RADC-TR-70-98
Technical Report
June 1970

MODERN CLASSICAL MULTIVARIABLE FEEDBACK CONTROL THEORY
PART I

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32

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FOREWORD

This phase report was prepared by Polytechnic Institute of Brooklyn under Contract F30602-69-C-0053, Project 8505, PIB No. PIBEP-70-054. Mr. Haywood E. Webb, Jr (EMBIS) was the RADC Project Engineer.

This report has been reviewed by the Office of Information (EMLS) and is releasable to the Clearinghouse for Federal Scientific and Technical Information.

This report has been reviewed and is approved.

Approved: HAYWOOD E. WEBB
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ABSTRACT

This report establishes rigorous stability criteria in terms of transfer matrices and in addition undertakes a careful and critical examination of the concepts of complete controllability and observability as applied to the closed loop. In particular it is empahsized that the latter properties are neither necessary nor desirable attributes of a practical multivariable feedback system.

MODERN CLASSICAL MULTIVARIABLE FEEDBACK CONTROL THEORY-PART I

1. Introduction and Notation. The objective of this report (which we project shall be issued in four parts) is to rewrite, extend and enlarge the relatively new field of classical multivariable feedback control in the light of the gains and insights provided by modern optimal control theory. Part I establishes rigorous stability criteria in terms of transfer matrices and in addition undertakes a careful and critical examination of the concepts of complete controllability and observability as applied to the closed loop. In particular it is emphasized that the latter properties are neither necessary nor desirable attributes of a practical multivariable feedback system. Part II (in preparation) is devoted to a new and general theory of broadband plant equalization via feedback. More specifically the following problem is posed and solved: Given the transfer matrix P(s) of a plant with asymptotically stable "hidden" modes what are the necessary and sufficient conditions for the matrix T(s) to be realizable as the input-output transfer description of a dynamical asymptotically stable closed loop incorporating the plant and an asymptotically stable controller and feedback sensor? In general the restrictions on T(s) are a direct consequence of the non-minimum-phase character of P(s) and may be expressed as generalized gain-bandwidth constraints. These constraints reveal the possible tradeoffs between dc gain (steady-state error) and transient response (overshoot, etcetera). In Part III the topology is delimited to conform with that in current use in present-day process control. For example, it is technologically expeditious to assign to each controlled variable its own controlling loop. This of course in no way implies a non-interacting control scheme. In fact this latter technique is shown to have some serious deficiencies both from a practical and theoretical point of view. In part IV we undertake to extend the above results to plants and feedback control arrangements with intrinsic time lags. Particular emphasis is attached to the problem of anolgue simulation which at the present writing appears to be susceptible to a direct and decisive solution. These considerations lead us naturally to the study of the decomposition theory of multivariable rational matrices.

As regards notation most of it is self-explanatory but nevertheless we point out that n-dimensional column-vectors are written $\underset{\sim}{x}, \underset{\sim}{a}$ or in the alternative fashion $\underset{\sim}{x} = (x_1, x_2, \ldots, x_n)'$ whenever it is desirable to exhibit the components explicitly. Moreover, A' and det A stand for the transpose of A and the determinant of A, respectively. By $O_{m,n}$, $\underset{\sim}{O}_n$ and 1_n we denote, in the same order, the $m \times n$ zero matrix, the n-dimensional zero column-vector and the $n \times n$ identity matrix.

Although most of the results in Part I have been known to the author for several years many of them have not been published by him prior to the appearance of this report. Thus we have been rather careful in the matter of credits and if Ref. 2 appears next to a theorem this means that to the best of the author's knowledge the theorem and its proof have appeared for the first time in Ref. 2.

2. The Basic Theorems. A schematic of the general feedback control arrangement (GFCA) studied in this paper is shown in Fig. 1.

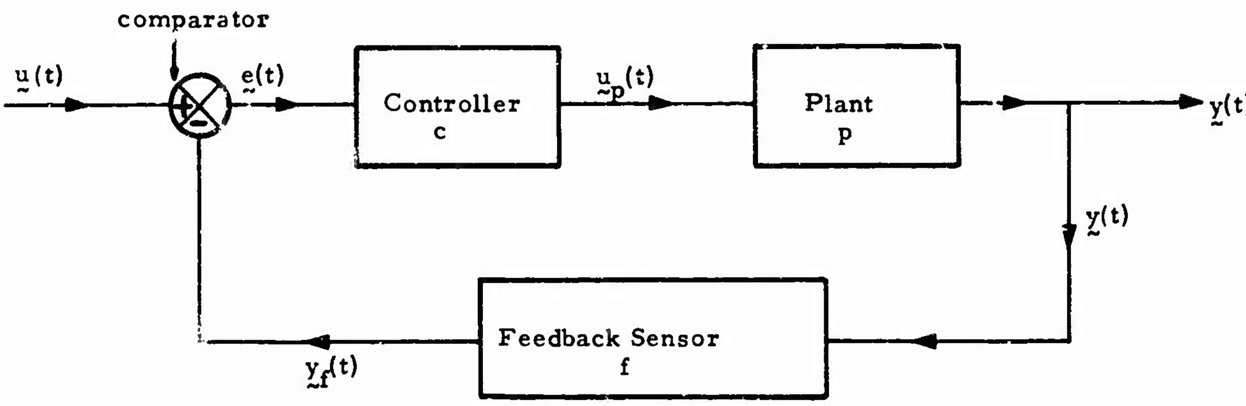

Fig. 1. The General Feedback Control Arrangement (GFCA)

Assumption 1 (A_1). The controller, plant and feedback sensor are assumed to be linear time-invariant dynamical systems possessing the respective state-variable descriptions

$$\dot{x}_c(t) = F_c x_c(t) + G_c u_c(t) \tag{1}$$

$$y_c(t) = H_c x_c(t) + J_c u_c(t) \; ; \tag{2}$$

$$\dot{x}_p(t) = F_p x_p(t) + G_p u_p(t) \tag{3}$$

$$y_p(t) = H_p x_p(t) + J_p u_p(t) \; ; \tag{4}$$

$$\dot{x}_f(t) = F_f x_f(t) + G_f u_f(t) \tag{5}$$

$$y_f(t) = H_f x_f(t) + J_f u_f(t) \; . \tag{6}$$

As usual, u, x, y are, in the same order, generic designations for "input", "state" and "output". Of course, in the situation depicted in Fig. 1, $u(t)$ is the actual GFCA reference input, $y(t)$ the corresponding output and $e(t)$ the feedback error. It follows from Eqs. (1)-(6) that the associated transfer matrices are given by

$$C(s) = J_c + H_c(s 1_{\nu_c} - F_c)^{-1} G_c = m \times r \text{ matrix;} \tag{7}$$

$$P(s) = J_p + H_p(s1_{\nu_p} - F_p)^{-1}G_p = n \times m \text{ matrix} \tag{8}$$

and

$$F(s) = J_f + H_f(s1_{\nu_f} - F_f)^{-1}G_f = r \times n \text{ matrix}, \tag{9}$$

where,

ν_c = dimensionality of controller = size of F_c;

ν_p = dimensionality of plant = size of F_p;

ν_f = dimensionality of feedback sensor = size of F_f.

Clearly, because of the particular interconnection of the blocks,

$$\underset{\sim}{u}_c(t) = \underset{\sim}{e}(t) = \underset{\sim}{u}(t) - \underset{\sim}{y}_f(t) , \tag{10}$$

$$\underset{\sim}{u}_p(t) = \underset{\sim}{y}_c(t) , \tag{11}$$

$$\underset{\sim}{u}_f(t) = \underset{\sim}{y}(t) . \tag{12}$$

The sizes of the column-vector functions of time $\underset{\sim}{u}(t)$, $\underset{\sim}{e}(t)$, $\underset{\sim}{u}_p(t)$, $\underset{\sim}{y}(t)$ and $\underset{\sim}{y}_f(t)$ are $r \times 1$, $r \times 1$, $m \times 1$, $n \times 1$ and $r \times 1$, respectively.

Let the Laplace transform of $\underset{\sim}{a}(t)$ be denoted by $\hat{\underset{\sim}{a}}(s)$. Then, by a straightforward analysis we find that*

$$\hat{\underset{\sim}{y}}(s) = T(s)\hat{\underset{\sim}{u}}(s) \tag{13}$$

and

$$\hat{\underset{\sim}{e}}(s) = E(s)\hat{\underset{\sim}{u}}(s) \tag{13a}$$

where

$$T(s) = [1_n + P(s)C(s)F(s)]^{-1}P(s)C(s) \tag{14}$$

$$= P(s)C(s)[1_r + F(s)P(s)C(s)]^{-1} \tag{15}$$

and

$$E(s) = [1_r + F(s)P(s)C(s)]^{-1} . \tag{16}$$

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* T(s) is n×r and E(s) is r×r.

Observe that (14)-(16) make sense if and only if

$$\det[1_r + F(s)P(s)C(s)] \neq 0 . \tag{17}$$

Conversely, if condition (17) is satisfied, then T(s), as defined by (15), is the unique $n \times r$ rational matrix function of the complex variable $s = \sigma + j\omega$ relating $\hat{y}(s)$ to $\hat{u}(s)$ in the manner prescribed by (13). Furthermore, it should be evident that if (17) is violated $e(t)$, $y_c(t)$, $v(t)$ and $y_f(t)$ are not uniquely determined by the input $u(t)$ for a specified initial state $x_c(0)$, $x_p(0)$, $x_f(0)$. Now even granting (17) it does not follow that the GFCA of Fig. 1 is dynamical, i.e., it is not necessarily true that

$$\lim_{s \to \infty} T(s) = \text{finite matrix} . \tag{18}$$

The possession of a pole by T(s) at $s = \infty$ means that GFCA is performing differentiation as well as integration. More to the point, suppose the input is composed of r Heaviside unit step functions $u_{-1}(t)$ of arbitrary amplitudes:

$$u(t) = u_o u_{-1}(t) ,$$

u_o an arbitrary constant $r \times 1$ vector. Then,

$$\hat{y}(s) = \frac{1}{s} \cdot T(s) u_o$$

and invoking the Laplace transform initial value theorem it is seen that $y(0^+)$ is well-defined for every choice of u_o only if (18) is valid. If this is indeed the case,

$$y(0^+) = T(\infty) u_o .$$

Theorem 1. Under assumption A_1, the GFCA shown in Fig. 1 is dynamical if and only if*

$$0 \neq \det[1_r + F(\infty)P(\infty)C(\infty)] = \det[1_n + P(\infty)C(\infty)F(\infty)] . \tag{19}$$

Or equivalently, in view of the identifications

$$F(\infty) = J_f ,$$

$$P(\infty) = J_p , \tag{20}$$

$$C(\infty) = J_c ,$$

---

*If A is $n \times r$ and B is $r \times n$, $\det(1_n + AB) = \det(1_r + BA)$.

if and only if

$$0 \neq \det(1_r + J_f J_p J_c) = \det(1_n + J_p J_c J_f) \ . \tag{21}$$

Proof. Suppose (19) holds. Then (17) is certainly true and T(s), as defined by (15), is meaningful. Moreover, (19) and A_1 taken together imply that both

$$[1_r + F(s)P(s)C(s)]^{-1}$$

and T(s) are regular at $s = \infty$. Inversely, if GFCA is dynamical then of necessity $\underset{\sim}{y}(t)$ is uniquely determined by $\underset{\sim}{u}(t)$ and the initial state. Hence (17) must hold and T(s), as given by (15), makes sense. In addition, $T(\infty)$ = finite matrix whence, because of assumption 1, F(s)T(s) is also regular at infinity. But

$$F(s)T(s) = 1_r - [1_r + F(s)P(s)C(s)]^{-1} \tag{22}$$

and consequently

$$\det[1_r + F(\infty)P(\infty)C(\infty)] \neq 0 \ , \quad \text{Q. E. D.}$$

Motivated by theorem 1 we introduce assumption 2.

Assumption 2(A_2). The GFCA of Fig. 1 satisfies the constraint

$$0 \neq \det[1_r + F(\infty)P(\infty)C(\infty)] = \det(1_r + J_f J_p J_c) \ . \tag{23}$$

The true significance of A_2 is perhaps best brought out by an examination of the state equations of GFCA when exhibited in terms of the coefficient matrices $F_{c,p,f}$, $G_{c,p,f}$, $H_{c,p,f}$, $J_{c,p,f}$ describing the dynamical behaviour of controller, plant and feedback sensor. Put

$$\underset{\sim}{x}_o(t) = \begin{bmatrix} \underset{\sim}{x}_c(t) \\ \hline \underset{\sim}{x}_p(t) \\ \hline \underset{\sim}{x}_f(t) \end{bmatrix} = (\nu_c + \nu_p + \nu_f) \times 1 \text{ vector} \ .$$

Then, after some straightforward algebra we find that

$$\dot{\underset{\sim}{x}}_o(t) = F_o \underset{\sim}{x}_o(t) + G_o \underset{\sim}{u}(t) \ , \tag{24}$$

$$\underset{\sim}{y}(t) = H_o \underset{\sim}{x}_o(t) + J_o \underset{\sim}{u}(t) \ , \tag{25}$$

where*

$$F_o = \left[\begin{array}{c|c|c} F_c - G_c(1_r + J_f J_p J_c)^{-1} J_f J_p H_c & -G_c(1_r + J_f J_p J_c)^{-1} J_f H_p & -G_c(1_r + J_f J_p J_c)^{-1} H_f \\ \hline G_p(1_m + J_c J_f J_p)^{-1} H_c & F_p - G_p(1_m + J_c J_f J_p)^{-1} J_c J_f H_p & -G_p(1_m + J_c J_f J_p)^{-1} J_c H_f \\ \hline G_f(1_n + J_p J_c J_f)^{-1} J_p H_c & G_f(1_n + J_p J_c J_f)^{-1} H_p & F_f - G_f(1_n + J_p J_c J_f)^{-1} J_p J_c H_f \end{array}\right], \tag{26}$$

$$H_o = (1_n + J_p J_c J_f)^{-1} [J_p H_c \,|\, H_p \,|\, -J_p J_c H_f]; \; J_o = (1_n + J_p J_c J_f)^{-1} J_p J_c \tag{27}$$

$$G_o = \left[\begin{array}{c} G_c \\ \hline G_p J_c \\ \hline G_f J_p J_c \end{array}\right] (1_r + J_f J_p J_c)^{-1} . \tag{28}$$

Of course,

$$T(s) = J_o + H_o (s 1_{\nu_o} - F_o)^{-1} G_o . \tag{28a}$$

For future reference we extract out two special cases.

Case a. Unity feedback; i.e., $F(s) = 1_n$, $r = n$, $\nu_f = 0$, $J_f = 1_n$.

$$F_o = \left[\begin{array}{c|c} F_c - G_c(1_n + J_p J_c)^{-1} J_p H_c & -G_c(1_n + J_p J_c)^{-1} H_p \\ \hline G_p(1_m + J_c J_p)^{-1} H_c & F_p - G_p(1_m + J_c J_p)^{-1} J_c H_p \end{array}\right] \tag{29}$$

$$H_o = (1_n + J_p J_c)^{-1} [J_p H_c \,|\, H_p]; \; J_o = (1_n + J_p J_c)^{-1} J_p J_c \tag{30}$$

$$G_o = \left[\begin{array}{c} G_c \\ \hline G_p J_c \end{array}\right] (1_n + J_p J_c)^{-1} . \tag{31}$$

Case b. Unity controller; i.e., $C(s) = 1_r$, $r = m$, $\nu_c = 0$, $J_c = 1_r$.

$$F_o = \left[\begin{array}{c|c} F_p - G_p(1_r + J_f J_p)^{-1} J_f H_p & -G_p(1_r + J_f J_p)^{-1} H_f \\ \hline G_f(1_n + J_p J_f)^{-1} H_p & F_f - G_f(1_n + J_p J_f)^{-1} J_p H_f \end{array}\right] \tag{32}$$

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* Bear in mind that

$$0 \neq \det(1_n + J_p J_c J_f) = \det(1_r + J_f J_p J_c) = \det(1_m + J_c J_f J_p) .$$

$$H_o = (1_n + J_p J_f)^{-1}[H_p | -J_p H_f]; \; J_o = (1_n + J_p J_f)^{-1} J_p \tag{33}$$

$$G_o = \left[\begin{array}{c} G_p \\ \hline G_f J_p \end{array}\right](1_r + J_f J_p)^{-1} . \tag{34}$$

Although (26) looks very formidable it may be recast in a remarkably simple form. By inspection,*

$$F_o = (F_c \dotplus F_p \dotplus F_f) - (G_c \dotplus G_p \dotplus G_f) L^{-1} \left[\begin{array}{c|c|c} J_f J_p & J_f & 1_r \\ \hline -1_m & J_c J_f & J_c \\ \hline -J_p & -1_n & J_p J_c \end{array}\right] (H_c \dotplus H_p \dotplus H_f) , \tag{35}$$

where

$$L = (1_r + J_f J_p J_c) \dotplus (1_m + J_c J_f J_p) \dotplus (1_n + J_p J_c J_f) . \tag{36}$$

With the aid of the identity

$$\left[\begin{array}{c|c|c} J_f J_p & J_f & 1_r \\ \hline -1_m & J_c J_f & J_c \\ \hline -J_p & -1_n & J_p J_c \end{array}\right] \begin{bmatrix} J_c & -1_m & O_{m,n} \\ O_{n,r} & J_p & -1_n \\ 1_r & O_{r,m} & J_f \end{bmatrix} = L \tag{37}$$

we can rewrite (35) as

$$F_o = (F_c \dotplus F_p \dotplus F_f) - (G_c \dotplus G_p \dotplus G_f) \begin{bmatrix} J_c & -1_m & O_{m,n} \\ O_{n,r} & J_p & -1_n \\ 1_r & O_{r,m} & J_f \end{bmatrix}^{-1} (H_c \dotplus H_p \dotplus H_f) . \tag{38}$$

To determine the stability properties of GFCA we must find the eigenvalues of F_o. That is, it is necessary to fix the location of the zeros of the polynomial $\Delta_o(s) \equiv \det(s1_{\nu_o} - F_o)$; i.e., the roots of

$$\det(s1_{\nu_o} - F_o) = 0, \quad \nu_o = \nu_c + \nu_p + \nu_f .$$

According to a well known result in the theory of partitioned matrices,

---

* $A \dotplus B$ denotes the "direct sum" of the two matrices A, B.

$$\det\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] = \det(A_{11}-A_{12}A_{22}^{-1}A_{21})\det A_{22} \tag{39}$$

$$= \det(A_{22}-A_{21}A_{11}^{-1}A_{12})\det A_{11} \tag{39a}$$

provided the inverses exists. Hence,

$$\Delta_o(s)=\det(s1_{\nu_o}-F_o)=\pm\eta^{-1}\cdot\det\left[\begin{array}{ccc|ccc} J_c & -1_m & O_{m,n} & H_c & O_{m,\nu_p} & O_{m,\nu_f} \\ O_{n,r} & J_p & -1_n & O_{n,\nu_c} & H_p & O_{n,\nu_f} \\ 1_r & O_{r,m} & J_f & O_{r,\nu_c} & O_{r,\nu_p} & H_f \\ \hline G_c & O_{\nu_c,m} & O_{\nu_c,n} & F_c-s1_{\nu_c} & O_{\nu_c,\nu_p} & O_{\nu_c,\nu_f} \\ O_{\nu_p,r} & G_p & O_{\nu_p,n} & O_{\nu_p,\nu_c} & F_p-s1_{\nu_p} & O_{\nu_p,\nu_f} \\ O_{\nu_p,r} & O_{\nu_p,m} & G_f & O_{\nu_f,\nu_c} & O_{\nu_f,\nu_p} & F_f-s1_{\nu_f} \end{array}\right] \tag{40}$$

where

$$\eta = \det(1_n+J_pJ_cJ_f) = \det\begin{bmatrix} J_c & -1_m & O_{m,n} \\ O_{n,r} & J_p & -1_m \\ 1_r & O_{r,m} & J_f \end{bmatrix} . \tag{40a}$$

Using (39) with $A_{22}=F_f-s1_{\nu_f}$ and recognizing that

$$F(s)=J_f+H_f(s1_{\nu_f}-F_f)^{-1}G_f$$

we get (all subscripts on the zero matrices are now dropped)

$$\Delta_o(s)=\pm\eta^{-1}\cdot\det\begin{bmatrix} J_c & -1_m & O & H_c & O \\ O & J_p & -1_n & O & H_p \\ 1_r & O & F(s) & O & O \\ G_c & O & O & F_c-s1_{\nu_c} & O \\ O & G_p & O & O & F_p-s1_{\nu_p} \end{bmatrix}\cdot\Delta_f(s) , \tag{41}$$

$$\Delta_f(s)=\det(s1_{\nu_f}-F_f) .$$

Repeating the process two more times in succession with $A_{22} = F_p - s1_{\nu_p}$ and $A_{22} = F_c - s1_{\nu_c}$, respectively, one obtains

$$\Delta_o(s) = \pm\eta^{-1} \cdot \det \begin{bmatrix} C(s) & -1_m & O \\ O & P(s) & -1_n \\ 1_r & O & F(s) \end{bmatrix} \cdot \Delta_f(s)\Delta_p(s)\Delta_c(s) , \tag{42}$$

$$\Delta_p(s) = \det(s1_{\nu_p} - F_p) ; \; \Delta_c(s) = \det(s1_{\nu_c} - F_c) .$$

To calculate the determinant appearing in (42) we simply interchange block columns to form the matrix

$$\begin{bmatrix} -1_m & O & C(s) \\ P(s) & -1_n & O \\ O & F(s) & 1_r \end{bmatrix} \tag{43}$$

and then apply (39) two last times to reach the very informative formula*

$$\Delta_o(s) = \eta^{-1} \cdot \det[1_n + P(s)C(s)F(s)] \cdot \Delta_c(s)\Delta_p(s)\Delta_f(s) , \tag{44}$$

$$\eta = \det(1_n + J_p J_c J_f) = \det[1_n + P(\infty)C(\infty)F(\infty)] .$$

The result embodied in (44) is undoubtedly one of the most important in the entire field of classical multivariable control. Our next theorem is immediate.

Theorem 2 [2, 3]. Under assumptions A_1, A_2, GFCA is asymptotically stable if and only if the scalar function

$$\det[1_n + P(s)C(s)F(s)] \cdot \Delta_c(s)\Delta_p(s)\Delta_f(s) \tag{45}$$

is devoid of zeros in $\text{Re}\, s \geq 0$.

Unfortunately the test for stability provided by theorem 2 cannot be carried out in terms of transfer matrices alone but requires knowledge of the three polynomials $\Delta_c(s)$, $\Delta_p(s)$, $\Delta_f(s)$, quantities which depend on the internal structure of GFCA. At the

* The sign ambiguity is removed by comparing coefficients of the highest powers of s.

present level of generality nothing can be done to remedy this situation*. However, if one additional constraint is imposed on the controller, plant and feedback sensor it is possible to eliminate this defect completely.

Assumption 3 (A_3). The controller, plant and feedback sensor are completely controllable and observable realizations of their respective transfer matrices C(s), P(s), F(s).

The following algebraic characterization of A_3 is well known [4]. Let $\psi_c(s)$, $\psi_p(s)$ and $\psi_f(s)$ be the characteristic denominators** of C(s), P(s) and F(s), respectively. Then, A_3 is valid if and only if

$$\begin{aligned} \Delta_c(s) &= \psi_c(s) , \\ \Delta_p(s) &= \psi_p(s) , \\ \Delta_f(s) &= \psi_f(s) . \end{aligned} \tag{46}$$

Theorem 3 [2, 3]. Under assumptions A_1-A_3, GFCA is asymptotically stable if and only if the scalar function

$$\det[1_n + P(s)C(s)F(s)]\cdot \psi_c(s)\psi_p(s)\psi_f(s) \tag{47}$$

is devoid of zeros in $\mathrm{Re}\, s \geq 0$.

Proof. Theorem 2 and (46), Q. E. D.

In general, independently of any considerations of controllability and observability,

$$\begin{aligned} \Delta_c(s) &= h_c(s)\psi_c(s) , \\ \Delta_p(s) &= h_p(s)\psi_p(s) , \\ \Delta_f(s) &= h_f(s)\psi_f(s) , \end{aligned} \tag{47a}$$

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* Physically speaking, this means that the stability of a system (an internal notion) cannot always be predicted from an examination of its several describing transfer matrices (an external notion).

** The characteristic denominator of a rational matrix A(s) (square or otherwise) is by definition the monic least common multiple of all denominators of all minors of A(s). It is understood of course that each minor is expressed as the ratio of two relatively prime polynomials.

where $h_c(s)$, $h_p(s)$ and $h_f(s)$ are three polynomials whose zeros determine the exponential growth rates of those "hidden" modes not detected in the poles of C(s), P(s) and F(s). The correctness of this assertion is certainly intuitively obvious but a strict demonstraion is available. Suppose, to be specific, that the plant is not completely controllable and observable. Then, there exists a nonsingular constant square matrix K, square matrices $\hat{F}_p$, F_{22}, F_{33} and constant matrices F_{13}, F_{21}, F_{23} such that [13, 14]

$$F_p = K^{-1} \cdot \overset{\begin{matrix} n_1 & n_2 & n_3 \end{matrix}}{\begin{bmatrix} \hat{F}_p & O & F_{13} \\ F_{21} & F_{22} & F_{23} \\ O & O & F_{33} \end{bmatrix}} \cdot K \qquad (47b)$$

where $n_1 + n_2 + n_3 = \nu_p$ and

$$\det(s1_{n_1} - \hat{F}_p) = \psi_p(s) . \qquad (47c)$$

Thus, from (47b) and (47c),

$$\Delta_p(s) = \det(s1_{\nu_p} - F_p) = \det(s1_{n_3} - F_{33})\det(s1_{n_2} - F_{22}) \cdot \det(s1_{n_1} - \hat{F}_p) = h_p(s)\psi_p(s)$$

where

$$h_p(s) = \det(s1_{n_3} - F_{33}) \cdot \det(s1_{n_2} - F_{22}) , \quad \text{Q. E. D.}$$

Corollary 1. If the "hidden" modes of controller, plant and feedback sensor, are all asymptotically stable then, under assumptions A_1, A_2, GFCA is asymptotically stable if and only if the scalar function (47) is devoid of zeros in $\operatorname{Re} s \geq 0$.

Proof. If the hidden modes of controller, plant and feedback sensor are all asymptotically stable, the three polynomials $h_c(s)$, $h_p(s)$, $h_f(s)$ are strict Hurwitz*. Now use theorem 2 and (47a), Q. E. D.

We are now in a position to describe a truly significant multivariable extension of the classical Nyquist criterion for single-input, single-output feedback systems.

Theorem 4. Let the controller, plant and feedback sensor be asymptotically stable and define

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* A polynomial is strict Hurwitz if all its zeros lie in $\operatorname{Re} s < 0$.

$$G(s) = F(s)P(s)C(s) \tag{48}$$

to be the GFCA $r \times r$ open-loop gain matrix [15]. Let $\Gamma_i(s)$ equal the sum of all principal minors of G(s) formed with i rows and columns, $(i = 1, 2, \ldots, r)$, and put

$$\Gamma(s) = \sum_{i=1}^{r} \Gamma_i(s) . \tag{49}$$

Then, under assumptions A_1, A_2, GFCA is asymptotically stable if and only if the complex plot of $\Gamma(j\omega)$ as ω varies from $-\infty$ to $+\infty$ does not enclose the point $s = -1$ in the clockwise direction.

Proof. Since controller, plant and feedback sensor are asymptotically stable, the three polynomials $\Delta_c(s)$, $\Delta_p(s)$, $\Delta_f(s)$ are strict Hurwitz. Thus, from theorem 2, GFCA is asymptotically stable if and only if

$$\det[1_n + P(s)C(s)F(s)] = \det[1_r + F(s)P(s)C(s)]$$

is free of zeros in $\mathrm{Re}\, s \geq 0$. That is, if and only if $\det[1_r + G(s)]$ has no zeros in $\mathrm{Re}\, s \geq 0$. However, from a well known determinantal expansion,

$$\det[1_r + G(s)] = 1 + \Gamma(s) . \tag{50}$$

Clearly, because of assumption A_1 and the strict Hurwitz character of $\Delta_c(s)$, $\Delta_p(s)$, $\Delta_f(s)$, the three transfer matrices C(s), P(s), F(s) are analytic in $\mathrm{Re}\, s \geq 0$, $s = \infty$ included. Moreover,

$$1 + \Gamma(\pm j\infty) = \det(1_r + J_f J_p J_c) = \det[1_r + F(\infty)P(\infty)C(\infty)] \neq 0$$

and the standard Nyquist argument [15] is applicable to $\Gamma(s)$, Q. E. D.*

Corollary 1. Let C(s), P(s) and F(s) be analytic in $\mathrm{Re}\, s \geq 0$. Then, under assumptions A_1-A_3, GFCA is asymptotically stable if and only if the complex plot of $\Gamma(j\omega)$ as ω varies from $-\infty$ to $+\infty$ does not enclose the point $s = -1$ in the clockwise direction.

Proof. Under A_1-A_3, the analyticity of C(s), P(s) and F(s) in $\mathrm{Re}\, s \geq 0$ implies the asymptotic stability of controller, plant and feedback sensor, respectively. The rest now follows from theorem 4, Q. E. D.

---

* Of course, if for some ω, $-\infty < \omega < \infty$, $\Gamma(j\omega) = -1$, GFCA is not asymptotically stable.

If GFCA is asymptotically stable T(s) is certainly analytic in Re $s \geq 0$ (Eq. (28a)) but the converse is not necessarily true*. Nevertheless, in one special but very practical situation this converse is valid.

Corollary 2. Suppose controller, plant and feedback sensor are individually asymptotically stable and assumptions A_1, A_2 attain. Then GFCA is asymptotically stable if and only if T(s) is analytic in Re $s \geq 0$.

Proof. The "only if" part is obvious. From the assumptions, $\Delta_c(s)$, $\Delta_p(s)$ and $\Delta_f(s)$ are strict Hurwitz whence, C(s), P(s) and F(s) are analytic in Re $s \geq 0$. Suppose therefore that T(s) is analytic in Re $s \geq 0$. Then

$$T(s)F(s) = 1_n - [1_n + P(s)C(s)F(s)]^{-1}$$

is also analytic in Re $s \geq 0$ which implies $\det[1_n + P(s)C(s)F(s)] \neq 0$, Re $s \geq 0$, and GFCA is asymptotically stable (theorem 4), Q. E. D.

Under the conditions of theorem 4 an experimental procedure for measuring $G(j\omega)$ is readily described. Referring to Fig. 2 observe the break in the system between the comparator and controller. With the input $\underset{\sim}{u}(t)$ shorted apply to the r terminals

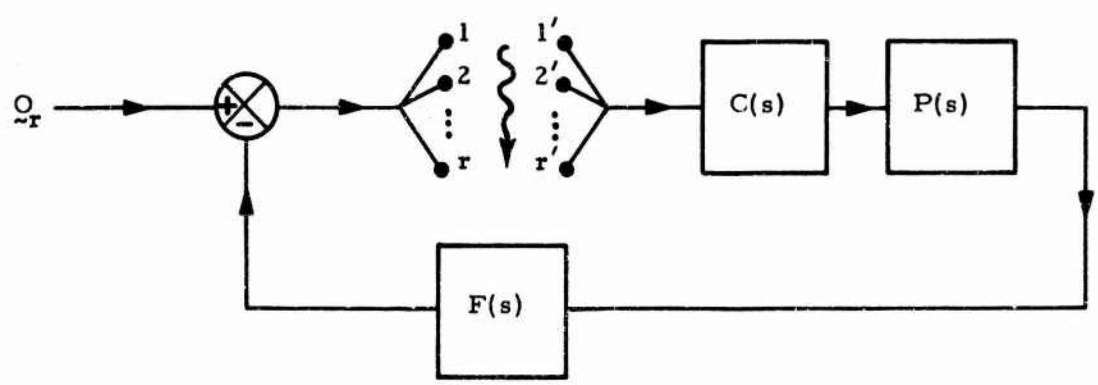

Fig. 2. Measurement scheme for $G(j\omega)$, the open-loop gain matrix.

1', 2', . . . , r' the r steady state drivers $e^{j\omega t}, 0, \ldots, 0$, respectively. Measure the r steady state responses** at the corresponding points 1, 2, . . . , r and amalgamate them into a column-vector $\underset{\sim}{a}_1(j\omega)e^{j\omega t}$. Setting

* A counterexample follows shortly.

** Because of the assumptions these steady-state responses exist.

$$\underset{\sim}{v}_1(j\omega) = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix},$$

we have

$$\underset{\sim}{a}_1(j\omega) = -G(j\omega)\underset{\sim}{v}_1(j\omega) .$$

Similarly, with terminal k′ excited by $e^{j\omega t}$ and the others shorted, let $\underset{\sim}{a}_k(j\omega)e^{j\omega t}$ denote the measured column-vector steady-state response at points 1, 2, . . . , r and put*

$$\underset{\sim}{v}_k(j\omega) = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 1 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \quad (k = 1, 2, \ldots, r) .$$

Obviously if $R(j\omega) = [\underset{\sim}{a}_1(j\omega), \underset{\sim}{a}_2(j\omega), \ldots, \underset{\sim}{a}_r(j\omega)]$,

$$G(j\omega) = -R(j\omega) \tag{51}$$

and the measurement of $G(j\omega)$ has been accomplished.** In the practical applications $C(s)$, $P(s)$, $F(s)$ and therefore $\Gamma(s)$ are functions of one or several parameters and it becomes necessary to study the complex plot of $\Gamma(j\omega)$ over the range of these parameters and it is perhaps not out of place here to mention that some interesting investigations along these lines have already been carried out by Curtiss [15].

A specialization of theorem 3 to the single-input, single-output case may not be without interest. For this situation $r = n = 1$, $F(s)$ is 1×1 (a scalar), $C(s)$ is $m \times 1$, $P(s)$ is $1 \times m$ and $P(s)C(s)$ is 1×1 (a scalar). Let

$$C(s) = \frac{\underset{\sim}{a}_c(s)}{g_c(s)} , \tag{52}$$

---

* $\underset{\sim}{v}_k$ has a 1 in the k^{th} row and zeros everywhere else.

** Transmission from the primed to the unprimed side of the break is governed by the transfer matrix $-F(s)P(s)C(s) = R(s)$ which is sometimes called the <u>return-ratio</u> matrix.

$$P(s) = \frac{\underset{\sim}{a}_p'(s)}{g_p(s)}, \tag{53}$$

$$\theta(s) = \underset{\sim}{a}_p'(s)\underset{\sim}{a}_c(s)$$

and

$$F(s) = \frac{a_f(s)}{g_f(s)} \tag{54}$$

where $\underset{\sim}{a}_c(s)$, $\underset{\sim}{a}_p(s)$ are two $m \times 1$ polynomial vectors and $g_c(s)$, $g_p(s)$, $g_f(s)$, $a_f(s)$, $\theta(s)$, are five scalar polynomials. Subject to the understanding that corresponding numerators and denominators are relatively prime, assumptions A_1-A_3 are satisfied if and only if

1. no entry in $\underset{\sim}{a}_c(s)$ has degree exceeding degree $g_c(s)$, no entry in $\underset{\sim}{a}_p(s)$ has degree exceeding $g_p(s)$ and no entry in $a_f(s)$ has degree exceeding degree $g_f(s)$.
2. $\lim_{s \to \infty} \frac{\theta(s)\, a_f(s)}{g_p(s)g_c(s)g_f(s)} \neq -1$

   and
3. $g_{c,p,f}(s) = \Delta_{c,p,f}(s)$.

Theorem 3, Corollary 2. A single-input, single-output GFCA satisfying assumptions A_1-A_3 is asymptotically stable if and only if the polynomial

$$g_p(s)g_c(s)g_f(s) + \theta(s)a_f(s) \tag{55}$$

is strict Hurwitz.

Proof. Theorem 3 and the equalities

$$g_{c,p,f}(s) = \Delta_{c,p,f}(s), \quad \text{Q. E. D.}$$

Observe that the zeros of (55) do not coincide with those of

$$1 + \frac{\theta(s)a_f(s)}{g_c(s)g_p(s)g_f(s)} \tag{56}$$

unless $\theta(s)a_f(s)$ is relatively prime to $g_c(s)g_p(s)g_f(s)$. Thus stability cannot always be checked by simply locating the zeros of $1 + P(s)C(s)F(s)$ or the poles of

$$T(s) = \frac{P(s)C(s)}{1+P(s)C(s)F(s)} .$$

As an example suppose $\varphi(s)$ is a strict Hurwitz polynomial and consider the single-input, single-output arrangement of Fig. 3. To apply (52) we must examine the

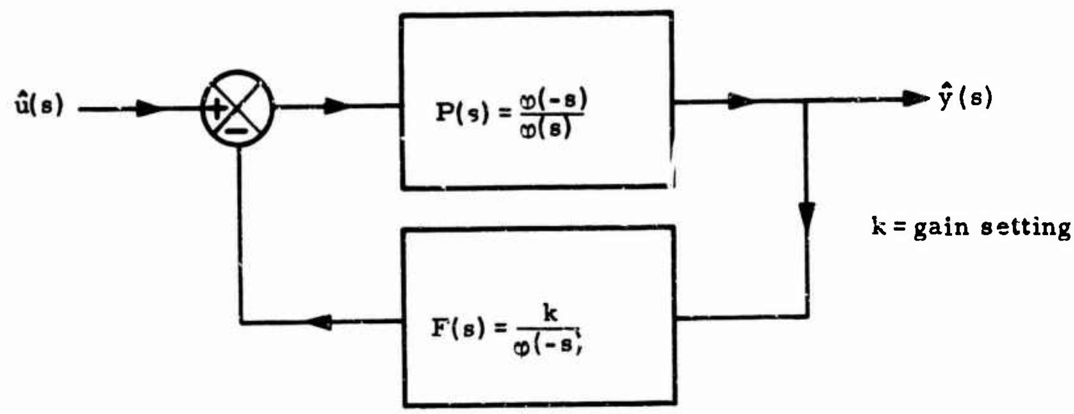

Fig. 3. Externally stable but internally unstable feedback arrangement.

roots of the equation

$$\varphi(s)\varphi(-s) + k\varphi(-s) = 0 ;$$

or

$$\varphi(-s) = 0 ; \ \varphi(s) + k = 0 .$$

Obviously, all the zeros of $\varphi(-s)$ are in Re $s > 0$ and the feedback circuit of Fig. 2 has modes which blow up exponentially as $t \to \infty$ irrespective of the gain setting k. However

$$1 + P(s)F(s) = \frac{\varphi(s) + k}{\varphi(s)}$$

and

$$T(s) = \frac{k}{\varphi(s) + k} .$$

Hence, for sufficiently small $|k|$, $1 + P(s)F(s)$ has no zeros and $T(s)$ no poles in Re $s > 0$.

To illustrate the use of theorem 3 we choose $r = n = 2$, unity controller ($C = 1_2$),

$$P(s) = \begin{bmatrix} k \cdot \frac{s-1}{s-2} & \frac{1}{s+2} \\ 2 & 1 \end{bmatrix} , \ k \text{ real,}$$

and

$$F(s) = \left[\begin{array}{c|c} \frac{1+2s}{1+s} & \frac{3s+1}{2(1+s)} \\ \hline 1 & 1 \end{array}\right] .$$

Obviously*,

$$\psi_p(s) = (s-2)(s+2) = s^2-4 ,$$

$$\psi_f(s) = s + 1$$

and

$$\det [1_2 + P(\infty)F(\infty)] = \det \left[\begin{array}{c|c} 2k+1 & \frac{3}{2}k \\ \hline 5 & 5 \end{array}\right] = \frac{5}{2}(k+2) .$$

$$\therefore \ \eta = \det [1_2 + P(\infty)F(\infty)] \neq 0 \quad \text{if } k \neq -2 .$$

By straightforward calculation,

$$\eta \Delta_o(s) = \det [1_2 + P(s)F(s)]\psi_p(s)\psi_f(s) = (s+2)(3s+2)[(2+k)s-(4+k)].$$

The zeros of $\Delta_o(s)$ are

$$s = -2, \ -2/3, \ \frac{4+k}{2+k} .$$

The latter lies in Re $s < 0$ only if

$$-4 < k < -2 \tag{57}$$

which is therefore the stability range for the parameter k**.

3. Controllability and Observability of GFCA.

As we have already seen the asymptotic stability of GFCA does not always follow from the analyticity of $T(s)$ in Re $s \geq 0$ even under assumption A_3. Generally speaking, the reason for this failure is due to the fact that complete controllability and observability of controller, plant and feedback sensor do not always guarantee that of GFCA when viewed as a system with input $\underset{\sim}{u}(t)$ and output $\underset{\sim}{y}(t)$. Thus it is possible for GFCA to

---

* det $F(s) \equiv 1/2$.

** It has been assumed in accordance with the requirements of theorem 3, that plant and feedback sensor are completely controllable and observable.

possess unstable modes of $\underset{\sim}{x}_o(t)$ which are either uncontrollable, unobservable or both and are not detectable in T(s).* To examine this matter in depth let us rephrase assumption A_3 in terms of McMillan degrees.**

Assumption 3 (A_3).

$$\nu_c = \delta(C) ,$$

$$\nu_p = \delta(P) , \qquad (58)$$

$$\nu_f = \delta(F) .$$

In view of this assumption, GFCA constitutes a realization of T(s) employing exactly $\delta(C) + \delta(P) + \delta(F)$ energy storage elements and since a minimum of $\delta(T)$ energy storage elements is required in any realization of T(s) [4, 5, 6, 11] it must follow that

$$\delta(T) \leq \delta(C) + \delta(P) + \delta(F) . \qquad (59)$$

A direct proof of (59) is easily inferred from the identity

$$\begin{bmatrix} 1_n & PC \\ -F & 1_r \end{bmatrix}^{-1} = \begin{bmatrix} (1_n + PCF)^{-1} & -T \\ -F(1_n + PCF)^{-1} & (1_r + FPC)^{-1} \end{bmatrix} \qquad (60)$$

Consequently,*** taking degrees of both sides,

$$\delta(T) \leq \delta(PC) + \delta(F) \leq \delta(C) + \delta(P) + \delta(F), \text{ Q. E. D.}$$

---

* T(s) is determined solely by the completely controllable and observable part of GFCA [5, 6, 7].

**The McMillan degree of a rational matrix A(s) is denoted by $\delta(A)$ or $\delta[A(s)]$. The reader is assumed to have a good understanding of this important concept and a study of one of the accessible treatments in Refs. 4, 8-11 is strongly advised.

***For any two rational matrices A(s) and B(s), $\delta(A) = \delta(A^{-1})$, $\delta(AB) \leq \delta(A) + \delta(B)$, $\delta(A \dotplus B) = \delta(A) + \delta(B)$ and $\delta(K) = 0$ if and only if K is a constant matrix. Furthermore, the degree of a matrix is greater or equal to the degree of any one of its submatrices.

It is not difficult to show that subject to assumptions A_1-A_3 the condition

$$\delta(FPC) = \delta(PCF) = \delta(C) + \delta(P) + \delta(F) \quad (61)$$

is sufficient for the complete controllability and observability of GFCA when viewed as a system with input $\underset{\sim}{u}(t)$ and output $\underset{\sim}{y}(t)$.*

<u>Proof.</u> Considered as a system with input $\underset{\sim}{u}(t)$ and output $\underset{\sim}{e}(t)$, GFCA is described by the transfer matrix (16):

$$E(s) = [1_r + F(s)P(s)C(s)]^{-1} .$$

Since $\delta(E) = \delta(FPC)$,

$$\delta(FPC) = \delta(C) + \delta(P) + \delta(F) \quad (62)$$

certainly suffices to guarantee the controllability of GFCA with input $\underset{\sim}{u}(t)$ and output $\underset{\sim}{y}(t)$. By duality [6], the <u>observability</u> of GFCA with input $\underset{\sim}{u}(t)$ and output $\underset{\sim}{y}(t)$ is equivalent to the <u>controllability</u> of the <u>dual</u> dynamical system

$$\dot{\underset{\sim}{x}}_d(t) = F_o' \underset{\sim}{x}_d(t) + H_o' \underset{\sim}{u}_d(t) , \quad (63)$$

$$\underset{\sim}{y}_d(t) = G_o' \underset{\sim}{x}_d(t) + J_o' \underset{\sim}{u}_d(t) . \quad (64)$$

The reader should have no trouble in convincing himself that the dual GFCA, $(GFCA)_d$ say, is correctly represented in Fig. 4 provided

$$C'(s) \rightarrow (F_c', H_c', G_c', J_c') ,$$

$$P'(s) \rightarrow (F_p', H_p', G_p', J_p') ,$$

$$F'(s) \rightarrow (F_f', H_f', G_f', J_f')$$

where

$$C(s) \rightarrow (F_c, G_c, H_c, J_c) ,$$

$$P(s) \rightarrow (F_p, G_p, H_p, J_p) ,$$

$$F(p) \rightarrow (F_f, G_f, H_f, J_f) .$$

---

* This is lemma 1 of Chen's paper [2]. In a private communication to the author Chen has outlined a different proof based on some earlier results obtained in Ref. 1. Clearly, as far as controllability is concerned the choice of <u>output</u> point is immaterial.

Thus $C'(s), P'(s), F'(s)$ are all rea·ized minimally and A_3 is therefore also satisfied by $(GFCA)_d$.

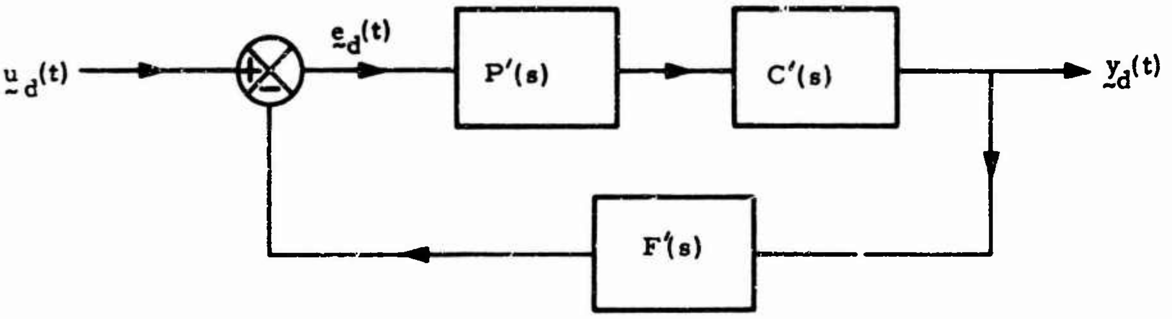

Fig. 4. A schematic of $(GFCA)_d$, the generalized feedl k control arrangement dual to that of Fig. 1.

Viewed as a system with input $\underset{\sim}{u}_d(t)$ and output $\underset{\sim}{e}_d(t)$, $(GFCA)_d$ has the transfer matrix description

$$E_d(s) = [1_n + F'(s)C'(s)P'(s)]^{-1}. \tag{65}$$

Hence controllability of $(GFCA)_d$ and concomitantly, the observability of GFCA with output $\underset{\sim}{y}(t)$ is assured if

$$\delta(F'C'P') = \delta(C') + \delta(P') + \delta(F');$$

or, since a matrix and its transpose have the same degree, if

$$\delta(PCF) = \delta(C) + \delta(P) + \delta(F) . \tag{66}$$

Taking (62) and (66) together we get (61), Q. E. D.

Instead of working with the criterion (61) which is rather intractable anyway it appears advantageous at this stage to introduce four realistic and simplifying restrictions from the very outset.

R_1. The number of plant inputs m equals or exceeds the number of plant outputs n; i. e., $m \geq n$.

That R_1 is physically correct is obvious and that it is satisfied in almost all practical situations is attested to by the abundance of examples described in the literature*.

R_2. The number of system input variables equals the number of output variables n; i. e., $\underset{\sim}{u}(t)$ and $\underset{\sim}{y}(t)$ have the same dimension and $r = n$.

* See, for example, the interesting book by Meerov [12].

Since $\underset{\sim}{u}(t)$ serves as the reference vector or "setpoint" for the plant output $\underset{\sim}{y}(t)$, the plausibility of R_2 should be apparent.

R_3. The plant transfer matrix P(s) is non-degenerate. That is, P(s) has normal rank equal to the number of its rows*:

$$\text{normal rank } P = n \; .$$

If R_3 is false there exists at least one polynomial dependence between the rows of P(s). This implies in turn the existence of a fixed nontrivial set of polynomials $a_1(s), a_2(s), \ldots, a_n(s)$, independent of $\hat{\underset{\sim}{u}}(s)$, such that

$$a_1(s)\hat{y}_1(s) + a_2(s)\hat{y}_2(s) + \ldots + a_n(s)\hat{y}_n(s) \equiv 0 \; , \tag{67}$$

where the $\hat{y}_i(s)$ are the components of $\hat{\underset{\sim}{y}}(s)$, $(i = 1, 2, \ldots, n)$. Hence, at least one transformed output variable may be eliminated in terms of the others and therefore becomes superfluous.

R_4. The $m \times n$ controller transfer matrix C(s) is chosen so that the square $n \times n$ matrix P(s)C(s) has normal rank n; i.e.,

$$\det [P(s)C(s)] \not\equiv 0 \; . \tag{68}$$

Choosing C(s) so that $\det[P(s)C(s)] \equiv 0$ leads to an identically singular T(s):

$$\det T(s) \equiv 0 \; . \tag{69}$$

Again this implies a polynomial dependence of the type (67) on the n transformed output variables and GFCA is deficient from the point of view of output controllability. Stated differently, under (69) there exists a very rich class of reference inputs (including even some of the "step" type) which GFCA cannot follow. These practical considerations coupled with the relatively simple theory that results, is, in our opinion, ample justification for pursuing the consequences of a design procedure modeled around R_1-R_4.

Definition 1. A GFCA satisfying assumptions A_1-A_3 and strictions R_1-R_4 is said to be standard.

Theorem 5. A standard GFCA is completely controllable and observable if and only if

---

* A rational matrix A(s) is said to have normal rank equal to k if 1) at least one $k \times k$ minor does not vanish identically but 2) all minors of orders larger than k do vanish identically.

1. $\delta[(PC)^{-1}+F]=\delta[(PC)^{-1}]+\delta(F)$ and
2. $\delta(PC)=\delta(P)+\delta(C)$.

Proof. Invoking (68),

$$T=PC(1_n+FPC)^{-1}=[(PC)^{-1}+F]^{-1} .$$

$$\therefore\ \delta(T)=\delta[(PC)^{-1}+F]\leq\delta[(PC)^{-1}]+\delta(F)=\delta(PC)+\delta(F)\leq\delta(C)+\delta(P)+\delta(F) . \tag{70}$$

Thus, to achieve equality in (70) between the extreme left and right members it is necessary and sufficient to have both

$$\delta[(PC)^{-1}+F]=\delta[(PC)^{-1}]+\delta(F)$$

and

$$\delta(PC)=\delta(P)+\delta(C),\quad \text{Q. E. D.}$$

Corollary 1. A sufficient set of conditions for a standard GFCA to be completely controllable and observable are

S_1. $(PC)^{-1}$ and F(s) have no common poles.
S_2. $\delta(PC)=\delta(P)+\delta(C)$.

In the single-input, single-output case ($r=n=1$) S_1 and S_2 are also necessary.

Proof. It is a property of McMillan degree that $\delta(A+B)=\delta(A)+\delta(B)$ whenever A(s) and B(s) are devoid of common poles. Now let $r=n=1$. Then C(s) is $m\times 1$, P(s) is $1\times m$, PC is 1×1, F(s) is 1×1. Thus $(PC)^{-1}$ and F(s) are two scalar rational functions a(s) and b(s), say. If a(s) and b(s) admit $s=s_o$ as a common pole of respective orders $r_1\geq 1$ and $r_2\geq 1$, it is clear that $s=s_o$, as a pole of $a(s)+b(s)$, cannot have order exceeding

$$\max(r_1,r_2)<r_1+r_2$$

and $\delta(a+b)=\delta(a)+\delta(b)$ is impossible, Q. E. D.*

Corollary 2. A standard GFCA with unity controller ($C(s)=1_n$; $n=m=r$) is completely controllable and observable if and only if

$$\delta(P^{-1}+F)=\delta(P^{-1})+\delta(F) . \tag{71}$$

---

* And so we have the classical rule forbidding the cancellation of any zero of the net feedforward transfer function by a pole of the net feedbackward transfer function. Incidentally, this shows that (61) is not even necessary in the single-input, single-output case!

Moreover, to achieve equality in (72) it suffices that $P^{-1}(s)$ and $F(s)$ be devoid of common poles.

Despite the extensive analysis presented above the constraint of complete controllability and observability is not necessarily an important nor even desirable attribute of a GFCA! This statement is undoubtedly disconcerting and puzzling but consider that the main objective of GFCA is not to control or observe the internal state $\underset{\sim}{x}_o(t)$ but to drive the plant output $\underset{\sim}{y}(t)$ to the "set point" $\underset{\sim}{u}(t)$. And in fact it seems safe to say that most successful multivariable control systems are not designed under this constraint which, when superimposed on the other requirements of steady-state performance, transient response and asymptotic stability, presents insurmountable algebraic difficulties. What is of paramount importance is that GFCA be asymptotically stable. As we know from theorem 4, corollary 2, there are many practical situations in which this is ascertainable purely from an examination of the analyticity properties of $T(s)$ in $\text{Re}\, s \geq 0$ even if GFCA is not completely controllable and observable. In any case theorems 2-4 and their accompanying corollaries provide an adequate apparatus for checking the stability of GFCA irrespective of other considerations. Also, as is well known, the use of redundant energy storage elements may improve both the sensitivity and reliability of GFCA, permit the use of more realistic topologies and even facilitate the synthesis problem of minimizing one or several cost indices. Nevertheless the effort spent in this section is not wasted because it has revealed insights which not only deepen our understanding but also provide the conceptual background essential to the development of perspective.

5. <u>The Problem of Plant Equalization</u>. The given datum is the $n \times m$ transfer matrix $P(s)$ of a dynamical plant with asymptotically stable hidden modes*.

<u>Definition 2.</u> The $n \times r$ matrix $T(s)$ is said to be <u>realizable</u> for $P(s)$ if for some choice of <u>asymptotically stable dynamical</u> controller and feedback sensor the resultant GFCA of Fig. 1 incorporating the given plant is a <u>dynamical asymptotically stable</u> system possessing the prescribed input-output transfer description $T(s)$.

Analytically the question of realizability reduces to the following: Given a $P(s)$ which is finite at $s = \infty$ and a $T(s)$, find, if possible, two real rational matrices

* Thus, the stability properties of the plant are completely summarized by $\psi_p(s)$, the characteristic denominator of $P(s)$ (See theorem 3, corollary 1).

C(s) and F(s) both analytic in $\mathrm{Re}\, s \geq 0$, $s = \infty$ included, such that*

1. $T(s) = [1_n + P(s)C(s)F(s)]^{-1} P(s)C(s)$;
2. $\det [1_n + P(\infty)C(\infty)F(\infty)] \neq 0$ and
3. $\det [1_n + P(s)C(s)F(s)] \psi_p(s) \neq 0$, $\mathrm{Re}\, s \geq 0$. (72)

---

* It is intended that C(s) and F(s) shall be realized minimally; i.e., as completely controllable and observable dynamical systems. Thus $\Delta_c(s) = \psi_c(s)$ and $\Delta_f(s) = \psi_f(s)$ are strict Hurwitz and 3 reduces to the stability criterion of theorem 2. Part II (in preparation) presents a complete solution for the standard GFCA together with a full discussion of its theoretical and practical implications.

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UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Polytechnic Institute of Brooklyn 333 Jay Street Brooklyn, NY 11201	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP

3. REPORT TITLE

MODERN CLASSICAL MULTIVARIABLE CONTROL THEORY PART I

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Phase Report

5. AUTHOR(S) (First name, middle initial, last name)

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6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
June 1970	25	15

8a. CONTRACT OR GRANT NO. F30602-69-C-0053 b. PROJECT NO. 8505 c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) PIBEP-70-054 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) RADC-TR-70-98

10. DISTRIBUTION STATEMENT

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11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Rome Air Development Center Griffiss AFB, NY 13440

13. ABSTRACT

This report establishes rigorous stability criteria in terms of transfer matrices and in addition undertakes a careful and critical examination of the concepts of complete controllability and observability as applied to the closed loop. In particular it is emphasized that the latter properties are neither necessary nor desirable attributes of a practical multivariable feedback system.

DD FORM 1 NOV 65 1473

UNCLASSIFIED

Security Classification

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14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Control System Feedback Stability Observability Controllability Nyquist Plot						

UNCLASSIFIED
Security Classification