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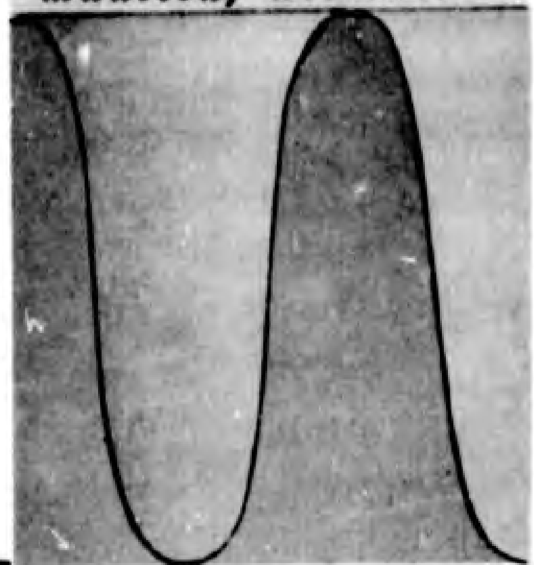

FUNDAMENTAL CONCEPTS AND PROBLEMS OF OPTIMAL CONTROL THEORY

L. C. Young

Orientation Lecture Series #9

MATHEMATICS RESEARCH CENTER

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This is a series of three introductory lectures, and one somewhat specialized lecture, on the mathematical theory of optimization for those control problems which concern systems with a continuously varying time.

Prerequisites

The introductory lectures assume familiarity with differential equations, vector notation and some further maturity of outlook, which may be acquired in courses such as advanced calculus, probability, or in a number of courses in applied mathematics. The specialized lecture requires in addition some knowledge of topology and real variables, although the illustrations from everyday life may make it understandable to some degree without this.

Objectives

(a) In the introductory lectures, it is pointed out that the theory of control often deals with problems which were, at one time, believed impossible of solution, and therefore that the existence of a solution can hardly be regarded as physically obvious. Mathematically this makes it necessary to study existence theorems, and also to admit, as solutions, controls which depend on time in a highly discontinuous and irregular manner. This irregular behaviour of solutions is illustrated by a number of examples. It represents a major difference between modern Optimal Control Theory and the classical Calculus of Variations. The central principle is now the Maximum Principle of Pontrjagin.

This principle, which differs only slightly from corresponding statements established earlier by McShane and Hestenes, is here described in terms of maximizing the instantaneous "performance" by choice of the controls at each instant of time. Two examples are given, to which the Principle is applied and a complete solution is obtained in an elementary manner.

(b) In the specialized lecture, an introduction is given to the material covered by MRC technical summary report #654. This deals with the adaptation of a method for obtaining complete solutions in certain optimization problems. The basic difficulties arise from the highly discontinuous and irregular nature of the solutions, and it is shown that the method, which goes back to Carathéodory in 1905, or even earlier, can indeed be adapted to this type of situation. The Carathéodory method has been used for some time by Bellman under the name of principle of dynamic programming; however in that form it has been open, according to Pontrajagin, to serious objections, since it makes assumptions which are hardly ever satisfied, even in the simplest control problems. These objections no longer apply to the improved form of the method, introduced here.

FIRST LECTURE

§1. Introduction: The New Freedom

Optimal Control is a modern development of the theory of constrained minima in the Calculus of Variations; this is a very old theory, which studies problems such as those of Lagrange and Bolza, and also isoperimetric problems such as the problem of Dido. These various problems are roughly equivalent, and those of Optimal Control do not differ from them in any essential way.

In a typical problem of Optimal Control, we seek the smallest cost of the operation of putting a projectile on a given moving target, such as the Moon, when the projectile's motion is controlled by a differential equation

$$\dot{x} = g(t, x, u) \quad , \tag{1.1}$$

where x is a vector describing the state (or phase) of the projectile at the time t, while $\dot{x}$ denotes the rate of change of this state, and where u is a control variable on some dial, or system of dials, so that it can be thought of as a vector whose components vary on certain segments, which means that u varies in a unit cube, or on some fixed parallelopiped. Both x and u are functions of time, and when we need to stress this we shall write $x(t)$ and $u(t)$. In the case of a particle, the state is usually its position and velocity, while for a rigid body it might specify also the way in which it spins. The cost to be minimized is an integral of the form

$$\int f(t, x, u)\, dt \quad ; \tag{1.2}$$

this depends on the functions $x(t)$, $u(t)$, i.e. on the whole motion of the projectile and on the manner in which it will have been controlled during the motion.

In the special case where f reduces to the constant unity, the cost reduces to the time taken to reach the target. (Time is money.)

The fact that u varies only in some fixed parallelopiped U, rather than in the whole corresponding Euclidean space, means that our constraints consist partly of inequalities, just as in elementary equilibrium problems of Statics, when the reaction of a constraint has to remain non-negative. This has to be borne in mind, but does not really complicate the issue.

The real difference between Optimal Control and the classical theory lies not in any such minor difference in the nature of the constraints, but in a new element of freedom, which consists in a much freer interpretation of what we accept as a solution. Ours is an age of greater freedom; it is made possible by better technical means, and in particular by better, and more flexible, mathematical tools. These tools make all the difference here, and their importance is comparable to that of the engineering advances.

The reason for this is that, in a sense, in the problems of Optimal Control, we are skirting rather close to the impossible. For a Moonshot a hundred years ago, conceived as a single shot affair where you just shoot the projectile at the Moon without controlling it further, it would have been necessary to hit on the way a certain extremely small area out in space. The whole operation became much more feasible when engineers provided the greater freedom of a radar-controlled rocket.† Similarly the greater freedom and flexibility rendered possible by better mathematical tools is needed even in some of the simplest of Optimal Control problems.

† Actually, we are no longer concerned just with hitting the Moon, but with making a soft landing of our projectile on it.

A radar-controlled rocket is governed by a differential equation of the type (1), in which u can be a sharply discontinuous function u(t). The function g is, in practice, continuously differentiable; however, if we write

$$g^*(t,x) = g(t,x,u(t)),$$

this means that the motion is governed by

$$\dot{x} = g^*(t,x), \tag{1.3}$$

where g^* is still nice and smooth in x for constant t, but sharply discontinuous in t for constant x.

Differential equations (1.3) with right hand sides subject to such lopsided conditions were studied over 50 years ago in Carathéodory's Real Functions, but no one dreamt of their ever having practical use. The theory uses Lebesgue integration, but is otherwise no more difficult than the elementary theory. It assumes g^* smooth in x for constant t, measurable in t for constant x, and subject to $|g^*| \leq \varphi$, where φ is some fixed integrable function of t only. A solution x(t) of (1.3) is interpreted to mean an absolutely continuous function which satisfies (1.3) almost everywhere. This is more elaborate than we need, but who would refuse the free use of a readily available Rolls? So why not take advantage of our new freedom?

Of course the rules are slightly different, and there is the slight danger of an accident. We must take extra precautions.

§2. Perron's Paradox and the Triangle Joke

When we may be skirting the impossible, there is one elementary precaution that we must insist on, at least temporarily: we must on no account assume that a problem has a solution. Otherwise we come up against what is known as

Perron's paradox. As I say in my book † [14, p. 22], in the Middle Ages an important part was played by the Jester; a little joke that seemed so harmless, could, as its real meaning began to sink in, topple kingdoms. It is just such little jokes that can play havoc with a mathematical theory; we call them paradoxes.

Perron's paradox runs as follows: "Let N be the largest positive integer. If N were greater than 1, we would have N less than its square, contrary to the definition of N as largest. Therefore N must be 1."

The implications are devastating. In virtually every problem one customarily begins by saying: "Let X be the desired solution", and one then proceeds in some way to calculate X. If the calculation leads to just one value for X, one thinks the problem solved. Unfortunately this is just what it is not. In particular, in Optimal Control problems, we must prove the existence of a solution before we can apply the classical arguments for deriving necessary conditions. This is the price we must pay for tackling problems which come close to the impossible.

The best way to begin is perhaps to look at a few special examples, and I shall take these first from the Calculus of Variations itself, where they are simpler. The examples are connected with a bit of tomfoolery that used to be thought great fun in the High Schools, and according to which a pupil who wanted to show off his cleverness would claim to prove that in any triangle any two sides are together equal to the third. Lebesgue refers to this in his little book "In the Margin of the Calculus of Variations" [7], and I partly repeat it in my book [14, p. 152].

† References will be found at the end of these lectures.

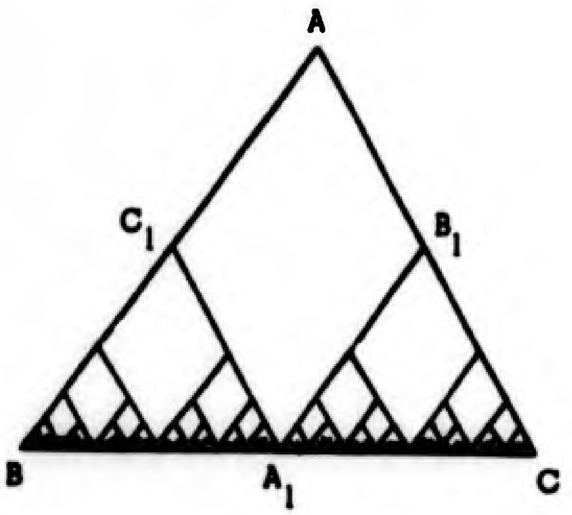


Figure 1. Tomfoolery

Let ABC be any triangle (see Figure 1). If A_1, B_1, C_1 are the mid-points of the sides, we have

$BA + AC = BC_1 + C_1A_1 + A_1B_1 + B_1C$.

On each of the triangles BC_1A_1 , A_1B_1C, proceed as on ABC . We obtain a broken line, formed of eight segments, and equal to $BA + AC$. By continuing in this way, we obtain a sequence of broken lines, which stray less and less from the side BC, and which still have as length the sum of the other two sides of our original triangle. It is therefore claimed, by the pupil who wants to show off, that the segment BC, the geometrical limit of our broken lines, must also have as length the sum of the other two sides $BA + AC$.

Lebesgue goes on to say that, when he was in High School, his schoolfellows thought this just a joke; however he himself was quite disturbed by it, since he could see no difference between this type of argument and the proofs he was getting every day in his geometry classes, in regard to the lengths and areas of curved figures.

§3. Zermelo's Sailing Problem

Actually the same basic constructions can be given a highly practical interpretation. We consider for this purpose a particular case of the so-called Zermelo navigation problem, which can be formulated equally as a problem of the Calculus of Variations and as one of Optimal Control. We prefer here to state it simply as a practical problem, such as anyone who likes to go sailing might meet, and we therefore suppress the analytical details.

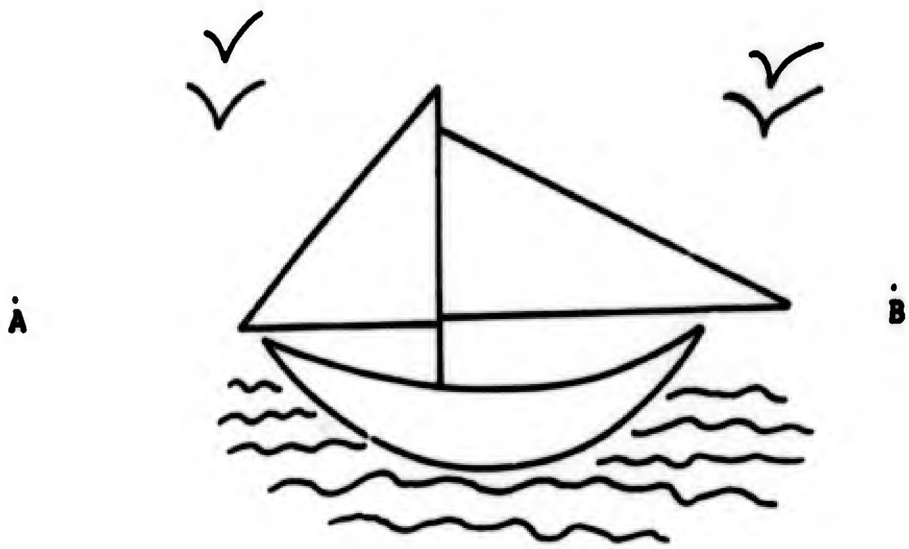


Figure 2. The Zermelo navigation problem

The problem is that of sailing against the wind from a point A to a point B, where A, B are mid-stream points on a river, and B is downstream (see Figure 2). This is a problem frequently met by sailing enthusiasts.

If we disregard the current for a moment, and suppose the river wide enough, every sailor knows that the way to proceed is to make a simple tack, i.e. follow a broken line AXB, where AX, XB make, with AB, at A and at B, a certain same angle α, which is the most favorable (see Figure 3). He can thus use the wind to sail against the wind, in two installments, at the most favorable angle α. This takes him from

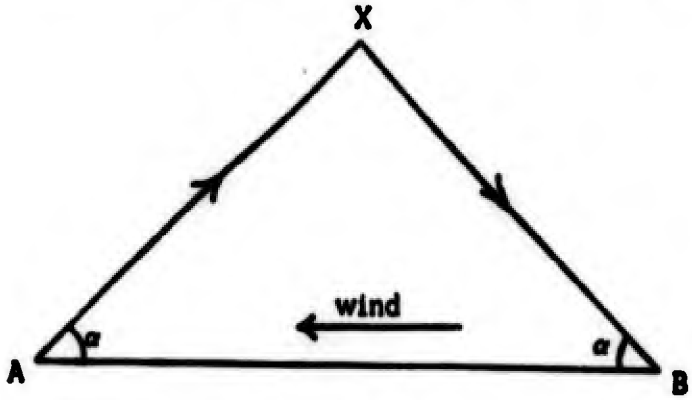


Figure 3. A simple tack

A to B in least time (if the river is wide enough and there is no current).

Of course, in the special case just considered, it would cost nothing to tack more than once at the same optimal angle. Another solution is therefore found in which he follows a zigzag path of four segments, and yet another with eight, or sixteen, and so on (see Figure 4), as in the case of the triangle joke, or in fact with any even number of segments. Generally, to sail in least time from A to B, a sailor can tack any number ν of times, and follow a zigzag path, consisting of 2ν equal segments, alternately parallel to AX and XB. (We neglect any delay in changing the direction at the corners.)

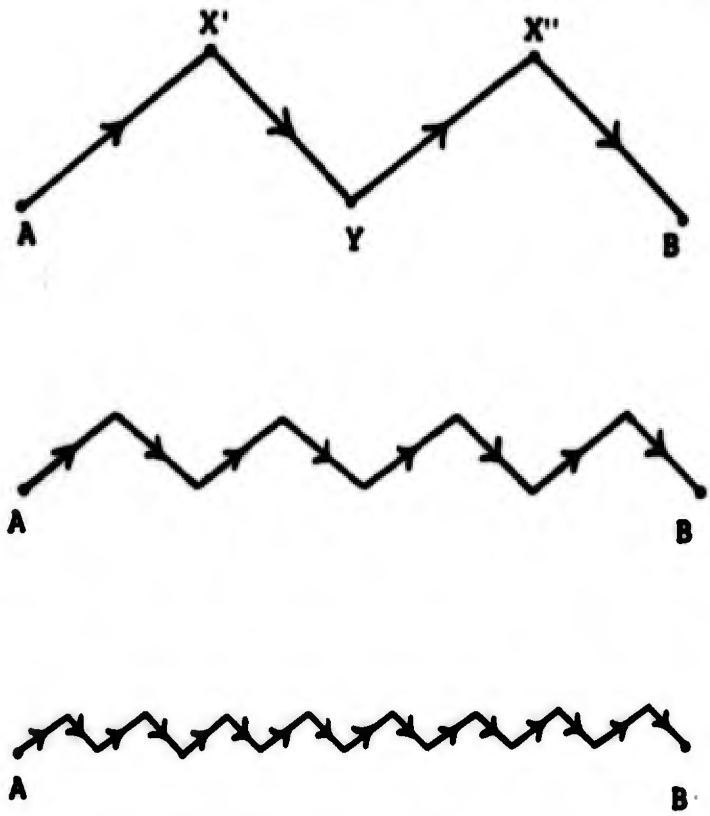


Figure 4. Multiple tacks

It will be observed that the broken lines thus described are just those that we had occasion to construct in our previous triangle jest. Now we shall similarly proceed to the limit in an appropriate sense, and this will enable us to take account of the current and of the limited width of our river.

Clearly, if the river is very narrow, the only difference will be that ν must be large enough for our broken line to remain in the river.

In regard to the current, this superposes on the motion already described a certain amount of drifting downstream. This decreases the time from A to B by a constant amount if the river flows with uniform velocity; however, we shall suppose, as seems more in accordance with what actually occurs, that the current is strongest in midstream.

In that case, the mathematically best course is obtained by tacking an extremely large number of times, so as to remain close to the straight line AB. The sails are then constantly shifted from one position to another, and the boat keeps changing its direction from that parallel to AX to that parallel to XB. In fact, ideally, the boat should stay on AB itself, by tacking infinitely often, but so that the direction of the boat is not that of AB (in which case the wind would simply blow it backwards); instead, this direction should continue to alternate between the parallel to AX and the parallel to XB.

With the classical restrictions as to what is a mathematically acceptable course for our sailing boat, this ideal solution is unattainable, although we can approximate to it as closely as we please by a finite zigzag; in that case our problem of minimum has no solution, and to such problems we could never apply the classical methods of Euler and Lagrange.

This is where the new freedom of modern mathematics, which allows so-called weak solutions, makes all the difference. We can regard our sailing problem as a control problem, in which the control u is the angle determining the boat's direction, so that u (in radians) lies between π and $-\pi$. Our new freedom then admits not only motions and controls $x(t)$, $u(t)$ derived via (1.1) from discontinuous controls $u(t)$, but also those derived from much more general so-called "chattering" controls. In our case, such a chattering can be obtained, for instance, by assigning to the two favorable values $u = \pm\alpha$ equal weights, or equal probabilities, $\frac{1}{2}$; this is the nearest we can get to describing a state of affairs in which u changes at each instant t from α to $-\alpha$ and back.

With the new freedom provided by chattering controls, the ideal solution becomes attainable, not only in this problem, but in many others, and once we know this, we are in a position to say once more: "Let X be a solution". This is the first big step in solving a problem.

§4. Maxwell's Problem

Our second example illustrates another aspect of our new freedom, which is, from the practical point of view, almost equally important. In my book [14, p. 287], this example is introduced in the form of an anecdote, by which an experimentalist thought to poke fun at mathematicians.

Two brothers, an engineer and a mathematics teacher, went into partnership to buy an old hunting lodge, at the top of a small mountain, and decided to build a road to it from the nearest railway station in the valley, and to convert it into a motel.

The mathematics teacher had once attended a course on the Calculus of Variations, so he studied his old notes and applied them to the problem of finding the road along which a station-wagon could drive in least time from the railway station to the lodge. Taking account of the fact that the station-wagon doesn't pull too well up a steep hill, he formulated an appropriate problem of the Calculus of Variations, and by working hard managed to solve it. He found that there was exactly one solution, given by a curved road winding its way all around the mountain, in and out of gorges, past waterfalls, and so on, and that the time it would take the station-wagon to drive along this road, if it were built, would be just 36 minutes. He also went to great trouble to prove that, if the road were built along any other rectifiable curve, leading from the railway station to the lodge, the driving time would be greater. Finally he made an estimate of the cost, including a couple of bridges, and he found that the road would cost two hundred thousand dollars to build, at the very least.

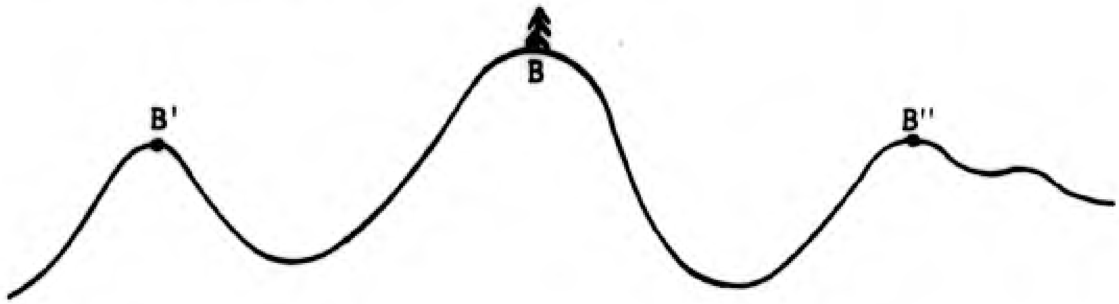
Meanwhile the engineer had proceeded routinely to map the usual zigzag road up. He found that, on this road, the station-wagon would take 37 minutes, and that the road would cost ninety thousand dollars to build.

All this was recounted by the experimentalist during a coffee break, and the mathematicians present were not slow to put the story in its true light. The problem of a road up a mountain, along which to reach the summit in least time, is a well known one, usually referred to as the problem of Maxwell. It is strongly affected by the freedom we allow ourselves as to what is accepted as a solution.

The problem of Maxwell dates from a period when people had not forgotten how to walk, and before exertion had been eliminated from sport by golf-cars, ski-lifts and television sets. It is usually formulated, not for a car, but for a pedestrian mountain-climber, or equivalently an old-fashioned ski-er, in the days when much of a ski-er's time was occupied in going up a steep hill, and it was even unsporting to remove the skis for this. To negotiate a steep upward slope on skis, one places the skis alternately at a certain optimal angle (see Figure 5). Similarly, to avoid slipping back, a pedestrian places his feet alternately at such an angle, and roads for motorists zigzag at this angle to the line of slope. All this corresponds to the art of tacking which we discussed in sailing, and

Figure 5. Left and right ski positions

the whole set-up is very similar. Maxwell's problem is to find a way up a mountain in least time, under these circumstances, from A in the valley, to B at the summit (see Figure 6).



As in our first problem, we suppress the analytical details, and the problem then differs from the shortest distance problem only in those parts of the terrain, where the greatest slope exceeds a certain constant, which is the most favorable slope for gaining altitude without slipping back.

Figure 6. The ski-er's version of Maxwell's problem

In the case of a convex hill, it was observed by Maxwell that there is a solution, which simply winds its way round the hill at the most favorable slope, until the terrain flattens down sufficiently for the best path to terminate along a geodesic to reach B. However, in general, such a procedure, if it is possible, may lead up the wrong mountain (see Figure 6).

Mountain-climers, ski-ers, and road-builders, have all come up with a second, more reliable, solution. We first draw the geodesic from A to B.

The ski-er simply follows this geodesic, except that, where it is too steep for normal progress, he uses the procedure described above (see Figure 5), of placing his skis alternately at an angle to one side, or to the other. The pedestrian proceeds similarly, while the road-builder has to content himself with an approximate solution, consisting of a zigzag road, which bears the same relation to the geodesic as the sailor's zigzag did to the straight line AB in our sailing problem. Thus ideally, these solutions should consist in following the geodesic, but with a direction which constantly alternates between the two most favorable ones for gaining altitude.

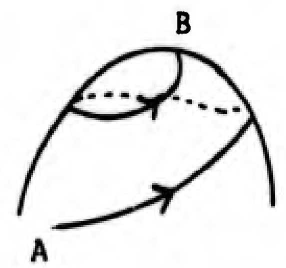


Figure 7. Maxwell's solution

The classical Calculus of Variations did not recognize this second solution. Therefore the mathematics teacher in our anecdote found only Maxwell's optimal route. There should have been a hidden solution, which our modern freedom would have recognized, along which the driving time would have been again exactly 36 minutes. The engineer's road was simply a workable approximation to this. The anecdote really shows that our new freedom is highly practical, even in those problems which apparently could be solved without it.

§5. An Explicit Example

I shall give one more example of a similar nature, but this time in its analytical form, without any background in real life. We ask for the minimum of the cost-integral

$$\int_0^1 (1+x^2)(1+(1-u^2)^2)\,dt \qquad (5.1)$$

for real-valued absolutely continuous functions $x(t)$, subject to the end-conditions

$$x(0) = x(1) = 0$$

and to the controlled differential equation

$$\dot{x} = u \quad .$$

Here u can take arbitrary real values, but if preferred we can write

$$u = \tan(\pi v/2) \quad ,$$

where v lies between -1 and 1.

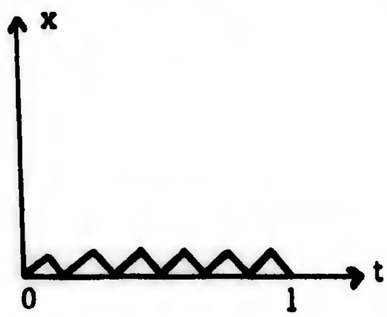


Figure 8. Graph of $x(t)$

The solution is very simple. On the broken line of Figure 8, we have alternately $u = 1$ and $u = -1$, on successive parts of a subdivision of the unit t-interval into an even number of subintervals, of equal lengths ϵ. For the corresponding function $x(t)$, which takes values between 0 and ϵ, the integrand never exceeds $1 + \epsilon^2$, and therefore the integral cannot exceed this either.

On the other hand, for every $x(t)$, the integrand is at least 1, and only takes this value when we have both $x = 0$ and $u = \pm 1$. It follows that the infimum of our integral is 1, and that this value is approached when $\epsilon \to 0$ in the broken line of our figure. The solution is not $x = 0$, with $\dot{x} = u = 0$, but with u alternating between 1 and -1 at every instant of time t.

DISCUSSION OF FIRST LECTURE

I(i) What possible relation can this new freedom have to reality? How could one conceive, for instance, of a sailing boat tacking all the time infinitesimally?

These questions indicate a very natural prejudice.

Because, in actual fact, no sailing boat can follow an infinitesimal zigzag, we find it difficult to conceive of its doing so. Yet we have no difficulty in conceiving of a sailing boat which follows a straight line, although no sailing boat ever constructed is capable of doing so. It does not require much sailing experience to realize that a boat will sway and roll, and even zigzag slightly, however much we may try to keep it on a straight course.

We have chosen examples from the unsophisticated world of sailing boats, ski-ers and mountain-climbers, and such examples do not bear too close an analysis. We have used the rather naive conventions of language, whereby persons walk, or climb, along continuous paths instead of by a series of discontinuous steps. We have completely ignored the finite size of a boat and the time lost by drifting while changing the positions of the sails. Similarly we have ignored the finite size of feet, skis, cars, and the width of roads. The conventions are of the same general nature as those by which a black pencil-smear on a piece of paper is referred to as a line, and a blob as a point. It is only by means of conventions such as these that anything in the real world can be expressed in mathematical terms.

In regard to conceptual difficulties, of the type referred to in the questions above, we tried to overcome these in the lecture, by regarding the infinitesimal zigzag as a kind of limiting situation, to be approximated by the more easily grasped case of a finite zigzag. The difficulty here is that secondary phenomena, ignored by our conventions, seriously interfere with such an approximation, except perhaps in the case of a very small boat, sailing, not on a river, but across the ocean. We can easily list a number of such phenomena, some of which we have already mentioned, for instance the time lost in shifting sails. In ordinary everyday life, these secondary phenomena cannot be ignored except in rather straightforward cases. This is why sailing, for instance, is an art, not a mathematical problem: in a race, the momentary shelter from the wind, provided by a few trees, may make it worthwhile to risk almost running aground.

However things are very different in the context of modern high precision physics and engineering, and this is more the sort of context to which our new freedom is intended to apply. Here at least, our "limiting situations", if we still think of them in this way, can be approximated to a very much higher degree. It takes much less time to flick a switch, and to alter in so doing the motion of a rocket out in space, even allowing for the time delay in a radar control, than to shift sails in a wind. If the flick of a switch is too slow, physicists have enormously faster techniques, and radar delay can be avoided by transmitting instructions slightly early, or else by automation. In this context, an infinitesimal zigzag need not even be thought of as a kind of limiting situation: we practically have one right there, whenever we turn on an alternating current.

We shall have more to say on these matters after the next lecture.

I(ii) <u>What precisely is the additional flexibility in a Moonshot, and how is the motion governed by a first order equation, such as</u> (1.1), <u>rather than by a second order equation involving acceleration?</u>

To answer the second part first, as it is most easily disposed of, there is no real difference between a second order equation and a first order one, when we use appropriate vector notations. In our case, for mathematical convenience, x denotes a point in phase space, so that it represents the position and velocity. (If the projectile is taken to be a rigid body, rather than a particle, other variables, such as spin velocities, are included in the symbol x.) In phase space, the differential equations are automatically of the first order. It is also worth noting that the target (for a soft landing) is given by positions <u>with zero velocity</u>, which lie on the surface of the Moon.

Passing to the first part of the question, we may recall that Jules Verne quite correctly described what might happen to a Moonshot in his day, as the result of a minute initial inaccuracy. The flexibility in a modern Moonshot, as compared with the sort of thing that Jules Verne or H. G. Wells had in mind, arises from the presence of u in the differential equation (1.1). This makes it possible to steer the projectile back onto the desired path (in phase space) if there is any such initial inaccuracy. However it does much more than this, for it ensures that there will be a much wider class of trajectories which terminate on the target, and that on these we can slow down our projectile so that it arrives with zero speed.

This additional flexibility does not depend on allowing $u(t)$ to be discontinuous, nor on allowing it to enjoy the new freedom, but if we do not allow this we exclude limiting situations which may be important in optimization problems.

SECOND LECTURE

§6. Rigidity and the Cost-Barrier

The new freedom, which we need in Optimal Control when we might otherwise be skirting the impossible, introduces into our problems another fundamental difference. This is connected with a phenomenon, which partly caused some of the difficulties of the classical Calculus of Variations, in problems involving constraints, a phenomenon known as "rigidity".

A standard example of rigidity may be found, for instance, in Carathéodory [8]. It concerns the constraint expressed by the differential equation

$$\dot{X} = \sqrt{1 + \dot{x}^2}. \tag{6.1}$$

Here our n-dimensional vector, which we generally denote by x, has been written instead (x, X); it is now a two-dimensional one, and x, X are now the components along the two axes. We shall generally use this notation when n is two, and then revert to x being a vector for general n. Our problem is thus in 3-space, when we include the time t as a variable; and we take the axis of X to be vertical, while those of t, x are to be thought of as horizontal. The relevant constraint can also be expressed by the control differential equations

$$\dot{x} = u, \quad \dot{X} = \sqrt{1 + u^2}, \tag{6.2}$$

where the control value varies on the real line.

Suppose we fix a point A of 3-space, and consider the curves C of the form (t, x(t), X(t)) through A, which satisfy the constraint given above (see Figure 9). We remark firstly that the differential equation determines X(t), as a function of t, when we know x(t), i.e. when we know the projection C^* of C in the (t, x) plane; thus C is determined by C^* (see Figure 9). Secondly we remark that the difference of X at the ends of C is, by an elementary formula, simply the length of C^*.

Now choose a particular C^* which reduces to a segment C_0^* through the projection of A. Then C will be a corresponding segment C_0 through A, we denote by B its second extremity.

Clearly C_0 is then the only curve C, subject to our differential equation, which joins A to B. To see this, note that for any such C, C^* would have to possess the same length as C_0^*, since this is the difference of the X-coordinates of A, B; and at the same time C^* would have to have the same ends as C_0^*; however, no second curve C^* joining the ends of the segment C_0^* can possess the same length as this segment.

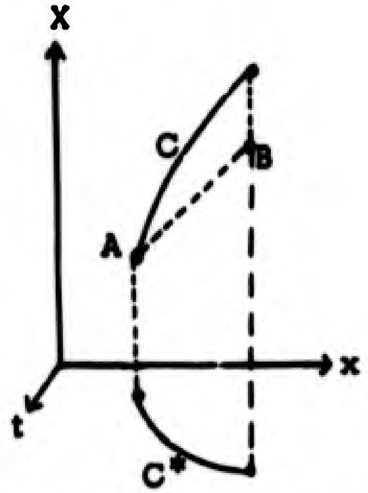


Figure 9. A typical path

Thus, if we formulate a problem of minimum for the curves through A, B which satisfy our differential equation above, we would be *ipso facto* seeking the minimum of a set comprising only one element. This is a trivial minimum, and we need no conditions whatever to find it. For this reason it becomes disturbing when we seek to prove that certain conditions are necessary.

Thus rigidity may cause inconvenience to a professional mathematician, who is trying to develop an algorithm, or a method, but I suspect that some people, who may have had teachers in mathematics who were hard task-masters, would view this as a kind of poetic justice. I therefore would like to draw attention to a related phenomenon, which might affect them in a more practical way.

Suppose that our problem of Optimal Control, in the form in which it was first formulated at the beginning of the previous lecture, were to concern the cost of putting a projectile on the Moon, with zero arrival-speed, by certain technical means, about which others know much more than I do. Suppose, in fact, that, by solving this problem, we had found that it was feasible to carry out such a program at a minimum cost of (say) a quarter of a billion dollars. We may well imagine that some congressman might want to question us on this matter of cost, and that he might say: "Look here, that is a lot of money, what could you do with half that ?"

Our answer might be: "For that sum, we could not actually put our projectile softly on the Moon; however we could bring it softly to a destination as close as you please to the Moon, missing it by less than a hair's breadth."

To illustrate this possibility, I shall modify slightly, not Carathéodory's example, but the last example considered in the previous lecture. This is the example in which we wrote down the explicit analytical details. We shall take the same cost-integral (5.1), but we consider it for trajectories involving an extra variable X, subject to

$$\dot{X} = x^2, \quad X(0) = X(1) = 0. \tag{6.3}$$

In the problem thus modified, only one controlled trajectory satisfies the end-conditions; this is the trajectory given by setting $x = X = u = 0$ for all t, and the cost-integral is then 2. This is again the phenomenon of rigidity. However, at the same time, we see that, when $x(t)$, $u(t)$ are the functions defining the zigzag trajectory produced in the previous lecture, and we complete the pair to a tripple by the adjunction of the appropriate $X(t)$ subject to $X(0) = 0$ and to the differential equation, but not to the end-condition at $t = 1$, we obtain, for our modified problem a trajectory whose second extremity is as close as we wish to the required terminal point, while the cost-integral for it approximates the much smaller value, unity, which we had in the previous version of the problem, as the ideal solution.

We may well imagine the astonishment of our friend the congressman, if he were answered as stated above. And indeed it is precisely the artificiality of a problem that admits of such an answer that should open our eyes to the clear and absolute necessity of replacing the whole setting by something more in keeping with the demands of common sense.

For these reasons we shall cease to dignify our original formulation by the name of the problem: henceforth, we refer to it merely as the pre-problem. The control-values and the trajectories, that are admitted in it, will be spoken of as conventional control-values and conventional trajectories. The pre-problem becomes a true problem only when it has been enlarged by our new freedom. This means that we retain the original end-conditions, but instead of conventional controls and trajectories we admit more general ones, that we term chattering controls and relaxed trajectories. (The names sliding controls, generalized, weak, relaxed, controls are also used, with corresponding names for the trajectories, and in differential games one speaks of "mixed strategies".)

Actually, the phenomenon we have been describing, is very similar in character to the non-existence of a solution, for in it we are really again skirting the impossible: we are trying to break a cost-barrier, when it can only be broken with the new freedom. There is a clear parallel in Sport. To climb Everest for the first time meant skirting the impossible in the first way: to climb it at all was long thought to be beyond the capacity of man, although in fact what it required was better organization, better planning, better techniques. To break a cost-barrier of a quarter of a billion dollars in our Moonshot would be more like running, for the first time, a four minute mile. This is a second way to skirt the impossible: anyone can run a mile in an hour (or even walk it for that matter); however, to run it in four minutes was, for a long time, believed to be impossible, and the dangers involved in fast running are brought to mind if we remember the Athenian runner from Salamis who died as he reached Athens with the news of victory. In actual fact, to run the first four minute mile required elaborate new techniques and training. Generally, it may be said similarly that, in skirting the impossible, in any of its forms, there is invariably a same basic need for greater freedom of action, provided by improved techniques or methods.

§7. The Relaxed Problem

I have tried, by these examples, to give an intuitive idea of the way in which the pre-problem has to be enlarged. Instead of a control $u(t)$ whose values lie in the dial,† or elementary figure, U, we allow more generally a control presenting at each t a sort of indeterminacy. This more general control, which we term a chattering control, is only given to a certain probability. At each instant of time, we specify, not a definite value $u(t)$, but a probability, i.e. a unit measure $v(t)$ on the whole figure U. In particular, if the whole probability is concentrated at a single value $u(t)$, the chattering control may be thought to reduce to a conventional one, namely to $u(t)$ itself. In our examples, $v(t)$ was divided equally between two

†By a dial we mean a segment, or circular arc, on which a pointer, or arrow, is attached, which can be made to single out any individual position. A finite set of such dials is equivalent, for our purposes, to the figure determined by the Cartesian product, with a corresponding pointer.

distinct values in U. Similarly, one can conceive of cases where it is represented by different weights attached to two values in U, or to a finite number of such values, or indeed to the whole of U.

The enlargement of the pre-problem lies in the admission of these more general interpretations of the notion of control. If $v(t)$ is a chattering control, we must still explain in what degree it determines a trajectory $x(t)$ and a corresponding cost. This can be done in complete analogy to what we described at the beginning of the previous lecture. We write $g(t, x, v)$, $f(t, x, v)$ for the mean value (as function of u in U, for fixed t, x) of $g(t, x, u)$, $f(t, x, u)$, where the mean value is taken with respect to the unit measure, or probability, v. In the case of a measure concentrated at a single u, this mean value coincides with the result of substituting the particular u for v in g or f; similarly, if v consists in attaching weights w_i to a finite number of values u_i, where $\sum w_i = 1$, the symbols $g(t, x, v)$, $f(t, x, v)$ will stand for the finite sums

$$(7.1) \qquad \sum w_i g(t, x, u_i), \quad \sum w_i f(t, x, u_i) .$$

In the general case, these sums have to be replaced by integrals in the unit measure v. With this interpretation of symbols, a relaxed trajectory $x(t)$ is related to a corresponding chattering control $v(t)$ by the control differential equation

$$(7.2) \qquad \dot{x}(t) = g(t, x(t), v(t))$$

or, as we prefer to write for brevity

$$\dot{x} = g(t, x, v). \tag{7.3}$$

Similarly, the cost-integral is now

$$\int f(t, x(t), v(t))\, dt, \tag{7.4}$$

or simply

$$\int f(t, x, v) dt. \tag{7.5}$$

For instance, in the example at the end of the previous lecture, where v consisted of equal probabilities $\frac{1}{2}$ at the values $u = \pm 1$, we find that

$$\begin{aligned} g(t, x, v) &= \tfrac{1}{2} g(t, x, -1) + \tfrac{1}{2} g(t, x, 1) = \tfrac{1}{2} \cdot (-1) + \tfrac{1}{2} \cdot (1) = 0, \\ f(t, x, v) &= \tfrac{1}{2} f(t, x, -1) + \tfrac{1}{2} f(t, x, 1) = \tfrac{1}{2} \cdot (1 + x^2) + \tfrac{1}{2} \cdot (1 + x^2). \end{aligned} \tag{7.6}$$

In this case the control differential equation (4) becomes $\dot{x} = 0$ so that $x(t) = \text{const.} = x(0) = 0$; the cost-integral then reduces to the integral of $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 1 = 1$, which is 1. In the general case the calculations are carried out similarly, and this will be understood in what follows.

Thus the new freedom leads us from (1.1) and (1.2) in the pre-problem to the entirely similar formulae (7.3) and (7.4) of the relaxed problem, the only change being the substitution of v for u. This means that the variable is in a space which is not U but the space of certain probabilities, or unit measures, over U. Any such v(t) can then be used just as well as a conventional u(t) to determine a relaxed trajectory x(t); in fact this is done by the same lopsided Carathéodory method of which I spoke in the first lecture. Once this has been done, we are in a position to evaluate the corresponding cost-integral.

From the formal point of view, the difference between the pre-problem and the corresponding relaxed version appears minimal: it lies in a more general interpretation of what we formerly called a dial, or system of dials, U. This causes no special difficulties. What has changed is that we have security: we need not fear so much to be skirting the impossible. In my book [14, p. 307] I prove in this respect a general existence theorem which should set our minds at rest. This means that we can once again attack problems in the time-honored way, by saying: "Let X be the solution." At the same time the new formulation eliminates the equally obnoxious possibility of a cost break-through: we can assure our congressman that, with chattering controls, there is no substantial saving in missing by a hair's breadth, so that we might just as well hit our objective fair and square.

I shall not repeat here the proofs of these basic results which set our minds at rest on matters that would otherwise by very ticklish. However I will indicate briefly how one goes about constructing such proofs, because these are ideas that permeate much of Optimal Control. The key to the method lies in the remark that our proofs would be greatly simplified if we could somehow remove the variable x from the functions g and f. This is really why, in the theory of differential equations, one makes lop-sided assumptions: the functions g and f are assumed to be rather smooth in x, but we need much less about the dependence on t, u. In my book [14, p. 300], the removal of the variable x is brought about by what I call a key lemma on approximations. This can be used in other contexts too; for instance it greatly simplifies stochastic control existence theory, where it avoids going into very elaborate spaces.

Of course there are problems in which no solution exists. Our existence theory can only assert that, with our new freedom, if there is at least one relaxed trajectory of the problem which satisfies our end conditions, then there is at least one for which the cost-integral assumes its smallest value. This is what we need to pass on to the central principle of the theory of Optimal Control, which is known as the Maximum Principle.

§8. Description of the Maximum Principle

The Maximum Principle is a necessary condition to be satisfied by any solution $x(t)$, $u(t)$ or $x(t)$, $v(t)$ of our problem. The principle apllies to the pre-problem, or to the relaxed problem. However, it becomes more powerful in the relaxed version, and the proof is then much more natural. Of course proofs keep getting simpler, so that we cannot assert that the proof of the relaxed version will always remain more natural, but this seems to be so at the moment. We shall not prove the principle here; a proof is of course given in my book [14, p. 308] and better proofs will doubtless be appearing. However, I shall try to describe the principle, and to explain how it arises.

Historically, the final form of the Maximum Principle is due to the Russian mathematician Pontrjagin and his associates [10]; it differs slightly from earlier formulations of necessary conditions by McShane and Hestenes [8], which were closer to the classical context of Euler equations and Weierstrass condition. In fact the principle can be regarded as the culmination of some 150 years of attempts to formulate properly the so-called multiplier rule.

There are two basic ideas.

One of these is the classical idea of Euler and Lagrange, of incorporating a problem with constraints into a wider problem without constraints. This is what leads to the so-called multipliers. The method is used in the elementary maxima and minima of the differential calculus, or equivalently, in equilibrium problems of Statics. In the latter, the multipliers are the reactions of the constraints, and the basic principle states that the given system must be in equilibrium, irrespective of the constraints, when we regard it as acted on by these reactions as well as by the external forces considered. We recall here the well known complication that the reactions of the constraints are, generally, not uniquely determined by the problem, for instance in the equilibrium of a table standing on more than three legs.

The other basic idea takes us even further back, perhaps to the first glimmerings of human intelligence. When we want to minimize a cost-integral of the type (1.2) or (7.5), our first natural reaction is to minimize at each instant of time, when the trajectory has reached the point t, $x(t)$, the integrand f in its dependence on the control value u or v. This may be termed the penny-pinching method. Unfortunately, the control values chosen in this way determine a trajectory via the differential equation (1.1) or (7.3), which may simply fail to have any relation to the desired target. Therefore one tries, not to minimize at each instant the function f, i.e. the cost, but to maximize a different quantity, which somehow represents the progress towards a goal, and which we may loosely term the instantaneous performance.

In the classical Calculus of Variations, Lagrange developed the multiplier rule for the so-called Lagrange problem, namely that of the minimum of an integral of the form

$$\int F(t, x, \dot{x})dt , \tag{8.1}$$

for functions $x(t)$ subject to prescribed end-conditions and to constraints expressed by differential equations of the form

$$G(t, x, \dot{x}) = 0 . \tag{8.2}$$

Lagrange widened the problem by writing

$$F^* = F + \sum \lambda G, \tag{8.3}$$

where the λ are functions of t called multipliers. He then considered, without regard to the constraints, the minimum of the integral

(8.4) $$\int F^*(t, x, \dot{x}, \lambda)dt,$$

and as I shall explain in the discussion, he was led to the equations

(8.5) $$\frac{d}{dt}(F^*_{\dot{x}}) - F^*_x) = 0 .$$

These are familiar under the name of Lagrange's equations in Mechanics, but in the Calculus of Variations one prefers to call all such equations Euler's. When everything is considered in (t, x)-space, the symmetry of the variables leads to a further equation for the component t, namely

(8.6) $$\frac{d}{dt}(F^* - \dot{x}F^*_{\dot{x}}) = F^*_t .$$

(Here $\dot{x}F^*_{\dot{x}}$ means the scalar product of the vectors $\dot{x}$ and $F^*_{\dot{x}}$). As I explain in several places in my book [14, p. 30], the expression $F^* - \dot{x}F^*_{\dot{x}}$ is what corresponds in the t-direction to the momentum $F^*_{\dot{x}}$. This expression is important, and keeps cropping up; when we change its sign, its value is that of the so-called Hamiltonian.

Although the original argument by which Lagrange thought to have proved the necessity of the equation (8.5) has a certain plausibility, its conclusion is false, on account of the phenomenon of rigidity; in fact, in a rigid case, the minimum is trivial for every choice of F, and therefore F cannot then enter into any necessary condition. For this reason, in the modern version of multipliers, there is an additional "self-multiplier" λ_0 and F^* is then

(8.7) $$\lambda_0 F + \sum \lambda G.$$

We then add the stipulation that $\lambda_0 \geq 0$ and that the system of multipliers λ_0, λ is not identically 0.

In our case, we are concerned with a problem of Optimal Control, expressed as that of the minimum of (1.2) or (7.5), subject to (1.1) or (7.3). It is usual then to introduce some minor changes in the notation, so as to increase symmetry. We write y for the vector with the components λ, but in place of λ_0 we now write $-y_0$. Further F is now f, G will be taken to be the difference $\dot{x} - g$, and all products of vectors will be understood to mean scalar products. Thus

(8.8) $$F^* = -y_0 f + y \cdot (\dot{x} - g), \quad F^*_{\dot{x}} = y, \quad F^* - \dot{x} F^*_{\dot{x}} = -(y_0 f + yg) .$$

In accordance with the remark made higher up, we call the expression $y_0 f + yg$ Hamiltonian, and denote it by $\mathcal{H}$, or in full by $\mathcal{H}(t, x, u, y, y_0)$. We shall need to calculate the following partial derivatives

(8.9) $$\mathcal{H}_y = g, \quad \mathcal{H}_x = y_0 f_x + y g_x = -F^*_x, \quad \mathcal{H}_t = y_0 f_t + y g_t = -F^*_t ,$$

so that the constraint equation (1.1) or (7.3), and the Euler equation (8.5) take, respectively, the forms

(8.10) $$\dot{x} = \mathcal{H}_y , \quad \dot{y} = -\mathcal{H}_x ,$$

while (8.6) becomes $(d/dt)\mathcal{H} = \mathcal{H}_t$. This last may be written out in full

(8.11) $$\mathcal{H}_t + \mathcal{H}_{y_0}\dot{y}_0 + \mathcal{H}_y \dot{y} + \mathcal{H}_x \dot{x} + \mathcal{H}_u \dot{u} = \mathcal{H}_t ,$$

which reduces by (8.10) to

(8.12) $$\mathcal{H}_{y_0} \cdot \dot{y}_0 + \mathcal{H}_u \cdot \dot{u} = 0;$$

however, in the classical framework, the vector x corresponds to what is now the vector (x, u), so that we would have equations similar to (8.5) in u, which then clearly reduce to $F^*_u = 0$, because $\dot{u}$ does not appear in F^*; since F^* and $-\mathcal{H}$ only differ in the term $y\dot{x}$ which does not contain u, this may be written

(8.13) $$\mathcal{H}_u = 0,$$

so that (8.12) now becomes $f \cdot \dot{y}_0 = 0$, which means, in practice, that $\dot{y}_0 = 0$. We shall therefore add to (8.10) the requirement that y_0 be a non-positive constant. This is, however, the only place where (8.13) will be used; instead we shall have a much stronger condition that I shall formulate in a moment.

By analogy with the Hamiltonian theory in Mechanics and in the Calculus of Variations, we term (8.10) the canonical Euler equations, and y the conjugate vector to x; the scalar $\mathcal{H}$ is similarly conjugate to t. The classical theory also formulated certain so-called transversality conditions, to be satisfied at the ends of our trajectories, when these ends are allowed to vary on smooth loci. In Optimal Control the most important case, to which I shall here confine myself, is that in which the trajectories start from a fixed point and end on a locus of x-space, termed the target; the initial time, or the final time, is also fixed, in fact, usually, the final time is $t = 0$, and in that case the initial time is usually quite unrestricted on the negative real axis. The transversality conditions then state that, at the final time $t = 0$, the conjugate vector y (which corresponds to a momentum in Mechanics) is orthogonal to the target, and that at the initial time, which is unrestricted, the Hamiltonian $\mathcal{H}$ vanishes.

To the classical conditions, consisting of the Euler conditions together with the transversality conditions, a further condition, of a rather different kind, was added by Weierstrass [12]. We shall not go into details here. All these conditions involve, when properly formulated, the same system of multipliers, and therefore, in our case, the same system of y, y_0. However this last fact was not realized until quite late, and the first complete proof in this sense was given by McShane, some thirty years ago [8]. For this reason, McShane can be considered to be the real founder of the necessary conditions now combined under the name of the Maximum Principle. Nevertheless this principle involves an important point not found in the McShane formulation. We said earlier that the Maximum Principle is based on two ideas. So far we have used only one of them, the idea of multipliers.

In making the transition to the second basic idea, we must remark on the fundamental difference in character between the space coordinates, which are the components of the vector x, and the control coordinates, which are those of the vector u. The vector x defines actual positions on our trajectories; it can only vary continuously, since it satisfies a differential equation such as (1.3). The vector u, or more generally the point v in the space of unit measures over U, can be varied much more freely, and has no intrinsic significance, in so far as we could easily control it by means of a second system of dials, as in the remote control of a television set, or a garage door. It is therefore natural to treat u quite differently from x.

At any particular instant t, the canonical point $x(t)$, $y(t)$ and the constant non-positive value of y_0 will be determined by continuity from their previous neighboring values; however, the value of $u(t)$ or $v(t)$ we are free to choose, since it is not bound by the past. We desire to make this choice so as to maximize some instantaneous performance; the Maximum Principle asserts that we must do so, and that the relevant instantaneous performance is the Hamiltonian

$$\mathcal{H}(t, x(t), u, y(t), y_0) = y_0 f + yg, \tag{8.14}$$

regarded as a function of u; alternatively, if we allow chattering controls, we shall have the corresponding function of v. In this part of the principle, our new freedom plays a fundamental part; it is true that in practice no one can make entirely independent choices of the values of a function $u(t)$ for all the different values of t; however the use of chattering controls here simulates sufficiently this arbitrariness. For instance, in the example at the end of the previous lecture, we could not have chosen u to be alternately 1 and -1; we did the next best thing, which was to introduce a probability $\frac{1}{2}$ for each of these values. Thus our new freedom really plays here a fundamental part, and it is not surprising that it also makes possible much simplified proofs.

As already stated, the proof of the principle may be found in by book [14, p. 319]. Here we shall merely formulate its precise statement, which summarizes the discussion above.

THE MAXIMUM PRINCIPLE. Let $x(t)$, $u(t)$, or $x(t)$, $v(t)$, minimize (1.2) subject to (1.1) or (7.5) subject to (7.3) and subject to end-conditions of the type described above. Then there exist $y(t)$ and a non-positive constant y_0, such that the Euler equations (8.10) are satisfied, and also the transversality conditions described above, and such that, further, the Hamiltonian (8.14) as function of u, or its extension as function of v, assumes its maximum for $u = u(t)$, or for $v = v(t)$.

DISCUSSION OF SECOND LECTURE

II(i). <u>How was Lagrange wrong in deriving</u> (8.5), <u>and how does it imply</u> (8.6)?

As to the latter, we should say more precisely that (8.2) and (8.5) together imply (8.6). In fact

$$\frac{d}{dt}(F^* - \dot{x}F^*_{\dot{x}}) - F^*_t = \dot{\lambda}F^*_\lambda + \dot{x}F^*_x + (F^*_{\dot{x}})\frac{d}{dt}\dot{x} - \frac{d}{dt}(\dot{x}F^*_{\dot{x}})$$

$$= \dot{\lambda}G + \dot{x}F^*_x - \dot{x}\frac{d}{dt}F^*_{\dot{x}} = \dot{\lambda}G + \dot{x}(F^*_x - \frac{d}{dt}F^*_{\dot{x}}),$$

and this vanishes when (8.2) and (8.5) hold. In mechanics $F^*_{\dot{x}}$ is the "momentum in x" and (8.5) states that its rate of change is F^*_x; (8.6) is a similar statement for the "momentum in t" given by $F^* - \dot{x}F^*_{\dot{x}}$. Unfortunately, in the calculus of variations these equations are incorrect.

To see this by the argument sketched in the lecture, let m be the number of (scalar) equations (8.2), so that m is less than the dimension of x-space. We suppose x(t) to be a rigid solution of (8.2). Corresponding to the m equations (8.2), the expressions $G_x - (d/dt)G_{\dot{x}}$ then become m vector-valued functions b(t), so that if a(t) is similarly formed from $F_x - (d/dt)F_{\dot{x}}$, (8.5) asserts that $a(t) + \sum \lambda(t)b(t) = 0$. This means that the vector a(t) lies in the at most m-dimensional subspace of x-space, determined by the vectors b(t). Therefore if we choose a(t) to be never in this subspace, (8.5) cannot be satisfied; an appropriate F is obtained by choosing $F(t, x, \dot{x}) = a(t)x$. Yet in this last case $x(t)$ certainly provides a minimum of (8.1) subject to (8.2), since it is the only solution of (8.2) with the given ends.

As to how the error occurred, part of Lagrange's argument is routine and unexceptionable. This is the part that could have established (for smooth enough F, G) the following proposition (X): *Suppose that* (8.4) *assumes its minimum, for a sufficiently smooth pair* $x(t)$, $\lambda(t)$, *in the class of analogous pairs with the same end-values on the given interval* T; *then* (8.2) *and* (8.5) *must hold*. We need for this a standard lemma of Euler: in order that a smooth vector-valued function $a^*(t)$ vanish identically in T, it is sufficient that $\int_T a^*(t)\xi(t)dt = 0$ for every smooth vector-valued $\xi(t)$, which vanishes at the ends of T. (A little reflexion shows that the condition that $\xi(t)$ vanishes at the ends can be removed without affecting the truth of the lemma, in which case its conclusion is made evident by choosing $\xi(t) = a^*(t)$.) To derive (X), it is therefore enough to show that its premises imply

$$\text{(II(i).1)} \qquad \int_T \{\xi(t)(F_x^* - \frac{d}{dt} F_{\dot{x}}^*) + \sum \mu(t)G\}dt = 0,$$

for all smooth choices of a vector $\xi(t)$ and of scalars $\mu(t)$, which vanish at the ends of T.

For this purpose, consider for real α the functions $x(t) + \alpha\xi(t)$, $\lambda(t) + \alpha\mu(t)$ which have the same end-values as $x(t)$, $\lambda(t)$. In place of (8.4), our integral is then

$$I(\alpha) = \int_T F^*(t, x(t) + \alpha\xi(t), \dot{x}(t) + \alpha\dot{\xi}(t), \lambda(t) + \alpha\mu(t))dt.$$

The hypotheses of (X) clearly imply that $I(\alpha)$ attains its minimum for $\alpha = 0$. Hence

$$0 = I'(0) = \int_T (\xi(t)F^*_x + \dot{\xi}(t)F^*_{\dot{x}} + \mu(t)F^*_\lambda)dt,$$

and to complete the proof of X we need only remark that this last expression coincides with the left hand side of (II(i).1), since we have, by integration by parts

$$\int_T \dot{\xi}(t)F^*_{\dot{x}}dt = -\int_T \xi(t)\frac{d}{dt}(F^*_{\dot{x}})dt.$$

Unfortunately, the premises of (X) are very different from the hypothesis on which the proof of (8.5) should, if possible, have been based, though the appearance of (8.2) in the conclusion of (X) may well confuse the reader, unless he is a logician: in fact, if he thinks long enough, he may find himself repeating Lagrange's mistakes, which many amateurs in the calculus of variations still find themselves doing. The rather chastening cure is a reference to the pityless critical discussion in the old book of Bolza [2].

II(ii). Is is really within the realm of possibilities, in a soft Moonshot, that a cost-estimate might be out by a factor 2, because of a restriction to the pre-problem?

In present day outer space rocketry, the basic differential equations (1.1) are linear in u, so that all such complications are excluded. On the other hand, the propulsion systems are still very clumsy, and much research is being devoted to improving them. Therefore it is not unthinkable that in the future more complex forms of (1.1) will be relevant. (In that case I am sure that the engineers would soon spot any such factor 2, without its reaching the cost-estimate stage.) On the other hand, there are other areas of control theory, particularly those which border on the theory of games and on war strategies and tactics, where the considerations I have been presenting are more immediately pertinent. Since it is simpler to illustrate the state of affairs with reference to a Moonshot, I shall suppose, for the sake of argument, that we are in sometime in the future, that the basic control equations (1.1) are non-linear in u, and that we do have a theoretical discrepancy, by a factor 2, between the cost-estimates for the pre-problem and the relaxed problem.

In that case, our question is still not fully answered, since these cost-estimates should refer, not to idealized mathematical controlled trajectories, but to those actually obtainable in practice, by operating concrete control mechanisms. In other words, they should be engineers' costs, not mathematicians' cost-integrals, and the question is then how closely can these approximate to one another. (We are concerned, of course, only with the cost of transfer to the Moon of the proposed projectile, after all equipment has been set up and tested.)

In this basic question of approximation, we are back where we were after the first lecture: in principle, it is like asking whether a sailing boat can follow, to a sufficient approximation (i. e. without affecting too much the time taken), the course corresponding to an infinitesimal zigzag. In this respect, existing control mechanisms for rockets are no more designed for rapid back and forth switching than sails are. Therefore a feasible physical approximation to a mathematical relaxed trajectory might well give rise to a cost-integral several times larger, leaving no saving at all as compared with the conventional solution of the pre-problem. In practice, however, this is not at all what would happen, and this shows, I think, that the basic question is not one of mathematics anymore, but one of engineering.

What would happen is simply this: if anything could be gained by redesigning control mechanisms for ultra-rapid back and forth switching, the engineers would at once set to work and construct such mechanisms, just as one could produce a mechanical device on a piano, for executing certain trills and tremolos (though God forbid that we should thus ruin our beautiful old instruments!)

I am not competent to pass judgement on this question of approximation, which must be left to the engineers. However, I expect we would all guess that the approximation could really be reasonably close. This would thus mean an affirmative answer to the question originally stated.

II(iii). How does transversality arise, and what does it mean to Optimal Control?

We may answer the first part in terms of the shortest distance problem between two non-intersecting and non-concentric circles in the plane; this is a variational problem in which we seek a minimum for the length of admissible curves joining the two circles. As may be easily verified (see [14], p. 7), the solutions of the Euler equation are the straight lines; this corresponds to the obvious fact that we need only consider those admissible curves which reduce to single segments joining our two circles. However, since there are still many such segments, the Euler equation does not suffice to solve our problem. We need a further condition, obtained by noting from elementary geometry that our shortest segment must be orthogonal to both circles (see Fig. 10); this means that it must lie on the line joining the two centers.

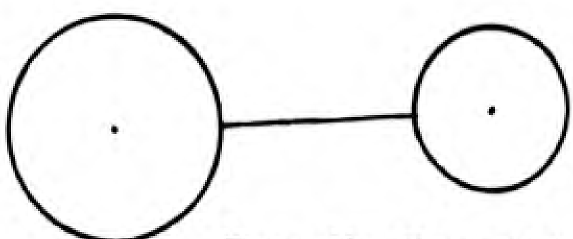


Figure 10. Shortest distance

In more general variational problems, this orthogonality has to be replaced by what we term transversality. For instance, if we modify the above by a change of scale on only one of the two axes in the plane, right angles are not preserved. Our circles become appropriate ellipses, and the length-integral is replaced by the integral of a function F of the form of the square root of a quadratic in $\dot{x}$ with constant coefficients. The ellipses are, of course, no longer orthogonal to the minimizing segment joining them, which is obtained by our change of scale transformation from the shortest segment joining the circles. In other words, they are not orthogonal to the velocity-vector $\dot{x}$ along this minimizing segment. However, it is remarkable that they are orthogonal instead to the so-called momentum-vector $F_{\dot{x}}$, and this is the assertion of transversality.

In problems without constraints, transversality provides a second necessary condition, independent of the condition expressed by the Euler equations of the type (8.5), (8.6), and it can be proved by an argument of the kind used in our discussion of II(i). However, we have seen that such an argument no longer applies to problems with constraints, so that it has no validity in Optimal Control. In addition, it is important to observe that transversality is no longer a condition independent of the Euler equation: this is because both conditions now involve the multipliers λ together with the self-multiplier λ_0, and these multipliers may well be, to a large extent, indeterminate, just like the reactions of constraints in a problem of Statics. In combining the Euler equation with transversality, an essential part of the assertion is the existence of a system of multipliers which satisfy both conditions simultaneously. This is one of the reasons why the Maximum Principle, which incorporates both the Euler equation and transversality, cannot be established by trivial considerations, such as those in our discussion of II(i).

In this connection, we wish to draw attention to one rather strange, and really very remarkable fact. We have stressed here the importance of our new freedom in problems of Optimal Control, and in particular the need to allow discontinuities. We have also stressed the indeterminacy of the multipliers, and therefore, in Optimal Control, that of the momentum vector $(y, -\mathcal{H})$. It would be natural to expect a close connection between these two phenomena. The question whether nature is continuous or not is undecidable in modern physics in general, and in this sense our freedom only affects mathamatical convenience. However, there is one case where discontinuity must be accepted in nature, and that is the case of indeterminacy. If a table stands on four legs and two are shortened, the reactions of the constraints vary in a discontinuous manner, depending on which is shortened first.

Thus there seems to be a strong reason for suspecting that it is just because of the indeterminacy of the momentum that we need discontinuity in Optimal Control theory. Very likely this is indeed the case, and yet, if we look at the Euler equations, these state that, in addition to the constraint equation, we have $\dot{y} = -\mathcal{H}_x$, $-\dot{\mathcal{H}} = -\mathcal{H}_t$, so that the momentum-vector $(y, -\mathcal{H})$ is certainly continuous, on any trajectory which provides a solution to our problem.

THIRD LECTURE

§9. Additional Comments.

This third and final lecture of the introductory series can be given either before or after the more specialized lecture, in which I adapt the Carathéodory theory to the complete solution of Optimal Control problems.[†]

As already explained, the new freedom, which consists in introducing chattering controls, plays an important part in the existence theory and in the proof of the Maximum Principle. However, from the formal point of view, it causes only a small change in the statement of the problem, the substitution of v for u. This means that the tangible control values u on our dials, which represent something very concrete, are replaced by values v in a much less tangible space of measures. The same procedure can be carried further, and we can simply think of v as being a point in some very abstract space, or just as some sort of a label to distinguish a function of (t, x) from another. For instance, some writers in Control let u vary from the outset in a Hilbert space, or a still more general one, of infinite dimensions. This might seem rather far-fetched, but for the very basic role played by our new freedom.

The examples I have given show that chattering controls do occur in the solution of quite simple problems, and that they can be suitably approximated by conventional controls, which give rise to zigzag trajectories, i.e. by what engineers term bang-bang controls. In fact chattering controls are simply limiting cases of these bang-bang affairs.

[†]Here the specialized lecture follows after this third lecture and discussion.

Of course there are also many problems in which these limiting cases are not needed as actual solutions. This is partly because control-mechanisms are designed with an eye on simplicity of operation, just as car-manufacturers design cars so as to avoid a speed-wobble. However even in problems in which the functions f, g are linear functions of u, there are cases where one needs to include chattering controls in order to obtain a solution.

When chattering controls are not needed for the actual solution of a particular problem, we should think of them as forming part of the general reserve of troups and back-room planners that make it possible for the front-line troups to attack successfully. Without them there would be neither the security of existence theorems, nor the planning represented by the Maximum Principle. Similarly, in building a house, we do not use the foundations to live, but the house might blow away without them.

Before I leave the topic of existence theorems and the like, I would like to say that this replacing of u by a unit measure, or by a mere label which varies in some very abstract space, whereas u itself can be touched, might well cause a practical person to wonder how one can really control a differential equation in this way, not just in the examples that I have given, but in the very general and very abstract situation in which I seem to be placing myself. This would indeed be a serious difficulty, but for a rather beautiful theorem of McShane and Warfield [9], which I call in my book [14, p. 292] the "half-way map" theorem. It is just in such abstract situations that mathematicians seem to find some of the nicest theorems.

I should also say that what I have called the new freedom is needed not only in Control problems, but also in more general problems such as those of Game theory. One can regard a problem of Control as a Game theory problem with only one player. These things were studied extensively by von Neumann in the last war, but there is still much work being done, for instance at the MRC. In addition the recent MRC Technical Summary Report #895 by Drs. Becker and Mandrekar [1] deals with corresponding questions of Stochastic Control.

However existence theorems and necessary conditions are by no means all we may need to solve problems of Optimal Control. If we know that a solution exists, and that only one controlled trajectory satisfies the Maximum Principle, and leads from the appropriate point to the target, then, of course, this is the solution. However, this would depend on some sort of uniqueness theorem, and such theorems are hard to come by. In fact, in the classical Calculus of Variations, no such uniqueness is given in any book except mine [14], and there it is no easy matter. This uniqueness theory was developed by Carathéodory [3], but he did not include it in his book.

What we need to confirm solutions of problems are sufficient conditions. Indeed, in my book [14], the uniqueness theorems are based on sufficient conditions, and in any case they apply only to the classical Calculus of Variations, not to Optimal Control. Sufficient conditions are the topic of the specialized lecture to which I referred to at the beginning of this lecture. Here I shall treat only quite simple examples, which may help to illustrate what is involved.

As in the specialized lecture, I shall confine myself to autonomous time-optimal problems, i.e. to the case in which f is the constant unity and g is independent of the variable t; thus g has the form $g(x, u)$. It is customary to write H for yg , and to term this the Hamiltonian in place of $\mathcal{H}$; this amounts to subtracting the constant y_0 . In the specialized lecture I consider only the case in which this last constant is different from 0 ; however, this restriction will not be needed here. The question of relaxing this restriction in the general case is a difficult one, but some light may be thrown on it by our examples.

When we look for sufficient conditions, we do not really need existence theorems; nor do we need necessary conditions. What we do need is to be able to guess right. Our task is then to justify our guess. Thus, in a sense, the search for sufficient conditions can be carried out at a more elementary level. If we use the Maximum Principle, or any other convenient rule, to form a guess as to what the solutions might be, we need not justify this use; it would be equally permissible to consult some witch-doctor, or some fake medium, provided that we subsequently prove our guess to be correct. Actually we shall use the Maximum Principle here, although one must look elsewhere for the luxury of its proof.

Now the Maximum Principle is not a sufficient condition; it would be extremely naive to suppose that it is, since it involves, apart from the transversality at the ends, only local conditions, whereas the minimum we seek is in the large. In fact, in the classical Calculus of Variations, examples abound in which such local conditions fail to provide a minimum in the large. For instance, on the surface of a sphere, a great circle arc which exceeds a semi-circle satisfies all desirable local conditions, without providing the shortest path joining its ends on the sphere. In the book of Pontrjagin and

and his associates [10], examples are treated in which certain controlled trajectories are shown to satisfy the Maximum Principle, and it is then assumed, without further justification, that these are the solutions. This is just what we must not do. Here such trajectories will only be regarded as suspected solutions, or guesses. When we have obtained them we can forget how we did so, and merely concentrate on proving that they solve our problem: for this we can forget entirely about Hamiltonians and conjugate variables, which will not here enter into this crucial part of our discussion. (This is not, however, the case in the corresponding discussion given in the specialized lecture.)

§10. Two Time-Optimal Problems.

We are now concerned in determining, if possible, a controlled trajectory $x(t)$, $u(t)$ subject to the differential equation

$$\dot{x} = g(x, u), \tag{10.1}$$

such that the moving point $x(t)$ has a given initial position and reaches, in least time, a given target, consisting of a sufficiently smooth set of points. Strictly this is still only the pre-problem, but we shall ignore for simplicity any other than conventional solutions. The Maximum Principle then asserts (or, for our purposes we need only that it gests) that, for any such optimal pair $x(t)$, $u(t)$, there exists at least one nowhere vanishing conjugate vector-valued function $y(t)$, such that

$$
(10.2)\begin{cases}
\text{(i)} & \text{for the arguments } t, x(t), u(t), y(t), \text{ our new Hamiltonian } H \\
& \text{regarded as function of } t \text{ is equal to a non-negative constant,} \\
& \text{and we have the canonical Euler equations} \\
& \qquad \dot{x} = H_y, \qquad \dot{y} = -H_x; \\
\text{(ii)} & \text{at the terminal time, which we take to be } t = 0, \text{ the vector } y(t) \\
& \text{is normal to the target, and so directed that } y\dot{x} \geq 0; \\
\text{(iii)} & \text{for each fixed } t \text{ and for the values } x(t), y(t) \text{ of } x, y \text{ the quantity} \\
& H, \text{ considered as function of } u, \text{ assumes its maximum when} \\
& u = u(t).
\end{cases}
$$

To see that this is all contained in the general form of the Maximum Principle, we choose, in accordance with the latter, $y(t), y_0$. On our controlled canonical trajectory $x(t)$, $u(t)$, $y(t)$, the old Hamiltonian $\mathcal{H}$ satisfies, as we saw above, $(d/dt)\mathcal{H} = \mathcal{H}_t$, and here the right hand side clearly vanishes; thus $\mathcal{H}$ is constant along the trajectory; its value is thus 0 , since it vanishes at the initial time by transversality. Hence, along our controlled canonical trajectory, the new Hamiltonian $H = \mathcal{H} - y_0$ equals the non-negative constant $-y_0$. In particular, at $t = 0$, we have $y\dot{x} = yg = H \geq 0$. Further $y(t)$ can never vanish, since we would then have also $y_0 = \mathcal{H} - yg = 0$, in which case the pair (y_0, y) would vanish, contrary to the Maximum Principle. The other assertions are as before.

We shall consider two well known and very simple examples. They are among those treated, for instance, in the book of Pontrjagin and his associates [10, pp. 23-35], in the incomplete manner I referred to above. Full details will be found in my book [14, p. 233]. In both examples, the values of the control u will be real and restricted to $-1 \leq u \leq 1$, while the point x will lie in the plane and we therefore write (x, X) in place of x. Both problems may be regarded as controlled acceleration problems on the x-axis, for a particle whose initial position and velocity on this axis are given. The acceleration $\ddot{x}$ is u in the first problem, and $u - x$ in the second. In the first problem, which is that of a soft landing, we seek to bring the particle to rest at the origin; in the second, to reduce its energy

$$\frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2 \tag{10.3}$$

to a given level, or in particular, to 0. In both cases, this is to be done in least time. In both problems, we write X for the velocity $\dot{x}$.

The trajectories in phase-space, i.e. in the (x, X)-plane, are given by the following differential equations. In the soft landing problem, or problem A,

$$\dot{x} = X, \qquad \dot{X} = u; \tag{10.4}$$

in the second problem, problem B,

$$\dot{x} = X, \qquad \dot{X} = u - x. \tag{10.5}$$

Problem B concerns the controlled slowing of an oscillator.

Our vector-valued function g is thus (X, u) in problem A, $(X, u-x)$ in problem B. Hence, writing (y, Y) for the conjugate vector, we find that

$$H = yX + Yu \quad \text{in (A)}, \qquad H = yX + Y(u - x) \quad \text{in (B)}. \tag{10.6}$$

The Euler equations (8.10) therefore give

$$\begin{aligned} \dot{y} &= 0 \quad, & \dot{Y} &= -y \quad \text{in (A)}, \\ \dot{y} &= Y \quad, & \dot{Y} &= -y \quad \text{in (B)}. \end{aligned} \tag{10.7}$$

Thus in (A) $y = \text{const.}$, $Y = -yt + \text{const.}$; while in (B) we find that

$y + iY = a\cos(t-t_0) + i\,a\sin(t-t_0)$, where $a > 0$ and t_0 are constants. The control u for which H is maximal is given in both cases by

(10.8) $$u = \operatorname{sgn}(Y) = \pm 1 .$$

Thus the suspected optimal control changes sign only once in (A), since Y is a linear function; while in (B) it changes back and forth when $t-t_0$ is increased by π. On the intervals of constancy of u, thus arising, the trajectory arc is given by integrating the constraint equations. In (A) they are $\dot{x} = X$, $\dot{X} = u$, so that we obtain parabolic arcs

(10.9) $$x = \tfrac{1}{2} u\tau^2 + \text{const.}, \quad X = u\tau ,$$

where $u = \pm 1$ and $\tau = t +$ const. ; in (B) we obtain circular arcs of center $(u, 0)$; this is seen by combining the constraint equations in the form

(10.10) $$\dot{z} = -iz , \quad \text{where } z = x - u + iX ,$$

so that their solution is

(10.11) $$z = r\,e^{-i(t-\gamma)},$$

where $r > 0$ and γ are constant. We shall have in both problems two systems of such arcs, corresponding to the values ± 1 of the control u, and we pass from one to the other when u changes sign.

In (A) the arcs lie on the two systems of parabolas

(10.12) $$\tfrac{1}{2} X^2 - ux = \text{const.} \quad (u = \pm 1) ;$$

we speak of an upward parabola when $u = 1$, since X increases with . ; while, for $u = -1$, it will be a downward one. Two of our parabolic arcs constitute in this case smooth suspected optimal trajectories. They are the ones which terminate at the origin (for $t = 0$), and they are given by

(10.13) $$x = \tfrac{1}{2} ut^2 , \quad X = ut \quad (t \le 0) ,$$

where $u = \pm 1$; we denote them by α and β (see Fig 11). Clearly $\alpha + \beta$ divides the (x, X)-plane in two domains, and clearly also, any initial point above $\alpha + \beta$ can be joined to β by just one downward parabola; similarly, any initial point below $\alpha + \beta$ can be joined to α by just one upward parabola. It follows that, for each position of an initial point P in the plane, there is just one suspected optimal trajectory from P which terminates at the origin for $t = 0$. It consists, for P above $\alpha + \beta$, of a downward arc leading to a point Q of β , together with the final portion $Q0$ of β; for P below $\alpha + \beta$, it consists of an upward arc leading to a point Q of α together with a final portion of α . It is easy to calculate explicitly the coordinates of Q for given P , and hence to find the time taken to reach the origin along the suspected optimal trajectory from P . In this way, we find that the suspected least time from $P = (x, X)$ to the origin has the value

$$(10.14) \quad \begin{cases} T(x, X) = X + \sqrt{2X^2 + 4x} & \text{for } P \text{ above } \alpha + \beta , \\ T(x, X) = -X + \sqrt{2X^2 - 4x} & \text{for } P \text{ below } \alpha + \beta . \end{cases}$$

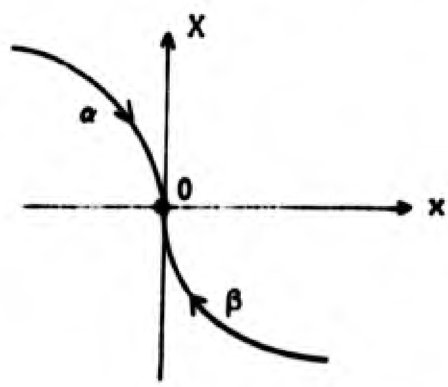


Figure 11. The smooth optimal trajectories

All this is done in detail in my book [14, pp. 235-239], and except for the final evaluation of the suspected least time, it is explained and beautifully illustrated in the book of Pontrjagin and his associates [10].

The preceding sentence applies also to the more elaborate calculations which arise in (B), and we shall therefore sketch these as briefly as we can. Generally speaking, the proper fitting together of the relevant smooth arcs of our suspected optimal trajectories is of the same nature as certain questions in the study of so-called generalized "spline functions", which are occupying an active group of MRC mathematicians at the present time. In our case, the discussion can be based on considerations of elementary plane geometry, and therefore the diagrams in the book of Pontrjagin and others [10] are particularly helpful. It is convenient to follow our suspected optimal trajectories backwards from the target, which consists, according to our hypotheses, of a circle

(10.15) $$x^2 + X^2 = R^2 .$$

This means that the terminal position reached by the moving point on our trajectory is that of a point Q given by

(10.16) $$x + iX = Re^{i\alpha} .$$

For simplicity we suppose $R \neq 0$, although our conclusions extend without difficulty to the excluded case. The transversality then requires the vector (y, Y) to be directed along the inward radial direction at Q, at any rate unless x is tangent there to the target. (Since our trajectory arcs lie on circles of center $(u, 0)$, this last can only occur when Q is on the x-axis.) Hence by setting $t = 0$ in our previous expression for $y + iY$, we find that α is the angle t_0, so that

(10.17) $$u(t) = \text{sgn}\,(\sin(t - \alpha)) .$$

This expression for $u(t)$ remains valid in the excluded cases $\alpha = 0$, and $\alpha = -\pi$. It follows that, for $0 \leq \alpha < \pi$, $u(t) = -1$ for small negative t , while in the complementary interval of α it is 1 . Thus the arcs on which the terminal point is on the upper half of the target (including the point at which $\alpha = 0$) lie on circles of center $(-1,0)$, while those which end on the lower half lie on circles of center $(1,0)$. By proceeding backwards from Q to the successive points at which $u(t)$ changes sign, and by using some elementary properties of triangles and circular arcs, we find the following state of affairs (see Fig. 12): Let Γ denote the disc enclosed by our target, and let Γ' and Γ'' denote, respectively, loci in the second and fourth quadrant, consisting of the upper halves of unit circles of centers at the points $-R-1$, $-R-3, \ldots$, and of lower halves of unit circles of centers at the points $R+1$, $R+3, \ldots$,

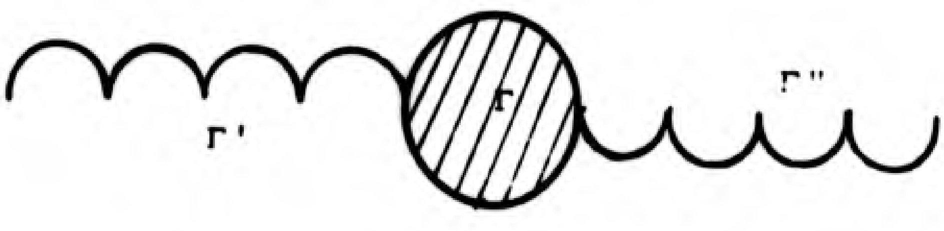


Figure 12. Boundary of control domains

then on our suspected optimal trajectories, which end on the target, $u(t)$ is always -1 when the point $(x(t), X(t))$ lies above $\Gamma' + \Gamma + \Gamma''$, and 1 when it lies below it. Moreover it is easy to see that through any given initial point (x, X) there is exactly one suspected optimal trajectory to the target: it consists of circular arcs alternately lying in the two domains.

We have completed in both our problems the preliminary stage which is that of finding all possible conventional suspected optimal trajectories leading to the target from the various initial positions in the phase space. This preliminary stage can be compared to the rounding up of suspects in a murder trial. It is all of no avail without positive proof of guilt. In our case we can make sure that we have not omitted hidden solutions, in which $u(t)$ is replaced by a chattering control $v(t)$: this is easily done, for we see that this would require $Y(t)$ to vanish on a set of t of positive Lebesgue measure, contrary to the differential equations for (y, Y). Since we have only one suspect leading from a given point to the target, it is now possible to derive from our existence theory and our necessary conditions that this suspect is, in fact, the sole solution with that initial point. However, uniqueness is anything but typical in such problems, so that such a method of completing the discussion would apply only in rather limited cases. This method also uses rather elaborate machinery, and we cannot in all honesty maintain that the existence of a solution is "Physically obvious" in view of the examples considered in the first lecture; in other words, we cannot properly use short cuts with this method, even when the rather rare phenomenon of uniqueness is present, as it is here.

Generally speaking, in problems of Optimal Control, we must now apply the sufficiency theory which I develop in the special lecture referred to. However, there are cases where a much simpler method of completing the solution is possible, and this is what I wish to illustrate here.

Returning to problem A, we calculate for any point (x, X) above or below $\alpha + \beta$, the expression

(10.18) $$E(x, X, u) = 1 + XT_x + uT_X ,$$

except on $\alpha + \beta$, where we set $E(x, X, u) = 0$; here T is given by (10.14). Along any trajectory $x(t), X(t), u(t)$ this expression becomes a function $E(t)$ and we have

(10.19) $$E(t) = \frac{d}{dt}(t + T) ;$$

for the right hand side is

(10.20) $$1 + \dot{x}T_x + \dot{X}T_X ,$$

and the constraints are $\dot{x} = X$, $\dot{X} = u$. This applies on $\alpha + \beta$ as well, because these arcs can only be described towards the origin, and then $t + T$ vanishes on them. (In fact, if we write the equation of α or β in the form $X^2 = 2u_0 x$, differentiation gives $X\dot{X} = u_0\dot{x}$, which reduces by the constraints to $Xu = u_0 X$, and so to $u = u_0$.)

For $-1 \leq u \leq 1$, we find by straightforward differentiation that E has the value

(10.21) $$\begin{aligned} &(1 + u)\left(1 + \frac{2X}{\sqrt{2X^2 + 4x}}\right) \quad \text{above } \alpha + \beta , \\ &(1 - u)\left(1 - \frac{2X}{\sqrt{2X^2 - 4x}}\right) \quad \text{below } \alpha + \beta . \end{aligned}$$

Each of these expressions is non-negative in its domain, and indeed strictly positive unless u has the corresponding optimal value -1, or 1. By symmetry, we need only verify this for the first expression, and we do so by inspection unless (x, X) lies in the fourth quadrant above β; in this excepted case, we have however $2x > X^2$, and therefore

(10.22) $$|2X| < \sqrt{2X^2 + 4x} ,$$

which again makes the assertion above evident.

Since E(t) is thus non-negative, it follows that $t + T$ increases as we increase t from the initial value to the final value 0 . Hence at the initial point the suspected least time T is never greater than the time-difference $-t$ along the trajectory; and indeed we see that it can only equal this when the trajectory is one of our suspected optimal ones.

A similar discussion is possible in problem B , but it involves more elaboration; it is given in my book [14, pp. 239-242].

DISCUSSION OF THIRD LECTURE

III(1). How are the two problems of the lecture related to Space Mathematics?

They are typical, except for being over-simplified, mainly by restriction to one dimension. The dimension of our phase-space is, of course, double that of the underlying real space, in the case of a projectile which can be regarded as a particle. Thus actual problems of Space Mathematics would be analogues of our problems in four, or six, dimensional phase-space. The main other simplifications that we have made concern the form of (1.1) and (1.2). We have taken (1.2) to be simply the time elapsed, whereas in a practical problem one usually seeks to minimize the fuel consumption; and we have taken g in (1.1) to be at most linear in x, as well as in u, and to be independent of t, whereas, in Space Mathematics, non-linear functions of x arise from the inverse square law; of course, such non-linear terms can be approximated by linear ones in suitable ranges of x, so that by treating the linear case one can get a first idea of what a solution looks like. In this sense, problem B may be thought of as a highly simplified analogue of a re-entry problem; variants of problem B with two controls are also relevant to this, and they will be found in my book (vol. II, Chap. I, pp. 253-260).

Problem A is simplified even further by making g altogether independent of x, and strictly speaking this problem corresponds, not to a vertical soft landing, but to a space rendezvous (see Space Mathematics [11], vol. 3, pp. 276-301, especially p. 294). The absence of x from g makes it possible to look at this case in a very intuitive manner, and it is perhaps worth reviewing the problem from this point of view. We can largely forget about phase-space, and just think of the motion as restricted to a straight line, on which the acceleration u is subject to $-1 \leq u \leq 1$. (This means that, on this line, rocket thrust can be in either direction up to a certain maximum, and the units of distance and time are such that the resulting maximum acceleration is unity.) Our deduction from the Maximum Principle that $u = \pm 1$ means that nothing is gained by using a partial acceleration: if one is to come to rest at the origin in a minimum time, one should always be applying maximum acceleration in one direction or the other.

Case 1. Suppose that initially $x > 0$, $X > 0$, i.e. one is to the right of the origin and going away from it. One applies negative acceleration ($u = -1$) until one has not only stopped going away from the origin but built up a negative velocity back toward it, and one continues doing this until the crucial last possible moment, when, by reversing the direction of the acceleration (making $u = 1$), one can just barely stop without running past the origin.

Case 2. Next suppose that initially $x \leq 0$, $X > 0$; three subcases then arise. Case 2a. It may be that by applying maximum leftward acceleration (slowing down as much as possible) one can just barely stop at the origin: the initial phase point (x, X) is then on α, and one should proceed as just indicated. Case 2b. Or the initial velocity X may be greater than in case 2a; then one again begins by applying maximum leftward acceleration, but one cannot help running past the origin, and one proceeds after that as in case 1. Case 2c. Or again the initial velocity X may be less than in case 2a; then one applies maximum rightward acceleration (speeds up) until the crucial moment, when, by reversing the direction of acceleration (slowing down) one can just avoid running past the origin.

Other cases proceed similarly.

III (ii). What is this "half-way map" theorem of McShane and Warfield, and how does it affect differential equations such as (7.2) ?

The theorem extends earlier results of Filippov [5] and Warga, and is concerned with a theoretically troublesome question of measurability of implicit functions. (The term "measurability", as used by mathematicians, has nothing to do with physical measurement.) The fact is that, since no one has ever been able to define, completely and unambiguously, a non-measurable function, it is perhaps natural to restrict our new freedom slightly by requiring functions such as $u(t)$ or $v(t)$ to be not quite arbitrary, but to be subject to the requirement of measurability. The main purpose of the half-way map theorem is to make clear that this added requirement does not affect the generality of the solutions to (1.1) or (7.2). From the point of view of the applications, this is important because one can approximate to measurable functions by rather simple functions, which correspond more closely to the physical reality.

The general form of the half-way map involves certain rather minor restrictions, and oddly enough, the famous continuum hypothesis. We are given a measurable map f (unpaved but passable road) from a space C to a space A, and a continuous map g (good paved road) for an intermediate space B to the space A. The basic question is whether there exists a measurable map h from C to B, which can be combined with the continuous map g to make a map gh from C to A, fully equivalent to the original map f (but less unpleasant up to thc intermediate spot B). For more details, we refer to my book.

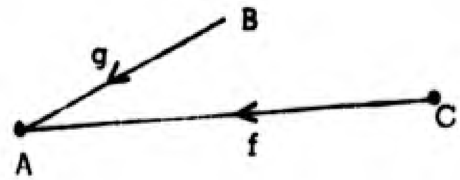


Figure 13. Half-way map theorem.

SPECIALIZED LECTURE†

§11. An old method in variational problems.

The subject of this lecture is taken from a Mathematics Research Center report of May 1966 [13], to appear in my book [14, Vol. II, Ch. II] on the Calculus of Variations and Optimal Control. The method, according to Carathéodory [4], goes back to Huygens, Traité de la Lumière, and it is also variously known as that of geodesic fields or coverings, of Cauchy characteristics, of Bernouilli's brachistochrone. It consists, in its most elementary form, in verifying certain hunches; which is how it is used in a chapter of Hardy-Littlewood-Polya [6].

In the simplest variational problem $\int L(t, x, \dot{x})dt = \text{Min.}$, with fixed ends,‡ say the brachistochrone‡ problem in the lower half-plane, from the origin to some point P_0, we take as suspects the downward vertical together with the simple arches of cycloids from 0 (see Fig. 14). They cover the half-plane simply, and we can compute the value $S(P)$ of $\int L$ along a suspect

†This specialized lecture will appear (with only trivial changes) under the title: "Strengthening Carathéodory's method to apply in control problems", in the proceedings of the Workshop on Calculus of Variations and Optimal Control, held in honor of M.R. Hestenes at UCLA in July 1968.

‡In optimal control t usually denotes the time. However in the calculus of variations t will just be a real variable, one of the co-ordinates of a point (t, x). Here x denotes in both cases a vector, which may reduce to a real number, and x may become a function $x(t)$, when the context so indicates.

‡ Although brachistochrone merely means shortest time, and there are any number of shortest time problems, it has become traditional to understand by the brachistochrone problem that of determining the shape of a wire, in a vertical plane, such that a bead can slide on it in least time from one given point to another, under gravity. (Of Course the second point must not be higher than the first.) In this problem, with the t-axis horizontal and the x-axis pointing downward, the function $L(t, x, \dot{x})$ has the form $x^{-1/2}\sqrt{1+\dot{x}^2}$.

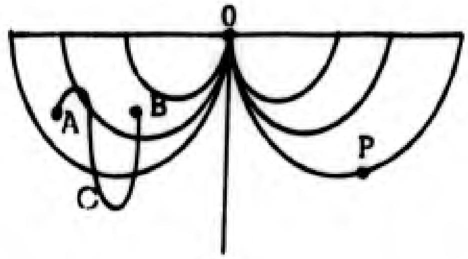


Figure 14. Covering of the half-plane by arcs of cycloids.

from 0 to any given point P . This is a "nice" function of $P = (t, x)$, we can differentiate it and set

(11.1) $$\varepsilon = L(t, x, \dot{x}) - S_t - S_x \dot{x} .$$

Then

(11.2) $$\int_C L\, dt = S(B) - S(A) + \int_C \varepsilon\, dt ,$$

for any curve C joining in the half-plane the points A, B. The minor miracle is that $\varepsilon \geq 0$. By taking $A = 0$, $B = P_0$, we derive, for any curve C joining 0 to P_0 , that $\int_C L\, dt$ is not less than $S(P_0)$, which is the integral along our suspect.

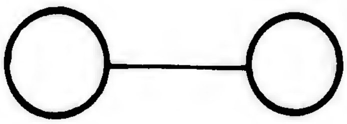


Figure 15. A non-domain.

§12. Synchronization.

In its primitive simplicity, the method demands, not that the covering be one-to-one, but only that S(P) be single-vlaued: if two suspects intersect, we need the synchronization condition that, at the intersection, the

the corresponding values of our integral should agree. Also, the set covered need not be a domain, for instance it could be two disks joined by a segment (see Fig. 15); this sort of thing can easily be imagined in problems of optical instruments, where the method originated.

Intersecting suspects occur already in the minimal surface of revolution problem. This is a problem $\int L\,dt = \text{Min.}$,† in the upper half-plane, between two end-points, one of which we may take to be (0, 1). There are two kinds of suspects, catenaries issuing from (0, 1), and broken lines each consisting of a portion of the horizontal axis and a pair of vertical segments (see Fig. 16). Two suspects of the second kind have a common initial arc, consisting of a pair of segments of the two axes. This already violates one-to-oneness. Each catenary will meet many suspects of the second kind. We put in a barrier, (dotted in the figure) and allow only catenaries above the barrier, and suspects of the second kind below the barrier. Now each catenary meets a single suspect of the second kind, at the barrier. As one-to-oneness is still violated along the barrier, we must have the synchronization condition along the barrier; this determines the location of the barrier uniquely.

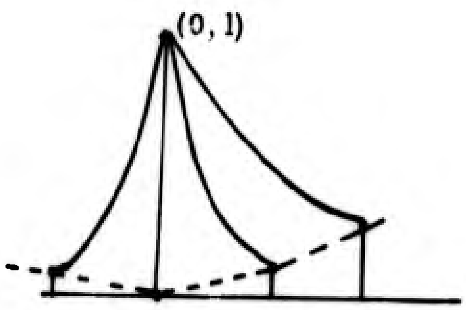


Figure 16. Intersecting suspects

†L is now the function $x\sqrt{1+\dot{x}^2}$, and the t-axis is again horizontal, while the x-axis is now directed upward.

§13. Carathéodory's modification of the method.

The method works only if the miracles occur, and can be verified. Suspects must be chosen suitably, as they are in police matters, on grounds which need not stand up in Court, but which are plausible enough, at least. We choose them by a "rule of thumb", which is the Euler equation, but we then still need the minor miracle, to build up additional incontrovertible evidence. This may mean a lot of work, such as computations using implicit functions, all of it useless if the miracle declines to appear.

For this reason, the method underwent changes at the hands of Carathéodory and others, so that miracles do not appear in the version given in books. Instead, assumptions, or precautions, ensure the same results by virtue, not of miracles, but of theorems. They demand a certain regularity of the problem, one-to-oneness of the covering, and second order smoothness of the corresponding map. These are drastic requirements, rarely fulfilled in the large, even in simple problems such as that of the minimal surface of revolution, and this is why the minima studied in classical works are mainly local, rather than global.

§14. Difficulties in adapting the method to control problems.

In optimal control, one could imitate this classical treatment, by making equally drastic assumptions, except that we now require, not local minima, but the actual absolute minimum. In practice, in control problems, such assumptions are hardly ever satisfied even in the simplest cases, and we shall see why shortly. Therefore, if we proceed, as some have done, in total disregard of these inconvenient facts of life, we may well establish theorems in little more than the empty set of control problems.

Nevertheless, the question is whether more up-to-date procedures can give us the best of both worlds: the generality of the primitive method of Huygens, but without miracles, and without unrealistic assumptions. We shall start by enumerating the difficulties, and in so doing, we limit ourselves to time-optimal problems, which are the only ones considered here.

(a) Suspects (chosen as will be explained) may be rather wild. They can be "generalized" trajectories and controls - like the things engineers approximate when building mountain roads.

(b) Different suspects may have whole arcs in common, and more rarely they may intersect at single points.

(c) There is normally no sort of second order smoothness in the large.

(d) The sets covered by suspects, and indeed those covered by arbitrary trajectories, subject to the constraints, may fail to contain a domain.

Another difficulty, characteristic of control problems, will come from the classical construction which replaces that given above for $S(P)$. This quantity was defined by an integral of the form $\int (y\,dx - H\,dt)$, the "Hilbert independence integral," which in control theory will be simply $\int y\,dx$. In control problems, the integrand y is many-valued, like the reactions of constraints in Statics.

To overcome these difficulties will require, besides measure-theoretic and other techniques, certain general topological concepts, of interest for their own sake. The concepts are of two kinds, those for replacing local one-to-oneness, and those for passing from local to global results.

§15. The first concepts.

We have seen that one-to-oneness may be violated in two ways: by suspects with common arcs, and by suspects which meet at single points. In control problems, the former is connected with the many-valued nature of the Hilbert integrand, which may vary with certain additional parameters ρ. This necessitates here what we term a nice projection (non-precipitous). When a set S^* of pairs (σ, ρ) is projected onto the corresponding set S of σ, we term the projection nice, if, given $(\sigma_0, \rho_0) \in S^*$, and given any sufficiently small rectifiable curve γ issuing from σ_0, there exists on γ a continuous function $\rho(\sigma)$ such that $\rho(\sigma_0) = \rho_0$ and that $(\sigma, \rho(\sigma)) \in S^*$ for all $\sigma \in \gamma$. Fig. 17 illustrates a precipitous (not nice) projection.

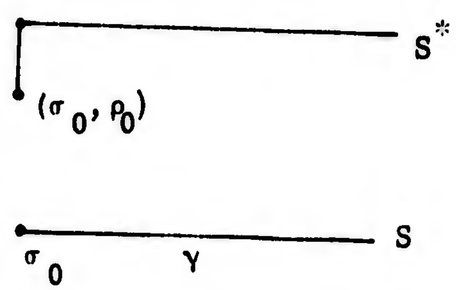


Figure 17. A precipitous projection.

The second way, in which one-to-oneness may be violated, necessitates what we term a descriptive map (in the MRC report, a satisfactory one.) We can think of it as a road map which does not fool us as to distances: if we are at q and there is a short road marked on the map, then there must correspond to it at least one short road for us from q, not, as often happens, only roads dragging on indefinitely. (Of course, the scenery may make it worthwhile.)

A map $f: Q \to f(Q) = P$ is termed descriptive if, given any rectifiable curve $C \subset P$, issuing from $f(q)$, there exists a rectifiable curve $\Gamma \subset Q$, issuing from q, such that every small arc of C, issuing from $f(q)$ is the image under f of a small arc of Γ, issuing from q. Fig. 18 illustrates a "collapsed" map, which is not descriptive. In it, the plane is mapped into itself, by pushing together two half-planes separated by a strip, and by mapping points of this strip into the points of the same altitude on the line separating the half-planes in their new positions. The example suggests a possible connection between the notion of descriptive map and that of light map, as used in topology.

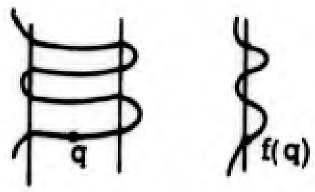


Figure 18. A collapsed map

§16. Further concepts.

The main concept for passing from local to global results is that of unimpaired union. We reduce it first to a second concept, that of a repairable decomposition, defined a little further on. A set R, with a class of subsets P, is the unimpaired union of these P, if it is their union, and if, further, there exists a repairable decomposition of R into (at most countably many) sets R_ν, such that each P is the union of those R_ν which are its subsets. Everything depends on what decompositions of R into disjoint sets R_ν will be termed repairable.

Intuitively, we must think of repairing something which has been rudely shattered; actually, we shall be concerned with a property of certain curves, and our hope is that, if this property holds for each curve situated in anyone of the R_ν, then it will hold for every curve in R itself. The property happens to be preserved by certain elementary operations on curves:

(a) fusion, the adding of two curves C_1, C_2, placed end to end; (see Fig. 19);

(b) embellishment, the addition, to a curve C_1, of a closed curve C_2, which has at least one intersection with C_1 (see Fig. 20);

(c)(d) the inverse operations, cutting and trimming.

Here (a) and (c) are performed a finite number of times; (b) and (d) countably often, if desired. Therefore we term the decomposition of R into the R_ν repairable, if every rectifiable curve in R can be derived from the rectifiable curves in the various R_ν, by finite fusion and cutting, countable embellishment and trimming. The decomposition of a plane into a line and two open half-planes is repairable, that into rational points and irrational points, is not; however, neither of these facts is obvious.

Figure 19. Fusion

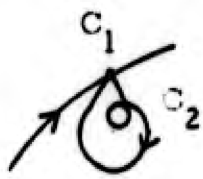


Figure 20. Embellishment

§17. Statement of the problem.

Our basic problem is an autonomous time-optimal control problem, with a given set as a target. The letters t, x, u denote the time, a point of n-space, and a control value which is just a label, while $x(t)$ will always be an absolutely continuous function, and $u(t)$ an arbitrary one, with points or labels as values. We suppose a given function $g(x, u)$ uniformly smooth in x, with values in n-space, and we term admissible trajectory, a pair $x(t)$, $u(t)$, defined for $t \leq 0$ and subject almost everywhere to the differential equation

$$\dot{x} = g(x(t), u(t)) \tag{17.1}$$

and to the condition: $x(0)$ on the target. We want to be sure that our trajectories get to the target at $t = 0$, but we allow them to start anywhere. We denote by t_1 the initial time on a trajectory; it will vary with the trajectory itself, and it will be < 0. The problem is to determine, for a given value of $x(t_1)$, the smallest value of $-t_1$ in the class of all admissible trajectories.

§18. Lines of flight.

We now describe, without justifying it, the rule for selecting suspects among the admissible trajectories. This rule is a strengthened form of Pontrjagin's Maximum Principle (with the initial multiplier unity) together with any further convenient procedure for reducing still more the class of suspects, until it conforms to assumptions that we shall state here in due course. We therefore suspect at most those admissible $x(t)$, $u(t)$ for each of which there exists at least one so-called "conjugate" vector function $y(t)$, which never vanishes, is absolutely continuous, and satisfies

$$\begin{cases} \dot{y}(t) = -y(t)g_x(x(t), u(t)) \text{ for almost all } t, \\ y(t)g(x(t), u) \leq 1 \text{ for all } u, \text{ with equality when } u = u(t), \\ y(0) \text{ normal to the target at the point } x(0). \end{cases} \tag{18.1}$$

Both (17.1) and (18.1) imply some restrictions on $u(t)$, but these are always indirect: they concern composite functions of t formed by means of $u(t)$. This is natural since the u are just labels. Also (18.1) implies that the target has normals; however, this is to be suitably interpreted, for instance every direction is normal if the target is only one point.

From now on, suspected trajectories will be termed lines of flight. The corresponding trios $x(t), y(t), u(t)$ will be termed canonical lines of flight. Arcs of such lines will be termed arcs of flight, or canonical arcs of flight. The first main assumption about lines, or arcs, of flight is that of <u>synchronization:</u> if two of them meet at a point x, they do so at the same time t.

§19. Local smoothness.

The assumptions in this section will be considered local. We shall suppose arcs of flight nicely grouped into families, dependent on some parameter† σ, while the corresponding $y(t)$ similarly depend on σ and also on an additional parameter ρ. (This is because $y(t)$ is not uniquely determined by the functions $x(t), u(t)$.) This grouping is something like a police-filing of suspects, corresponding to the parameter σ, and for each suspect a further filing of witnesses and so forth, corresponding to the parameter ρ. In our case σ, ρ are Euclidean parameters, the set of values of σ is supposed open, while that of (σ, ρ) is to have a nice projection onto it. The relevant families of arcs in these sets will be defined by functions

(19.1) $$x(t,\sigma),\ y(t,\sigma,\rho),\ u(t,\sigma)\quad t^-(\sigma) \le t \le t^+(\sigma),\quad t \neq -\infty .$$

The functions t^-, t^+ are to be continuous (except in the case of t^-, where

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†All parameters are understood to be vector-valued unless the context implies the contrary.

it is $-\infty$, and this set of values of σ, i.e. the set $t^-(\sigma) = -\infty$, is to be open); they satisfy $-\infty \leq t^- < t^+ \leq 0$. In addition, the local restriction of t^+ to small segments parallel to the σ-axes is to be of bounded variation on each such segment.

We shall denote by S^-, S, S^+ and by S^{*-}, S^*, S^{*+} the sets of (t, σ) and (t, σ, ρ), for which σ, ρ are as before, while t is subject to the corresponding condition $-\infty < t^-(\sigma) = t$, or $t^-(\sigma) < t < t^+(\sigma)$, or $t = t^+(\sigma)$. The unions will be similarly written $[S]$ and $[S^*]$. The images under the map $x(t, \sigma), u(t, \sigma)$ and that defined by (19.1) will be written E^-, E, E^+ and E^{*-}, E^*, E^{*+}; the corresponding unions will be $[E]$, $[E^*]$.

Subject to assumptions that we shall list below, the family of functions of t given by (19.1), without, or with, the middle term $y(t, \sigma, \rho)$, will be termed a spray of flights, or a canonical spray of flights. The sets E^-, E, E^+ or E^{*-}, E^*, E^{*+} will be termed its source, its flight corridor, its destination. The arcs themselves will be taken as open arcs, situated in E or E^*, but possessing end-points situated in the source, or destination, except that there is no initial end if $t^- = -\infty$. Finally, when $(t, \sigma) \in S$ and when x is a point of E sufficiently near to the value $x(t, \sigma)$, we write $h(t, \sigma)$, $g(x, t, \sigma)$, $g_x(x, t, \sigma)$ for the expressions

$$g[x(t, \sigma), u(t, \sigma)], \; g[x, u(t, \sigma)], \; g_x[x, u(t, \sigma)].$$

Our further assumptions concerning the functions in (19.1) are:

(a) the maps $S^- \to E^-$, $S \to E$, defined by $x(t, \sigma)$, are descriptive;

(b) $x(t, \sigma)$ is first order smooth in $[S]$;

(c) $y(t, \sigma, \rho)$ is continuous in $[S^*]$; and finally

(d) (a composite assumption, since the u are elusive) the function $h(t, \sigma)$ and for each fixed $x \in E$ the function $g(x, t, \sigma)$ [when (t, σ) is near to the values at which $x(t, \sigma)$ takes the value x], are first order smooth in S and satisfy at $x = x(t, \sigma)$ the relation

$$\frac{\partial h}{\partial \sigma} = g(x, t, \sigma) \frac{\partial x}{\partial \sigma} + \frac{\partial g(x, t, \sigma)}{\partial \sigma} . \tag{19.2}$$

§20. The main theorem.

A finite or countably sequence $\sum_1, \sum_2, \ldots$ of sprays of flights will be termed a chain of flights, and the corresponding sequence of canonical sprays $\sum_1^*, \sum_2^*, \ldots,$ a canonical chain, if, for $r = 1, 2, \ldots$ they "fit together" in inverse order, so that the source of each $\sum_r^*$ contains the destination of $\sum_{r+1}^*$. (The canonical sprays fit, not merely their projections, the $\sum_r$.) If the destination of $\sum_1$ is on the target, we speak of a chain of flights to the target. By the constituent sets of a canonical chain, of flights, we mean the sources and flight-corridors of the individual sprays. Further we term concourse of flights, any set of chains of flight to the target; the corresponding set of canonical chains will be termed a canonical concourse. A constituent set of any member of a concourse, or canonical concourse, will be called a constituent set of this concourse, or canonical concourse. The union of the constituent sets of a concourse will be termed its zone.

Our basic assumption will be that the set R covered by our lines of flight is the zone of a concourse of flights, and moreover the unimpaired union of the constituent sets of this concourse; and that the set $R^\#$ covered, in (x, y)-space, by our canonical lines of flight, is the countable union of a selection of the corresponding canonical constituent sets. (No unimpaired union assumption is made for $R^{\#}$.)

We shall need a final, rather mild, assumption, which necessitates a few more definitions. We denote by $T(x)$, for $x \epsilon R$, and term flight-time from x, the length of the time-interval for a line of flight issuing from the point x. We term "of bounded flight-time," any set of points, or curve, on which $T(x)$ is bounded. We denote, further, by $Y(x)$, for $x \epsilon R$, and term momentum range at x, the set of values of the conjugate vector y, for which $(x, y) \epsilon R^{\#}$; we term momentum in R, any function $y(x)$, $x \epsilon R$, such that, for each x, $y(x) \epsilon Y(x)$. Our final assumption is the existence in R of at least one momentum $y(x)$ which is bounded in each bounded subset of R of bounded flight-time.

Under these various assumptions, our main result is as follows:

(20.1) THEOREM. Let x be any point of R. Then the flight-time $T(x)$ is the least time for transferring this point to the target along an admissible trajectory in R.

This is the basic existence theorem provided by the method.

§21. Some indications of the proof of (20.1).

We shall describe the proof briefly; details can be found in the MRC report [13], or the forthcoming book [14, Vol. II, Ch. II]. At a point $x \epsilon R$, we term direction of univalence, a direction θ on which all the vectors $y \epsilon Y(x)$ have the same projection $y\theta$. We term curve of univalence, a rectifiable curve $C \subset R$, such that, at almost all points of C, the tangent has a direction of univalence. We term set of univalence, a subset A of R such that all rectifiable curves $C \subset A$, of bounded flight-time, are curves of univalence.

Given in R , a rectifiable curve C , of bounded flight-time, and a momentum $y(x)$, we define the Hilbert integral

$$\int_C y(x)dx = \int y(x) \frac{dx}{ds} ds ,$$

whenever $y(x)\, dx/ds$ is (say) a bounded measurable function of the arc-length s along C . In the case of univalence of C , this integral, if it exists, does not depend on the choice of $y(x)$. We term exact, a set A of univalence, such that, for every rectifiable curve C , of bounded flight-time, in A , the Hilbert integral exists for each momentum $y(x)$ and satisfies the relation

$$\int_C y(x)\, dx = T(x_1) - T(x_2) ,$$

where x_1, x_2 are the initial and final points of C . With the method of Section 11, the proof of (20.1) reduces in virtue of the second condition in (18.1), to a corollary of the fundamental result, that, with the same assumptions, R is exact. This in turn, reduces, with surprising ease, to a much weaker assertion of relative exactness for the constituent sets of a chain of flights to the target.

Relative exactness, together with relative univalence, arises when we consider, for a given spray of flights $\sum$, only those values of $y(x)$ for which there exist (t, σ, ρ) such that $x = x(t, \sigma)$, $y = y(t, \sigma, \rho)$. This turns out to be equivalent to a still more restricted definition, in which everything takes place in the variables t, σ, ρ.

Now, since the intersection of R with the target is exact (this is the proper interpretation, with $T = 0$, of the last condition in (18.1)), everything now reduces, by an easy induction, to the following result:

MALUS'S THEOREM. <u>Let</u> $\sum$ <u>be a spray of flights with a relative exact destination</u> E^+. <u>Then</u> $\sum$ <u>possesses a relative exact source</u> E^- <u>and a relative exact flight corridor</u> E.

(What is crucial for this, is the equivalence with the more restricted exactness.)

<u>Note</u>. In the MRC report [13] an unnecessary countability assumption is made concerning the set of chains defining a concourse. Its removal requires only minor changes in wording, when the results are as stated here.

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