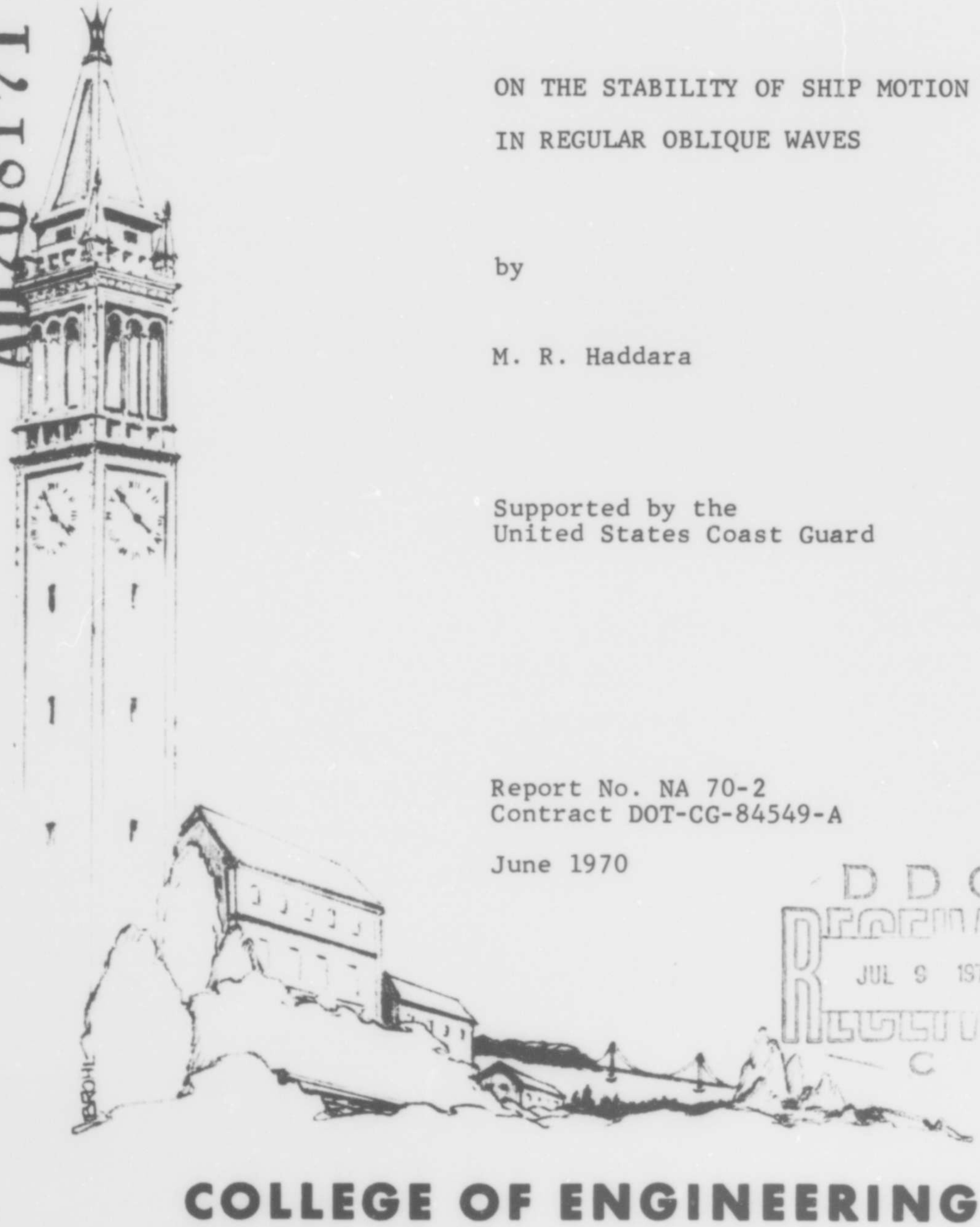

ON THE STABILITY OF SHIP MOTION IN REGULAR OBLIQUE WAVES

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M. R. Haddara

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ABSTRACT

The nonlinear equations of motion of a ship sailing in regular oblique waves are formulated and an analytical method is developed to obtain an approximate solution for them. The stability of the steady-state solution is investigated by examining the behavior of the solution of the equations of the first variation of the equations of motion. This study shows that the existence of unstable motion is predicted for certain cases if we take account of nonlinearities in the equations of motion. This instability is very sensitive to variations in the nonlinear damping coefficient in roll.

CONTENTS

LIST OF SYMBOLS

$OXYZ$	Coordinate axes fixed rigidly in the ship.
$\hat{O}\hat{X}\hat{Y}\hat{Z}$	Coordinate axes moving with the average motion of the ship.
$\bar{O}\bar{X}\bar{Y}\bar{Z}$	Coordinate axes fixed in space.
m	Mass of the ship.
u, v, w	Components of the velocity of the center of gravity of the ship along ox, oy, oz-axes respectively.
ϕ, θ, ψ	Rotation about ox, oy, oz-axes respectively.
p, q, r	Components of the angular velocity $\vec{\Omega}$, in the system oxyz.
I_x, I_y, I_z	Moments of inertia of the ship about ox, oy, oz-axes respectively.
U	Constant average forward velocity of the ship.
ω	Frequency of encounter, $\omega = \sigma + kU\cos\delta$.
σ	Wave frequency, $\sigma = \sqrt{gk}$.
k	Wave number, $k = 2\pi/\lambda$.
λ	Wave length.
δ	Angle between $\hat{o}\hat{x}$ and $\bar{o}\bar{x}$.
Φ	Velocity potential.
g	Gravitational acceleration.

I. INTRODUCTION

Until recently, the problem of ship motion in waves has been simplified by assuming that the following problems can be dealt with separately:

(1) The coupled motion of heave and pitch in head seas.

(2) The coupled motion of sway and yaw.

(3) The uncoupled rolling motion.

It has been shown by Grim [1956] that sway is strongly coupled to roll through linear coupling terms. However, it was not until 1967 that the problem of coupled sway, roll, and yaw motions had been considered by both Tasai and Webster. Tasai [1967] used a strip method to evaluate the hydrodynamic forces in the linear equations of motion. He obtained numerical solutions for several different ships having zero forward speed. His results show that the peak of roll response occurs in beam seas at zero forward speed, since this is the position of maximum roll moment and the exciting frequency is unaffected by heading.

Observations of ships at sea show that the worst roll usually occurs in quartering seas, when the ship is operating at design speed. This is a result of the effects of forward motion on the frequency of encounter of the wave. It is usually found that predominant wave components coincide with the ship's natural frequency in a quartering sea.

Webster [1967] used a semi-empirical approach to formulate the equations of motion. His main concern was to study the theory of motion control by means of active tank stabilizers. The problem of stability of motion in waves was not discussed by Tasai, while Webster considered only the stability of a linear system.

While these studies were mainly concerned with the linear equations of motion, Weinblum [1950] pointed out that even for moderate roll angles the restoring moment is a nonlinear function of the roll angle. Another important nonlinearity in the roll equations is the damping in which viscous effects play a large role.

It has also been shown by Abkowitz [1964] and others that certain features of the coupled sway-yaw motion can only be described by including nonlinear terms in the differential equations.

The importance of the effect of nonlinear terms in the equations of motion was illustrated in the work of Paulling and Rosenberg [1959]. They showed that unstable rolling motion is possible as a result of nonlinear static coupling with either heave or pitch.

In the present work attention is focused on the problem of nonlinear ship motion in oblique waves. The effect of certain nonlinear terms on the motion and its stability is investigated and discussed.

II. MATHEMATICAL FORMULATION

Coordinate System

Three sets of coordinate axes will be used and are shown in Figure 1. The first, $\bar{o}\bar{x}\bar{y}\bar{z}$, is fixed in space and is used to describe the wave surface. The second, $\hat{o}\hat{x}\hat{y}\hat{z}$, moves with the average motion of the ship. The time dependent motion of the ship is described with respect to this system. The third, oxyz, is fixed in the ship, and is used to describe the hull surface. The system oxyz has been chosen such that the origin of the coordinates is located at the center of gravity of the ship with the x-axis directed towards the bow, the y-axis to starboard, and the z-axis downwards. The x, y, and z-axes are assumed to coincide with the principal axes of inertia of the ship.

Equations of Motion

One of the main objectives of this work is to investigate the effects of the nonlinear sway-roll-yaw coupling on the stability of the resultant motion. In order to focus attention on the nonlinear coupling in these modes, which are known to be strongly coupled, we eliminate the symmetrical motions of surge, heave, and pitch. Therefore, we will consider coupled motion of small time-dependent sway, roll, and yaw motions superimposed on a constant average forward motion.

Using the laws of Newtonian dynamics one can write the equations of motion of the ship in the oxyz coordinate system as

$$
\begin{aligned}
m(\dot{v} + rU) &= Y\,, \\
I_x \dot{p} &= K\,, \\
I_z \dot{r} &= N\,.
\end{aligned}
\qquad (1)
$$

Furthermore, we shall express the right hand side of equations (1) as

$$
\begin{aligned}
Y &= Y_{HYD} + Y(t) \\
K &= K_{HYD} + K(t) \\
N &= N_{HYD} + N(t)
\end{aligned}
$$

where Y(t), K(t), N(t) are the linear sway force, roll moment, and yaw moment acting on the ship as a result of the pressure distribution in the wave, and Y_{HYD}, K_{HYD}, and N_{HYD} are the force and moment acting on the ship as a result of its motion (oscillations). Assuming that the hydrodynamic forces acting at a given instant on a ship sailing in unrestricted waters are analytic functions of the motion of the ship at the instant, one can express these forces in their Taylor series expansion. Equations (1) then become

$$
\begin{aligned}
(m - Y'_{\dot{v}})\dot{v} - Y'_{\dot{p}}\dot{p} - Y'_{\dot{r}}\dot{r} = {} & Y'_v v + Y'_p p + Y'_R r + Y'_{vvv} v^3 \\
& + Y'_{ppp} p^3 + Y'_{rrr} r^3 + Y'_{vpp} vp^2 + Y'_{vrr} vr^2 \\
& + Y'_{pvv} pv^2 + Y'_{prr} pr^2 + Y'_{rvv} rv^2 + Y'_{rpp} rp^2 \\
& + Y'_{vpr} vpr + Y'_s \sin \omega t + Y'_c \cos \omega t
\end{aligned}
\qquad (2)
$$

$$
\begin{aligned}
-K'_{\dot v}\dot v + (I_x - K'_{\dot p})\dot p - K'_{\dot r}\dot r = {} & K'_v v + K'_p p + K'_r r + K'_{vvv} v^3 \\
& + K'_{ppp} p^3 + K'_{rrr} r^3 + K'_{vpp} vp^2 + K'_{vrr} vr^2 \\
& + K'_{pvv} pv^2 + K'_{prr} pr^2 + K'_{rvv} rv^2 \\
& + K'_{rpp} rp^2 + K'_{vpr} vpr + K'_s \sin \omega t \\
& + K'_c \cos \omega t + K'_\phi \phi + K'_{\phi\phi\phi} \phi^3
\end{aligned}
$$

$$
\begin{aligned}
-N'_{\dot v}\dot v - N'_{\dot p}\dot p + (I_z - N'_{\dot r})\dot r = {} & N'_v v + N'_p p + N'_r r + N'_{vvv} v^3 \\
& + N'_{ppp} p^3 + N'_{rrr} r^3 + N'_{vpp} vp^2 + N'_{vrr} vr^2 \\
& + N'_{pvv} pv^2 + N'_{prr} pr^2 + N'_{rvv} rv^2 \\
& + N'_{rpp} rp^2 + N'_{vpr} vpr + N'_s \sin \omega t \\
& + N'_c \cos \omega t .
\end{aligned}
$$

where the subscripts notation is used to denote derivatives of the force with respect to the variable appearing as a subscript. The prime is merely used to distinguish the present form of the coefficients from a later form; the wave exciting forces are expressed in their final form as harmonic functions (see appendix B for detailed derivation).

Note that the following simplifications have been made in equations (2):

(1) Symmetry of the ship with respect to the centerline plane has been made use of to eliminate the even-ordered terms in Taylor's Series.

(2) The cross-coupling between acceleration and velocity and the second or higher order

acceleration terms are neglected. These assumptions were introduced by Abkowitz [1964] and confirmed experimentally by Strøm-Tejsen [1965] for coupled sway-yaw motion. It is reasonable to extend these assumptions to our case, since derivatives with respect to acceleration play only a minor roll in rolling motion.

We can remove the acceleration coupling between the three equations (2) and write them as

$$\begin{aligned}
\dot{v} = {} & Y_v v + Y_p p + Y_R r + Y_{vvv} v^3 + Y_{ppp} p^3 + Y_{rrr} r^3 + Y_{vpp} v p^2 \\
& + Y_{vrr} v r^2 + Y_{pvv} p v^2 + Y_{prr} p r^2 + Y_{rvv} r v^2 + Y_{rpp} r p^2 + Y_{vpr} v p r \\
& + Y_\phi \phi + Y_{\phi\phi\phi} \phi^3 + Y_s \sin \omega t + Y_c \cos \omega t \\[1ex]
\dot{p} = {} & K_v v + K_p p + K_r r + K_{vvv} v^3 + K_{ppp} p^3 + K_{rrr} r^3 + K_{vpp} v p^2 \\
& + K_{vrr} v r^2 + K_{pvv} p v^2 + K_{prr} p r^2 + K_{rvv} r v^2 + K_{rpp} r p^2 + K_{vpr} v p r \\
& + K_\phi \phi + K_{\phi\phi\phi} \phi^3 + K_s \sin \omega t + K_c \cos \omega t \\[1ex]
\dot{r} = {} & N_v v + N_p p + N_r r + N_{vvv} v^3 + N_{ppp} p^3 + N_{rrr} r^3 + N_{vpp} v p^2 \\
& + N_{vrr} v r^2 + N_{pvv} p v^2 + N_{prr} p r^2 + N_{rvv} r v^2 + N_{rpp} r p^2 \\
& + N_{vpr} v p r + N_\phi \phi + N_{\phi\phi\phi} \phi^3 + N_s \sin \omega t + N_c \cos \omega t
\end{aligned} \tag{3}$$

Equations (3) have unique solutions for specified initial conditions: $\phi = v = p = r = 0 \quad , \ t = 0.$

III. MOTION PREDICTION

It is not possible at present to obtain an exact solution to the equations (3), and we will have to resort to approximate methods. Before we attempt to find an approximate solution, it is desirable to rewrite the equations in a suitable form. We shall assume that the nonlinear damping and the nonlinear restoring moment terms are proportional to a small parameter β .

Physically this means that the nonlinearities in the equations of motion are assumed to be small. This is a reasonable assumption, since we will consider only moderate angles of roll.

Using the following change of variables:

$$x_1 = y \quad , \quad x_2 = \phi \quad , \quad x_3 = \psi \quad ,$$
$$\dot{x}_1 = v \quad , \quad \dot{x}_2 = p \quad , \quad \dot{x}_3 = r \quad .$$

we can rewrite the equations of motion as

$$\ddot{x}_k + \sum_{j=1}^{3} a_{kj}\dot{x}_j + \omega_k^2 x_2 = \beta\left[\sum_{i,j,m}^{3} d_{kijm}\,\dot{x}_i\,\dot{x}_j\,\dot{x}_m + \gamma_k x_2^3\right]$$
$$+ X_{k0}\sin(\omega t + \theta_k) \quad , \; k = 1, 2, 3 \qquad (4)$$

To find an approximate solution for (4) we shall use a modification to a method developed by Struble [1962]. His method combines the classical perturbation technique and the method of slowly varying parameters. In this method, one uses the solution of the free linear undamped equation as a generating solution, and then introduces a perturbation to obtain the solution for the nonlinear equation.

Since our equations are forced equations, it is better to modify this approach slightly and use the solution of the linear damped forced equations as a generating solution.

This modification to Struble's procedure has been tested by using it to obtain the well known solution of Duffing's equation. In Appendix C we compare solutions of Duffing's equation obtained by Struble's method, the modified method, and a method developed by Hsu [1960]. Hus's method is known to give a good approximation. From this comparison it is clear that the suggested modification has the following advantages over Struble's method:

(1) The approximation is better. It is shown in Appendix C that Struble's solution can be obtained from the modified method solution after making certain approximations.

(2) The modified method gives a single solution which is valid for a wide range of values of the forcing frequency, while two solutions are needed in the original method.

(3) A wider class of problems can be attacked, since we do not require the exciting force to be small, of the same order of magnitude as the

nonlinearity. This makes the method particularly suitable for the case treated in the present study.

Using the above mentioned modified method (see Appendix A for details of the method) one can write the steady state solution to equations (4) in the form

$$x_k = A_k \cos \omega t + B_k \sin \omega t + S_k \sin 3\omega t + C_k \cos 3\omega t \quad , k = 1,2,3 . \tag{5}$$

where A_K, B_K, S_K and C_k are constants to be determined from the solution of a set of six nonlinear coupled algebraic equations as explained in Appendix A. The solution given by (5) was evaluated for three models: a DTMB Series 60, Block Coefficient = 0.60, a DTMB Series 60, Block 0.70 model, and a Mariner class model. These ships were chosen because experimental values for certain nonlinear hydrodynamic coefficients are available for them and they are representatives of modern seagoing ships. There are also some experimental results for the response in roll and yaw in regular waves for the Series 60 models which can be used to check the theory.

The total wave exciting forces are calculated according to the scheme illustrated in Appendix B. The results of these calculations appear in Figures 2-4. Figure 2 shows good agreement between the calculated values for the sway force and experimental values obtained by Chey [1964].

The predicted solution is compared with a solution obtained by the numerical integration of the equations of motion. The numerical integration solution is obtained by using a fourth-order Runge-Kutta routine in which the step size is adjusted to give minimum truncation error. The result of the comparison which shows good agreement is given in Figure 11.

The predicted responses for the Series 60 models are also compared with available experimental results. Those for the Block 60 model are taken from Lewis and Numata [1960], and those for the Block 70 model are obtained from Yamanouchi [1966]. The results are given in Figures 5 through 10. The agreement is good for the roll response, but not as good for the yaw response. This seems to be a result of using the strip method to calculate a part of the wave-yaw moment.

The roll response curves display a peculiar double-peaked characteristic. The first peak in the response curve coincides with the peak in the exciting moment, which occurs for beam seas. The second peak corresponds to resonant roll motion at a frequency of encounter equal to the natural frequency of roll. This usually occurs in a quartering sea.

The effect of systematic variations in the moments of inertia of the ship on the response in roll and yaw is studied. The results of this investigation are shown in Figures 14 and 16 for the Series 60, Block 60 model and in Figure 17 and 18 for the Mariner model. The curves show that both roll and yaw motions are sensitive to variations in the moments of inertia but the degree of sensitivity varies according to the heading. Figure 20 shows the effect of systematically varying the metacentric height on the rolling motion amplitude. It is clear that decreasing the metacentric height may decrease the roll amplitude at one heading but will cause the roll amplitude to increase at another heading.

IV. STABILITY ANALYSIS

The analysis of the stability of the steady-state solution of the equations of motion will be considered here. Let us explain first what we mean by stability.

Let $x_i(t)$ be a solution of the equations of motion. We shall say that $x_i(t)$ is stable if, for a given $\varepsilon > 0$ and t_o, there exists $\eta = \eta(\varepsilon, t_o)$ such that any solution $y_i(t)$ for which $|x_i(t_o)-y_i(t_o)| < \eta$ satisfies $|x_i(t)-y_i(t)| < \varepsilon$ for $t \geqslant t_o$. If no such η exists, then the solution $x_i(t)$ is unstable.

In other words a motion of the ship is considered to be stable if it satisfies the following condition: any perturbation superimposed on this solution dies out with time after the cause of this perturbation has ceased to exist. If the perturbation grows up with time then the motion is considered unstable.

Let the periodic solution of (4) be denoted by

$$X_{io} = X_{io}(t) \qquad , \; i = 1, 2, 3. \tag{6}$$

which we may consider as a nonperturbed solution. Let $x_i(t)$ be the solution corresponding to an initial value $x_i(t_o) \neq 0$; this will be called the perturbed solution. Between the two there exists the following relation:

$$X_i(t) = X_{io}(t) + \xi_i \qquad , \; i = 1, 2, 3. \tag{7}$$

where $|\xi_i|$ are small, so that one can neglect their higher powers. We shall assume that the linear damping in equation (4) is small, and such that we can replace a_{kj} by βe_{kj}. Then, substituting (7) into (4) we get the following variational equations:

$$\ddot{\xi}_k + \omega_k^2 \xi_2 = \beta\Big[\sum_{j=1}^{3} e_{kj}\dot{\xi}_j + \sum_{i,j,m}^{3} d_{kijm}\{\dot{x}_{i0}\dot{x}_{j0}\dot{\xi}_m + \dot{x}_{j0}\dot{x}_{m0}\xi_i$$

$$+ \dot{x}_{m0}\dot{x}_{i0}\dot{\xi}_j\} + 3\gamma_k x_{20}^2 \xi_2\Big] \quad , k = 1,2,3 \qquad (8)$$

Thus we are able to reduce the problem of the stability of the periodic solution of (4) to the problem of the stability of the trivial solution of equation (8). This later problem can be solved by studying the behavior of the nontrivial solution of (8) with small initial values. If this nontrivial solution remains bounded as time increases, then the trivial solution is stable, and consequently the motion will be stable.

Equations (8) are three coupled linear differential equations with periodic coifficients of a type usually referred to as Hill's equation. The literature on such linear differential equations is abundant and in it one can find problems of similar nature attacked by one of two methods. One may find the behavior of the nontrivial solution by formulating the characteristic equation and determining the characteristic exponents. This usually involves intricate mathematical analyses (see, for example, Valeev [1961]); or, one may attempt to find the boundaries between regions of stability or instability by seeking the conditions under which periodic solutions for (8) may exist.

A third method, and the one which will be used here to determine the behavior of the nontrivial solution of (8) as a function of time, is the method introduced by Struble [1962], a modification of which was used to solve the equations of motion in the previous section. The method has been generalized and extended to the case of n -coupled equations by Hsu [1963]. As mentioned before, the method combines the method of slowly varying parameters of Krylov-Bogoliubov-Mitropolsky and the classical perturbation method of Poincare. This gives the method greater flexibility, and allows us to gain more insight into the resonance-producing mechanism by which unstable solutions are induced. There is no need, as in the solution of the equations of motion, to modify the analysis; the analysis as developed by Hsu [1963] is directly applicable to the case considered here. In the following we will highlight the main features involved in this method, and for details of the method the reader is referred to Hsu's paper.

Using equations (5) one can expand x_{20}^2 as follows:

$$x_{20}^{2} = g^{(0)} + \sum_{m=2,4,6} \left\{ g_c^{(m)} \cos m\omega t + g_s^{(m)} \sin m\omega t \right\} \qquad (9)$$

substituting (9) into (8), we get

$$\ddot{\xi}_k + n_k^2 \xi_2 = \beta \Big[\sum_{j=1}^{3} e_{kj} \dot{\xi}_j + \sum_{j,i,m}^{3} d_{kijm} \big\{ \dot{x}_{io} \dot{x}_{jo} \dot{\xi}_m + \dot{x}_{jo} \dot{x}_{mo} \dot{\xi}_i$$

$$+ \dot{x}_{mo} \dot{x}_{io} \dot{\xi}_j \big\} + 3 \gamma_k \sum_m \big\{ g_c^{(m)} \cos m\omega t + g_s^{(m)} \sin \omega m t \big\} \Big] ,$$

$$n_k^2 = \omega_k^2 - 3 \gamma_k g^{(0)} \quad , \qquad k = 1, 2, 3 \qquad (10)$$

For $\beta = 0$, equation (10) has a solution of the form

$$\xi_k = \rho_k \left[G \cos n_2 t + H \sin n_2 t \right] ,$$

$$\dot{\xi}_k = n_2 \rho_k \left[- G \sin n_2 t + H \cos n_2 t \right] , \quad k = 1, 2, 3 . \qquad (11)$$

where

$$\rho_k = \left(\frac{n_k}{n_2} \right)^2$$

The solution (11) is used to generate a solution for equations (8) when $\beta \neq 0$. This solution is written in the form

$$\xi_k = \rho_k \left[G(t) \cos n_2 t + H(t) \sin n_2 t \right] + \sum_{m=1}^{N} \beta^m \xi_{km} ,$$

$$\dot{\xi}_k = n_2 \rho_k \left[- G(t) \sin n_2 t + H(t) \cos n_2 t \right] + \sum_{m=1}^{N} \beta^m \dot{\xi}_{km} ,$$

$$k = 1, 2, 3 . \qquad (12)$$

The form of this solution indicates that for a first order theory the behavior of ξ_1 and ξ_3 depends on the behavior of ξ_2. That is, for a first order theory, the behavior of ξ_2, and consequently that of G, H and ξ_{21} is sufficient to indicate whether or not the motion is stable. Two cases have to be considered.

(1) Nonresonance case:

Here the natural frequency, n_2 of the equation of the first variation is not approximately equal to $m\omega/2$. In this case the following expression can be derived:

$$G = C_1 \; EXP[\beta K_2/2n_2]$$

$$H = C_2 \; EXP[\beta K_2/2n_2] \tag{13}$$

$$\xi_{21} = \sum_m \left\{ -\frac{N_{1s}^{(m)}}{(2n_2+m\omega)} \sin(n_2+m\omega)t - \frac{N_{1c}^{(m)}}{(2n_2+m\omega)} \cos(n_2+m\omega)t \right.$$

$$\left. + \frac{N_{2s}^{(m)}}{(2n_2-m\omega)} \sin(n_2-m\omega)t + \frac{N_{2c}^{(m)}}{(2n_2-m\omega)} \cos(n_2-m\omega)t \right\} \tag{14}$$

where $K_2 = n_2 [K_o + \sum_{j=1}^{3} \rho_j \; e_{2j}]$, K_o is a function of the nonlinear hydrodynamic damping coefficients, the motion of the ship, and the exciting frequency. C_1 and C_2 are constants of integration. $N_{1s}^{(m)}$, $N_{1c}^{(m)}$, $N_{2s}^{(m)}$ and $N_{2c}^{(m)}$ are functions of G, H, $g_s^{(m)}$, and $g_c^{(m)}$.

It is clear from (13) that the motion, in this case, will be stable only if K_2 is negative. K_2 can be expressed approximately as follows:

$$K_2 = \frac{n_2 \omega^2}{2} [I_1 P_1^2 + I_2 P_2^2 + I_3 P_3^2] + n_2 \sum_{j=1}^{3} \rho_j e_{2j}$$

where P_1, P_2, and P_3 are the amplitudes of sway, roll, and yaw motion respectively. Approximate expressions for I_1, I_2, and I_3 can also be given as

$$I_1 = 0.3\,K_{vvv} + K_{pvv},$$
$$I_2 = 0.1\,K_{vpp} + 3\,K_{ppp},$$
$$I_3 = 0.1\,K_{vrr} + K_{prr}.$$

I_2 is usually less than zero, but I_1 and I_3 may take either negative or positive values. Thus unstable motion can only occur if I_3 is a large positive quantity, or I_2 is small quantity (i.e. K_{ppp} is small). The instability is most likely to occur in a quartering sea where roll amplitude and yaw amplitude are approximately of the same magnitude. It can be shown easily that this instability is peculiar to the coupled nonlinear equations.

Consider the case of uncoupled nonlinear roll. In such case K_2 will be given by $K_2 = \frac{3n_2\omega^2}{2} P_2^2 K'_{ppp} + n_2K'_p$, since K'_{ppp} and K'_p are both negative, then K_2 can never become positive. Therefore, unstable motion is impossible in this case.

(2) Resonance Case:

When the natural frequency, n_2, is nearly equal to $\frac{m\omega}{2}$ the expressions for G and H take different form as follows:

$$G = e^{qt}\left[C_3 e^{-i\nu t} + C_4 e^{i\nu t}\right]$$

$$H = e^{qt}\left[C_5 e^{-i\nu t} + C_6 e^{i\nu t}\right] \tag{14}$$

where $\nu = \frac{s\omega}{2} - n_2$ and s is the particular value of m at which resonance occurs. C_3, C_4, C_5, and C_6 are constants of integration. The expression for ξ_{21} will be the same as in (14) except that the summation is carried over $m \neq s$. The exponent q in (14) is given by

$$q = \frac{\beta K_2}{2n_2} \pm \sqrt{\left(\frac{\beta}{2n_2}\right)^2 \left\{b_s^2 + C_s^2\right\} - \nu^2}$$

where b_s and C_s are functions of the nonlinear restoring moment in roll and the motion of the ship. In this case it is clear from (14) that unstable motion is possible if q becomes positive.

In order to study the mechanism that produces unstable motion further, the hydrodynamic coefficients which appear in the expressions for I_2 and I_3 for the Series 60, Block 60 model are varied about their nominal, measured values. It was found that the published coefficients for this form resulted in stable motion. However, by varying certain of the coefficients, it was possible to produce a modified model which has unstable motion at certain headings. The hydrodynamic coefficients for this model are given in Table 1. For this modified model the following study is performed:

i. The analytical method is used to predict possible headings at which unstable motion may occur.

ii. The equations of the first variation in the motion are then integrated numerically, at the particular heading at which unstable motion was predicted to occur. Here the periodic response calculated analytically is used for the time-dependent coefficients in the equations. A non-zero initial condition of 5° is chosen for the perturbed angle of roll, with all the other variables having zero initial values. The result of this process yields the perturbed motion of the model as a function of time. The perturbed roll angle for 15° heading is plotted in Figure 12.

iii. The third step is to integrate the nonlinear equations of motion numerically, with the same non-zero initial condition as in (ii). This gives the total motion of the model as a function of time. Figure 13 shows the rolling motion of the modified model for 15° heading.

These three methods indicate strongly unstable motion for the modified model at this heading. The logical question to be asked now is whether this model can be considered a representative model. We can answer this question with confidence only after similar stability studies have been carried out for many different ships. However, it is clear that a ship with small nonlinear damping may very possibly have an unstable motion.

A study of the effect on the stability index of systematic variations in the moments of inertia and the metacentric height of the ship is carried out for the Series 60 model and the Mariner model. The results of this study appear in Figures 15, 19, and 21. The results show that the requirements for larger stability index conflicts with the requirements for comfortable rolling motion.

Also the sensitivity of the stability index to systematic variations in some of the nonlinear hydrodynamic coefficients is calculated. Figure 22 shows the results. These results indicate that the stability is most sensitive to variations in the coefficient of the nonlinear damping in roll. A 50% reduction in the value of this coefficient causes the motion of the Series 60, Block 0.60 model to become unstable. The importance of this result lies in the fact that the value of this coefficient is determined during the process of ship design. It is affected by the design of section shapes and the design of the antiroll apparatus (bilge keels, fins, or antiroll tanks). It therefore, appears desirable to carry out a study of the stability of motion as a part of the design process.

V. CONCLUSIONS AND COMMENTS

(1) The mathematical model developed in this work for ship motion in regular oblique waves seems realistic. When combined with the method of solution introduced, it gives results that agree well with the available experimental values for roll response.

(2) The agreement with experimental results is not as good, in the case of yaw response, as in roll. This may be a result of a deficiency in the strip method which is used to calculate the yaw moment correction.

(3) A study of the stability of the coupled nonlinear equations revealed the following results for the ships considered here:

   i. Unstable motion, which is not revealed by a linear mathematical model, can be predicted by taking into consideration nonlinearities in the equations of motion.

   ii. This instability is most sensitive to variations in the nonlinear damping coefficient in roll.

(4) Response and stability are sensitive to variations in the moments of inertia and the metacentric height. It may be desirable to consider the motion and stability when estimating these parameters for a new design.

(5) Since our final goal is to understand ship motion in a realistic sea, this study should be extended to cover the following topics:

   i. Study of ship motion and stability in a random sea.

   ii. Study of the motion of a ship having six degrees of freedom in regular and irregular waves.

(6) More precise experimental and theoretical studies of wave exciting forces and moments are very much needed. Ship motion and consequently stability are very sensitive to small variations in wave forces and moments.

VI. REFERENCES

Abkowitz, M. A.

Lectures on ship hydrodynamics -- steering and maneuverability. Hydro- and Aerodynamics Laboratory, Lyngby, Denmark, Report No. Hy-5, May (1964).

Anon.

Nomenclature for treating the motion of a submerged body through a fluid. Technical and Research Bulletin No. 1-5, The Society of Naval Architects and Marine Engineers, New York, N. Y., April (1950).

Bogoliubov, N. N. and Mitropolsky, Y. A.

Asymptotic methods in the theory of nonlinear oscillations. Translated from Russian. Hindustan Publishing Corp. (India), Delhi-6 (1961).

Chey, Y.

Experimental determination of wave exciting forces and moments acting on a ship model running in oblique waves. Davidson Laboratory Report 1046, Oct. (1964).

Chislett, M. S. and Strom-Tejsen, J.

Planar motion mechanism tests and full-scale steering and maneuvering predictions for a mariner class vessel. International Shipbuilding Progress, vol. 12, no. 129, May (1965), pp. 201-224.

Eda, H. and Crane, L. C. Jr.

Steering characteristics of ships in calm water and waves. Transactions S.N.A.M.E. vol. 73, pp. 135-177 (1965).

Grim, O.

Die hydrodynamicschen Kräfte beim Roll versuch. Schiffstechnik, 3, pp. 147-151 (1956).

Haddara, M. R.

Stability of Ship Motion in Oblique Seas. Ph.D thesis, Department of Naval Architecture, University of California, Berkeley (1970).

Hsu, C. S.

On the application of elliptic functions in nonlinear forced oscillations. Quarterly of Applied Mathematics, 17, pp. 393-407 (1960).

On the parametric excitation of a dynamic system having multiple degrees of freedom. Journal of Applied Mechanics, 17, pp. 367-372 (1963).

Lewis, E. V. and Numata, E.

Ship motions in oblique seas. Transactions S.N.A.M.E. vol. 68, pp. 510-547 (1960).

Paulling, J. R. and Rosenberg, R. M.

On unstable ship motion resulting from nonlinear coupling. Journal of Ship Research, vol. 3, pp. 36-46 (1959).

Struble, R. A. and Yionoulis, S. M.

General perturbational solution of the harmonically forced duffing equation. Arch. Rational Mech. Anal., vol 9, pp. 422-438 (1962).

Tasai, F.

On the swaying, yawing, and rolling motions of ships in oblique waves. International Shipbuilding Progress, vol. 14, no. 153, May (1967).

Valeev, K. G.

On Hill's Method in the theory of linear differential equations with periodic coefficients. Determination of the characteristic exponents. Journal of Applied Mathematics and Mechanics, translation of Prikladnaia Matematika i Mekhnika, vol. 24, pp. 460-466 (1961).

Webster, W. C.

Analysis of the control of activated antiroll tanks. Transactions S.N.A.M.E., vol 75, pp. 296-331 (1967).

Wehausen, J. V.

Ship Motion. Lecture Notes, University of California (1964).

Wehausen, J. V. and Laitone, E.V.

Surface waves. Hadbuch der Physik, vol. IX, Springer-Verlag, Berlin (1960).

Weinblum, G. and St. Denis, M.

On the motions of ships at sea. Trans. S.N.A.M.E., vol. 58, pp. 184-248 (1950).

Yamanouchi, Y. and Ando, S.

Experiments on a Series 60, C_B 0.70 ship model in oblique regular waves. Papers of Ship Research Institute, no. 18, Oct. (1966).

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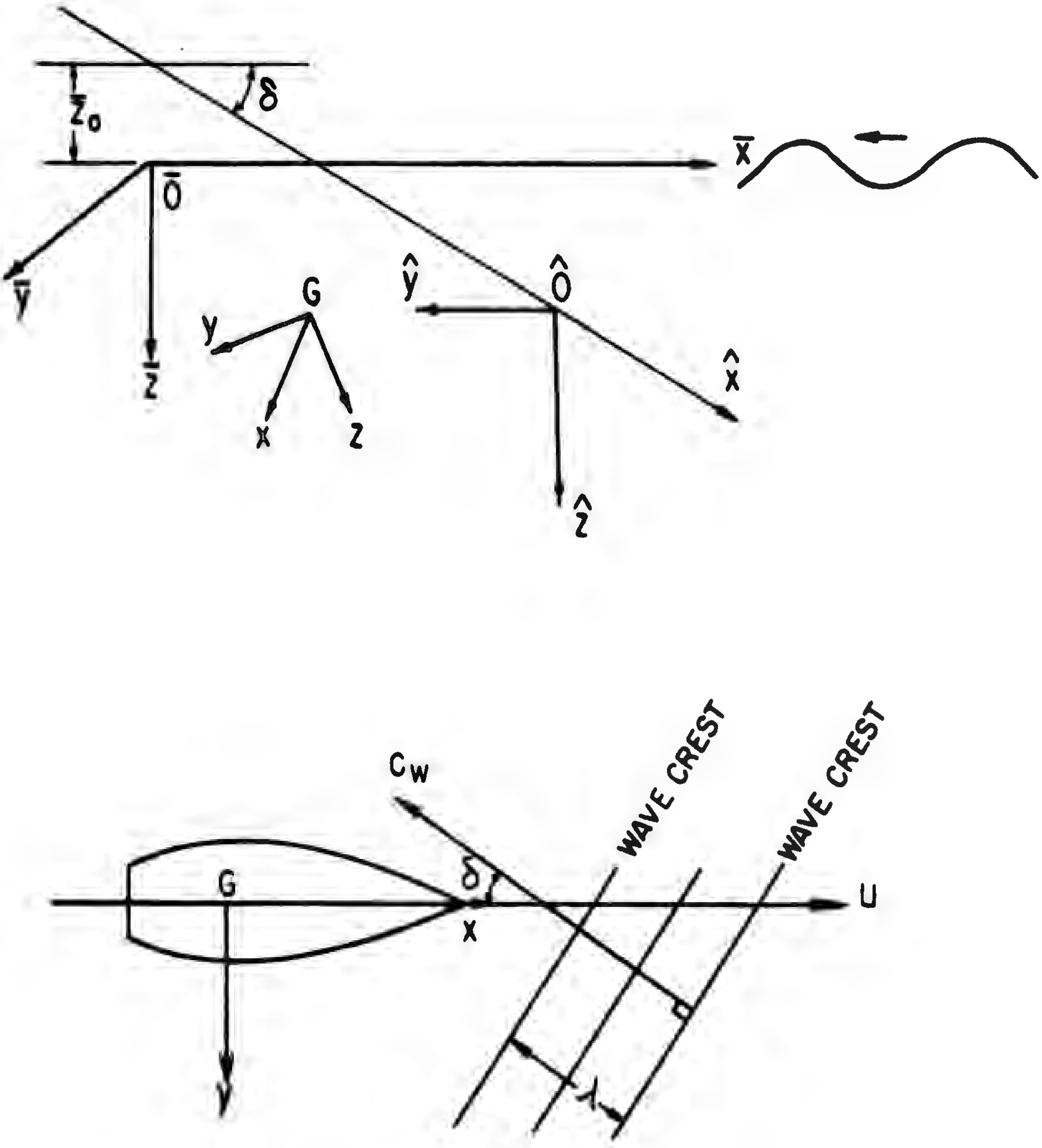

HEADING = 180°-δ

FIG. 1 COORDINATE AXES

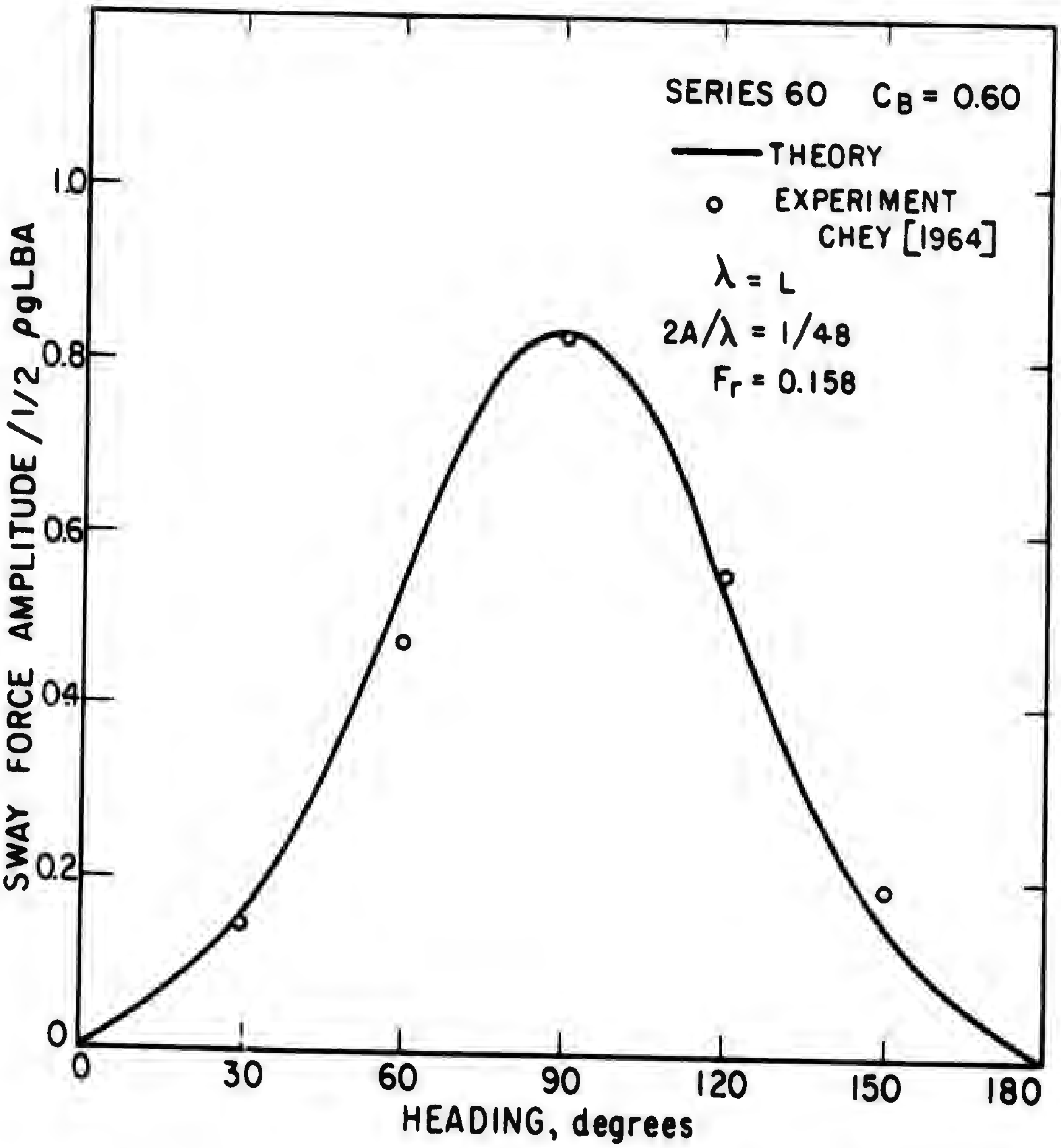

FIG 2 VARIATION OF AMPLITUDE OF WAVE SWAY FORCE vs HEADING, THEORY AND EXPERIMENT

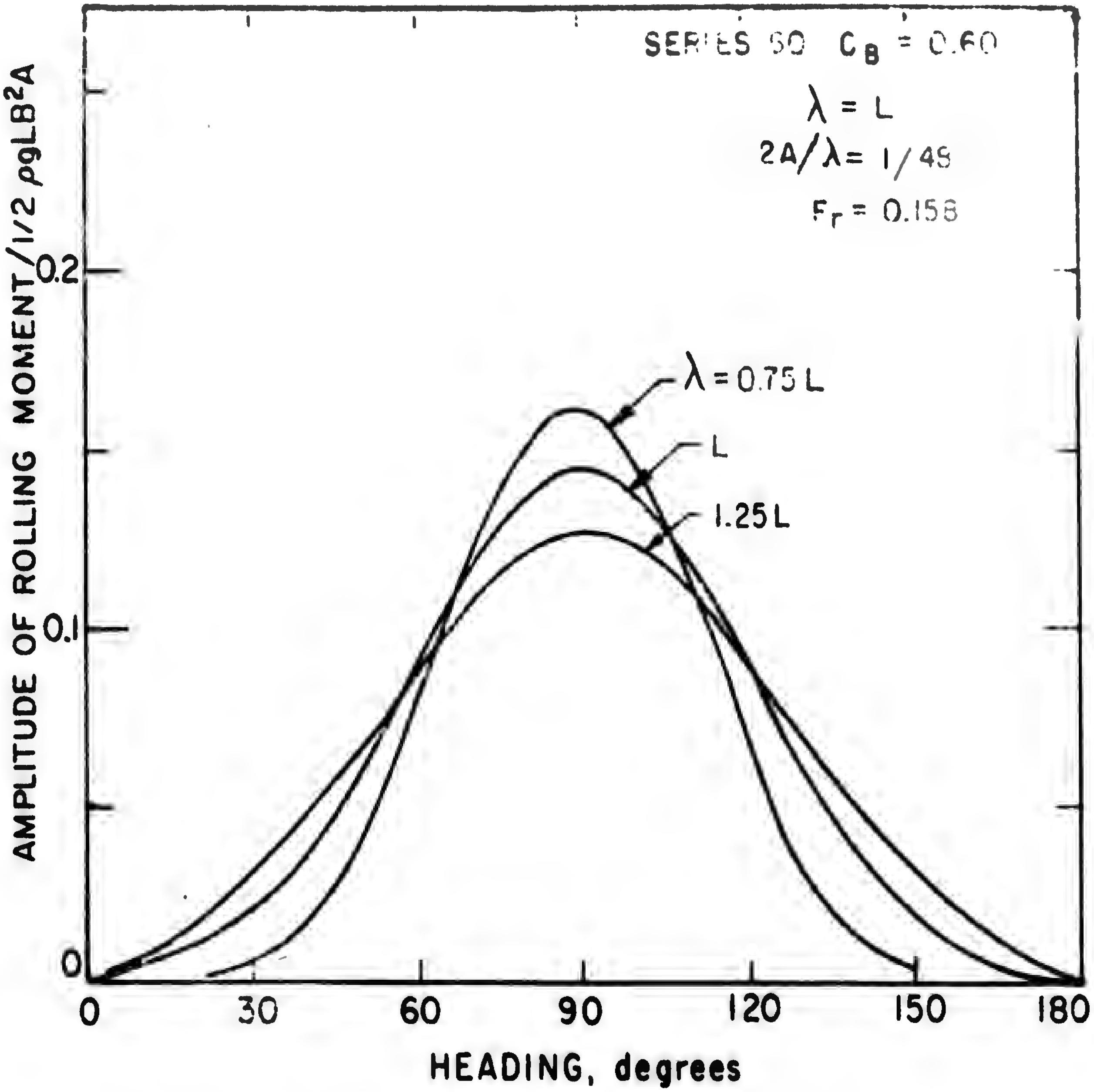

FIG 3 VARIATION OF AMPLITUDE OF WAVE ROLL MOMENT VS HEADING AS A FUNCTION OF WAVE LENGTH

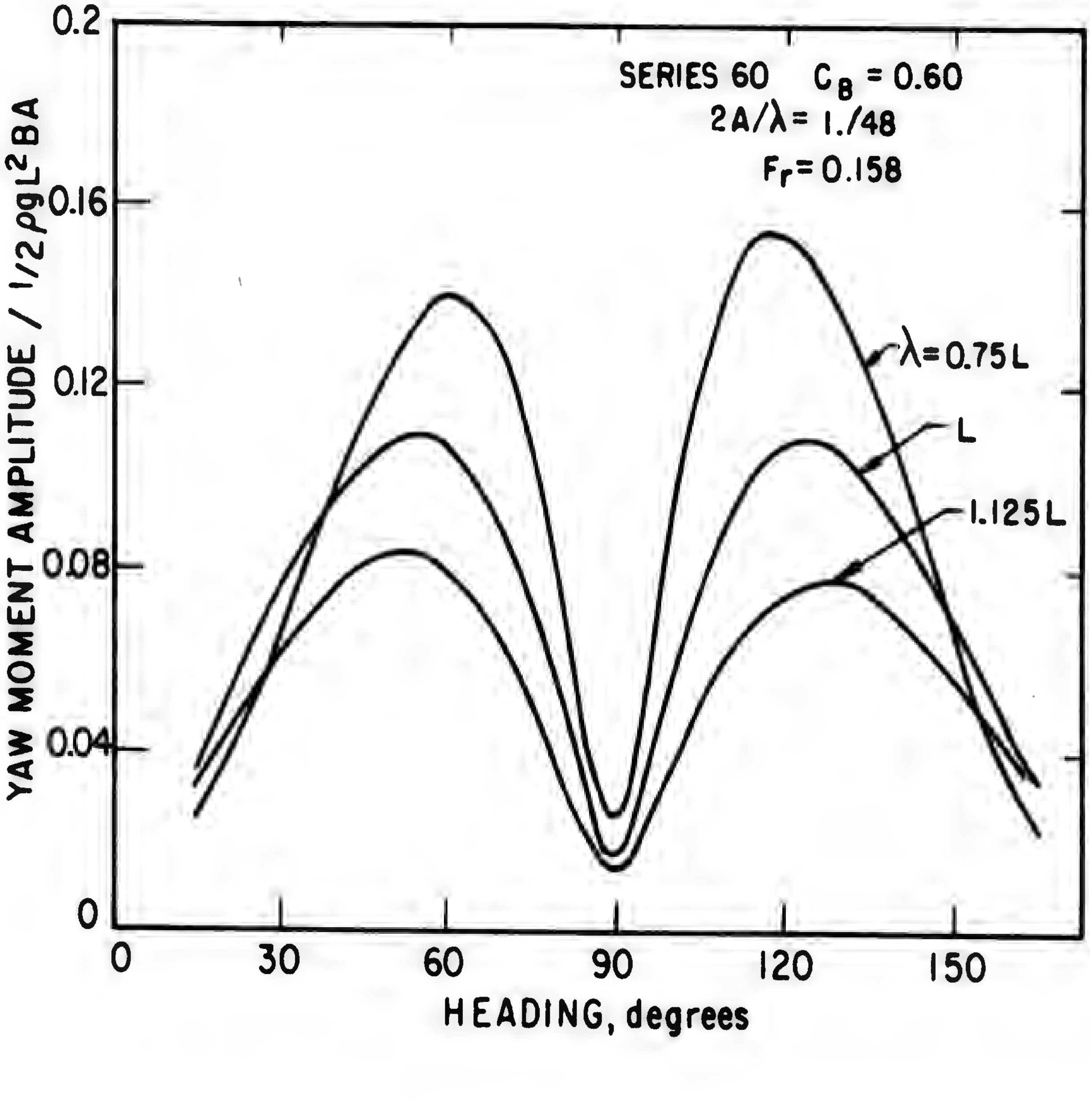

FIG 4 AMPLITUDE OF WAVE YAW MOMENT

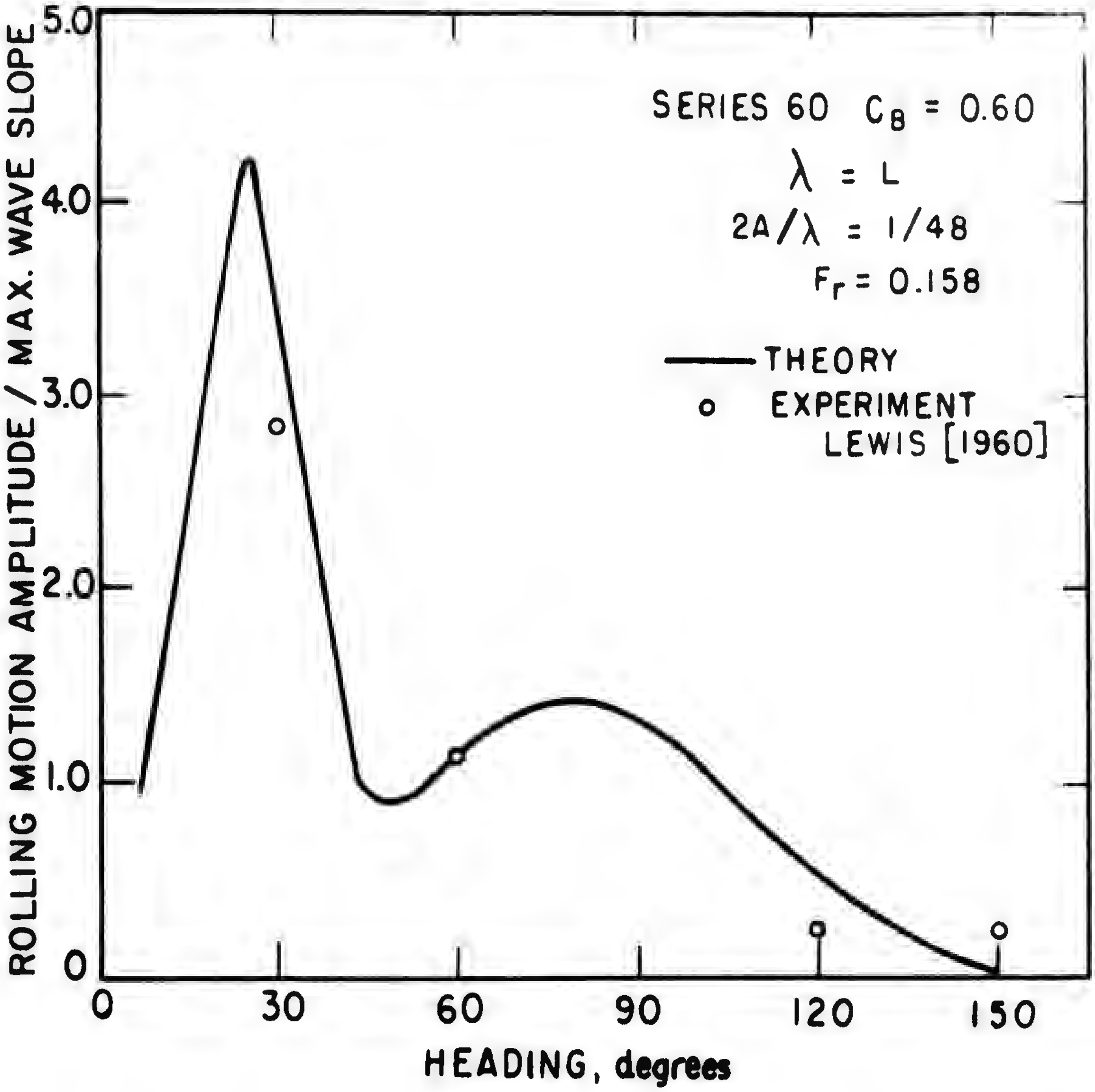

FIG. 5 ROLL RESPONSE, THEORY AND EXPERIMENT

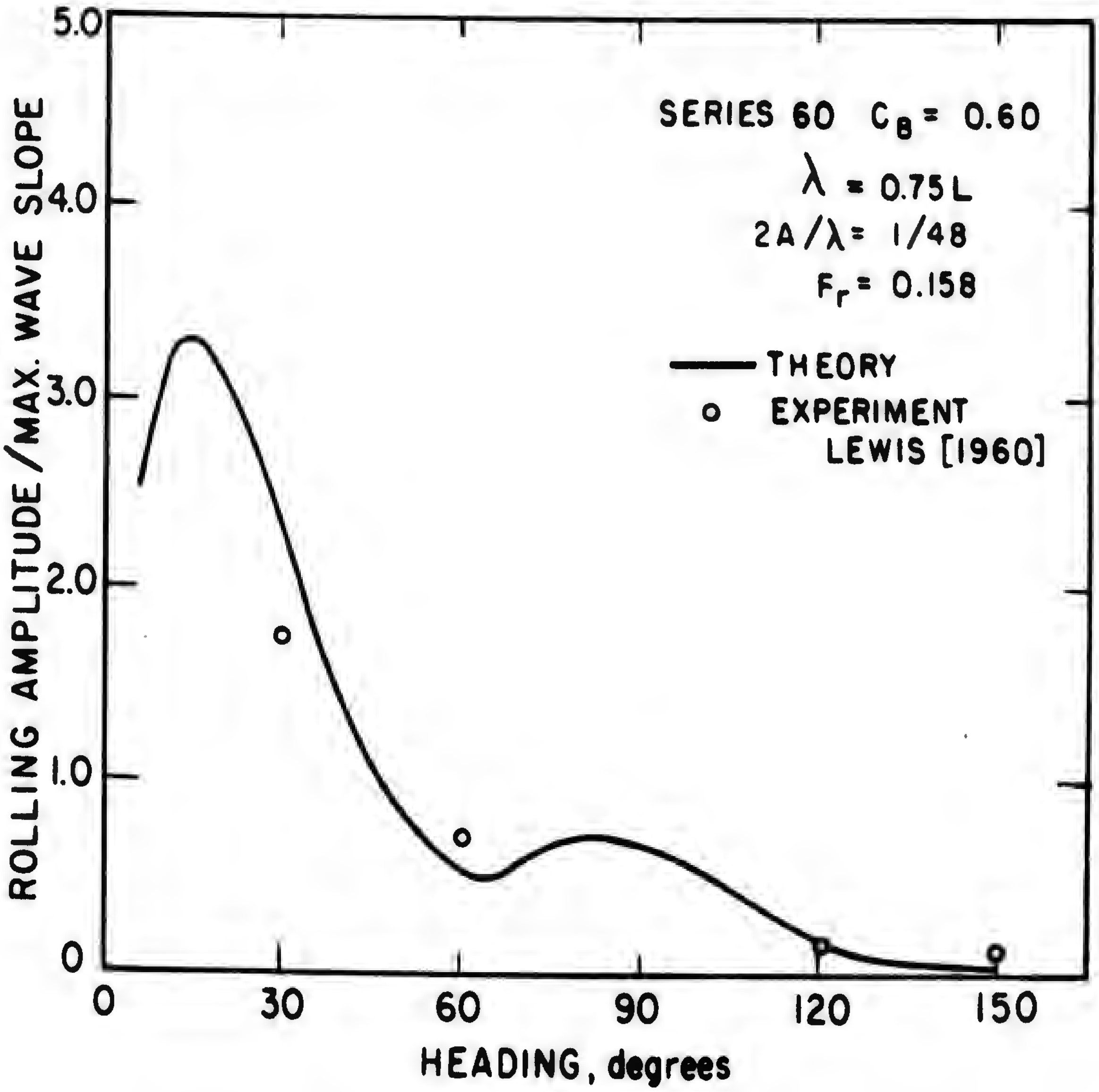

FIG. 6 ROLL RESPONSE, THEORY AND EXPERIMENT

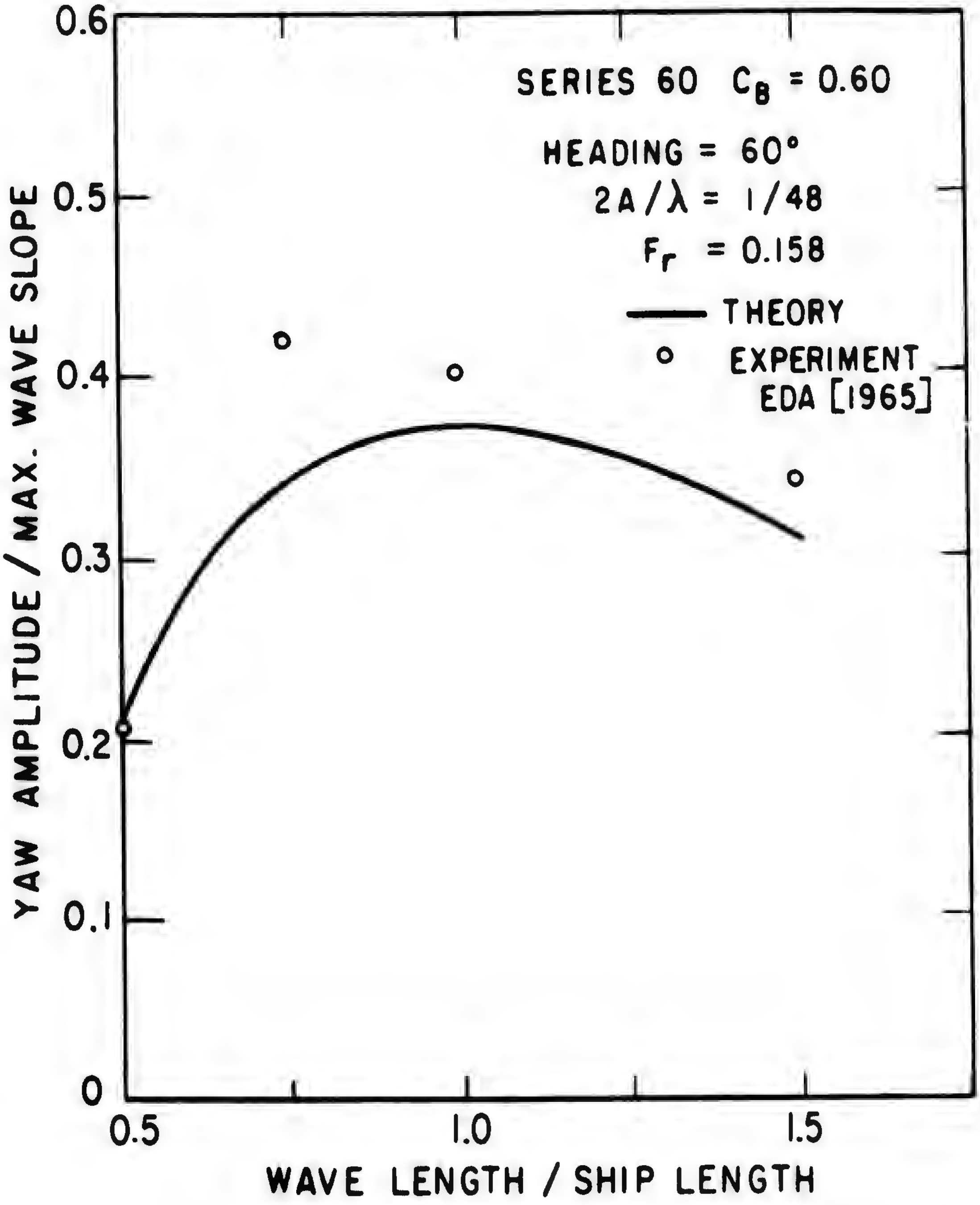

FIG. 7 YAW RESPONSE

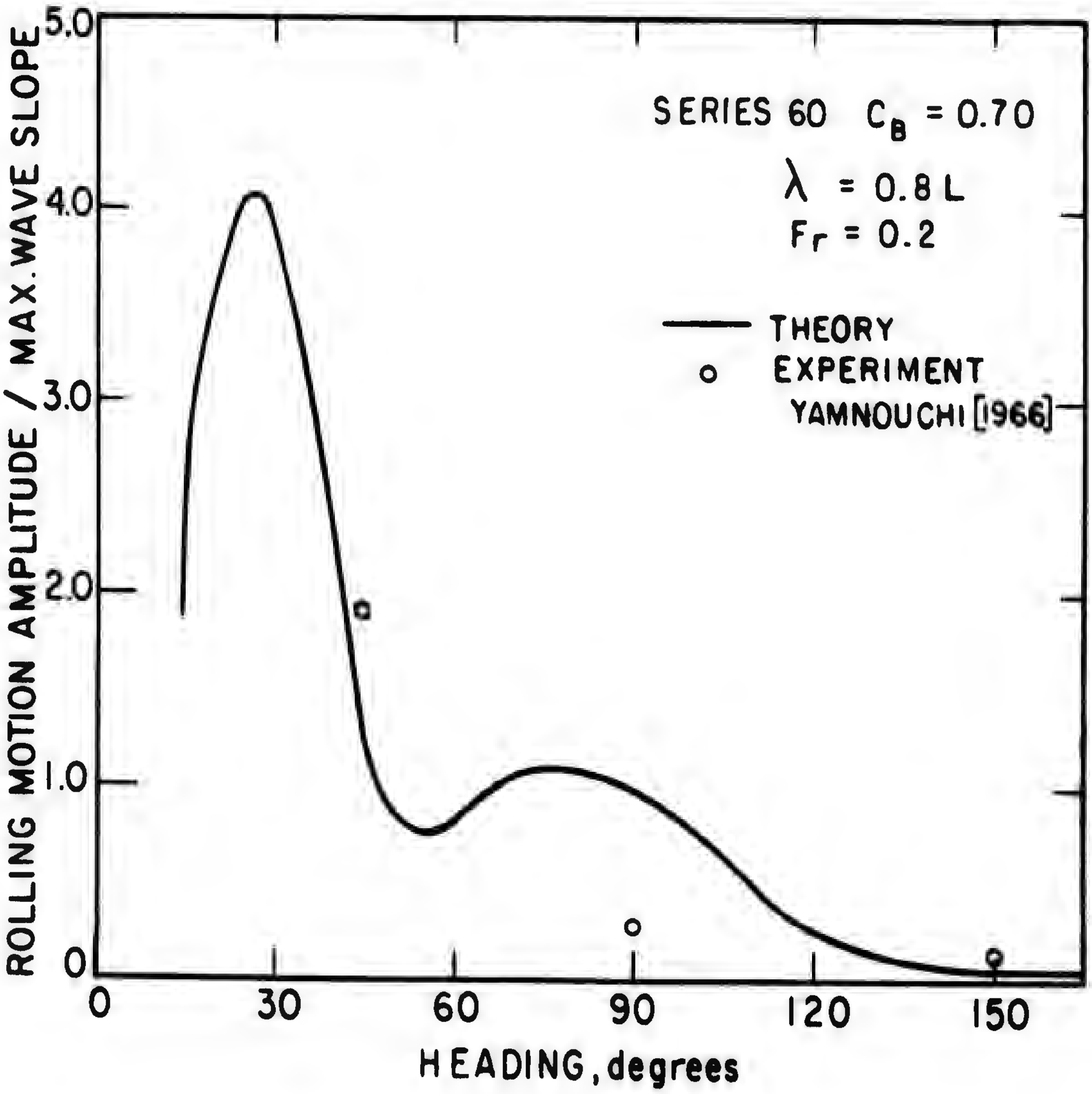

FIG. 8 ROLL RESPONSE, THEORY AND EXPERIMENT

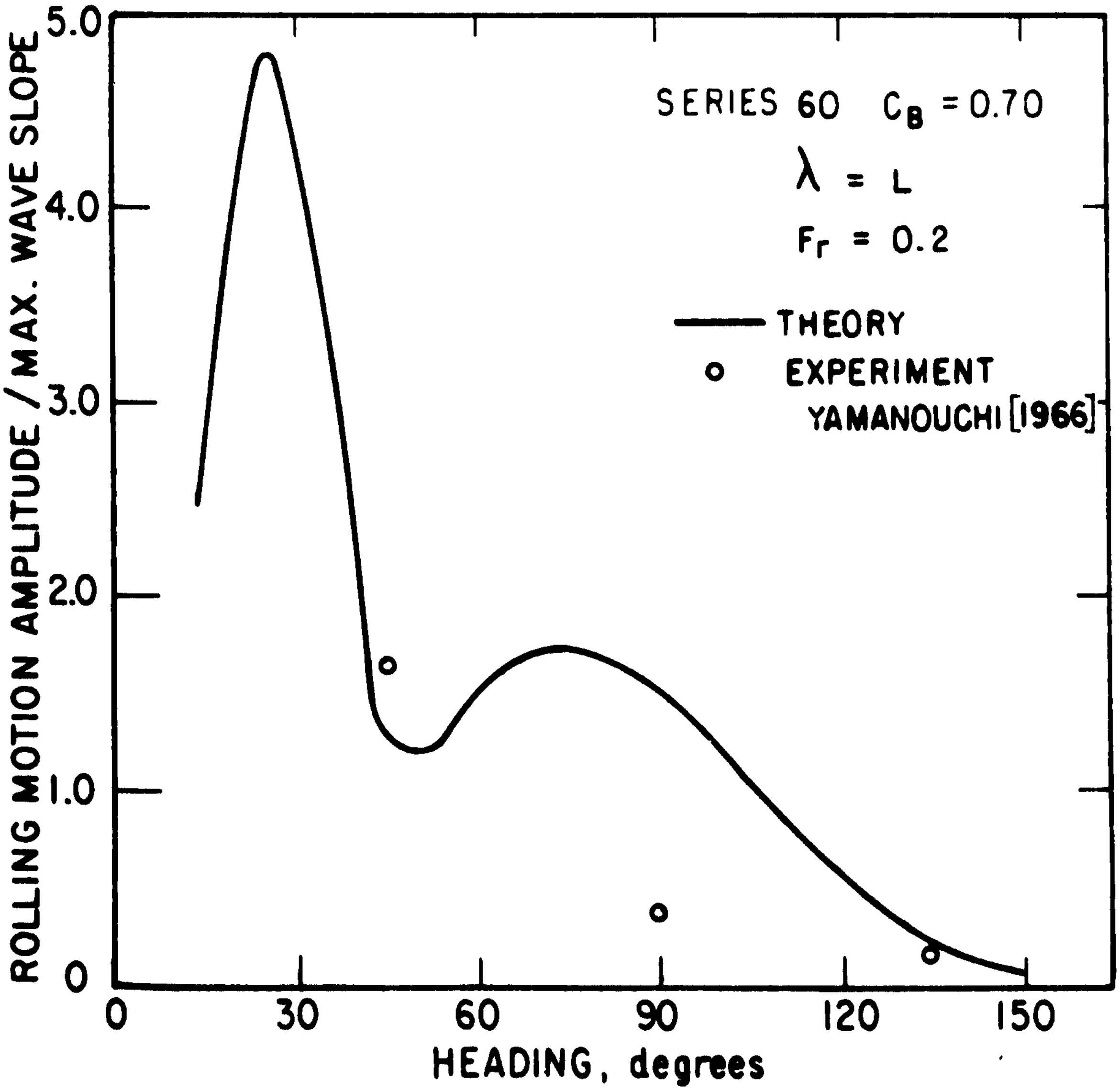

FIG.9 ROLL RESPONSE, THEORY AND EXPERIMENT

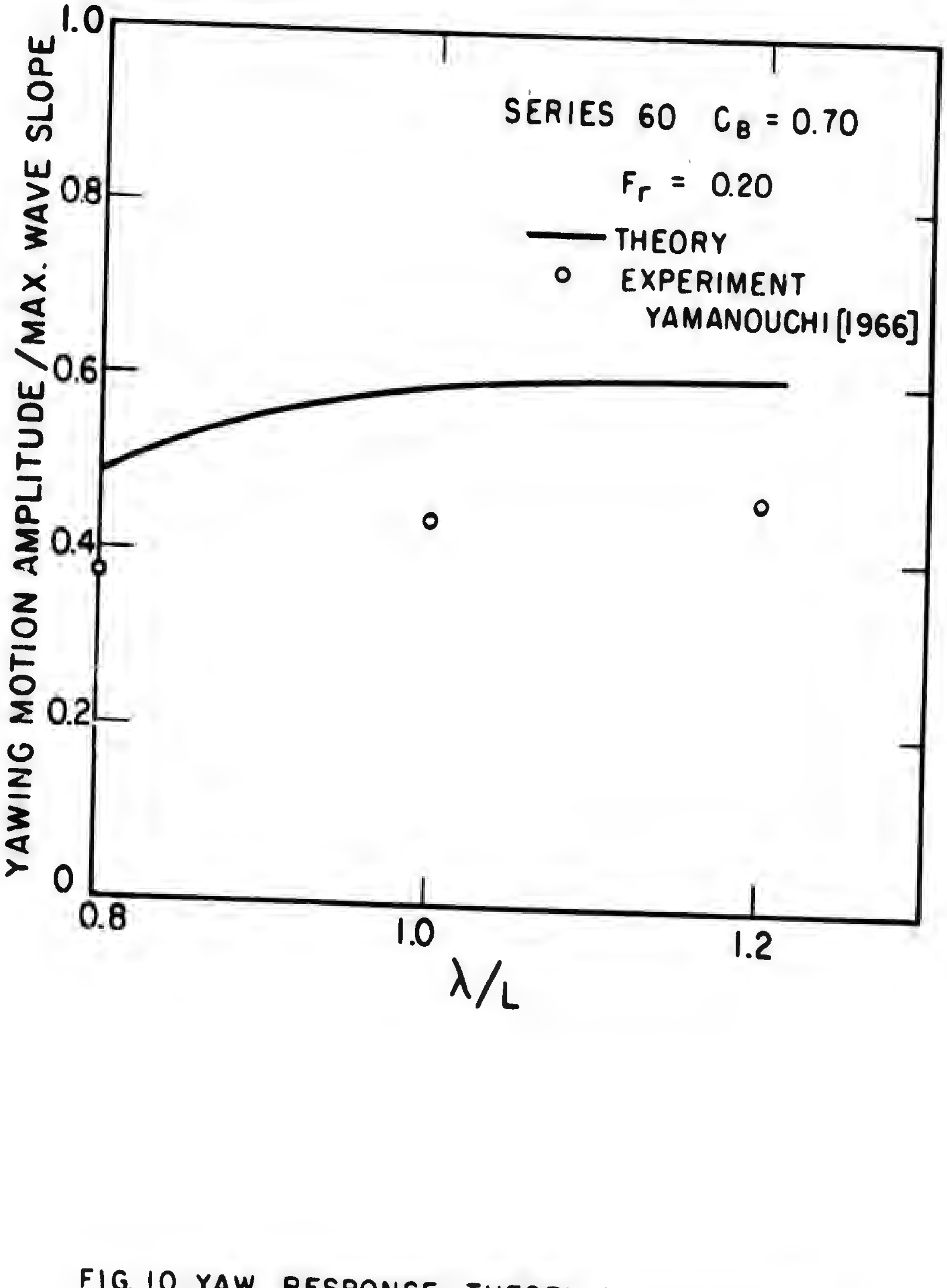

FIG. 10 YAW RESPONSE, THEORY AND EXPERIMENT

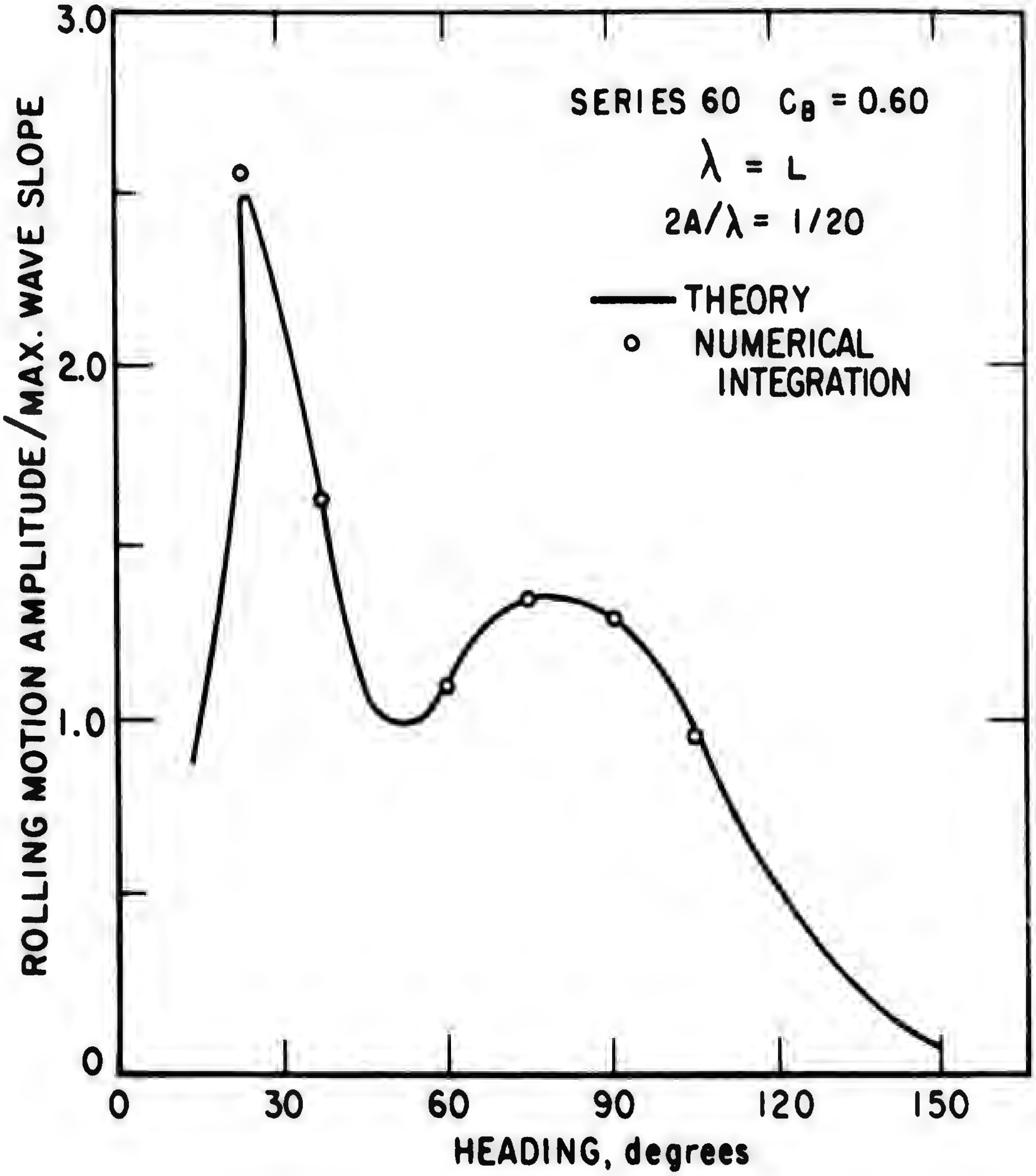

FIG. 11 ROLL RESPONSE

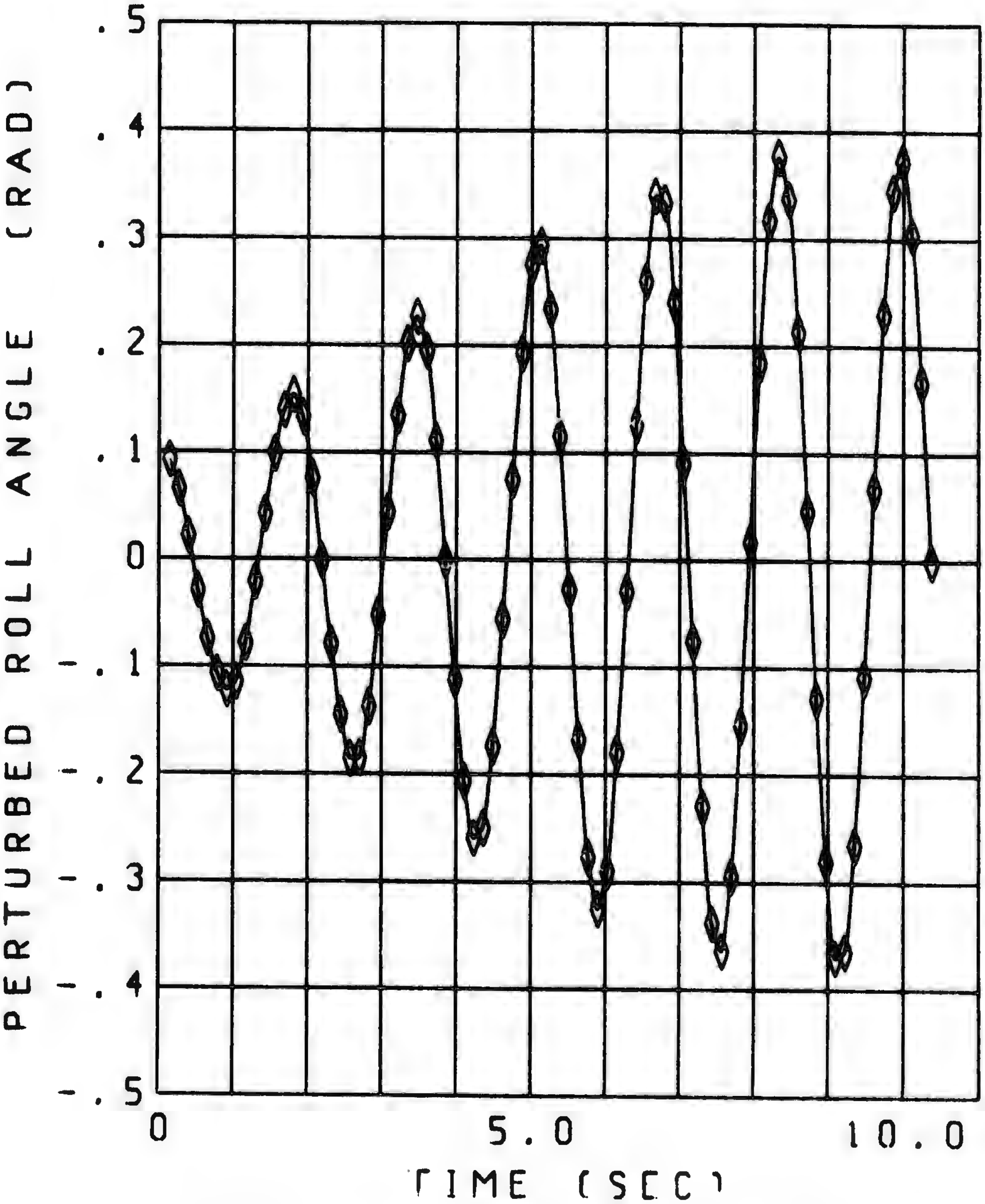

FIG.12 UNSTABLE MOTION

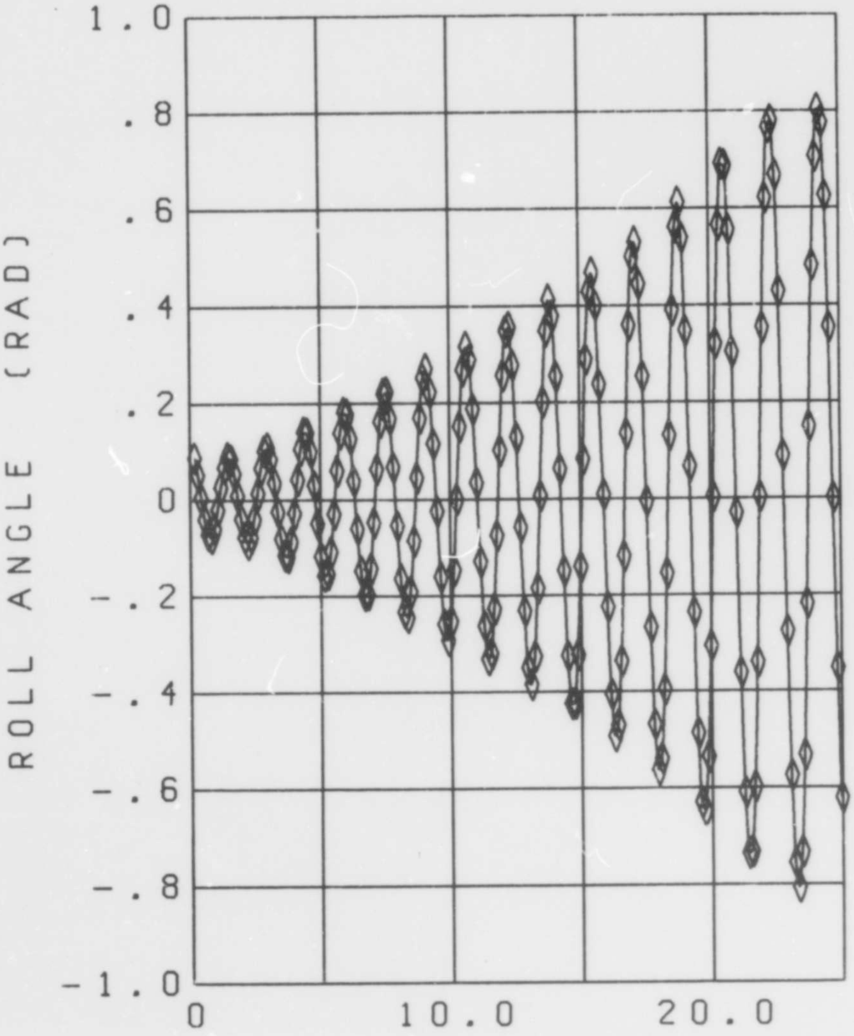

FIG.13 UNSTABLE MOTION

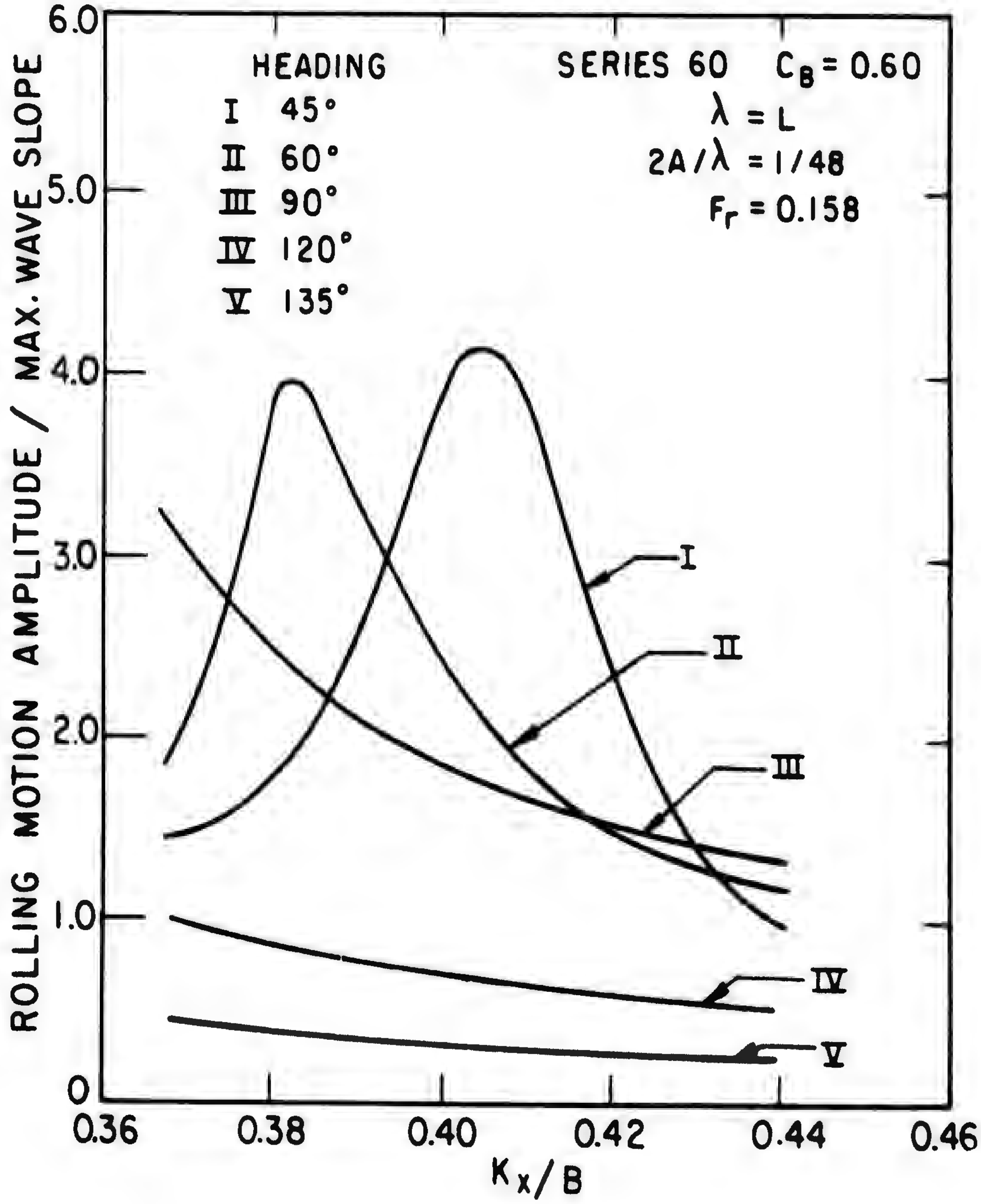

FIG. 14 EFFECT OF GYRADIUS ON ROLL RESPONSE

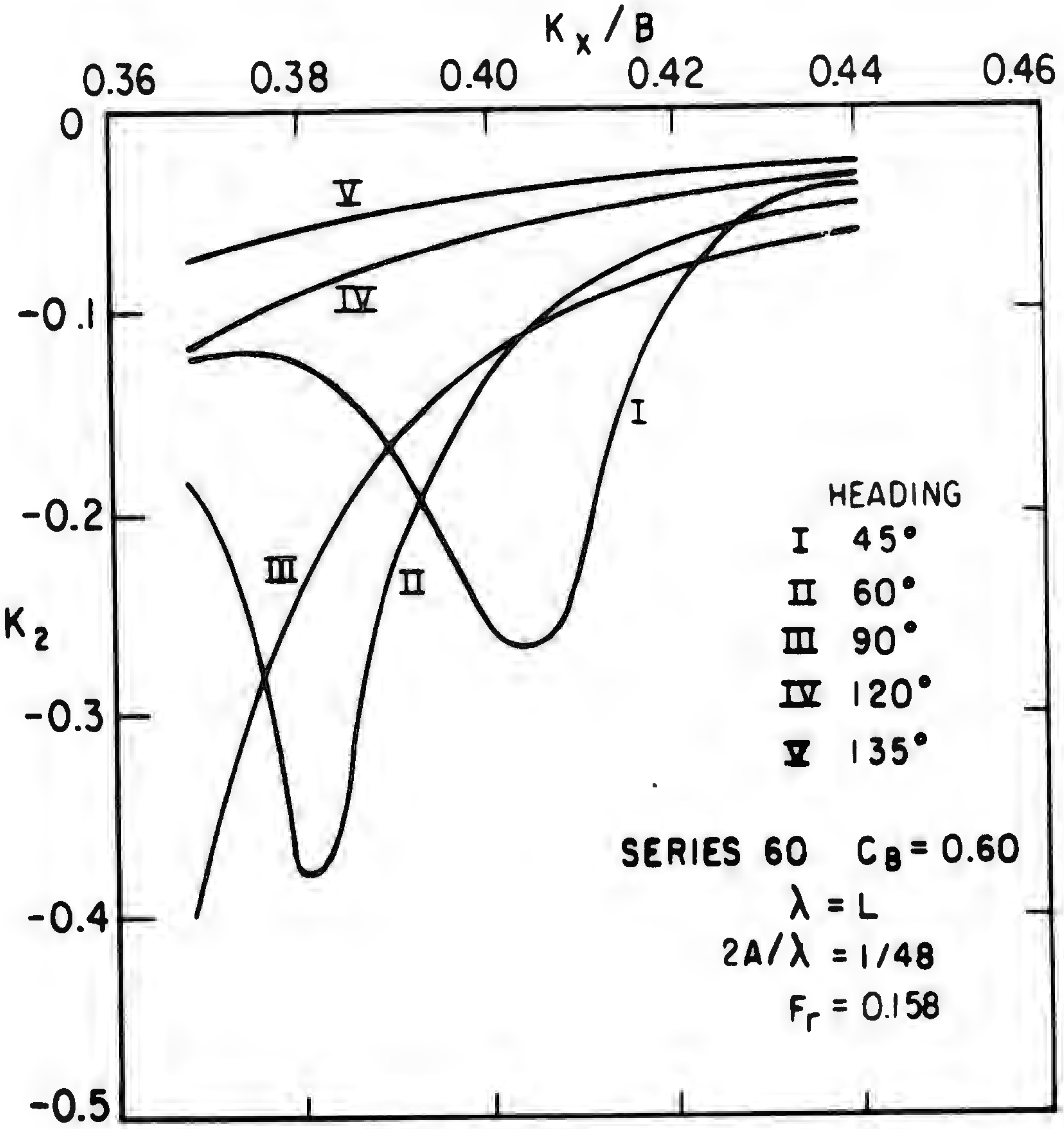

FIG. 15 EFFECT OF GYRADIUS ON STABILITY

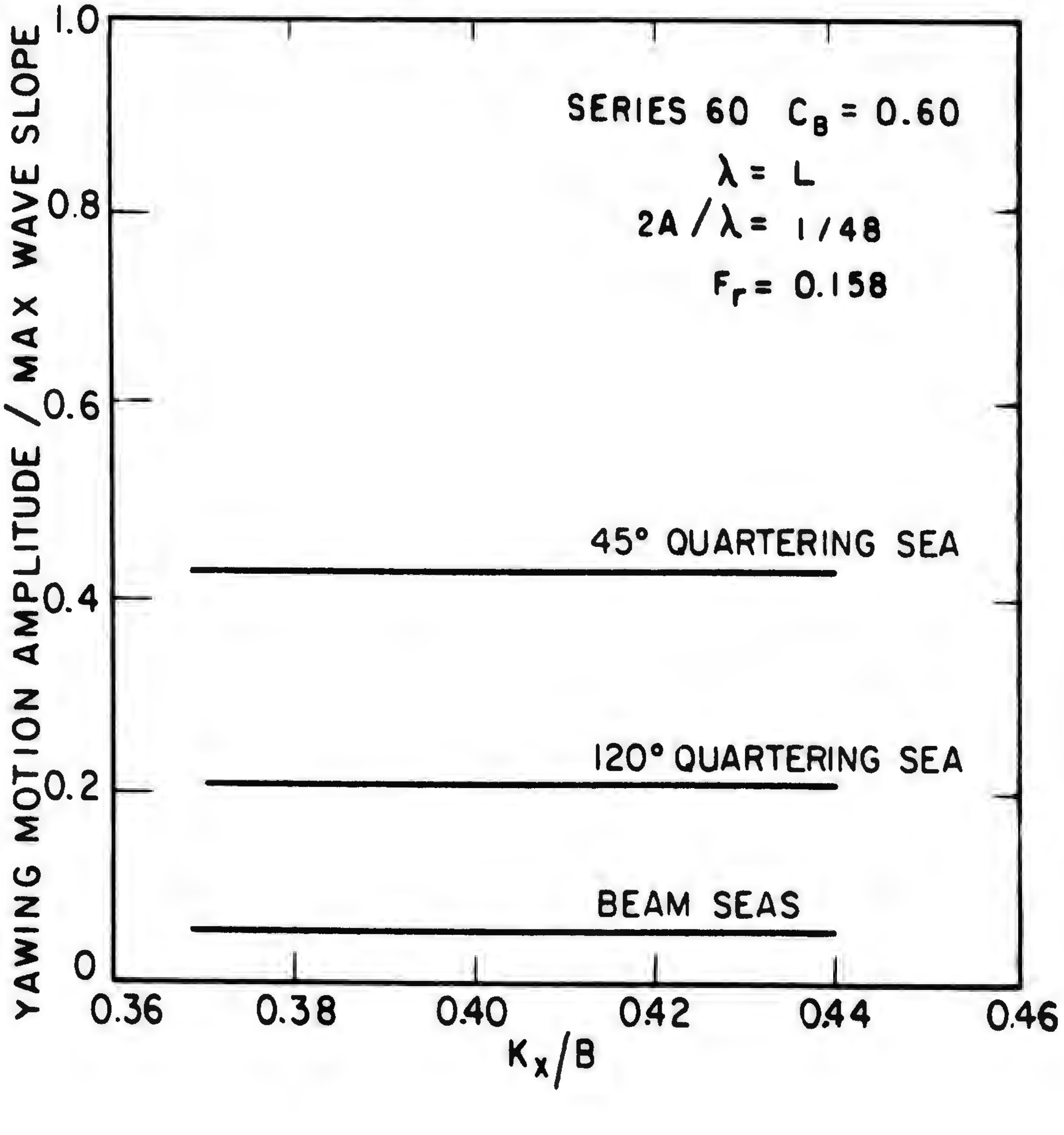

FIG. 16 EFFECT OF GYRADIUS ON YAW RESPONSE

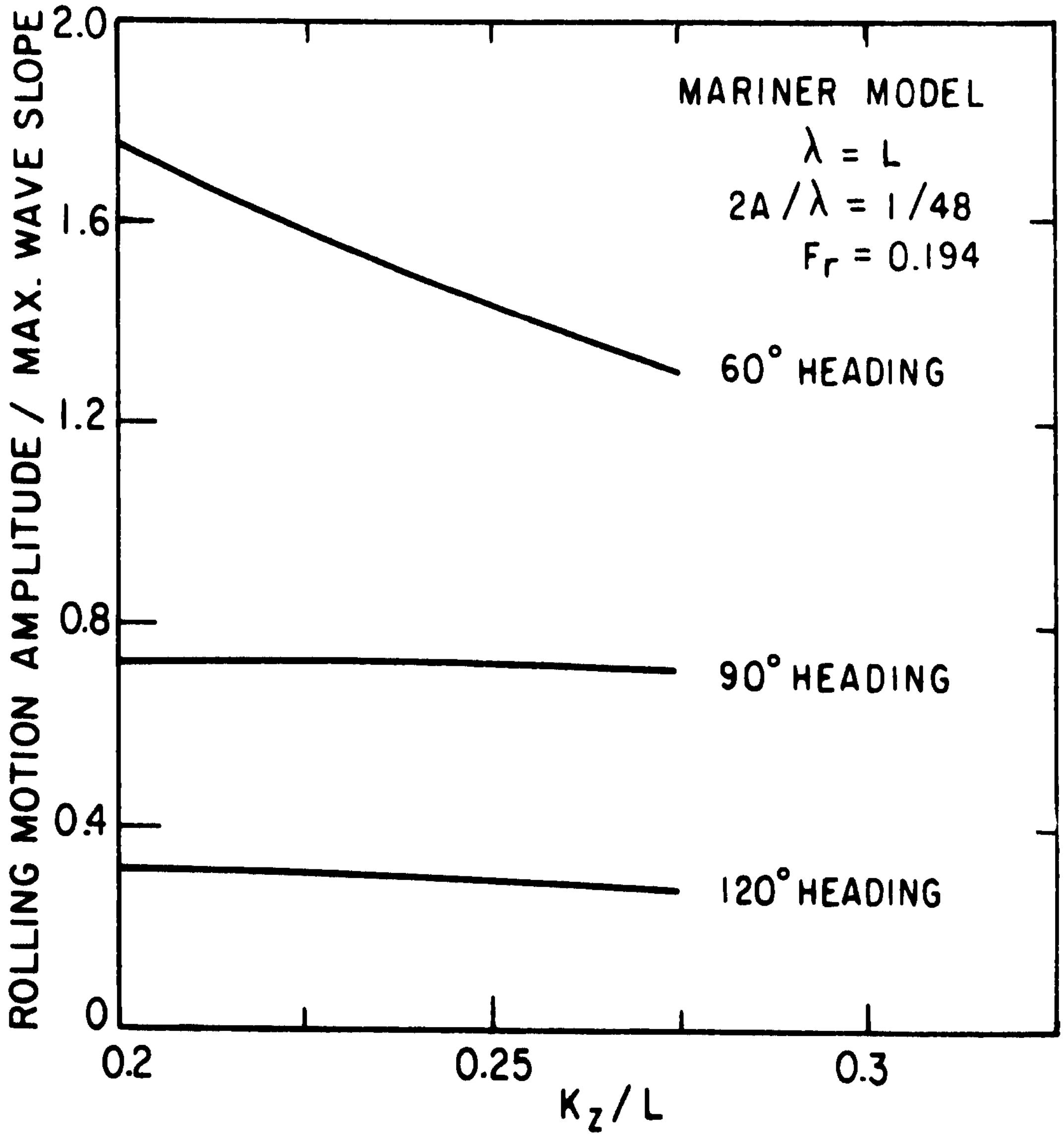

FIG. 17 EFFECT OF LONGITUDINAL GYRADIUS ON ROLL RESPONSE

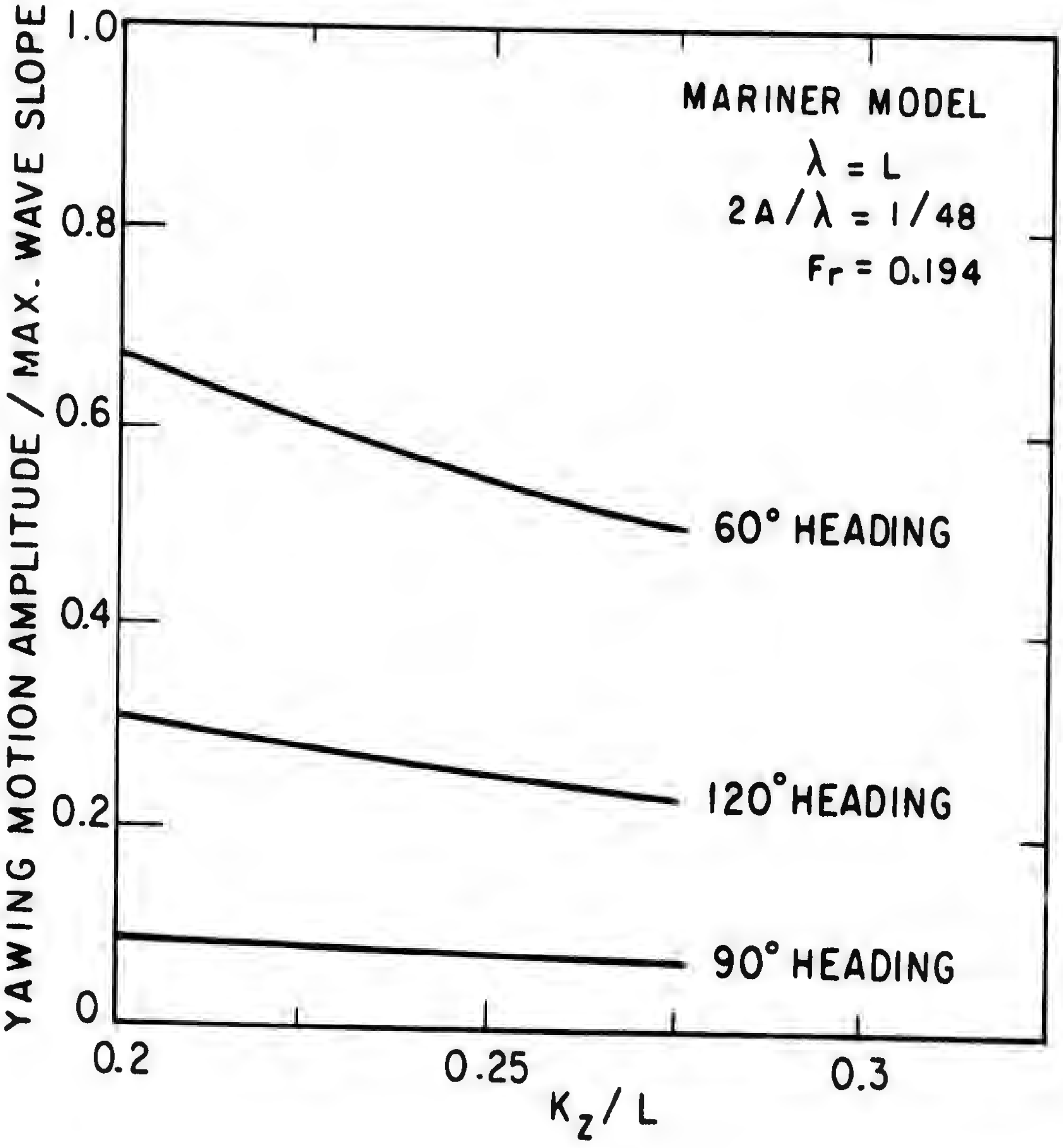

FIG. 18 EFFECT OF LONGITUDINAL GYRADIUS ON YAW RESPONSE

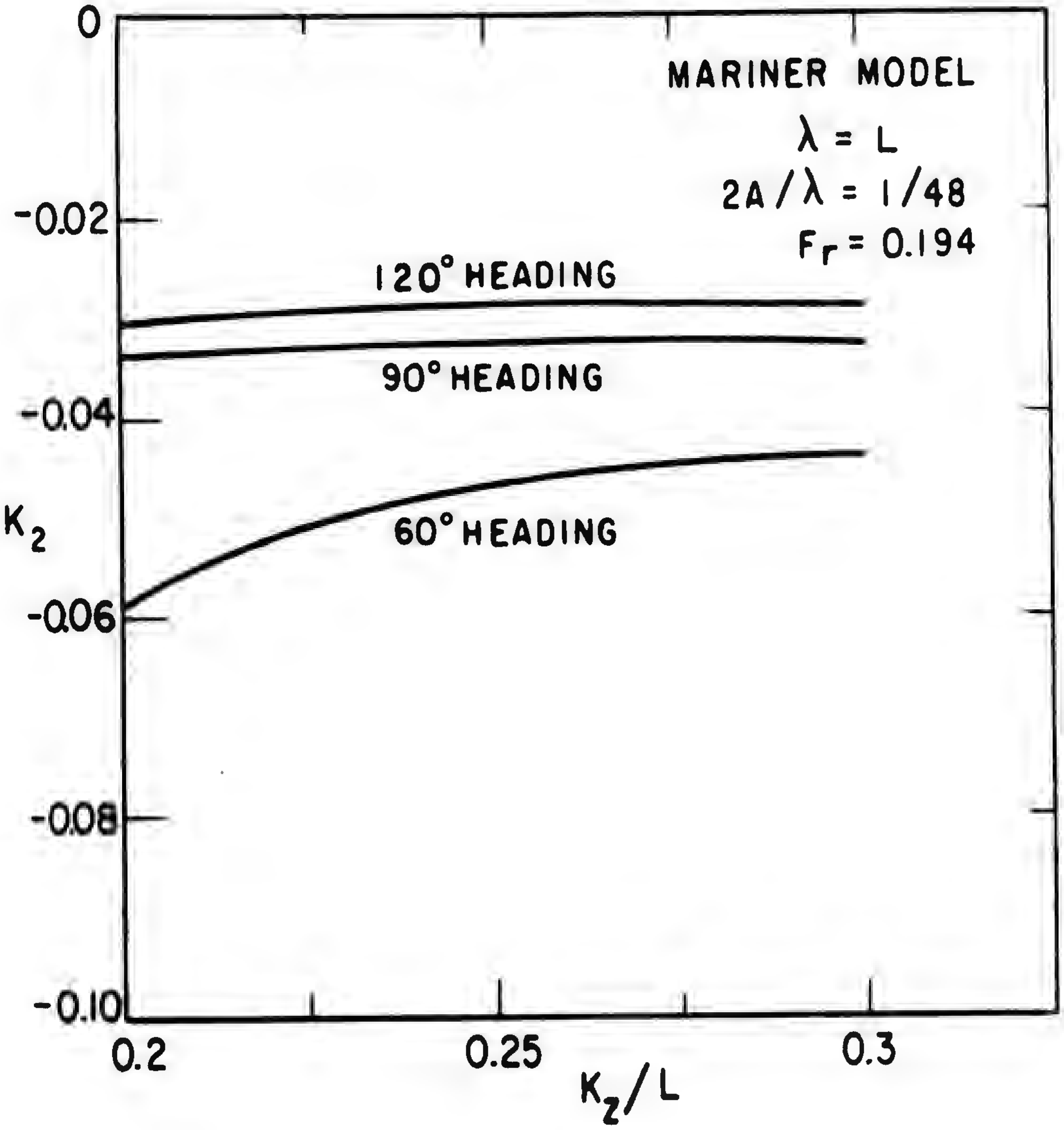

FIG. 19 EFFECT OF LONGITUDINAL GYRADIUS ON STABILITY INDEX

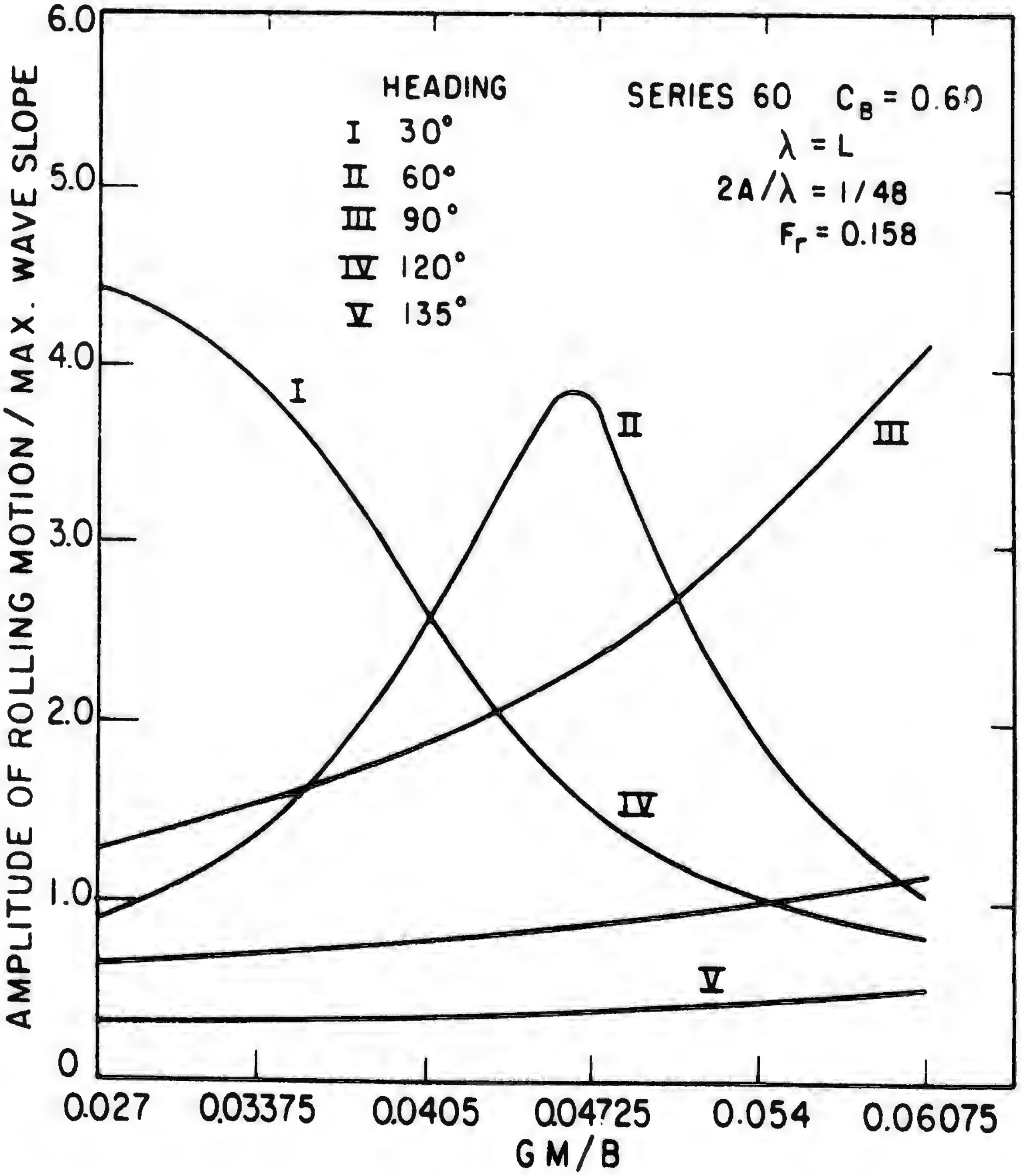

FIG. 20 EFFECT OF GM ON ROLL RESPONSE

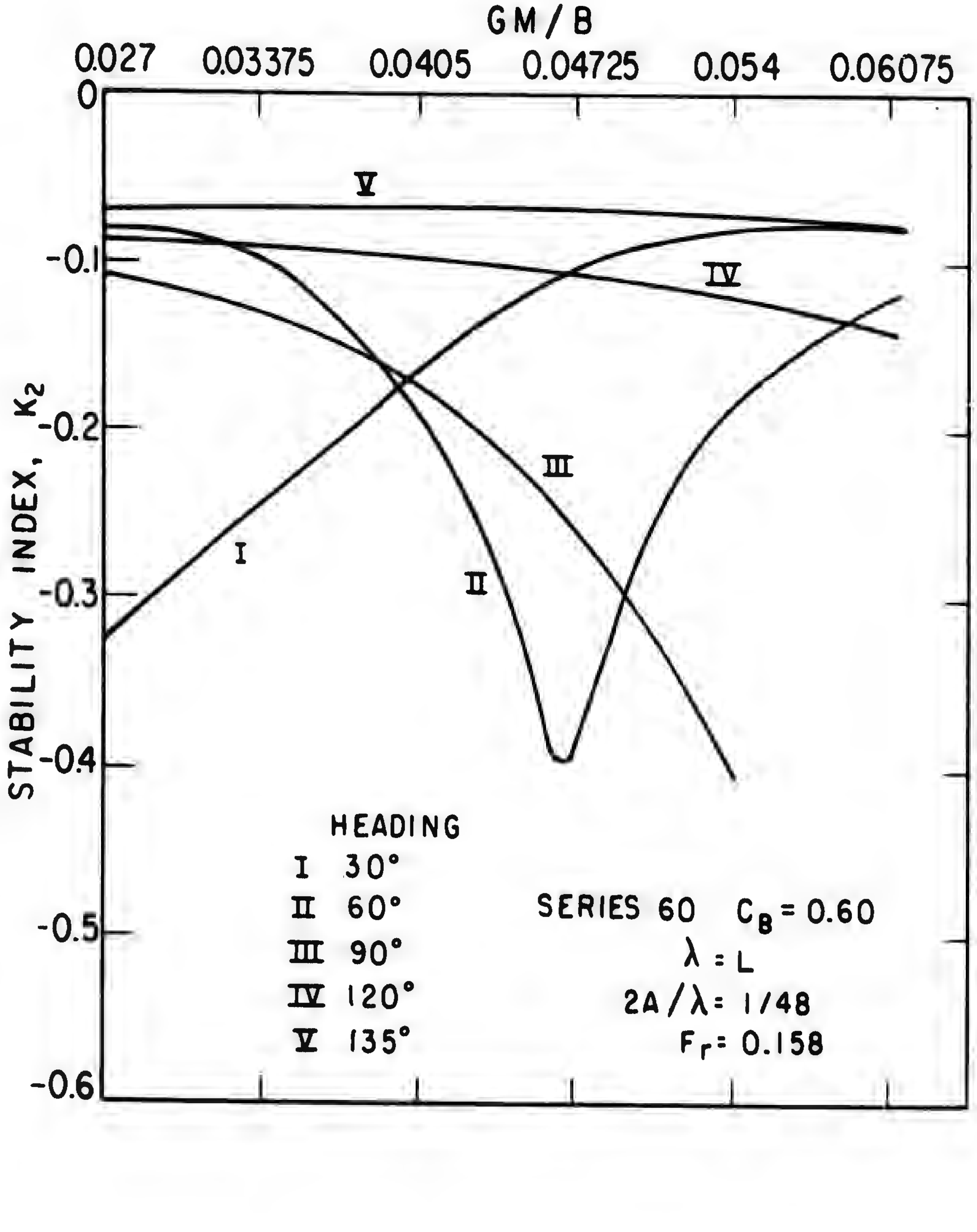

FIG. 21 EFFECT OF GM ON STABILITY

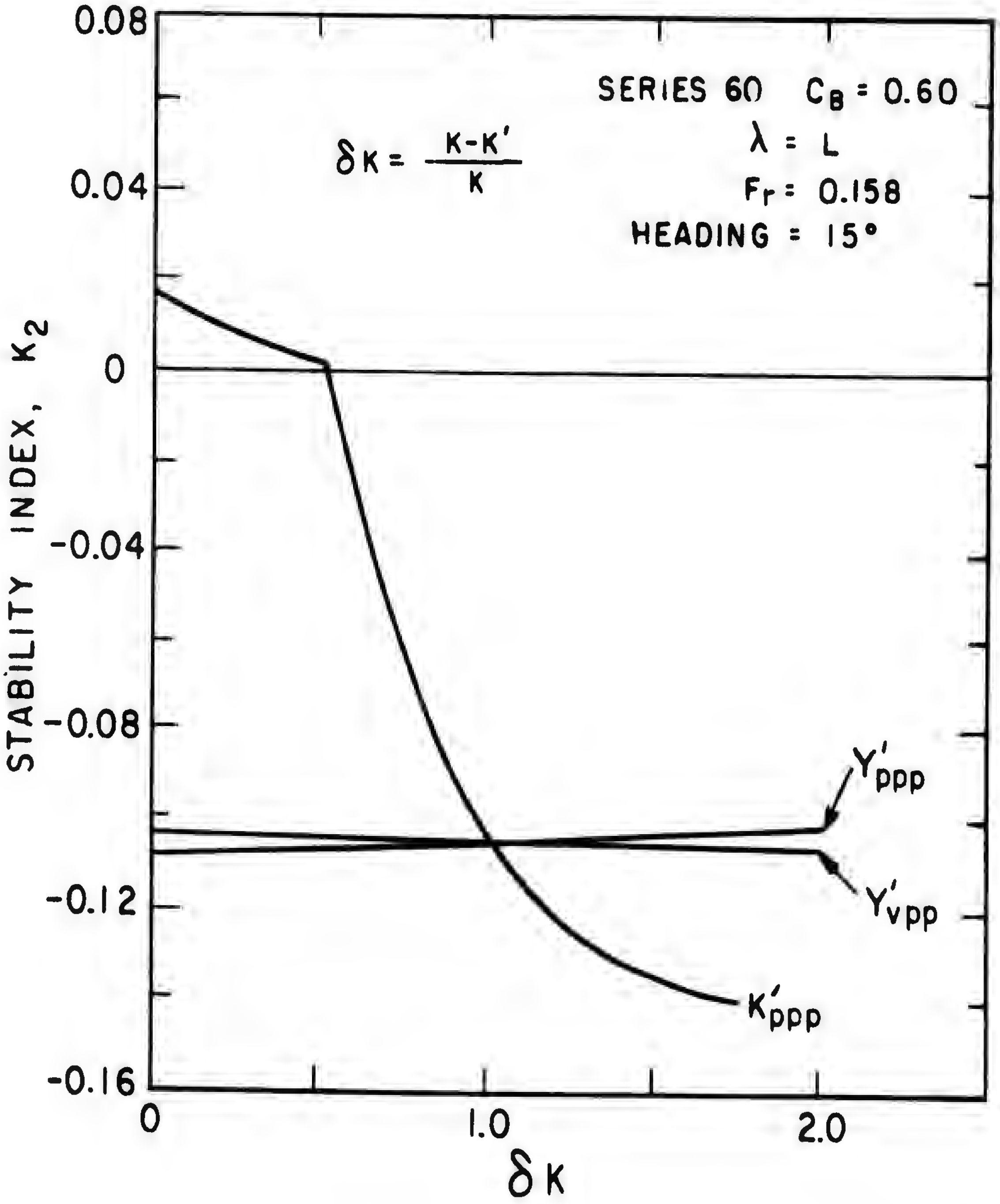

FIG. 22 VARIATION OF K_2 vs PERCENTAGE CHANGE IN HYDRODYNAMIC COEFFICIENTS

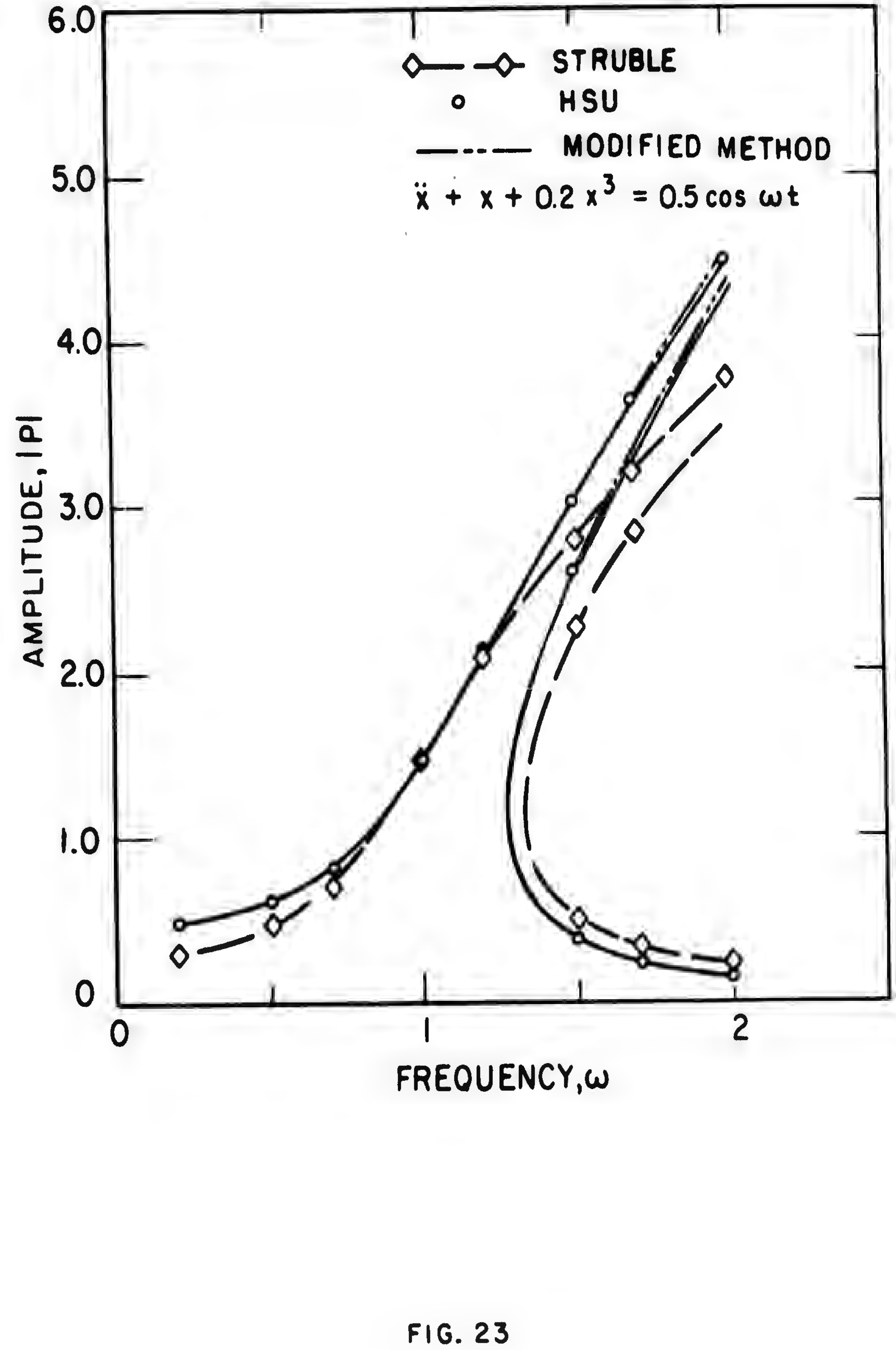
6.0
5.0
4.0
3.0
2.0
1.0
0
0
1
2
AMPLITUDE, |P|
FREQUENCY, ω
STRUBLE
HSU
MODIFIED METHOD
$\ddot{x} + x + 0.2\,x^3 = 0.5 \cos \omega t$

FIG. 23

TABLE I

Hydrodynamic Coefficients for the Modified Model

Coefficient	Original Model	Modified Model
Y'_{prr}	0.4723	1.4168
Y'_{vrr}	-11.9663	-5.9832
Y'_{ppp}	0.0037	0.0112
Y'_{vpp}	-0.026	0.0
K'_{prr}	-1.88	0.0
K'_{vpp}	0.1035	0.3104
K'_{vrr}	13.0517	39.155

APPENDIX A

Solution of the Equations of Motion

In this appendix we will give the details of the solution of the equations of motion using the modified method.

Solution of the Equation of Motion

If we put $\beta = 0$, in equations (4) we get

$$\ddot{x}_k + \sum_{j=1}^{3} a_{kj}\dot{x}_j + \omega_k^2 x_2 = X_{k0}\sin(\omega t + \theta_k) \quad , \quad k = 1, 2, 3. \qquad \text{(A-1)}$$

These are three linear coupled differential equations. They have a steady state solution of the form

$$x_k = A_k \cos \omega t + B_k \sin \omega t \quad ,$$

$$\dot{x}_k = -\omega A_k \sin \omega t + \omega B_k \cos \omega t \, , \quad k = 1, 2, 3. \qquad \text{(A-2)}$$

Now for $\beta \neq 0$, we shall take

$$x_k = A_k(t)\cos\omega t + B_k(t)\sin\omega t + \sum_{m=1}^{N} \beta^m x_{km} \, ,$$

$$\dot{x}_k = -\omega\left[A_k(t)\sin\omega t - B_k(t)\cos\omega t\right] + \sum_{m=1}^{N} \beta^m \dot{x}_{km} \quad , \quad k = 1, 2, 3 \qquad \text{(A-3)}$$

as one possible form of the solution, where

$$A_k = A_k(t) \quad , \quad B_k = B_k(t) \quad , \quad k = 1, 2, 3 .$$

are slowly varying functions of time. If we differentiate the first line in (A-3) to obtain $\dot{x}_k$, and equate the resulting expression to the assumed form for $\dot{x}_k$ in the second line of (A-3), we get

$$\frac{dA_k}{dt} \cos \omega t + \frac{dB_k}{dt} \sin \omega t = 0 \ , k = 1, 2, 3. \qquad \text{(A-4)}$$

Substituting (A-3) into (4) we get

$$\begin{aligned}
&\left(-\omega \frac{dA_k}{dt} - \omega^2 B_k - \omega \sum_{j=1}^{3} a_{kj} A_j + \omega_k^2 B_2\right) \sin \omega t \\
&+\left(\omega \frac{dB_k}{dt} - \omega^2 A_k + \omega \sum_{j=1}^{3} a_{kj} B_j + \omega_k^2 A_2\right) \cos \omega t \\
&+ \beta \left(\ddot{x}_{k1} + \sum_{j=1}^{3} a_{kj} \dot{x}_j + \omega_k^2 x_{21} \right) \\
&= \beta \Big[\omega^3 \sum d_{kijm} (-A_i \sin \omega t + B_i \cos \omega t)(-A_j \sin \omega t + B_j \cos \omega t)(-A_m \sin \omega t + B_m \cos \omega t) \\
&\qquad + \gamma_k (A_2 \cos \omega t + B_2 \sin \omega t)^3 + X_{k0} \sin(\omega t + \theta_k) \ , k = 1, 2, 3. \qquad \text{(A-5)}
\end{aligned}$$

In writing equation (A-5) we retained terms only through the first order of β . If we now use the following trigonometric identities

$$\sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$$

$$\cos^3 x = \frac{3}{4} \cos x + \frac{1}{4} \cos 3x$$

and rearrange, we obtain for equation (A-5)

$$\left(-\omega \frac{dA_k}{dt} - \omega^2 B_k - \omega \sum_{j=1}^{3} a_{kj} A_j + \omega_k^2 B_2 - \beta I_{ks}\right) \sin \omega t$$

$$+\left(\omega \frac{dB_k}{dt} - \omega^2 A_k + \omega \sum_{j=1}^{3} a_{kj} B_j + \omega_k^2 A_2 - \beta I_{kc}\right) \cos \omega t$$

$$+\beta \left(\ddot{x}_{k1} + \sum_{j=1}^{3} a_{kj} \dot{x}_{j1} + \omega_k^2 x_{21}\right)$$

$$= X_{k0} \sin(\omega t + \theta_k) + \beta\left(R_{ks} \sin 3\omega t + R_{kc} \cos 3\omega t\right),$$

$$k = 1, 2, 3 \qquad \text{(A-6)}$$

where I_{ks}, I_{kc}, R_{ks}, and R_{kc} are nonlinear functions of A_k and B_k. The form of equations (A-6) suggests separating that equation into the following two equations:

$$\left(-\omega \frac{dA_k}{dt} - \omega^2 B_k - \omega \sum_{j=1}^{3} a_{kj} A_j + \omega_k^2 B_2 - \beta I_{ks}\right) \sin \omega t$$

$$+\left(\omega \frac{dB_k}{dt} - \omega^2 A_k + \omega \sum_{j=1}^{3} a_{kj} B_j + \omega_k^2 A_2 - \beta I_{kc}\right) \cos \omega t$$

$$= X_{k0} \sin(\omega t + \theta_k), \quad k = 1, 2, 3 \qquad \text{(A-7)}$$

and

$$\ddot{x}_{k1} + \sum_{j=1}^{3} a_{kj} x_{j1} + \omega_k^2 x_{21} = R_{ks} \sin 3\omega t + R_{kc} \cos 3\omega t,$$

$$k = 1, 2, 3 \qquad \text{(A-8)}$$

This division has been made such that all the fundamental harmonic terms are grouped together.

Equations (A-7) are three nonlinear coupled equations. We can solve for $\frac{dA_k}{dt}$, $\frac{dB_k}{dt}$ in (A-7) by the use of the relations offered by (A-4). The resulting expressions for

$\frac{dA_k}{dt}$ and $\frac{dB_k}{dt}$ can be written as

$$\frac{dA_k}{dt} = \frac{1}{\omega} f_k^{(1)}(A_k, B_k, t) \quad ,$$

$$\frac{dB_k}{dt} = \frac{1}{\omega} f_k^{(2)}(A_k, B_k, t) \quad ,$$

$$k = 1, 2, 3 \qquad \text{(A-9)}$$

where $f_k^{(1)}$ and $f_k^{(2)}$ are nonlinear algebraic functions of A_k and B_k .

Now, we recall that, in writing down the solution (A-3) we assumed that A_k and B_k are slowly varying functions of time. Hence we can replace the right-hand side of equations (A-9) by their average over one cycle. This gives

$$\frac{dA_k}{dt} = \frac{1}{2\pi\omega}\int_0^{2\pi} f_k^{(1)}(A_k, B_k, t)\, dt = \frac{1}{2\omega}\overline{f_k^{(1)}},$$

$$\frac{dB_k}{dt} = \frac{1}{2\pi\omega}\int_0^{2\pi} f_k^{(2)}(A_k, B_k, t)\, dt = \frac{1}{2\omega}\overline{f_k^{(2)}},$$

$$k = 1, 2, 3 \qquad \text{(A-10)}$$

where

$$\overline{f_k^{(1)}} = \left\{-\omega^2 B_k + \omega_k^2 B_2 - \omega \sum_{j=1}^{3} a_{kj} A_j - \beta I_{ks} - X_{k0}\cos\theta_k\right\}$$

$$\overline{f_k^{(2)}} = \left\{\omega^2 A_k - \omega_k^2 A_2 - \omega \sum_{j=1}^{3} a_{kj} B_j + \beta I_{kc} + X_{k0}\sin\theta_k\right\}$$

In writing (A-10) we have used the main assumption involved in the theory of slowly varying parameters (for details see Bogoliubov and Mitropolsky [1955]). Since we are mainly interested in the steady state solution of the equations of motion, we will consider only the case when

$$\frac{dA_k}{dt} = \frac{dB_k}{dt} = 0 \qquad , \; k = 1, 2, 3 .$$

Then equations (A-11) and (A-12) become

$$-\omega^2 B_k + \omega_k^2 B_2 - \omega \sum_{j=1}^{3} a_{kj} A_j - \beta I_{ks} - X_{ko} \cos \theta_k = 0,$$

$$\omega^2 A_k - \omega_k^2 A_2 - \omega \sum_{j=1}^{3} a_{kj} B_j + \beta I_{kc} + X_{ko} \sin \theta_k = 0, k = 1,2,3. \qquad \text{(A-11)}$$

These are six nonlinear algebraic equations which can be solved numerically for values of A_k and B_k . After solving for A_k and B_k the solution of (4) can be written as

$$x_k = A_k \cos \omega t + B_k \sin \omega t + S_k \sin 3\omega t + C_k \cos 3\omega t$$

where S_k and C_k are functions of A_k and B_k and have to be determined from the solution of equation (A-8).

APPENDIX B

Wave Exciting Forces

Exciting forces acting on a ship sailing in waves may be obtained by integrating the pressure distribution in the wave over the wetted surface of the hull. If only the pressure distribution in the incident wave is considered then the forces are said to be calculated according to the Froude-Krylov hypothesis. This however neglects the effect of the presence of the ship hull on the pressure distribution. In the present work we shall use an approximate method to express the effect of the ship as a correction to be added to the Froude-Krylov forces.

Let us first calculate the pressure distribution in the incident waves. We shall assume that the fluid is inviscid and the flow is irrotational. These assumptions imply that we can describe the flow by a velocity potential Φ , which for infinitesimal, deep water waves, take the form

$$\Phi = -\frac{gA}{\sigma} e^{-k\bar{z}} \cos(k\bar{x} + \sigma t) \tag{B-1}$$

where A is the wave amplitude,

σ is the wave frequency,

k is the wave number.

Details of the derivation of (B-1) can be found in Wehausen [1960].

Using Bernoulli's law, we can get the pressure distribution in the wave which can be expressed in the hull coordinate system as

$$p(x,y,\zeta,t) = \rho g(\zeta-\zeta_0+\varphi y) - \rho g A\, e^{-k(\zeta-\zeta_0)} \sin(k_1 x - k_2 y + \omega t) \tag{B-2}$$

where

$$k_1 = k\cos\delta \;, \quad k_2 = k\sin\delta \;, \quad \omega = \sigma + kU\cos\delta \,.$$

The sway force Y_f, roll moment, K_f, and yaw moment N_f, calculated according to Froude-Krylov hypothesis are given by (see Wehausen [1964]).

$$\begin{aligned} Y_f &= -\iint_S p\, dx\, d\zeta \;, \\ K_f &= \iint_S p[\zeta\, d\zeta - y\, dy]\, dx \;, \\ N_f &= \iint_S p[y\, dy - x\, dx]\, d\zeta \,. \end{aligned} \tag{B-3}$$

where S is the wetted surface of the hull. Performing the integration one can express these forces as

$$\begin{aligned} Y_f &= Y_{fc}\cos\omega t + Y_{fs}\sin\omega t \;, \\ K_f &= K_{fc}\cos\omega t + K_{fs}\sin\omega t \;, \\ N_f &= N_{fc}\cos\omega t + N_{fs}\sin\omega t \;, \end{aligned} \tag{B-4}$$

where

$$Y_{fc} = 2\rho g A \frac{k_2}{k} \int_{l_1}^{l_2} dx \int_0^b dy\, (e^{-kF}-1) \quad \cos k_1 x \cos k_2 y$$

$$Y_{fs} = -2\rho g A \frac{k_2}{k} \int_{l_1}^{l_2} dx \int_0^b dy\, (e^{-kF}-1) \quad \sin k_1 x \cos k_2 y$$

$$K_{fc} = -2\rho g A \int_{l_1}^{l_2} dx \int_0^b dy\, [y e^{-kF} \cos k_1 x \sin k_2 y$$

$$+ \frac{k_2}{k}\left\{\left(F+\frac{1}{k}\right)e^{-kF} - \frac{1}{k} + \zeta_0 (e^{-kF}-1)\right\} \cos k_1 x \cos k_2 y\,]$$

$$K_{fs} = 2\rho g A \int_{l_1}^{l_2} dx \int_0^b dy \, [\, y e^{-kF} \sin k_1 x \sin k_2 y$$

$$+ \frac{k_2}{k} \{ (F + \frac{1}{k}) e^{-kF} - \frac{1}{k} + \zeta_0 (e^{-kF} - 1) \} \sin k_1 x \cos k_2 y \,]$$

$$N_{fc} = 2\rho g A \int_{l_1}^{l_2} dx \int_0^b dy \left[\frac{K_1}{K} \{ y (e^{-kF} - 1) \sin k_1 x \sin k_2 y \} \right.$$

$$\left. + \frac{K_2}{K} \{ x (e^{-kF} - 1) \cos k_1 x \cos k_2 y \} \right]$$

$$N_{fs} = 2\rho g A \int_{l_1}^{l_2} dx \int_0^b dy \left[\frac{K_1}{K} \{ y (e^{-kF} - 1) \cos k_1 x \sin k_2 y \} \right.$$

$$\left. - \frac{K_2}{K} \{ x (e^{-kF} - 1) \sin k_1 x \cos k_2 y \} \right]$$

and the ship's hull is given by

$$\zeta = F(x, y)$$

The corrections to the Froude-Krylov forces are calculated as follows:

Let us denote the components of the velocity and the acceleration of a water particle on the surface in the direction of the y-axis by u_y and $\dot{u}_y$. Then u_y and $\dot{u}_y$ are given by

$$u_y = -\sigma A \, e^{-k(\zeta - \zeta_0)} \sin (k_1 x - k_2 y + \omega t) \sin \delta \qquad \text{(B-5)}$$

$$\dot{u}_y = -\sigma^2 A \, e^{-k(\zeta - \zeta_0)} \cos (k_1 x - k_2 y + \omega t) \sin \delta \qquad \text{(B-6)}$$

The wave slope and its derivatives with respect to time can be expressed as

$$\alpha = kA \cos(k_1 x - k_2 y + \omega t) \tag{B-7}$$

$$\dot{\alpha} = -\sigma kA \sin(k_1 x - k_2 y + \omega t) \tag{B-8}$$

$$\ddot{\alpha} = -\sigma^2 kA \cos(k_1 x - k_2 y + \omega t) \tag{B-9}$$

The corrections to the sway force, Y_{CR} , the roll moment, K_{CR}, and the yaw moment, N_{CR} , can be expressed as

$$\begin{aligned} Y_{CR} &= -\int_{l_1}^{l_2} \left[Y_{uD}\,\bar{u}_y + Y_{\dot{u}D}\,\bar{\dot{u}}_y + Y_{pD}\,\bar{\dot{\alpha}} + Y_{\dot{p}D}\,\bar{\ddot{\alpha}} \right] dx \\ K_{CR} &= -\int_{l_1}^{l_2} \left[K_{uD}\,\bar{u}_y + K_{\dot{u}D}\,\bar{\dot{u}}_y + K_{pD}\,\bar{\dot{\alpha}} + K_{\dot{p}D}\,\bar{\ddot{\alpha}} \right] dx \\ N_{CR} &= \int_{l_1}^{l_2} x \, dY_{CR} \end{aligned} \tag{B-10}$$

where Y_{uD} , $Y_{\dot{u}D}$, Y_{pD} , and $Y_{\dot{p}D}$ are the two-dimensional values for the sway damping coefficient, the sway added mass, the sway-roll damping coupling, and the sway-roll inertia coupling coefficient respectively, and K_{uD} , $K_{\dot{u}D}$, K_{pD} , and $K_{\dot{p}D}$ are the two-dimensional roll-sway damping coupling coefficient, the roll-sway inertia coupling coefficient, the roll damping coefficient, and the roll added moment of inertia respectively.

A bar over the variable denotes the averaging of the variable over the cross-sectional area. Substituting in (B-10) and manipulating, one can rewrite it as

$$Y_{CR} = Y_{CRC} \cos \omega t + Y_{CRS} \sin \omega t$$

$$K_{CR} = K_{CRC} \cos \omega t + K_{CRS} \sin \omega t$$

$$N_{CR} = N_{CRC} \cos \omega t + N_{CRS} \sin \omega t$$

(B-11)

where

$$Y_{CRC} = -\int_{l_1}^{l_2} dx \Big[\frac{2\sigma A}{k\partial} \sin \delta \Big\}\Big\{ \int_0^b (e^{-kF} - 1) \cos k_2 y \Big\}\Big\}\Big\{ Y_{uD} \sin k_1 x + \sigma Y_{\dot{u}D} \cos k_1 x \Big\}$$

$$- \frac{\sigma A}{b} \sin k_2 b \Big\{ Y_{pD} \sin k_1 x + \sigma Y_{\dot{p}D} \cos k_1 x \Big\} \Big]$$

$$Y_{CRS} = -\int_{l_1}^{l_2} dx \Big[\frac{2\sigma A}{k\partial} \sin \delta \Big\{ \int_0^b (e^{-kF} - 1) \cos k_2 y \Big\}\Big\{ Y_{uD} \cos k_1 x - \sigma Y_{\dot{u}D} \sin k_1 x \Big\}$$

$$- \frac{\sigma A}{b} \sin k_2 b \Big\{ Y_{pD} \cos k_1 x - \sigma Y_{\dot{p}D} \sin k_1 x \Big\} \Big]$$

$$K_{CRC} = -\int_{l_1}^{l_2} dx \Big[\frac{2\sigma A}{k\partial} \sin \delta \Big\}\Big\{ \int_0^b (e^{-kF} - 1) \cos k_2 y \, dy \Big\}\Big\}\Big\{ K_{uD} \sin k_1 x + \sigma K_{\dot{u}D} \cos k_1 x \Big\}$$

$$- \frac{\sigma A}{b} \sin k_2 b \Big\{ K_{pD} \sin k_1 x + \sigma K_{\dot{p}D} \cos k_1 x \Big\} \Big]$$

$$K_{CRS} = -\int_{l_1}^{l_2} dx \Big[\frac{2\sigma A}{k\partial} \sin \delta \Big\}\Big\{ \int_0^b (e^{-kF} - 1) \cos k_2 y \, dy \Big\}\Big\{ K_{uD} \cos k_1 x - \sigma K_{\dot{u}D} \sin k_1 x \Big\}$$

$$- \frac{\sigma A}{b} \sin k_2 b \Big\{ K_{pD} \cos k_1 x - \sigma K_{\dot{p}D} \sin k_1 x \Big\} \Big]$$

$$N_{CRC} = \int_{l} x \, dY_{CRC}$$

$$N_{CRS} = \int_{l} x \, dY_{CRS} \, .$$

and a is the area of the cross-sections of the ship.

APPENDIX C

Solution of Duffing's Equation

The equation of motion of a single degree of freedom spring-mass system with nonlinear restoring force can be written as

$$\ddot{x} + x + \beta x^3 = X_0 \cos \omega t \tag{C-1}$$

This equation is usually known as Duffing's equation. In order to find the solution of this equation using Struble's method one has to assume that the excitation is small of the order of magnitude of β. Then we can write

$$X_0 = \beta F_0 \quad , \quad |\beta| < 1$$

and (C-1) becomes

$$\ddot{x} + x + \beta x^3 = \beta F_0 \cos \omega t \tag{C-2}$$

Applying Struble's method to C-2 (see Struble [1960]) one can express the relation between the amplitude and frequency of the steady-state motion of the mass as

$$\omega = 1 + \frac{3}{8} \beta P^2 - \beta \frac{F_0}{2P} \tag{C-3}$$

while if we use the modified method to C-1 we get

$$\omega^2 = 1 + \frac{3}{4} \beta P^2 - \frac{X_0}{P} \tag{C-4}$$

It is easy to show that (C-3) is only an approximation of (C-4). Substitute βF_0 for X_0 in (C-4) and

Take the square root and we obtain

$$\omega = 1 + \frac{\beta}{2}\left(\frac{3}{4}P^2 - \frac{F_0}{P}\right)$$

which is (C-3). This is not unexpected since Struble starts with the linear undamped free vibration as a generating solution to the nonlinear equation, while in the modified method the generating solution is the forced damped linear solution.

A third solution to the same equation (C-1), is given by Hsu [1960]. Hsu's method is known to give good approximation. This is given by

$$\omega^2 = 1 + \frac{7\omega^2-1}{9\omega^2-1}\beta P^2 - \frac{X_0}{P} \qquad \text{(C-5)}$$

It is clear that (C-5) coincides with (C-4) if we replace the term $\frac{7\omega^2-1}{9\omega^2-1}$ by 3/4, which is sufficiently accurate for the range of ω considered here. The relations (C-3) - (C-5) are plotted for certain specific examples and the results are shown in Fig. (23).

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