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ON THE EXACT DISTRIBUTIONS OF THE TRACES OF $S_1(S_1 + S_2)^{-1}$ and $S_1S_2^{-1}$

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PROJECT NO. 7071

AIR FORCE SYSTEMS COMMAND
United States Air Force

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Corrections to

"On the Exact Distributions of the Traces of $S_1(S_1+S_2)^{-1}$ and $S_1S_2^{-1}$", ARL 70-0241

P. R. Krishnaiah and T. C. Chang

1. Multiply the left side of Eq. (2.1) by $\nu!$.
2. Delete $\nu!$ on the right side of Eqs. (2.5) and (2.6).
3. On lines 12-13 of page 3, replace "the number of factors a_{ij} $(i<j)$" with "the number of terms in the summation".

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ON THE EXACT DISTRIBUTIONS OF THE TRACES OF $S_1(S_1 + S_2)^{-1}$ and $S_1S_2^{-1}$

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AEROSPACE RESEARCH LABORATORIES
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
WRIGHT-PATTERSON AIR FORCE BASE, OHIO

FOREWORD

This report was prepared for Applied Mathematics Research Laboratory, Aerospace Research Laboratories by P. R. Krishnaiah and T. C. Chang under Project 7071, "Research in Applied Mathematics". Part of the work of T. C. Chang was performed at the Aerospace Research Laboratories while in the capacity of an Ohio State University Research Foundation Visiting Research Associate under Contract F 33615 C 1758. The present affiliation of T. C. Chang is the University of Cincinnati.

In this report, the authors derived the exact distributions of the traces of two random matrices that arise in multivariate statistical analysis.

ABSTRACT

In this paper, the authors derived the exact distributions of the traces of $S_1S_2^{-1}$ and $S_1(S_1 + S_2)^{-1}$ where S_1 and S_2 are independently distributed as central Wishart matrices and the expected values of these matrices are the same. The method used involves expressing the Laplace transformations in terms of the linear combinations of products of certain double integrals and then taking the inverse Laplace transformations.

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1. Introduction. The distributions of the traces of $S_1(S_1 + S_2)^{-1}$ and $S_1 S_2^{-1}$ are quite useful in the applications of certain test procedures (e.g; see [6, 10, 12]) in multivariate analysis. (Unless otherwise specified, we will assume that S_1 and S_2 are independently distributed positive definite central Wishart matrices of order p x p and with n_1 and n_2 degrees of freedom respectively.) The exact distribution of V was derived by Nanda [9] for $p = 2, 3, 4$, and $m = 0$ when V denotes the trace of $S_1(S_1 + S_2)^{-1}$ and $m = (n_1 - p - 1)/2$. Using the method of Nanda [9], Pillai and Jayachandran [13] derived the exact density of V when $p = 3$, $m = 1, 2, 3$ and $p = 4$, $m = 1$; they have also constructed exact percentage points for $p = 3, 4$ and a few values of m. The exact density of T_0^2 where $T_0^2 = n_2 U$ and U denotes the trace of $S_1 S_2^{-1}$ was derived by Hotelling [6] for $p=2$. Constantine [1] derived the non-null density of T_0^2 for $T_0^2 < 1$ in terms of the zonal polynomials. Pillai and Chang [11] obtained an expression for the density of U for $p = 3$ by transformation of variables. Recently, Pillai and Young [14] gave an expression for the distribution of T_0^2 as a linear combination of the inverse Laplace transformations of certain pseudo-determinants when m is an integer; explicit expressions for the inverse Laplace transformations of these pseudo-determinants are evaluated in [14] for $p = 3$, $m = 0(1)5$, and $p = 4$, $m = 0, 1, 2$. Exact percentage points of the distributions

of T_0^2 are available in the literature (see [4],[5],[14])for p=2,3,4 and for some values of m. Davis [3] has shown that the density of T_0^2 satisfies an ordinary linear homogeneous differential equation of order p. The nonnull densities of U and V were derived by Pillai and Jayachandran [12] for p=2.

In this paper, the authors expressed the exact distributions of V and U, for any p and m, as linear combinations of the inverse Laplace transformations of the products of certain double integrals. The motivation behind considering the distribution of V in this paper is that it is not known* for general p and m. The method considered here is different from the one considered by Nanda [9] and it has the advantages over Nanda's method as p and m increase. In the case of the distribution of U, the inverse Laplace transformations of the products of double integrals encountered here are easier to evaluate than the inverse Laplace transformations of the pseudo-determinants that arise in Pillai and Young [14]; this is the motivation behind considering the distribution of U in this paper. It should be pointed out here, however, that the complexity of the explicit evaluation of the inverse Laplace transformations encountered in this paper and [14] increases as p and m increase. The authors have discussed about the explicit evaluation of the inverse Laplace transformations encountered in this paper and illustrated them for p=2 and m=1.

*Khatri and Pillai (Ann. Math. Statist. 39 215-226) gave an expression for the noncentral distribution of V in terms of zonal polynomials but it is not of much practical use since it is valid only when V<1.

2. Preliminaries. The inverse and determinant of a square matrix M is denoted by M^{-1} and $|M|$ respectively. Also, let $L(t;f)$ denote the Laplace transformation of f.

If $\Lambda = (a_{ij})$ is a skew-symmetric matrix (that is, $\Lambda = -\Lambda'$) and Λ is of even order (say 2ν), then $|\Lambda|^{1/2}$ is known as a pfaffian. It is also known (see [2 , p. 521]) that the pfaffian can be expressed as a polynomial as follows:

(2.1) $$|\Lambda|^{1/2} = \sum \pm a_{i_1 i_2} a_{i_3 i_4} \cdots a_{i_{2\nu-1} i_{2\nu}}$$

where the summation is over all permutations $i_1, \ldots, i_{2\nu}$ of $1, 2, \ldots, 2\nu$ subject to the restrictions $i_1 < i_2$, $i_3 < i_4, \ldots, i_{2\nu-1} < i_{2\nu}$ and the sign is positive or negative according as the permutation is even or odd; here we note that the number of factors $a_{ij}(i < j)$ in (2.1) is equal to $1.3\ldots(2\nu - 1)$. We need the following notations in the sequel.

Let

(2.2) $$\rho(t,\psi;q,r,L,N) = \int \cdots \int_{L \le x_1 \le \ldots \le x_q \le N} \prod_{i=1}^{q} \{x_i^r \psi(t,x_i)\} \prod_{i>j}^{q} (x_i - x_j) \prod_{i=1}^{q} dx_i ,$$

$$F_s^u(t) = \int_L^N F_s(t,\theta)\theta^u \psi(t,\theta)d\theta \; , \quad F_s(t,\theta) = \int_L^\theta \psi(t,x)x^s dx,$$

and

(2.3) $$f_s^u(t) = F_s^u(t) - F_u^s(t).$$

In addition, let

$$(2.4)\quad \Delta(t,\psi;2\nu,r,L,N) = \begin{vmatrix} 0 & f_r^{r+1}(t) & \cdots & f_r^{r+2\nu-1}(t) \\ f_{r+1}^{r}(t) & 0 & & f_{r+1}^{r+2\nu-1}(t) \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ f_{r+2\nu-1}^{r}(t) & f_{r+2\nu-1}^{r+1}(t) & \cdots & 0 \end{vmatrix}^{1/2}$$

and let $G_u(t,\psi;2\nu+1, r,L,N)$ denote the determinant $|A_u|^{1/2}$ where $A_u = (a_{ij})_{i,j=1,2,\ldots,u-1,u+1,\ldots,2\nu+1}$, and $a_{ij} = f_{i+r-1}^{j+r-1}(t)$. We need the following lemma in the sequel.

Lemma 2.1 Let $\psi(t,x)$ be a function of x such that the integral given in (2.2) exists, and let $L < N$ and $r > 0$ be real constants. Also, let t be real or complex. Then

$$(2.5)\quad \rho(t,\psi;q,r,L,N) = \Delta(t,\psi;2\nu,r,L,N)/\nu!$$

when $q = 2\nu$, and

$$(2.6)\quad \rho(t,\psi;q,r,L,N) = \frac{1}{\nu!}\sum_{i=0}^{2\nu}(-1)^i F_{r+i}(t,N)G_{i+1}(t,\psi;2\nu+1,r,L,N)$$

when $q = 2\nu + 1$.

Mehta [8] proved the above lemma when $q = 2\nu$ and $\psi(t;x) = \exp(-x^k)$ where k is any positive integer. Using the method in [8], Krishnaiah and Chang [7] established that the lemma is true in general.

We need the following lemmas also in the sequel.

Lemma 2.2. Let

$$(2.7)\quad R(t;r,r+k) = \int_0^1\int_0^\theta \exp\{-t(x+\theta)\}(x^r\theta^{r+k} - x^{r+k}\theta^r)\,dx\,d\theta$$

where k is a non-negative integer, $r \geq 0$. Then

$$(2.8)\quad R(t;r,r+k) = \sum_{i=1}^{k} \frac{1}{(r+i)_i}[2(1/2)^{i-1}(2r+k+i)_{i-1}I(2t;2r+k+1)$$

$$-\exp(-t)(r+k)_{i-1}I(t;r+k-i+1)]$$

where

$$I(at;b) = \int_0^1 \exp(-atx)x^b\,dx \qquad \text{for } b > -1,$$

and

$$(x)_n = x(x-1)\ldots(x-n+1),\ (x)_o = 1.$$

Lemma 2.3. Let

$$(2.9)\quad S(t;r,r+k) = \int_0^1\int_0^\theta \exp\{-t(\tfrac{1}{x}+\tfrac{1}{\theta})\}(x^r\theta^{r+k} - x^{r+k}\theta^r)\,dx\,d\theta$$

where k is a non-negative integer, $r \geq 0$ and $Re(t) > 0$. Then

$$(2.10)\quad S(t;r,r+k) = \sum_{i=1}^{k} \frac{\exp(-2t)}{(r+k+1)_i}[(r+i)_{i-1}g(t;r+i+1,1)$$

$$-(1/2)^{i-1}(2r+k+2)_{i-1}g(t;2r+k+3,2)]$$

where $Re(t)$ denotes the real value of t, and

$$(2.11) \quad g(t;a,b) = \int_0^\infty \exp(-tz)/(1 + \frac{z}{b})^a dz.$$

The above two lemmas can be proved using integration by parts.

3. Distribution of the trace of $S_1(S_1 + S_2)^{-1}$. Let S_1 and S_2 be independently distributed as central Wishart matrices with n_1 and n_2 degrees of freedom respectively and let $E(S_1/n_1) = E(S_2/n_2) = \Sigma$. Also, let $\theta_p > \theta_{p-1} > \cdots > \theta_1$ be the latent roots of $S_1(S_1 + S_2)^{-1}$. Then, it is known (see Roy [15]) that the joint density of $\theta_1, \ldots, \theta_p$ is given by

$$(3.1) \quad g(\theta_1, \ldots, \theta_p) = C(p,r,m) \prod_{i=1}^{p} \{\theta_i^r (1 - \theta_i)^m\} \prod_{i>j}^{p} (\theta_i - \theta_j)$$

$$0 < \theta_1 < \cdots < \theta_p < 1$$

where

$$C(p,r,m) = \frac{\pi^{\frac{1}{2}p^2} \Gamma_p(r + m + p + 1)}{\Gamma_p((2r + p + 1)/2)\Gamma_p((2m + p + 1)/2)\Gamma_p(p/2)},$$

$$\Gamma_p(a) = \pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma(a - \frac{1}{2}(i - 1)),$$

$$r = (n_1 - p - 1)/2 \text{ and } m = (n_2 - p - 1)/2.$$

Then the Laplace transformation of $V = \sum_{i=1}^{p} \theta_i$ is given by

$$\begin{aligned} \mathcal{L}(V;t) &= E\,\{\exp(-tV)\} \\ &= C(p,r,m)\,\rho(t;\psi_1;p,r,0,1) \end{aligned} \tag{3.2}$$

where $\psi_1(t,x) = \exp(-tx)(1-x)^m$. Using Lemma 2.1 and Equation (2.1), we obtain

$$\text{(3.2a)}\quad \mathcal{L}(V;t) = \frac{C(p,r,m)}{q!} \sum \pm\, b_{i_1 i_2}(t) b_{i_3 i_4}(t) \cdots b_{i_{2q-1}, i_{2q}}(t)$$

$$\text{when } p = 2q,$$

and

$$\text{(3.2b)}\quad \mathcal{L}(V;t) = \frac{C(p,r,m)}{q!} \sum_{u=0}^{2q} (-1)^u B_{r+u}(t,1)$$

$$\{\textstyle\sum_u \pm\, b_{i_1 i_2}(t) b_{i_3 i_4}(t) \cdots b_{i_{2q-1}, i_{2q}}(t)\}$$

$$\text{when } p = 2q + 1.$$

In the above equations b_{ij} is equivalent to $f_{i+r-1}^{j+r-1}(t)$ when $L = 0$, $N = 1$ and $\psi(t,x) = \psi_1(t,x)$. Also, in Equation (3.2a), the summation is over all permutations $i_1,\ldots i_{2q}$ of $1, 2,\ldots,2q$ subject to the restrictions $i_1 < i_2, i_3 < i_4, \ldots, i_{2q-1} < i_{2q}$ and the sign is positive or negative according as the permutation is even or odd. In Equation (3.2b), $\sum_u$ denotes the summation over all permutations $i_1,\ldots,i_{2q}$ of $1, 2,\ldots,u$, $u+2,\ldots, 2q+1$ subject to the restriction

$i_1 < i_2, i_3 < i_4, \ldots, i_{2q-1} < i_{2q}$ and the sign is positive or negative according as the permutation is even or odd. Also $B_{r+u}(t,1)$ is equivalent to the value of $F_{r+u}(t,1)$ when $L = 0$ and $\psi(t,x) = \psi_1(t,x)$. Now, taking the inverse Laplace transformation of (3.2a) and (3.2b) the density function of V is given by

(3.3a) $$f(V) = \frac{C(p,r,m)}{q!} \sum \pm h(i_1,\ldots,i_{2q};V)$$

when $p = 2q$

and

(3.3b) $$f(V) = \frac{C(p,r,m)}{q!} \sum_{u=0}^{2q} (-1)^u$$

$$\{\sum_u \pm h_u^*(i_1,\ldots,i_{2q};V)\}$$

when $p = 2q + 1$

where $h(i_1,\ldots,i_{2q};V)$ is the inverse Laplace transformation of $b_{i_1 i_2}(t)\ldots b_{i_{2q-1},i_{2q}}(t)$ and $h_u^*(i_1,\ldots,i_{2q};V)$ is the inverse Laplace transformation of $B_{r+u}(t,1)b_{i_1 i_2}(t)\ldots b_{i_{2q-1},i_{2q}}(t)$, and the summations $\sum$ and $\sum_u$ are defined in the same way as in (3.2a) and (3.2b) respectively. The signs $\pm$ in (3.3a) and (3.3b) depend in the same way as in (3.2a) and (3.2b) respectively. We will now discuss as to how to evaluate $h(i_1,\ldots i_{2q};V)$. (The evaluation of $h_u^*(i_1,\ldots,i_{2q};V)$ requires slight change.) When $\psi(t,x) = \psi_1(t,x)$, we know that

$$(3.4)\quad f_i^j(t) = \sum_{a_1=0}^{\infty} \sum_{a_2=0}^{\infty} \binom{m}{a_1}\binom{m}{a_2}(-1)^{a_1+a_2}$$

$$\times \int_0^1 \int_0^\theta \exp\{-t(x+\theta)\}(x^{i+a_1}\theta^{j+a_2} - x^{j+a_1}\theta^{i+a_2})\,dx\,d\theta.$$

The above equation can be rewritten as follows :

$$(3.5)\quad f_i^j(t) = \sum_{a=0}^{\infty} \binom{m}{a}^2 R(t;i+a,\ j+a)$$

$$+ \sum_{a_1<a_2} \binom{m}{a_1}\binom{m}{a_2}(-1)^{a_1+a_2}[R(t;i+a_1,j+a_2)$$

$$+ R(t;i+a_2,j+a_1)]$$

where $R(t;r,r+k)$ is defined by (2.7) and $R(t;r,r+k) = -R(t;r+k,r)$. Now, using (2.8), (3.3a), (3.3b) and (3.5), we observe that the evaluation of $f(V)$ depends upon finding the inverse Laplace transformations of the linear combinations of terms of the form

$$\prod_{i=1}^{[\frac{p+1}{2}]} \exp\{-c_i t\} I(\alpha_i t;\beta_i)$$

where $\alpha_i = 1$ or 2, $\beta_i > 0$ and $c_i(\ 0 \leq c_i \leq [\frac{p+1}{2}])$ are integers, and $[(p+1)/2$ denotes the integral part of $(p+1)/2$. Let $I^*(s;\ \alpha_i,\ \beta_j,\ c_k)$ be the inverse Laplace transformation of $I(\alpha_i\ t;\ \beta_j)\exp(-c_k t)$. Then

$$(3.6)\qquad I^*(s;\ \alpha_i,\ \beta_j,\ c_k) = (1/\alpha_i)^{\beta_j+1}(s-c_k)^{\beta_j}$$

$$c_k < s < c_k + \alpha_i$$

since

$$(3.7)\quad I(\alpha_i t;\beta_j)\exp(-c_k t) = (1/\alpha_i)^{\beta_j+1}\int_{c_k}^{c_k+\alpha_i} \exp(-ty)(y - c_k)^{\beta_j}dy.$$

In general, when $p > 2$, the inverse Laplace transformation of $\prod_{i=1}^{[(p+1)/2]} \exp\{-c_i t\} I(\alpha_i t;\beta_i)$ involves the evaluation of $[\frac{p+1}{2}]$-fold convolution integral. We will illustrate the evaluation of $f(V)$ when $p = 2$ and $m = 1$. In this special case, the density of $V = \theta_1 + \theta_2$ is given by (3.3a)

$$(3.8)\qquad f(V) = C(2,r,1)h(1,2;V).$$

But, $h(1,2;V)$ is the inverse Laplace transformation of $b_{1,2}(t)$. By (3.5)

$$(3.9)\quad b_{1,2}(t) = f_r^{r+1}(t) = R(t;r,r+1) + R(t;r+1,r+2) - R(t;r,r+2).$$

Finally applying lemma (2.2) and Equation (3.6) we have

$$(3.10)\quad f(V) = \begin{cases} C(2,r,1)(\phi_1(V)) & 0 < V < 1 \\ C(2,r,1)(\phi_1(V) + \phi_2(V)) & 1 \leq V < 2 \end{cases}$$

where

$$\phi_1(V) = \frac{1}{r+1}(1/2)^{2r+2}(V^{2r+2} - V^{2r+3}) + \frac{1}{r+2}(1/2)^{2r+4}V^{2r+4},$$

$$\phi_2(V) = \frac{1}{r+1}(V-1)^{r+2} - \frac{1}{r+2}(V-1)^{r+2}.$$

4. Distribution of the trace of $S_1S_2^{-1}$. Let S_1 and S_2 be as defined in the previous section. Then the joint density of the latent roots $\lambda_1,\ldots,\lambda_p$ of $S_1S_2^{-1}$ is known (see Roy [15]) to be of the form

$$(4.1)\quad f(\lambda_1,\ldots,\lambda_p) = C(p,m,r)\prod_{i=1}^{p}\{\lambda_i^m(1+\lambda_i)^{-(m+r+p+1)}\}\prod_{i>j}(\lambda_i-\lambda_j)$$

$$0 < \lambda_1 < \ldots < \lambda_p < \infty$$

where r,m and C(p,m,r) are defined in the previous section. Now, let $U = \sum_{i=1}^{p}\lambda_i$. Then the Laplace transformation of U is given by

$$(4.2)\quad L(U;t) = \int\cdots\int_{R_1} \exp\{-t\sum_{i=1}^{p}\lambda_i\} f(\lambda_1,\ldots,\lambda_p)\prod_{i=1}^{p} d\lambda_i$$

where $R_1 : \{0 \leq \lambda_1 \leq \ldots \leq \lambda_p < \infty\}$ and $Re(t) > 0$. As in [14], we make the transformations

$$x_i = 1/(1+\lambda_{p-i+1}),\ i = 1,\ldots,p.$$

Then, we obtain

$$(4.3)\quad L(U;t) = C(p,m,r)\exp(pt)\rho(t,\psi_2;p,r,0,1)$$

where $\psi_2(t,x) = \exp(-t/x)(1-x)^m$. Using Lemma 2.1 and Equation (2.1), we obtain

$$(4.3a)\quad L(U;t) = \frac{C(p,m,r)\exp(pt)}{q!}\sum \pm c_{i_1i_2}(t)c_{i_3i_4}(t)\ldots c_{i_{2q-1},i_{2q}}(t)$$

$$\text{when } p = 2q$$

and

(4.3b) $$L(U;t) = \frac{C(p,m,r)\exp(pt)}{q!} \sum_{u=0}^{2q} (-1)^u C_{r+u}(t,1)$$

$$\times \{\textstyle\sum_u \pm c_{i_1 i_2}(t) c_{i_3 i_4}(t) \ldots c_{i_{2q-1}, i_{2q}}(t)\}$$

$$\text{when } p = 2q + 1.$$

In the above equations $c_{ij}(t)$ is equivalent to $f_{i+r-1}^{j+r-1}(t)$ when $L = 0$, $U = 1$ and $\psi(t,x) = \psi_2(t,x)$. The summation and the signs $\pm$ in (4.3a) are defined in the same way as in (3.2a) whereas the summation $\sum_u$ and the signs $\pm$ in (4.3b) are defined in the same way as in (3.2b). Also $C_{r+u}(t,1)$ is equivalent to $F_{r+u}(t,1)$ when $L = 0$ and $\psi(t,x) = \psi_2(t,x)$. The density function of U is given by

(4.4a) $$f(U) = \frac{C(p,m,r)}{q!} \sum \pm J(i_1, \ldots, i_{2q}; U)$$

$$\text{when } p = 2q$$

and

(4.4b) $$f(U) = \frac{C(p,m,r)}{q!} \sum_{u=0}^{2q} (-1)^u$$

$$\times \{\textstyle\sum_u \pm J_u^*(i_1, \ldots, i_{2q}; U)\}$$

$$\text{when } p = 2q + 1$$

where $J(i_1,\ldots,i_{2q};U)$ is the inverse Laplace transformation of $\exp(pt)c_{i_1 i_2}(t)\ldots c_{i_{2q-1},i_{2q}}(t)$ and $J_u^*(i_1,\ldots,i_{2q};U)$ is the inverse Laplace transformation of $C_{r+u}(t,1)\exp(pt)c_{i_1 i_2}(t)\ldots c_{i_{2q-1},i_{2q}}(t)$, and the summations and the signs $\pm$ in (4.4a) and (4.4b) are defined in the same way as in (4.3a) and (4.3b) respectively. The evaluation of terms of the form $J(i_1,\ldots,i_{2q};U)$ is discussed below. (Again the evaluation of $J_u^*(i_1,\ldots,i_{2q};U)$ requires only slight change.)

When $\psi(t,x) = \psi_2(t,x)$, it is seen that

$$(4.5)\qquad f_i^j(t) = \sum_{a=0}^{\infty}\binom{m}{a}^2 S(t;i+a,\,j+a) + \sum_{a_1<a_2}\binom{m}{a_1}\binom{m}{a_2}(-1)^{a_1+a_2}[S(t;i+a_1,\,j+a_2) + S(t;i+a_2,\,j+a_1)]$$

where $S(t;r,r+k)$ is defined by (2.9) and $S(t;r,r+k) = -S(t;r+k,r)$. Now using (2.10), (4.4a), (4.4b) and (4.5), we observe that the evaluation of $f(U)$ depends upon finding the inverse Laplace transformations of the linear combinations of terms of the form $\prod_{i=1}^{[(p+1)/2]} g(t;\alpha_i,\beta_j)$ where $g(t,\alpha_i,\beta_i)$ is defined by (2.11).

Let us denote the inverse Laplace transformation of $g(t;\alpha_i,\beta_i)$ by $g^*(s;\alpha_i,\beta_i)$. Then the inverse Laplace transformation of $\prod_{i=1}^{k} g(t;\alpha_i,\beta_i)$ is the convolution

$$g^*(s;\alpha_1,\beta_1) * g^*(s;\alpha_2,\beta_2) * \dots * g^*(s;\alpha_k,\beta_k).$$

In particular

$$g^*(s;\alpha,\beta) = \beta^\alpha/(\beta + s)^\alpha \qquad 0 < s < \infty \tag{4.6}$$

and

$$g^*(s;\alpha_1,\beta_1) * g^*(s;\alpha_2,\beta_2) = \int_0^s \frac{\beta_1^{\alpha_1}\beta_2^{\alpha_2}\,dx}{(\beta_1 + x)^{\alpha_1}(\beta_2 + s - x)^{\alpha_2}}.$$

For illustration , we write below explicitly the density of U for $p = 2$ and $m = 1$. In this special case, the density of $U = \lambda_1 + \lambda_2$ is given by (4.4a)

$$f(u) = C(2,1,r)J(1,2;u) \tag{4.7}$$

But $J(1,2;u)$ is the inverse Laplace transformation of $\exp(2t)c_{12}(t)$. By (4.5)

$$c_{12}(t) = f_r^{r+1}(t) = S(t;r,r+1) + S(t;r+1,r+2) - S(t;r,r+2). \tag{4.8}$$

Finally applying lemma 2.3 and Equation (4.6) we have

$$(4.9)\quad f(u) = \frac{C(2,1,r)}{r+2}\left[(1+u)^{-(r+2)} - (1+\frac{u}{2})^{-(2r+4)}\right]$$

$$- \frac{C(2,1,r)}{r+3}\left[(1+u)^{-(r+2)} - 2(1+\frac{u}{2})^{-(2r+5)} + (1+\frac{u}{2})^{-(2r+6)}\right],$$

where $0 < u < \infty$.

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In this paper, the authors derived the exact distributions of the traces of $S_1 S_2^{-1}$ and $S_1 (S_1 + S_2)^{-1}$ where S_1 and S_2 are independently distributed as central Wishart matrices and the expected values of these matrices are the same. The method used involves expressing the Laplace transformations in terms of the linear combinations of products of certain double integrals and then taking the inverse Laplace transformations.

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