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HYBRID SIMULATION OF AN OPTIMAL STOCHASTIC CONTROL SYSTEM

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by

Edgar C. Tacker and Thomas D. Linton
Louisiana State University
Baton Rouge, Louisiana
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HYBRID SIMULATION OF AN OPTIMAL STOCHASTIC CONTROL SYSTEM[1]

by

Edgar C. Tacker, Associate Professor, Departments of Chemical and Electrical Engineering and Director, Hybrid Simulation Laboratory, Louisiana State University, Baton Rouge, Louisiana 70803

and

Thomas D. Linton, Research Associate, Department of Electrical Engineering Louisiana State University, Baton Rouge, Louisiana 70803

Summary

Hybrid computation techniques as they apply to simulating stochastic control systems are discussed. By means of studying a particular system it is shown that a controller optimized on a deterministic basis can be considerably inferior to one optimized by explicitly including the ever-present stochastic effects in the controller design.

1. Introduction

In order that results of the past decade[2] in control and estimation theory applicable to on-line computer control of physical processes become more widely disseminated among practicing engineers, it is important that motivational simulation case studies be performed and reported on. In particular such studies should give easily understood examples wherein system performance is shown to be appreciably better when the control law is derived by explicitly taking the ever-present system uncertainties and disturbances into account. These studies should also provide a tutorial function, with the paper being organized so as to attempt to motivate the reader to study further in the areas of optimal control and stochastic processes. These studies should also include as much as possible in the way of discussing how practical implementation constraints interact with the mathematical constraints of present day control theoretic results.

Since virtually all of the available literature in the area of stochastic system simulation deals with digital simulation, (for instance, see [18-25]), it seemed appropriate that the authors should report on the use of hybrid computers for this purpose.[3] An EAI680-XDS Σ5 hybrid computing system was used in the simulation study.

The above two paragraphs summarize the primary motivations for writing this paper. The first paragraph also suggests a figure of merit by which this paper may be judged.

In the following exposition it will often prove inconvenient to give mathematically precise formulations, and we shall not hesitate to lower the level of precision or generality for the sake of clarity or to make analytical or experimental progress. For instance, a stochastic control system will be modeled by a nonlinear dynamic system driven by a white noise process. The model differential equation (2.1) is improperly posed mathematically, and is "non-physical" as well. The Ito formulation [13,15,27-31] that interprets equations such as (2.1) as stochastic integral equations is one means by which the procedure can be placed on a sound mathematical basis. This procedure also converts (2.1) to a physically meaningful formulation since the resulting "integrated" white noise (Brownian motion, Wiener process) furnishes an excellent model for various random disturbances present in nature [32].

2. Problem Formulation

Consider the following mathematical model of a stochastic control system

$$\dot{x} = f(x) + B\,(u + v) \qquad (2.1)$$

$$y = Hx + w \qquad \text{on } t \in [0,T], \qquad (2.2)$$

where the n x_i's are the system state variables, the m y_i's are the system output variables, the r u_i's are the system control variables, B and H are constant n x r and m x n matrices respectively, and f is a nonlinear function satisfying an appropriate Lipschitz condition that will insure existence of a unique solution to the deterministic unforced initial-value problem [33]. The control u is to be determined as a function of the m (m < n) measurable variables y_i; refer to Figure 2.1.

The r "input noise" variables v_i and the m "measurement noise" variables w_i are zero mean Gaussian white noise stochastic processes with covariance matrices

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2. References [1-17] are representative samples from the available literature. The papers by Kalman are especially recommended.
3. A forthcoming publication [26] will discuss both hybrid and digital simulations of stochastic control systems and will make some quantitative comparisons on their relative effectiveness.

$$\text{cov}[v(t), v(\tau)] = Q\,\delta(t-\tau) \qquad (2.3)$$

$$\text{cov}[w(t), w(\tau)] = R\,\delta(t-\tau) \quad \text{for all } t \text{ and } \tau$$

$$\text{cov}[v(t), w(\tau)] = 0$$

where δ is the Dirac delta function, Q is a symmetric positive semi-definite constant r x r matrix, and R is a symmetric positive definite constant m x m matrix. The initial state $x(0)$ is a zero mean Gaussian random vector with covariance $\text{cov}[x(0), x(0)] = P_{x(0)}$, and is independent of both the v and w processes. Note that the above formulation allows one to account for modeling uncertainties as well as for disturbance inputs.

We shall assume that the given data for the problem includes a specification of a nominal control function $u^{(n)}(t)$ on [0,T] that in the absence of noise leads to a very acceptable system performance (the optimization criterion used in making this choice may well have been formulated on the basis of subjective as well as objective considerations). The function $u^{(n)}(t)$ is assumed to be sufficiently regular so that with v = 0 and x (0) specified the mathematical model equation (2.1) admits a unique solution, the nominal (state) trajectory $x^{(n)}(t)$ on [0,T]. For convenience in the simulation study we shall assume that

$$u^{(n)}(t) = 0 \text{ on } [0,T]. \qquad (2.4)$$

Thus variations u about the nominal control, $u^{(n)}$, will from now on simply be denoted by u(i.e. $u(t) = u^{(n)}(t) + \delta u(t)$ for any t in [0,T].)

The noise signals v and w will tend to make the system state deviate from the desired nominal trajectory; v will enter directly via (2.1) while the effect due to w is introduced indirectly through the fact that u is a function of the measured variables y_i. A control law is sought that will keep these deviations reasonably small while not requiring excessive control "effort". The criterion by which system performance is judged should not be based on effects due to individual sample functions of the noise processes, but rather should be based on a consideration of the entire ensemble of such sample functions.

With the above thoughts in mind, the performance of the stochastic control system as modeled by equations (2.1) through (2.4) will be evaluated by means of the cost functional

$$J_E(u) = E\left\{ J(u) \right\}, \text{ where} \qquad (2.5)$$

$$J(u) = \int_0^T \Big(u'(t)\, Su(t) + e'(t)\, Ne(t)\Big)dt \qquad (2.6)$$

and

$$e(t) = x(t) - x^{(n)}(t) \text{ on } [0,T]. \qquad (2.7)$$

The objective is then to determine a control u that minimizes the cost functional (2.5). There are to be no explicit constraints[4] on u. S is a symmetric positive definite constant r x r matrix, N is a symmetric positive semi-definite constant n x n matrix, "prime" denotes "transpose", and E denotes mathematical expectation (i.e., for any integrable random variable Z defined on the underlying probability space (Ω,F,P), $E\left\{ Z \right\} = \int_\Omega Z(w)\, dP(w)$).

Referring to Figure 2.1, consider repeating this control experiment a number of times, each experimental run (trial) being of T seconds duration. Intuitively, in a particular trial the chance mechanism underlying the noise processes will "select" a particular sample function ("waveform") from both the v and w ensembles. These particular noise inputs will cause the "waveforms" u(t) and e(t) on [0,T] to be generated. The number J(u) in (2.6) can be computed during the run. The term $\int_0^T u'(t)\, Su(t)\, dt$ measures the "control energy" expended during the run, and the term $\int_0^T e'(t)\, Ne(t)\, dt$ measures the "distance" from the nominal trajectory. If $J_E(u)$ is small, then this implies that both the average control energy $E\left\{\int_0^T u'(t)\, Su(t)\, dt\right\}$ and the average distance $E\left\{\int_0^T e'(t)\, Ne(t)\, dt\right\}$ are small. $J_E(u)$ being small does not guarantee that every J(u) is small; however, it does indicate that the probability is small that individual J(u)'s take on large values. The above two statements follow directly from the fact that both terms in J(u) are nonnegative functionals.

3. Separation, Linearization and Other Considerations

We have now discussed the criterion by which system performance will be judged; however, no mention as yet has been made on the design of Block II of Figure 2.1. Present day deterministic optimal control formulations [11,12,17] deal either with open-loop policies (which we shall not discuss) or with feedback policies, that in general require that the true system state x(t) be available for every t in [0,T]. However, it is clear from (2.2) that this information is never available. To progress past this theoretical hurdle, a most reasonable strategem is to "separate" the problem into an estimation phase and a control phase -- refer to Figure 3.1. Block II-e is designed by using optimal (stochastic) estimation theory[5], and at

4. That is, there are no equality or inequality constraints on u.
5. See section 4 of this paper.

time t in [0,T] this block outputs the value $\hat{x}(t)$, which is the "best" estimate of the true system state based on the noisy observations of y over the time interval [0,t]. Block II-c is designed by using optimal (deterministic) control theory[6] and at each time t in [0,T] this block outputs the value $u^0(t)$, which is the "best" control input based on the assumption that $\hat{x}(t) = x(t)$. This "separation principle" has been widely used [12,16,17].

If the function f in (2.1) were linear in x, then an application of the separation principle would lead to an overall optimal system (see references [14,34] for proofs of the appropriate separation theorems). This fact immediately leads one to consider linearizing the plant model (2.1) about the nominal trajectory $x^{(n)}$. This results in the following linear time-varying system

$$\dot{e} = A(t)\, e + B\,(u + v) \tag{3.1}$$

$$A(t) = \left.\frac{\partial f}{\partial x}\right|_{x = x^{(n)}(t)} = \left[\begin{matrix} \frac{\partial f_1}{\partial x_1} & \cdots\cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_n}{\partial x_n} & \cdots\cdots & \frac{\partial f_n}{\partial x_n} \end{matrix}\right]_{\substack{x_1 = x_1^{(n)}(t) \\ \vdots \\ x_n = x_n^{(n)}(t)}} \tag{3.2}$$

$$z = He + w, \text{ where } e \text{ is as defined in (2.7).} \tag{3.3}$$

For reasonably small disturbances, these equations give an adequate representation[7] of the plant. Figure 3.2 gives a block diagram of the resulting overall optimal linear system.

4. The Linear (Stochastic) Optimal Estimation Problem[8]

Consider the linear system described by Equations (3.1) - (3.3). Assume further that the estimation objective is to find an estimate $\hat{e}(t)$ that is a linear function of the form $\hat{e}(t) = \int_0^t h(t,\tau) z(\tau) d\tau$ (i.e., $\hat{e}$ is the output of a linear filter with $h(t,\tau)$ as the impulse response function) that minimizes the cost function $I(\hat{e})$ given by (4.1)

$$I(\hat{e}) = E\left\{ \left[\tilde{e}(t)\right]' \left[\tilde{e}(t)\right] \right\} \text{(mean square error)} \tag{4.1}$$

where

$$\tilde{e}(t) = e(t) - \hat{e}(t) \text{ (estimation error)} \tag{4.2}$$

This defines a "minimum variance estimation problem". The "Kalman-Bucy algorithm" that realizes the solution to this problem is defined by Equations (4.3) - (4.7).

$$\frac{d\hat{e}}{dt} = A(t)\,\hat{e} + Bu + K(t)\,\tilde{z}(t), \quad \text{with } \hat{e}(0) = 0 \tag{4.3}$$

where

$$\tilde{z}(t) = z(t) - H\,\hat{e}(t) \tag{4.4}$$

The "Kalman gains" K(t) are obtained from

$$K(t) = P(t)\, H' R^{-1}, \text{ where } P(t) \tag{4.5}$$

is the unique solution to the Riccati equation (called the "variance equation").

$$\frac{dP}{dt} = A(t)\, P + PA'(t) - P\, H' R^{-1} HP + BQB' \tag{4.6}$$

where[9]

$$P(0) = \mathrm{cov}[e(0), e(0)] = P_{e(0)} \tag{4.7}$$

Figure 4.1 gives the simulation block diagram for the optimal estimator (Kalman filter).

5. The Linear (Deterministic) Optimal Control Problem

Consider the linear system described by Equation (3.1) with v = 0. Assume further that the control objective is to find a control input u(t) that is a linear function on the values of e over the time interval [0,t] that minimizes the cost function J(u) given by (2.6). This defines the "linear (deterministic) regulator problem". This is an especially tractable problem, and it is well known [2,11,12] that there exists a unique optimal control law u^0 given by

$$u^0(t) = -S^{-1} B' D(t)\, e(t) \qquad t \in [0,T], \tag{5.1}$$

where the symmetric n x n matrix D(t) is the unique solution to the Riccati equation

$$\frac{dD}{dt} = -D(t)A(t) - A'(t)D(t) + D(t)\, BS^{-1}B' D(t) - N \tag{5.2}$$

satisfying the boundary condition

6. See Section 5 of this paper.
7. Later sections of this paper will give some discussion on the adequacy of this representation.
8. As mentioned in Section 1, our formulations can often be generalized considerably. For more general formulations on Kalman filtering see references [1,3,13].

9. For the case at hand (see Section 2) $P(0) = P_{x(0)}$.

$$D(T) = 0 \tag{5.3}$$

Also, it turns out that D(t) is positive definite for all $0 \le t < T$.

Corresponding to the final value D(T) there is an initial value D(0) that may be pre-computed by a reverse-time integration of (5.2). The analog computer can then be used to generate the optimal control as a continuous function of e(t) by using (5.1) with D(t) being the (continuously generated) solution of (5.2) with initial condition D(0).

In accordance with the separation principle discussed in Section 3, the simulation diagram for Block II-c is described by (5.1) with $\hat{e}(t)$ replacing e(t), and D(t) is generated as described above using the initial value D(0). Figure 5.1 gives the simulation block diagram for the optimal controller (Kalman controller) used in this simulation study.

6. Numerical Specification of the System Model

To proceed further with the simulation study, it becomes necessary to specify a particular mathematical model. In accordance with the desire that the example be easily understood, a third order plant with a single nonlinear term seemed an appropriate choice.

The particular system configuration is defined by using the following data:

T = 1 second,

$$f(x) = \begin{bmatrix} x_2 \\ x_3 \\ -x_1^2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad S = [0.1],$$

$$Q = [q], \quad R = \begin{bmatrix} r_{11} & 0 \\ 0 & r_{22} \end{bmatrix}$$

$$\text{and} \quad P_{x(0)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k \end{bmatrix} \tag{6.1}$$

where the statistical parameters $q>0$, $r_{11}>0$, $r_{22}>0$, and $k \ge 0$ are to be varied in the course of the system study.

In scalar form the model is defined on the time interval [0,1] by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 + u + v \quad \text{(plant)} \\ \dot{x}_3 &= -x_1^2 \end{aligned} \tag{6.2}$$

$$\begin{aligned} y_1 &= x_1 + x_2 + w_1 \\ y_2 &= x_2 + x_3 + w_2 \end{aligned} \quad \text{(measurement process)} \tag{6.3}$$

Notice that the plant described by (6.2) is an unstable system.[10] Intuitively, this can be seen by setting $u = 0$, $v = 0$ and then investigating the equation $\dot{x}_3 = -x_1^2$. If x_1 ever differs from zero, then x_3 will monotonically decrease with time from that instant on. Although we are not concerned with asymptotic behavior (since T = 1 second and the magnitude of the eigenvalues of A(t) of (6.5) are approximately unity for the given $x^{(n)}(t)$), we must still be concerned with compuational instabilities. There are several sources of error that accompany any computer implementation; e.g., digital (roundoff and discretization errors, etc.), analog (reference variations, component tolerances, etc.), and linkage (quantization and time skew in converters, etc.). An unstable system is often very sensitive to such errors, and even over a short period of time these errors could accumulate to an intolerable level. For these reasons, it seems plausible that the system (6.2) would provide a good test case for these studies.

The perturbation equations for (6.2) about the nominal response $x^{(n)}(t)$ are given by

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= e_3 + u + v \\ \dot{e}_3 &= [-2x_1^{(n)}(t)]\, e_1 - (e_1)^2 \end{aligned} \tag{6.4}$$

Thus the linearized perturbation equations are given by

$$\dot{e} = A(t)e + B(u + v), \text{ where} \tag{6.5}$$

$$A(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ a(t) & 0 & 1 \end{bmatrix} \quad \text{and}$$

$$a(t) = -2x_1^{(n)}(t) \tag{6.5a}$$

Similarly, the perturbation equations for (6.3) are given by

$$\begin{aligned} z_1 &= e_1 + e_2 + w_1 \quad \text{and} \\ z_2 &= e_2 + e_3 + w_2 \end{aligned} \tag{6.6}$$

For future reference we now show that the linear system (6.5) - (6.6) is uniformly observable[11] on [0,1]. Using the notation of reference

10. The linearized plant described by (6.5) is also unstable.

11. By a similar formulation one may also easily show that the linear system (6.5) is uniformly controllable on [0,1]. This fact would be of importance if one were to wish to study the asymptotic ($T \to \infty$) behavior of the stochastic control system.

[35], the observability matrix $\theta_0(t)$ for this system is given by

$\theta_0(t) = [H' \vdots \Delta_0 H' \vdots \Delta_0^2 H']$ where

$\Delta_0(\cdot) = (A(t) + \frac{d}{dt})(\cdot)$. This evaluates to

$$\theta_0(t) = \begin{bmatrix} 1 & 0 & 0 & a(t) & a(t) & a(t) + b(t) \\ 1 & 1 & 1 & 0 & 0 & a(t) \\ 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

where $b(t) = -2x_2^{(n)}(t)$.

From properties of the given nominal trajectory it is known that

$$\{t\epsilon[0,1]: x_1^{(n)}(t) = 1/2\} \cap$$

$$\{t\epsilon[0,1]: x_2^{(n)}(t) = 0\} = \emptyset,$$

i.e., $x_1^{(n)}$ does not pass through the value 1/2 at the same time $x_2^{(n)}$ passes through zero. Thus the rank of $\theta_0(t)$ is 3 for all t in [0,1], and uniform observability of the linear system follows.

7. Generation of the Noise Signals and Computation of $J_E(u)$

From a simulation point of view, one must generally be satisfied with generating signals whose time averages are of such a character that he can reasonably infer that the (real or imagined) underlying stochastic processes approximate the specified noise processes v and w.

Since it is desired to simulate the stochastic control system on a hybrid computer, it is convenient to supply the noise signals from a multichannel FM tape recorder (Precision Instruments 6200 series). An analog random noise generator (Solartron Model BO 1227) was used to obtain these recorded signals. The statistical properties of the generator were validated by estimating the autocorrelation function, the power spectral density (using both the discrete Fourier transform on the autocorrelation function data and the fast Fourier transform using the sample function data directly), and the first and second order density functions. See [36,37] for further details, and [38,39,40] for other methods.

The validation studies showed that the generator could accurately be modeled by a zero mean stationary Gaussian process with a pseudo-white spectrum (the power spectral density was approximately constant from dc to 2 khz). The effective correlation time for the process was of the order of 0.25 msec (i.e., $\tau \geq 0.25$ msec $\Rightarrow |R(\tau)| < e^{-1} \cdot R(0)$). Since the process is approximately Gaussian, this implies that samples taken from the process at intervals significantly greater than 0.25 msec can reasonably be assumed to be independent. Since the noise was recorded one channel at a time (a 45 minute process) the data on separate channels may safely be regarded as mutually independent. Further, since each simulation of the control system takes one second, one may assume that the "costs" J_i, where J_i is the cost

$$\int_0^T \{u'(t)\,Su(t) + e'(t)\,Ne(t)\}\,dt$$

of the i^{th} trial, are independent samples of the random variable J(u). Thus, by Kolmogorov's Law of Large Numbers [41] the "sample mean"

$$\hat{J}_E(N) = \frac{1}{N} \sum_{i=1}^{N} J_i$$

should furnish a good estimate of $J_E(u)$ for sufficiently large N. An acceptable index of convergence is

$$D_{N,N-1} = |\hat{J}_E(N) - \hat{J}_E(N-1)|;\text{ i.e.,}$$

computation of the average cost is terminated after run N whenever $D_{N,N-1}$ becomes and remains sufficiently small.

Although not needed for this study, it should be mentioned that noise processes with nonstationary covariance functions can sometimes be generated by using the stationary processes mentioned above, together with appropriate time-function generators. For instance, consider generating a zero mean nonstationary stochastic process with covariance matrix

$$R = \begin{bmatrix} r_{11}(t) & 0 \dots & 0 \\ 0 & r_{22}(t) & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ 0 & \dots & r_{mm}(t) \end{bmatrix} =$$

diag $\{r_{11}(t), \dots, r_{mm}(t)\}$.

Letting $w = \begin{bmatrix} w_1 \\ \cdot \\ \cdot \\ \cdot \\ w_m \end{bmatrix}$ be a zero mean stationary stochastic process with covariance matrix diag $\{1, \dots, 1\} = I_{m \times m}$, it is seen that the desired process may be simulated (within the accuracy of second moment characterization) by forming the products

$$w_i \cdot (r_{ii}(t))^{1/2} \text{ for } i = 1, \dots, m.$$

In more general cases it may be necessary to employ more advanced techniques such as using truncated orthogonal series expansions for the nonstationary stochastic processes (see references [40,41]).

8. Design of the Kalman Observer

Shortly we shall wish to determine the performance of the system of Figure 2.1 when the controller is designed on a deterministic basis. As will be shown below, this will involve using the configuration of Figure 3.2 where Block II-e is a "deterministic filter".

In Section 10 the performance of this system will be compared with the optimal linear system described in Section 6 as various statistical parameters are varied. It should be noted that this deterministic element was "optimized" in so far as possible so as to be fair in making comparisons, i.e., one should, if possible, compare optimal systems.

Consider the formulation of Section 2 where $v = 0$, $w = 0$, and $y = x$. If this were true, then every state variable would always be available to the controller of Figure 2.1. It would then be very reasonable to evaluate the performance of this deterministic system by means of the cost functional of Equation (2.6). Using the results of Section 5, the "best" controller to use would be of the form of Figure 5.1, where $\hat{e}$ would be replaced by the actual state e.

The formulation of the last paragraph does not quite fit our needs since m is actually less than n; i.e., even in the noise-free case e is not available to the controller of Figure 2.1. Consider the linearized plant equations (6.5) - (6.6). In Section 6 it was shown that this systime is uniformly observable on [0,1]. Thus, as can be shown [4,12,13], the state variables e_i can be "reconstructed" (theoretically in an arbitrarily short amount of time) by passing the output signal z through the linear dynamical system ("observer") of Figure 8.1. The output of the observer satisfies [12]

$$h(t) = e(t) + \varphi(t,t_o)(h(0) - e(0)) \qquad (8.1)$$

where $\varphi(t,t_o)$ is the state transition matrix of the linear system

$$\dot{n} = (A(t) - G(t)H)n \qquad (8.2)$$

By appropriate choice of the matrix G(t) (the only free parameter in the observer structure) the "transient response" of (8.2) can be made to decay to zero very rapidly. This will mean that the second term on the right hand side of Equation (8.1) will also decay to zero very rapidly (to accentuate this effect the value of h(0) should also be set equal to the a priori best estimate of e(0)). From Equation (8.1), the appropriate cost functional for the observer is clearly the amount of time it takes for this decay to occur.

It should be noted that the Kalman observer is simply a special case of the Kalman filter (compare Figures 5.1 and 8.1). In Figure 8.1 the inputs are noise-free and the Kalman gains G(t) are chosen as described above instead of via the variance equation. This fact makes the Kalman observer more convenient for our purposes than other formulations [42,12] since one can quite simply (say by means of setting or resetting a sense switch) change between the stochastic and deterministic filters, i.e., only the Kalman gains need be changed.

Using the parameter values of Section 6 one finds that

$$L(t) = A(t) - G(t)H = \begin{bmatrix} -g_{11} & 1-(g_{11}+g_{12}) & -g_{12} \\ -g_{21} & -(g_{21}+g_{22}) & 1-g_{22} \\ a(t)-g_{31} & -(g_{31}+g_{32}) & -g_{32} \end{bmatrix} \qquad (8.3)$$

where

$G(t) = (g_{ij})$ and a(t) is as defined in (6.5a). A very acceptable rate of decay is obtained by choosing G(t) to be the constant matrix

$$G = \begin{bmatrix} 1 & -11.8 \\ 30.84 & 1 \\ -3 & -2.84 \end{bmatrix} \qquad (8.5)$$

This resulted from assuming $a(t) \equiv -3$, then setting the characteristic equation for L(t) equal to $(\lambda+10)^3$, equating coefficients, and determining a solution of the resulting nonlinear algebraic equations for the g_{ij}'s. Since a(t) actually takes on values in the range (-4.35, -2), the corresponding variations of the eigenvalues of L(t) were computed. Some sample values are given in Table 8.1.

a	λ_1	λ_2, λ_3
-2.0	-3.02	$-13.49 \pm j4.97$
-2.5	-4.56	$-12.72 \pm j4.03$
-3.0	-10.00	$-10.00 \pm j0.03$
-3.5	-14.66	$-7.6 \pm j4.71$
-4.0	-15.75	$-7.13 \pm j6.05$
-4.35	-16.28	$-6.86 \pm j6.75$

Table 8.1

As anticipated, this observer performed very well in the absence of noise (see Figure 10.2).

9. Hybrid Computer Implementation

The simulation block diagram of Figure 9.1 illustrates how the problem has been divided between the analog and digital computers. This particular[12] division of tasks emphasizes the analog computer in that the main responsibility of the digital computer is to precompute and store the Kalman gains, and then during each run to send these gains to the analog computer via multiplying DAC's.

In computing estimates of the cost $J_E(u)$ of (2.5) it is clearly desirable to speed the solution rate as much possible. A speed-up by a factor of 10 was decided upon--the execution time of the digital interrupt program used to send the Kalman gains to the analog computer precluded a significantly larger speed-up. This speed-up involves changing the time scale of all analog integrators and increasing the tape recorder speed by a factor of 10. In the following discussion one should now consider that T = 0.1 sec and that the correlation time of the noise generator is approximately 25 μsec. With this speed-up, if N = 1,000 trials are used in computing $\hat{J}_E(u)$, then this will require 120 sec of real-time (including initial condition time). Hence, to obtain a plot of $\hat{J}_E(u)$ for 20 values of a statistical parameter, approximately 40 minutes of real-time is required.

The flowgraph of Figure 9.2 describes the role that the digital computer plays in the simulation study. Some specific comments pertaining to this figure follow.

Block B1 (Off-Line): An analog clock signal is used to cause an external interrupt every 0.4 msec. The interrupt servicing routine reads the ADC and stores the resulting value $a(t_n)$ in an array.

Block B2 (Off-Line): The variances of the three noise signals v, w_1, and w_2 as they were recorded are 1.0, 1.1, and 1.0. To obtain signals with the variances desired, it is necessary to attentuate the signals from the recorder. For example, to obtain a signal with variance $b \le 1$ from v, one must arrange to multiply v by $(b/1.1)^{1/2}$ since

$$E\{\left((b/1.1)^{1/2}v\right)^2\} = \frac{b}{1.1}E\{v^2\} = \frac{b}{1.1}(1.1) = b.$$

The noise signals v, w_1, and w_2 are patched into multiplying DAC's and the coefficients sent from the digital computer are $q^{1/2}$, $(r_{11}/1.1)^{1/2}$, and $r_{22}^{1/2}$.

12. See reference [26] for other divisions of tasks for this and related problems.

Block B3 (Off-Line): Let

$$L_p(t) = \left[\begin{array}{c|c} -A'(t) & HR^{-1}H \\ \hline BQB' & A(t) \end{array}\right]$$

and let $\theta(t,t_0)$ be the transition matrix of the sixth order system

$$\dot{c} = L_p(t)c.$$

The solution of the variance equation (4.6) is then given by [3]

$$P(t) = [\theta_{21}(t,0) + \theta_{22}(t,0)\cdot P(0)]\cdot$$
$$[\theta_{11}(t,0) + \theta_{12}(t,0)P(0)]^{-1}$$

where the $\theta_{ij}(t,0)$ are defined by

$$\theta(t,0) = \left[\begin{array}{c|c} \theta_{11}(t,0) & \theta_{12}(t,0) \\ \hline \theta_{21}(t,0) & \theta_{22}(t,0) \end{array}\right]$$

From the known character of A(t) of equation (6.5a) $L_p(t)$ is known to be approximatley constant over each time interval $[t_{n-1},t_n]$, where $\Delta t = 0.4$ msec and $t_n = n\cdot\Delta t$. Thus $\theta(t_n,t_{n-1})$ can be approximated very closely by

$$\exp\,[L(t_{n-1})\cdot\Delta t],$$

which in turn can be approximated very closely by the truncated series

$$I + L(t_{n-1})\cdot\Delta t + \frac{1}{2}\left(L(t_{n-1})\cdot\Delta t\right)^2 +$$
$$\frac{1}{6}\left(L(t_{n-1})\Delta t\right)^3.$$

Hence the solution to the variance equation can be obtained recursively by

$$P(t_n) = [\theta_{21}(t_n,t_{n-1}) + \theta_{22}(t_n,t_{n-1})P(t_{n-1})]\cdot$$
$$[\theta_{11}(t_n,t_{n-1}) + \theta_{12}(t_n,t_{n-1})P(t_{n-1})]^{-1}$$

The Kalman gains are then computed from

$$K(t_n) = P(t_n)H'R^{-1}.$$

As a validation check for the above discretization procedure the variance equation (4.6) was also integrated using Euler's method. The results checked quite closely when $\Delta t < 0.1$ msec was used with the Euler approximation.

Block B4 (Real-Time): The Kalman gains are supplied to the analog computer using the off-line results of Block B3. These gains must be updated every 0.4 msec. An analog clock signal is used to cause an external interrupt every 0.4 msec. The interrupt servicing routine causes the multiplying DAC registers to be updated according to the latest values provided by the digital computer. At the end of each run $i = 1, \ldots$ the cost J_i is converted and added to the accumulated cost

$$\sum_{k=1}^{i-1} J_k .$$

Comments

(1) In this study, the primary objective was to attempt to generate more interest in applying stochastic control theory to the design of physical control systems. It was desired to use these theories in a context for which they are rigorously true. This leads to the use of the linearized plant model in Figure 9.1.

To apply the above results to controlling a physical process, one could employ a configuration such as in Figure 9.1[13] wherein the linearized plant is replaced by the physical process (assuming, of course, availability of appropriate tranducers, etc.). A configuration of this type was used in a simulation mode (i.e., the nonlinear plant equations were used). The same tests were run on both the linear system and the nonlinear system. Based on cost comparisons (see Section 10) the performance of these two systems differed by no more than 5-10%. This consistency of results served to some extent to validate the procedures used in this study.

The experimentation using the nonlinear plant model in the simulation mode provides useful information on how the actual plant would perform in this same configuration. In particular, there would be an opportunity to study (view a succession of sample functions on an oscilloscope) the behavior of the (inaccessible) plant state variables. This allows one to gain some intuitive understanding of how the control system performs[14].

(2) As noted in Section 5, the solution $D(t)$ of the Riccati equation is generated by using analog computer circuits with $D(0)$ as initial condition. For most choices of the "weight factors" S and N in (2.6) the solution $D(t)$ is sensitive with respect to the value $D(0)$. By trial and error an acceptable value for $D(0)$ can be found prior to initiating the first simulation run. If the task of generating the controller gains $D_{ij}(t_n)$ were assigned to the digital computer, then the various gains could be obtained by a single pre-computation involving a reverse-time solution of the Riccati equation. Thus, in this event, the sensitivity problem would be averted (at the cost of storage space and execution time).

(3) Certain choices for the weighting factors S and N and statistical parameters q, r_{11}, and r_{22} can lead to unpleasant amplitude scaling problems. In the authors' opinion this does not offset the hybrid computer's tremendous advantage with respect to solution speed. These high data rates not only are useful in cost calculations, but they also allow one to conveniently study various sample functions via oscilloscopic displays. Refer to [26] for further details.

10. Simulation Results

Several polaroid pictures were taken to illustrate various aspects of the performance of the stochastic control system. The pictures are shown in Figures 10.1 through 10.7.

Figure 10.1 shows a sample function from each of the three "noises" v, w_1, and w_2. The time scale is such that only one tenth of a computer run is shown.

Figure 10.2 illustrates the behavior of the deterministic observer when no noise is present. Figure 10.3 illustrates the behavior of the deterministic observer in the presence of noise. Clearly the presence of noise severely degrades the performance of the observer.

Figure 10.4 illustrates the behavior of the Kalman filter for the same corditions as in Figure 10.3. The output of the Kalman filter is much less noisy than the deterministic observer and thus provides a much better estimate of $e_1(t)$.

Figure 10.5 shows one of the Kalman gains as it is being sent from the digital computer. This illustrates the discretization used in computing $K(t)$.

Figure 10.6 shows how the two components of the cost accumulate with time. Four different runs are shown and there is a relatively large variation in both components. As mentioned in Section 2, the top trace,

$$100 \int_0^t (e_1^2 + e_2^2 + e_3^2)dt$$

is the "distance" from the nominal trajectory, and the bottom trace,

$$100 \int_0^t u^2 dt$$

---

13. One could obviate the need for generating the nominal trajectory by linearizing about the previous best estimate of the state--the "extended Kalman filter" approach [18,20-25].
14. This becomes of increasing value as the noise levels increase.

is the "control energy". Note that in computing the total cost, the distance is weighted 10 times greater than the control energy.

Figure 10.7 illustrates the smoothing effect of the plant (top 2 waveforms) and the Kalman filter (bottom 2 waveforms).

Cost Comparisons

Consider Figure 3.2. The following notation will be used in describing the cost comparisons (even though as mentioned in Section 8 S_d is only a sub-optimal system).

Block II-e	System of Figure 3.2
Kalman Filter	S_s: Optimal Stochastic System
Kalman Observer	S_d: Optimal Deterministic System

The objective now is to compare the performance of these systems S_s and S_d for various choice of the statistical parameters q, r_{11}, r_{22}, and k.

Figure 10.8 gives a plot of $\hat{J}_E(u)$ vs. q for fixed values of r_{11}, r_{22}, and k for both S_s and S_d. Note that the cost for both S_s and S_d had a noticeable positive trend with increasing q. For every value of q used, the cost for S_s was markedly lower than S_d--the range of percent cost increase incurred by using S_d instead of S_s was from 32% to 63%.

Figure 10.9 gives a plot of $\hat{J}_E(u)$ vs. $r_{11} = r_{22}$ for fixed values of q and k for both S_s and S_d. Note that while the cost for S_d had a noticeable positive trend with increasing $r_{11} = r_{22}$, the cost for S_s was virtually insensitive to these variations. This is plausible since the Kalman observer simply tries to follow its input as rapidly as possible regardless of how large the variance of the measurement noise w becomes, while the Kalman filter was designed so that its gains are adjusted as r_{11} and r_{22} are varied. This allows the Kalman filter to virtually ignore the variations due to this measurement noise. Note that for small measurement noise S_d is almost as good as S_s, but that for moderate noise levels the percent cost increment incurred by using S_d instead of S_s increased to 49%.

Comments

(4) Although the complexity of the system used in this simulation study is not great, and several arbitrary choices were made; e.g., in choosing the weighting matrices in (2.6), the cost comparisons given in Figures 10.8 and 10.9 do indicate that a significant improvement in performance can be obtained by explicitly taking the system uncertainties into account when optimizing the design of the controller of Figure 2.1.

(5) An even greater (sometimes by an order of magnitude or more) improvement in system performance is theoretically possible if "nonlinear filtering" techniques are used in designing the system controller. See references [13,43] for a discussion of this intriguing area of applied research.

References

1. Kalman, R. E., "A New Approach to Linear Filtering and Prediction Problems", Journal of Basic Engineering, March, 1960.

2. Kalman, R. E., "Contributions to the Theory of Optimal Control", Reimpreso del Boletin de la Sociedad Matematica Mexicana, 1960.

3. Kalman, R. E. and R. S. Bucy, "New Results in Linear Filtering and Prediction Theory", Journal of Basic Engineering, March, 1961.

4. Kalman, R. E., "On the General Theory of Control", Proceedings of the 1st International Congress on Automatic Control, Moscow, USSR, 1960.

5. Pontryagin, L. S., et. al, "The Mathematical Theory of Optimal Processes", Interscience Publ., New York, 1962.

6. Dreyfus, S. E., "Some Types of Optimal Control of Stochastic Systems", SIAM Journal on Control, Ser. A, Vol. 2, No. 1, 1962.

7. Wonham, W. M., "Stochastic Problems in Control", Tech. Rept. TR63-14, RIAS, Martin Company, Baltimore, 1963.

8. Lee, R. C. K., "Optimal Estimation, Identification and Control", M.I.T. Press, Cambridge, Mass., 1964.

9. Bryson, A. E., Jr., and D. E. Johansen, "Linear Filtering for Time-Varying Systems Using Measurements Containing Colored Noise", IEEE Transactions on Automatic Control, Vol. AC-10, No. 1, January, 1965.

10. Kushner, H. J., "On the Status of Optimal Control and Stability for Stochastic Systems", Proc. of the IEEE, 1966.

11. Athans, M. and P. L. Falb, "Optimal Control", McGraw-Hill, New York, 1966.

12. Sage, A. P., "Optimum Systems Control", Prentice-Hall, Englewood Cliffs, N. J., 1968.

12a. Aoki, M., "Optimization of Stochastic Systems", Academic Press, New York, 1967.

13. Bucy, R. S. and P. D. Joseph, "Filtering for Stochastic Processes and Applications to Guidance", Interscience Publ., New York, 1968.

14. Wonham, W. M., "On the Separation Theorem of Stochastic Control", SIAM Journal on

Control, Vol. 6, p. 312, 1968.

15. Kailath, T. and T. Frost, "Mathematical Modeling of Stochastic Processes", Stochastic Problems in Control, American Society of Mechanical Engineers, New York, 1968.

16. Meditch, J. S., "Stochastic Optimal Linear Estimation and Control", McGraw-Hill, New York, 1969.

17. Bryson, A. E., Jr. and Y. C. Ho, "Applied Optimal Control: Optimization, Estimation and Control," Blaisdell Publ., Waltham, Mass., 1969.

18. Schmidt, S. F., "State Space Techniques Applied to the Design of a Space Navigation System", Preprints for the 1962 JACC, New York City.

19. Kalman, R. E., T. S. Englar, R. S. Bucy, "Fundamental Study of Adaptive Control Systems", Wright-Patterson Air Force Base Technical Report No. ASD-TDR-61-27, 1962.

20. Peschon, J. and R. E. Larson, "Analysis of an Intercept System", Final Report, Stanford Research Institute, Menlo Park, Calif., 1965.

21. Larson, R. E., R. M. Dressler, R. S. Ratner, "Application of the Extended Kalman Filter to Ballistic Trajectory Estimation" Final Report, Stanford Research Institute, Menlo Park, Calif., 1967.

22. Schmidt, S. F., J. D. Weinburg, J. S. Lukesh, "Case Study of Kalman Filtering in the C-5 Navigation Systems", Preprints of 1968 Case Studies in Automatic Control. University of Michigan, 1968.

23. Wells, C. H., "Application of the Extended Kalman Estimator to the Nonlinear Well Stirred Reactor Problem", Preprints for the 1969 JACC, Boulder, Colo.

24. Wells, C. H., and R. E. Larson, "Application of Combined Optimum Control and Estimation Theory to Direct Digital Control", Proceedings of the IEEE, January, 1970.

25. Wells, C. H., "Optimum Estimation of Carbon and Temperature in a BOF", Preprints for the 1970 JACC, Atlanta, Ga.

26. Tacker, E. C. and T. D. Linton, "Hybrid and Digital Simulation of Optimal Stochastic Control Systems", Proceedings of the 1970 AICA/IFIP Congress, Munich, Germany, September, 1970.

27. Ito, K., "On Stochastic Differential Equations", Mem. American Math. Society, No. 4, pp. 1-51, 1951.

28. Wong, E. and M. Zakai, "On the Relation Between Ordinary and Stochastic Differential Equations and Applications to Stochastic Problems in Control Theory", Proceedings of the 3rd IFAC Congress, London, 1966.

29. Doob, J. L., "Stochastic Processes", John Wiley & Sons, New York, 1953.

30. Skorokhod, A. V., "Studies in the Theory of Random Processes", Addison-Wesley, Reading, Mass., 1965.

31. McKean, H. P., Jr., "Stochastic Integrals", Academic Press, 1969.

32. Wax, N. (Editor), "Selected Papers on Noise and Stochastic Processes", Dover Publ., N. Y., 1954.

33. Coddington, E. A. and N. Levinson, "Theory of Ordinary Differential Equations", McGraw-Hill, New York, 1955.

34. Joseph, P. D. and J. T. Tou, "On Linear Control Theory", Trans. AIEE pt. II, Vol. 80, p. 193, 1961.

35. D'Angelo, H., "Linear Time-Varying Systems: Analysis and Synthesis", Allyn and Bacon Publ., Boston, 1970.

36. Tacker, E. C. and T. D. Linton, "Hybrid Computation of Autocorrelation Functions", SIMULATION, March, 1970.

37. Tacker, E. C. and T. D. Linton, "Hybrid Computational Methods in Time Series Analysis", Manuscript in preparation.

38. Korn, G. A., "Random Process Simulation and Measurements", McGraw-Hill, New York, 1966.

39. Bendat, J. S., A. G. Piersol, "Measurement and Analysis of Random Data", John Wiley & Sons, New York, 1966.

40. Shreider, Y. A. (Editor), "The Monte Carlo Method", Pergamon Press, New York, 1966.

41. Loève, M., "Probability Theory (3rd Edition)", Van Nostrand, Princeton, New Jersey, 1963.

42. Luenberger, D. G., "Observers for Multivariable Systems", IEEE Transactions on Automatic Control, Vol. AC-11, No. 2, pp. 190-197, April, 1966.

43. "Sequential Processing Techniques for Trajectory Estimation", NASA CR-1360, NASA, Washington, D. C., October, 1969.

Captions

Figure 2.1. The Stochastic Control System.

Figure 3.1. The Stochastic Control System with "Separated" Controller.

Figure 3.2. Optimal Linear Stochastic Control System.

Figure 3.3. Nonlinear Plant Simulation.

Figure 4.1 Optimal Linear Filter

Figure 5.1. Optimal Linear Controller

Figure 8.1. Deterministic Observer

Figure 9.1. The Simulation Block Diagram.

Figure 9.2. The Simulation Flow Graph.

Figure 10.1. v, w_1, w_2 vs. t. $q = .5$, $r_{11} = 1$, $r_{22} = .1^2$ Vertical scale: 2 volts/division; time scale: 1msec/division.

Figure 10.2. $e_1(t)$, $\hat{e}_1(t)d$ vs. t. With no noise the deterministic observer (with $e_1(0) \neq \hat{e}_1(0)d$) converges rapidly to e(t). Vertical scale: 0.1 volts/division; time scale: 10 msec/division.

Figure 10.3. $e_1(t)$, $\hat{e}_1(t)_d$ vs. t. With $q = .5$, $r_{11} = r_{22} = .01$, the deterministic observer has a very noisy output. Vertical scale: .01 volts/division; time scale: 10 msec/division.

Figure 10.4. $e_1(t)$ $\hat{e}_1(t)$ vs. t. With the same noise statistics as in Figure 10.3. the Kalman filter provides a less noisy output than the deterministic observer. $P_{11}(0) = .1$ was used to account for initial error. Vertical scale: .01 volts/division; time scale: 10 msec/division.

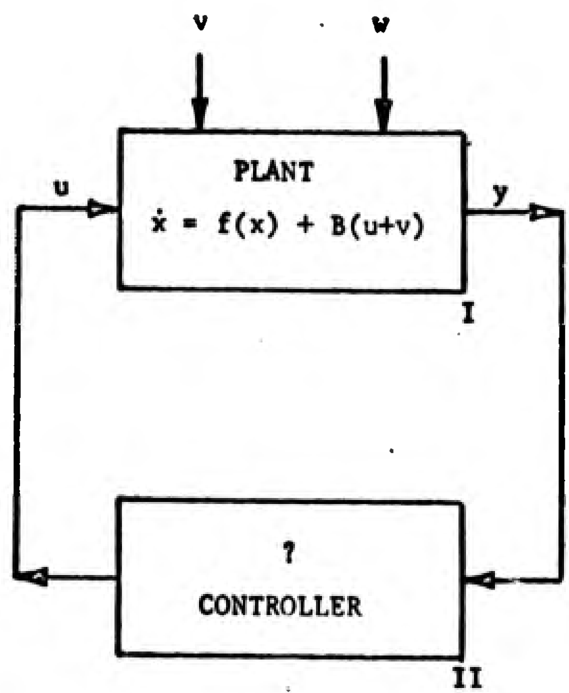
Figure 10.5. $K_{12}(t)$ vs. t. With $q = .5$, $r_{11} = r_{22}$ .1, and $p(0) = 0$, $K_{12}(t)$ is the most rapidly changing Kalman gain. This figure illustrates the discretization of the Kalman gains. Vertical scale: .25 volts/division; time scale: 10 msec/division.

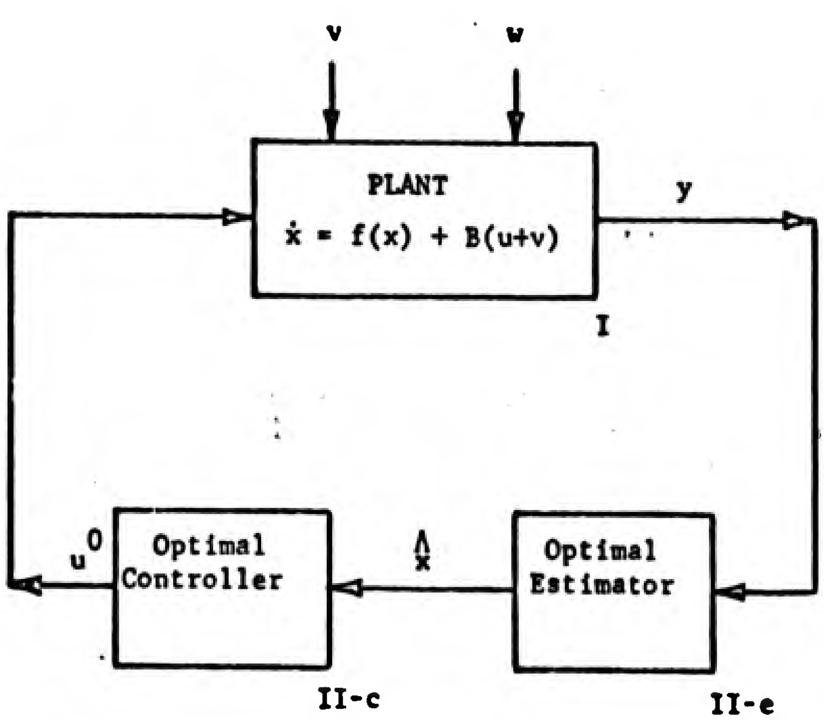
Figure 10.6. $100 \int_0^t (e_1^2 + e_2^2 + e_3^2)\, dt$, $100 \int_0^t u^2 dt$ vs. t. Note that since $D(T) = 0$, then $u(T) = 0$, causing $\int_0^t u^2 dt$ to level off as t nears T. Vertical scale: .02 volts/division; time scale: 50 msec/division.

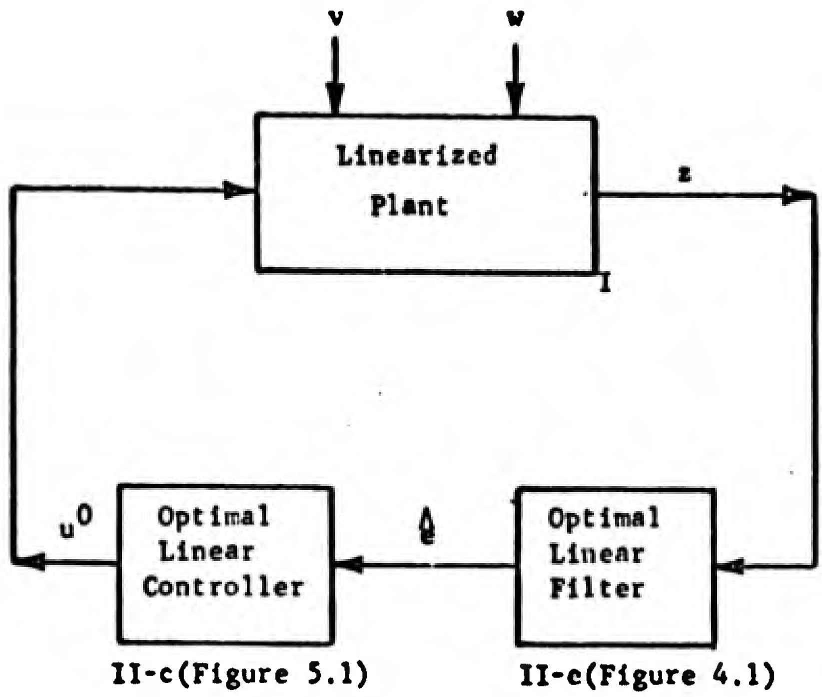
Figure 10.7. $(u+v)$, e_1, z_1, $\hat{e}_1$ vs. t. $q = 0.5$, $r_{11} = r_{22} = 0.1$. Vertical scales: 2 volts/division for u + v .02 volts/division for $e_1(t)$ and $\hat{e}_1(t)$, 1 volt/division for z_1; time scale: 10 msec/division.

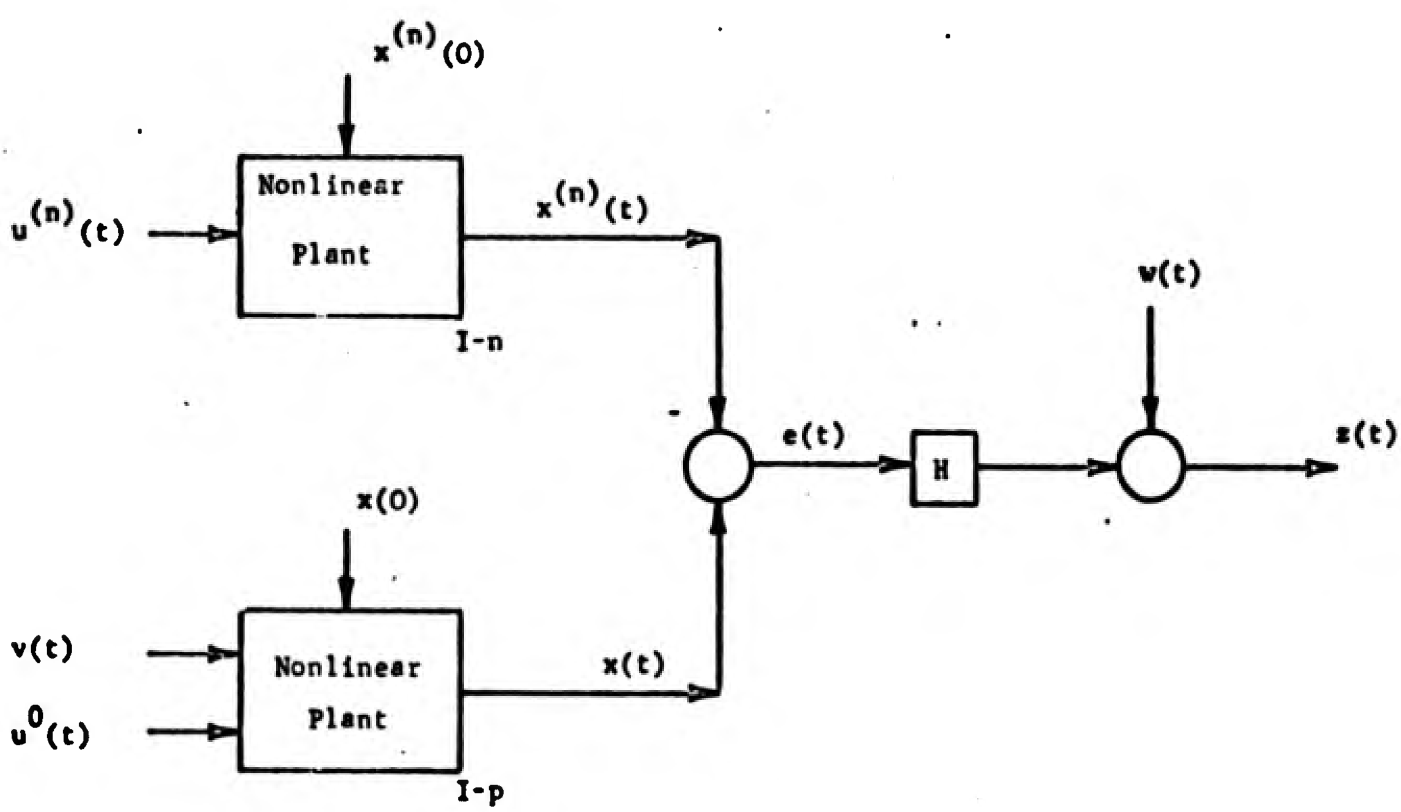
Figure 10.8. Cost Comparison for Various Input Noise Levels.

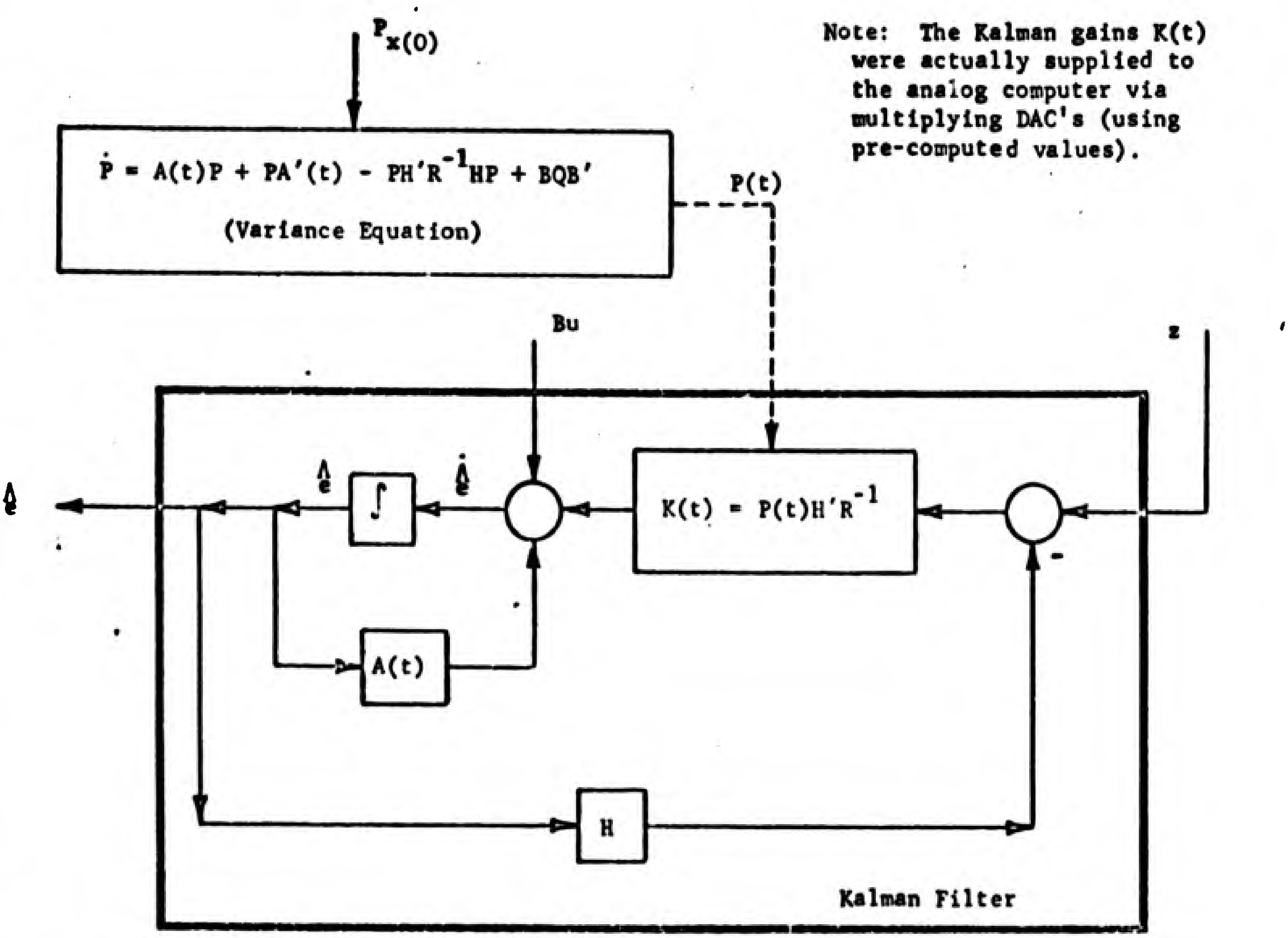
Figure 10.9. Cost Comparison for Various Measurement Noise Levels.

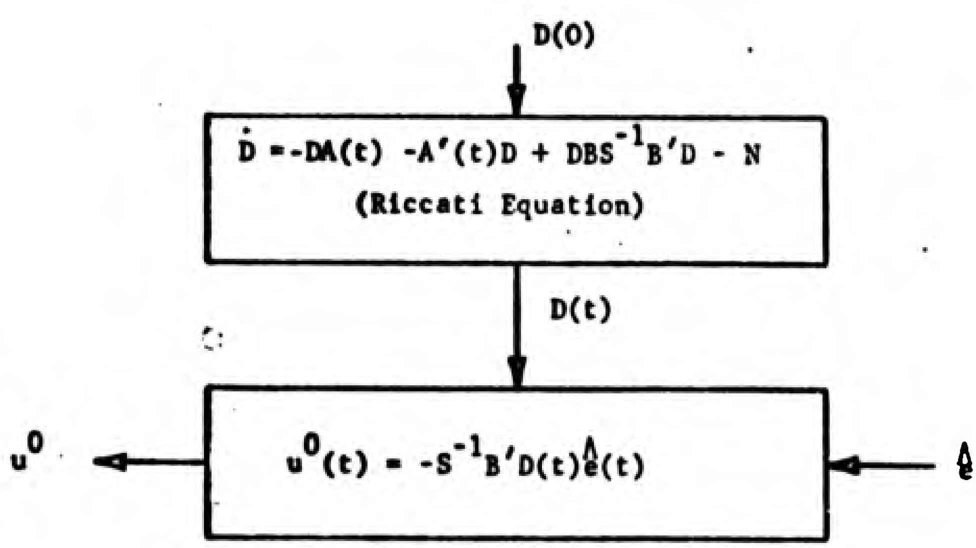

v
w
PLANT
ẋ = f(x) + B(u+v)
u
y
I
?
CONTROLLER
II

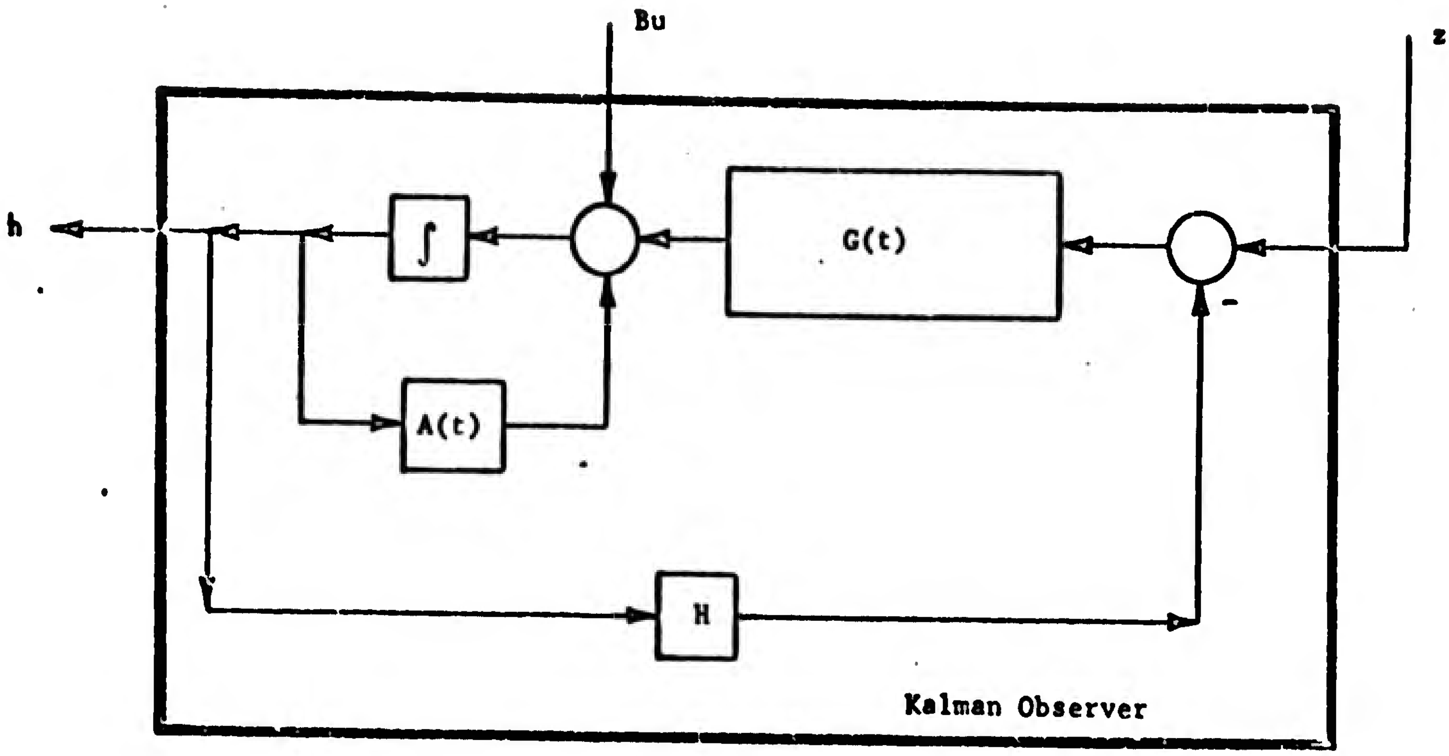

v
w
PLANT
ẋ = f(x) + B(u+v)
y
I
u0
Optimal
Controller
x̂
Optimal
Estimator
II-c
II-e

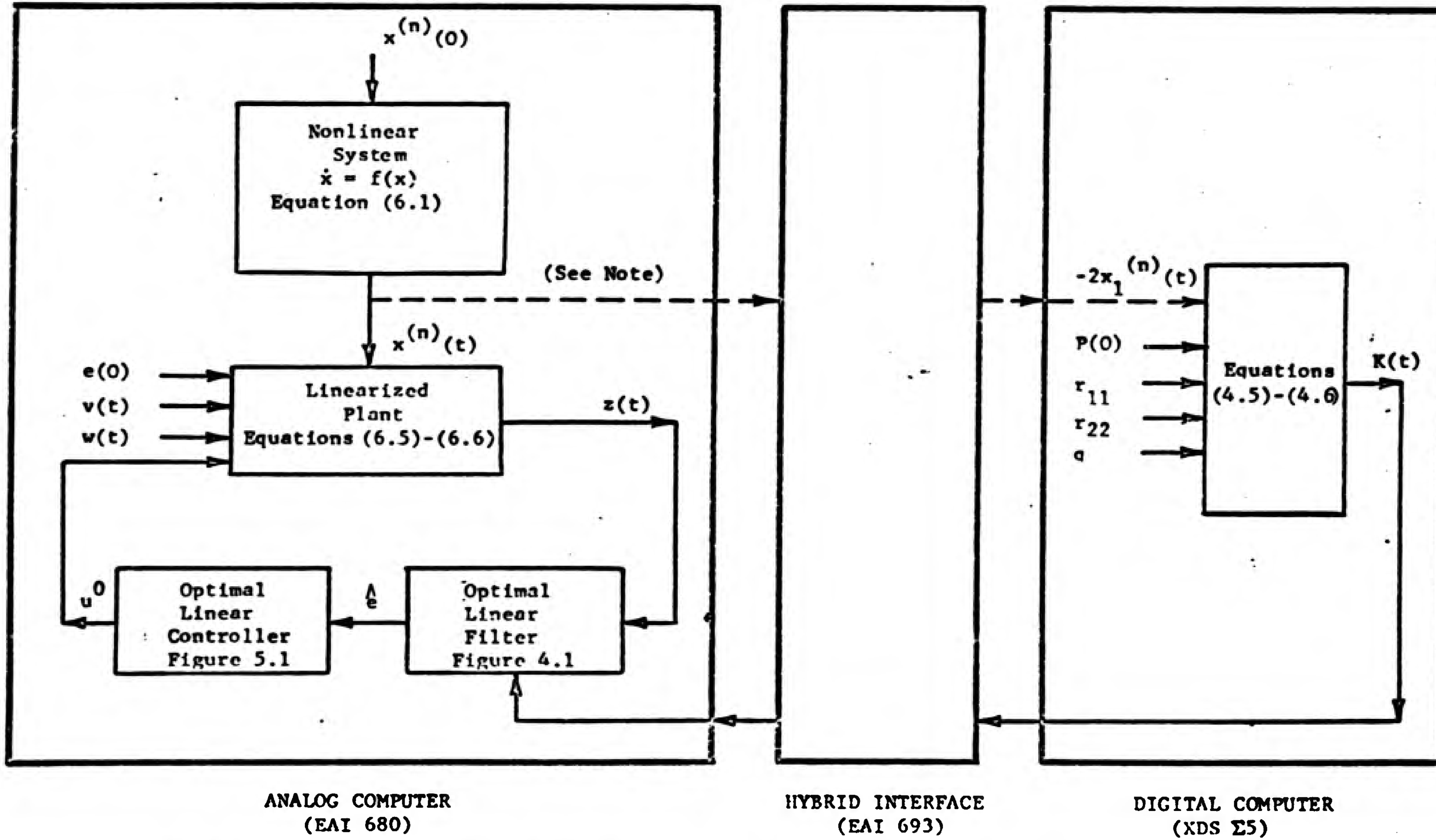

v
w
Linearized
Plant
z
I
u^0
Optimal
Linear
Controller
$\hat{e}$
Optimal
Linear
Filter
II-c(Figure 5.1)
II-c(Figure 4.1)


$x^{(n)}(0)$
$u^{(n)}(t)$
Nonlinear
Plant
$x^{(n)}(t)$
I-n
v(t)
e(t)
H
z(t)
x(0)
v(t)
$u^0(t)$
Nonlinear
Plant
x(t)
I-p

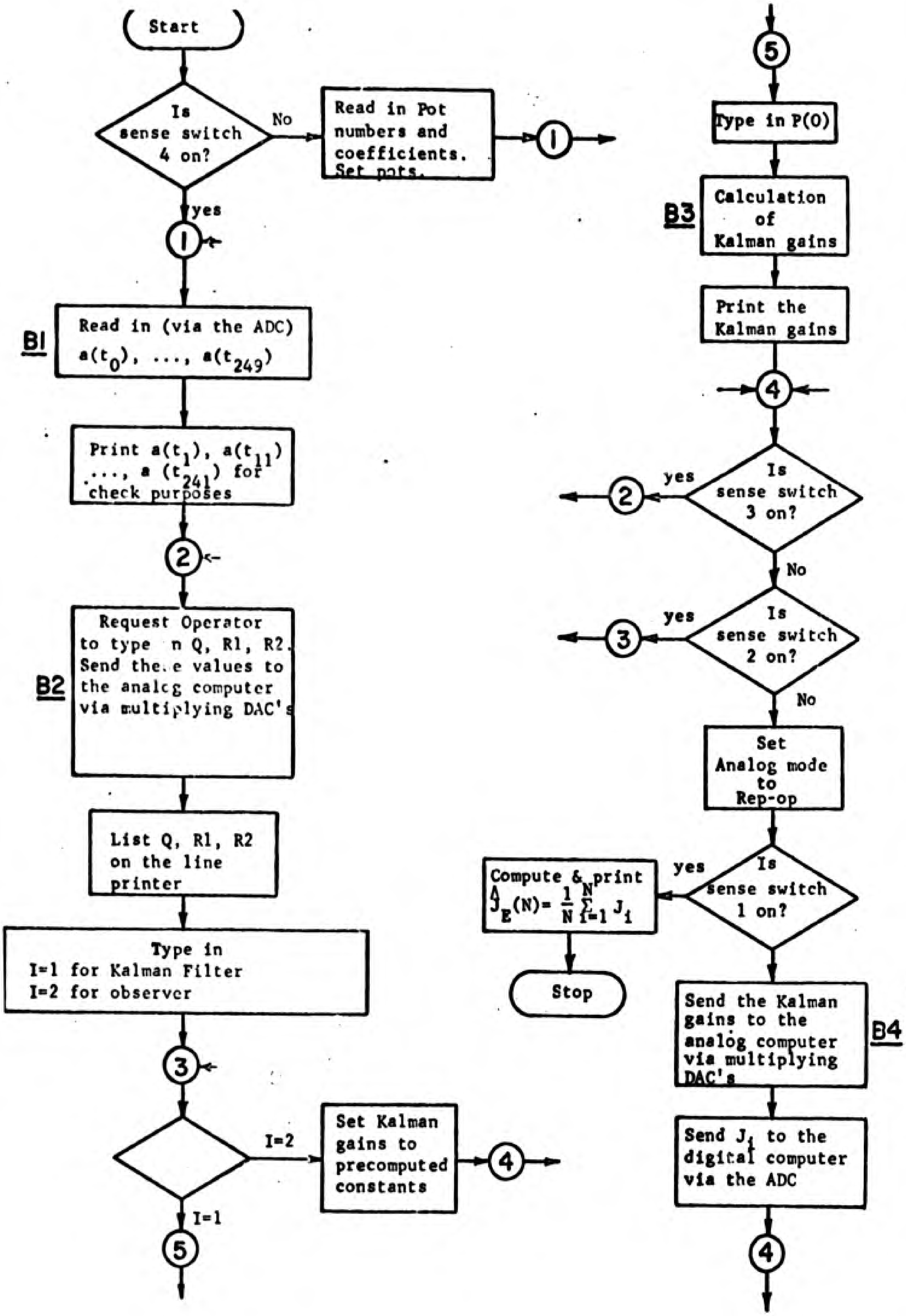

$P_{x(0)}$
Note: The Kalman gains K(t) were actually supplied to the analog computer via multiplying DAC's (using pre-computed values).
$\dot{P} = A(t)P + PA'(t) - PH'R^{-1}HP + BQB'$
(Variance Equation)
P(t)
Bu
z
$\hat{e}$
$\dot{\hat{e}}$
$K(t) = P(t)H'R^{-1}$
A(t)
H
Kalman Filter


D(0)
$\dot{D} = -DA(t) - A'(t)D + DBS^{-1}B'D - N$
(Riccati Equation)
D(t)
u^0
$u^0(t) = -S^{-1}B'D(t)\hat{e}(t)$
$\hat{e}$

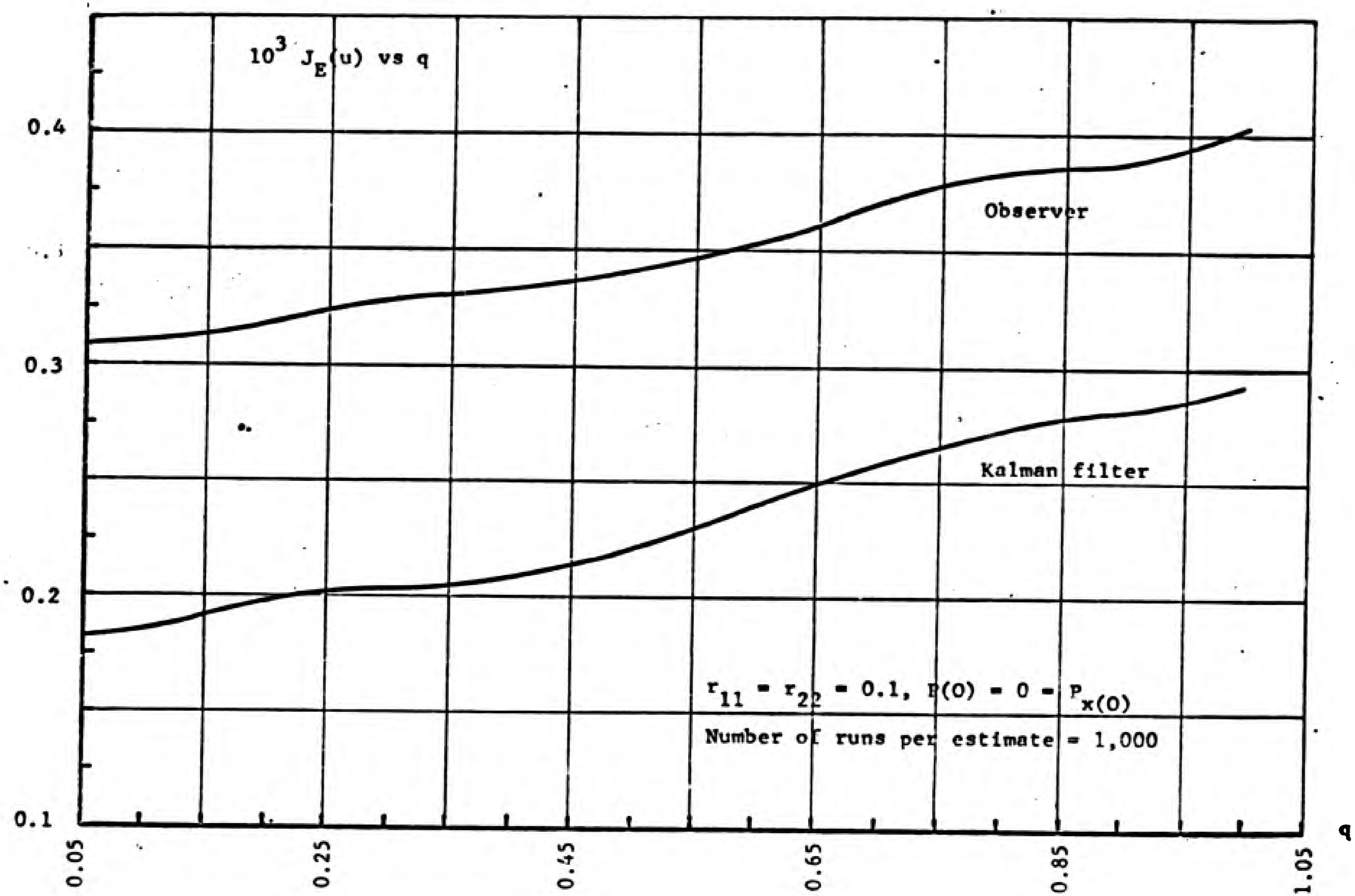

Bu
z
h
∫
G(t)
-
A(t)
H
Kalman Observer



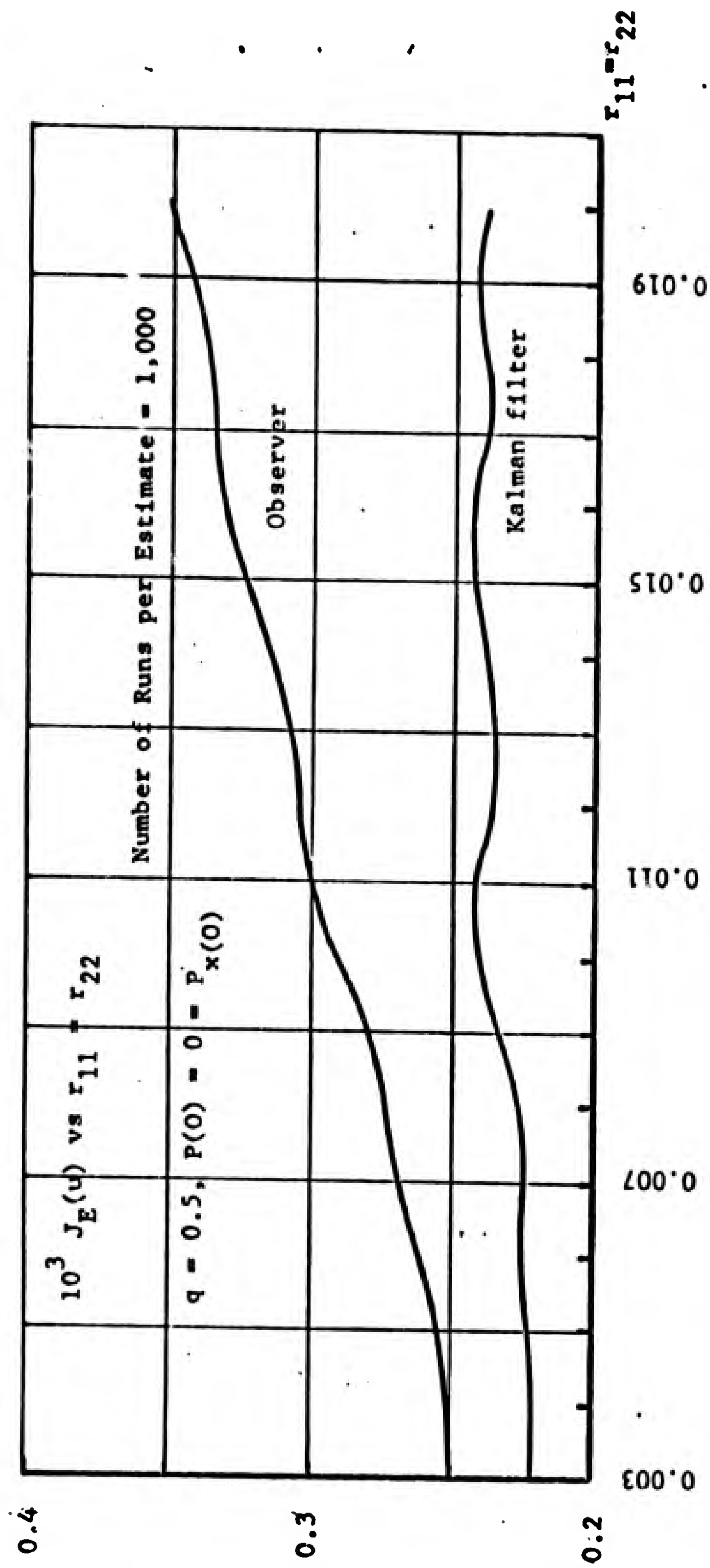
NOTE: This (dashed) data line is used only while pre-computing the Kalman gains $K_{ij}(t_m)$, m = 1, ..., 250.



$\hat{J}_E(N) = \frac{1}{N}\sum_{i=1}^{N} J_i$



10^3 J_E(u) vs q
Observer
Kalman filter
r_11 = r_22 = 0.1, P(0) = 0 = P_x(0)
Number of runs per estimate = 1,000
0.4
0.3
0.2
0.1
0.05
0.25
0.45
0.65
0.85
1.05
q



10^3 $J_E(u)$ vs $r_{11} = r_{22}$
q = 0.5, P(O) = 0 = $P_x(O)$
Number of Runs per Estimate = 1,000
Observer
Kalman filter
0.4
0.3
0.2
0.003
0.007
0.011
0.015
0.019
$r_{11}=r_{22}$

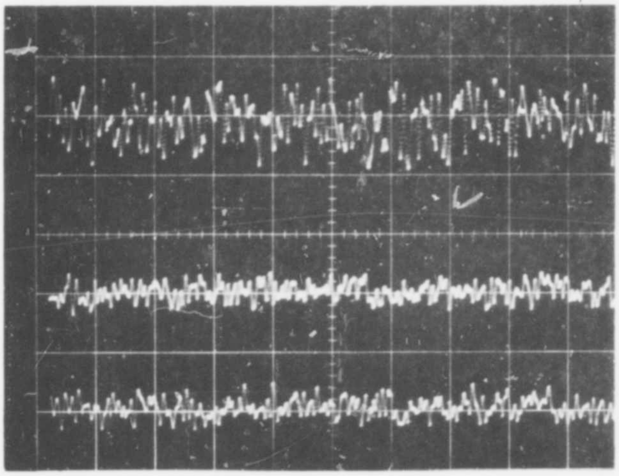


Figure 10.1.

v, w_1&w_2 vs. t. q = .5, r_{11} = 1, r_{22} = .1 Vertical scale: 2 volts/ division; time scale: 1msec/division.

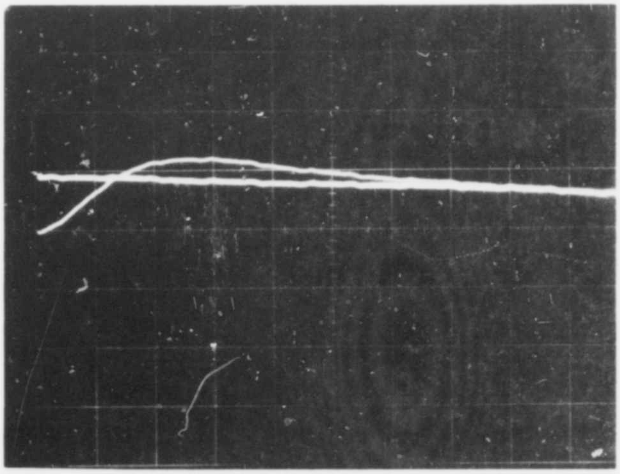


Figure 10.2.

$e_1(t)$&$\hat{e}_1(t)d$ vs. t. With no noise the deterministic observer (with $e_1(0) \neq \hat{e}_1(0)d$) converges rapidly to e(t). Vertical scale: 0.1 volts/ division; time scale: 10 msec/ division.

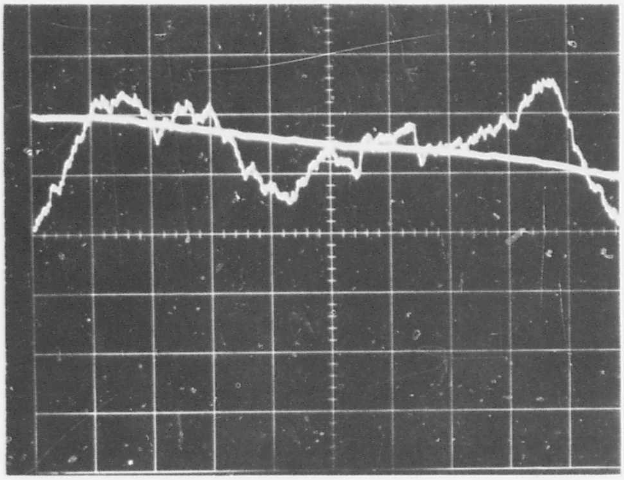


Figure 10.3.

$e_1(t)$&$\hat{e}_1(t)_d$ vs. t. With q = .5, $r_{11} = r_{22} = .01$, the deterministic observer has a very noisy output. Vertical scale: .01 volts/division; time scale: 10 msec/division.

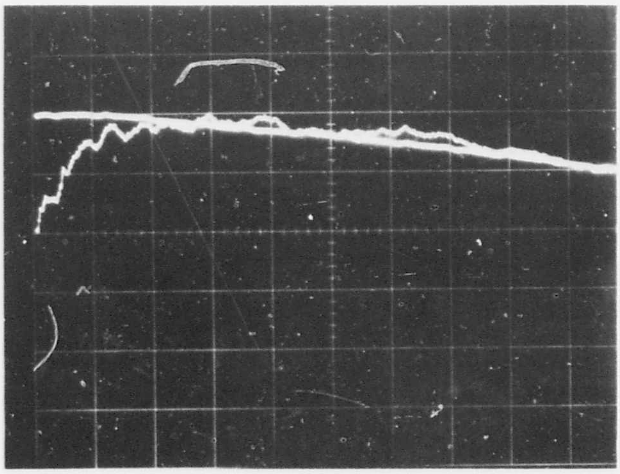


Figure 10.4.

$e_1(t)$&$\hat{e}_1(t)$ vs. t. With the same noise statistics as in Figure 10.3 the Kalman filter provides a less noisy output than the deterministic observer. $p_{11}(0) = .1$ was used to account for initial error. Vertical scale: .01 volts/division; time scale: 10 msec/division.

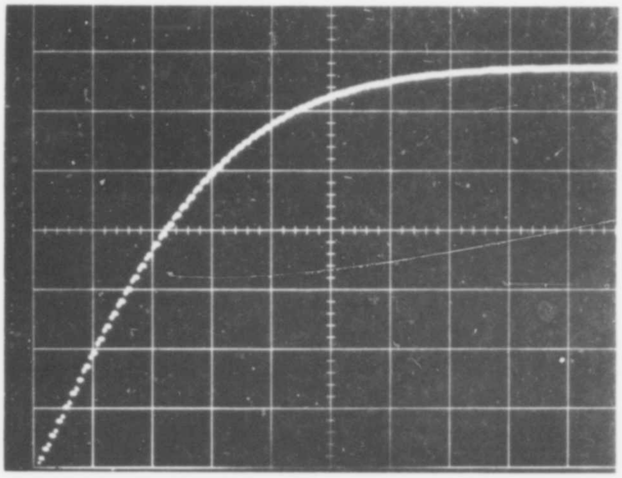


Figure 10.5.

$K_{12}(t)$ vs. t. With q = .5, r_{11} = r_{22} .1, and p(0) = 0, $K_{12}(t)$ is the most rapidly changing Kalman gain. This figure illustrates the discretization of the Kalman gains. Vertical scale: .25 volts/division; time scale: 10 msec/division.

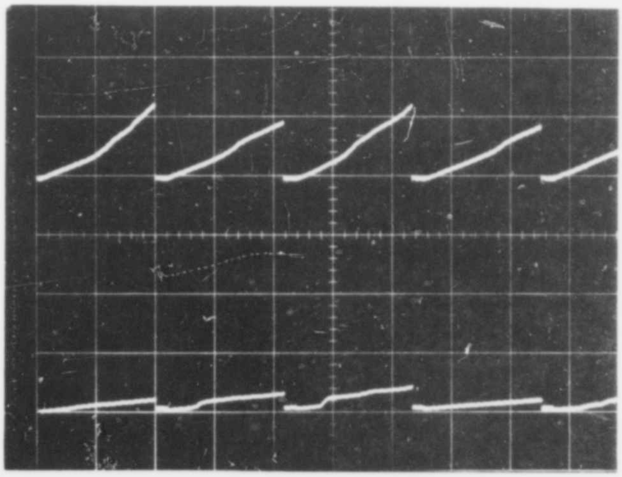


Figure 10.6.

$100 \int_0^t (e_1^2 + e_2^2 + e_3^2)\, dt$ & $100 \int_0^t u^2 dt$ vs. t. Note that since D(T) = 0, then u(T) = 0, causing $\int_0^t u^2 dt$ to level off as t nears T. Vertical scale: .02 volts/division; time scale: 50 msec/division.

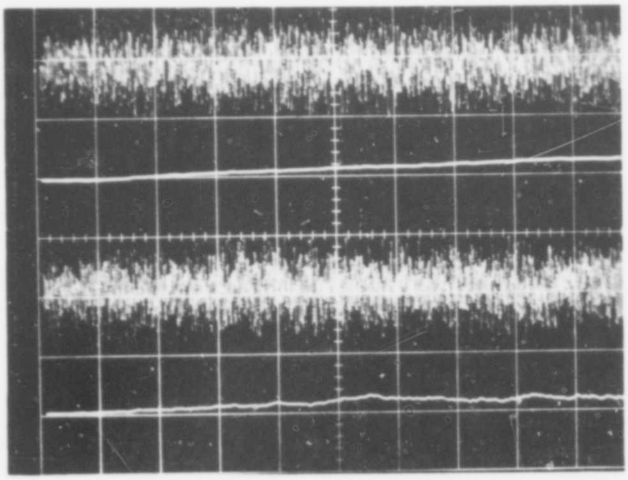


Figure 10.7.

$(u+v)$, e_1, z_1 & $\hat{e}_1$ vs. t.

$q = 0.5$, $r_{11} = r_{22} = 0.1$. Vertical scales: 2 volts/division for $u + v$ .02 volts/division for $e_1(t)$ and $\hat{e}_1(t)$, 1 volt/division for z_1; time scale: 10 msec/division.

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13. ABSTRACT

By means of studying a particular system it is shown that a controller optimized on a deterministic basis can be considerably inferior to one optimized by explicitly including system uncertainties and disturbances. Hybrid computation techniques as they apply to simulating stochastic control systems are discussed. The model equations are formulated in continuous-time and the analog computer assumes the major burden of the simulation workload. This study together with the one reported in reference (26) serve to illustrate how the high-speed capabilities of hybrid computers can be effectively employed in the design by simulation of stochastic control systems,

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