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SUMMARY

Hamilton-Jacobi's equation for the problem of two bodies in bipolar coordinates is solved in closed form; Stumpff's transcendental functions are used to express a complete solution of it in a universal form. This characteristic function is shown to be the source of the analytical expressions necessary to solve the two-point boundary version of the problem of two bodies. In particular, it establishes immediately Lambert's equation in its universal form.

INTRODUCTION

The problem of two bodies is separable not only in spherical coordinates and in parabolic coordinates, but also in bipolar coordinates. The latter separation introduces elements that do not appear well suited to solve the problem of initial values. But we show here how a complete solution of Hamilton's Jacobi's equation in bipolar coordinates yields immediately the *characteristic function*, and that the two-point boundary version of the problem centers around that basic function. In particular it yields immediately Lambert's equation as well as the dual relations that determine the velocities at the boundaries. To illustrate how convenient a vantage point the characteristic function is for the two-point boundary version, we show how it readily produces Godal's compatibility conditions for the terminal velocities, the eccentric anomalies at the boundaries and Battin's formula for the semi-latus rectum.

Throughout this survey, we use systematically the trigonometry of universal functions. The formulas come out uniformized in the neighborhood of the zero energy. In this manner, we bypass unpleasant discussions case by case according to the type of Keplerian orbits.

The present paper does not aim at being exhaustive. It suggests the characteristic function in bipolar coordinates as the entity around which the recent literature could be organized; it shows how this re-arrangement would work in a few instances. A thorough survey should be left to a textbook.

1. Bipolar coordinates

Consider a pair (O,Q) of distinct points in the plane; indicate by α (> 0) the distance OQ. Set at O a Cartesian frame of reference Oxy: the axis Ox is the line OQ oriented positively from O to Q. Call C the middle point of O and Q. Let $C\bar{x}\bar{y}$ be the translated of the coordinate system Oxy.

For any point P in the plane, consider the oriented angle w from OQ to OP. It makes convenient notations in what follows if we introduce the function

$$\zeta \equiv \zeta(w) = \begin{cases} +1 & \text{if } 0 \le w < \pi \quad (\text{mod. } 2\pi), \\ -1 & \text{if } \pi \le w < 2\pi \quad (\text{mod. } 2\pi). \end{cases} \tag{1}$$

Now construct the transformation

$$\begin{aligned} \bar{x} &= \frac{1}{2}\alpha\xi\eta, \\ \bar{y} &= \zeta\alpha[(\xi^2-1)(1-\eta^2)]^{1/2} \end{aligned} \tag{2}$$

from the plane $(\bar{x},\bar{y})$ to the strip

$$\xi \ge 1, \qquad -1 \le \eta \le +1 \tag{3}$$

in the plane (ξ,η). The mapping is 2 to 1. To each point (ξ,η) in the strip (3) correspond two points $(\bar{x},\bar{y})$ symmetrically located with respect to the $\bar{x}$-axis. Figure 1 suggests several noteworthy details concerning the transformation (1).

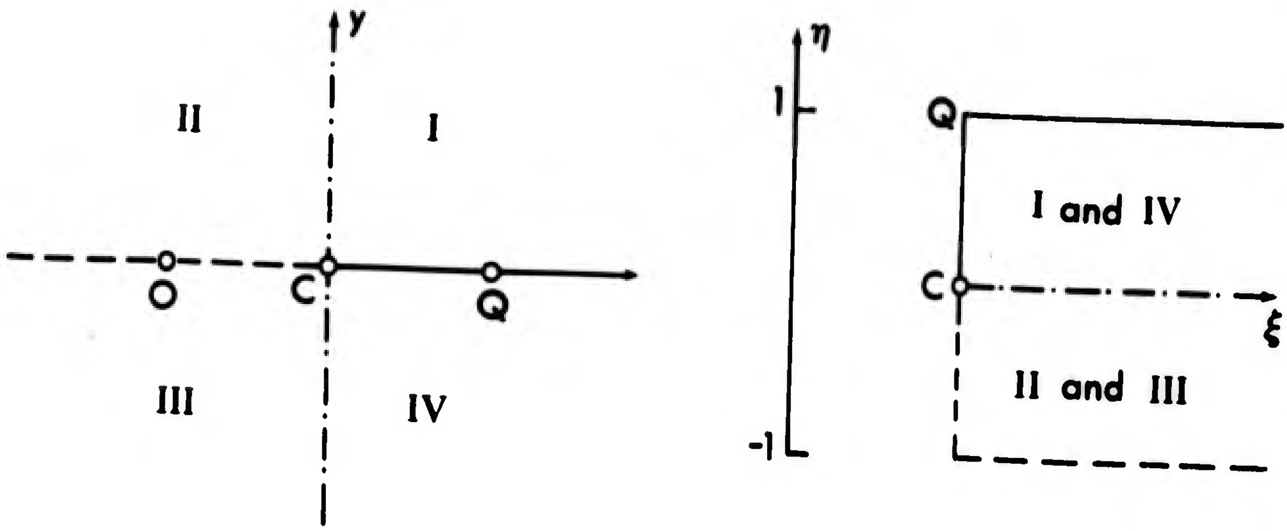

Fig. 1. The mapping from Cartesian to bipolar coordinates.

Designate by r and c the distances from P to O and Q respectively; the length

$$s = 1/2(r + c + \alpha)$$

is the half-perimeter of the triangle OQP. With these notations, the equations of the transformation (1) imply the relations

$$\begin{aligned} 2r &= \alpha(\xi + \eta), & 2(s - r) &= \alpha(1 - \eta), \\ 2c &= \alpha(\xi - \eta), & 2(s - c) &= \alpha(1 + \eta), \\ 2s &= \alpha(\xi + 1), & 2(s - \alpha) &= \alpha(\xi - 1), \end{aligned} \tag{4}$$
$$\alpha^2[(\xi^2 - 1)(1 - \eta^2)]^{1/2} = 4[s(s - \alpha)(s - r)(s - c)]^{1/2}.$$

The first two relations in the first column of the list (4) have caused (ξ,η) to be called the *bipolar* or *biradial* coordinates of P with respect to the base pair (O,Q).

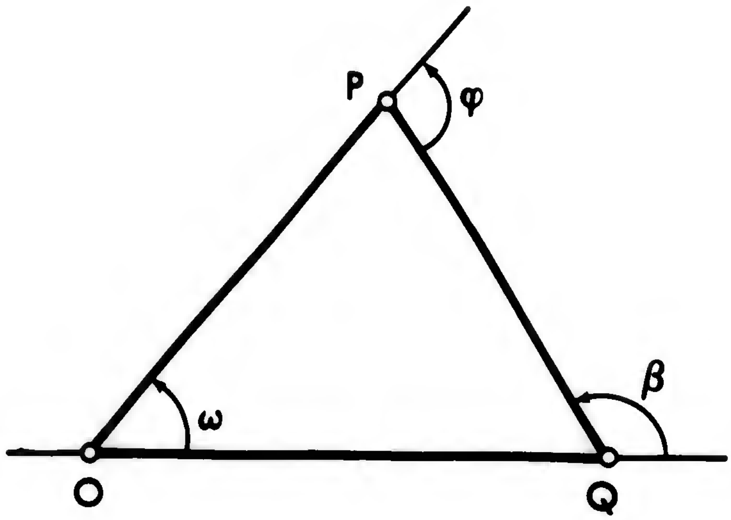

Fig. 2. The fundamental triangle.

Figure 2 displays the orientation adopted for the triangle OQP, and it defines two oriented exterior angles β and ϕ respectively at the vertices Q and P. From the identities

$$c^2 = \alpha^2 + r^2 - 2\alpha r \cos w,$$

$$c \cos \beta = r \cos w - \alpha, \qquad c \cos \phi = \alpha \cos w - r, \tag{6}$$

$$c \sin \beta = r \sin w, \qquad c \sin \phi = \alpha \sin w,$$

there follows that

$$
\begin{aligned}
\cos\frac{1}{2}w &= \zeta\left[\frac{s(s-c)}{\alpha r}\right]^{1/2} &&= \zeta\left[\frac{(\xi+1)(1+\eta)}{2(\xi+\eta)}\right]^{1/2} \\
\sin\frac{1}{2}w &= \left[\frac{(s-\alpha)(s-r)}{\alpha r}\right]^{1/2} &&= \left[\frac{(\xi-1)(1-\eta)}{2(\xi+\eta)}\right]^{1/2} \\
\cos\frac{1}{2}\beta &= \zeta\left[\frac{(s-\alpha)(s-c)}{\alpha c}\right]^{1/2} &&= \zeta\left[\frac{(\xi-1)(1+\eta)}{2(\xi-\eta)}\right]^{1/2} \\
\sin\frac{1}{2}\beta &= \left[\frac{s(s-r)}{\alpha c}\right]^{1/2} &&= \left[\frac{(\xi+1)(1-\eta)}{2(\xi-\eta)}\right]^{1/2} \\
\cos\frac{1}{2}\phi &= \zeta\left[\frac{(s-r)(s-c)}{rc}\right]^{1/2} &&= \zeta\left[\frac{1-\eta^2}{\zeta^2-\eta^2}\right]^{1/2} \\
\sin\frac{1}{2}\phi &= \left[\frac{s(s-\alpha)}{rc}\right]^{1/2} &&= \left[\frac{\xi^2-1}{\xi^2-\eta^2}\right]^{1/2}.
\end{aligned} \tag{7}
$$

Let it also be recalled Heron's formula for the area Δ of the oriented triangle OQP,

$$\Delta = \zeta[s(s-\alpha)(s-r)(s-c)]^{1/2}. \tag{8}$$

It implies that

$$2\Delta = \alpha r \sin w = \alpha c \sin \beta = rc \sin \phi. \tag{9}$$

These elementary equalities will be used throughout in this survey without being referred to explicitly. Notice that the orientation given to the exterior angles β and ϕ has been selected in order to cause the duality between identities involving β and the corresponding ones involving ϕ.

2. The problem of two bodies in bipolar coordinates

With respect to the inertial frame Oxy, the problem of two bodies is described by the Lagrangian function

$$\mathcal{L} = \frac{1}{2}(\dot{x}^2+\dot{y}^2) + \frac{\mu}{r} . \tag{10}$$

We propose to express it in the bipolar coordinates defined in Section 1. Obviously

$$\dot{x} = d\bar{x}/dt = \frac{1}{2}\alpha(\dot{\xi}\eta+\xi\dot{\eta}),$$

$$\dot{y} = d\bar{y}/dt = \frac{1}{2}\zeta\alpha\,\frac{(1-\eta^2)\xi\dot{\xi}-(\xi^2-1)\eta\dot{\eta}}{[(\xi^2-1)(1-\eta^2)]^{1/2}} . \tag{11}$$

A few manipulations yield that

$$V^2 = \dot{x}^2 + \dot{y}^2 = \frac{1}{4}\alpha^2(\xi^2-\eta^2)[\dot{\xi}^2(\xi^2-1)^{-1} + \dot{\eta}^2(1-\eta^2)^{-1}], \tag{12}$$

$$G = x\dot{y} - \dot{x}y = r^2\sin w[\dot{\xi}(\xi^2-1)^{-1} - \dot{\eta}(1-\eta^2)^{-1}], \tag{13}$$

$$\mathcal{L} = \frac{1}{8}\alpha^2(\xi^2+\eta^2)[\dot{\xi}^2(\xi^2-1)^{-1} + \dot{\eta}^2(1-\eta^2)^{-1}] + \mu r^{-1}. \tag{14}$$

The transformed Lagrangian belongs to Liouville's class of separable systems.

The moments conjugate to the time derivatives $\dot{\xi}$ and $\dot{\eta}$ are

$$p_\xi = \partial\mathcal{L}/\partial\dot{\xi} = \frac{1}{4}\alpha^2\dot{\xi}(\xi^2-\eta^2)(\xi^2-1)^{-1},$$

$$p_\eta = \partial\mathcal{L}/\partial\dot{\eta} = \frac{1}{4}\alpha^2\dot{\eta}(\xi^2-\eta^2)(1-\eta^2)^{-1}. \tag{15}$$

Note the dimensions of these moments: length^2/time or angular momentum per unit of mass. They have interesting kinematic meanings. Put

$$A = (p_\xi + p_\eta)/\alpha,$$
$$B = (p_\xi - p_\eta)/\alpha, \qquad (16)$$

then, on extracting $\dot\xi$ and $\dot\eta$ from (15) and substituting them into (11), check that

$$\dot x = A \cos w + B \cos \beta,$$
$$\dot y = A \sin w + B \sin \beta. \qquad (17)$$

Now let $\hat{\underset{\sim}{c}}$ and $\hat{\underset{\sim}{x}}$ be the unit vectors along the sides QP and PO of the oriented triangle OQP. Decompose the velocity vector $\dot{\underset{\sim}{x}}$ into the sum

$$\dot{\underset{\sim}{x}} = -\rho\hat{\underset{\sim}{x}} + \chi\hat{\underset{\sim}{c}}; \qquad (18)$$

ρ (resp. χ) is called the *central* (resp. the *chordal*) component of the velocity with respect to the oriented triangle OQP. Comparison of (17) and (18) results in the identities

$$\rho = -A, \qquad \chi = B. \qquad (19)$$

On introducing the conjugate moments into (13), we derive that

$$G = (p_\xi - p_\eta)\left(\frac{f}{c}\right)\sin w, \qquad (20)$$

or even more concisely that

$$G = 2\Delta B/c. \qquad (21)$$

The components of Laplace's vector

$$\underset{\sim}{L} = \dot{\underset{\sim}{x}} \times \underset{\sim}{G} - \mu\underset{\sim}{x}/r$$

in the Cartesian frame Oxy are

$$P = G\dot{y} - \mu x/r,$$
$$Q = -G\dot{x} - \mu y/r$$

and they transform accordingly into the functions

$$P = 2\Delta B(A \sin w + B \sin \beta)/c - \mu \cos w,$$
$$Q = -2\Delta B(A \cos w + B \cos \beta)/c - \mu \sin w. \qquad (22)$$

Formulas (21) and (22) are essential in characterizing Keplerian motions. Once the central and chordal components A and B of the velocity are known, formula (21) yields the semi-latus rectum

$$p = G^2/\mu = 4\Delta^2 B^2/\mu c^2 \qquad (23)$$

whereas formulas (13) produce the argument g of perigee with respect to Ox and the eccentricty e:

$$\mu e = (P^2+Q^2)^{1/2}, \qquad (24_1)$$
$$\mu e \cos g = P, \qquad (24_2)$$
$$\mu e \sin g = Q. \qquad (24_3)$$

3. Hamilton-Jacobi's equation in bipolar coordinates

The Hamiltonian function dual of the Lagrangian (14) is

$$\mathcal{H} = \frac{2}{\alpha^2(\xi^2-\eta^2)} [(\xi^2-1)p_\xi^2 + (1-\eta^2)p_\eta^2] - \frac{2\mu}{\alpha(\xi+\eta)} . \tag{25}$$

It gives rise to Hamilton-Jacobi's partial differential equation

$$(\xi^2-1)(\frac{\partial W}{\partial \xi})^2 - \mu\alpha\xi(1 + \frac{h\alpha}{2\mu}\xi) = (\eta^2-1)(\frac{\partial W}{\partial \eta})^2 - \mu\alpha\eta(1 + \frac{h\alpha}{2\mu}\eta) \tag{26}$$

where h denotes the constant of energy. Equation (26) admits a complete solution

$$W \equiv W(\xi,\eta;\alpha,h)$$

in the form of a sum

$$W = W_1 + W_2 \tag{27}$$

such that

$$W_1 \equiv W_1(\xi,-;\alpha,h),$$
$$W_2 \equiv W_2(-,\eta;\alpha,h).$$

The components are solutions of the ordinary differential equations

$$(\xi^2-1)(\frac{dW_1}{d\xi})^2 - \mu\alpha\xi(1 + \frac{h\alpha}{2\mu}\xi) = -\mu\alpha(1 + \frac{h\alpha}{2\mu}),$$

$$(\eta^2-1)(\frac{dW}{d\eta})^2 - \mu\alpha\eta(1 + \frac{h\alpha}{2\mu}\eta) = -\mu\alpha(1 + \frac{h\alpha}{2\mu}).$$

The term artificially introduced in the right hand members will result in the equations taking the simpler form

$$(\xi+1)\left(\frac{dW_1}{d\xi}\right)^2 - \mu\alpha\left[1 + \frac{h\alpha}{2\mu}(\xi+1)\right] = 0,$$

$$(\eta+1)\left(\frac{dW_2}{d\eta}\right)^2 - \mu\alpha\left[1 + \frac{h\alpha}{2\mu}(\eta+1)\right] = 0; \qquad (28)$$

they define implicitly W_1 (resp. W_2) as an algebraic function of ξ (resp. of η). In order to uniformize the various determinations of these functions, we introduce two state variables

$$\sigma \equiv \sigma(\xi,\eta;\alpha,h) \quad \text{and} \quad \delta \equiv \delta(\xi,\eta;\alpha,h)$$

through the implicit equations

$$\frac{1}{2}\sigma U_1\left(\frac{1}{2}\sigma;-2h\right) = \sqrt{\frac{s}{2\mu}}, \qquad (29_1)$$

$$\frac{1}{2}\delta U_1\left(\frac{1}{2}\delta;-2h\right) = \sqrt{\frac{s-c}{2\mu}} \qquad (29_2)$$

involving Stumpff's universal functions (Stumpff 1947, 1959) in Sconzo's notations (Sconzo 1967). Note the dimensions of σ and δ: time/length.

The definitions (4) imply that

$$\xi + 1 = \frac{2\mu}{\alpha}\sigma^2 U_2(\sigma;-2h), \qquad (30_1)$$

$$\eta + 1 = \frac{2\mu}{\alpha}\delta^2 U_2(\delta;-2h), \qquad (30_2)$$

hence, by virtue of the fundamental identities involving universal functions

$$1 + \frac{h\alpha}{2\mu}(\xi+1) = U_0^2\left(\frac{1}{2}\sigma;-2h\right) = \frac{1}{2}[1 + U_0(\sigma;-2h)], \qquad (31_1)$$

$$1 + \frac{h\alpha}{2\mu}(\eta+1) = U_0^2\left(\frac{1}{2}\delta;-2h\right) = \frac{1}{2}[1 + U_0(\delta;-2h)]. \qquad (31_2)$$

By differentiating the formulas (30) to obtain that

$$\frac{d\xi}{d\sigma} = \frac{2\mu}{\alpha}\,\sigma U_1(\sigma;-2h), \tag{32$_1$}$$

$$\frac{d\eta}{d\sigma} = \frac{2\mu}{\alpha}\,\delta U_1(\delta;-2h), \tag{32$_2$}$$

one can write that

$$\frac{dW_1}{d\xi} = \frac{\alpha}{2\mu}\,[\sigma U_1(\sigma;-2h)]^{-1}\,\frac{dW_1}{d\sigma}\,, \tag{33$_1$}$$

$$\frac{dW_2}{d\eta} = \frac{\alpha}{2\mu}\,[\delta U_1(\delta;-2h)]^{-1}\,\frac{dW_1}{d\delta} \tag{33$_2$}$$

so that the differential equations (28) become

$$\left(\frac{dW_1}{d\sigma}\right)^2 = \mu^2[1 + U_0(\sigma;-2h)]^2, \tag{34$_1$}$$

$$\left(\frac{dW_2}{d\delta}\right)^2 = \mu^2[1 + U_0(\delta;-2h)]^2. \tag{34$_2$}$$

Carrying out the elementary quadratures and combining the double signs implied by the square roots finally yields the complete solution

$$\boxed{W = \mu[\sigma + \sigma U_1(\sigma;-2h)] \pm \mu[\delta + \delta U_1(\delta;-2h)]} \tag{35}$$

In what follows, unless otherwise explicitly stated, wherever a double sign appears in a formula, the upper sign will correspond to the sign + in (35), the lower sign to the - in (35).

4. Canonical constants of integration related to bipolar coordinates

The complete solution (35) defines a completely canonical transformation from the bipolar state variables $(\xi, \eta, p_\xi, p_\eta)$ to the phase coordinates (Γ, τ) and the phase momenta (α, h) through the set of implicit equations

$$p_\xi = \frac{\partial W}{\partial \xi}, \quad (36_1) \qquad \Gamma = \frac{\partial W}{\partial \alpha}, \quad (36_3)$$

$$p_\eta = \frac{\partial W}{\partial \eta}, \quad (36_2) \qquad \tau = -t + \frac{\partial W}{\partial h}. \quad (36_4)$$

Let us examine the equations expressing the conjugate moments in terms of the canonical constants h and α. In view of Eq. (33) and (34),

$$p_\xi = \frac{1}{2}\alpha \frac{U_0(\frac{1}{2}\sigma;-2h)}{\frac{1}{2}\sigma U_1(\frac{1}{2}\sigma;-2h)}, \qquad (37_1)$$

$$p_\eta = \pm\frac{1}{2}\alpha \frac{U_0(\frac{1}{2}\delta;-2h)}{\frac{1}{2}\delta U_1(\frac{1}{2}\delta;-2h)}, \qquad (37_2)$$

so that, on account of (29_1) and (29_2), the conjugate moments are the *energetic functions* (Kustaanheimo 1963)

$$p_\xi \equiv p_\xi(h,\alpha,s) = \alpha\sqrt{\frac{1}{2}(h + \frac{\mu}{s})}, \qquad (38_1)$$

$$p_\eta \equiv p_\eta(h,\alpha,s-c) = \pm\alpha\sqrt{\frac{1}{2}(h + \frac{\mu}{s-c})}. \qquad (38_2)$$

Returning from (38_2) to Eq. (15), we may now interpret the double sign contained in the complete solution W: the upper (*resp.* lower) sign in W corresponds to η increasing (*resp.* decreasing) in the time interval $t-t_0$.

There results from Eq. (16) that the central and chordal components of the velocity at P are the *energetic functions*

$$A \equiv A(h,s,s-c) = \sqrt{\frac{1}{2}(h + \frac{\mu}{s})} \pm \sqrt{\frac{1}{2}(h + \frac{\mu}{s-c})}\ , \qquad (39_1)$$

$$B \equiv B(h,s,s-c) = \sqrt{\frac{1}{2}(h + \frac{\mu}{s})} \mp \sqrt{\frac{1}{2}(h + \frac{\mu}{s-c})}\ . \qquad (39_2)$$

We calculate the partial derivative of (36_3) in several steps. From the first two relations in the list (6), we deduce that

$$\frac{\partial c}{\partial \alpha} = -\cos \beta,$$

hence, that

$$\frac{\partial s}{\partial \alpha} = \frac{1}{2}(1 + \frac{\partial c}{\partial \alpha}) = \sin^2 \frac{1}{2}\beta,$$

$$\frac{\partial (s-c)}{\partial \alpha} = \frac{1}{2}(1 - \frac{\partial c}{\partial \alpha}) = \cos^2 \frac{1}{2}\beta.$$

Therefrom, addressing ourselves to the Eq. (29), we find that

$$\frac{\partial \sigma}{\partial \alpha} = \frac{1}{2}\,\frac{1}{\sqrt{2\mu s}}\,\frac{1}{U_0(\frac{1}{2}\sigma;-2h)}\,\sin^2 \frac{1}{2}\beta,$$

$$\frac{\partial \delta}{\partial \alpha} = \frac{1}{2}\,\frac{1}{\sqrt{2\mu(s-c)}}\,\frac{1}{U_0(\frac{1}{2}\delta;-2h)}\cdot\cos^2 \frac{1}{2}\beta.$$

By composition of partial differentiations, we conclude from (34) and (35) that

$$\Gamma = \frac{1}{2}\mu \frac{\sin^2\frac{1}{2}\beta}{\sqrt{2\mu s}} \frac{1+U_0(\sigma;-2h)}{U_0(\frac{1}{2}\sigma;-2h)} \pm \frac{1}{2}\mu \frac{\cos^2\frac{1}{2}\beta}{\sqrt{2\mu(s-c)}} \frac{1+U_0(\delta;-2h)}{U_0(\frac{1}{2}\delta;-2h)}$$

$$= \frac{\sin^2\frac{1}{2}\beta}{\sqrt{\frac{2s}{\mu}}} U_0(\tfrac{1}{2}\sigma;-2h) \pm \frac{\cos^2\frac{1}{2}\beta}{\sqrt{\frac{2(s-c)}{\mu}}} U_0(\tfrac{1}{2}\delta;-2h).$$

We use Eq. (29) and (37) to arrive at a simpler expression

$$\Gamma = 2\,\frac{1}{\alpha}(p_\xi \cdot \sin^2\tfrac{1}{2}\beta + p_\eta \cdot \cos^2\tfrac{1}{2}\beta)$$

$$= \frac{1}{\alpha}[(p_\xi+p_\eta) - (p_\xi-p_\eta)\cos\beta]$$

or, through the definitions (16), at the final expression

$$\Gamma = A - B\cos\beta. \tag{40}$$

By reworking this expression as the sum

$$\Gamma = (A-B)\cos^2(\beta/2) + (A+B)\sin^2(\beta/2)$$

and using some of the expressions in the list (7), we can display the constant of integration Γ as the *energetic function*

$$\Gamma = \frac{(s-\alpha)(s-c)}{\alpha c}\sqrt{2(h + \frac{\mu}{s-c})} + \frac{s(s-r)}{\alpha c}\sqrt{2(h + \frac{\mu}{s})}\ . \tag{41}$$

The set of elements (h,Γ,α) may seem unusual to determine a Keplerian orbit. Although we do not propose them to solve the initial value version of the problem of two bodies, we ought to outline how they could be used to that purpose.

We easily check that

$$\Gamma^2 + B^2\sin^2\beta = 2(h + \frac{\mu}{\alpha}). \quad (42)$$

Together with (40), this relation will be used to express the quantities A and B as functions of the constants h,Γ,α and of the parametrizing angle β. We put

$$\Phi = \sqrt{2(h + \frac{\mu}{\alpha})}$$

and find that

$$A = \Gamma + \frac{1}{2}\,\Phi(\text{cotan}\,\frac{\beta}{2} - \tan\frac{\beta}{2}),$$

$$B = \frac{1}{2}\,\Phi(\text{cotan}\,\frac{\beta}{2} + \tan\frac{\beta}{2}).$$

Then the definitions (16) are called to express the conjugate moments p_ξ and p_η in the same way, and one goes to the equations (38) in order to calculate the half-perimeter s and the length s-c in terms of the elements h,α,Γ and of the parametrizing angle β. At this stage, we could produce the central distance r, and the anomaly w, as well as the velocity through the formulas (17). There would remain to relate the angle β to the time, and this we shall undertake at least in principle in the next section.

5. Lambert's equation

The equation (36_4) is somewhat difficult to eliminate. Its right hand member is the sum of three terms

$$\frac{\partial W}{\partial h} = \left(\frac{\partial W}{\partial \sigma}\right)_h \cdot \frac{\partial \sigma}{\partial h} + \left(\frac{\partial W}{\partial \delta}\right)_h \cdot \frac{\partial \delta}{\partial h} + \left(\frac{\partial W}{\partial h}\right)_{\sigma,\delta},$$

where $(\partial W/\partial \sigma)_h$ indicates the partial derivative of W with respect to σ for h being kept fixed, $(\partial W/\partial \delta)_h$ the partial derivative with respect to δ for h fixed, and $(\partial W/\partial h)_{\sigma,\delta}$ the partial derivative with respect to h for σ and δ being kept fixed. We make use of the partial derivative (see Goodyear 1966)

$$\frac{\partial}{\partial k} U_1(z;k) = \frac{1}{2} z^2[U_3(z;k)-U_2(z;k)]$$

to deduce from (35) and the definitions (29) that

$$\left(\frac{\partial W}{\partial h}\right)_{\sigma,\delta} = -\mu\sigma^3[U_3(\sigma;-2h)-U_2(\sigma;-2h)] \mp \mu\delta^3[U_3(\delta;-2h)-U_2(\delta;-2h)],$$

$$[1+U_0(\sigma;-2h)] \cdot \frac{\partial \sigma}{\partial h} = \sigma^3[2U_3(\sigma;-2h)-U_2(\sigma;-2h)],$$

$$[1+U_0(\delta;-2h)] \cdot \frac{\partial \delta}{\partial h} = \delta^3[2U_3(\delta;-2h)-U_2(\delta;-2h)].$$

Therefrom we conclude that (36_4) is the equation

$$\mu^{-1}(t-\tau) = \sigma^3 U_3(\sigma;-2h) \pm \delta^3 U_3(\delta;-2h). \tag{43}$$

Such a universal expression is interesting for its conciseness, but it is remote from more familiar expressions and it is not convenient for numerical applications. In order to remedy these definitions, the definitions (29) will be rewritten as

$$\sigma = \frac{\sqrt{\frac{2s}{\mu}}}{U_0(\frac{1}{2}\,\sigma;-2h)}\,, \quad (44_1) \qquad \delta = \frac{\sqrt{\frac{2(s-c)}{\mu}}}{U_1(\frac{1}{2}\,\delta;-2h)} \quad (44_2)$$

and it will be recalled (see Appendix) that

$$U_3(z;k) = \frac{1}{6}\,U_1^3(\frac{1}{2}\,z;k) \times F(1,3;\tfrac{5}{2};q) \quad (45)$$

where

$$q = k(\frac{1}{4}\,z)^2 U_1^2(\frac{1}{4}\,z;k) \quad (46)$$

and F is a hypergeometric function. Accordingly the variables (as suggested by Lancaster *et al* 1966)

$$\sigma^* = -2h(\frac{1}{4}\,\sigma)^2 U_1^2(\frac{1}{4}\,\sigma;-2h), \quad (47_1)$$

$$\delta^* = -2h(\frac{1}{4}\,\delta)^2 U_1^2(\frac{1}{4}\,\delta;-2h) \quad (47_2)$$

transform Eq. (43) into the relation

$$6\sqrt{\mu}\,(t-\tau) = (2s)^{3/2}F(1,3;\tfrac{5}{2};\sigma^*) \pm [2(s-c)]^{3/2}F(1,3;\tfrac{5}{2};\delta^*). \quad (48)$$

At first sight the definitions (44) may seem artificial; yet the quantities σ^* and δ^* have the advantage of being dimensionless. Moreover they are in simple relation with the half-perimeter s and the constant of energy h. Indeed there results from (47) that

$$1-\sigma^* = U_0^2(\frac{1}{4}\,\sigma;-2h),$$

$$1-\delta^* = U_0^2(\frac{1}{4}\,\delta;-2h),$$

so that

$$\sigma^*(1-\sigma^*) = -\frac{1}{4}\frac{h}{\mu s}\,, \quad (49_1)$$

$$\delta^*(1-\delta^*) = -\frac{1}{4}\frac{h}{\mu(s-c)}\,. \quad (49_2)$$

Incidentally, (47_1) and (47_2) establish σ^* and δ^* as energetic functions in the sense of Kustaanheimo. Their dependence on h for a fixed half-perimeter s and a fixed chord c is suggested in Fig. 3. Besides the equations (47) are easy to invert. Let us recall (see Appendix) that

$$\frac{1}{U_1(z;k)} = F(1,1;\tfrac{3}{2};q) \tag{50}$$

with the argument q as defined in (46). Consequently there follows from (44) that

$$\sigma = \sqrt{\frac{2s}{\mu}}\, F(1,1;\tfrac{3}{2};\sigma^*), \tag{51_1}$$

$$\delta = \sqrt{\frac{2(s-c)}{\mu}}\, F(1,1;\tfrac{3}{2};\delta^*). \tag{51_2}$$

At this point, we use Gauss' quadratic transformation formula for hypergeometric functions, namely

$$F(a,b;\tfrac{1}{2}a + \tfrac{1}{2}b + \tfrac{1}{2};z) = F(\tfrac{1}{2}a,\tfrac{1}{2}b;\tfrac{1}{2}a + \tfrac{1}{2}b + \tfrac{1}{2};4z - 4z^2)$$

in order to transform (51_1), (51_2) and (48) into the expressions valid for all types of Keplerian motions

$$\sigma = \sqrt{\frac{2s}{\mu}}\, F(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};-\frac{hs}{\mu}), \tag{52_1}$$

$$\delta = \sqrt{\frac{2(s-c)}{\mu}}\, F(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};-\frac{h(s-c)}{\mu}), \tag{52_2}$$

$$\boxed{6\sqrt{\mu}\,(t-\tau) = (2s)^{3/2}F(\tfrac{1}{2},\tfrac{3}{2};\tfrac{5}{2};-\frac{hs}{\mu}) \pm (2(s-c))^{3/2}F(\tfrac{1}{2},\tfrac{3}{2};\tfrac{5}{2};-\frac{h(s-c)}{\mu})}\ . \tag{53}$$

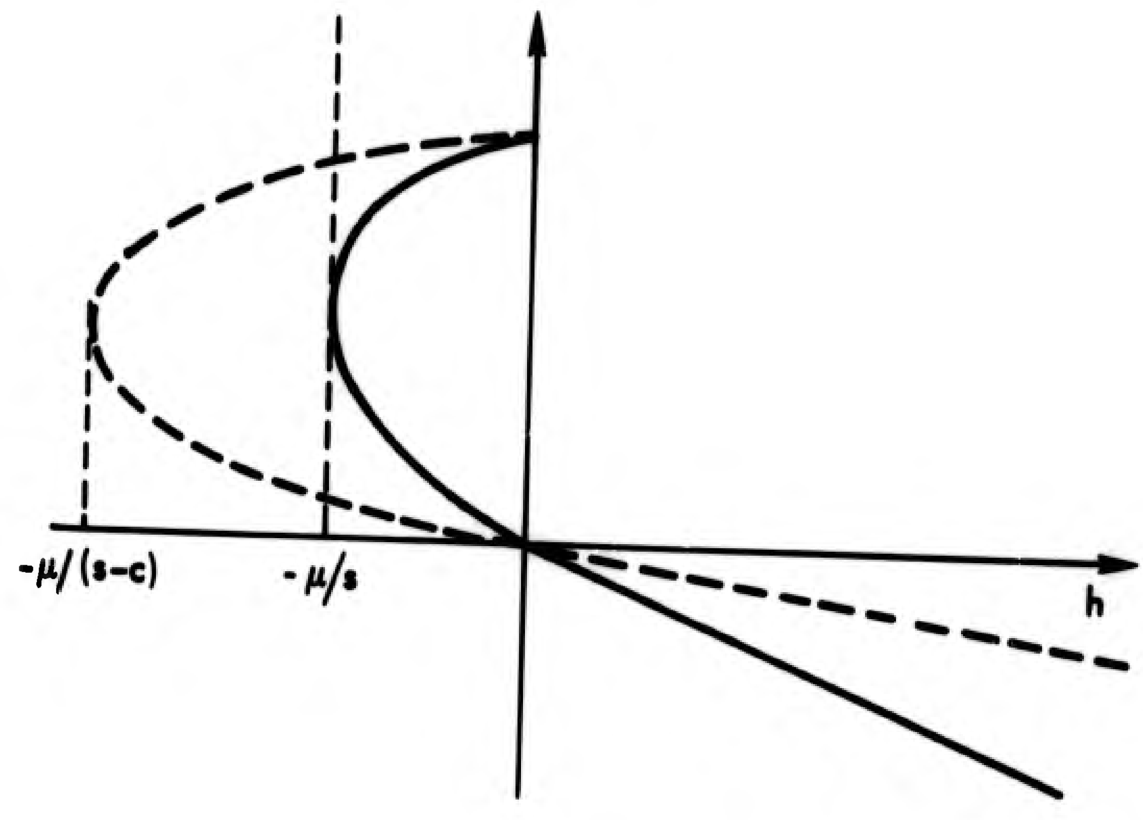

Fig. 3. The variables σ^* (in solid) and δ^* (in dotted) as functions of the energy h.

Making $h = 0$ in (53) will produce Euler's well-known equation

$$6\sqrt{\mu}\,(t-\tau) = (2s)^{3/2} \pm (2(s-c))^{3/2}. \tag{54}$$

Notice also that the expressions (52_1) and (52_2) constitute the explicit solutions of the implicit equations (29_1) and (29_2).

The universal form of Lambert's equation (53) leads to equivalent expressions to be found in the recent literature of the problem of two bodies. For instance, let us define the "modified mean motion" $\nu > 0$ such that

$$\nu^2(2s)^3 = \mu,$$

the "shape factor"

$$\eta = \zeta\sqrt{1 - \frac{c}{s}}$$

and the dimensionless energetic function

$$z = -\frac{hs}{\mu}.$$

With these notations, Lambert's equation (53) becomes

$$\nu(t-\tau) = \frac{1}{6}\,F(\tfrac{1}{2},\tfrac{3}{2};\tfrac{5}{2};z) \pm \frac{1}{6}\,\zeta^3\eta^3 F(\tfrac{1}{2};\tfrac{3}{2};\tfrac{5}{2};\eta^2 z).$$

Then the hypergeometric function F is replaced by its development in power series of z so that

$$F(\tfrac{1}{2},\tfrac{3}{2};\tfrac{5}{2};z) = 6\sum_{n\geq 0} a_n z^n,$$

and Lambert's equation takes the form proposed by Gedeon (1963):

$$\nu(t-\tau) = \sum_{n\geq 0} a_n(1 \pm \eta^{2n+3})z^n.$$

Other selections of the "shape factor" η and of the dimensionless energetic quantity z will lead to equivalent versions of (53) (see for instance Battin 1968).

6. The characteristic function

The major, if not the only, advantage of the bipolar coordinates for the problem of two bodies, is that they introduce in a direct and elementary way what Hamilton calls the *characteristic* function. Simplicity through the bipolar coordinates contrasts eminently with the difficulties caused by the polar coordinates in the transition from a complete solution of Hamilton-Jacobi's equation to the characteristic function (see for instance Lure 1962).

Assume that Q is the initial position P_0 of the particle at the initial time t_0, so that α is the initial distance r_0 from the center of attraction O. Thus

$$\alpha = r_0 = \sqrt{x_0^2+y_0^2} ,$$

$$c = \sqrt{(x-x_0)^2+(y-y_0)^2} ,$$

hence s and s-c are functions of the initial and terminal coordinates, so are also the bipolar coordinates (ξ,η) and the uniformizing variables σ and δ. Therefore the complete solution

$$\mathfrak{A} \equiv \mathfrak{A}(x,y,x_0,y_0;h) = W(\sigma(x,y,x_0,y_0;h),\delta(x,y,x_0,y_0;h);h)$$

in which the uniformizing variables ought to be expressed in terms of the initial and final coordinates, is a characteristic function of the problem of two bodies.

Accordingly the components of the initial and terminal velocities are given by the partial derivatives

$$\dot{x}_0 = \frac{\partial \mathcal{A}}{\partial x_0}, \quad (55_1) \qquad \dot{x} = \frac{\partial \mathcal{A}}{\partial x} \qquad (55_3)$$

$$\dot{y}_0 = \frac{\partial \mathcal{A}}{\partial y_0}, \quad (55_2) \qquad \dot{y} = \frac{\partial \mathcal{A}}{\partial y} \qquad (55_4)$$

whereas the time it takes for the particle to travel from P_0 to P is given by the derivative

$$t - t_0 = \frac{\partial \mathcal{A}}{\partial h}. \qquad (56)$$

Eq. (56) which constitutes Lambert's equation properly speaking has been amply analyzed in the preceding section.

We compute the partial derivative in (55_1) by the chain rule

$$\frac{\partial \mathcal{A}}{\partial x_0} = \frac{\partial W}{\partial \sigma} \cdot \frac{\partial \sigma}{\partial x_0} + \frac{\partial W}{\partial \delta} \cdot \frac{\partial \delta}{\partial x_0}.$$

Since

$$2 \frac{\partial s}{\partial x_0} = \frac{x_0}{r_0} + \frac{x_0 - x}{c},$$

$$2 \frac{\partial (s-c)}{\partial x_0} = \frac{x_0}{r_0} - \frac{x_0 - x}{c},$$

the definitions (29) imply that

$$2\mu\sigma U_1(\sigma;-2h) \cdot \frac{\partial \sigma}{\partial x_0} = \frac{x_0}{r_0} + \frac{x_0 - x}{c},$$

$$2\mu\delta U_1(\delta;-2h) \cdot \frac{\partial \delta}{\partial x_0} = \frac{x_0}{r_0} - \frac{x_0 - x}{c},$$

and therefore that

$$\frac{\partial \mathcal{A}}{\partial x_0} = A \frac{x_0}{r_0} + B \frac{x_0 - x}{c}.$$

Similar expressions are found for the derivatives in (55_2), (55_3) and (55_4). The conclusion is that, in any inertial frame of reference located at the center of attraction, if the initial and final positions P_0 and P as well as the level of energy h are assigned, then the components of the velocity at the initial position P_0 are

$$\boxed{\begin{aligned} \dot{x}_0 &= -A\frac{x_0}{r_0} - B\frac{x_0-x}{c} \\ \dot{y}_0 &= -A\frac{y_0}{r_0} - B\frac{y_0-y}{c} \end{aligned}} \tag{57}$$

whereas the components of the velocity at the final position P are

$$\boxed{\begin{aligned} \dot{x} &= A\frac{x}{r} + B\frac{x-x_0}{c} \\ \dot{y} &= A\frac{y}{r} + B\frac{y-y_0}{c} \end{aligned}} \tag{58}$$

These formulas known already to Jacobi (Dziobek 1888) have been rediscovered lately by Godal (1961). It is of some interest to relate Godal's geometric approach to the analytical deduction made here.

Consider the special inertial frame whose x-axis goes through the initial position P_0; in this coordinate system, we have

$$x_0 = r_0, \qquad y_0 = 0,$$
$$x = r\cos w, \qquad y = r\sin w,$$
$$x-x_0 = c\cos\phi_0, \qquad y-y_0 = c\sin\phi_0.$$

(Note that to render the notations more symmetric, we have decided to indicate by ϕ_0 the exterior angle at $P_0 = Q$ that has been labeled β in Fig. 2.) The dual relations (57) and (58) become

$$\dot{x}_0 = -A + B\cos\phi_0,$$
$$\dot{y}_0 = \qquad B\sin\phi_0, \qquad (59)$$

and

$$\dot{x} = A\cos w + B\cos\phi_0,$$
$$\dot{y} = B\sin w + B\sin\phi_0. \qquad (60)$$

As expected, we recover in (60) the decomposition in *central* and *chordal* components already performed in the formulas (17). The formulas (59) indicate that the velocity vector $\dot{\mathbf{x}}_0$ at the initial position P_0 may be decomposed into the sum

$$\dot{\mathbf{x}}_0 = \rho_0\hat{\mathbf{x}}_0 + \chi_0\hat{c},$$

with $\hat{\mathbf{x}}_0$ being the unit vector in the direction OP_0; ρ_0 (resp. χ_0) is the central (resp. the chordal) component of the initial velocity. According to (59),

$$\rho_0 = -A, \qquad \chi_0 = B. \qquad (61)$$

Then out of formulas (19), (39) and (61) result Godal's conditions of compatibility of terminal positions and velocities:

(i) The chordal components of the terminal velocities relative to the oriented triangle OP_0P are equal, more precisely

$$\rho = \rho_0 = \sqrt{\tfrac{1}{2}(h + \tfrac{\mu}{s})} \pm \sqrt{\tfrac{1}{2}(h + \tfrac{\mu}{s-c})} \;;$$

(ii) The central components of the terminal velocities relative to the oriented triangle OP_0P are equal, more precisely

$$\chi = \chi_0 = \sqrt{\tfrac{1}{2}(h + \tfrac{\mu}{s})} \mp \sqrt{\tfrac{1}{2}(h + \tfrac{\mu}{s-c})} \;;$$

(iii) The product of the chordal and central components for the velocity either at P_0 or at P depends solely on the geometry of the triangle of the boundary positions, and not on the Keplerian orbit connecting them; more precisely

$$\rho\chi = \rho_0\chi_0 = -AB = \frac{1}{2}\,\frac{\mu c}{s(s-c)} = \frac{\mu c}{2\Delta}\tan\frac{w}{2}\,. \tag{62}$$

Incidentally, the element $-\Gamma$ that we discussed in Section 4 receives now a direct interpretation: it is the radial component $\dot{x}_0$ of the initial velocity in association with the triangle OP_0P.

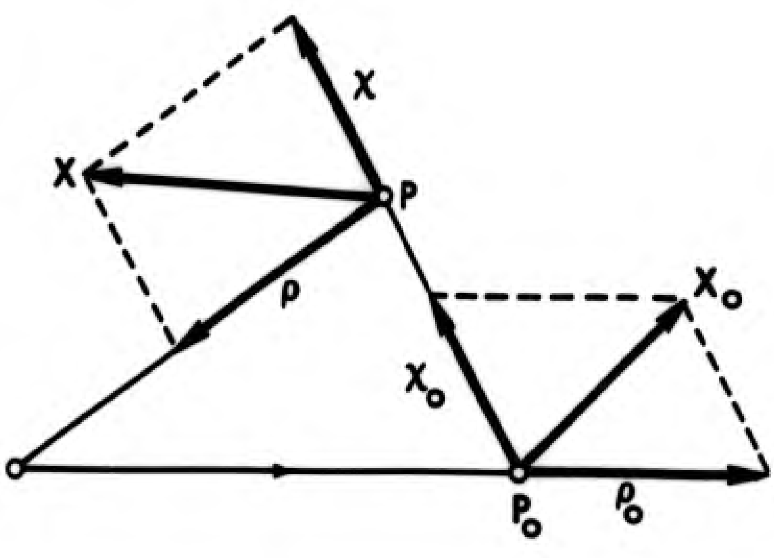

Fig. 4. Central and chordal components of the velocities at the boundaries.

7. <u>The eccentric anomalies at the boundaries</u>

The state variables σ and δ are related in a simple way to the universal eccentric anomalies of P_0 and P.

Indeed, for a Keplerian orbit going through P_0 and P, in the apsidal frame of reference (the axis Ox_p being the line of apsides oriented positively from the center of attraction to the periapsis), the coordinates of P_0 and P_1 are respectively

$$x_p(t_0) = r_p - \mu\varepsilon_0^2 U_2(\varepsilon_0;-2h),$$

$$y_p(t_0) = \sqrt{\mu p} \cdot \varepsilon_0 U_1(\varepsilon_0;-2h),$$

$$x_p(t) = r_p - \mu\varepsilon^2 U_2(\varepsilon;-2h),$$

$$y_p(t) = \sqrt{\mu p} \cdot \varepsilon U_1(\varepsilon;-2h),$$

where ε is the universal eccentric anomaly (dimension: time/length) defined by the differential form

$$dt = r d\varepsilon$$

and r_p is the distance from the center of attraction to the periapsis, thus

$$r_p = \frac{p}{1+e} .$$

Also the radial distances r_0 and r, expressed in terms of the eccentric anomaly, are

$$r_0 = r_P + \mu e \varepsilon_0^2 U_2(\varepsilon_0;-2h),$$

$$r = r_P + \mu e \varepsilon^2 U_2(\varepsilon;-2h).$$

In application of the fundamental identity relating the universal functions U_0 and U_2, there results that

$$-\frac{2h}{\mu} r_0 = 1 - eU_0(\varepsilon_0;-2h),$$

$$-\frac{2h}{\mu} x_P(t_0) = U_0(\varepsilon_0;-2h) - e,$$

$$-\frac{2h}{\mu} r = 1 - eU_0(\varepsilon;-2h),$$

$$-\frac{2h}{\mu} x_P(t) = U_0(\varepsilon;-2h) - e.$$

These formulas are uniformized versions of familiar ones in the various classes of Keplerian motions.

With the help of the variables

$$S = \frac{1}{2}(\varepsilon+\varepsilon_0), \qquad D = \frac{1}{2}(\varepsilon-\varepsilon_0), \tag{63}$$

we deduce that

$$-\frac{h}{\mu}(r+r_0) = 1 - eU_0(S;-2h) \cdot U_0(D;-2h),$$

$$x_P(t) - x_P(t_0) = -2\mu S U_1(S;-2h) \cdot DU_1(D;-2h),$$

$$y_P(t) - y_P(t_0) = \sqrt{2\mu p}\, U_0(S;-2h) \cdot DU_1(D;-2h).$$

There follows in particular that

$$-\frac{h}{\mu} c^2 = 2\mu D^2 U_1^2(D;-2h) \cdot [1 - e^2 U_0(S;-2h)].$$

The "modified" anomaly $\bar{S}$ such that

$$U_0(\bar{S};-2h) = eU_0(S;-2h) \tag{64}$$

transforms the sum $r_0 + r$ and the chord c into the expressions

$$-\frac{h}{\mu}(r+r_0) = 1 - U_0(\bar{S};-2h) \cdot U_0(D;-2h),$$

$$c = 2\mu S^*U_1(\bar{S};-2h) \cdot DU_1(D;-2h).$$

Consequently we find that

$$s = \frac{1}{2}\,\mu(S^*+D)^2U_1^2(\tfrac{1}{2}(S^*+D);-2h),$$

$$s-c = \frac{1}{2}\,\mu(S^*-D)^2U_1^2(\tfrac{1}{2}(S^*-D);-2h).$$

In view of the definitions (29), we obtain the identities

$$\sigma^2U_1^2(\tfrac{1}{2}\,\sigma;-2h) = (\bar{S}+D)^2U_1^2(\tfrac{1}{2}(\bar{S}+D);-2h), \tag{65_1}$$

$$\delta^2U_1^2(\tfrac{1}{2}\,\delta;-2h) = (\bar{S}-D)^2U_1^2(\tfrac{1}{2}(\bar{S}-D);-2h) \tag{65_2}$$

that relates the variables σ and δ to the eccentric anomalies ε_0 and ε at the boundaries. They constitute uniformized interpretations of formulas that are ordinarily established separately for the various classes of Keplerian motions.

The formulas (65) are essential in discussing the types of Keplerian orbits that are solutions to the two-point boundary problem. It is not difficult to reconstruct around them--and thereby uniformize--the analysis given case by case by Plummer (1918).

8. The semi-latus rectum

Battin (1964) has evaluated the semi-latus rectum of the solutions in a discussion case by case. It is of interest to show how the present approach yields a more direct way of obtaining Battin's formulas and of uniformizing for all classes of Keplerian orbits.

There follows from Eq. (29) that

$$\frac{1}{2}(h + \frac{\mu}{s}) = \frac{1}{4} \frac{U_0^2(\frac{1}{2}\sigma;-2h)}{(\frac{\sigma}{2})^2 U_1^2(\frac{\sigma}{2};-2h)} ,$$

$$\frac{1}{2}(h + \frac{\mu}{s-c}) = \frac{1}{4} \frac{U_0^2(\frac{1}{2}\delta;-2h)}{(\frac{\delta}{2})^2 U_1^2(\frac{1}{2}\delta;-2h)} .$$

Hence, in view of (38),

$$p_\xi = \frac{1}{2}\alpha \frac{U_0(\frac{1}{2}\sigma;-2h)}{\frac{1}{2}\sigma U_1(\frac{1}{2}\sigma;-2h)} ,$$

$$p_\eta = \pm\frac{1}{2}\alpha \frac{U_0(\frac{1}{2}\delta;-2h)}{\frac{1}{2}\delta U_1(\frac{1}{2}\delta;-2h)} .$$

The theorems of addition for universal functions will then be used in (16) to express the quantities A and B in terms of σ and δ:

$$A = \frac{\mu}{\sqrt{s(s-c)}} \frac{\sigma\pm\delta}{2} U_1(\frac{\sigma\pm\delta}{2};-2h), \qquad (66_1)$$

$$B = -\frac{\mu}{\sqrt{s(s-c)}} \frac{\sigma\mp\delta}{2} U_1(\frac{\sigma\mp\delta}{2};-2h). \qquad (66_2)$$

We can thus conclude from (23) by way of (66_2) that

$$p = 4\mu \frac{(s-r_0)(s-r)}{c^2} \left(\frac{\sigma\mp\delta}{2}\right)^2 U_1^2\left(\frac{\sigma\mp\delta}{2};-2h\right). \tag{67}$$

This is the uniformized version of Battin's formula.

Appendix

If

$$q = k\left(\frac{z}{2}\right)^2 U_1^2\left(\frac{z}{2};k\right),$$

then

$$\frac{U_3(2z;k)}{U_1^3(z;k)} = \frac{1}{6}\, F(1,3;\tfrac{5}{2};q),$$

and

$$\frac{1}{U_1(z;k)} = F(1,1;\tfrac{3}{2};q)$$

Indeed let us put

$$Q \equiv Q(z;k) = U_3(2z;k)/U_1^3(z;k).$$

Then we differentiate this definition with respect to z to derive that

$$8z^3U_1^3(z)\,\frac{dQ}{dz} + 24z^2U_1^2(z)U_0(z)Q = 8z^2U_2(2z).$$

But

$$2U_2(2z) = U_1^2(u),$$

hence the preceding identity becomes

$$2zU_1(z)\,\frac{dQ}{dz} + 6U_0(z)\cdot Q = 1. \tag{1}$$

Now we readily check that

$$1-q = U_0^2(\tfrac{z}{2}),$$

$$1-2q = U_0(z),$$

$$q(1-q) = \frac{1}{4} kz^2U_1^2(z),$$

$$\frac{dq}{dz} = \frac{1}{2} kzU_1(z).$$

These relations are used to transform (1) into

$$4q(1-q) \frac{dQ}{dq} + 6(1-2q)Q = 1. \qquad (2)$$

On differentiating (2) with respect to q, we arrive at the relation

$$q(1-q) \frac{d^2Q}{dq^2} + (\tfrac{5}{2} - 5q) \frac{dQ}{dq} - 3Q = 0.$$

This is actually Gauss' equation

$$x(1-x) \frac{d^2y}{dx^2} + [c - (a+b+1)x] \frac{dy}{dx} - aby = 0$$

in which

$$a = 3, \quad b = 1, \quad c = 5/2.$$

Consequently, because Q is holomorphic at $z = 0$ and $Q(0) = 1/6$, the function $Q(q)$ is identical to the hypergeometric series

$$Q = \frac{1}{6} F(3,1;\tfrac{5}{2};q).$$

The second half of the proposition is proved in the same way. Let us put

$$Q \equiv Q(z;k) = 1/U_1(z;k),$$

and let us differentiate this definition with respect to z so as to obtain that

$$zU_1(z)\frac{dQ}{dz} + U_0(z)\cdot Q = 1. \tag{3}$$

We use the identities in q to put (3) in the form

$$2q(1-q)\frac{dQ}{dq} + (1-2q)\cdot Q = 1.$$

By differentiating it with respect to q, we obtain that

$$q(1-q)\frac{d^2Q}{dq^2} + [\tfrac{3}{2} - 3q]\frac{dQ}{dq} - Q = 0.$$

But this is Gauss' equation for

$$a = 1, \quad b = 1, \quad c = 3/2;$$

moreover Q is holomorphic at $z = 0$ and $Q(0) = 1$. Hence we conclude that

$$Q = F(1,1;\tfrac{3}{2};q).$$

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