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Allocation of Importance: An Axiom System

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Abstract

The proposed axiom system yields utility functions for different orderings of multi-dimensional objects, functions that, for each ordering, are expressed as sums of real-valued functions, called <u>constituent functions</u>, weighted by their corresponding <u>importance measures</u>. A constituent function is a utility function on each dimension that is transformed into the fixed interval [0, 1]. Importance measures are positive real numbers that add to one. It is shown that if the axioms hold, the constituent functions are constant for all different orderings, while only the importance measures dictate the change of ordering. The axioms are stated in terms of a person's choice behavior under various circumstances.

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ALLOCATION OF IMPORTANCE: AN AXIOM SYSTEM[1]

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1. Introduction

This article presents a model about people's choices among multi-dimensional objects (person, events, and things) for which all of the relevant dimensions are known. Usually, consideration of such decisions rests solely on the person's utility function for each of the dimensions. However, such treatment overlooks a second fundamental factor that must accompany utility; the person must decide how much each dimension will be weighted in the final determination of his choice. The weight judgment for each dimension will be called its importance measure.

Any theory of this type must face the fact that choice orderings over the same objects frequently change when the goal of the decision situation changes. For example, a college senior evaluating job offers might order them one way when geographical proximity to his fiancee was desirable and another way after the engagement was broken and proximity was undesirable. In most discussions of utility the functions are defined for a particular ordering of objects. Therefore, even a minor change in the choice ordering necessitates reconstruction of the entire utility function. This assumed stability of utility functions is unreasonable from a psychological viewpoint, and the necessity to reconstruct utility functions for all possible circumstances is unreasonable from a practical viewpoint. The present model provides a solution to these problems by assuming that changes of choice ordering are due solely to changes in the distribution of importance measures on the dimensions that define the objects.

The basic structure of the model is an extension of additive utility theory (Debreu, 1960; Luce and Tukey, 1964); utility functions are expressed as a sum of real-valued functions each weighted by their corresponding importance measures. These real-valued functions, called constituent functions, are assumed to be specific to each of the dimensions of the objects. Thus, each object would retain its particular dimension values from one decision goal to another, but the ordering of the objects may well change if the importance measures change in response to changes in goals.

The model consists of an axiom system and the theorems derived from the axioms. The former assume specific relations among different orderings of objects, the later yield utility functions in the form of sums of constituent functions weighted by importance measures where only the latter will be assumed to change.

The weighted-sum form has been adopted in a number of other psychological theories (e.g., Anderson, 1964, 1967, 1970; Brunswik, 1943; Hammond, 1965; Hursch, Hammond, and Hursch, 1964). In these theories response measures were assumed to be described as weighted sums of cues, or stimuli, where the weights were called "cue-utilization coefficients" or "stimulus weights." However, these theories differ fundamentally from the present model. The former do not deal with changes in orderings, while the latter prescribes exactly what these changes should be under various circumstances. Moreover, the weights in former are merely parameters that are estimated by statistical analysis of the data, while in the latter importance measures are a subject's independent, conscious, judgments.

Expression of utility function as weighted sums of constituent functions has been proposed by a number of authors (e.g., Churchman, Ackoff, and Arnoff, 1957; Fishburn, 1965). However, they did not deal with different utility functions at different times for a single person; thus it is not possible to use their constituent functions for a variety of decision goals and to account for different orderings by the same person.

Recently, Micko and Fischer (1970) presented a scaling model which is based upon a notion similar to that presented in the present model. They assumed that a person has a multidimensional psychological space in which a similarity judgment between a pair of objects is represented as a distance between the objects. The distance function is assumed to be a weighted sum of the differences along each dimension defining the objects. Weights are called "importance scores" or "attention scores" and are assumed to be different under different circumstances and from one person to another. Moreover, Micko and Fischer elaborated on the assumption that only the differences in importance distributions yield differences in the observed orderings of objects. However, the basic postulates presented by Micko and Fischer were stated in terms of analytic-geometric structures of psychological space, and their concern was to investigate the possibility of scaling repeated response data. By contrast, the axioms of the present model are stated in terms of the person's choice behavior, and the concern is to prescribe the consistent relations between the allocations of importance and subsequent choices among objects.

The organization of the presentation is as follows: Section two contains the model's assumptions without getting into the mathematical details. The axiom system is in section three, and at the end of that section the theorem that justifies the expression of utility function in the form of a weighted sum is presented. Section four contains a proof of this theorem.

2. Assumptions

The model deals with m different orderings, $\leqslant_k$, $k = 1, \ldots, m$, of the set of n-dimensional objects, $X = \prod_{i=1}^{n} X_i$. Here each object in X is an n-tuple of the form $(x_1, \ldots, x_n)$, where x_i is a component of the set X_i for dimension i. For every a, $b \in X$, the relation $a \leqslant_k b$ may be interpreted as "a is not preferred to b," or equivalently, "b is preferred to, or indifferent to, a." These orderings are considered to be given by a single subject under m different circumstances, i.e., with different goals.

A utility function $U_k(x)$ for any ordering $\leqslant_k$ is defined such that, for every x, $y \in X$,

$$x \leqslant_k y \text{ if and only if } U_k(x) \leq U_k(y).$$

The purpose of the model is to show the conditions required for the utility function $U_k(x)$ to be expressed as

$$U_k(x) = w_k^{(1)} u^{(1)}(x_1) + \ldots\ldots + w_k^{(n)} u^{(n)}(x_n), \qquad (1)$$

where $x = (x_1, \ldots, x_n) \in X = \prod_{i=1}^{n} X_i$, $w_k^{(i)} > 0$ for all i, $\sum_{j=1}^{n} w_k^{(j)} = 1$, and $0 \leq u^{(i)}(x_i) \leq 1$ for all i.

Here, $w_k^{(i)}$, the importance measures, are assumed to be positive real numbers that add to one. The $u^{(i)}(x_i)$, the constituent functions, are assumed to be real-valued functions ranging from zero to one. Note that the subscript k does not appear in the notation for constituent functions; the constituent functions are assumed to be common to all the different orderings, $\precsim_k$, $k = 1, \ldots, m$.

In what follows we shall examine some of the major assumptions required for representing utility functions in the form of Eq. 1. These assumptions will be included later in he axiom system.

Additive Utility

The discussion starts with the assumption that a utility function $U_k(x)$ for every $\precsim_k$ is additive. That is, $U_k(x)$ is expressed as

$$U_k(x) = f_k^{(1)}(x_1) + \ldots\ldots + f_k^{(n)}(x_n), \tag{2}$$

where $f_k^{(i)}(x_i)$ is a real-valued function defined over the i^{th} dimension under the ordering $\precsim_k$. The utility function $U_k(x)$ thus defined is unique up to an increasing linear transformation (see, e.g., Debreu, 1960; Luce and Tukey, 1964).

Invariance of Ordering within Dimension

Consider a pair of objects, a and b of X, such that

$$a = (x_1, \ldots, x_{j-1}, a_j, x_{j+1}, \ldots, x_n), \quad (3)$$

$$b = (x_1, \ldots, x_{j-1}, b_j, x_{j+1}, \ldots, x_n). \quad (4)$$

If Eq. 1 holds, then

$$U_k(a) - U_k(b) = w_k^{(j)}[u^{(j)}(a_j) - u^{(j)}(b_j)]. \quad (5)$$

In other words, in order for Eq. 1 to hold, it is necessary that Eq. 5 hold for the a and b as defined in Eqs. 3 and 4. Moreover, in Eq. 5 the $w_k^{(j)}$ must be positive and $[u^{(j)}(a_j) - u^{(j)}(b_j)]$ must be constant for all k. Therefore, it is implied in Eq. 5 that, if a $\lesssim_k$ b (i.e., $U_k(a) \leq U_k(b)$) for some k, the same ordering must hold for any other k. That is, the ordering of components within each dimension must remain the same under any circumstances.

It should be noted that the last requirement is asserted only for those objects that differ among themselves on only one common dimension (see Eqs, 3 and 4). For a set of arbitrarily chosen objects, the ordering may change under different circumstances.

<u>Bounded Utilities</u>

Suppose that the set X_i of components for dimension i has a least upper bound, b_i^o, and a greatest lower bound, a_i^o; i.e., $b_i^o = \mathrm{Sup}\{X_i\}$ and $a_i^o = \mathrm{Inf}\{X_i\}$. Such b_i^o and a_i^o will be uniquely determined for every i, because, from the previous assumption, the ordering of conponents within each dimension remains the same under any circumstances.

Using the b_i^o and a_i^o, we re-write Eq. 2 in a form similar to Eq. 1 by the following procedure (Eqs. 6 - 8):

Define, for every $i \in \{1, \ldots, n\}$,

$$\phi_k^{(i)}(x_i) = \frac{f_k^{(i)}(x_i) - f_k^{(i)}(a_i^o)}{f_k^{(i)}(b_i^o) - f_k^{(i)}(a_i^o)}, \tag{6}$$

therefore,

$$f_k^{(i)}(x_i) = [f_k^{(i)}(b_i^o) - f_k^{(i)}(a_i^o)]\, \phi_k^{(i)}(x_i) + f_k^{(i)}(a_i^o).$$

Substituting into Eq. 2,

$$U_k(x) = \sum_{i=1}^{n} [f_k^{(i)}(b_i^o) - f_k^{(i)}(a_i^o)]\, \phi_k^{(i)}(x_i) + \sum_{i=1}^{n} f_k^{(i)}(a_i^o).$$

Define further,

$$w_k^{(i)} = \frac{f_k^{(i)}(b_i^o) - f_k^{(i)}(a_i^o)}{\sum_{j=1}^{n} [f_k^{(j)}(b_j^o) - f_k^{(j)}(a_j)]}. \tag{7}$$

Thus we obtain,

$$U_k(x) = \alpha\,[w_k^{(1)}\phi_k^{(1)}(x_1) + \ldots + w_k^{(n)}\phi_k^{(n)}(x_n)] + \beta,$$

where $\alpha = \sum_{j=1}^{n} [f_k^{(j)}(b_j^o) - f_k^{(j)}(a_j^o)]$ and $\beta = \sum_{j=1}^{n} f_k^{(j)}(a_j^o)$.
Since α is a positive constant and β is also a constant, the utility function $U_k(x)$ will preserve the ordering even if α and β are dropped from the equation; i.e.,

$$U_k(x) = w_k^{(1)}\phi_k^{(1)}(x_1) + \ldots\ldots + w_k^{(n)}\phi_k^{(n)}(x_n). \qquad (8)$$

Equation 8 is quite similar to Eq. 1. In fact, the $w_k^{(i)}$ in Eq. 8 satisfy all the requirements for the importance measures; i.e., $w_k^{(i)} > 0$ and $\sum_{j=1}^{n} w_k^{(j)} = 1$. Moreover, the $\phi_k^{(i)}(x_i)$ satisfy some of the requirements for constituent functions, $u^{(i)}(x_i)$; e.g., $0 \leq \phi_k^{(i)}(x_i) \leq 1$. However, the $\phi_k^{(i)}(x_i)$ cannot be called constituent functions because the $\phi_k^{(i)}(x_i)$ are dependent upon the choice of k, while the constituent functions are not. Therefore, the problem is to obtain equivalence between $\phi_k^{(i)}(x_i)$ and $\phi_{k^*}^{(i)}(x_i)$ for all different k, $k^* \in \{1, \ldots, m\}$.

<u>Invariance of Constituent Functions</u>

To obtain fixed constituent functions requires introduction of the notion of a quarternary relation D_k between two utility differences. That is, it is assumed that a person can tell whether the difference in utility between object x and object y is less than, or greater than, the difference between another pair of objects, z and w. If the former difference (including the sign) is less than the latter, the quarternary relation is denoted,

$$xy \; D_k \; zw, \text{ or equivalently, } wz \; D_k \; yx.$$

Sometimes the two differences might be of the same magnitude (including the sign), in which case,

$$xy \; I_k \; zw, \text{ or equivalently, } yx \; I_k \; wz$$

$$(\text{but not } yx \; I_k \; zw).$$

Now, to return to the assumption required for fixed constituent functions. Consider three objects, a, b, and c, that differ among themselves on only one arbitrarily chosen dimension j; i.e.,

$$a = (x_1, \ldots, x_{j-1}, a_j, x_{j+1}, \ldots, x_n),$$

$$b = (x_1, \ldots, x_{j-1}, b_j, x_{j+1}, \ldots, x_n),$$

$$c = (x_1, \ldots, x_{j-1}, c_j, x_{j+1}, \ldots, x_n).$$

Then assume that, for any k, $k^* \in \{1, \ldots, m\}$,

$$ab\ I_k\ bc \text{ if and only if } ab\ I_{k^*}\ bc.$$

In other words, it is assumed that a midpoint between any pair of components on the utility scale for a particular dimension remains to be the midpoint under any circumstances. This assumption, together with other minor assumptions, guarantees the invariance of the constituent functions. Thus, defining $\phi_k^{(i)}(x_i) = u^{(i)}(x_i)$, Eq. 8 becomes equivalent to Eq. 1.

<u>Utility Differences</u>

Recall that, in order to define the indifference relation I_k, it is presupposed that a person could make judgments about utility differences. Unfortunately, this presupposition cannot be justified sheerly on the basis of the existence of additive utility functions, becuase additive utilities are based upon binary relations $\precsim_k$ (e.g., Debreu, 1960), while utility differences are based upon quarternary relations D_k (e.g., Suppes and Winet, 1955) and these two have been treated separately. The model presented here, on the other hand, requires that the two kinds of orderings, D_k and $\precsim_k$, be related to each other. In order to link quarternary relations with binary relations, binary relations $\precsim_k$, k = 1, ..., m, are defined over a product of products, $Y = X \times X$, where $X = \prod_{i=1}^{n} X_i$. Additive utility functions are now assumed to exist for $\precsim_k$ over Y, instead of X.

An intuitive interpretation of $\precsim_k$ over $Y = X \times X$ is that a person always chooses among a number of pairs of objects.

Now, for a pair of objects, $a = (a_1, \ldots, a_n) \in X$ and $b = (b_1, \ldots, b_n) \in X$, let [a, b] be the combined set of the ordered components $(a_1, \ldots, a_n, b_1, \ldots, b_n)$; therefore, $[a, b] \in Y = X \times X$. Then define D_k and $\precsim_k$ over $X = \prod_{i=1}^{n} X_i$ such that, for every a, b, c, $d \in X$,

(1) ab D_k cd if and only if $[a, d] \precsim_k [c, b]$,

(2) $a \precsim_k b$ if and only if ab D_k bb, or equivalently, $[a, b] \precsim_k [b, b]$.

In the axiom system, these definitions and the related axioms will yield a utility function $U_k(x)$ such that, for every a, b, c, $d \in X$,

$a \precsim_k b$ if and only if $U_k(a) \leq U_k(b)$,

ab D_k cd if and only if $U_k(a) - U_k(b) \leq U_k(c) - U_k(d)$,

where $U_k(x)$ is expressed as in Eq. 1.

3. Axiom System

Let X_i, $i = 1, \ldots, n$, be sets, X be the Cartesian product $\prod_{i=1}^{n} X_i$, and Y be the Cartesian product $X \times X = \prod_{j=1}^{2n} Y_j$. For a pair of elements of X, $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$, let $[a, b]$ be the combined set of the ordered elements $(a_1, \ldots, a_n, b_1, \ldots, b_n)$. Here, it must be allowed that $a = b$; i.e., the choice of a must be independent of the choice of b. Therefore, $[a, b] \in Y$. An element of Y is also denoted $(y_1, \ldots, y_{2n})$. Let $\precsim_k$, $k = 1, \ldots, m$, be binary relations over Y.

<u>Assumption 1</u>. There is a real-valued function χ_k for every $\precsim_k$, $k \in \{1, \ldots, m\}$, on $Y = \prod_{j=1}^{2n} Y_j$ such that for all $s_y = (y_1, \ldots, y_{2n}) \in Y$ and $s_z = (z_1, \ldots, z_{2n}) \in Y$,

$$s_y \precsim_k s_z \quad \text{if and only if} \quad \chi_k(s_y) \leq \chi_k(s_z),$$

where $\chi_k(s_y) = \sum_{j=1}^{2n} \lambda_k^{(j)}(y_j)$ and $\chi_k(s_z) = \sum_{j=1}^{2n} \lambda_k^{(j)}(z_j)$. Moreover, if any other functions ψ_k and $\mu_k^{(j)}$ also satisfy these relations, they are related to the χ_k and $\lambda_k^{(j)}$ by increasing linear transformations; i.e.,

$$\psi_k = \zeta \cdot \chi_k + \xi, \text{ and } \quad \mu_k^{(j)} = \zeta \cdot \lambda_k^{(j)} + \xi_j,$$

where $\xi = \sum_{j=1}^{2n} \xi_j$ and $\zeta > 0$.

Different sets of axioms for Assumption 1 have been proposed by a number of authors (Debreu, 1960; Krantz, 1964; Luce, 1966; Luce and Tukey, 1964; Tversky, 1967). Any one of the sets of axioms can be substituted for Assumption 1.

Definition 1. For every a, b, c, d $\in X$, ab D_k cd if and only if $[a, d] \precsim_k [c, b]$.

Definition 2. For every a, b, c, d $\in X$, ab I_k cd if and only if ab D_k cd and cd D_k ab.

Definition 3. For every a, b $\in X$, a $\precsim_k$ b if and only if ab D_k bb, or equivalently, $[a, b] \precsim_k [b, b]$.

Axiom 1. For every a, b $\in X$, aa I_k bb.

Theorem 1. If Assumption 1 and Axiom 1 hold, there exists a real-valued function F_k for every $\precsim_k$, $k \in \{1, \ldots, m\}$, over $X = \prod_{i=1}^{n} X_i$ such that for all $a = (a_1, \ldots, a_n) \in X$ and $b = (b_1, \ldots, b_n) \in X$,

(1) a $\precsim_k$ b if and only if $F_k(a) \leq F_k(b)$,

(2) ab D_k cd if and only if $F_k(a) - F_k(b) \leq F_k(c) - F_k(d)$,

where $F_k(a) = \sum_{i=1}^{n} f_k^{(i)}(a_i)$ for all $a = (a_1, \ldots, a_n) \in X$.

Moreover, if any other functions G_k and $g_k^{(i)}$ also satisfy these relations, they are related to the F_k and $f_k^{(i)}$ by increasing linear transformations; i.e.,

$$G_k = \alpha \cdot F_k + \beta \text{ and } g_k^{(i)} = \alpha \cdot f_k^{(i)} + \beta_i,$$

where $\beta = \sum_{j=1}^{n} \beta_j$ and $\alpha > 0$.

<u>Proof</u>: Let y and z be elements of $Y = \prod_{j=1}^{2n} Y_j = X \times X$. From Assumption 1, there exists a real-valued function χ_k such that

$$y \precsim_k z \text{ if and only if } \chi_k(y) \leq \chi_k(z). \tag{9}$$

Define F_k and $\bar{F}_k$ such that for every $y = [a, d]$ and $z = [c, b]$, $a, b, c, d \in X$,

$$\chi_k(y) = F_k(a) + \bar{F}_k(d) \text{ and } \chi_k(z) = F_k(c) + \bar{F}_k(b). \tag{10}$$

Here, for every $x = (x_1, \ldots, x_n) \in X$, $F_k(x) = \sum_{i=1}^{n} f_k^{(i)}(x_i)$ and $\bar{F}_k(x) = \sum_{i=1}^{n} \bar{f}_k^{(i)}(x_i)$.

On the other hand, from Axiom 1, for every $p, q \in X$, $pp \; I_k \; qq$, i.e., $[p, q] \precsim_k [q, p]$ and $[q, p] \precsim_k [p, q]$. Therefore, $F_k(p) + \bar{F}_k(q) = F_k(q) + \bar{F}_k(p)$, i.e., for every $p \in X$, $\bar{F}_k(p) - F_k(p) = \text{constant}$. That is, for every $p \in X$,

$$\bar{F}_k(p) = F_k(p) + \text{constant}. \tag{11}$$

From Def. 1, for every $y = [a, d]$ and $z = [c, b]$, $ab\ D_k\ cd$ if and only if $y \preccurlyeq_k z$. Therefore, from (9), (10), and (11),

$$ab\ D_k\ cd \text{ if and only if } F_k(a) - F_k(b) \leq F_k(c) - F_k(d),$$

where $F_k(a) = \sum_{i=1}^{n} f_k^{(i)}(a_i)$ for all $a = (a_1, \ldots, a_n) \in X$. Moreover, from Def. 3, $a \preccurlyeq_k b$ if and only if $ab\ D_k\ bb$, i.e., $F_k(a) \leq F_k(b)$. From the later part of Assumption 1, the F_k and $f_k^{(i)}$ are unique up to increasing linear transformations.

<u>Axiom 2 (Bounded utilities)</u>. There exist a least upper bound and a greatest lower bound in every X_i, $i = 1, \ldots, n$.

<u>Axiom 3 (Invariance of ordering within dimension)</u>. For every $u, v \in \{1, \ldots, m\}$, and for all $a, b \in X$ such that $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ where $a_i = b_i$ for $i \neq j$, $j \in \{1, \ldots, n\}$,

$$a \preccurlyeq_u b \text{ if and only if } a \preccurlyeq_v b.$$

<u>Axiom 4 (Existence of midpoints)</u>. For every $k \in \{1, \ldots, m\}$, for all $a, b \in X$ such that $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ where $a_i = b_i$ for $i \neq j$, $j \in \{1, \ldots, n\}$, there exists $c = (c_1, \ldots, c_n) \in X$ such that $c_i = a_i = b_i$ for $i \neq j$, and

$$ac\ I_k\ cb.$$

Axiom 5 (Invariance of midpoints). For all a, b, $c \in X$ such that $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$, and $c = (c_1, \ldots, c_n)$ where $a_i = b_i = c_i$ for $i \neq j$, $j \in \{1, \ldots, n\}$, for every $u, v \in \{1, \ldots, m\}$,

$$ac\ I_u\ cb \text{ if and only if } ac\ I_v\ cb.$$

Theorem 2. (1) If Assumption 1 and Axioms 1 - 5 hold, there exist

(i) a real-valued function U_k over $X = \prod_{i=1}^{n} X_i$ bounded between zero and one,

(ii) a set of positive real numbers $w_k^{(i)}$, $i = 1, \ldots, n$, such that $\sum_{j=1}^{n} w_k^{(j)} = 1$, and

(iii) a set of real-valued functions $u^{(i)}$, $i = 1, \ldots, n$, bounded between zero and one,

that satisfy the following relations: For every $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, $x, y \in X$,

$$x \precsim_k y \text{ if and only if } U_k(x) \leq U_k(y),$$

where

$$U_k(x) = \sum_{i=1}^{n} w_k^{(i)}\ u^{(i)}(x_i) \quad \text{for all } x \in X.$$

(2) Moreover, if there is another function V_k such that

$$x \precsim_k y \quad \text{if and only if} \quad V_k(x) \leq V_k(y),$$

then the V_k is related to the U_k by an increasing linear transformation.

4. Proof of Theorem 2

Let $F_k(x) = \sum_{i=1}^{n} f_k^{(i)}(x_i)$ be the function over all $x = (x_1, \ldots, x_n) \in X = \prod_{i=1}^{n} X_i$, that satisfies the relations in Theorem 1. Define $a_i^o = \mathrm{Inf}\{X_i\}$ and $b_i^o = \mathrm{Sup}\{X_i\}$ for every X_i.

From Axiom 4, there exist a positive integer e and a series of points $\{c_i^r\}$, $c_i^r \in X_i$, $r = 0, 1, \ldots, m'$, $m' = 2^e$, such that $c_i^0 = a_i^o$, $c_i^{m'} = b_i^o$, $a_i^o < c_i^r < c_i^{r+1} \leq b_i^o$ for $0 < r \leq m'$, and

$$f_k^{(i)}(c_i^{r-1}) - f_k^{(i)}(c_i^r) = f_k^{(i)}(c_i^r) - f_k^{(i)}(c_i^{r+1}), \qquad (12)$$

for $0 < r < m'$. From Axiom 5, if Eq. 12 holds for some k, it holds for all $k = 1, \ldots, m$. Equation 12 also can be espressed as,

$$f_k^{(i)}(c_i^1) - f_k^{(i)}(a_i^o) = \frac{1}{2^e}\,[f_k^{(i)}(b_i^o) - f_k^{(i)}(a_i^o)], \qquad (13)$$

$$f_k^{(i)}(c_i^r) - f_k^{(i)}(a_i^o) = r\,[f_k^{(i)}(c_i^1) - f_k^{(i)}(a_i^o)]. \qquad (14)$$

Here again, Eqs. 13 and 14 hold for all $k = 1, \ldots, m$.

Define a function $\phi_k^{(i)}$ on every $x \in X_i$ by

$$\phi_k^{(i)}(x_i) = \frac{f_k^{(i)}(x_i) - f_k^{(i)}(a_i^o)}{f_k^{(i)}(b_i^o) - f_k^{(i)}(a_i^o)}. \qquad (15)$$

In order to prove that $\phi_k^{(i)}(x_i) = \phi_{k^*}^{(i)}(x_i)$ for k, $k^* \in \{1, \ldots, m\}$, suppose that $\phi_k^{(i)}(x_i) \neq \phi_{k^*}^{(i)}(x_i)$. Without loss of generality, assume that

$$\phi_k^{(i)}(x_i) = \phi_{k^*}^{(i)}(x_i) + \mathcal{E}, \text{ where } \epsilon > 0. \tag{16}$$

(A) Suppose that $x_i \in X_i$ is a member of the series $\{c_i^r\}$; that is, there exists an integer s such that for all k = 1, ..., m,

$$f_k^{(i)}(x_i) - f_k^{(i)}(a_i^o) = s\,[f_k^{(i)}(c_i^1) - f_k^{(i)}(a_i^o)].$$

Substituting this equation and Eq. 13 into Eq. 15,

$$\phi_k^{(i)}(x_i) = \frac{s}{2^e},$$

for all k = 1, ..., m. Therefore, for every k, $k^* \in \{1, \ldots, m\}$, $\phi_k^{(i)}(x_i) = \phi_{k^*}^{(i)}(x_i)$. This contradicts the initial supposition, Eq. 16.

(B) Suppose that $x_i \in X_i$ is not a member of the series $\{c_i^r\}$. In this case, there must be an integer t and elements c_i^t, $c_i^{t+1} \in X_i$, such that for all k = 1, ..., m,

$$f_k^{(i)}(c_i^t) - f_k^{(i)}(a_i^o) = t\,[f_k^{(i)}(c_i^1) - f_k^{(i)}(a_i^o)], \tag{17}$$

$$f_k^{(i)}(c_i^{t+1}) - f_k^{(i)}(a_i^o) = (t+1)\,[f_k^{(i)}(c_i^1) - f_k^{(i)}(a_i^o)], \tag{18}$$

$$f_k^{(i)}(c_i^t) < f_k^{(i)}(x_i) < f_k^{(i)}(c_i^{t+1}). \tag{19}$$

From Eqs. 17 and 18 for k, $k^* \in \{1, \ldots, m\}$, substituted in Eq. 15,

$$\phi_k^{(i)}(c^t) = t/2^e \qquad \text{and} \quad \phi_{k^*}^{(i)}(c^{t+1}) = (t+1)/2^e.$$

Therefore, $\phi_{k^*}^{(i)}(c_i^{t+1}) - \phi_k^{(i)}(c_i^t) = 1/2^e$. From Axiom 4, it is possible to choose e such that $1/2^e < \varepsilon$. In this case,

$$\phi_{k^*}^{(i)}(c_i^{t+1}) - \phi_k^{(i)}(c_i^t) < \varepsilon. \tag{20}$$

From (19) for k and k^*,

$$\phi_k^{(i)}(c_i^t) < \phi_k^{(i)}(x_i) < \phi_k^{(i)}(c_i^{t+1}),$$

$$\phi_{k^*}^{(i)}(c_i^t) < \phi_{k^*}^{(i)}(x_i) < \phi_{k^*}^{(i)}(c_i^{t+1}).$$

Therefore, $\phi_k^{(i)}(c_i^t) + \phi_{k^*}^{(i)}(x_i) < \phi_k^{(i)}(x_i) + \phi_{k^*}^{(i)}(c_i^{t+1})$. Thus, $\phi_{k^*}^{(i)}(x_i) - \phi_k^{(i)}(x_i) < \phi_{k^*}^{(i)}(c_i^{t+1}) - \phi_k^{(i)}(c_i^t) < \varepsilon$ (from Eq. 20). This contradicts the initial supposition, Eq. 16. Therefore, for all k, $k^* \in \{1, \ldots, m\}$, for every $x_i \in X_i$, there exists a function $u^{(i)}(x_i)$ such that

$$\phi_k^{(i)}(x_i) = \phi_{k^*}^{(i)}(x_i) = u^{(i)}(x_i).$$

Now, define $w_k^{(i)} = [f_k^{(i)}(b_i^o) - f_k^{(i)}(a_i^o)] / \sum_{j=1}^{n} [f_k^{(j)}(b_j^o) - f_k^{(j)}(a_j^o)]$, $\alpha = \sum_{j=1}^{n} [f_k^{(j)}(b_j^o) - f_k^{(j)}(a_j^o)]$, and $\beta = \sum_{j=1}^{n} f_k^{(j)}(a_j^o)$. Then the $F_k(x)$ for all $x = (x_1, \ldots, x_n) \in X$ can be re-written; $F_k(x) = \alpha \cdot U_k(x) + \beta$, where $U_k(x) = \sum_{i=1}^{n} w_k^{(i)}\ u^{(i)}(x_i)$ (see Sec. 2). From Theorem 1, the U_k also preserves the ordering of X. Noticing that $w_k^{(i)} > 0$, $\sum_{j=1}^{n} w_k^{(j)} = 1$, and $0 \leq u^{(i)}(x_i) \leq 1$ for every $x_i \in X_i$, the first part of Theorem 2 is proved. The second part is immediate from the later part of Theorem 1.

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The proposed axiom system yields utility functions for different orderings of multi-dimensional objects, functions that, for each ordering, are expressed as sums of real-valued functions, called constituent functions, weighted by their corresponding importance measures. A constituent function is a utility function on each dimension that is transformed into the fixed interval (0,1). Importance measures are positive real numbers that add to one. It is shown that if the axioms hold, the constituent functions are constant for all different orderings, while only the importance measures dictate the change of ordering. The axioms are stated in terms of a person's choice behavior under various circumstances.

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