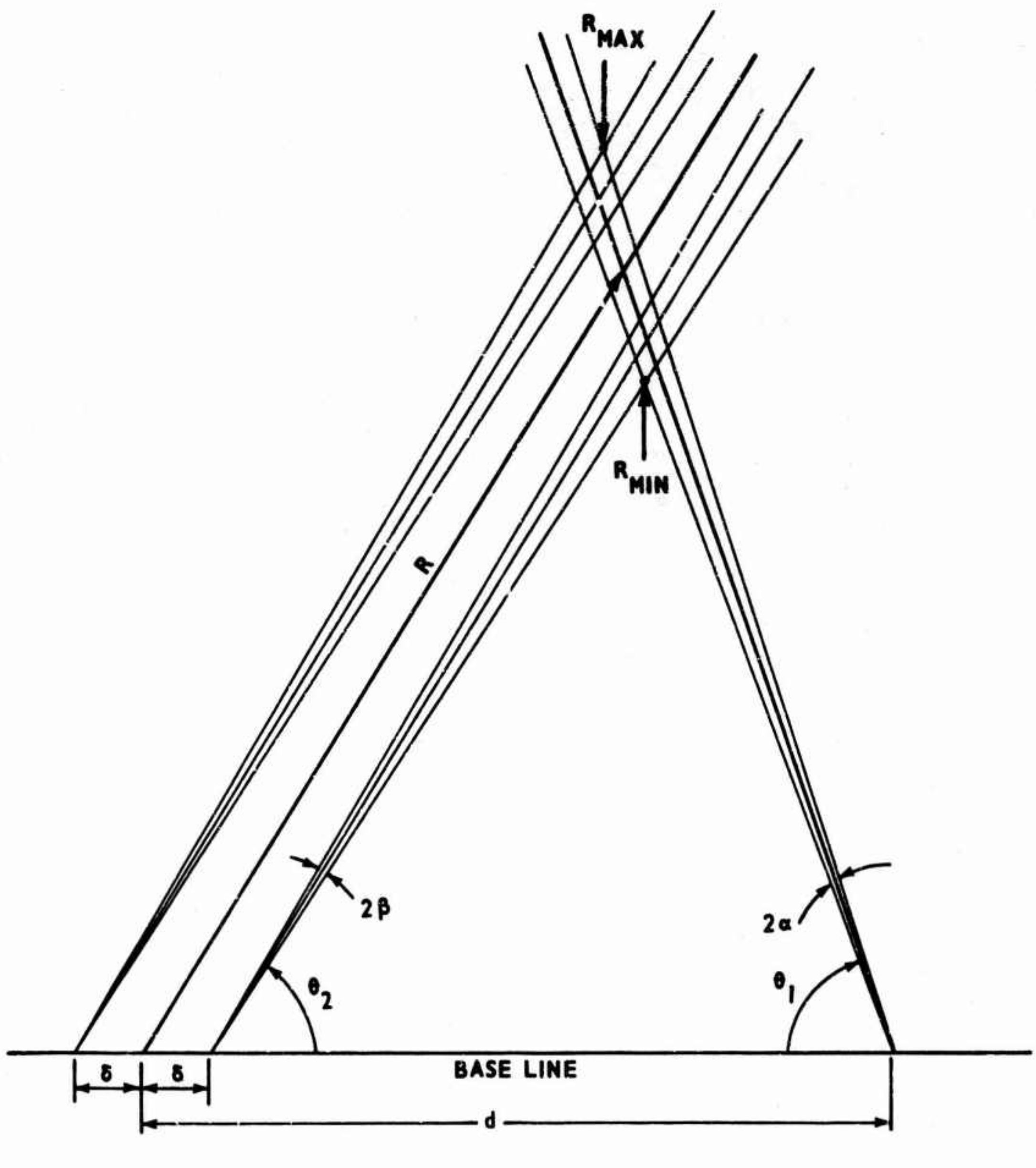

FIGURE 4. INCREMENTAL ERROR GEOMETRY

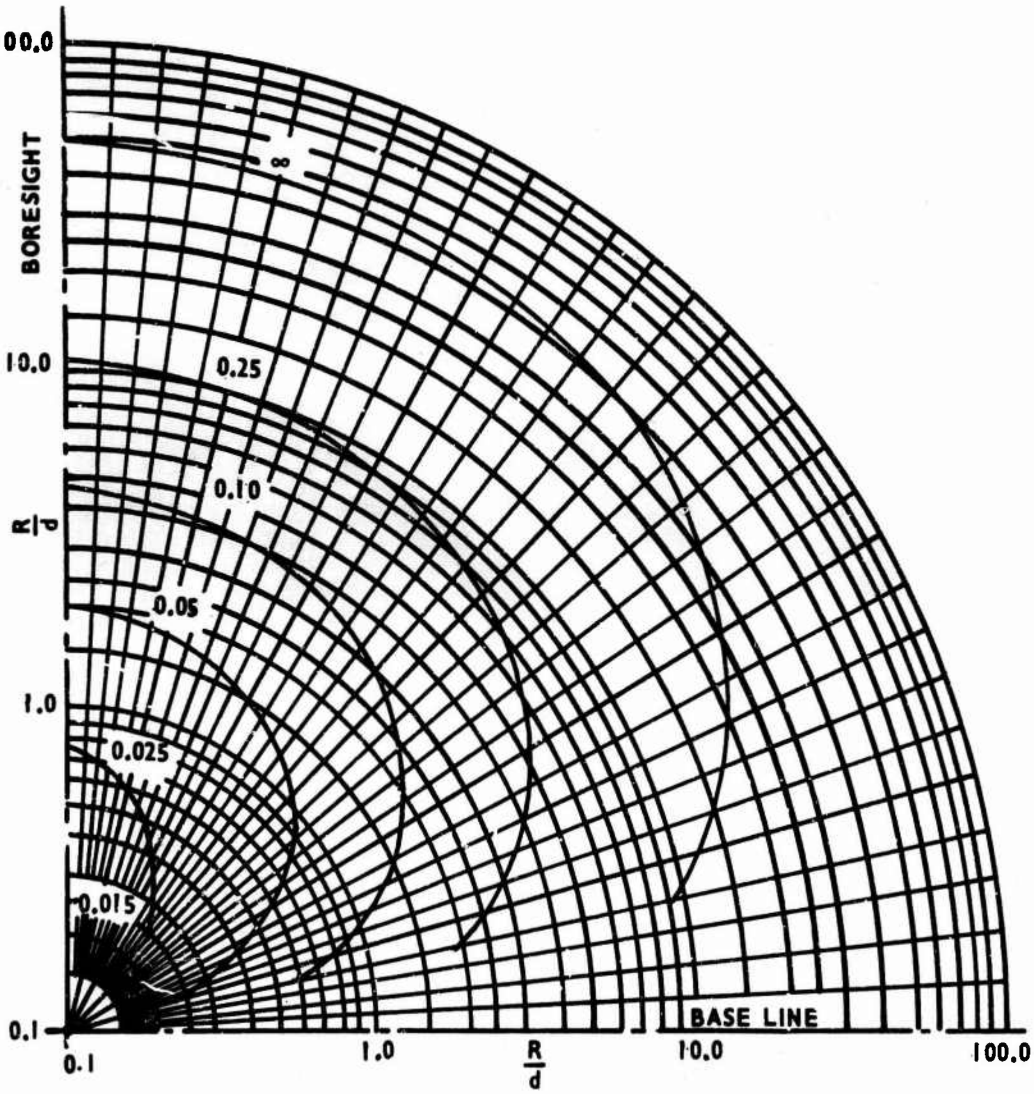

FIGURE 5. INCREMENTAL ERROR: CASE I σ_θ = 0.01 RADIAN

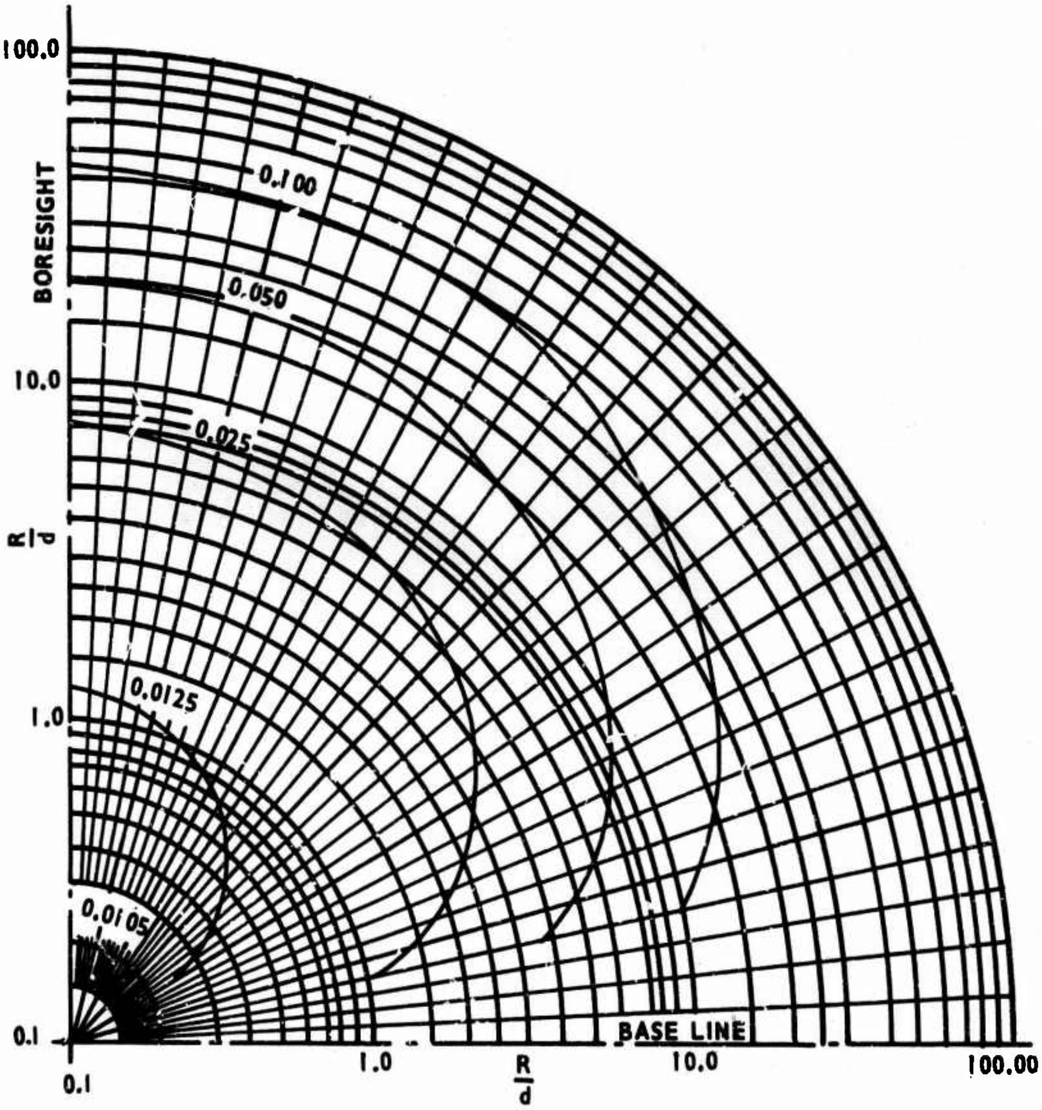

FIGURE 6. INCREMENTAL ERROR: CASE II σ_{θ} = 0.001 RADIAN

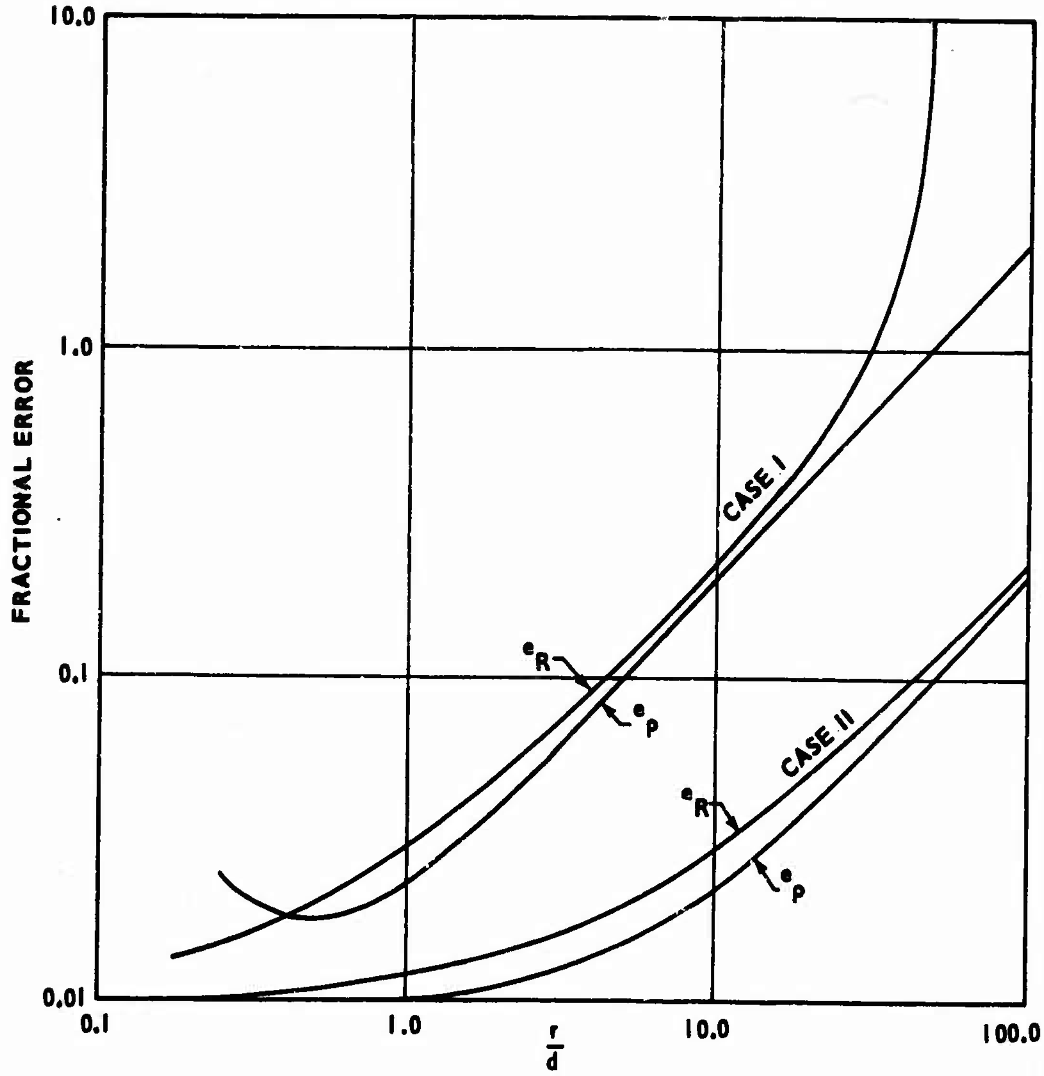

FIGURE 7. INCREMENTAL VERSUS DIFFERENTIAL ERROR ALONG BORESIGHT

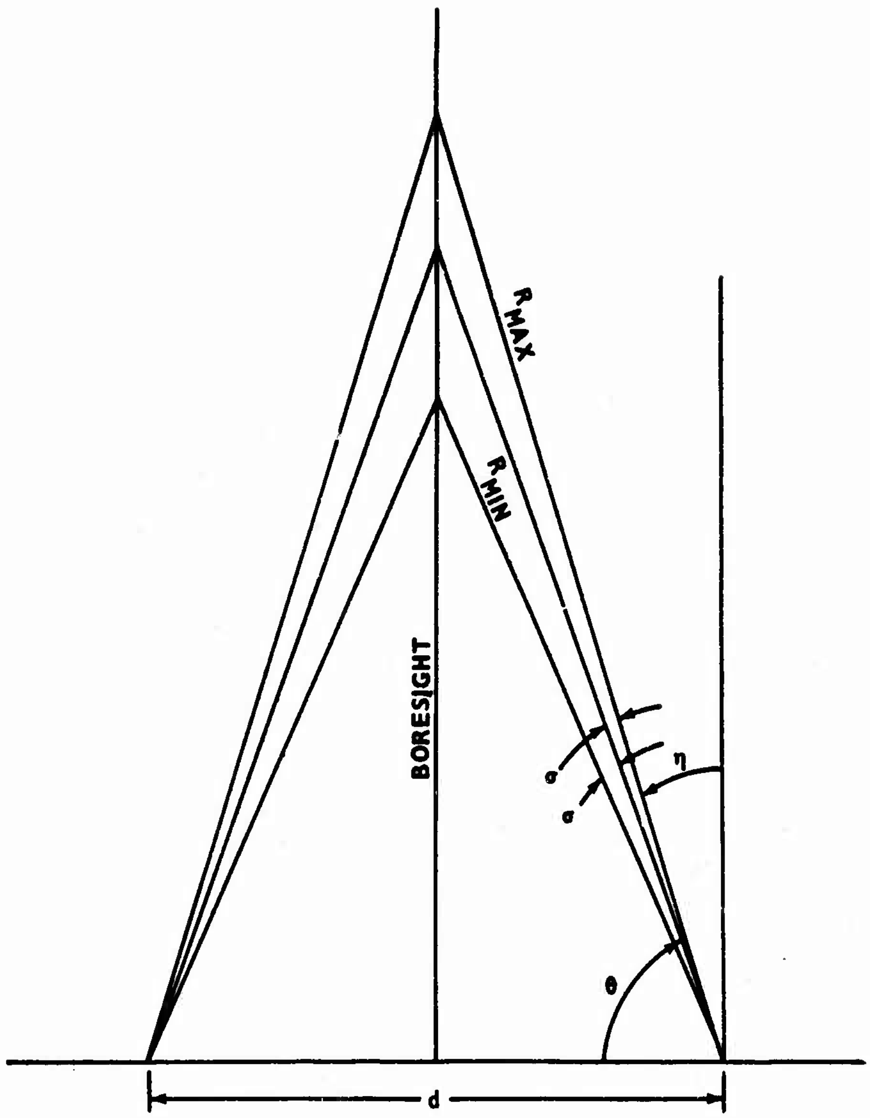

FIGURE 8. LARGE RANGE GEOMETRY USED TO OBTAIN AN ESTIMATE OF e_R

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Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Action Group II Research and Engineering Directorate (Provisional) U. S. Army Missile Command Redstone Arsenal, Alabama 35805	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP NA

3. REPORT TITLE

ERRORS IN RANGE ESTIMATION BY TRIANGULATION

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Report

5. AUTHOR(S) (First name, middle initial, last name)

Captain Nels A. Broste

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
22 January 1970	21	0

8a. CONTRACT OR GRANT NO.	9a. ORIGINATOR'S REPORT NUMBER(S)
b. PROJECT NO.	RE-TR-70-3
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	AD__________

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13. ABSTRACT

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14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Range estimation errors Triangulation Gaussian random errors Incremental analysis						