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EXTENSION OF RESULTS CONCERNING PARAMETER ESTIMATORS, TOLERANCE BOUNDS AND WARRANTY PERIODS FOR WEIBULL MODELS

NANCY R. MANN
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Contract No. AF33(615)-2818
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UNITED STATES AIR FORCE
WRIGHT-PATTERSON AIR FORCE BASE, OHIO

FOREWORD

The research documented by the three papers contained in this report was sponsored by the Aerospace Research Laboratories, Office of Aerospace Research, United States Air Force, Wright-Patterson Air Force Base, Ohio. The work was performed under Contract AF33(615)-2818, technically monitored by Dr. H. Leon Harter, and was a part of Project 7071, Research in Applied Mathematics.

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ABSTRACT

This report consists of three papers, all of which apply to the two-parameter Weibull distribution. The first paper gives tables for obtaining estimates of the shape parameter and of distribution percentage points. This paper provides a more extensive range of values than are available in the published article to which the tables apply.

The second and third papers extend previous results. The second applies to the calculation of warranty periods from sample data for a lot to be manufactured in the future. Warranty periods to be calculated from three ordered observations are derived for orderings of the three observations not considered earlier. Investigation is made of the expected squared deviations of the calculated log warranty periods from the first failure in the lot to be manufactured in the future. On the basis of this investigation, a tabulation is given for lots of size n, $n = 10, 11, \ldots, 25$ and samples of size m, $m = 2, 3, \ldots, n-3$.

The third paper considers Taylor series approximations to the distribution of three-order-statistic estimators of reliable life. By use of this approach, published tables of values for obtaining three-order-statistic confidence bounds on reliable life are extended.

TABLE OF CONTENTS

A. TABLES FOR OBTAINING ESTIMATES OF WEIBULL PARAMETERS BASED ON A FEW ORDERED OBSERVATIONS

INTRODUCTION AND SUMMARY

The table given here was calculated in conjunction with the work described in [4]. The values listed and the corresponding order numbers of Weibull variates enable one to obtain, for censored samples of size n, estimates of the Weibull shape parameter for $n = 2, 3, \ldots, 22$ and of population percentage points for $n = 2, 3, \ldots, 19$. The mean squared errors, and in some cases the efficiencies with respect to best linear invariant estimators (see [1]), are shown. The tabulated values are also given in [4] for $n = 2, 3, \ldots, 15$, $m = 2, 3, \ldots, n$, where m is the censoring number. The additional values listed here were excluded from the published paper in order to keep the article (which also includes tables for obtaining confidence bounds) of a reasonable length.

THE MODEL FOR THE ESTIMATORS

It is assumed that for the random variable T, which represents failure time,

$$P[T \le t] = \begin{cases} 1 - \exp[-(t/\delta)^{1/b}], & t \ge 0 \\ 0 & \text{otherwise,} \end{cases} \qquad (1)$$

where the parameters δ and b are both positive. It is also assumed that a sample of n failure times has been randomly selected from a population with distribution given by (1), and that only the first m of the n sample values are observed. A survival proportion γ is specified, and an estimator is required for t_γ, the time at which 100γ percent of the population to which T belongs will have survived.

From the m ordered observed logarithms of failure times, $X_1, X_2, \ldots, X_m$, the three observations X_ν, X_p, and X_q are chosen, for each combination of m, n, and R, such that $X_\nu + C_x(X_q - X_p)$ has smallest expected squared error among estimators of $\ell n(t_\gamma)$ of this form. The estimator of t_γ is then given by $\exp[X_\nu + C_x(X_q - X_p)]$.

Table A.1 gives values of ν, p, q and C_x for each combination of n, m, and γ, $n = 2, 3, \ldots, 19$, $m = 2, 3, \ldots, n$, $\gamma = .90$, .95 and exp(-1) (the value of γ corresponding to $t_\gamma = \delta$). Also given are values of p, q, and C_b for obtaining estimates of the shape parameter b based on estimators of the form $C_b(X_q - X_p)$. These estimators have smallest mean squared error among invariant two-order-statistic estimators of b, and are given for $n = 2, 3, \ldots, 22$, $m = 2, 3, \ldots, n$. Values of mean squared error are included in the tables, and for the estimators of b and $u = \ell n\, \delta$, the efficiencies with respect to best linear invariant estimators are given. Values of the variances and covariances of $X_1, X_2, \ldots, X_n$, $n = 1, 2, \ldots, 25$, given in [3] were used to calculate mean squared error corresponding to the combinations of values of ν, p and q given, as well as all other possible combinations, for each combination of values of γ (if applicable), n and m. Estimators of the type tabulated here, along with their asymptotic approximations, are discussed in [2].

REFERENCES

[1] MANN, NANCY R., 1967. Tables for obtaining the best linear invariant estimates of parameters of the Weibull distribution. Technometrics 9, 629-645.

[2] MANN, NANCY R., 1968. Point and interval estimation procedures for the two-parameter Weibull and extreme-value distributions. Technometrics 10, 231-256.

[3] MANN, NANCY R., 1968. Results on Statistical Estimation and Hypothesis Testing With Application to the Weibull and Extreme-Value Distributions. Aerospace Research Laboratories Report ARL 68-0068, Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio.

[4] MANN, NANCY R., 1970. Estimators and exact confidence bounds for Weibull parameters based on a few ordered observations. To appear in Technometrics, August, 1970.

TABLE A.I - VALUES FOR OBTAINING ESTIMATES $X_v + C_x(X_q - X_p)$ AND $C_b(X_q - X_p)$ OF WEIBULL PARAMETERS

		Parameter									
		$x_{.90}$					$x_{.95}$				
n	m	v	p	q	C_x	MSE	v	p	q	C_x	MSE
2	2	2	1	2	-1.058997	2.59389	2	1	2	-1.362321	4.10248
3	2	2	1	2	-0.851412	1.91563	2	1	2	-1.176854	3.23617
	3	2	1	3	-0.586716	1.87465	2	1	3	-0.839134	2.78687
4	2	2	1	2	-0.700471	1.50541	2	1	2	-1.035518	2.65523
	3	2	1	2	-0.700471	1.50541	2	1	3	-0.765496	2.42022
	4	2	1	4	-0.400595	1.41576	2	1	4	-0.625564	2.12955
5	2	2	1	2	-0.582903	1.23860	2	1	2	-0.923354	2.24501
	3	2	1	2	-0.582903	1.23860	2	1	3	-0.687803	2.11888
	4	2	1	2	-0.582903	1.23860	2	1	4	-0.582109	1.94040
	5	3	1	5	-0.508226	1.15613	2	1	5	-0.500256	1.75673
6	2	2	1	2	-0.486886	1.05618	2	1	2	-0.830807	1.94086
	3	2	1	2	-0.486886	1.05618	2	1	3	-0.618060	1.88092
	4	2	1	2	-0.486886	1.05618	2	1	4	-0.529613	1.76003
	5	3	1	5	-0.470518	1.03984	2	1	5	-0.470001	1.64272
	6	3	1	6	-0.409589	0.96846	3	1	6	-0.604811	1.47400
7	2	2	1	2	-0.405849	0.92762	2	1	2	-0.752187	1.70708
	3	2	1	2	-0.405849	0.92762	2	1	3	-0.556437	1.69059
	4	2	1	2	-0.405849	0.92762	2	1	4	-0.479440	1.60588
	5	2	1	2	-0.405849	0.92762	3	1	5	-0.655529	1.51899
	6	3	1	6	-0.382373	0.89318	3	1	6	-0.590506	1.40102
	7	3	1	7	-0.339855	0.84263	3	1	7	-0.525219	1.27531
8	2	2	1	2	-0.335801	0.83553	2	1	2	-0.683921	1.52256
	3	2	1	2	-0.335801	0.83553	2	1	2	-0.683921	1.52256
	4	2	1	2	-0.335801	0.83553	2	1	4	-0.433311	1.47578
	5	2	1	2	-0.335801	0.83553	3	1	5	-0.617409	1.40265
	6	3	1	6	-0.347597	0.81979	3	1	6	-0.562948	1.31341
	7	3	1	7	-0.317821	0.78979	3	1	7	-0.515367	1.22722
	8	3	1	8	-0.286401	0.75305	3	1	8	-0.464290	1.13232
9	2	2	1	2	-0.274147	0.76927	2	1	2	-0.623635	1.37398
	3	3	1	3	-0.461608	0.76596	2	1	2	-0.623635	1.37398
	4	3	1	3	-0.461608	0.76596	2	1	4	-0.391187	1.36551
	5	3	1	3	-0.461608	0.76596	3	1	5	-0.580986	1.30027
	6	3	1	6	-0.313060	0.75539	3	1	6	-0.533155	1.22970
	7	3	1	7	-0.289386	0.73565	3	1	7	-0.493673	1.16379
	8	4	1	8	-0.385222	0.71194	3	1	8	-0.456969	1.09857
	9	4	1	9	-0.349296	0.67106	3	1	9	-0.415618	1.02465
10	2	2	1	2	-0.219109	0.72192	2	1	2	-0.569681	1.25251
	3	3	1	3	-0.413681	0.69213	2	1	2	-0.569681	1.25251
	4	3	1	3	-0.413681	0.69213	2	1	2	-0.569681	1.25251
	5	3	1	3	-0.413681	0.69213	3	1	5	-0.546883	1.21050
	6	3	1	3	-0.413681	0.69213	3	1	6	-0.503828	1.15341
	7	3	1	7	-0.260593	0.68663	3	1	7	-0.469386	1.10069
	8	4	1	8	-0.362633	0.66630	3	1	8	-0.439179	1.05053
	9	4	1	9	-0.337476	0.64020	3	1	9	-0.409828	0.99976
	10	4	1	10	-0.308425	0.60840	5	2	10	-0.773350	0.93356

TABLE A.I - CONTINUED

Parameter

n	m	$x_{.90}$ ν	p	q	C_x	MSE	$x_{.95}$ ν	p	q	C_x	MSE
11	2	2	1	2	-0.169416	0.68882	2	1	2	-0.520867	1.15200
	3	3	1	3	-0.370435	0.63234	2	1	2	-0.520867	1.15200
	4	3	1	3	-0.370435	0.63234	2	1	2	-0.520867	1.15200
	5	3	1	3	-0.370435	0.63234	3	1	5	-0.515121	1.13165
	6	3	1	3	-0.370435	0.63234	3	1	6	-0.475789	1.08485
	7	3	1	3	-0.370435	0.63234	3	1	7	-0.444958	1.04164
	8	4	1	8	-0.338609	0.62341	3	1	8	-0.418762	1.00122
	9	4	1	9	-0.318363	0.60475	3	1	9	-0.394782	0.96205
	10	4	1	10	-0.298255	0.58442	4	1	10	-0.474819	0.91823
	11	4	1	11	-0.274310	0.55926	5	2	11	-0.704002	0.84403
12	2	2	1	2	-0.124131	0.66668	2	1	2	-0.476310	1.06809
	3	3	1	3	-0.331046	0.58359	2	1	2	-0.476310	1.06809
	4	3	1	3	-0.331046	0.58359	2	1	2	-0.476310	1.06809
	5	3	1	3	-0.331046	0.58359	3	1	5	-0.485540	1.06208
	6	3	1	3	-0.331046	0.58359	3	1	6	-0.449256	1.02340
	7	3	1	3	-0.331046	0.58359	3	1	7	-0.421229	0.98747
	8	3	1	3	-0.331046	0.58359	3	1	8	-0.397891	0.95409
	9	4	1	9	-0.297708	0.57066	3	1	9	-0.377206	0.92238
	10	4	1	10	-0.281649	0.55606	4	1	10	-0.463884	0.88806
	11	4	1	11	-0.265255	0.53999	5	2	11	-0.725546	0.83827
	12	6	2	12	-0.513560	0.51474	5	2	12	-0.646604	0.77291
13	2	2	1	2	-0.082540	0.65313	2	1	2	-0.435331	0.99755
	3	3	1	3	-0.294890	0.54368	2	1	2	-0.435331	0.99755
	4	3	1	3	-0.294890	0.54368	2	1	2	-0.435331	0.99755
	5	3	1	3	-0.294890	0.54368	2	1	2	-0.435331	0.99755
	6	3	1	3	-0.294890	0.54368	3	1	6	-0.424233	0.96828
	7	3	1	3	-0.294890	0.54368	3	1	7	-0.398496	0.93814
	8	3	1	3	-0.294890	0.54368	3	1	8	-0.377369	0.91014
	9	4	1	9	-0.277075	0.53927	3	1	9	-0.359021	0.68382
	10	4	1	10	-0.263523	0.52830	4	1	10	-0.449676	0.85452
	11	4	1	11	-0.250527	0.51673	5	2	11	-0.723853	0.81929
	12	6	2	12	-0.520353	0.50163	5	2	12	-0.666153	0.76993
	13	6	2	13	-0.466646	0.47356	5	2	13	-0.598076	0.71511
14	2	2	1	2	-0.044090	0.64641	2	1	2	-0.397405	0.93793
	3	3	1	3	-0.261480	0.51095	2	1	2	-0.397405	0.93793
	4	3	1	3	-0.261480	0.51095	2	1	2	-0.397405	0.93793
	5	3	1	3	-0.261480	0.51095	2	1	2	-0.397405	0.93793
	6	3	1	3	-0.261480	0.51095	3	1	6	-0.400641	0.91872
	7	3	1	3	-0.261480	0.51095	3	1	7	-0.376841	0.89328
	8	3	1	3	-0.261480	0.51095	3	1	8	-0.357516	0.86957
	9	4	1	9	-0.257016	0.51080	3	1	9	-0.340967	0.84737
	10	4	1	10	-0.245258	0.50240	4	1	10	-0.434353	0.82090
	11	4	1	11	-0.234359	0.49368	4	1	11	-0.415184	0.79199
	12	6	2	12	-0.511046	0.48355	5	2	12	-0.665149	0.75515
	13	6	2	13	-0.472683	0.46340	5	2	13	-0.615841	0.71388
	14	6	2	14	-0.426618	0.43983	6	2	14	-0.631626	0.66307
15	2	2	1	2	-0.008344	0.64518	2	1	2	-0.362110	0.88736
	3	3	1	3	-0.230434	0.48414	2	1	2	-0.362110	0.88736
	4	3	1	3	-0.230434	0.48414	2	1	2	-0.362110	0.88736
	5	3	1	3	-0.230434	0.48414	2	1	2	-0.362110	0.88736
	6	3	1	3	-0.230434	0.48414	3	1	6	-0.378374	0.87402
	7	3	1	3	-0.230434	0.48414	3	1	7	-0.356254	0.85248
	8	3	1	3	-0.230434	0.48414	3	1	8	-0.338450	0.83227
	9	3	1	3	-0.230434	0.48414	3	1	9	-0.323364	0.81336
	10	4	1	10	-0.227369	0.47865	4	1	10	-0.418780	0.78847
	11	4	1	11	-0.217966	0.47194	4	1	11	-0.401632	0.76383
	12	5	1	12	-0.285203	0.46280	5	2	12	-0.656417	0.73606
	13	6	2	13	-0.464520	0.44860	5	2	13	-0.615262	0.70211
	14	6	2	14	-0.431893	0.43174	6	2	14	-0.652388	0.66347
	15	6	2	15	-0.391932	0.41173	6	2	15	-0.591923	0.61929

TABLE A.I - CONTINUED

Parameter

n	m	$x_{.90}$ ν	p	q	C_x	MSE	$x_{.95}$ ν	p	q	C_x	MSE
16	2	2	1	2	0.025052	0.64843	2	1	2	-0.329107	0.84438
	3	3	1	3	-0.201441	0.46225	2	1	2	-0.329107	0.84438
	4	4	1	4	-0.319256	0.46041	2	1	2	-0.329107	0.84438
	5	4	1	4	-0.319256	0.46041	2	1	2	-0.329107	0.84438
	6	4	1	4	-0.319256	0.46041	3	1	6	-0.357320	0.83356
	7	4	1	4	-0.319256	0.46041	3	1	7	-0.336686	0.81530
	8	4	1	4	-0.319256	0.46041	3	1	8	-0.320199	0.79801
	9	4	1	4	-0.319256	0.46041	4	1	9	-0.420626	0.78031
	10	4	1	10	-0.210061	0.45704	4	1	10	-0.403358	0.75770
	11	5	1	11	-0.281638	0.45112	4	1	11	-0.387737	0.73636
	12	5	1	12	-0.270748	0.44248	5	2	12	-0.643987	0.71539
	13	6	2	13	-0.451262	0.43260	5	2	13	-0.607700	0.68631
	14	6	2	14	-0.424558	0.41939	6	2	14	-0.654410	0.65425
	15	6	2	15	-0.396476	0.40512	6	2	15	-0.611105	0.62060
	16	7	2	16	-0.421551	0.38570	6	2	16	-0.557021	0.58209
17	2	2	1	2	0.056385	0.65534	2	1	2	-0.298120	0.80781
	3	3	1	3	-0.174249	0.44450	2	1	2	-0.298120	0.80781
	4	4	1	4	-0.294324	0.43348	2	1	2	-0.298120	0.80781
	5	4	1	4	-0.294324	0.43348	2	1	2	-0.298120	0.80781
	6	4	1	4	-0.294324	0.43348	3	1	6	-0.337376	0.79684
	7	4	1	4	-0.294324	0.43348	3	1	7	-0.318073	0.78136
	8	4	1	4	-0.294324	0.43348	3	1	8	-0.302750	0.76653
	9	4	1	4	-0.294324	0.43348	4	1	9	-0.404359	0.74865
	10	4	1	4	-0.294324	0.43348	4	1	10	-0.388278	0.72875
	11	5	1	11	-0.266024	0.43038	4	1	11	-0.373876	0.71006
	12	5	1	12	-0.256299	0.42336	4	1	12	-0.360601	0.69224
	13	5	1	13	-0.247121	0.41624	5	2	13	-0.596734	0.66891
	14	6	2	14	-0.412628	0.40581	6	2	14	-0.649216	0.64109
	15	7	2	15	-0.457699	0.39452	6	2	15	-0.613252	0.61319
	16	7	2	16	-0.428767	0.38029	6	2	16	-0.574778	0.58397
	17	7	2	17	-0.392211	0.36334	6	2	17	-0.526015	0.55011
18	2	2	1	2	0.085895	0.66529	2	1	2	-0.268917	0.77671
	3	3	1	3	-0.148648	0.43025	3	1	3	-0.450347	0.76843
	4	4	1	4	-0.270861	0.41013	3	1	3	-0.450347	0.76843
	5	4	1	4	-0.270861	0.41013	3	1	3	-0.450347	0.76843
	6	4	1	4	-0.270861	0.41013	3	1	6	-0.318442	0.76340
	7	4	1	4	-0.270861	0.41013	3	1	7	-0.300348	0.75029
	8	4	1	4	-0.270861	0.41013	3	1	8	-0.286067	0.73757
	9	4	1	4	-0.270861	0.41013	4	1	9	-0.388699	0.71924
	10	4	1	4	-0.270861	0.41013	4	1	10	-0.373632	0.70162
	11	4	1	4	-0.270861	0.41013	4	1	11	-0.360243	0.68513
	12	5	1	12	-0.242105	0.40557	4	1	12	-0.348027	0.66949
	13	5	1	13	-0.233935	0.39977	6	2	13	-0.673350	0.65095
	14	6	2	14	-0.398609	0.39208	6	2	14	-0.640427	0.62621
	15	7	2	15	-0.447829	0.38249	6	2	15	-0.608785	0.60214
	16	7	2	16	-0.424212	0.37120	6	2	16	-0.576962	0.57793
	17	7	2	17	-0.398699	0.35890	6	2	17	-0.542488	0.55233
	18	7	2	18	-0.366031	0.34409	7	2	18	-0.553953	0.52032
19	2	2	1	2	0.113781	0.67776	2	1	2	-0.241305	0.75032
	3	3	1	3	-0.124465	0.41900	3	1	3	-0.426536	0.72976
	4	4	1	4	-0.248706	0.38987	3	1	3	-0.426536	0.72976
	5	4	1	4	-0.248706	0.38987	3	1	3	-0.426536	0.72976
	6	4	1	4	-0.248706	0.38987	3	1	3	-0.426536	0.72976
	7	4	1	4	-0.248706	0.38987	3	1	7	-0.283443	0.72180
	8	4	1	4	-0.248706	0.38987	4	1	8	-0.390159	0.70875
	9	4	1	4	-0.248706	0.38987	4	1	9	-0.373640	0.69191
	10	4	1	4	-0.248706	0.38987	4	1	10	-0.359460	0.67625
	11	4	1	4	-0.248706	0.38987	4	1	11	-0.346936	0.66160
	12	5	1	12	-0.228284	0.38911	4	1	12	-0.335597	0.64777
	13	5	1	13	-0.220925	0.38434	6	2	13	-0.660104	0.63281
	14	6	2	14	-0.383604	0.37867	6	2	14	-0.629664	0.61066
	15	7	2	15	-0.435692	0.37018	6	2	15	-0.600978	0.58941
	16	7	2	16	-0.415246	0.36083	6	2	16	-0.573056	0.56853
	17	7	2	17	-0.394501	0.35108	6	2	17	-0.544651	0.54735
	18	7	2	18	-0.371838	0.34038	7	2	18	-0.572422	0.52332
	19	8	2	19	-0.390443	0.32677	8	3	19	-0.673036	0.49356

TABLE A.I - CONTINUED

					Parameter							
		u						b				
n	m	v	p	q	C_x	MSE	Eff	p	q	C_b	MSE	Eff
2	2	2	1	2	-0.110731	0.65713	1.000	1	2	0.421383	C.41584	1.00C
3	2	2	1	2	0.166001	0.79546	1.000	1	2	0.452110	0.45006	1.000
	3	3	2	3	-0.396648	0.41488	0.970	1	3	0.350665	C.27081	0.946
4	2	2	1	2	0.346974	1.01478	1.000	1	2	0.465455	C.46439	1.000
	3	3	1	2	0.031378	0.42435	0.997	1	3	0.371137	C.29950	C.940
	4	4	2	4	-0.375081	0.31285	0.935	1	4	0.312531	C.20710	C.887
5	2	2	1	2	0.481434	1.24921	1.000	1	2	0.472962	C.47231	1.000
	3	3	1	3	0.146216	0.49048	1.000	1	3	0.389791	C.31354	0.938
	4	4	3	4	-0.235201	0.29202	0.955	1	4	0.335587	C.23032	C.878
	5	5	3	5	-0.552811	0.25741	0.895	1	5	0.288032	0.17139	0.833
6	2	2	1	2	0.588298	1.48102	1.000	1	2	0.477782	0.47734	1.000
	3	3	1	3	0.247077	0.57891	0.994	1	3	0.357357	C.32201	C.937
	4	4	1	3	0.063932	0.31555	1.000	1	4	0.347283	0.24272	0.875
	5	5	3	5	-0.271675	0.22804	0.980	1	5	0.3C8507	C.19063	0.823
	6	6	3	6	-0.501931	0.21684	0.878	2	6	0.416799	C.14459	0.806
7	2	2	1	2	0.676894	1.70468	1.000	1	2	0.481141	C.48C82	1.000
	3	3	1	3	0.329189	0.67654	0.987	1	3	0.402423	0.32769	0.936
	4	4	1	4	0.133388	0.35451	0.997	1	4	0.354577	C.25063	0.873
	5	5	4	5	-0.093679	0.23510	0.992	1	5	0.319310	C.2C150	C.818
	6	6	3	6	-0.287819	0.19086	0.957	2	6	0.463312	C.16169	C.789
	7	6	3	6	-0.287819	0.19086	0.850	2	7	0.383738	C.12123	0.811
8	2	2	1	2	0.752513	1.91861	1.000	1	2	0.483616	0.48338	1.000
	3	3	1	3	0.398458	0.77724	0.980	1	3	0.406063	0.33179	C.935
	4	4	1	4	0.201217	0.40312	0.987	1	4	0.359613	0.25615	0.871
	5	5	1	4	0.075596	0.25225	0.999	1	5	0.326253	C.20869	0.816
	6	6	4	6	-0.179944	0.18735	0.993	2	6	0.489583	0.17198	C.780
	7	7	4	7	-0.407911	0.16502	0.940	2	7	0.423677	0.13519	0.793
	8	7	4	7	-0.407911	0.16502	0.857	2	8	0.359276	C.10521	C.808
9	2	2	1	2	0.818444	2.12272	1.000	1	2	0.485517	C.48533	1.0C0
	3	3	1	3	0.458356	0.87809	[illegible]	1	3	0.4C8807	0.33488	C.935
	4	4	1	4	0.259135	0.45718	[illegible]	1	4	0.363315	C.26024	0.871
	5	5	1	5	0.124489	0.27802	[illegible]	1	5	0.331157	0.21386	C.814
	6	6	1	4	0.029635	0.19700	[illegible]	2	6	0.5C7207	C.17902	C.775
	7	7	4	7	-0.220786	0.15825	[illegible]	2	7	0.446651	C.14395	0.763
	8	8	4	8	-0.394833	0.14638	[illegible]	2	8	0.394442	C.11693	0.789
	9	8	4	8	-0.394833	0.14638	0.856	2	9	0.34C293	0.09355	C.799

TABLE A.I - CONTINUED

		u						b				
n	m	v	p	q	C_x	MSE	Eff	p	q	C_b	MSE	Eff
10	2	2	1	2	0.876869	2.31744	1.000	1	2	0.487022	0.48687	1.000
	3	3	1	3	0.511110	0.97766	0.971	1	3	0.410951	0.33729	0.935
	4	4	1	4	0.309703	0.51398	0.965	1	4	0.366158	0.26339	0.870
	5	5	1	5	0.174750	0.30930	0.981	1	5	0.334829	0.21778	0.814
	6	6	1	5	0.080570	0.21057	0.996	2	6	0.520042	0.18419	0.772
	7	7	6	7	-0.167531	0.16216	0.991	2	7	0.462330	0.15013	0.777
	8	8	5	8	-0.331256	0.13812	0.970	2	8	0.414944	0.12452	0.779
	9	9	5	9	-0.489346	0.13235	0.904	2	9	0.371833	0.10360	0.781
	10	9	5	9	-0.489346	0.13235	0.850	2	10	0.325037	0.08467	0.788
11	2	2	1	2	0.929310	2.50340	1.000	1	2	0.488243	0.48812	1.000
	3	3	1	3	0.558234	1.07519	0.967	1	3	0.412674	0.33923	0.934
	4	4	1	4	0.354586	0.57198	0.956	1	4	0.368413	0.26591	0.870
	5	5	1	5	0.218952	0.34406	0.967	1	5	0.337690	0.22085	0.813
	6	6	1	6	0.117898	0.22908	0.988	2	6	0.529875	0.18816	0.769
	7	7	1	5	0.044263	0.16962	0.997	2	7	0.473518	0.15476	0.773
	8	8	6	8	-0.230044	0.13788	0.991	2	8	0.429121	0.12598	0.772
	9	9	5	9	-0.333816	0.12329	0.958	2	9	0.390406	0.11027	0.771
	10	10	5	10	-0.466387	0.12054	0.892	2	10	0.353725	0.09345	0.771
	11	10	5	10	-0.466387	0.12054	0.847	3	11	0.384682	0.07708	0.782
12	2	2	1	2	0.976872	2.68127	1.000	1	2	0.489255	0.48915	1.000
	3	3	1	3	0.600807	1.17035	0.964	1	3	0.414090	0.34083	0.934
	4	4	1	4	0.394936	0.63026	0.948	1	4	0.370246	0.26795	0.869
	5	5	1	5	0.258423	0.38089	0.954	1	5	0.339986	0.22333	0.812
	6	6	1	6	0.157411	0.25120	0.975	2	6	0.537679	0.19132	0.766
	7	7	1	7	0.076885	0.18087	0.993	2	7	0.482505	0.15839	0.770
	8	8	1	4	0.015464	0.14379	0.987	2	8	0.435721	0.13415	0.768
	9	9	6	9	-0.260261	0.12101	0.986	2	9	0.403385	0.11515	0.764
	10	10	5	10	-0.334910	0.11208	0.943	2	10	0.370750	0.09936	0.760
	11	11	5	11	-0.449690	0.11146	0.877	3	11	0.425563	0.08443	0.768
	12	11	5	11	-0.449690	0.11146	0.839	3	12	0.368021	0.07009	0.784
13	2	2	1	2	1.020377	2.85170	1.000	1	2	0.490105	0.49002	1.000
	3	3	1	3	0.639626	1.26298	0.962	1	3	0.415273	0.34216	0.934
	4	4	1	4	0.431584	0.68829	0.941	1	4	0.371767	0.26565	0.869
	5	5	1	5	0.294091	0.41890	0.942	1	5	0.341873	0.22538	0.812
	6	6	1	6	0.192866	0.27569	0.960	2	6	0.544035	0.19390	0.766
	7	7	1	7	0.112772	0.19510	0.983	2	7	0.490112	0.16131	0.768
	8	8	1	6	0.052967	0.14946	0.996	2	8	0.448026	0.13746	0.764
	9	9	7	9	-0.137899	0.12366	0.991	2	9	0.413180	0.11894	0.759
	10	10	6	10	-0.277551	0.10855	0.977	2	10	0.382747	0.10379	0.753
	11	11	6	11	-0.410539	0.10286	0.932	3	11	0.450493	0.08572	0.757
	12	11	6	11	-0.410539	0.10286	0.871	3	12	0.405232	0.07649	0.770
	13	11	6	11	-0.410539	0.10286	0.838	3	13	0.354029	0.06446	0.782

TABLE A.I - CONTINUED

		Parameter											
		u						b					
n	m	v	p	q	C_x	MSE	Eff	p	q	C_b	MSE	Eff
14	2	2	1	2	1.060461	3.01528	1.000	1	2	0.490832	0.49076	1.000
	3	3	1	3	0.675294	1.35307	0.960	1	3	0.416277	0.34329	0.934
	4	4	1	4	0.465151	0.74571	0.935	1	4	0.373049	0.27109	0.869
	5	5	1	5	0.326628	0.45753	0.931	1	5	0.343450	0.22709	0.8[illegible]2
	6	6	1	6	0.225037	0.30173	0.945	2	6	0.549320	0.19604	0.765
	7	7	1	7	0.145098	0.21175	0.969	2	7	0.496035	0.16372	0.766
	8	8	1	8	0.078835	0.15875	0.990	2	8	0.454743	0.14016	0.762
	9	9	1	6	0.027176	0.12815	0.993	2	9	0.420919	0.12197	0.755
	10	10	7	10	-0.191857	0.10900	0.991	2	10	0.391872	0.10724	0.748
	11	11	6	11	-0.288415	0.09896	0.967	3	11	0.468391	0.09375	0.749
	12	12	6	12	-0.401321	0.09532	0.920	3	12	0.428009	0.08120	0.759
	13	12	6	12	-0.401321	0.09532	0.868	3	13	0.388245	0.07012	0.770
	14	12	6	12	-0.401321	0.09532	0.839	3	14	0.342073	0.05984	0.779
15	2	2	1	2	1.097617	3.17257	1.000	1	2	0.491458	0.49139	1.000
	3	3	1	3	0.708282	1.44063	0.958	1	3	0.417139	0.34427	0.934
	4	4	1	4	0.496112	0.80232	0.930	1	4	0.374145	0.27232	0.869
	5	5	1	5	0.356541	0.49638	0.922	1	5	0.344791	0.22855	0.811
	6	6	1	6	0.254491	0.32878	0.931	2	6	0.553788	0.19785	0.765
	7	7	1	7	0.174530	0.23010	0.953	2	7	0.500998	0.16574	0.765
	8	8	1	8	0.108640	0.17013	0.978	2	8	0.460306	0.14241	0.760
	9	9	1	8	0.055161	0.13380	0.994	2	9	0.427224	0.12447	0.752
	10	10	9	10	-0.091901	0.11320	0.982	2	10	0.399131	0.11004	0.744
	11	11	7	11	-0.223342	0.09810	0.987	3	11	0.482202	0.09696	0.743
	12	12	7	12	-0.360479	0.09119	0.957	3	12	0.444452	0.08483	0.751
	13	13	7	13	-0.465857	0.08907	0.908	3	13	0.409253	0.07434	0.759
	14	13	7	13	-0.465857	0.08907	0.863	3	14	0.373799	0.06488	0.767
	15	13	7	13	-0.465857	0.08907	0.837	3	15	0.331709	0.05596	0.775
16	2	2	1	2	1.132243	3.32404	1.000	1	2	0.492005	0.49195	1.000
	3	3	1	3	0.738962	1.52576	0.956	1	3	0.417889	0.34511	0.934
	4	4	1	4	0.524840	0.85800	0.926	1	4	0.375093	0.27339	0.869
	5	5	1	5	0.384222	0.53520	0.914	1	5	0.345944	0.22982	0.811
	6	6	1	6	0.281656	0.35645	0.919	2	6	0.557616	0.19940	0.764
	7	7	1	7	0.201558	0.24963	0.937	2	7	0.505222	0.16746	0.763
	8	8	1	8	0.135854	0.18324	0.963	2	8	0.464997	0.14431	0.758
	9	9	1	9	0.079844	0.14164	0.986	2	9	0.432478	0.12656	0.750
	10	10	1	7	0.036361	0.11603	0.994	2	10	0.405082	0.11236	0.741
	11	11	9	11	-0.180948	0.09998	0.990	3	11	0.493316	0.09959	0.739
	12	12	7	12	-0.243652	0.08964	0.980	3	12	0.457208	0.08775	0.745
	13	13	7	13	-0.358380	0.08472	0.947	3	13	0.424485	0.07765	0.751
	14	14	7	14	-0.451642	0.08351	0.900	3	14	0.393323	0.06872	0.756
	15	14	7	14	-0.451642	0.08351	0.860	3	15	0.361331	0.06051	0.763
	16	14	7	14	-0.451642	0.08351	0.837	3	16	0.322619	0.05267	0.769

TABLE A.I - CONTINUED

		Parameter u						Parameter b				
n	m	v	p	q	C_x	MSE	Eff	p	q	C_b	MSE	Eff
17	2	2	1	2	1.164659	3.47016	1.000	1	2	0.452486	0.45244	1.000
	3	3	1	3	0.767633	1.60854	0.955	1	3	0.418546	0.34585	0.934
	4	4	1	4	0.551636	0.91267	0.922	1	4	0.375921	0.27432	0.868
	5	5	1	5	0.409981	0.57382	0.907	1	5	0.346946	0.23091	0.811
	6	6	1	6	0.306864	0.38448	0.908	2	6	0.560934	0.20074	0.763
	7	7	1	7	0.226552	0.26999	0.922	2	7	0.508862	0.16895	0.763
	8	8	2	8	0.254643	0.19709	0.949	2	8	0.465013	0.14595	0.757
	9	9	1	9	0.105216	0.15101	0.973	2	9	0.436935	0.12835	0.748
	10	10	1	9	0.059226	0.12128	0.991	2	10	0.410069	0.11432	0.739
	11	11	1	6	0.017899	0.10319	0.988	3	11	0.502514	0.10178	0.736
	12	12	9	12	-0.224808	0.09018	0.990	3	12	0.467526	0.09016	0.741
	13	13	8	13	-0.313117	0.08275	0.973	3	13	0.436357	0.08032	0.744
	14	14	7	14	-0.356487	0.07937	0.937	3	14	0.407533	0.07174	0.748
	15	15	7	15	-0.440367	0.07888	0.889	3	15	0.379594	0.06401	0.752
	16	15	7	15	-0.440367	0.07888	0.854	4	16	0.405979	0.05651	0.762
	17	15	7	15	-0.440367	0.07888	0.833	4	17	0.358272	0.04948	0.768
18	2	2	1	2	1.195128	3.61130	1.000	1	2	0.492912	0.49287	1.000
	3	3	1	3	0.794540	1.68907	0.954	1	3	0.419127	0.34651	0.934
	4	4	1	4	0.576740	0.96630	0.919	1	4	0.376651	0.27514	0.868
	5	5	1	5	0.434067	0.61210	0.900	1	5	0.347826	0.23188	0.811
	6	6	1	6	0.330380	0.41269	0.898	2	6	0.563839	0.20191	0.763
	7	7	2	7	0.411152	0.29036	0.910	2	7	0.512035	0.17025	0.762
	8	8	2	8	0.292618	0.21123	0.938	2	8	0.472491	0.14736	0.756
	9	9	2	9	0.198368	0.16087	0.962	2	9	0.440769	0.12990	0.747
	10	10	1	10	0.080305	0.12800	0.982	2	10	0.414319	0.11600	0.737
	11	11	1	9	0.040790	0.10625	0.994	3	11	0.510287	0.10365	0.733
	12	12	10	12	-0.089524	0.09291	0.984	3	12	0.476106	0.09219	0.737
	13	13	9	13	-0.250553	0.08250	0.987	3	13	0.446002	0.08255	0.740
	14	14	8	14	-0.317907	0.07705	0.965	3	14	0.418652	0.07420	0.742
	15	15	8	15	-0.411692	0.07469	0.928	3	15	0.392928	0.06679	0.744
	16	16	8	16	-0.490992	0.07461	0.882	4	16	0.429070	0.05962	0.752
	17	16	8	16	-0.490992	0.07461	0.851	4	17	0.392778	0.05282	0.763
	18	16	8	16	-0.490992	0.07461	0.831	4	18	0.348697	0.04664	0.767
19	2	2	1	2	1.223869	3.74782	1.000	1	2	0.493292	0.49325	1.000
	3	3	1	3	0.819887	1.76745	0.953	1	3	0.419644	0.34709	0.934
	4	4	1	4	0.600353	1.01889	0.916	1	4	0.377298	0.27587	0.868
	5	5	1	5	0.456684	0.64998	0.895	1	5	0.348605	0.23274	0.811
	6	6	1	6	0.352417	0.44092	0.889	2	6	0.566404	0.20295	0.763
	7	7	2	7	0.448167	0.31064	0.901	2	7	0.514825	0.17139	0.761
	8	8	2	8	0.328135	0.22590	0.927	2	8	0.475538	0.14861	0.755
	9	9	2	9	0.232927	0.17134	0.952	2	9	0.444106	0.13125	0.746
	10	10	2	10	0.154428	0.13529	0.973	2	10	0.417992	0.11746	0.735
	11	11	1	10	0.062094	0.11105	0.988	3	11	0.516558	0.10527	0.731
	12	12	1	8	0.025437	0.09499	0.991	3	12	0.483387	0.09393	0.734
	13	13	10	13	-0.156647	0.08387	0.990	3	13	0.454055	0.08443	0.736
	14	14	9	14	-0.267238	0.07630	0.983	3	14	0.427719	0.07626	0.737
	15	15	8	15	-0.321170	0.07229	0.957	3	15	0.403397	0.06908	0.738
	16	16	8	16	-0.404586	0.07064	0.920	4	16	0.446112	0.06217	0.744
	17	16	8	16	-0.404586	0.07064	0.878	4	17	0.414390	0.05567	0.754
	18	16	8	16	-0.404586	0.07064	0.849	4	18	0.381146	0.04966	0.763
	19	16	8	16	-0.404586	0.07064	0.832	4	19	0.340147	0.04417	0.766

TABLE A.I - CONTINUED

n	m	Parameter b p	q	C_b	MSE	Eff
20	2	1	2	0.493634	0.49360	1.000
	3	1	3	0.420107	0.34761	0.934
	4	1	4	0.377877	0.27652	0.868
	5	1	5	0.349299	0.23350	C.811
	6	2	6	0.568686	0.20387	0.762
	7	2	7	0.517299	0.17240	0.761
	8	2	8	0.478227	0.14971	0.754
	9	2	9	0.447C38	0.13243	C.745
	10	2	10	0.421202	C.11874	0.734
	11	3	11	0.522758	0.10667	0.729
	12	3	12	0.489660	0.09544	C.732
	13	3	13	0.460913	C.08605	0.733
	14	3	14	0.435317	0.07802	0.733
	15	3	15	0.411963	0.07100	0.733
	16	4	16	0.459606	0.06428	0.738
	17	4	17	0.430374	0.05801	C.746
	18	4	18	0.401479	C.05228	0.754
	19	4	19	0.370804	0.04692	C.762
	20	4	20	0.332454	0.04201	0.763
21	2	1	2	0.493942	0.49391	1.000
	3	1	3	0.420525	0.34808	0.934
	4	1	4	0.378398	0.27711	C.868
	5	1	5	0.349922	0.23419	C.811
	6	2	6	0.570729	0.2C470	0.762
	7	2	7	0.519508	0.17330	0.760
	8	2	8	0.480621	C.15068	C.753
	9	2	9	0.449638	0.13349	0.744
	10	2	10	0.424033	0.11988	0.733
	11	3	11	0.527854	0.1C791	0.727
	12	3	12	0.495134	0.09676	0.730
	13	3	13	0.466845	0.08747	0.730
	14	3	14	0.441810	0.07954	C.730
	15	3	15	0.419163	0.07264	0.729
	16	4	16	0.470725	0.06608	C.733
	17	4	17	0.443063	0.C5997	0.739
	18	4	18	0.416544	0.C5444	0.746
	19	4	19	0.390017	C.C4934	0.753
	20	4	20	0.361534	C.04453	C.760
	21	4	21	0.325486	0.04010	C.76C
22	2	1	2	0.494223	0.49419	1.000
	3	1	3	0.420903	0.34851	0.934
	4	1	4	0.378868	C.27764	C.868
	5	1	5	0.350484	0.23481	C.810
	6	2	6	0.572570	0.20544	0.762
	7	2	7	0.521493	C.17412	C.760
	8	2	8	0.482766	0.15156	0.753
	9	2	9	0.451960	0.13443	0.743
	10	2	10	0.426551	C.12089	0.732
	11	3	11	0.532371	0.10901	C.726
	12	3	12	C.499960	0.09793	C.728
	13	3	13	0.472037	0.08871	C.728
	14	3	14	0.447442	C.08C87	C.727
	15	3	15	0.425334	0.07407	0.726
	16	4	16	0.480128	0.06763	0.729
	17	4	17	0.453545	0.06164	C.734
	18	4	18	0.428523	0.05627	C.740
	19	4	19	0.404275	C.C5136	0.745
	20	4	20	0.379761	0.04677	C.751
	21	5	21	0.393310	C.C4236	0.758
	22	4	22	0.319137	C.C3840	C.756

B. WARRANTY PERIODS BASED ON THREE ORDERED SAMPLE OBSERVATIONS FROM A WEIBULL POPULATION

SUMMARY

Further investigation is made of the method derived in [5] for calculating from Weibull failure data warranty periods for lots to be manufactured in the future. A tabulation is given of warranty periods associated with lots of size n, $n = 10, 11, \ldots, 25$, for failure data obtained from samples of size m, $m = 2, 3, \ldots, n-3$. The specified assurance level is .95.

INTRODUCTION

It is assumed that a sample of m items has been subjected to life test at a fixed level of stress under constant environmental conditions until s, with $s \leq m$, failures have been observed. We consider a lot consisting of n items from the same population as those in our preliminary sample of size m and we suppose that this lot will be subjected to the level of stress and environmental conditions maintained during the life test on the sample. Specifically, we assume that under the conditions specified for stress and environment, that the failure times in the lot of size n and those in the size m sample are from the same population. We suppose that this population consists of identically distributed two-parameter Weibull variates so that for any positive random variate T from this population

$$P[T \leq t] = 1 - \exp[-(t/\xi)^{\kappa}], \quad \xi > 0, \quad \kappa > 0. \tag{1}$$

We specify a level of assurance α_o that no failures will occur in the lot prior to the expiration of a warranty period which must be determined from the sample. In [6], results of [3] motivate the derivation of a parameter-free expression for the warranty period of the form $\exp[X_{r,m} + \nu(X_{p,m}-X_{q,m})]$, where $X_{i,m}$, $i=1, 2, \ldots, k$, is the logarithm of the $\underline{i}$th observed failure time in the size m sample with $X_{i,m} < X_{i+1,m}$, $i=1, 2, \ldots, s-1$, and $p < q$. For $r \leq p$, Taylor series expansions in $\sigma = 1/(\nu+a)$, a, $\nu > 0$ are derived in [5] for determining α_o as a function of ν, m, n, p, q and r. Linear and quadratic approximations for ν are also given as functions of α_o, m, n, p, q and r. For $r=1$, $p=2$, $q=3$ and for $r=p=1$, $q=2$ the linear approximations for ν are shown to be good for large α_o, and the quadratic approximations agree with the exact values which can be evaluated by numerical integration combined with iterative techniques to two significant figures for α_o equal to .75 and above.

In the following, expressions for α_o which are Taylor-series expansions in $\sigma = 1/\nu$, $\nu > 0$ have been derived for the two cases $r > q$ and $p < r < q$. For the case $r = q$, the expression can be obtained from that corresponding to $r = p$ (since $X_{q,m} + \nu(X_{p,m}-X_{q,m}) = X_{p,m} + (\nu-1)(X_{p,m}-X_{q,m})$). Then, since $\alpha_o = P[Z_{1,n} > X_{r,m} + \nu(X_{p,m}-X_{q,m})]$, where $Z_{1,n}$ is the logarithm of the first failure time in the lot of size n, one can for fixed α_o, m and n evaluate $E\left\{\left[(Z_{1,n}-X_{r,m}-\nu(X_{p,m}-X_{q,m})\right]^2\right\}$, the expected squared deviation of log warranty period from $Z_{1,n}$, for various combinations of values of r, p and q. (In these evaluations, one uses tables of values of the means and variances of the logarithms of the reduced parameter-free ordered observations, such as those appearing in [2].) In this way a "best" combination of three ordered failure times can be determined for each combination of values of α_o, m and n. The criterion of the mean squared deviation of the "forecaster" from the function of observations which is "forecast" (in our case, $Z_{1,n}$) is suggested by Tiago de Oliveira and Littauer [9] and is similar to the criterion suggested by Harter [1] for confidence bounds.

Comparisons of the mean squared deviations of log warranty periods from $Z_{1,n}$ are made in certain cases following the derivations given below for the Taylor-series expansions for evaluating ν for specified α_o, m and n for the two cases not treated in [5]. Also discussed following these derivations is the precision of the Taylor-series expansions with respect to the exact values to be obtained by numerical integration combined with a Newton-Raphson iterative procedure. Tables are given for $n = 10, 11, \ldots, 25$, $m = 2, 3, \ldots, n-3$, $\alpha_0 = .95$.

DERIVATION OF EXPRESSIONS FOR ν CORRESPONDING TO SPECIFIED ASSURANCE LEVEL

In the following derivations, reference is made several times to results appearing in [6]. It should be noted, however, that many of the mathematical details of the proof of these results are omitted from [6], but can be found in the report [5] bearing the same title, on which [6] is based.

In [6], it is assumed that $X_{1,m}, X_{2,m}, \ldots X_{s,m}$, $X_{i,m} \le X_{i+1,m}$, $i=1, 2, \ldots, s-1$, are the logarithms of ordered observations from a size m sample of failure times from a two-parameter Weibull distribution, and $\exp[X_{r,m} + \nu(X_{p,m}-X_{q,m})]$, $p < q \le s$, is to be used to specify a warranty period for a size n lot of items with failure times each distributed identically to those in the preliminary sample. It is then shown that α_0, the probability that no failure occurs before time $\exp[X_{r,m} + \nu(X_{p,m}-X_{q,m})]$, is given by $\int_0^\infty \Phi(u/n)e^{-u}\,du$, where $\Phi(u) = P[Y_r + \nu(Y_p-Y_q) \le \ln u]$ and $Y_i = (X_{i,m}-\ln \xi)/(1/\kappa)$, $i=1, 2, \ldots, s$. We wish to evaluate α_0 for specified ν for the case $X_{r,m} \ge X_{q,m} \ge X_{p,m}$. We therefore consider the joint density $f(y_p, y_q, y_r)$ of Y_p, Y_q and Y_r which are ordered observations from size m samples selected from the reduced (parameter-free) first asymptotic distribution of smallest values.

For $Y_p < Y_q < Y_r$,

$$f(y_p,y_q,y_r) = C^* \sum_{i=1}^{p} \sum_{j=1}^{q-p} \sum_{k=1}^{r-q} K_{i,j,k}$$

$$\cdot \exp(y_p+y_q+y_r-C_7e^{y_p}-C_8e^{y_q}-C_9e^{y_r}), \qquad -\infty < y_r < \infty,$$
$$-\infty < y_q \le y_r,$$
$$-\infty < y_p \le y_q$$

where $C^* = \dfrac{m!}{(p-1)!(q-p-1)!(r-q-1)!(m-r)!}$,

$K_{i,j,k} = (-1)^{i+j+k-1}\binom{p-1}{i-1}\binom{q-p-1}{j-1}\binom{r-q-1}{k-1}$, $C_7 = q-p+i-j$, $C_8 = r-q+j-k$ and $C_9 = m-r+k$. We let $z_1 = y_r$, $z_2 = y_q - y_r$ and $z_3 = y_p - y_q$ so that $y_r = z_1$, $y_q = z_1 + z_2$ and $y_p = z_1 + z_2 + z_3$, with $-\infty < z_1 < \infty$, $-\infty < z_2 \le 0$ and $-\infty < z_3 \le 0$. Then, since the Jacobian of this set of transformations is equal to 1,

$$f(z_1,z_3) = C^* \sum_{i=1}^{p} \sum_{j=1}^{q-p} \sum_{k=1}^{r-q} K_{i,j,k} \exp(3z_1+z_3)$$

$$\frac{\exp(-C_9e^{z_1})\left\{1-\exp\left[-C_8e^{z_1}-C_7e^{(z_1+z_3)}\right]\left[1+C_8e^{z_1}+C_7e^{(z_1+z_3)}\right]\right\}}{\left[C_8e^{z_1}+C_7e^{(z_1+z_3)}\right]^2}$$

Now, since

$$P[Y_r + \nu(Y_p-Y_q) \le \ln u] = P[(Y_p-Y_q) \le (\ln u-Y_r)/\nu] \qquad (2)$$

and $P[(Y_p-Y_q) \le 0] = 1$, the probability (2) (call it $I(u)$) is given by

$$I(u) = C^* \sum_{i=1}^{p} \sum_{j=1}^{q-p} \sum_{k=1}^{r-q} K_{i,j,k} \left\{ \int_u^\infty \frac{\exp(-C_9 s)[1-\exp(-C_8 s)]\, ds}{C_7 C_8} \right.$$

$$- \left. \int_u^\infty \frac{\exp(-C_9 s)\left\{1-\exp[-C_8 s - C_7 s (u/s)^{\frac{1}{\nu}}]\right\} ds}{C_7[C_8 + C_7 (u/s)^{\frac{1}{\nu}}]} \right\}$$

$$+ \left\{1 - \frac{m!}{(r-1)!(m-r!)} \sum_{\ell=1}^{r} (-1)^{\ell-1} \binom{r-1}{\ell-1} \frac{\exp[-(m-r+\ell)u]}{(m-r+\ell)}\right\} .$$

Here, the last term gives the probability that Y_r is less than $\ln u$.

Next, in order to obtain the integral $J(i,j,k,\ell) = \int_0^\infty I(u/n) e^{-u} du$ which gives the probability γ_o, we make the change of variable $t = (s/u)^{\frac{1}{\nu}}$. Then

$$J(i,j,k,\ell) = C^* \sum_{i=1}^{p} \sum_{j=1}^{q-p} \sum_{k=1}^{r-q} K_{i,j,k}$$

$$\cdot \int_0^\infty \left\{ \int_1^\infty \frac{u\nu t^{\nu-1} \exp[-C_9 (u/n) t^\nu][1-\exp[-C_8 (u/n) t^\nu]\, dt}{n\, C_7 C_8} \right.$$

$$- \int_1^\infty \frac{u\nu t^{\nu-1} \exp[-C_9 (u/n) t^\nu]\left\{1-\exp[-C_8 (u/n) t^\nu - C_7 (u/n) t^{\nu-1}]\right\} dt}{n\, C_7 (C_8 + C_7 t^{-1})}$$

$$+ 1 - \frac{m!}{(r-1)!(m-r)!} \sum_{\ell=1}^{r} (-1)^{\ell-1} \binom{r-1}{\ell-1} \frac{\exp[-(m-r+\ell)(u/n)]}{(m-r+\ell)} \Bigg\} \exp(-u) du .$$

Changing the order of integration, we obtain

$$J(i,j,k,\ell) = C^* \sum_{i=1}^{p} \sum_{j=1}^{q-p} \sum_{k=1}^{r-q} K_{i,j,k}$$

$$\cdot \int_1^\infty \left\{ \frac{n\,\nu\,t^{\nu-1}}{C_7 C_8 (C_9 t^\nu + n)^2} - \frac{n\,\nu\,t^{\nu-1}}{C_7 C_8 [(C_8+C_9)t^\nu + n]^2} \right.$$

$$\left. - \frac{\nu t^{\nu-1}}{nC_7(C_8+C_7 t^{-1})(C_9 t^\nu/n+1)^2} + \frac{\nu t^{\nu-1}}{nC_7(C_8+C_7 t^{-1})[(C_8+C_9)t^\nu/n + C_7 t^{\nu-1}/n+1]^2} \right\} dt$$

$$+ 1 - \sum_{\ell=1}^{r} B_\ell \frac{n}{(m-r+\ell)[(m-r+\ell)+n]},$$

where $B_\ell = \frac{m!}{(r-1)!(m-r)!} (-1)^{\ell-1} \binom{r-1}{\ell-1}$.

We now let y equal $t^{-\nu}$ and $\sigma = 1/\nu$. Then

$$J(i,j,k,\ell) = L(\sigma; i,j,k,\ell) = C^* \sum_{i=1}^{p} \sum_{j=1}^{q-p} \sum_{k=1}^{r-q} K_{i,j,k}$$

$$\cdot \left\{ \frac{[(C_8+C_9)(C_8+C_9+n) - C_9(C_9+n)]n}{C_7 C_8 C_9 (C_8+C_9)(C_9+n)(C_8+C_9+n)} - \int_0^1 \frac{dy}{nC_7 C_8 [1+(C_7/C_8)y^\sigma](C_9/n+y)^2} \right.$$

$$\left. + \int_0^1 \frac{dy}{nC_7 C_8 [1+(C_7/C_8)y^\sigma][(C_8+C_9)/n + C_7 y^\sigma/n+y]^2} \right\}$$

$$+ 1 - \sum_{\ell=1}^{r} B_\ell \frac{n}{(m-r+\ell)[m-r+\ell+n]} \quad .$$

As in [6], we let $J(i,j,k,\ell) = L(\sigma;i,j,k,\ell)$ be approximated by $L(0;i,j,k,\ell) + \sigma L^{(1)}(0;i,j,k,\ell) + \frac{\sigma^2}{2} L^{(2)}(0;i,j,k,\ell)$ (with $L^{(i)}(0;i,j,k,\ell)$ equal to the ith derivative of $L(0;i,j,k,\ell)$, which is appropriate for σ sufficiently small.

From results in [6] (see 2.9.1) we find

$$L(0;i,j,k,\ell) = C^* \sum_{i=1}^{p} \sum_{j=1}^{q-p} \sum_{k=1}^{r-q} K_{i,j,k} \left\{ \frac{n}{C_7(C_7+C_8)(C_7+C_8+C_9)(n+C_7+C_8+C_9)} - \frac{n}{C_7(C_7+C_8)C_9(n+C_9)} \right.$$

$$\left. + \frac{n[n+C_8+C_9+C_9]}{C_7C_9(C_8+C_9)(C_9+n)(C_8+C_9+n)} \right\} + 1 - \sum_{\ell=1}^{r} L_\ell \frac{n}{(m-r+\ell)[m-r+\ell+n]},$$

which can be shown to be equal to 1. It can also be seen from (2.9.2) and (2.9.3) that

$$L^{(1)}(0;i,j,k,\ell) = C^* \sum_{i=1}^{p} \sum_{j=1}^{q-p} \sum_{k=1}^{r-q} K_{i,j,k}$$

$$\cdot \left\{ - \frac{1}{C_7(C_7+C_8)C_9} \quad \frac{C_7}{(C_7+C_8)} \ln\left(\frac{C_9+n}{C_9}\right) \right.$$

$$\left. + \frac{1}{C_7(C_7+C_8)(C_7+C_8+C_9)} \left[\left(\frac{C_7}{C_7+C_8} + \frac{C_7}{C_7+C_8+C_9} \right) \ln\left(\frac{C_7+C_8+C_9+n}{C_7+C_8+C_9} \right) + \frac{C_7}{C_7+C_8+C_9} \quad \frac{n}{n+C_7+C_8+C_9} \right] \right\}$$

and

$$L^{(2)}(0;i,j,k,\ell) = C^* \sum_{i=1}^{p} \sum_{j=1}^{q-p} \sum_{k=1}^{r-q} K_{i,j,k}$$

$$\cdot \left\{ \frac{-1}{C_7(C_7+C_8)C_9} \quad d_1 H_{1,1}(C_9/n) \right.$$

$$\left. + \frac{1}{C_7(C_7+C_8)(C_7+C_8+C_9)} \left[e_1 H_{1,1}\left(\frac{C_7+C_8+C_9}{n} \right) + e_2 \ln\left(\frac{C_7+C_8+C_9+n}{C_7+C_8+C_9} \right) + e_3 \frac{n}{n+C_7+C_8+C_9} \right] \right\}$$

where

$$d_1 = 2\frac{C_7(C_8-C_7)}{(C_7+C_8)^2} ,$$

$$e_1 = 2\frac{C_8-C_7}{C_7+C_8}\left[\frac{C_7}{C_7+C_8} + \frac{C_7}{C_7+C_8+C_9}\right] - 4\left(\frac{C_7}{C_7+C_8+C_9}\right)^2 ,$$

$$e_2 = -2\frac{C_7}{C_7+C_8+C_9}\left[\frac{C_8-C_7}{C_7+C_8} - 3\frac{C_7}{C_7+C_8+C_9}\right] ,$$

$$e_3 = 2\left(\frac{C_7}{C_7+C_8+C_9}\right)^2$$

and

$$H_{1,1}(\delta) = -\tfrac{1}{2}(\ell n\,\delta)^2 - \pi^2/6 - \text{Dilog}\,(1+\delta).$$

Values of Dilog $(1+\delta)$, which is defined for $0 \le x \le 2$ by

$$\text{Dilog}\,(x) = -\int_1^x \frac{\ell n\, v}{1-v}\,dv = \sum_{k=1}^{\infty}(-1)^k(x-1)^k/k^2 ,$$ can be determined from values

tabulated by Stegun [8] and more extensively by Powell [7]. We recall that $C_7 = q-p+i-j$, $C_8 = r-q+j-k$ and $C_9 = m-r+k$.

A similar development for $p < r < q$ yields, for the integral $M(\sigma;i,j,k,\ell)$ corresponding to $L(\sigma;i,j,k,\ell)$,

$$M(\sigma;i,j,k,\ell) = C^{**}\sum_{i=1}^{p}\sum_{j=1}^{r-p}\sum_{k=1}^{q-r}(-1)^{i+j+k-1}\binom{p-1}{i-1}\binom{r-p-1}{j-1}\binom{q-r-1}{k-1}$$

$$\cdot\left\{\frac{n}{C_{10}C_{12}(C_{11}+C_{12})(C_{11}+C_{12}+n)} - \int_0^1 \frac{n\,dy}{C_{10}(C_{12}+C_{10}y^{\sigma})(C_{11}+C_{12}+C_{10}y^{\sigma}+ny)^2}\right. \qquad (3)$$

$$\left. - \int_0^1 \frac{n\,dy}{C_{12}(C_{10}+C_{12}y^{-\sigma})(C_{12}y^{-\sigma}+C_{10}+C_{11}+ny)^2}\right\} + 1 - \sum_{\ell=1}^{r} B_\ell \frac{n}{(m-r+\ell)(m-r+\ell+n)}$$

with $B_\ell = \frac{m!}{(r-1)!(m-r)!} (-1)^{r-1} \binom{r-1}{\ell-1}$ and $\sigma = 1/\nu$. Here C^{**} is equal to $\frac{n!}{(p-1)!(r-p-1)!(q-r-1)!(m-q)!}$, $C_{10} = r-p+i-j$, $C_{11} = q-r+j-k$ and $C_{12} = m-q+k$.

For small σ, one can approximate $M(\sigma;i,j,k,\ell)$ by $M(0;i,j,k,\ell) + \sigma M^{(1)}(0;i,j,k,\ell) + (\sigma^2/2)M^{(2)}(0;r,j,k,\ell)$, with $M^{(i)}(0;i,j,k,\ell)$ the ith derivative of $M(\sigma;i,j,k,\ell)$ evaluated at $\sigma = 0$.

Since $P[(Y_p-Y_q) \le 0]$ is equal to 1 (see the expression following (2) and let $\sigma = 1/\nu$ be equal to 0) and since $\int_0^\infty e^{-u}\,du = 1$, $M(\sigma;i,j,k,\ell)$ $= \int_0^\infty P[(Y_p-Y_q) \le (\ell n(u/n)-Y_r)/\nu]e^{-u}\,du$ must be equal to 1 for σ equal to zero. One can also demonstrate that this is so by letting σ be equal to zero in (3). And one can observe from (2.9.2) and (2.9.3) in [6] that $M^{(1)}(0;i,j,k,\ell) = 0$ and

$$M^{(2)}(0;i,j,k,\ell) = C^{**}\sum_{i=1}^{p}\sum_{j=1}^{r-p}\sum_{k-1}^{q-r}(-1)^{i+j+k-1}\binom{p-1}{i-1}\binom{r-p-1}{j-1}\binom{q-r-1}{k-1}$$

$$\cdot \frac{1}{(C_{10}+C_{11}+C_{12})^3}\left[4H_{1,1}\left(\frac{C_{10}+C_{11}+C_{12}}{n}\right) - 6\,\ell n\left(\frac{C_{10}+C_{11}+C_{12}+n}{C_{10}+C_{11}+C_{12}}\right) - 2\,\frac{n}{n+C_{10}+C_{11}+C_{12}}\right].$$

INVESTIGATION OF PRECISION OF APPROXIMATIONS AND EXPECTED SQUARED DEVIATION OF LOG WARRANTY PERIOD FROM $Z_{1,n}$

A comparison of calculations using the results derived above and results previously derived in [5] and [6] reveals that the precision of the new approximations is of the order tabulated in [5] and [6]. It remains then, to determine a most efficient combination of ordered observations (one such that log warranty period has smallest expected squared deviation from $Z_{1,n}$) for given n, m, s and α_0.

A preliminary investigation shows that for $s = 3$ and n large (so that ν is a large positive number), a combination of ordered observations such that $r=p=1$, $q=s$ is most efficient. This combination is, in fact, many times more efficient than the combination $r=1$, $p=2$, $q=3$ calculated in [6]. Also, from results in [3], we see that the expected squared deviation of $X_{r,m} + C_x(X_{p,m}-X_{q,m})$ from $x_R = \ln t$, for values of t corresponding to $\exp[-(t/\xi)^{\eta}] = .90, .95$, is smallest for $r=q$ (or equivalently for $r=p$) for moderate positive values of $-C_x$ and smallest for $p < r < q$ for small positive values of $-C_x$. Combinations such that r is greater than q are in no case considered in [3] most efficient estimators of x_R, nor do they in [4] yield approximately most accurate exact confidence bounds for x_R and $t_R = \exp(x_R)$. We therefore have eliminated from consideration combinations such that r is greater than q. Furthermore, we note that the Taylor series approximation for $M(\sigma;i,j,k,\ell)$ will not yield values of ν for $r-p > 1$ or $q-r > 1$, nor will $L(\sigma;i,j,k,\ell)$ yield values of ν for $r=p$ or $r=q$ if $q-p > 2$. Hence, we restrict attention to cases such that $r < p$ for $q \geq 4$. Investigation of untabulated computer output obtained in conjunction with evaluation of the tables in [3] reveals that one can find an expected squared deviation of $X_{r,m} - C_x(X_{q,m}-X_{p,m})$ from x_R corresponding to values of r less than p, close to the minimum value for $m \geq 4$.

Given in Table B.I are most efficient combinations of values of r, p and q (for $r \leq p$, $q = 2, 3$, $r < p$, $q \geq 4$), along with corresponding values of ν, for $n = 10, 11, \ldots, 25$, $s = m = 2, 3, \ldots, n-3$ and $\alpha_0 = .95$. Smaller values of α_0 have not been tabulated since tabulated results indicate that precision of the approximations increases as α_0 increases for fixed s, m and n and, naturally, tends to decrease as ν decreases (or σ increases). A few spot checks reveal that the approximations given are correct to within one unit in the first decimal place.

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TABLE B.I - VALUES FOR CALCULATING WARRANTY PERIODS OF THE FORM $X_{r,m} + \nu(X_{p,m}-X_{q,m})$ FROM WEIBULL DATA FOR LOTS OF SIZE n AND ASSURANCE LEVEL .95

n	m	r	p	q	ν
10	2	1	1	2	25.7
10	3	1	1	3	4.4
10	4	1	1	3	5.7
10	5	1	1	3	6.1
10	6	1	1	3	6.2
10	7	3	4	6	14.4
11	2	1	1	2	26.6
11	3	1	1	3	4.6
11	4	1	1	3	5.9
11	5	1	1	3	6.4
11	6	1	1	3	6.5
11	7	3	4	6	14.7
11	8	4	5	7	13.1
12	2	1	1	2	27.5
12	3	1	1	3	4.8
12	4	1	1	3	6.1
12	5	1	1	3	6.6
12	6	1	1	3	6.7
12	7	3	4	6	15.0
12	8	3	4	6	11.9
12	9	4	5	7	7.5
13	2	1	1	2	28.2
13	3	1	1	3	4.9
13	4	1	1	3	6.3
13	5	1	1	3	6.8
13	6	1	1	3	6.9
13	7	3	4	6	15.3
13	8	3	4	6	12.0
13	9	4	5	7	7.5
13	10	7	8	10	17.4
14	2	1	1	2	29.0
14	3	1	1	3	5.1
14	4	1	1	3	6.5
14	5	1	1	3	7.0
14	6	1	1	3	7.2
14	7	3	4	6	15.6
14	8	3	4	6	12.2
14	9	4	5	7	7.4
14	10	7	8	10	17.7
14	11	5	6	8	12.3
15	2	1	1	2	29.7
15	3	1	1	3	5.2
15	4	1	1	3	6.7
15	5	1	1	3	7.2
15	6	1	1	3	7.4
15	7	3	4	6	15.8
15	8	3	4	6	12.3
15	9	4	5	7	7.4
15	10	7	8	10	17.9
15	11	5	6	8	12.8
15	12	4	5	7	10.9
16	2	1	1	2	30.3
16	3	1	1	3	5.3
16	4	1	1	3	6.8
16	5	1	1	3	7.4
16	6	1	1	3	7.5
16	7	3	4	6	16.1
16	8	3	4	6	12.4
16	9	3	4	6	9.2
16	10	7	8	10	18.2
16	11	5	6	8	13.2
16	12	4	5	7	11.4
16	13	4	5	7	13.2
17	2	1	1	2	30.9
17	3	1	1	3	5.5
17	4	1	1	3	7.0
17	5	1	1	3	7.5
17	6	1	1	3	7.7
17	7	3	4	6	16.3
17	8	3	4	6	12.6
17	9	3	4	6	9.2
17	10	7	8	10	18.4
17	11	5	6	8	13.6
17	12	4	5	7	11.8
17	13	4	5	7	13.6
17	14	4	5	7	14.9

TABLE B.I (continued) - VALUES FOR CALCULATING WARRANTY PERIODS

OF THE FORM $X_{r,m} + \nu(X_{p,m}-X_{q,m})$ FROM WEIBULL DATA

FOR LOTS OF SIZE n AND ASSURANCE LEVEL .95

n	m	r	p	q	ν
18	2	1	1	2	31.5
18	3	1	1	3	5.6
18	4	1	1	3	7.1
18	5	1	1	3	7.7
18	6	1	1	3	7.9
18	7	3	4	6	16.5
18	8	3	4	6	12.7
18	9	3	4	6	9.2
18	10	7	8	10	18.7
18	11	4	5	7	9.3
18	12	4	5	7	12.2
18	13	4	5	7	14.1
18	14	4	5	7	15.4
18	15	4	5	7	16.2
19	2	1	1	2	32.0
19	3	1	1	3	5.7
19	4	1	1	3	7.2
19	5	1	1	3	7.8
19	6	1	1	3	8.0
19	7	3	4	6	16.7
19	8	3	4	6	12.8
19	9	3	4	6	9.3
19	10	7	8	10	18.9
19	11	5	6	8	14.4
19	12	4	5	7	12.6
19	13	4	5	7	14.5
19	14	4	5	7	15.8
19	15	4	5	7	16.7
19	16	4	5	7	17.3
20	2	1	1	2	32.5
20	3	1	1	3	5.8
20	4	1	1	3	7.4
20	5	1	1	3	8.0
20	6	1	1	3	8.2
20	7	3	4	6	17.0
20	8	3	4	6	13.0
20	9	3	4	6	9.3
20	10	7	8	10	19.1
20	11	5	6	8	14.7
20	12	4	5	7	13.0
20	13	4	5	7	14.9
20	14	4	5	7	16.2
20	15	4	5	7	17.1
20	16	4	5	7	17.8
20	17	4	5	7	18.2
21	2	1	1	2	33.0
21	3	1	1	3	5.9
21	4	1	1	3	7.5
21	5	1	1	3	8.1
21	6	1	1	3	8.3
21	7	3	4	6	17.1
21	8	3	4	6	13.1
21	9	3	4	6	9.3
21	10	7	8	10	19.3
21	11	4	5	7	10.3
21	12	4	5	7	13.4
21	13	4	5	7	15.3
21	14	4	5	7	16.6
21	15	4	5	7	17.6
21	16	4	5	7	18.2
21	17	4	5	7	18.7
21	18	4	5	7	19.0
22	2	1	1	2	33.5
22	3	1	1	3	6.0
22	4	1	1	3	7.6
22	5	1	1	3	8.2
22	6	1	1	3	8.5
22	7	3	4	6	17.3
22	8	3	4	6	13.2
22	9	3	4	6	9.4
22	10	7	8	10	19.5
22	11	4	5	7	10.6
22	12	4	5	7	13.7
22	13	4	5	7	15.7
22	14	4	5	7	17.0
22	15	4	5	7	18.0
22	16	4	5	7	18.6
22	17	4	5	7	19.1
22	18	4	5	7	19.4
22	19	4	5	7	19.6

TABLE B.I (continued) - VALUES FOR CALCULATING WARRANTY PERIODS

OF THE FORM $X_{r,m} + \nu(X_{p,m}-X_{q,m})$ FROM WEIBULL DATA

FOR LOTS OF SIZE n AND ASSURANCE LEVEL .95

n	m	r	p	q	ν
23	2	1	1	2	33.9
23	3	1	1	3	6.0
23	4	1	1	3	7.7
23	5	1	1	3	8.4
23	6	1	1	3	8.6
23	7	3	4	6	17.5
23	8	3	4	6	13.3
23	9	3	4	6	9.4
23	10	7	8	10	19.7
23	11	4	5	7	10.9
23	12	4	5	7	14.0
23	13	4	5	7	16.0
23	14	4	5	7	17.4
23	15	4	5	7	18.3
23	16	4	5	7	19.0
23	17	4	5	7	19.5
23	18	4	5	7	19.9
23	19	3	4	6	9.0
23	20	3	4	6	9.1

n	m	r	p	q	ν
24	2	1	1	2	34.3
24	3	1	1	3	6.1
24	4	1	1	3	7.8
24	5	1	1	3	8.5
24	6	1	1	3	8.7
24	7	3	4	6	17.7
24	8	3	4	6	13.5
24	9	3	4	6	9.5
24	10	7	8	10	19.9
24	11	4	5	7	11.2
24	12	4	5	7	14.3
24	13	4	5	7	16.3
24	14	4	5	7	17.7
24	15	4	5	7	18.7
24	16	4	5	7	19.4
24	17	4	5	7	19.9
24	18	3	4	6	9.2
24	19	3	4	6	9.4
24	20	3	4	6	9.5
24	21	3	4	6	9.5

n	m	r	p	q	ν
25	2	1	1	2	34.7
25	3	1	1	3	6.2
25	4	1	1	3	7.9
25	5	1	1	3	8.6
25	6	1	1	3	8.9
25	7	3	4	6	17.9
25	8	3	4	6	13.6
25	9	3	4	6	9.5
25	10	7	8	10	20.1
25	11	4	5	7	11.4
25	12	4	5	7	14.6
25	13	4	5	7	16.6
25	14	4	5	7	18.1
25	15	4	5	7	19.1
25	16	4	5	7	19.8
25	17	3	4	6	9.2
25	18	3	4	6	9.5
25	19	3	4	6	9.7
25	20	3	4	6	9.8
25	21	3	4	6	9.9
25	22	3	4	6	9.9

C. EXTENSION OF TABLES FOR OBTAINING THREE-ORDER-STATISTIC CONFIDENCE BOUNDS ON RELIABLE LIFE FOR A WEIBULL MODEL WITH PROGRESSIVE CENSORING

SUMMARY

In this paper we consider a Taylor series approximation to the distribution of a three-order-statistic estimator of reliable life for a two-parameter Weibull model. The approximating method will not yield results in certain cases considered, but in others the method gives values agreeing well with exact values for obtaining confidence bounds on reliable life. Tables given in [6] and [7] are extended.

INTRODUCTION

In the following we consider a two-parameter Weibull population of failure times and confidence bounds for a percentage point of this population which corresponds to a specified survival proportion γ. The percentage point of the population is sometimes called the reliable life corresponding to the specified survival proportion (or reliability), and the confidence bounds on reliable life are often referred to as tolerance limits or tolerance bounds.

In [6], tolerance boundr based on three ordered observations from a size n sample which may be censored at the $\underline{m}$th ordered observation are derived for a two-parameter Weibull population. In [5], similar tolerance bounds based on three ordered observations are derived under the assumption that the sample of Weibull variates may be censored progressively at each observation. That is, in order to approximate more closely a model corresponding to many real life collections of failure data, it is assumed in [5] that in life testing one or more items may be removed from life test at the time of any failure. In [6], removal of n-m items from test occurs only at the time of the $\underline{m}$th failure.

In [6], the three-order-statistic tolerance bounds for singly censored data are shown to be highly efficient compared with those based on all m ordered observations which can be obtained for certain combinations of n and m (see Johns and Lieberman [3]) and those which could be obtained for other combinations of n and m were the necessary tables available. In [4] several comparisons are given which demonstrate the efficiency of three-order-statistic point estimators for estimating reliable life. No such comparisons could be made for the progressive-censoring model at the time of the publication of [5]. Since then, however, best linear invariant estimators and their mean squared errors have been determined for this model (see [7]), and results of comparisons are consistent with the results for the single-censoring model.

In the two papers in which the tolerance bounds based on three ordered observations are derived, some tables for obtaining the bounds from sample data are given. In both cases computer calculation of the entries for the tables required numerical integration, which is extremely expensive in terms of computer time if much precision is required. Also, the numerical integration must be performed with progressively higher precision as m increases. Hence, the tables in [6] include no values of n or m greater than 12. Furthermore, in [5] 462 values for obtaining tolerance bounds were calculated and compared in terms of power of a corresponding test in order to determine which were the appropriate 57 values (all possible censoring patterns for $n=2,3,\ldots,6$, $m=2,3,\ldots,n$) to be actually tabulated for obtaining the "best" of the three-order-statistic bounds for $\gamma = .95$ and confidence level equal to .90.

In the following, a technique, suggested in [8] and based on Taylor series expansions, is used to derive the values from which the bounds can be obtained. The calculation of these derived values then requires no numerical integration and can be accomplished with high precision in a fraction of the computer time required previously. Thus, extension of the tables in [5] and [6] becomes feasible without the use of an inordinate amount of computer time.

The bounds to be obtained from tabulated values of ν corresponding to specified values of n, m, γ and confidence level are of the form $\exp[X_{r,n} + \nu(X_{q,n}-X_{p,n})]$ where $X_{i,m}$ is the logarithm of the ith observed failure time in the sample of size n with $X_{i,n} \le X_{i+1,n}$, $i=1,2\ldots,m-1$ and $X_{p,n} < X_{q,n} \le X_{m,n}$. In [5] and [6], the values of ν tabulated on the basis of comparisons of power functions or mean squared errors of corresponding point estimators pertain to bounds for which $p \le r < q$. Furthermore, the tabulated values of p, q and r correspond to those for the best linear invariant three-order-statistic point estimators in the single-censoring case (see [5] and [6]). For this reason, in deriving the Taylor series expansions for obtaining the bounds, we first assume such a relationship between p, q and r (since the derivation varies accordingly as the order of r with respect to the other two indices varies).

DERIVATION OF THE TOLERANCE BOUNDS

If T has the two-parameter Weibull distribution, then $X = \ln T$ has the first asymptotic distribution of smallest values with

$$P[X \le x] = 1 - \exp\left[-\exp\left(\frac{x-u}{b}\right)\right], \quad b > 0 \tag{1}$$

We let $X_{1,n}, X_{2,n}, \ldots, X_{m,n}$, with $X_{i,n} \le X_{i+1,n}$, $i=1,2,\ldots,m-1$, represent the first m of n sample observations and we note that

$$X_{r,n} + \frac{X_{r,n}-x_\gamma}{X_{q,n}-X_{p,n}}(X_{q,n}-X_{p,n}), \quad p < q < m, \tag{2}$$

is identically equal to x_γ. If $x_\gamma = u + b \ln\ln(1/\gamma)$ (that is,

$\gamma = \exp\left[-\exp\left(\frac{x_\gamma - u}{b}\right)\right]$), then x_γ is the reliable life corresponding to the survival proportion γ. If we define Y_i to be $(X_{i,n}-u)/b$, then $V = (X_{r,n}-x_\gamma)/(X_{q,n}-X_{p,n})$ is equal to $(Y_r-\ln\ln(1/\gamma))/(Y_q-Y_p)$ and can be shown to have a parameter-free distribution. If a lower confidence bound at level $1-\alpha$ is found for V and this bound is substituted for V in (2), a $(1-\alpha)$-level lower confidence bound is thus defined for x_γ. The confidence bounds derived in [5] and [6] are of this form. As indicated earlier, the tabulated values of the percentage points of V all correspond to the case $p \leq r < q$, however the derivation has not been given anywhere for $p=r$ or for $p < r < q$, which case yields the most complicated expression for the distribution of V. In [5], the distribution of V is derived for $r < p < q$ and in [6] the expression for the exact distribution of V is given but not derived for $p < r < q$. We therefore give here the complete derivation of the exact distribution of $V = (Y_r-\ln\ln(1/\gamma))/(Y_p-Y_q)$ for $p < r < q \leq m$, and then proceed to determine a Taylor series approximation for this exact expression which must be evaluated by means of numerical integration. It will be shown that the Taylor series approach gives excellent approximations in some cases and in certain other cases fails entirely to yield an approximate value.

The joint density of Y_p, Y_r and Y_q is, from (1) above, given by

$$f(y_p,y_r,y_q) = \sum_{i=1}^{p} \sum_{j=1}^{r-p} \sum_{k=1}^{q-r} K_{i,j,k}$$

$$\cdot \exp\left(y_p+y_r+y_q-C_{3,i,j}\ell^{y_p}-C_{4,j,k}\ell^{y_r}-C_{5,k}\ell^{y_q}\right)$$

$$-\infty < y_p \le y_r,$$

$$-\infty < y_r < \infty,$$

$$y_r \le y_q < \infty .$$

Here, for the general progressive-censoring model (see [5]) where r_i represents the number of items removed from life test at the time of the ith failure, $K_{i,j,k}$ is given by

$$n\prod_{j=2}^{q}\left[n-\sum_{i=1}^{j-1}(r_i+1)\right](-1)^{i+j+k+1}C_1(i,0,p)C_2(i,p)C_1(j,p,r)C_2(j,r)C_1(k,p,q)C_2(k,q),$$

with

$$C_1(\ell,s,\nu) = \left[\prod_{\mu=1}^{\nu-s-\ell}\sum_{i=\nu-\ell-\mu+1}^{\nu-\ell}(r_i+1)\right]^{-1}, \quad \ell = 1,2,\ldots, \nu-s-1, \ \nu-s \ge 2$$

and

$$C_2(\ell,\nu) = \left[\prod_{\mu=1}^{j-1}\sum_{i=\nu-\ell+1}^{\nu-\ell+\mu}(r_i+1)\right]^{-1}, \quad \ell = 2,3,\ldots, \nu, \ \nu \ge 2,$$

$$C_1(\nu-s,s,\nu) = C_2(1,\nu) = 1.$$

The constants $C_{3,i,j}$, $C_{4,j,k}$ and $C_{5,k}$ are given by

$$C_{3,i,j} = \sum_{i=p-i+1}^{r-j}(r_i+1), \quad i = 1,2,\ldots, p, \ j = 1,2,\ldots, r-p,$$

$$C_{4,j,k} = \sum_{i-r-j+1}^{q-k}(r_i+1), \quad j = 1,2,\ldots, r-p, \ k = 1,2,\ldots, q-r,$$

and

$$C_{5,k} = \sum_{i=q-k+1}^{m} (r_i+1), \quad k = 1,2,\ldots, q-r.$$

For the single-censoring model, $K_{i,j,k}$ reduces to

$$\frac{n!\ (-1)^{i+j+k+1}}{(p-1)!(r-p-1)!(q-r-1)!(n-q)!} \binom{p-1}{i-1} \binom{r-1}{j-1} \binom{q-1}{k-1} \text{ with } C_{3,i,j} = r-p+i-j,$$

$C_{4,j,k} = q-p+j-k$ and $C_{5,k} = n-q+k$. Hereafter we write $C_{3,i,j}$, $C_{4,j,k}$ and $C_{5,k}$ each with a single subscript.

If we let $z_1 = y_r$, $z_2 = y_p - y_r$ and $z_3 = y_q - y_p$, then $y_r = z_1$, $y_p = z_1 + z_2$ and $y_q = z_1 + z_2 + z_3$ with $-\infty < z_1 < \infty$, $-z_3 \le z_2 \le 0$ and $0 < z_3 < \infty$ and the Jacobian of the transformation is [illegible]. Therefore, the joint density of $Z_1 = Y_r$ and $Z_3 = Y_q - Y_p$ is given by

$$\sum_{i=1}^{p} \sum_{j=1}^{r-p} \sum_{k=1}^{q-p} K_{i,j,k}$$

$$\cdot\left\{ \frac{\exp(3z_1-z_3)\ \exp\left[C_3 e^{z_1-z_3} - (C_4+C_5)e^{z_1}\right] \left[1 + C_3 e^{z_1-z_3} + C_5 e^{z_1}\right]}{(C_3 e^{z_1+z_3} + C_5 e^{z_1})^2} \right.$$

$$\left. - \frac{\exp(3z_1+z_3)\ \exp\left[-(C_3+C_4)e^{z_1} - C_5 e^{z_1+z_3}\right] \left[1 + C_3 e^{z_1} + C_5 e^{z_1+z_3}\right]}{(C_3 e^{z_1} + C_5 e^{z_1+z_3})^2} \right\}.$$

We now let $s = \exp[z_1 - \ln\ln(1/\gamma)]$, $v = \frac{-\ln s}{z_3}$ and $\xi = \ln(1/\gamma)$. Then, the probability density function of $V = (\ln\ln(1/\gamma) - Y_1)/(Y_q - Y_p)$ is

$$\int_s \sum_{i=1}^{p} \sum_{j=1}^{r-p} \sum_{k=1}^{q-r} K_{i,j,k}\, \xi^3 \frac{\ell n\, s}{v^2}$$

$$\left\{ \frac{s^{2+1/v} \exp\left[- \xi(C_4+C_5)\, s - \xi\, C_3 s^{1+1/v}\right] (1+\xi\, C_3 s^{1+1/v} + \xi\, C_5 s)}{(\xi C_3 s^{1+1/v} + \xi C_5 s)^2} \right.$$

$$\left. - \frac{s^{2-1/v} \exp\left[- \xi(C_3+C_4)\, s - \xi\, C_5 s^{1-1/v}\right] (1+\xi\, C_3 s^{1-1/v})}{(\xi C_3 s + \xi C_5 s^{1-1/v})^2} \right\} ds$$

$0 \le s < \infty$

$0 \le v < \infty$ for $0 \le s \le 1$

and $-\infty < v \le 0$ for $1 \le s < \infty$.

Hence,

$$P(V \le \nu) = \int_1^{\infty} \sum_{i=1}^{p} \sum_{j=1}^{r-p} \sum_{k=1}^{q-r} K_{i,j,k} \left\{ - \frac{\xi \exp\left[-\xi(C_3+C_4+C_5)s\right]}{C_3 C_5} \right. \tag{3}$$

$$\left. + \frac{\xi\, s \exp\left[-\xi(C_3+C_4)s - \xi\, C_5 s^{1-1/\nu}\right]}{C_5(C_3 s+C_5 s^{1-1/\nu})} + \frac{\xi\, s \exp\left[-\xi(C_4+C_5)s - \xi\, C_3 s^{1+1/\nu}\right]}{C_3(C_3 s^{1+1/\nu}+C_5 s)} \right\} ds$$

for $-\infty < \nu \le 0$. If $0 \le \nu < \infty$, $P(V > \nu)$ is given by the negative of the expression above, except that the limits of integration for s are 0 and 1 rather than 1 and ∞, respectively.

Now, in order to obtain a Taylor-series representation for the distribution of V we let σ be equal to $1/(\nu-1)$. Since, in the ranges of γ and α of interest ν will be negative, σ will range from -1 to 0. We then express (3) as a function of σ,

$$L(\sigma) = \sum_{i=1}^{p} \sum_{j=1}^{r-p} \sum_{k=1}^{q-r} K_{i,j,k} \left\{ - \frac{\exp\left[-\xi(C_3+C_4+C_5)\right]}{C_3C_5(C_3+C_4+C_5)} \right.$$

$$\left. + \int_1^\infty \left\{ \frac{\xi\, s \exp\left[-\xi(C_3+C_4)s - \xi\, C_5 s^{1/(\sigma+1)}\right]}{C_5(C_3 s + C_5 s^{1/(\sigma+1)})} + \frac{\xi\, s \exp\left[-\xi(C_4+C_5)s - \xi\, C_3 s^{(2\sigma+1)/(\sigma+1)}\right]}{C_3(C_3 s^{(2\sigma+1)/(\sigma+1)} + C_5 s)} \right\} ds \right\}.$$

Tnen, for $L^{(i)}(\sigma)$ the <u>i</u>th derivative of $L(\sigma)$ with respect to σ,

$$L^{(1)}(\sigma) = \sum_{i=1}^{p} \sum_{j=1}^{r-p} \sum_{k=1}^{q-r} K_{i,j,k}$$

$$\cdot \left\{ - \int_1^\infty \frac{\ln s}{(\sigma+1)^2} \frac{s^{\frac{3\sigma+2}{\sigma+1}} \xi^3 \exp\left[-\xi(C_4+C_5)s - \xi\, C_3 s^{\frac{2\sigma+1}{\sigma+1}}\right] (1 + \xi\, C_3 s^{\frac{2\sigma+1}{\sigma+1}} + \xi\, C_5 s) ds}{(\xi\, C_3 s^{\frac{2\sigma+1}{\sigma+1}} + \xi\, C_5 s)^2} \right. \tag{4}$$

$$\left. + \int_1^\infty \frac{\ln s}{(\sigma+1)^2} \frac{s^{\frac{\sigma+2}{\sigma+1}} \xi^3 \exp\left[-\xi(C_3+C_4)s - \xi\, C_5 s^{1/(\sigma+1)}\right] (1 + \xi\, C_3 s + \xi C_5 s^{1/(\sigma+1)}) ds}{(\xi\, C_3 s + \xi\, C_5 s^{1/(\sigma+1)})^2} \right\}$$

and

$$L^{(2)}(\sigma) = \sum_{i=1}^{p} \sum_{j=1}^{r-p} \sum_{k=1}^{q-r} K_{i,j,k}$$

$$\cdot \left\{ \int_1^\infty \frac{\ln s}{(\sigma+1)^4} \frac{s^{\frac{3\sigma+2}{\sigma+1}} \xi^3 \exp\left[-\xi(C_4+C_5)s - \xi\, C_3 s^{\frac{2\sigma+1}{\sigma+1}}\right]}{(\xi\, C_3 s^{\frac{2\sigma+1}{\sigma+1}} + \xi\, C_5 s)^3} \right.$$

$$\cdot \left\{ \left[2\xi C_3 s^{\frac{2\sigma+1}{\sigma+1}} \ln s + (\xi C_3 s^{\frac{2\sigma+1}{\sigma+1}} \ln s + 2(\sigma+1) - \ln s)(\xi C_3 s^{\frac{2\sigma+1}{\sigma+1}} + \xi C_5 s) \right] (1 + \xi C_3 s^{\frac{2\sigma+1}{\sigma+1}} + \xi C_5 s) \right\} ds$$

$$+ \int_1^\infty \frac{\ln s}{(\sigma+1)^4} \frac{s^{\frac{\sigma+2}{\sigma+1}} \xi^3 \exp\left[-\xi(C_3+C_4)s - \xi\, C_5 s^{1/(\sigma+1)}\right]}{(\xi\, C_3 s + \xi\, C_5 s^{1/(\sigma+1)})^3} \tag{5}$$

$$\left. \cdot \left\{ \left[2\xi C_5 s^{1/(\sigma+1)} \ln s + (\xi C_5 s^{1/(\sigma+1)} \ln s - 2(\sigma+1) - \ln s)(\xi C_3 s + \xi C_5 s^{1/(\sigma+1)}) \right] (1 + \xi C_3 s + \xi C_5 s^{1/(\sigma+1)}) \right\} ds \right\}.$$

Thus,

$$L(0) = \sum_{i=1}^{p} \sum_{j=1}^{r-p} \sum_{k=1}^{q-r} K_{i,j,k} \left\{ \frac{-\exp\left[-\xi(C_3+C_4+C_5)\right]}{C_3C_5(C_3+C_4+C_5)} + \frac{\exp\left[-\xi(C_3+C_4+C_5)\right]}{C_3C_5(C_3+C_4+C_5)} \right\} = 0 \tag{6}$$

and

$$L^{(1)}(0) = \sum_{i=1}^{p} \sum_{j=1}^{r-p} \sum_{k=1}^{q-r} K_{i,j,k} \left\{ - \int_1^{\infty} \frac{\ln s \ \xi \exp\left[-\xi(C_3+C_4+C_5)\right] (1 + \xi C_3 s + \xi C_5 s) \, ds}{(C_3+C_5)^2} + \int_1^{\infty} \frac{\ln s \ \xi \exp\left[-\xi(C_3+C_4+C_5)\right] (1 + \xi C_3 s + \xi C_5 s) \, ds}{(C_3+C_5)^2} \right\} = 0. \tag{7}$$

If we let $\eta = \xi(C_3+C_4+C_5)$ and let $\Phi_{r,s}(\eta) = \int_{\eta}^{\infty} x^r (\ln x)^s \, e^{-x} dx$, then

$$L^{(2)}(0) = \sum_{i=1}^{p} \sum_{j=1}^{r-p} \sum_{k=1}^{q-r} K_{i,j,k} \left\{ \frac{\xi^3\left[\Phi_{2,2}(\eta) - 2\ln \eta \ \Phi_{2,1}(\eta) + (\ln \eta)^2 e^{-\eta}(2+2\eta+\eta^2)\right]}{\eta^3} + \frac{\xi^2\left[\Phi_{1,2}(\eta) - 2\ln \eta \ \Phi_{1,1}(\eta) + (\ln \eta)^2 (1+\eta) e^{-\eta}\right]}{(C_3+C_5)\ \eta^2} \right\}. \tag{8}$$

A method of evaluation of $\Phi_{2,1}(\eta)$ and $\Phi_{2,2}(\eta)$ is given in the Appendix.

A similar development for $r = p$ yields $L(0) = 0$,

$$L^{(1)}(0) = \sum_{i=1}^{p} \sum_{j=1}^{q-p} G_{i,j} \frac{-\xi^2[\Phi_{1,1}(\eta) - \ln \eta(1+\eta)e^{-\eta}]}{\eta^2} \tag{9}$$

and

$$L^{(2)}(0) = \sum_{i=1}^{p} \sum_{j=1}^{q-p} G_{i,j} \left\{-C_6\xi^3 \frac{\Phi_{2,2}(\eta)-2\ln \eta\, \Phi_{2,1} + (\ln \eta)^2(2+2\eta+\eta^2)e^{-\eta}}{\eta^3}\right.$$

$$+ \frac{\xi^2[\Phi_{1,2}(\eta)-2\ln \eta\, \Phi_{1,1}(\eta) + (\ln \eta)^2(1+\eta)e^{-\eta}]}{\eta^2} \tag{10}$$

$$\left. + \frac{2\xi^2[\Phi_{1,1}(\eta)- \ln(\eta)(1+\eta)e^{-\eta}]}{\eta^2}\right\},$$

where $G_{i,j} = \frac{n!}{(p-1)!(q-p-1)!(n-q)!}\binom{p-1}{i-1}\binom{q-p-1}{j-1}$, $C_6 = q-p+i-j$ and $\eta = (n-p+i)\xi$ for the single censoring model.

At this point we observe some rather peculiar phenomena concerning the Taylor series approximations for the cases $p < r < q$ and $r=p$. First note that no matter whether σ is equal to $1/(\nu-1)$ or $1/(\nu-c)$, $c > 0$, the values of $L(0)$, $L^{(1)}(0)$ and $L^{(2)}(0)$ will be as given by (6), (7) and (8). Furthermore, for $r - p > 1$ or $q - r > 1$ the first term of (8) is equal to zero. Also, for $r=p$, (9) is equal to zero for $q - p > 1$ and (10) is also equal to zero for $q - p > 2$. We therefore consider the Taylor series approximation for $r < p$ in order to determine if the difficulties mentioned above can be avoided in this way. For $r < p$ we obtain $L(0) = 0$,

$$L^{(1)}(0) = \sum_{i=1}^{r}\sum_{j=1}^{p-r}\sum_{k=1}^{q-r} H_{i,j,k}\left\{\frac{-\xi^2[\Phi_{1,1}(\eta)-\ln\eta(1+\eta)e^{-\eta}]}{(C_8+C_9)\eta^2} + \frac{-\xi[\Phi_{0,1}(\eta)-\ln\eta e^{-\eta}]}{(C_8+C_9)^2\eta}\right\}$$

and

$$L^{(2)}(0) = -\sum_{i=1}^{r}\sum_{j=1}^{p-r}\sum_{k=1}^{q-r} H_{i,j,k}\left\{C_9\xi^3\frac{[\Phi_{2,2}(\eta)-2\ln\eta\,\Phi_{2,1}(\eta) + (\ln\eta)^2e^{-\eta}(2+2\eta+\eta^2)]}{(C_8+C_9)\eta^3}\right.$$

$$+\frac{(2C_9-C_8)\xi^2[\Phi_{1,2}(\eta)-2\ln\eta\,\Phi_{1,1}(\eta) + (\ln\eta)^2(1+\eta)e^{-\eta}]}{(C_8+C_9)^2\eta^2}$$

$$-\frac{(C_8-C_9)\xi[\Phi_{0,2}(\eta)-2\ln\eta\,\Phi_{0,1}(\eta) + (\ln\eta)^2e^{-\eta}]}{(C_8+C_9)^3\eta}$$

$$\left.-\frac{2\xi^2[\Phi_{1,1}(\eta)-\ln\eta(1+\eta)e^{-\eta}]}{(C_8+C_9)\eta^2} - \frac{2\xi[\Phi_{0,1}(\eta)-\ln\eta\,e^{-\eta}]}{(C_8+C_9)^2\eta}\right\},$$

where $H_{i,j,k} = \frac{n!}{(r-1)!(p-r-1)!(q-r-1)!(n-q)!}\binom{r-1}{i-1}\binom{p-r-1}{j-1}\binom{q-r-1}{k-1}$,

$\Phi_{0,1}(y) = \int_y^\infty \ln x\, e^{-x}dx$, $\Phi_{0,2}(y) = \int_y^\infty (\ln x)^2e^{-x}dx$ and for the single censoring model $C_8 = q-p+j-k$, $C_9 = n-q+k+1$ and $\eta = (n-r+i+1)\xi$.

For $r < p$, $L^{(1)}(0)$ and $L^{(2)}(0)$ behave well and yield approximations good to within about 5 units in the first decimal place as long as r is equal to 1. For $1 < r < p$, both $L^{(1)}(0)$ and $L^{(2)}(0)$ are identically equal to zero. One can observe from unpublished computer output obtained in conjunction with work on [6] that, for $m \geq 4$, it is possible to find a combination of r, p and q such that $r < p$ and the mean squared deviation of the estimator $X_{r,n} + C_x(X_{q,n}-X_{p,n})$ from x_γ is close to the minimum for three-order-statistic estimators. Furthermore, for $\gamma = .95$ and $n \geq 7$, the combination $r = p = 1$, $q = 2$ (equivalent to $r=q=2$, $p=1$) has smallest mean squared error among estimators with $X_{r,n} = X_{1,n}$. For n as large as 15, the mean squared deviation of $X_{1,n} + C_x(X_{2,n}-X_{1,n})$ from x_γ is only 1.43 times that based on the most efficient combination of ordered observations.

In Table C.I, values of r, p, q, and ν for obtaining confidence bounds of the form $X_{r,n} + \nu(X_{q,n}-X_{p,n})$ are given for all possible censorings for $n = 2, 3, \ldots, 9$ and for single censoring for $n = 10, 11, \ldots, 15$, $\gamma = .95$, $\alpha = .10$, $.20$. From the published tables in [6] and additional calculations made during this study, one can observe that the tabulated values may be in error by as much as 4 units in the first decimal place. It might therefore be worthwhile to evaluate $L^{(3)}(0)$ for $r = p$ during subsequent investigations. This would involve the derivation of an expression for $\Phi_{0,3}$.

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TABLE C.I - VALUES FOR OBTAINING A (1-α)-LEVEL CONFIDENCE BOUND ON THE 5 PERCENT POINT OF A WEIBULL POPULATION

					Number Censored at ith Observation									α = .10	α = .20
n	m	r	p	q	1	2	3	4	5	6	7	8	9	ν	ν
2	2	2	1	2	0	0	0	0	0	0	0	0	0	-14.0	-7.2
3	2	2	1	2	1	0	0	0	0	0	0	0	0	-8.7	-4.8
3	2	2	1	2	0	1	0	0	0	0	0	0	0	-15.1	-7.5
3	3	3	1	3	0	0	0	0	0	0	0	0	0	-4.6	-3.2
4	2	2	1	2	2	0	0	0	0	0	0	0	0	-6.2	-3.6
4	2	2	1	2	1	1	0	0	0	0	0	0	0	-10.6	-5.5
4	2	2	1	2	0	2	0	0	0	0	0	0	0	-14.6	-7.1
4	3	1	2	3	1	0	0	0	0	0	0	0	0	-10.3	-4.7
4	3	1	2	3	0	1	0	0	0	0	0	0	0	-8.6	-4.1
4	3	1	2	3	0	0	1	0	0	0	0	0	0	-16.0	-6.9
4	4	1	3	4	0	0	0	0	0	0	0	0	0	-13.8	-6.3
5	2	2	1	2	3	0	0	0	0	0	0	0	0	-4.8	-2.8
5	2	2	1	2	2	1	0	0	0	0	0	0	0	-8.0	-4.3
5	2	2	1	2	1	2	0	0	0	0	0	0	0	-10.9	-5.5
5	2	2	1	2	0	3	0	0	0	0	0	0	0	-13.8	-6.5
5	3	1	2	3	2	0	0	0	0	0	0	0	0	-8.1	-3.7
5	3	1	2	3	1	1	0	0	0	0	0	0	0	-6.7	-3.1
5	3	1	2	3	0	2	0	0	0	0	0	0	0	-6.0	-2.8
5	3	1	2	3	1	0	1	0	0	0	0	0	0	-12.4	-5.4
5	3	1	2	3	0	1	1	0	0	0	0	0	0	-11.1	-4.9
5	3	1	2	3	0	0	2	0	0	0	0	0	0	-15.9	-6.4
5	4	1	3	4	1	0	0	0	0	0	0	0	0	-11.2	-5.1
5	4	1	3	4	0	1	0	0	0	0	0	0	0	-10.4	-4.8
5	4	1	3	4	0	0	1	0	0	0	0	0	0	-9.0	-4.2
5	4	1	3	4	0	0	0	1	0	0	0	0	0	-16.8	-7.1
5	5	1	3	5	0	0	0	0	0	0	0	0	0	-4.6	-3.0

TABLE C.I (continued) - VALUES FOR OBTAINING A (1-α)-LEVEL CONFIDENCE BOUND ON THE 5 PERCENT POINT OF A WEIBULL POPULATION

					Number Censored at ith Observation									α = .10	α = .20
n	m	r	p	q	1	2	3	4	5	6	7	8	9	ν	ν
6	2	2	1	2	4	0	0	0	0	0	0	0	0	-3.9	-2.3
6	2	2	1	2	3	1	0	0	0	0	0	0	0	-6.3	-3.5
6	2	2	1	2	2	2	0	0	0	0	0	0	0	-8.6	-4.5
6	2	2	1	2	1	3	0	0	0	0	0	0	0	-10.7	-5.3
6	2	2	1	2	0	4	0	0	0	0	0	0	0	-12.8	-6.0
6	3	1	2	3	3	0	0	0	0	0	0	0	0	-6.6	-3.0
6	3	1	2	3	2	1	0	0	0	0	0	0	0	-5.4	-2.5
6	3	1	2	3	1	2	0	0	0	0	0	0	0	-4.8	-2.2
6	3	1	2	3	0	3	0	0	0	0	0	0	0	-4.4	-2.1
6	3	1	2	3	2	0	1	0	0	0	0	0	0	-10.0	-4.3
6	3	1	2	3	1	1	1	0	0	0	0	0	0	-8.8	-3.9
6	3	1	2	3	0	2	1	0	0	0	0	0	0	-8.1	-3.7
6	3	1	2	3	1	0	2	0	0	0	0	0	0	-12.7	-5.1
6	3	1	2	3	0	1	2	0	0	0	0	0	0	-11.6	-4.9
6	3	1	2	3	0	0	3	0	0	0	0	0	0	-15.0	-5.7
6	4	1	3	4	2	0	0	0	0	0	0	0	0	-9.3	-4.2
6	4	1	3	4	0	2	0	0	0	0	0	0	0	-8.2	-3.9
6	4	1	3	4	1	1	0	0	0	0	0	0	0	-8.6	-3.8
6	4	1	3	4	1	0	1	0	0	0	0	0	0	-7.4	-3.4
6	4	1	3	4	0	1	1	0	0	0	0	0	0	-7.0	-3.2
6	4	1	3	4	0	0	2	0	0	0	0	0	0	-6.4	-3.0
6	4	1	3	4	1	0	0	1	0	0	0	0	0	-13.8	-5.8
6	4	1	3	4	0	1	0	1	0	0	0	0	0	-13.1	-5.6
6	4	1	3	4	0	0	1	1	0	0	0	0	0	-11.8	-5.1
6	4	1	3	4	0	0	0	2	0	0	0	0	0	-17.1	-6.7
6	5	1	3	5	1	0	0	0	0	0	0	0	0	-3.7	-2.4
6	5	1	3	5	0	1	0	0	0	0	0	0	0	-3.5	-2.2
6	5	1	3	5	0	0	1	0	0	0	0	0	0	-3.1	-1.9
6	5	1	3	5	0	0	0	1	0	0	0	0	0	-4.0	-2.5
6	5	1	3	5	0	0	0	0	1	0	0	0	0	-6.1	-4.0
6	6	1	4	6	0	0	0	0	0	0	0	0	0	-4.6	-2.9
7	2	2	1	2	5	0	0	0	0	0	0	0	0	-3.2	-2.0
7	2	2	1	2	4	1	0	0	0	0	0	0	0	-5.2	-3.0
7	2	2	1	2	3	2	0	0	0	0	0	0	0	-7.0	-3.8
7	2	2	1	2	2	3	0	0	0	0	0	0	0	-8.7	-4.5
7	2	2	1	2	1	4	0	0	0	0	0	0	0	-10.3	-5.0
7	2	2	1	2	0	5	0	0	0	0	0	0	0	-11.9	-5.5

TABLE C.I (continued) - VALUES FOR OBTAINING A (1-α)-LEVEL CONFIDENCE BOUND ON THE 5 PERCENT POINT OF A WEIBULL POPULATION

n	m	r	p	q	1	2	3	4	5	6	7	8	9	α = .10 ν	α = .20 ν
7	3	2	1	2	4	0	0	0	0	0	0	0	0	-5.2	-3.0
7	3	2	1	2	3	1	0	0	0	0	0	0	0	-7.0	-3.8
7	3	2	1	2	2	2	0	0	0	0	0	0	0	-8.7	-4.5
7	3	2	1	2	1	3	0	0	0	0	0	0	0	-10.3	-5.0
7	3	2	1	2	0	4	0	0	0	0	0	0	0	-11.9	-5.5
7	3	2	1	2	3	0	1	0	0	0	0	0	0	-7.0	-3.8
7	3	2	1	2	2	1	1	0	0	0	0	0	0	-8.7	-4.5
7	3	2	1	2	1	2	1	0	0	0	0	0	0	-10.3	-5.0
7	3	2	1	2	0	3	1	0	0	0	0	0	0	-11.9	-5.5
7	3	2	1	2	2	0	2	0	0	0	0	0	0	-8.7	-4.5
7	3	2	1	2	1	1	2	0	0	0	0	0	0	-10.3	-5.0
7	3	2	1	2	0	2	2	0	0	0	0	0	0	-11.9	-5.5
7	3	2	1	2	1	0	3	0	0	0	0	0	0	-10.3	-5.0
7	3	2	1	2	0	1	3	0	0	0	0	0	0	-11.9	-5.5
7	3	2	1	2	0	0	4	0	0	0	0	0	0	-11.9	-5.5
7	4	2	1	2	3	0	0	0	0	0	0	0	0	-7.0	-3.8
7	4	2	1	2	2	1	0	0	0	0	0	0	0	-8.7	-4.5
7	4	2	1	2	1	2	0	0	0	0	0	0	0	-10.3	-5.0
7	4	2	1	2	0	3	0	0	0	0	0	0	0	-11.9	-5.5
7	4	2	1	2	2	0	1	0	0	0	0	0	0	-8.7	-4.5
7	4	2	1	2	1	1	1	0	0	0	0	0	0	-10.3	-5.0
7	4	2	1	2	0	2	1	0	0	0	0	0	0	-11.9	-5.5
7	4	2	1	2	1	0	2	0	0	0	0	0	0	-10.3	-5.0
7	4	2	1	2	0	1	2	0	0	0	0	0	0	-11.9	-5.5
7	4	2	1	2	0	0	3	0	0	0	0	0	0	-11.9	-5.5
7	4	2	1	2	2	0	0	1	0	0	0	0	0	-8.7	-4.5
7	4	2	1	2	1	1	0	1	0	0	0	0	0	-10.3	-5.0
7	4	2	1	2	0	2	0	1	0	0	0	0	0	-11.9	-5.5
7	4	2	1	2	1	0	1	1	0	0	0	0	0	-10.3	-5.0
7	4	2	1	2	0	1	1	1	0	0	0	0	0	-11.9	-5.5
7	4	2	1	2	0	0	2	1	0	0	0	0	0	-11.9	-5.5
7	4	2	1	2	1	0	0	2	0	0	0	0	0	-10.3	-5.0
7	4	2	1	2	0	1	0	2	0	0	0	0	0	-11.9	-5.5
7	4	2	1	2	0	0	1	2	0	0	0	0	0	-11.9	-5.5
7	4	2	1	2	0	0	0	3	0	0	0	0	0	-11.9	-5.5
7	5	2	1	2	2	0	0	0	0	0	0	0	0	-8.7	-4.5
7	5	2	1	2	1	1	0	0	0	0	0	0	0	-10.3	-5.0
7	5	2	1	2	0	2	0	0	0	0	0	0	0	-11.9	-5.5
7	5	2	1	2	1	0	1	0	0	0	0	0	0	-10.3	-5.0
7	5	2	1	2	0	1	1	0	0	0	0	0	0	-11.9	-5.5
7	5	2	1	2	0	0	2	0	0	0	0	0	0	-11.9	-5.5
7	5	2	1	2	1	0	0	1	0	0	0	0	0	-10.3	-5.0

Number Censored at ith Observation: columns 1–9.

TABLE C.I (continued) - VALUES FOR OBTAINING A (1-α)-LEVEL CONFIDENCE BOUND ON THE 5 PERCENT POINT OF A WEIBULL POPULATION

					Number Censored at ith Observation									α = .10	α = .20
n	m	r	p	q	1	2	3	4	5	6	7	8	9	ν	ν
7	5	2	1	2	0	1	0	1	0	0	0	0	0	-11.9	-5.5
7	5	2	1	2	0	0	1	1	0	0	0	0	0	-11.9	-5.5
7	5	2	1	2	0	0	0	2	0	0	0	0	0	-11.9	-5.5
7	5	2	1	2	1	0	0	0	1	0	0	0	0	-10.3	-5.0
7	5	2	1	2	0	1	0	0	1	0	0	0	0	-11.9	-5.5
7	5	2	1	2	0	0	1	0	1	0	0	0	0	-11.9	-5.5
7	5	2	1	2	0	0	0	1	1	0	0	0	0	-11.9	-5.5
7	5	2	1	2	0	0	0	0	2	0	0	0	0	-11.9	-5.5
7	6	2	1	2	1	0	0	0	0	0	0	0	0	-10.3	-5.0
7	6	2	1	2	0	1	0	0	0	0	0	0	0	-11.9	-5.5
7	6	2	1	2	0	0	1	0	0	0	0	0	0	-11.9	-5.5
7	6	2	1	2	0	0	0	1	0	0	0	0	0	-11.9	-5.5
7	6	2	1	2	0	0	0	0	1	0	0	0	0	-11.9	-5.5
7	6	2	1	2	0	0	0	0	0	1	0	0	0	-11.9	-5.5
7	7	2	1	2	0	0	0	0	0	0	0	0	0	-11.9	-5.5
8	2	2	1	2	6	0	0	0	0	0	0	0	0	-2.8	-1.7
8	2	2	1	2	5	1	0	0	0	0	0	0	0	-4.4	-2.6
8	2	2	1	2	4	2	0	0	0	0	0	0	0	-5.9	-3.2
8	2	2	1	2	3	3	0	0	0	0	0	0	0	-7.2	-3.8
8	2	2	1	2	2	4	0	0	0	0	0	0	0	-8.5	-4.3
8	2	2	1	2	1	5	0	0	0	0	0	0	0	-9.8	-4.7
8	2	2	1	2	0	6	0	0	0	0	0	0	0	-11.0	-5.0
8	3	2	1	2	5	0	0	0	0	0	0	0	0	-4.4	-2.6
8	3	2	1	2	4	1	0	0	0	0	0	0	0	-5.9	-3.2
8	3	2	1	2	3	2	0	0	0	0	0	0	0	-7.2	-3.8
8	3	2	1	2	2	3	0	0	0	0	0	0	0	-8.5	-4.3
8	3	2	1	2	1	4	0	0	0	0	0	0	0	-9.8	-4.7
8	3	2	1	2	0	5	0	0	0	0	0	0	0	-11.0	-5.0
8	3	2	1	2	4	0	1	0	0	0	0	0	0	-5.9	-3.2
8	3	2	1	2	3	1	1	0	0	0	0	0	0	-7.2	-3.8
8	3	2	1	2	2	2	1	0	0	0	0	0	0	-8.5	-4.3
8	3	2	1	2	1	3	1	0	0	0	0	0	0	-9.8	-4.7
8	3	2	1	2	0	4	1	0	0	0	0	0	0	-11.0	-5.0
8	3	2	1	2	3	0	2	0	0	0	0	0	0	-7.2	-3.8
8	3	2	1	2	2	1	2	0	0	0	0	0	0	-8.5	-4.3
8	3	2	1	2	1	2	2	0	0	0	0	0	0	-9.8	-4.7
8	3	2	1	2	0	3	2	0	0	0	0	0	0	-11.0	-5.0
8	3	2	1	2	2	0	3	0	0	0	0	0	0	-8.5	-4.3
8	3	2	1	2	1	1	3	0	0	0	0	0	0	-9.8	-4.7
8	3	2	1	2	0	2	3	0	0	0	0	0	0	-11.0	-5.0
8	3	2	1	2	1	0	4	0	0	0	0	0	0	-9.8	-4.7
8	3	2	1	2	0	1	4	0	0	0	0	0	0	-11.0	-5.0
8	3	2	1	2	0	0	5	0	0	0	0	0	0	-11.0	-5.0

TABLE C.I (continued) - VALUES FOR OBTAINING A (1-α)-LEVEL CONFIDENCE

BOUND ON THE 5 PERCENT POINT OF A WEIBULL POPULATION

					Number Censored at ith Observation									α = .10	α = .20
n	m	r	p	q	1	2	3	4	5	6	7	8	9	ν	ν
8	4	2	1	2	4	0	0	0	0	0	0	0	0	-5.9	-3.2
8	4	2	1	2	3	1	0	0	0	0	0	0	0	-7.2	-3.8
8	4	2	1	2	2	2	0	0	0	0	0	0	0	-8.5	-4.3
8	4	2	1	2	1	3	0	0	0	0	0	0	0	-9.8	-4.7
8	4	2	1	2	0	4	0	0	0	0	0	0	0	-11.0	-5.0
8	4	2	1	2	3	0	1	0	0	0	0	0	0	-7.2	-3.8
8	4	2	1	2	2	1	1	0	0	0	0	0	0	-8.5	-4.3
8	4	2	1	2	1	2	1	0	0	0	0	0	0	-9.8	-4.7
8	4	2	1	2	0	3	1	0	0	0	0	0	0	-11.0	-5.0
8	4	2	1	2	2	0	2	0	0	0	0	0	0	-8.5	-4.3
8	4	2	1	2	1	1	2	0	0	0	0	0	0	-9.8	-4.7
8	4	2	1	2	0	2	2	0	0	0	0	0	0	-11.0	-5.0
8	4	2	1	2	1	0	3	0	0	0	0	0	0	-9.8	-4.7
8	4	2	1	2	0	1	3	0	0	0	0	0	0	-11.0	-5.0
8	4	2	1	2	0	0	4	0	0	0	0	0	0	-11.0	-5.0
8	4	2	1	2	3	0	0	1	0	0	0	0	0	-7.2	-3.8
8	4	2	1	2	2	1	0	1	0	0	0	0	0	-8.5	-4.3
8	4	2	1	2	1	2	0	1	0	0	0	0	0	-9.8	-4.7
8	4	2	1	2	0	3	0	1	0	0	0	0	0	-11.0	-5.0
8	4	2	1	2	2	0	1	1	0	0	0	0	0	-8.5	-4.3
8	4	2	1	2	1	1	1	1	0	0	0	0	0	-9.8	-4.7
8	4	2	1	2	0	2	1	1	0	0	0	0	0	-11.0	-5.0
8	4	2	1	2	1	0	2	1	0	0	0	0	0	-9.8	-4.7
8	4	2	1	2	0	1	2	1	0	0	0	0	0	-11.0	-5.0
8	4	2	1	2	0	0	3	1	0	0	0	0	0	-11.0	-5.0
8	4	2	1	2	2	0	0	2	0	0	0	0	0	-8.5	-4.3
8	4	2	1	2	1	1	0	2	0	0	0	0	0	-9.8	-4.7
8	4	2	1	2	0	2	0	2	0	0	0	0	0	-11.0	-5.0
8	4	2	1	2	1	0	1	2	0	0	0	0	0	-9.8	-4.7
8	4	2	1	2	0	1	1	2	0	0	0	0	0	-11.0	-5.0
8	4	2	1	2	0	0	2	2	0	0	0	0	0	-11.0	-5.0
8	4	2	1	2	1	0	0	3	0	0	0	0	0	-9.8	-4.7
8	4	2	1	2	0	1	0	3	0	0	0	0	0	-11.0	-5.0
8	4	2	1	2	0	0	1	3	0	0	0	0	0	-11.0	-5.0
8	4	2	1	2	0	0	0	4	0	0	0	0	0	-11.0	-5.0
8	5	2	1	2	3	0	0	0	0	0	0	0	0	-7.2	-3.8
8	5	2	1	2	2	1	0	0	0	0	0	0	0	-8.5	-4.3
8	5	2	1	2	1	2	0	0	0	0	0	0	0	-9.8	-4.7
8	5	2	1	2	0	3	0	0	0	0	0	0	0	-11.0	-5.0
8	5	2	1	2	2	0	1	0	0	0	0	0	0	-8.5	-4.3
8	5	2	1	2	1	1	1	0	0	0	0	0	0	-9.8	-4.7
8	5	2	1	2	0	2	1	0	0	0	0	0	0	-11.0	-5.0
8	5	2	1	2	1	0	2	0	0	0	0	0	0	-9.8	-4.7
8	5	2	1	2	0	1	2	0	0	0	0	0	0	-11.0	-5.0

TABLE C.I (continued) - VALUES FOR OBTAINING A (1-α)-LEVEL CONFIDENCE BOUND ON THE 5 PERCENT POINT OF A WEIBULL POPULATION

					Number Censored at <u>i</u>th Observation									α = .10	α = .20
n	m	r	p	q	1	2	3	4	5	6	7	8	9	ν	ν
8	5	2	1	2	0	0	3	0	0	0	0	0	0	-11.0	-5.0
8	5	2	1	2	2	0	0	1	0	0	0	0	0	-8.5	-4.3
8	5	2	1	2	1	1	0	1	0	0	0	0	0	-9.8	-4.7
8	5	2	1	2	0	2	0	1	0	0	0	0	0	-11.0	-5.0
8	5	2	1	2	1	0	1	1	0	0	0	0	0	-9.8	-4.7
8	5	2	1	2	0	1	1	1	0	0	0	0	0	-11.0	-5.0
8	5	2	1	2	0	0	2	1	0	0	0	0	0	-11.0	-5.0
8	5	2	1	2	1	0	0	2	0	0	0	0	0	-9.8	-4.7
8	5	2	1	2	0	1	0	2	0	0	0	0	0	-11.0	-5.0
8	5	2	1	2	0	0	1	2	0	0	0	0	0	-11.0	-5.0
8	5	2	1	2	0	0	0	3	0	0	0	0	0	-11.0	-5.0
8	5	2	1	2	2	0	0	0	1	0	0	0	0	-8.5	-4.3
8	5	2	1	2	1	1	0	0	1	0	0	0	0	-9.8	-4.7
8	5	2	1	2	0	2	0	0	1	0	0	0	0	-11.0	-5.0
8	5	2	1	2	1	0	1	0	1	0	0	0	0	-9.8	-4.7
8	5	2	1	2	0	1	1	0	1	0	0	0	0	-11.0	-5.0
8	5	2	1	2	0	0	2	0	1	0	0	0	0	-11.0	-5.0
8	5	2	1	2	1	0	0	1	1	0	0	0	0	-9.8	-4.7
8	5	2	1	2	0	1	0	1	1	0	0	0	0	-11.0	-5.0
8	5	2	1	2	0	0	1	1	1	0	0	0	0	-11.0	-5.0
8	5	2	1	2	0	0	0	2	1	0	0	0	0	-11.0	-5.0
8	5	2	1	2	1	0	0	0	2	0	0	0	0	-9.8	-4.7
8	5	2	1	2	0	1	0	0	2	0	0	0	0	-11.0	-5.0
8	5	2	1	2	0	0	1	0	2	0	0	0	0	-11.0	-5.0
8	5	2	1	2	0	0	0	1	2	0	0	0	0	-11.0	-5.0
8	5	2	1	2	0	0	0	0	3	0	0	0	0	-11.0	-5.0
8	6	2	1	2	2	0	0	0	0	0	0	0	0	-8.5	-4.3
8	6	2	1	2	1	1	0	0	0	0	0	0	0	-9.8	-4.7
8	6	2	1	2	0	2	0	0	0	0	0	0	0	-11.0	-5.0
8	6	2	1	2	1	0	1	0	0	0	0	0	0	-9.8	-4.7
8	6	2	1	2	0	1	1	0	0	0	0	0	0	-11.0	-5.0
8	6	2	1	2	0	0	2	0	0	0	0	0	0	-11.0	-5.0
8	6	2	1	2	1	0	0	1	0	0	0	0	0	-9.8	-4.7
8	6	2	1	2	0	1	0	1	0	0	0	0	0	-11.0	-5.0
8	6	2	1	2	0	0	1	1	0	0	0	0	0	-11.0	-5.0
8	6	2	1	2	0	0	0	2	0	0	0	0	0	-11.0	-5.0
8	6	2	1	2	1	0	0	0	1	0	0	0	0	-9.8	-4.7
8	6	2	1	2	0	1	0	0	1	0	0	0	0	-11.0	-5.0
8	6	2	1	2	0	0	1	0	1	0	0	0	0	-11.0	-5.0
8	6	2	1	2	0	0	0	1	1	0	0	0	0	-11.0	-5.0
8	6	2	1	2	0	0	0	0	2	0	0	0	0	-11.0	-5.0
8	6	2	1	2	1	0	0	0	0	1	0	0	0	-9.8	-4.7
8	6	2	1	2	0	1	0	0	0	1	0	0	0	-11.0	-5.0
8	6	2	1	2	0	0	1	0	0	1	0	0	0	-11.0	-5.0
8	6	2	1	2	0	0	0	1	0	1	0	0	0	-11.0	-5.0
8	6	2	1	2	0	0	0	0	1	1	0	0	0	-11.0	-5.0
8	6	2	1	2	0	0	0	0	0	2	0	0	0	-11.0	-5.0

TABLE C.I (continued) - VALUES FOR OBTAINING A (1-α)-LEVEL CONFIDENCE BOUND ON THE 5 PERCENT POINT OF A WEIBULL POPULATION

					Number Censored at ith Observation									α = .10	α = .20
n	m	r	p	q	1	2	3	4	5	6	7	8	9	ν	ν
8	7	2	1	2	1	0	0	0	0	0	0	0	0	-9.8	-4.7
8	7	2	1	2	0	1	0	0	0	0	0	0	0	-11.0	-5.0
8	7	2	1	2	0	0	1	0	0	0	0	0	0	-11.0	-5.0
8	7	2	1	2	0	0	0	1	0	0	0	0	0	-11.0	-5.0
8	7	2	1	2	0	0	0	0	1	0	0	0	0	-11.0	-5.0
8	7	2	1	2	0	0	0	0	0	1	0	0	0	-11.0	-5.0
8	7	2	1	2	0	0	0	0	0	0	1	0	0	-11.0	-5.0
8	8	2	1	2	0	0	0	0	0	0	0	0	0	-11.0	-5.0
9	2	2	1	2	7	0	0	0	0	0	0	0	0	-2.4	-1.5
9	2	2	1	2	6	1	0	0	0	0	0	0	0	-3.8	-2.2
9	2	2	1	2	5	2	0	0	0	0	0	0	0	-5.0	-2.8
9	2	2	1	2	4	3	0	0	0	0	0	0	0	-6.1	-3.3
9	2	2	1	2	3	4	0	0	0	0	0	0	0	-7.2	-3.7
9	2	2	1	2	2	5	0	0	0	0	0	0	0	-8.2	-4.1
9	2	2	1	2	1	6	0	0	0	0	0	0	0	-9.2	-4.4
9	2	2	1	2	0	7	0	0	0	0	0	0	0	-10.2	-4.6
9	3	2	1	2	6	0	0	0	0	0	0	0	0	-3.8	-2.2
9	3	2	1	2	5	1	0	0	0	0	0	0	0	-5.0	-2.8
9	3	2	1	2	4	2	0	0	0	0	0	0	0	-6.1	-3.3
9	3	2	1	2	3	3	0	0	0	0	0	0	0	-7.2	-3.7
9	3	2	1	2	2	4	0	0	0	0	0	0	0	-8.2	-4.1
9	3	2	1	2	1	5	0	0	0	0	0	0	0	-9.2	-4.4
9	3	2	1	2	0	6	0	0	0	0	0	0	0	-10.2	-4.6
9	3	2	1	2	5	0	1	0	0	0	0	0	0	-5.0	-2.8
9	3	2	1	2	4	1	1	0	0	0	0	0	0	-6.1	-3.3
9	3	2	1	2	3	2	1	0	0	0	0	0	0	-7.2	-3.7
9	3	2	1	2	2	3	1	0	0	0	0	0	0	-8.2	-4.1
9	3	2	1	2	1	4	1	0	0	0	0	0	0	-9.2	-4.4
9	3	2	1	2	0	5	1	0	0	0	0	0	0	-10.2	-4.6
9	3	2	1	2	4	0	2	0	0	0	0	0	0	-6.1	-3.3
9	3	2	1	2	3	1	2	0	0	0	0	0	0	-7.2	-3.7
9	3	2	1	2	2	2	2	0	0	0	0	0	0	-8.2	-4.1
9	3	2	1	2	1	3	2	0	0	0	0	0	0	-9.2	-4.4
9	3	2	1	2	0	4	2	0	0	0	0	0	0	-10.2	-4.6
9	3	2	1	2	3	0	3	0	0	0	0	0	0	-7.2	-3.7
9	3	2	1	2	2	1	3	0	0	0	0	0	0	-8.2	-4.1
9	3	2	1	2	1	2	3	0	0	0	0	0	0	-9.2	-4.4
9	3	2	1	2	0	3	3	0	0	0	0	0	0	-10.2	-4.6

TABLE C.I (continued) - VALUES FOR OBTAINING A (1-α)-LEVEL CONFIDENCE BOUND ON THE 5 PERCENT POINT OF A WEIBULL POPULATION

					Number Censored at ith Observation									α = .10	α = .20
n	m	r	p	q	1	2	3	4	5	6	7	8	9	ν	ν
9	3	2	1	2	2	0	4	0	0	0	0	0	0	-8.2	-4.1
9	3	2	1	2	1	1	4	0	0	0	0	0	0	-9.2	-4.4
9	3	2	1	2	0	2	4	0	0	0	0	0	0	-10.2	-4.6
9	3	2	1	2	1	0	5	0	0	0	0	0	0	-9.2	-4.4
9	3	2	1	2	0	1	5	0	0	0	0	0	0	-10.2	-4.6
9	3	2	1	2	0	0	6	0	0	0	0	0	0	-10.2	-4.6
9	4	2	1	2	5	0	0	0	0	0	0	0	0	-5.0	-2.8
9	4	2	1	2	4	1	0	0	0	0	0	0	0	-6.1	-3.3
9	4	2	1	2	3	2	0	0	0	0	0	0	0	-7.2	-3.7
9	4	2	1	2	2	3	0	0	0	0	0	0	0	-8.2	-4.1
9	4	2	1	2	1	4	0	0	0	0	0	0	0	-9.2	-4.4
9	4	2	1	2	0	5	0	0	0	0	0	0	0	-10.2	-4.6
9	4	2	1	2	4	0	1	0	0	0	0	0	0	-6.1	-3.3
9	4	2	1	2	3	1	1	0	0	0	0	0	0	-7.2	-3.7
9	4	2	1	2	2	2	1	0	0	0	0	0	0	-8.2	-4.1
9	4	2	1	2	1	3	1	0	0	0	0	0	0	-9.2	-4.4
9	4	2	1	2	0	4	1	0	0	0	0	0	0	-10.2	-4.6
9	4	2	1	2	3	0	2	0	0	0	0	0	0	-7.2	-3.7
9	4	2	1	2	2	1	2	0	0	0	0	0	0	-8.2	-4.1
9	4	2	1	2	1	2	2	0	0	0	0	0	0	-9.2	-4.4
9	4	2	1	2	0	3	2	0	0	0	0	0	0	-10.2	-4.6
9	4	2	1	2	2	0	3	0	0	0	0	0	0	-8.2	-4.1
9	4	2	1	2	1	1	3	0	0	0	0	0	0	-9.2	-4.4
9	4	2	1	2	0	2	3	0	0	0	0	0	0	-10.2	-4.6
9	4	2	1	2	1	0	4	0	0	0	0	0	0	-9.2	-4.4
9	4	2	1	2	0	1	4	0	0	0	0	0	0	-10.2	-4.6
9	4	2	1	2	0	0	5	0	0	0	0	0	0	-10.2	-4.6
9	4	2	1	2	4	0	0	1	0	0	0	0	0	-6.1	-3.3
9	4	2	1	2	3	1	0	1	0	0	0	0	0	-7.2	-3.7
9	4	2	1	2	2	2	0	1	0	0	0	0	0	-8.2	-4.1
9	4	2	1	2	1	3	0	1	0	0	0	0	0	-9.2	-4.4
9	4	2	1	2	0	4	0	1	0	0	0	0	0	-10.2	-4.6
9	4	2	1	2	3	0	1	1	0	0	0	0	0	-7.2	-3.7
9	4	2	1	2	2	1	1	1	0	0	0	0	0	-8.2	-4.1
9	4	2	1	2	1	2	1	1	0	0	0	0	0	-9.2	-4.4
9	4	2	1	2	0	3	1	1	0	0	0	0	0	-10.2	-4.6
9	4	2	1	2	2	0	2	1	0	0	0	0	0	-8.2	-4.1
9	4	2	1	2	1	1	2	1	0	0	0	0	0	-9.2	-4.4
9	4	2	1	2	0	2	2	1	0	0	0	0	0	-10.2	-4.6
9	4	2	1	2	1	0	3	1	0	0	0	0	0	-9.2	-4.4

TABLE C.I (continued) - VALUES FOR OBTAINING A (1-α)-LEVEL CONFIDENCE BOUND ON THE 5 PERCENT POINT OF A WEIBULL POPULATION

					Number Censored at ith Observation									α = .10	α = .20
n	m	r	p	q	1	2	3	4	5	6	7	8	9	ν	ν
9	4	2	1	2	0	1	3	1	0	0	0	0	0	-10.2	-4.6
9	4	2	1	2	0	0	4	1	0	0	0	0	0	-10.2	-4.6
9	4	2	1	2	3	0	0	2	0	0	0	0	0	-7.2	-3.7
9	4	2	1	2	2	1	0	2	0	0	0	0	0	-8.2	-4.1
9	4	2	1	2	1	2	0	2	0	0	0	0	0	-9.2	-4.4
9	4	2	1	2	0	3	0	2	0	0	0	0	0	-10.2	-4.6
9	4	2	1	2	2	0	1	2	0	0	0	0	0	-8.2	-4.1
9	4	2	1	2	1	1	1	2	0	0	0	0	0	-9.2	-4.4
9	4	2	1	2	0	2	1	2	0	0	0	0	0	-10.2	-4.6
9	4	2	1	2	1	0	2	2	0	0	0	0	0	-9.2	-4.4
9	4	2	1	2	0	1	2	2	0	0	0	0	0	-10.2	-4.6
9	4	2	1	2	0	0	3	2	0	0	0	0	0	-10.2	-4.6
9	4	2	1	2	2	0	0	3	0	0	0	0	0	-8.2	-4.1
9	4	2	1	2	1	1	0	3	0	0	0	0	0	-9.2	-4.4
9	4	2	1	2	0	2	0	3	0	0	0	0	0	-10.2	-4.6
9	4	2	1	2	1	0	1	3	0	0	0	0	0	-9.2	-4.4
9	4	2	1	2	0	1	1	3	0	0	0	0	0	-10.2	-4.6
9	4	2	1	2	0	0	2	3	0	0	0	0	0	-10.2	-4.6
9	4	2	1	2	1	0	0	4	0	0	0	0	0	-9.2	-4.4
9	4	2	1	2	0	1	0	4	0	0	0	0	0	-10.2	-4.6
9	4	2	1	2	0	0	1	4	0	0	0	0	0	-10.2	-4.6
9	4	2	1	2	0	0	0	5	0	0	0	0	0	-10.2	-4.6

9	5	2	1	2	4	0	0	0	0	0	0	0	0	-6.1	-3.3
9	5	2	1	2	3	1	0	0	0	0	0	0	0	-7.2	-3.7
9	5	2	1	2	2	2	0	0	0	0	0	0	0	-8.2	-4.1
9	5	2	1	2	1	3	0	0	0	0	0	0	0	-9.2	-4.4
9	5	2	1	2	0	4	0	0	0	0	0	0	0	-10.2	-4.6
9	5	2	1	2	3	0	1	0	0	0	0	0	0	-7.2	-3.7
9	5	2	1	2	2	1	1	0	0	0	0	0	0	-8.2	-4.1
9	5	2	1	2	1	2	1	0	0	0	0	0	0	-9.2	-4.4
9	5	2	1	2	0	3	1	0	0	0	0	0	0	-10.2	-4.6
9	5	2	1	2	2	0	2	0	0	0	0	0	0	-8.2	-4.1
9	5	2	1	2	1	1	2	0	0	0	0	0	0	-9.2	-4.4
9	5	2	1	2	0	2	2	0	0	0	0	0	0	-10.2	-4.6
9	5	2	1	2	1	0	3	0	0	0	0	0	0	-9.2	-4.4
9	5	2	1	2	0	1	3	0	0	0	0	0	0	-10.2	-4.6
9	5	2	1	2	0	0	4	0	0	0	0	0	0	-10.2	-4.6
9	5	2	1	2	3	0	0	1	0	0	0	0	0	-7.2	-3.7
9	5	2	1	2	2	1	0	1	0	0	0	0	0	-8.2	-4.1
9	5	2	1	2	1	2	0	1	0	0	0	0	0	-9.2	-4.4
9	5	2	1	2	0	3	0	1	0	0	0	0	0	-10.2	-4.6
9	5	2	1	2	2	0	1	1	0	0	0	0	0	-8.2	-4.1
9	5	2	1	2	1	1	1	1	0	0	0	0	0	-9.2	-4.4
9	5	2	1	2	0	2	1	1	0	0	0	0	0	-10.2	-4.6

TABLE C.I (continued) - VALUES FOR OBTAINING A (1-α)-LEVEL CONFIDENCE BOUND ON THE 5 PERCENT POINT OF A WEIBULL POPULATION

n	m	r	p	q	Number Censored at ith Observation 1	2	3	4	5	6	7	8	9	α = .10 ν	α = .20 ν
9	5	2	1	2	1	0	2	1	0	0	0	0	0	-9.2	-4.4
9	5	2	1	2	0	1	2	1	0	0	0	0	0	-10.2	-4.6
9	5	2	1	2	0	0	3	1	0	0	0	0	0	-10.2	-4.6
9	5	2	1	2	2	0	0	2	0	0	0	0	0	-8.2	-4.1
9	5	2	1	2	1	1	0	2	0	0	0	0	0	-9.2	-4.4
9	5	2	1	2	0	2	0	2	0	0	0	0	0	-10.2	-4.6
9	5	2	1	2	1	0	1	2	0	0	0	0	0	-9.2	-4.4
9	5	2	1	2	0	1	1	2	0	0	0	0	0	-10.2	-4.6
9	5	2	1	2	0	0	2	2	0	0	0	0	0	-10.2	-4.6
9	5	2	1	2	1	0	0	3	0	0	0	0	0	-9.2	-4.4
9	5	2	1	2	0	1	0	3	0	0	0	0	0	-10.2	-4.6
9	5	2	1	2	0	0	1	3	0	0	0	0	0	-10.2	-4.6
9	5	2	1	2	0	0	0	4	0	0	0	0	0	-10.2	-4.6
9	5	2	1	2	3	0	0	0	1	0	0	0	0	-7.2	-3.7
9	5	2	1	2	2	1	0	0	1	0	0	0	0	-8.2	-4.1
9	5	2	1	2	1	2	0	0	1	0	0	0	0	-9.2	-4.4
9	5	2	1	2	0	3	0	0	1	0	0	0	0	-10.2	-4.6
9	5	2	1	2	2	0	1	0	1	0	0	0	0	-8.2	-4.1
9	5	2	1	2	1	1	1	0	1	0	0	0	0	-9.2	-4.4
9	5	2	1	2	0	2	1	0	1	0	0	0	0	-10.2	-4.6
9	5	2	1	2	1	0	2	0	1	0	0	0	0	-9.2	-4.4
9	5	2	1	2	0	1	2	0	1	0	0	0	0	-10.2	-4.6
9	5	2	1	2	0	0	3	0	1	0	0	0	0	-10.2	-4.6
9	5	2	1	2	2	0	0	1	1	0	0	0	0	-8.2	-4.1
9	5	2	1	2	1	1	0	1	1	0	0	0	0	-9.2	-4.4
9	5	2	1	2	0	2	0	1	1	0	0	0	0	-10.2	-4.6
9	5	2	1	2	0	1	1	1	1	0	0	0	0	-10.2	-4.4
9	5	2	1	2	1	0	1	1	1	0	0	0	0	-9.2	-4.6
9	5	2	1	2	0	0	2	1	1	0	0	0	0	-10.2	-4.6
9	5	2	1	2	1	0	0	2	1	0	0	0	0	-9.2	-4.4
9	5	2	1	2	0	1	0	2	1	0	0	0	0	-10.2	-4.6
9	5	2	1	2	0	0	1	2	1	0	0	0	0	-10.2	-4.6
9	5	2	1	2	0	0	0	3	1	0	0	0	0	-10.2	-4.6
9	5	2	1	2	2	0	0	0	2	0	0	0	0	-8.2	-4.1
9	5	2	1	2	1	1	0	0	2	0	0	0	0	-9.2	-4.4
9	5	2	1	2	0	2	0	0	2	0	0	0	0	-10.2	-4.6
9	5	2	1	2	1	0	1	0	2	0	0	0	0	-9.2	-4.4
9	5	2	1	2	0	1	1	0	2	0	0	0	0	-10.2	-4.6
9	5	2	1	2	0	0	2	0	2	0	0	0	0	-10.2	-4.6
9	5	2	1	2	1	0	0	1	2	0	0	0	0	-9.2	-4.4
9	5	2	1	2	0	1	0	1	2	0	0	0	0	-10.2	-4.6
9	5	2	1	2	0	0	1	1	2	0	0	0	0	-10.2	-4.6
9	5	2	1	2	0	0	0	2	2	0	0	0	0	-10.2	-4.6
9	5	2	1	2	1	0	0	0	3	0	0	0	0	-9.2	-4.4
9	5	2	1	2	0	1	0	0	3	0	0	0	0	-10.2	-4.6
9	5	2	1	2	0	0	1	0	3	0	0	0	0	-10.2	-4.6
9	5	2	1	2	0	0	0	1	3	0	0	0	0	-10.2	-4.6
9	5	2	1	2	0	0	0	0	4	0	0	0	0	-10.2	-4.6

TABLE C.I (continued) - VALUES FOR OBTAINING A (1-α)-LEVEL CONFIDENCE BOUND ON THE 5 PERCENT POINT OF A WEIBULL POPULATION

n	m	r	p	q	Number Censored at ith Observation 1	2	3	4	5	6	7	8	9	α = .10 v	α = .20 v
9	6	2	1	2	3	0	0	0	0	0	0	0	0	-7.2	-3.7
9	6	2	1	2	2	1	0	0	0	0	0	0	0	-8.2	-4.1
9	6	2	1	2	1	2	0	0	0	0	0	0	0	-9.2	-4.4
9	6	2	1	2	0	3	0	0	0	0	0	0	0	-10.2	-4.6
9	6	2	1	2	2	0	1	0	0	0	0	0	0	-8.2	-4.1
9	6	2	1	2	1	1	1	0	0	0	0	0	0	-9.2	-4.4
9	6	2	1	2	0	2	1	0	0	0	0	0	0	-10.2	-4.6
9	6	2	1	2	1	0	2	0	0	0	0	0	0	-9.2	-4.4
9	6	2	1	2	0	1	2	0	0	0	0	0	0	-10.2	-4.6
9	6	2	1	2	0	0	3	0	0	0	0	0	0	-10.2	-4.6
9	6	2	1	2	2	0	0	1	0	0	0	0	0	-8.2	-4.1
9	6	2	1	2	1	1	0	1	0	0	0	0	0	-9.2	-4.4
9	6	2	1	2	0	2	0	1	0	0	0	0	0	-10.2	-4.6
9	6	2	1	2	1	0	1	1	0	0	0	0	0	-9.2	-4.4
9	6	2	1	2	0	1	1	1	0	0	0	0	0	-10.2	-4.6
9	6	2	1	2	0	0	2	1	0	0	0	0	0	-10.2	-4.6
9	6	2	1	2	1	0	0	2	0	0	0	0	0	-9.2	-4.4
9	6	2	1	2	0	1	0	2	0	0	0	0	0	-10.2	-4.6
9	6	2	1	2	0	0	1	2	0	0	0	0	0	-10.2	-4.6
9	6	2	1	2	0	0	0	3	0	0	0	0	0	-10.2	-4.6
9	6	2	1	2	2	0	0	0	1	0	0	0	0	-8.2	-4.1
9	6	2	1	2	1	1	0	0	1	0	0	0	0	-9.2	-4.4
9	6	2	1	2	0	2	0	0	1	0	0	0	0	-10.2	-4.6
9	6	2	1	2	1	0	1	0	1	0	0	0	0	-9.2	-4.4
9	6	2	1	2	0	1	1	0	1	0	0	0	0	-10.2	-4.6
9	6	2	1	2	0	0	2	0	1	0	0	0	0	-10.2	-4.6
9	6	2	1	2	1	0	0	1	1	0	0	0	0	-9.2	-4.4
9	6	2	1	2	0	1	0	1	1	0	0	0	0	-10.2	-4.6
9	6	2	1	2	0	0	1	1	1	0	0	0	0	-10.2	-4.6
9	6	2	1	2	0	0	0	2	1	0	0	0	0	-10.2	-4.6
9	6	2	1	2	1	0	0	0	2	0	0	0	0	-9.2	-4.4
9	6	2	1	2	0	1	0	0	2	0	0	0	0	-10.2	-4.6
9	6	2	1	2	0	0	1	0	2	0	0	0	0	-10.2	-4.6
9	6	2	1	2	0	0	0	1	2	0	0	0	0	-10.2	-4.6
9	6	2	1	2	0	0	0	0	3	0	0	0	0	-10.2	-4.6
9	6	2	1	2	2	0	0	0	0	1	0	0	0	-8.2	-4.1
9	6	2	1	2	1	1	0	0	0	1	0	0	0	-9.2	-4.4
9	6	2	1	2	0	2	0	0	0	1	0	0	0	-10.2	-4.6
9	6	2	1	2	1	0	1	0	0	1	0	0	0	-9.2	-4.4
9	6	2	1	2	0	1	1	0	0	1	0	0	0	-10.2	-4.6
9	6	2	1	2	0	0	2	0	0	1	0	0	0	-10.2	-4.6
9	6	2	1	2	1	0	0	1	0	1	0	0	0	-9.2	-4.4
9	6	2	1	2	0	1	0	1	0	1	0	0	0	-10.2	-4.6
9	6	2	1	2	0	0	1	1	0	1	0	0	0	-10.2	-4.6
9	6	2	1	2	0	0	0	2	0	1	0	0	0	-10.2	-4.6
9	6	2	1	2	1	0	0	0	1	1	0	0	0	-9.2	-4.4

TABLE C.I (continued) - VALUES FOR OBTAINING A (1-α)-LEVEL CONFIDENCE BOUND ON THE 5 PERCENT POINT OF A WEIBULL POPULATION

					Number Censored at ith Observation									α = .10	α = .20
n	m	r	p	q	1	2	3	4	5	6	7	8	9	ν	ν
9	6	2	1	2	0	1	0	0	1	1	0	0	0	-10.2	-4.6
9	6	2	1	2	0	0	1	0	1	1	0	0	0	-10.2	-4.6
9	6	2	1	2	0	0	0	1	1	1	0	0	0	-10.2	-4.6
9	6	2	1	2	0	0	0	0	2	1	0	0	0	-10.2	-4.6
9	6	2	1	2	1	0	0	0	0	2	0	0	0	-9.2	-4.4
9	6	2	1	2	0	1	0	0	0	2	0	0	0	-10.2	-4.6
9	6	2	1	2	0	0	1	0	0	2	0	0	0	-10.2	-4.6
9	6	2	1	2	0	0	0	1	0	2	0	0	0	-10.2	-4.6
9	6	2	1	2	0	0	0	0	1	2	0	0	0	-10.2	-4.6
9	6	2	1	2	0	0	0	0	0	3	0	0	0	-10.2	-4.6
9	7	2	1	2	2	0	0	0	0	0	0	0	0	-8.2	-4.1
9	7	2	1	2	1	1	0	0	0	0	0	0	0	-9.2	-4.4
9	7	2	1	2	0	2	0	0	0	0	0	0	0	-10.2	-4.6
9	7	2	1	2	1	0	1	0	0	0	0	0	0	-9.2	-4.4
9	7	2	1	2	0	1	1	0	0	0	0	0	0	-10.2	-4.6
9	7	2	1	2	0	0	2	0	0	0	0	0	0	-10.2	-4.6
9	7	2	1	2	1	0	0	1	0	0	0	0	0	-9.2	-4.4
9	7	2	1	2	0	1	0	1	0	0	0	0	0	-10.2	-4.6
9	7	2	1	2	0	0	1	1	0	0	0	0	0	-10.2	-4.6
9	7	2	1	2	0	0	0	2	0	0	0	0	0	-10.2	-4.6
9	7	2	1	2	1	0	0	0	1	0	0	0	0	-9.2	-4.4
9	7	2	1	2	0	1	0	0	1	0	0	0	0	-10.2	-4.6
9	7	2	1	2	0	0	1	0	1	0	0	0	0	-10.2	-4.6
9	7	2	1	2	0	0	0	1	1	0	0	0	0	-10.2	-4.6
9	7	2	1	2	0	0	0	0	2	0	0	0	0	-10.2	-4.6
9	7	2	1	2	1	0	0	0	0	1	0	0	0	-9.2	-4.4
9	7	2	1	2	0	1	0	0	0	1	0	0	0	-10.2	-4.6
9	7	2	1	2	0	0	1	0	0	1	0	0	0	-10.2	-4.6
9	7	2	1	2	0	0	0	1	0	1	0	0	0	-10.2	-4.6
9	7	2	1	2	0	0	0	0	1	1	0	0	0	-10.2	-4.6
9	7	2	1	2	0	0	0	0	0	2	0	0	0	-10.2	-4.6
9	7	2	1	2	1	0	0	0	0	0	1	0	0	-9.2	-4.4
9	7	2	1	2	0	1	0	0	0	0	1	0	0	-10.2	-4.6
9	7	2	1	2	0	0	1	0	0	0	1	0	0	-10.2	-4.6
9	7	2	1	2	0	0	0	1	0	0	1	0	0	-10.2	-4.6
9	7	2	1	2	0	0	0	0	1	0	1	0	0	-10.2	-4.6
9	7	2	1	2	0	0	0	0	0	1	1	0	0	-10.2	-4.6
9	7	2	1	2	0	0	0	0	0	0	2	0	0	-10.2	-4.6

TABLE C.I (continued) - VALUES FOR OBTAINING A (1-α)-LEVEL CONFIDENCE

BOUND ON THE 5 PERCENT POINT OF A WEIBULL POPULATION

					Number Censored at ith Observation									α = .10	α = .20
n	m	r	p	q	1	2	3	4	5	6	7	8	9	ν	ν
9	8	2	1	2	1	0	0	0	0	0	0	0	0	-9.2	-4.4
9	8	2	1	2	0	1	0	0	0	0	0	0	0	-10.2	-4.6
9	8	2	1	2	0	0	1	0	0	0	0	0	0	-10.2	-4.6
9	8	2	1	2	0	0	0	1	0	0	0	0	0	-10.2	-4.6
9	8	2	1	2	0	0	0	0	1	0	0	0	0	-10.2	-4.6
9	8	2	1	2	0	0	0	0	0	1	0	0	0	-10.2	-4.6
9	8	2	1	2	0	0	0	0	0	0	1	0	0	-10.2	-4.6
9	8	2	1	2	0	0	0	0	0	0	0	1	0	-10.2	-4.6
9	9	2	1	2	0	0	0	0	0	0	0	0	0	-10.2	-4.6

TABLE C.I - VALUES FOR OBTAINING A (1-α)-LEVEL CONFIDENCE BOUND

ON THE 5 PERCENT POINT OF A WEIBULL POPULATION

n	m	r	p	q	α = .10 ν	α = .20 ν
10	2	2	1	2	-9.4	-4.2
10	3	2	1	2	-9.4	-4.2
10	4	2	1	2	-9.4	-4.2
10	5	2	1	2	-9.4	-4.2
10	6	2	1	2	-9.4	-4.2
10	7	2	1	2	-9.4	-4.2
10	8	2	1	2	-9.4	-4.2
10	9	2	1	2	-9.4	-4.2
10	10	2	1	2	-9.4	-4.2
11	2	2	1	2	-8.7	-3.8
11	3	2	1	2	-8.7	-3.8
11	4	2	1	2	-8.7	-3.8
11	5	2	1	2	-8.7	-3.8
11	6	2	1	2	-8.7	-3.8
11	7	2	1	2	-8.7	-3.8
11	8	2	1	2	-8.7	-3.8
11	9	2	1	2	-8.7	-3.8
11	10	2	1	2	-8.7	-3.8
11	11	2	1	2	-8.7	-3.8
12	2	2	1	2	-8.1	-3.4
12	3	2	1	2	-8.1	-3.4
12	4	2	1	2	-8.1	-3.4
12	5	2	1	2	-8.1	-3.4
12	6	2	1	2	-8.1	-3.4
12	7	2	1	2	-8.1	-3.4
12	8	2	1	2	-8.1	-3.4
12	9	2	1	2	-8.1	-3.4
12	10	2	1	2	-8.1	-3.4
12	11	2	1	2	-8.1	-3.4
12	12	2	1	2	-8.1	-3.4
13	2	2	1	2	-7.5	-3.0
13	3	2	1	2	-7.5	-3.0
13	4	2	1	2	-7.5	-3.0
13	5	2	1	2	-7.5	-3.0
13	6	2	1	2	-7.5	-3.0
13	7	2	1	2	-7.5	-3.0
13	8	2	1	2	-7.5	-3.0
13	9	2	1	2	-7.5	-3.0
13	10	2	1	2	-7.5	-3.0
13	11	2	1	2	-7.5	-3.0
13	12	2	1	2	-7.5	-3.0
13	13	2	1	2	-7.5	-3.0

TABLE C.I (continued) - VALUES FOR OBTAINING A (1-α)-LEVEL CONFIDENCE BOUND ON THE 5 PERCENT POINT OF A WEIBULL POPULATION

n	m	r	p	q	α = .10 ν	α = .20 ν
14	2	2	1	2	-6.9	-2.7
14	3	2	1	2	-6.9	-2.7
14	4	2	1	2	-6.9	-2.7
14	5	2	1	2	-6.9	-2.7
14	6	2	1	2	-6.9	-2.7
14	7	2	1	2	-6.9	-2.7
14	8	2	1	2	-6.9	-2.7
14	9	2	1	2	-6.9	-2.7
14	10	2	1	2	-6.9	-2.7
14	11	2	1	2	-6.9	-2.7
14	12	2	1	2	-6.9	-2.7
14	13	2	1	2	-6.9	-2.7
14	14	2	1	2	-6.9	-2.7
15	2	2	1	2	-6.4	-2.2
15	3	2	1	2	-6.4	-2.2
15	4	2	1	2	-6.4	-2.2
15	5	2	1	2	-6.4	-2.2
15	6	2	1	2	-6.4	-2.2
15	7	2	1	2	-6.4	-2.2
15	8	2	1	2	-6.4	-2.2
15	9	2	1	2	-6.4	-2.2
15	10	2	1	2	-6.4	-2.2
15	11	2	1	2	-6.4	-2.2
15	12	2	1	2	-6.4	-2.2
15	13	2	1	2	-6.4	-2.2
15	14	2	1	2	-6.4	-2.2
15	15	2	1	2	-6.4	-2.2

APPENDIX

We first define $\Phi_{r,s}(y) = \int_y^\infty x^r (\ln x)^s e^{-x}\, dx$ and then determine expressions for evaluating $\Phi_{0,1}(y)$ and $\Phi_{0,2}(y)$. In order to accomplish this, we integrate $\Psi_1(y) = \int_0^y \ln x\, e^{-x} dx$ by parts to obtain $\ln y(1-e^{-y}) - \int_0^y \frac{(1-e^{-x})dx}{x} =$

$\ln y(1-e^{-y}) - \sum_{i=1}^\infty (-1)^{i+1} \frac{y^i}{i\, i!}$.

Then, from [1], $\Phi_{0,1} = -\gamma - \Psi_1(y)$, where γ is Euler's constant, approximately equal to 0.57721566. If we then integrate $\Psi_2(y) = \int_0^y (\ln x)^2 (1-e^{-x})dx$ by parts, we obtain

$$(\ln y)^2 (1-e^{-y}) - \int_0^y \frac{2\ln x(1-e^{-x})dx}{x} . \tag{A.1}$$

Then, use of the result above and integration by parts applied to the right hand term of (A.1) yields

$$\Psi_2(y) = (\ln y)^2(1-e^{-y}) - 2 \ln y \sum_{i=1}^\infty \frac{(-1)^{i+1} x^i}{i\, i!} + 2 \sum_{i=1}^\infty \frac{(-1)^{i+1} x^i}{i^2\, i!}$$

and, from [1] and [2], $\Phi_{0,2} = \pi^2/6 + \gamma^2 - \Psi_2(y)$. One can then recursively determine an expression for $\Phi_{r,s}$, for $r=1, 2$ and any $s > 0$, as a function of $\Phi_{0,1}$ and $\Phi_{0,2}$.

Here, we find expressions for $\Phi_{1,1}$, $\Phi_{1,2}$, $\Phi_{2,1}$ and $\Phi_{2,2}$ which are needed to evaluate σ (see (8) and (10)).

In each case, integration by parts is used to obtain the desired result. First,

$$\Phi_{1,1} = \int_y^\infty x \, \ln x \, e^{-x} dx = y \ln y \, e^{-y} + \int_y^\infty e^{-x} dx + \int_y^\infty \ln x \, e^{-x} dx$$

$$= y \ln y \, e^{-y} + e^{-y} + \Phi_{0,1}.$$

Then,

$$\Phi_{1,2} = \int_y^\infty x(\ln x)^2 \, e^{-x} dx = y(\ln y)^2 \, e^{-y} + \int_y^\infty [2x \ln x + (\ln x)^2] e^{-x} dx$$

$$= y(\ln y)^2 \, e^{-y} + 2\Phi_{1,1} + \Phi_{0,2}.$$

Also,

$$\Phi_{2,1} = \int_y^\infty x^2 \ln x \, e^{-x} dx = y^2 \ln y \, e^{-y} + \int_y^\infty (x+2x \ln x) e^{-x} dx$$

$$= y^2 \ln y \, e^{-y} + e^{-y}(1+y) + 2\Phi_{1,1}$$

and

$$\Phi_{2,2} = \int_y^\infty x^2(\ln x)^2 \, e^{-x} dx = y^2(\ln y)^2 \, e^{-y} + 2\int_y^\infty [x \ln x + x(\ln x)^2] e^{-x} dx$$

$$= y^2(\ln y)^2 \, e^{-y} + 2\Phi_{1,1} + 2\Phi_{1,2} .$$

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13. ABSTRACT This report consists of three papers, all of which apply to the two-parameter Weibull distribution. The first paper gives tables for obtaining estimates of the shape parameter and of distribution percentage points. This paper provides a more extensive range of values than are available in the published article to which the tables apply.

The second and third papers extend previous results. The second applies to the calculation of warranty periods from sample data for a lot to be manufactured in the future. Warranty periods to be calculated from three ordered observations are derived for orderings of the three observations not considered earlier. Investigation is made of the expected squared deviations of the calculated log warranty periods from the first failure in the lot to be manufactured in the future. On the basis of this investigation, a tabulation is given for lots of size n, n = 10, 11, ..., 25 and samples of size m, m = 2, 3, ..., n-3.

The third paper considers Taylor series approximations to the distribution of three-order-statistic estimators of reliable life. By use of this approach, published tables of values for obtaining three-order-statistic confidence bounds on reliable life are extended.

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