PROC. SYMPOSIUM ON NONLINEAR ESTIMATION THEORY AND ITS APPLICATIONS, UNIV. OF CALIF. AT SAN DIEGO, SEPT., 1970

ON THE ELIMINATION OF MEAN STEADY-STATE ERRORS IN KALMAN FILTERS*

by

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ABSTRACT

The purpose of this note is to examine under what conditions one can eliminate steady state bias errors that arise when the nominal parameters used in the realization of a Kalman-Bucy filter are different from the actual plant parameters. It is shown, under suitable conditions, that by adding to the standard Kalman-Bucy filter a correction signal, which is obtained by integration of the innovations process (residual signal), leads to a design that has zero mean steady state error.

1. INTRODUCTION

It has been widely recognized that the Kalman-Bucy filter will yield biased estimates and exhibit long term instability (from the estimation point of view) whenever the nominal parameters used in the implementation of the Kalman-Bucy filter differ (even slightly) from the true parameters of the plant. Several investigators (see references [1] to [3]) have analyzed this problem in order to obtain qualitative estimates of the errors involved. The majority of the available results analyze the differences between the Riccati equations when the true and the nominal parameters are used.

The problem of doing accurate estimation when the plant parameters are not precisely known involves parameter estimation and, hence nonlinear filtering whose increase in complexity may be undesirable for certain applications. A popular technique that is often used is to increase the covariance noise of the plant white noise, so as to tell the mathematics that there is an additional uncertainty in the equations. The net effect is to increase the gain matrix that operates on the innovations or residual signals so that the "filter pays more attention to the observations".

The purpose of this note is to examine the existence of mean steady-state (bias) estimation errors when the plant and filter parameters are different. For this reason, it is easier to deal with the performance of steady state Kalman filters for linear, time-invariant plants subject to stationary white noise processes. It will be shown that, under certain conditions, zero mean steady-state estimation error can be obtained by adding to the standard steady-state Kalman filter a feedfoward channel that involves the integration of the innovations signals. In this manner, one can guarantee the long term stability of the estimating filter.

2. DEFINITIONS

In this section the assumptions are stated regarding the plant parameters and their nominal values.

2.1 PLANT EQUATIONS

The plant is assumed to be a linear, time-invariant, completely controllable and observable system with state $x(t)\epsilon R_n$, an output vector $y(t)\epsilon R_m$ and a constant input vector $u(t) = u$ for all $t, u\epsilon R_r$. The plant is driven by a zero mean, stationary, white noise process $\xi(t)\epsilon R_n$. Thus, the plant state equation is

(2.1) $\dot{x}(t) = Ax(t)+Bu+\xi(t);\ x(0)= x_o;\ y(t)= Cx(t)$

The initial state x_o is viewed as a random vector with known statistics

(2.2) $E\{x_o\}= \bar{x}_o\ ;\quad \text{cov}\,[x_o;\ x_o] = \Sigma_o$

The white noise $\xi(t)$ is assumed to have known statistics

(2.3) $E\{\xi(t)\}= 0;\ \text{cov}[\xi(t);\ \xi(\tau)]= \Xi\delta(t-\tau),$
$\Xi= \Xi'>0\ ;\quad \text{cov}\,[\xi(t);\ x_o]= 0$

The three constant matrices A,B, and C represent the true plant parameter matrices. It is assumed that their values are not precisely known. What is known to the designer is a triplet of constant matrices A_n, B_n, C_n which represents the nominal values of A,B, and C respectively. Hence, as far as the designer is concerned, the nominal plant is described by- instead of (2.1)- the equations

(2.4) $\dot{x}(t) = A_n x(t) + B_n u + \xi(t);\ x(0) =$
$y(t) = C_n x(t)$

It is assumed that the nominal plant is also completely controllable and observable, Furthermore, it is assumed that the true plant (2.1) and the nominal plant (2.4) are strictly stable, i.e.

(2.5) $\text{Re}\ \lambda_i[A] < 0;\quad \text{Re}\ \lambda_i[A_n]< 0$

2.2 OBSERVATION EQUATIONS

The measurement vector is $z(t)$

(2.6) $z(t) = y(t) + \theta(t) = C\,x(t) + \theta(t)$

$\theta(t)$ is a zero mean stationary white process with known covariance matrix

(2.7) $\text{cov}[\theta(t);\ \theta(\tau)]=\Theta\delta(t-\tau);\Theta=\Theta'>0$

*This research was conducted at the Decision and Control Sciences Group of the M.I.T. Electronic Systems Laboratory with support extended by the Air Force Office of Scientific Research under grant AFOSR 69-1724 (suppl. A)

It is also assumed that $\theta(t)$ is independent of $\xi(t)$ and x_o.

2.3 NOTATION

The vector $w(t)$ will be used to denote the estimate of the true state vector $x(t)$.

The residual signal (innovations process) will be denoted by $\mu(t)$

(2.8) $\mu(t) = z(t) - C_n w(t) = Cx(t) - C_n w(t) + \theta(t)$

The state estimation error will be denoted by $e(t)$

(2.9) $e(t) = x(t) - w(t)$

An over score will be used to denote conditional expectations; thus

(2.10) $\bar{e}(t) = E\{e(t) \mid z(\tau);\ 0 \le \tau \le t\}$

(2.11) $\bar{x}(t) = E\{x(t)\}$

(2.12) $\bar{w}(t) = E\{w(t) \mid z(\tau);\ 0 \le \tau \le t\}$

2.4 THE STANDARD STEADY-STATE KALMAN FILTER

The standard steady-state Kalman filter is a linear time invariant system that generates the estimate $w(t)$ of $x(t)$ according to the algorithm (which uses the nominal values of the plant parameters)

(2.13) $\dot{w}(t) = A_n w(t) + B_n u + G_n \mu(t);$
$w(0) = \bar{x}_o$

where G_n is an nxm gain matrix given by

(2.14) $G_n = \Sigma_n C_n' \theta^{-1}$

and where Σ_n is the symmetric and positive definite solution of the algebraic Riccati equation

(2.15) $0 = \Sigma_n A_n + A_n' \Sigma_n + \Xi - \Sigma_n C_n' \theta^{-1} C_n \Sigma_n.$

3. THE AUGMENTED STEADY-STATE FILTER

In this section, the standard steady state Kalman filter, described in 2.4, is changed somewhat so as to indicate means for removing the bias errors

3.1 THE BIAS CORRECTION TERM, v(t).

Consider the following linear system whose state vector $w(t)$ is the estimate of $x(t)$

(3.1) $\dot{w}(t) = A_n w(t) + B_n u + G_n \mu(t) + v(t);\ w(0) = \bar{x}_o$

Note that if $v(t) = 0$, then (3.1) coincides with the standard steady-state Kalman filter. In eq. (3.1) G_n is still described by (2.14) and (2.15) and $\mu(t)$ is the residual signal vector given by (2.8). The state estimation error $e(t)$ of (2.9) can easily be shown to satisfy the stochastic differential equation

(3.2) $\dot{e}(t) = [A_n - G_n C_n] e(t) + [A - A_n] x(t) + [B - B_n] u$
$- G_n [C - C_n] x(t) + \xi(t) - G_n \theta(t) - v(t)$

The conditional mean $\bar{e}(t)$ satisfies the equation

(3.3) $\dot{\bar{e}}(t) = [A_n - G_n C_n]\, \bar{e}(t) + [A - A_n]\bar{x}(t) + [B - B_n] u$
$- G_n [C - C_n] \bar{x}(t) - \bar{v}(t)$

while the mean state $\bar{x}(t)$ satisfies the equation

(3.4) $\dot{\bar{x}}(t) = A\bar{x}(t) + Bu$

3.2 MEAN STEADY STATE VALUES

Form (3.4) it follows that

(3.5) $\bar{x}_{ss} \stackrel{\Delta}{=} \lim_{t\to\infty} \bar{x}(t) = -A^{-1} Bu$

(3.6) $\lim_{t\to\infty} \dot{\bar{x}}(t) = 0$

Since the matrix $[A_n - G_n C_n]$ is strictly stable it is nonsingular. Hence, the mean steady-state estimation error is given by

(3.7) $\bar{e}_{ss} \stackrel{\Delta}{=} \lim_{t\to\infty} \bar{e}(t) = [A_n - G_n C_n]^{-1} \cdot G_n [C - C_n] \bar{x}_{ss} + \bar{v}(t)$
$+ A_n \bar{x}_{ss} + B_n u$

Ideally one should select $\bar{v}(t)$ so as to make $\bar{e}_{ss} = 0$. However, this is not possible because it implies knowledge of C and $\bar{x}_{ss}$ (which involves knowledge of A and B in view of (3.5)). Thus, all one can say is that $\bar{v}(t)$ must have the property that

(3.8) $\lim_{t\to\infty} \bar{v}(t) = \text{unknown constant}$

(3.9) $\lim_{t\to\infty} \dot{\bar{v}}(t) = 0$

This leads us to differentiate (3.3) to obtain the second-order vector equation

(3.10) $\ddot{\bar{e}}(t) = [A_n - G_n C_n] \dot{\bar{e}}(t) + [A - A_n] \dot{\bar{x}}(t)$
$- G_n [C - C_n]\, \dot{\bar{x}}(t) - \dot{\bar{v}}(t)$

3.3 STABILITY CONSIDERATIONS

What we are concerned with is the stability as $t\to\infty$ of (3.10). In view of (3.6), it suffices to study the stability of the equation

(3.11) $\ddot{\bar{e}}(t) = [A_n - G_n C_n]\, \dot{\bar{e}}(t) - \dot{\bar{v}}(t)$

If $\dot{\bar{v}}(t) = 0$ (as is the case with the standard steady-state Kalman filter), then (3.11) is not ASIL even though $[A_n - G_n C_n]$ is a strictly stable matrix.

However, if one sets

(3.12) $\dot{\bar{v}}(t) = L\, \bar{e}(t)$

where L is a positive definite matrix, then a slight extension of a result in Bellman's book[reference [4] p.246] can be used to show that the equation

(3.13) $\ddot{\bar{e}}(t) - [A_n - G_n C_n]\, \dot{\bar{e}}(t) + L\, \bar{e}(t) = 0$

is strictly stable, because all of the roots of the characteristic polynomial

(3.14) $\det\left(\lambda^2 I - \lambda [A_n - G_n C_n] + L\right)$

have negative real parts.

3.4 IMPLEMENTATION OF $\dot{\bar{v}}(t) = L\, \bar{e}(t)$

The difficulty lies in the implementation of the desired result. We shall examine first a special case.

(3.15) ASSUMPTION : rank C = rank $C_n = n$, i.e. C^{-1} and C_n^{-1} exist

Consider the following generation of $\dot{v}(t)$

(3.16) $\dot{v}(t) = M\ \mu(t) = MC\, x(t) - MC_n w(t) + M\theta(t)$

or,

(3.17) $\dot{v}(t) = MC_n\, e(t) + M(C - C_n) x(t) + M\theta(t)$

so that

(3.18) $\dot{v}(t) = MC_n \bar{e}(t) + M(C-C_n)\bar{x}(t)$

We can see that this does not satisfy the desired stability result because of the non-zero term $M(C-C_n)\bar{x}(t)$. However, if the C matrix is known, M can be selected appropriately. This is formalized in the following proposition.

(3.19) Proposition . Suppose that $C=C_n$ and rank $C_n = n$. Let $M=C^{-1}$. In this case

(3.20) $\dot{v}(t) = \bar{e}(t)$

and the equation

(3.21) $\ddot{\bar{e}}(t) = [A_n - G_n C_n]\dot{\bar{e}}(t) + \bar{e}(t) + [A-A_n]\dot{\bar{x}}(t)$

with $\lim_{t\to\infty} \dot{\bar{x}}(t) = 0$, has the property that

(3.22) $\lim_{t\to\infty} \bar{e}(t) = 0$

This implies that there is no mean steady state error.

The actual signal v(t) required in the augmented steady-state Kalman filter is simply obtained by the integration of $\dot{v}(t)$. Thus

(3.23) $$v(t) = \int_0^t M\,\mu(\tau)d\tau$$

3.5 DISCUSSION

It appears that if $C \neq C_n$, then there may always exist a steady state error. Means by which this can be eliminated, for example by estimating the constant value of $[C-C_n]x_{ss}$, are currently under investigation.

The assumption that $C=C_n$ is not too serious, in the opinion of this author, since in many applications one knows which state variables are being measured. However, the assumption that rank C = rank C_n=n is much more stringent since it implies that essentially the full state is being measured in the presense of noise

Suppose that $C=C_n$ and rank $C_n = m<n$. In this case, one can select the matrix M such that the matrix

(3.24) $L = MC$

is positive semidefinite.

In this case ($\dot{v}(t) = M\,\mu(t)$), there will be non-zero components of $\bar{e}(t)$ at steady state . The author conjectures that a Kalman like filter can ve designed that operates upon the residual signals $\mu(t)$ to construct an estimate $\hat{\bar{e}}(t)$of $\bar{e}(t)$ and that $\dot{v}(t)$ be generated by operating upon this estimate e(t). This is a subject for future consideration.

4. ACKNOWLEDGEMENT

The author wishe to thank Professor Jan C. Willems for his discussions on this problem.

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UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)

Massachusetts Institute of Technology
Electronic Institute of Technology
Cambridge, Massachusetts 02139

2a. REPORT SECURITY CLASSIFICATION

UNCLASSIFIED

2b. GROUP

3. REPORT TITLE

ON THE ELIMINATION OF MEAN STEADY-STATE ERRORS IN KALMAN FILTERS

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Scientific Interim

5. AUTHOR(S) (First name, middle initial, last name)

Michael Athans

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
September 1970	3	4

8a. CONTRACT OR GRANT NO.

AFOSR 69-1724

b. PROJECT NO.

9749-01

c. 61102F

681304

d.

9a. ORIGINATOR'S REPORT NUMBER(S)

9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

AFOSR 70-2677TR

10. DISTRIBUTION STATEMENT

1. This document has been approved for public release and sale; its distribut on is unlimited.

11. SUPPLEMENTARY NOTES

PROCEEDINGS
Symposium on Nonlinear Estimation Theory and its Applications, University of California, San Diego, September 1970

12. SPONSORING MILITARY ACTIVITY

Air Force Office of Scientific Research (SRM)
1400 Wilson Boulevard
Arlington, Virginia 22209

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