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FOREIGN TECHNOLOGY DIVISION

AIR FORCE SYSTEMS COMMAND

DYNAMICS MISSILES

by

K. A. Abgaryan and
I. M. Rapoport

FOREIGN TECHNOLOGY DIVISION

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DYNAMICS MISSILES

By: K. A. Abgaryan and I. M. Rapoport

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U. S. BOARD ON GEOGRAPHIC NAMES TRANSLITERATION SYSTEM

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Г г	*Г г*	G, g	У у	*У у*	U, u
Д д	*Д д*	D, d	Ф ф	*Ф ф*	F, f
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К к	*К к*	K, k	Ъ ъ	*Ъ ъ*	"
Л л	*Л л*	L, l	Ы ы	*Ы ы*	Y, y
М м	*М м*	M, m	Ь ь	*Ь ь*	'
Н н	*Н н*	N, n	Э э	*Э э*	E, e
О о	*О о*	O, o	Ю ю	*Ю ю*	Yu, yu
П п	*П п*	P, p	Я я	*Я я*	Ya, ya

* ye initially, after vowels, and after ъ, ь; e elsewhere. When written as ё in Russian, transliterate as yё or ё. The use of diacritical marks is preferred, but such marks may be omitted when expediency dictates.

FOLLOWING ARE THE CORRESPONDING RUSSIAN AND ENGLISH DESIGNATIONS OF THE TRIGONOMETRIC FUNCTIONS

Russian	English
sin	sin
cos	cos
tg	tan
ctg	cot
sec	sec
cosec	csc
sh	sinh
ch	cosh
th	tanh
cth	coth
sch	sech
csch	csch
arc sin	$\sin^{-1}$
arc cos	$\cos^{-1}$
arc tg	$\tan^{-1}$
arc ctg	$\cot^{-1}$
arc sec	$\sec^{-1}$
arc cosec	$\csc^{-1}$
arc sh	$\sinh^{-1}$
arc ch	$\cosh^{-1}$
arc th	$\tanh^{-1}$
arc cth	$\coth^{-1}$
arc sch	sech^{-1}
arc csch	csch^{-1}

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rot	curl
lg	log

In the book are discussed fundamentals of dynamics of rocket operating on solid and liquid propellants. There are listed equations of undisturbed and disturbed motion of rockets as bodies of variable composition, methods of linearization of equations and decomposition of linearized equations into separate groups. There are examined stability and controllability of the rocket, its transfer functions and dynamic characteristics as the object of control.

There are comprehensively examined questions of stabilization of the rocket taking into account the elasticity of its construction and mobility of liquid propellant in the tanks.

The book is intended for students of engineering colleges and can be useful for engineers of the aviation and rocket industry. Tables 2. Illustrations 56. Bibliography 27 names.

PREFACE

Educational textbook "Rocket Dynamics" is devoted to a comparatively narrow class of flight vehicles – ballistic rockets and carrier rockets of space apparatuses.

The course of dynamics of ballistic rockets and carrier rockets of space apparatuses can be divided into two parts. In the first of them are examined trajectories and flight programs of rockets[1], in the second – stability and controllability of rockets. The given educational textbook encompasses problems of the second part of the course of rocket dynamics and includes equations of motion of the rocket as a deformable material system, dynamic properties of the rocket as the object of automatic control, methods of investigation of the effectiveness of slave organs of the control system and possible ways of stabilization of rocket motion.

In contrast to available educational literature on aircraft dynamics in "Rocket Dynamics" during investigation of the motion of rockets considerable attention is given to the account of elasticity

[1]The book of R. F. Appazov, S. S. Lavrov, V. P. Mishin [1] is dedicated to this part of the course.

of its body and the mobility of propellant in tanks[1], and with investigation of stability and controllability along with methods based on quenching of coefficients of equations, engineering methods of the investigation are stated with allowance for the effect of variability of coefficients.

In the process of designing contemporary ballistic and space rockets at the first stage the rocket is considered as a nondeformable material system. At this stage there is solved the problem of the rational aerodynamic layout of the rocket, the required effectiveness of slave organs of the control system and there is examined the question of stabilization of rocket motion as a nondeformable body.

At the following stage there are approximately examined the frequency characteristics of the rocket as the object of automatic control taking into account the elasticity of its construction and mobility of liquid propellant.

Subsequently there is calculated the frequency characteristics of the rocket as the object of control, accurate within the scope of its accepted dynamic model. On the basis of these frequency characteristics there is designed the automatic stabilization control.

With distribution of material by chapters the authors tried as much as possible to adhere to the shown sequence of carrying out dynamic investigations of the rocket.

It is assumed that the reader has knowledge of fundamentals of the theory of automatic control, aerodynamics, and also with divisions of higher mathematics, being studied at engineering colleges. In accordance with effective curriculum all these disciplines precede the course of dynamics of flight vehicles.

Chapters I, II and Appendices are written by K. A. Abgarya, Chapters III and IV – I. M. Rapoport.

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[1] During account of mobility of liquids and elasticity of the construction there is assumed smallness of oscillatons of liquids and smallness of elastic oscillations, due to which the problems are solved in linear setting.

CHAPTER I

MOTION EQUATIONS OF ROCKET

In flight part of the mass, contained inside the rocket body, with passage of time is continuously separated from the rocket mainly as a result of operation of engines. In connection with this there appears the need to determine which particles should be included in the composition of the rocket and which are considered already separated from it. In this question there is some uncertainty: it is not clear at which moment the particle, moving in the gas jet flowing from the combustion chamber of the engine, "leaves" the rocket. This uncertainty can be removed by the introduction of a *control surface*, conditionally considered as the boundary of the rocket. As such a surface, which will further also be called the *shell* of the rocket, we conveniently take the closed surface, formed by the external surface of the rocket and outlet sections of engine nozzles. The composition of the rocket includes only those particles, which at a given moment are located inside this surface.

Through a certain part of the control surface there occurs outflow of mass, which leads to change of the composition of material particles contained in it. Therefore, during the study of motion the rocket must be considered as a material system of variable composition.

Classic theorems of dynamics of systems of constant composition — theorems about change of the momentum and change of angular momentum — are not directly applicable to systems of variable composition. However, by using these theorems, it is possible to formulate similar theorems for systems of variable composition and formulate the principle of compilation of motion equations for such systems, and specifically for rockets.

§ 1. Fundamental Theorems of Dynamics of System of Variable Composition

Let us examine closed surface S_0 (Fig. 1.1), which limits volume V, filled with various (solid, liquid, gaseous) particles. With the passage of time some particles exit from volume V, and others, conversely, enter it. The aggregate of material particles, contained in volume V, is the system of variable composition. Let us designate this system by Σ.

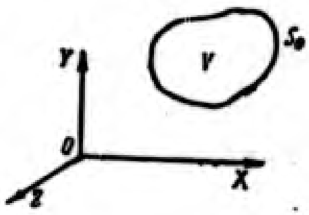

Fig. 1.1.

Let us assume surface S_0 and material particles move relative to some system of coordinates $OXYZ$. Surface S_0 in this case can also be deformed. Let us designate momentum and angular momentum of system Σ relative to point O by $\vec{K}$ and $\vec{L}$ respectively.

Along with system Σ let us examine system of constant composition Σ^*, consisting of only those material particles which at some fixed moment of time made up volume V. By $\vec{K}^*$ and $\vec{L}^*$ let us designate the momentum and angular momentum of system Σ^* relative to point O.

Systems of variable composition Σ and constant composition Σ^* at moment of time t coincide. Therefore,

$$\vec{K}^*(t)=\vec{K}(t),\ \vec{L}^*(t)=\vec{L}(t). \quad (1.1.1)$$

When $t_1 \neq t$ systems Σ and Σ^* will consist, generally speaking, of different particles. Because of this the momentum and angular momentum of system Σ will differ from momentum and angular momentum of system Σ^*, namely:

$$\begin{aligned}\vec{K}^*(t_1)&=\vec{K}(t_1)+\vec{k}(t_1),\\ \vec{L}^*(t_1)&=\vec{L}(t_1)+\vec{l}(t_1).\end{aligned} \quad (1.1.2)$$

Vectors $\vec{k}(t_1)$ and $\vec{l}(t_1)$ represent changes of the momentum and angular momentum of system Σ, connected with change of its composition for time $\Delta t = t_1 - t$.

Let us subtract (1.1.1) from (1.1.2) and divide the obtained difference by Δt. Transferring further to the limit when $t_1 \to t$ and considering in this case that

$$\vec{k}(t)=0,\ \vec{l}(t)=0, \quad (1.1.3)$$

we obtain

$$\frac{d\vec{K}^*}{dt}=\frac{d\vec{K}}{dt}+\frac{d\vec{k}}{dt},\quad \frac{d\vec{L}^*}{dt}=\frac{d\vec{L}}{dt}+\frac{d\vec{l}}{dt}. \quad (1.1.4)$$

Derivatives $d\vec{k}/dt$, $d\vec{l}/dt$ represent flow rates per second of the momentum and angular momentum through surface S_0 at moment of time t.

Relationships (1.1.4) have kinematic character and are valid for any system of coordinates (inertial or noninertial).

Let us suppose now that *OXYZ* – inertial system of coordinates. To system Σ^*, as to system of constant composition, there are applicable theorems of classic dynamics of change of the momentum and change of angular momentum, which are formulated so:

1. *Time derivative from the momentum of the system is equal to the sum of all external forces acting on the system.*

2. *Time derivative from the angular momentum of the system relative to some stationary center* (in this case relative to point O) *is equal to the sum of moments of all external forces acting on the system relative to the same center.*

On the basis of these theorems at fixed moment of time *t*

$$\frac{d\vec{K}^*}{dt}=\vec{F}, \quad \frac{d\vec{L}^*}{dt}=\vec{M}, \tag{1.1.5}$$

where $\vec{F}$ and $\vec{M}$ – respectively the principal vector and principal moment of all external forces, acting at moment of time *t* on system Σ^*, and therefore, on system Σ.

From equalities (1.1.4) and (1.1.5) follow equalities:

$$\frac{d\vec{K}}{dt}=\vec{F}-\frac{d\vec{k}}{dt}, \quad \frac{d\vec{L}}{dt}=\vec{M}-\frac{d\vec{l}}{dt}. \tag{1.1.6}$$

Relationships (1.1.6), obtained for given moment of time *t*, also remain valid for any other moment of time *t'*, if we consider that $\vec{F}$ and $\vec{M}$ – principal vector and principal moment of all external forces, acting on system Σ, and $d\vec{k}/dt$ and $d\vec{l}/dt$ – flow rates per second of the momentum and angular momentum through surface S_0 at considered moment of time *t'*.

Formulas (1.1.6) represent the mathematical record of theorems about the change of momentum and angular momentum of the system of variable composition.

§2. Migratory Motion of System of Variable Composition

Along with basic system *OXYZ* let us introduce another system of coordinates *Axyz* (Fig. 1.2) and let us represent the absolute motion of material particle with mass *dm* as compound, consisting of relative motion (motion of particle relative to system of coordinates *Axyz*) and migratory (motion of particle together with axes *Axyz* relative to inertial system *OXYZ*). Let us designate absolute, migratory and relative velocities of particles by $\vec{v}, \vec{v}_e, \vec{v}_r$, and accelerations – by $\vec{w}, \vec{w}_e, \vec{w}_r$ respectively. By $\vec{w}_K$ let us designate Coriolis acceleration of the particle. Finally, let us designate time derivative with respect to system *Axyz* (relative derivative) by symbol δ/dt.

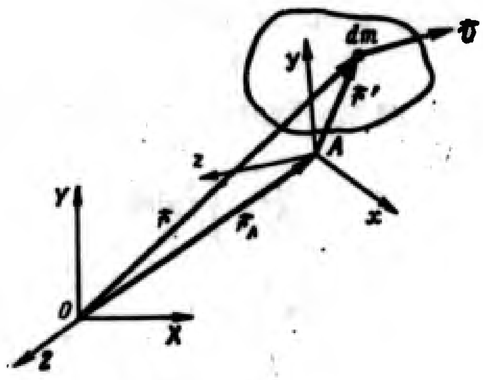

Fig. 1.2.

Momentum of the system of constant composition Σ^* in system *OXYZ* can be represented in the form

$$\vec{K}^* = \int_{\Sigma^*} \vec{v}\,dm. \tag{1.2.1}$$

Symbol $\int_{\Sigma^*}$ designates the sum over all particles of system Σ*. By differentiating (1.2.1) with respect to t, we obtain

$$\frac{d\vec{K}^*}{dt}=\int_{\Sigma^*}\vec{w}dm. \qquad (1.2.2)$$

According to theorem of summation of accelerations,

$$\vec{w}=\vec{w}_r+\vec{w}_e+\vec{w}_\kappa.$$

Therefore,

$$\frac{d\vec{K}^*}{dt}=\int_{\Sigma^*}\vec{w}_r dm+\int_{\Sigma^*}\vec{w}_e dm+\int_{\Sigma^*}\vec{w}_\kappa dm. \qquad (1.2.3)$$

Hence for moment of time t, when systems Σ and Σ* coincide, using (1.1.5), we obtain

$$\int_{\Sigma}\vec{w}_e dm=\vec{F}-\int_{\Sigma}\vec{w}_r dm+\vec{F}_\kappa, \qquad (1.2.4)$$

where $\vec{F}_\kappa=-\int_{\Sigma}\vec{w}_\kappa dm$ — principal moment of Coriolis *forces of inertia*, acting at moment of time t on system Σ.

Momentum of system Σ* in system of coordinates *Axyz* is equal to:

$$\vec{K}_r^*=\int_{\Sigma^*}\vec{v}_r dm.$$

Hence, taking into account that

$$\vec{w}_r=\frac{\delta\vec{v}_r}{dt},$$

we will have

$$\frac{\delta\vec{K}_r^*}{dt}=\int_{\Sigma^*}\vec{w}_r dm$$

and at moment of time t –

$$\frac{\delta\vec{K}_r^*}{dt}=\int_{\Sigma}\vec{w}_r dm. \qquad (1.2.5)$$

On the other hand, for this moment of time kinematic relationship (1.1.4) is valid, which in system $Axyz$ is written as:

$$\frac{\delta \vec{K}_r^*}{dt}=\frac{\delta \vec{K}_r}{dt}+\frac{\delta \vec{k}_r}{dt}. \qquad (1.2.6)$$

Here $\vec{K}_r$ – momentum of system of variable composition Σ in system $Axyz$; $\frac{\delta \vec{k}_r}{dt}$ — relative flow rate per second of the momentum through surface S_0 at moment of time t.

Taking into account (1.2.6) and (1.2.5) from (1.2.4) we find

$$\int_\Sigma \vec{w}_r dm=\vec{F}+\left(-\frac{\delta \vec{k}_r}{dt}\right)+\vec{F}_k+\left(-\frac{\delta \vec{K}_r}{dt}\right). \qquad (1.2.7)$$

Similar result can be obtained from the theorem of change of angular momentum of the system of variable composition.

Angular momentum of system Σ^* relative to point O is equal to (see Fig. 1.2):

$$\vec{L}^*=\int_{\Sigma^*}\vec{r}\times\vec{v}dm. \qquad (1.2.8)$$

Hence

$$\frac{d\vec{L}^*}{dt}=\frac{d}{dt}\int_{\Sigma^*}\vec{r}\times\vec{v}dm=\int_{\Sigma^*}\frac{d\vec{r}}{dt}\times\vec{v}dm+\int_{\Sigma^*}\vec{r}\times\vec{w}dm.$$

In the right side of the last equality the first integral is equivalent to zero, since

$$\frac{d\vec{r}}{dt}\times\vec{v}=\vec{v}\times\vec{v}=0,$$

this means

$$\frac{d\vec{L}^*}{dt}=\int_{\Sigma^*}\vec{r}\times\vec{w}dm=\int_{\Sigma^*}\vec{r}'\times\vec{w}dm+\int_{\Sigma^*}\vec{r}_A\times\vec{w}dm. \qquad (1.2.9)$$

But, considering (1.2.2),

$$\int_{\Sigma^*} \vec{r}_A \times \vec{w} dm = \vec{r}_A \times \int_{\Sigma^*} \vec{w} dm = \vec{r}_A \times \frac{d\vec{K}^*}{dt}.$$

Therefore,

$$\frac{d\vec{L}^*}{dt} = \int_{\Sigma^*} \vec{r}' \times \vec{w}_r dm + \int_{\Sigma^*} \vec{r}' \times \vec{w}_e dm + \int_{\Sigma^*} \vec{r}' \times \vec{w}_\kappa dm + \vec{r}_A \times \frac{d\vec{K}^*}{dt}. \qquad (1.2.10)$$

Hence at moment of time t, considering (1.1.5) we obtain

$$\int_{\Sigma} \vec{r}' \times \vec{w}_e dm = \vec{M} - \vec{r}_A \times \vec{F} - \int_{\Sigma} \vec{r}' \times \vec{w}_r dm + \vec{M}_{\kappa A}, \qquad (1.2.11)$$

where $\vec{M}_{\kappa A} = -\int_{\Sigma} \vec{r}' \times \vec{w}_\kappa dm$ — principal moment of Coriolis forces of inertia, acting at moment of time t on system Σ.

Principal moment of external forces relative to point A, acting at moment of time t on system Σ (as well as on system Σ^*), is equal to:

$$\vec{M}_A = \vec{M} - \vec{r}_A \times \vec{F}.$$

Therefore, (1.2.11) can be represented in the form

$$\int_{\Sigma} \vec{r}' \times \vec{w}_e dm = \vec{M}_A - \int_{\Sigma} \vec{r}' \times \vec{w}_r dm + \vec{M}_{\kappa A}. \qquad (1.2.12)$$

Angular momentum of system Σ^* relative to point A in system $Axyz$ is equal to:

$$\vec{L}^*_{rA} = \int_{\Sigma^*} \vec{r}' \times \vec{v}_r dm,$$

and its relative derivative

$$\frac{\delta \vec{L}^*_{rA}}{dt} = \frac{\delta}{dt} \int_{\Sigma^*} \vec{r}' \times \vec{v}_r dm = \int_{\Sigma^*} \vec{v}_r \times \vec{v}_r dm + \int_{\Sigma^*} \vec{r}' \times \vec{w}_r dm =$$
$$= \int_{\Sigma^*} \vec{r}' \times \vec{w}_r dm.$$

At fixed moment of time t

$$\frac{\delta \vec{L}^{*}_{rA}}{dt}=\int_{\Sigma} \vec{r}'\times\vec{w}_r dm. \qquad (1.2.13)$$

On the other hand,

$$\frac{\delta \vec{L}^{*}_{rA}}{dt}=\frac{\delta \vec{L}_{rA}}{dt}+\frac{\delta \vec{l}_{rA}}{dt}, \qquad (1.2.14)$$

where $\vec{L}_{rA}$ – angular momentum of system of variable composition relative to point A in system $Axyz$; $\frac{\delta \vec{l}_{rA}}{dt}$ – relative flow rate per second of the angular momentum through surface S_0.

From (1.2.12), (1.2.13) and (1.2.14) we obtain

$$\int_{\Sigma} \vec{r}'\times\vec{w}_e dm=\vec{M}_A+\left(-\frac{\delta \vec{l}_{rA}}{dt}\right)+\vec{M}_{\kappa A}+\left(-\frac{\delta \vec{L}_{rA}}{dt}\right). \qquad (1.2.15)$$

Equalities (1.2.7) and (1.2.15) describe migratory motion of medium Σ and represent another record of theorems of the momentum and angular momentum of the system of variable composition.

§3. Systems of Variable Composition with a Solid Shell. Principle of Solidification.

Formulas (1.2.7) and (1.2.15) can be given another form. For this let us examine some fictitious solid S, which would be obtained if the system of variable composition Σ at moment of time t as if solidified, i.e., motion of particles relative to system $Axyz$ would cease. Migratory accelerations of particles of system Σ are equal to absolute accelerations of particles of fictitious solid obtained in this way. Therefore, by designation $\vec{v}^{(S)}$ the velocity of particle of solid S, we will have

$$\frac{d\vec{v}^{(S)}}{dt}=\vec{w}_e.$$

At moment of time t the distribution of masses in the fictitious solid and in system Σ is identical. Therefore,

$$\int_{\Sigma} \vec{w}_r dm = \int_{S} \vec{w}_e dm = \int_{S} \frac{d\vec{v}^{(S)}}{dt} dm = \frac{d}{dt} \int_{S} \vec{v}^{(S)} dm = \frac{d\vec{K}^{(S)}}{dt},$$

where $\vec{K}^{(S)}$ – momentum of fictitious solid. Analogously

$$\int_{\Sigma} \vec{r}' \times \vec{w}_r dm = \int_{S} \vec{r}' \times \vec{w}_e dm = \int_{S} \vec{r}' \times \frac{d\vec{v}^{(S)}}{dt} dm = \frac{d\vec{L}_A^{(S)}}{dt} - \\ - \int_{S} \frac{d\vec{r}'}{dt} \times \vec{v}^{(S)} dm,$$

where $\vec{L}_A^{(S)} = \int_{S} \vec{r}' \times \vec{v}^{(S)} dm$ — angular momentum of fictitious solid relative to pole A.

Taking into account the obtained relationships equalities (1.2.7) and (1.2.15) can be replaced by equalities:

$$\frac{d\vec{K}^{(S)}}{dt} = \vec{F} + \left(-\frac{\delta\vec{P}_r}{dt}\right) + \vec{F}_\kappa + \left(-\frac{\delta\vec{K}_r}{dt}\right), \quad (1.3.1)$$

$$\frac{d\vec{L}_A^{(S)}}{dt} = \vec{M}_A + \left(-\frac{\delta\vec{l}_{rA}}{dt}\right) + \vec{M}_{\kappa A} + \left(-\frac{\delta\vec{L}_{rA}}{dt}\right) + \int_{S} \frac{d\vec{r}'}{dt} \times \vec{v}^{(S)} dm. \quad (1.3.2)$$

The center of mass of solid S is designated by C. Point C taken for the origin A of system of coordinates $Axyz$. In this instance instead of (1.3.2) we will have

$$\frac{d\vec{L}_C^{(S)}}{dt} = \vec{M}_C + \left(-\frac{\delta\vec{l}_{rC}}{dt}\right) + \vec{M}_{\kappa C} + \left(-\frac{\delta\vec{L}_{rC}}{dt}\right). \quad (1.3.3)$$

Actually, since $\vec{r}' = \vec{r} - \vec{r}_A$ (see Fig. 1.2) and for particles of fictitious solid

$$\frac{d\vec{r}}{dt} = \vec{v}^{(S)},$$

then

$$\int\limits_S \frac{d\vec{r}'}{dt} \times \vec{v}^{(S)} dm = \int\limits_S \vec{v}^{(S)} \times \vec{v}^{(S)} dm - \int\limits_S \frac{d\vec{r}_A}{dt} \times \vec{v}^{(S)} dm =$$

$$= -\frac{d\vec{r}_A}{dt} \times \int\limits_S \vec{v}^{(S)} dm = -\frac{d\vec{r}_A}{dt} \times \frac{d\vec{r}_C}{dt} \int\limits_S dm \to 0 \text{ when } A \to C.$$

By changing the moment of solidification t, we obtain various solid S, which are distinguished from one another by the amount and distribution of mass in space. Various bodies S can be considered as one fictitious solid, in which with the passage of time material particles appear or disappear, stationary relative to system of coordinates $Axyz$.

Let us now examine the system of variable composition, limited by solid shell S_0. Coordinate system $Axyz$ is permanently connected with solid shell S_0. In this case the absolute motion of shell S_0 and fictitious solid S coincides. Therefore, equations (1.3.1) and (1.3.2) [or (1.3.3)], representing the absolute motion of fictitious solid, at the same time represent the motion of solid shell S_0.

Relative derivatives in the right sides of equations (1.3.1), (1.3.2), (1.3.3) in this instance have the following physical meaning.

Let us assume $d\sigma$ – element of surface S_0.

Flow rate per second of mass through this element at moment of time t is equal to $\varrho v_{rn} d\sigma$. where ϱ – density of medium, and v_{rn} – standard component of relative velocity of medium. Flow rate per second of the momentum through element $d\sigma$ in the system, permanently connected with the shell, is equal to $\vec{v}_r \varrho v_{rn} d\sigma$. Relative flow rate per second of the momentum through surface S_0 is equal to:

$$\frac{\tilde{d}\vec{k}_r}{dt} = \iint\limits_{S_0} \varrho v_{rn} \vec{v}_r d\sigma, \qquad (1.3.4)$$

or

$$\frac{\delta \vec{k}_r}{dt} = m_{\text{сек}} \vec{u}, \qquad (1.3.5)$$

where $m_{\text{сек}} = \iint_{S_0} \varrho v_{rn} d\sigma$ — flow rate per second of mass through surface S_0; $\vec{u}$ - some average velocity of medium (relative to the shell).

Vector $\vec{v}_r \varrho v_{rn} d\sigma$ has dimensionality of force and is directed along the relative velocity of particle. Vector $-\vec{v}_r \varrho v_{rn} d\sigma$ has the opposite direction and is elementary *reactive force*. In accordance with this, vector $-\delta \vec{k}_r / dt$ is the principal vector of all reactive forces, and $-\delta \vec{l}_{rC}/dt$ — principal moment of all reactive forces relative to pole C.

Further let us suppose that at each point, fixed relative to the shell, the density of medium ρ and velocity of medium $\vec{v}_r$ do not change with time. Then the system of variable composition has constant mass, constant momentum $\vec{K}_r$ and constant angular momentum $\vec{L}_{rC}$ relative to system of coordinates $Oxyz$. This means,

$$\frac{\delta \vec{K}_r}{dt} = 0, \quad \frac{\delta \vec{L}_{rC}}{dt} = 0.$$

Generally, if inside the shell there is changed the density of medium and velocity of particle relative to the shell, there appear so-called *variation forces*. Vectors $-\delta \vec{K}_r / dt$ and $-\delta \vec{L}_{rC}/dt$ are the principal moment of these variation forces.

The following principle ensues from the above.

Principle of solidification for the system of variable composition with solid shell. Equations of motion of a solid shell of system of variable composition Σ at arbitrary moment of time t can be written in the form of motion equations of a solid (of constant composition), if we imagine that the system of variable composition Σ is solidified at moment of time t and that to the fictitious solid obtained in this way there are applied: 1) external forces, action on system Σ, 2) reactive forces, 3) Coriolis and 4) variation forces.

§4. Principle of Solidification for the Rocket

We will consider that the body of the rocket in flight is not deformed[1]. Then the principle of solidification, formulated in the previous paragraph, can be applied to the rocket. However, for a rocket it is expedient to change somewhat the formulation of this principle.

The point is that reactive force $-\delta\vec{k}_r/dt$ cannot be directly measured, therefore it is usually united with forces appearing as a result of atmospheric pressure on the rocket body and pressure in the gas jet in the nozzle exit section of the engine, and also with variation forces. The expediency of such interconnection can be observed on the following example.

Let us examine a rocket with solid shell and operating engine, fixed in a horizontal position on a testing stand.

Since system of coordinates $Oxyz$, connected with solid shell, is fixed, then

$$\vec{K}^{(S)}=0, \quad \vec{F}_{\text{к}}=0$$

and equation (1.3.1) in this case assumes the form

$$\vec{F}_{\text{т}}\left(-\frac{\delta\vec{k}_r}{dt}\right)+\left(-\frac{\delta\vec{K}_r}{dt}\right)=0. \tag{1.4.1}$$

[1]In actuality in flight under the action of external forces the rocket body is deformed and, thus, the control surface introduced above (shell of rocket) is also deformed. Account of the effect of deformation of the construction of the rocket on the character of its motion is cited in Chapter IV.

The rocket is affected by the following external forces: of 1) forces of weight, 2) reaction of supports, to which the rocket is attached, 3) forces of atmospheric pressure and pressure in the nozzle exit section, applied to the shell of the rocket.

Let us assume $\vec{G}$ – weight of rocket, and $\vec{P}$ – resultant of forces appearing during operation engine. The rocket is pressed to supports with force $\vec{G} + \vec{P}$. Reaction of supports, applied to the rocket, is equal to:

$$\vec{R} = -(\vec{G}+\vec{P}).$$

Resultant of forces of atmospheric pressure and pressures in nozzle exit section let us designate by $\vec{F}_*$. In quantity this resultant (Fig. 1.3) is equal to:

$$F_* = (p_a - p_H) S_a,$$

where p_a – pressure on nozzle section; p_H – atmospheric pressure; S_a – area of nozzle exit section.

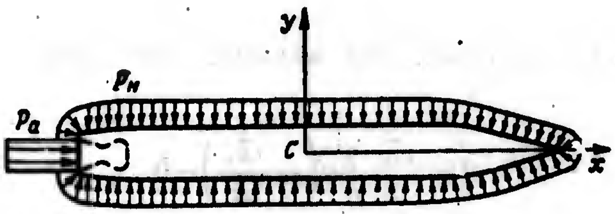

Fig. 1.3.

Thus, in the considered case the principal vector of all external forces is equal to:

$$\vec{F} = \vec{G} + \vec{R} + \vec{F}_* = -\vec{P} + \vec{F}_*.$$

By substituting this expression in (1.4.1), we obtain

$$\vec{P}=\left(-\frac{\delta\vec{K}_r}{dt}\right)+\vec{F}_e+\left(-\frac{\delta\vec{K}_r}{dt}\right). \tag{1.4.2}$$

Instruments on the stand measure the force of pressure of the rocket to the support, so that as a result of bench tests one can determine force P, which, as can be seen from the obtained expression, is the resultant of reactive forces, variation forces and external forces, caused by atmospheric pressure and gas pressure in the nozzle exit section.

Force P, determined by equality (1.4.2), is accepted to call *thrust force of the engine*, or simply *engine thrust*.

Having implied now under $\vec{F}$ and $\vec{M}_C$ respectively principal vector and principal of all external forces, except forces of atmospheric pressure and pressure in the nozzle exit section, equations (1.3.1) and (1.3.3) can be written so:

$$\frac{d\vec{K}^{(S)}}{dt}=\vec{F}+\vec{P}+\vec{F}_{\kappa}, \tag{1.4.3}$$

$$\frac{d\vec{L}_C^{(S)}}{dt}=\vec{M}_C+\vec{M}_{PC}+\vec{M}_{\kappa C}. \tag{1.4.4}$$

here $\vec{M}_{PC}=\left(-\frac{\delta\vec{l}_{rC}}{dt}\right)+\vec{M}_{eC}+\left(-\frac{\delta\vec{L}_{rC}}{dt}\right)$; by $\vec{M}_{*C}$ there is designated the principal moment of forces, caused by atmospheric pressure on the rocket body and gas pressure in the nozzle exit section of the engine, relative to center of mass C.

On the basis of equations (1.4.3) and (1.4.4) it is possible to formulate the following principle of compilation of equations of motion of a rocket.

Principle of solidification for the rocket. Motion equations of solid shell of a rocket at arbitrary moment of time t can be written in the form of motion equations of a solid (of constant composition), if we imagine that the rocket solidified at moment of time t and that to the fictitious solid obtained in this way there are applied: 1) external forces acting on the rocket (except force $\vec{F}_*$*), 2) thrust of reaction engine, 3) Coriolis force.*

§5. Motion Equations of the Center of Mass of a Rocket

Momentum of fictitious solid S, which is obtained as a result of solidification of the rocket at moment of time t, is equal to:

$$\vec{K}^{(S)} = \int_S \vec{v}^{(S)} dm = m\vec{V}_C^{(S)}, \tag{1.5.1}$$

where m — mass of fictitious solidification, equal to the mass of rocket at moment of time t; $\vec{V}_C^{(S)}$ — velocity of center of inertia of fictitious solid.

According to the principle of solidification in equation (1.4.3) the derivative of momentum $\vec{K}^{(S)}$ in time should be computed as in the case of a solid with constant mass, so that

$$\frac{d\vec{K}^{(S)}}{dt} = m\frac{d\vec{V}_C^{(S)}}{dt} = m\vec{w}_C^{(S)}, \tag{1.5.2}$$

where $\vec{w}_C^{(S)}$ — acceleration of center of mass of solid S.

By substituting (1.5.2) in (1.4.3), we obtain

$$m\vec{w}_C^{(S)} = \vec{F} + \vec{P} + \vec{F}_k. \tag{1.5.3}$$

With consumption of propellant the center of mass of the rocket is displaced relative to the body with certain velocity $\vec{V}_{Cr}$ and acceleration $\vec{w}_{Cr}$. Considering the motion of center of mass together with the rocket body as migratory motion, and motion relative to the body — as relative motion, for absolute velocity $\vec{V}_C$ and acceleration

$\vec{w}_C$ of the center of mass we will have the following expressions:

$$\vec{V}_C=\vec{V}_{Ce}+\vec{V}_{Cr}, \quad \vec{w}_C=\vec{w}_{Ce}+\vec{w}_{Cr}+2\vec{\omega}\times\vec{V}_{Cr}, \tag{1.5.4}$$

where $\vec{V}_{Ce}$, $\vec{w}_{Ce}$ – migratory velocity and acceleration of the center of mass of the rocket respectively; $\vec{\omega}$ – angular velocity of the rocket body.

At moment of time t (at moment of solidification of the rocket) the centers of mass of the rocket and body S coincide. Therefore, at this moment

$$\vec{V}_{Ce}=\vec{V}_C^{(S)}, \quad \vec{w}_{Ce}=\vec{w}_C^{(S)}. \tag{1.5.5}$$

By excluding $\vec{w}_C^{(S)}$ in equation (1.5.3) with the aid of equalities (1.5.4) and (1.5.5) we will have

$$m\vec{w}_C=\vec{F}+\vec{P}+\vec{F}_k+m\vec{w}_{Cr}+2m\vec{\omega}\times\vec{V}_{Cr}. \tag{1.5.6}$$

The last relationship is the motion equation of center of mass of the rocket in vector form. Components $m\vec{w}_{Cr}$ and $2m\vec{\omega}\times\vec{V}_{Cr}$ are caused by motion of the center of mass relative to the body and there dimensionality of force.

Subsequently for convenience we will write equations (1.5.6) in the form

$$m\vec{w}_C=\sum\vec{F}, \tag{1.5.7}$$

or

$$m\frac{d\vec{V}_C}{dt}=\sum\vec{F}, \tag{1.5.8}$$

by designating through $\Sigma\vec{F}$ the right side of equality (1.5.6) – sum of all forces applied to the rocket.

During practical investigations the vector motion equations of the rocket are replaced by an equivalent system of three scalar equations, projecting the vector equation to some three mutually perpendicular axes.

Let us project equation (1.5.8) to axes of system of rectangular coordinates *Cxyz*, origin of which (point *C*) coincides with the center of mass of the rocket.

If $d\vec{a}/dt$ is the derivative of some vector a relative to system *OXYZ*, $\delta\vec{a}/dt$ – derivative of this vector relative to system *Cxyz* and $\vec{\omega}$ – angular velocity of rotation of system *Cxyz* in system *OXYZ*, then

$$\frac{d\vec{a}}{dt}=\frac{\delta\vec{a}}{dt}+\vec{\omega}\times\vec{a}, \tag{1.5.9}$$

and projections of vector $d\vec{a}/dt$ to axes *Cx*, *Cy*, *Cz* are determined by formulas:

$$\begin{aligned}\left(\frac{d\vec{a}}{dt}\right)_x&=\frac{da_x}{dt}+\omega_y a_z-\omega_z a_y,\\ \left(\frac{d\vec{a}}{dt}\right)_y&=\frac{da_y}{dt}+\omega_z a_x-\omega_x a_z,\\ \left(\frac{d\vec{a}}{dt}\right)_z&=\frac{da_z}{dt}+\omega_x a_y-\omega_y a_x,\end{aligned} \tag{1.5.10}$$

where a_x, a_y, a_z – projections of vector a; ω_x, ω_y, ω_z – projections of vector $\vec{\omega}$ to axes *Cx*, *Cy*, *Cz* (Fig. 1.4).

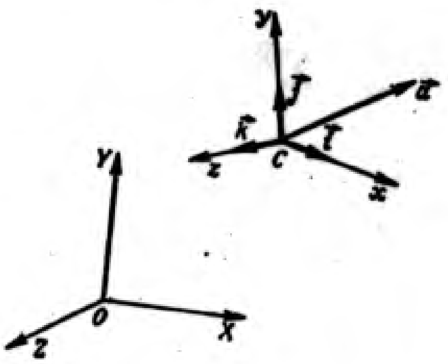

Fig. 1.4.

Actually, by differentiating expression

$$\vec{a}=a_x\vec{i}+a_y\vec{j}+a_z\vec{k},$$

we obtain

$$\frac{d\vec{a}}{dt}=\frac{da_x}{dt}\vec{i}+\frac{da_y}{dt}\vec{j}+\frac{da_z}{dt}\vec{k}+a_x\frac{d\vec{i}}{dt}+a_y\frac{d\vec{j}}{dt}+a_z\frac{d\vec{k}}{dt}. \qquad (1.5.11)$$

Inasmuch as $\vec{i}$, $\vec{j}$, $\vec{k}$ – unit vectors, their derivatives determine the velocities of the ends of these vectors, so that

$$\frac{d\vec{i}}{dt}=\vec{\omega}\times\vec{i},\quad \frac{d\vec{j}}{dt}=\vec{\omega}\times\vec{j},\quad \frac{d\vec{k}}{dt}=\vec{\omega}\times\vec{k}.$$

By considering still that relative derivative of vector $\vec{a}$ is equal to

$$\frac{\delta\vec{a}}{dt}=\frac{da_x}{dt}\vec{i}+\frac{da_y}{dt}\vec{j}+\frac{da_z}{dt}\vec{k},$$

from (1.5.11) we directly obtain formula (1.5.9).

By known rule of calculation of vector product

$$\vec{\omega}\times\vec{a}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \omega_x & \omega_y & \omega_z\\ a_x & a_y & a_z\end{vmatrix}=(\omega_y a_z-\omega_z a_y)\vec{i}+(\omega_z a_x-\omega_x a_z)\vec{j}+$$
$$+(\omega_x a_y-\omega_y a_x)\vec{k}.$$

By substituting the vector product by the provided expression in (1.5.9) and uniting the terms, containing the same unit vector, we obtain

$$\frac{d\vec{a}}{dt}=\left(\frac{da_x}{dt}+\omega_y a_z-\omega_z a_y\right)\vec{i}+\left(\frac{da_y}{dt}+\omega_z a_x-\omega_x a_z\right)\vec{j}+$$
$$+\left(\frac{da_z}{dt}+\omega_x a_y-\omega_y a_x\right)\vec{k}.$$

From this follow equalities (1.5.10).

By using formula (1.5.10), vector equation (1.5.8) can be replaced by the following three scalar equations:

$$m\left(\frac{dV_{Cx}}{dt}+\omega_y V_{Cz}-\omega_z V_{Cy}\right)=\sum F_x,$$
$$m\left(\frac{dV_{Cy}}{dt}+\omega_z V_{Cx}-\omega_x V_{Cz}\right)=\sum F_y, \quad (1.5.12)$$
$$m\left(\frac{dV_{Cz}}{dt}+\omega_x V_{Cy}-\omega_y V_{Cx}\right)=\sum F_z.$$

Here V_{Cx}, V_{Cy}, V_{Cz} – projections of velocity vector of center of mass of the rocket to axes Cx, Cy, Cz; ΣF_x, ΣF_y, ΣF_z – projections of all forces, applied to the rocket, to axes Cx, Cy, Cz.

§6. Equations of Rotary Motion of the Rocket

By omitting index "C" and writing the right side in the form $\Sigma \vec{M}$, let us present equation (1.4.4) in the form

$$\frac{d\vec{L}^{(S)}}{dt}=\sum \vec{M}. \quad (1.6.1)$$

Let us project equation (1.6.1) to axes of moving coordinate system.

Let us assume $L_x^{(S)}$, $L_y^{(S)}$, $L_z^{(S)}$ – projections of angular momentum of a fictitious solid to axes of moving system, and ΣM_x, ΣM_y, ΣM_z – projections of moments of forces, acting on the rocket, to axes of moving system. Then, by using formulas (1.5.10), instead of vector equation (1.6.1), we obtain the following equivalent system of three scalar equations:

$$\frac{dL_x^{(S)}}{dt}+\omega_y L_z^{(S)}-\omega_z L_y^{(S)}=\sum M_x,$$
$$\frac{dL_y^{(S)}}{dt}+\omega_z L_x^{(S)}-\omega_x L_z^{(S)}=\sum M_y, \quad (1.6.2)$$
$$\frac{dL_z^{(S)}}{dt}+\omega_x L_y^{(S)}-\omega_y L_x^{(S)}=\sum M_z.$$

If as axes of the system of coordinates there are selected the principal axes of inertia, then, as it is known,

$$L_x^{(S)}=J_x\omega_x,\ L_y^{(S)}=J_y\omega_y,\ L_z^{(S)}=J_z\omega_z,$$

where J_x, J_y, J_z – moments of inertia of fictitious solid ("solidified" rocket) relative to principal axes of inertia. In this case equations (1.6.2) take the form

$$\begin{aligned} J_x\frac{d\omega_x}{dt}+(J_z-J_y)\omega_y\omega_z&=\sum M_x,\\ J_y\frac{d\omega_y}{dt}+(J_x-J_z)\omega_z\omega_x&=\sum M_y,\\ J_z\frac{d\omega_z}{dt}+(J_y-J_x)\omega_x\omega_y&=\sum M_z. \end{aligned} \qquad (1.6.3)$$

(According to principle of solidification the derivatives of angular momentum and its projections are calculated as for a solid with constant moments of inertia).

Let us note that solid S and the rocket body have the same angular velocity of rotation $\vec{\omega}$ relative to the inertial system, inasmuch as motion of solid S coincides with motion of the rocket body.

Equations (1.6.3) are written for arbitrary, but fixed moment of time t. At different moments of time we will have solids S with different orientation of principal axes of inertia relative to the rocket body and with different moments of inertia relative to these axes. Therefore, equations (1.6.3) are projections of vector equation (1.6.1) to axes of coordinates, differently oriented relative to the rocket body. It would have been possible to consider rotation of principal axes of inertia relative to the rocket body and to write scalar equations of rotatory motion in the same system of coordinates. However, in flight the directions of principal axes of inertia of the rocket change insignificantly, therefore, subsequently, using equations (1.6.3), we will simply assume that directions of principal axes of inertia relative to the rocket body remain constant.

§7. Systems of Coordinates

During investigation of rocket motion there are applied rectangular right-hand systems of coordinates. In further sections there are used launching and earth-fixed systems of coordinates. It is possible to become familiar with other systems of coordinates, being applied in practice, for example, in book [11]. *Launching system of coordinates* $Ax_0y_0z_0$ (Fig. 1.5a).

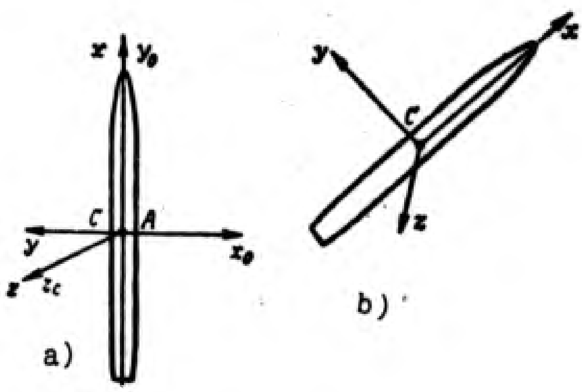

Fig. 1.5.

Axes of *launching* system of coordinates are earth-fixed. Origin of coordinate *A* is combined with the point of launch. Axis Ay_0 is directed upward along a vertical line – directly opposite the force of gravity. Two mutually perpendicular axes Ax_0, Az_0 are arranged in the horizontal plane so that the system of coordinates would be right-hand. Usually axis Ax_0 is directed along the sight line.

Launching system of axes of coordinates is subsequently considered as an inertial system.

Earth-fixed system of coordinates Cxyz (Fig. 1.5b).

The origin of launching system of coordinates C coincides with the center of mass of rocket. Three mutually perpendicular axes of system Cx, Cy, Cz from the right-hand system. As axes of coordinates of the launching system we take three mutually perpendicular principal axes of inertia of the rocket, assuming that the directions of principal axes of inertia relative to the rocket body remain constant. One of the principal axes of inertia of the rocket usually coincides or is very close to the longitudinal axis of the rocket. We will consider that the longitudinal axis of the rocket is the principal axis of inertia, and let us direct axis Cx along the longitudinal axis of the rocket.

At launch the rocket is placed vertically and axis Cx at the moment of launching coincides in direction with axis Ay_0 of the launching system. Let us direct axis Cz so that at the moment of launching it would be parallel to axis Az_0. Axis Cy in this case will be directed along axis Ax_0 to the opposite side. Relative position of launching and earth-fixed systems of coordinates at the moment of launching is shown in Fig. 1.5a.

Arrangment of the earth-fixed system of coordinates with respect to the launching system at arbitrary moment of time of flight of the rocket can be determined by six coordinates: three coordinates x_{0C}, y_{0C}, z_{0C} of origin of the earth-fixed system of coordinates and three angular coordinates.

Figure 1.6 shows axes of earth-fixed system $Cxyz$ and axes of launching system $Ax_0y_0z_0$, origin A of which by means of parallel migration coincides with the origin C of body axes. Three angles ϑ, ψ, γ uniquely determine the directions of axes Cx, Cy, Cz relative to the launching system of coordinates.

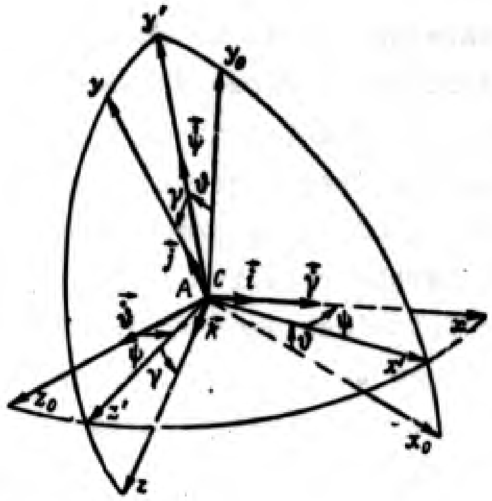

Fig. 1.6.

Angle ϑ (angle between axis Cx_0 of launching system and plane Cxz_0) is called *angle of pitch;* angle ψ (angle between axis Cx and plane Cx_0y_0) is called *angle of yaw;* finally, earth-fixed γ (earth-fixed between axis Cy and intersection line of planes Cyz and Cx_0y_0) is called *angle of roll.* Arrows show the positive directions of reading these angles.

Cosines of angles between body and launching axes of coordinates are numerically equal to corresponding projections of unit vectors $\vec{i}$, $\vec{j}$, $\vec{k}$, directed along body axes to launching axes. These projections are easily calculated by Fig. 1.6; values of projections of unit vectors $\vec{i}$, $\vec{j}$, $\vec{k}$ are listed in Table 1.1.

Table 1.1.

Unit vectors	Axes		
	Ax_0	Ay_0	Az_0
$\vec{i}$	$\cos\vartheta\cos\psi$	$\sin\vartheta\cos\psi$	$-\sin\psi$
$\vec{j}$	$\cos\vartheta\sin\psi\sin\gamma-\sin\vartheta\cos\gamma$	$\sin\vartheta\sin\psi\cdot\sin\gamma+\cos\vartheta\cos\gamma$	$\cos\psi\sin\gamma$
$\vec{k}$	$\cos\vartheta\sin\psi\cos\gamma+\sin\vartheta\sin\gamma$	$\sin\vartheta\sin\psi\cos\gamma-\cos\vartheta\sin\gamma$	$\cos\psi\cos\gamma$

§ 8. Kinematic Relationships

During flight the rocket, and together with it the axes of earth-fixed system of coordinates are continuously turned relative to the launching, which leads to continuous change of angles of pitch, yaw and roll. The rates of change these angles are equal to $\dot{\vartheta}, \dot{\psi}, \dot{\gamma}$. respectively. Vector of angular velocity $\vec{\dot{\vartheta}}$ are directed along axis Cz_0 (see Fig. 1.6); with rotation of body axes around axis Cz_0 angles ψ, γ remain constant and only angle ϑ. is changed, the plane, in which this angle is read, is perpendicular to the axis of rotation Cz_0. Angular velocity vector $\vec{\dot{\psi}}$ is directed along intermediate axis Cy'; with rotation of body axes around axis Cy' angles ϑ, γ remain constant and only angle ψ is changed, the plane in which angle ψ is read is perpendicular to axis of rotation Cy'. Angular velocity vector $\vec{\dot{\gamma}}$ is directed along body axis Cx; with rotation of body axes around axis Cx; angles ϑ, ψ remain constart and only angle γ is changed, the plane in which this angle is read is perpendicular to axis of rotation Cx. Arbitrary angular displacement of earth-fixed system of axes of coordinates relative to launching system around the axis passing through the origin of body axes can be represented as resulting displacement from rotary motions around three axes Cz_0, Cy' and Cx. Therefore, if ω – angular velocity of rotation of body axes relative to launching system of coordinates, then

$$\vec{\omega} = \vec{\dot{\vartheta}} + \vec{\dot{\psi}} + \vec{\dot{\gamma}}. \tag{1.8.1}$$

Let us project vector equality (1.8.1) to body axes Cx, Cy, Cz.

Direction cosines of vector $\vec{\dot{\vartheta}}$ in earth-fixed system of coordinates are equal $-\sin\psi$, $\cos\psi\sin\gamma$, $\cos\psi\cos\gamma$; vector $\vec{\dot{\psi}} - 0, \cos\gamma, -\sin\gamma$, vector $\vec{\dot{\gamma}} - 1, 0, 0$.

Considering this, we obtain the following group of kinematic equations, establishing the connection between projections to axes of earth-fixed system of angular velocity of rotation of the rocket with angular parameters, characterizing the position and motion of the rocket relative to launching system of coordinates:

$$\begin{aligned} \omega_x &= \dot{\gamma} - \dot{\vartheta}\sin\psi, \\ \omega_y &= \dot{\psi}\cos\gamma + \dot{\vartheta}\cos\psi\sin\gamma, \\ \omega_z &= -\dot{\psi}\sin\gamma + \dot{\vartheta}\cos\psi\cos\gamma. \end{aligned} \qquad (1.8.2)$$

The velocity of some particle of the rocket in launching system of coordinates $\vec{v}$ is made up of migratory velocity $\vec{v}_e$ (velocity of motion of particle together with body axes) and relative velocity $\vec{v}_r$ (velocity of motion of particle relative to body axes):

$$\vec{v} = \vec{v}_e + \vec{v}_r. \qquad (1.8.3)$$

Migratory and relative velocities are respectively equal to:

$$\vec{v}_e = \vec{V}_C + \vec{\omega} \times \vec{r}, \quad \vec{v}_r = \frac{\delta \vec{r}}{dt},$$

where $\vec{r}$ - radius vector of the considered point (Fig. 1.7).

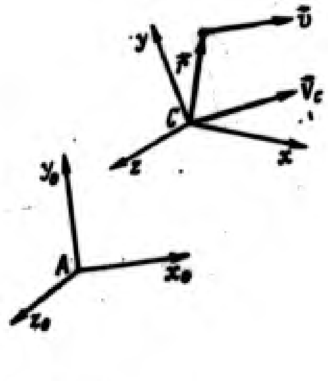

Fig. 1.7.

Hence

$$\vec{v}=\vec{V}_C+\vec{\omega}\times\vec{r}+\frac{\delta\vec{r}}{dt}. \qquad (1.8.4)$$

By projecting vector equality (1.8.4) to body axes, we obtain the second group of kinematic equations:

$$\begin{aligned} v_x&=V_{Cx}+\omega_y z-\omega_z y+\frac{dx}{dt},\\ v_y&=V_{Cy}+\omega_z x-\omega_x z+\frac{dy}{dt},\\ v_z&=V_{Cz}+\omega_x y-\omega_y x+\frac{dz}{dt}, \end{aligned} \qquad (1.8.5)$$

where x, y, z – projections of radius vector $\vec{r}$ to axes Cx, Cy, Cz.

§9. Forces Affecting the Rocket in Flight

The right sides of equations of motion of center of mass of the rocket include external forces, engine thrust, Coriolis forces, connected to relative motion of particles inside the rotating rocket body, and forces caused by displacements of center of mass of the rocket relative to the body. By external forces we imply forces of gravity and aerodynamic forces.

Bearing in mind that in subsequent sections the dynamic equations, represented by systems (1.5.12) and (1.6.3), will be written in projections to body axes, all forces and moments of these forces will also be projected to axes of earth-fixed system of coordinates.

9.1. Engine Thrust

For generality let us suppose that the rocket engine (power plant) is a cluster of a certain number of comparatively small engines. These engines, being sustainer engines, at the same time can be used for control of rocket flight. We will consider that all controls, connected to operation of the power plant, occupy a neutral position[1].

Thrust, being created by j-th engine, let us designate by $\vec{P}_j$, and its projections to body axes – by P_{jx}, P_{jy}, P_{jz}. Axial lines of engines with longitudinal axis of the rocket form angles, usually not exceeding several degrees. Therefore, cosine of the angle between vector $\vec{P}_j$ and axis Cx of earth-fixed system is practically equal to one. By η_j and ξ_j let us designate respectively the cosines of angles between vector $\vec{P}_j$ and axes Cy and Cz. Then

$$P_{jx} \approx P_j, \ P_{jy} = \eta_j P_j, \ P_{jz} = \xi_j P_j. \tag{1.9.1}$$

Total engine thrust of the rocket is equal to:

$$\vec{P} = \sum_j \vec{P}_j. \tag{1.9.2}$$

By projecting equality (1.9.2) to body axes, we obtain

$$\begin{aligned} P_x &= \sum_j P_{jx} \approx P, \\ P_y &= \sum_j \eta_j P_j, \\ P_z &= \sum_j \xi_j P_j. \end{aligned} \tag{1.9.3}$$

Radius vector of the point of intersection of the line of action of vector $\vec{P}_j$ with the plane of nozzle section of j-th engine is designated by $\vec{\varrho}_j$. Moment of engine thrust relative to center of mass of the rocket is equal to:

[1]Forces appearing with deflection of controls are considered in 9.4.

$$\vec{M}_p = \sum_j \vec{\varrho}_j \times \vec{P}_j. \qquad (1.9.4)$$

By projecting vector equality (1.9.4) to body axes, we will have

$$\begin{aligned} M_{px} &= \sum_j (y_j \zeta_j - z_j \eta_j) P_j, \\ M_{py} &= \sum_j (z_j - x_j \zeta_j) P_j, \\ M_{pz} &= \sum_j (x_j \eta_j - y_j) P_j, \end{aligned} \qquad (1.9.5)$$

where x_j, y_j, z_j – projections of radius vector $\vec{\varrho}_j$ to body axes.

9.2. Force of Gravity

Force of gravity, or weight of the rocket, $\vec{G}$ is expressed by known formula

$$\vec{G} = m\vec{g}, \qquad (1.9.6)$$

where m – mass of rocket; $\vec{g}$ – acceleration of the earth's attraction.

If g_x, g_y, g_z – projections of acceleration of earth's attraction $\vec{g}$ to axes Cx, Cy, Cz, then projections of weight of the vehicle to the same axes will be equal to:

$$G_x = mg_x, \; G_y = mg_y, \; G_z = mg_z. \qquad (1.9.7)$$

Vector $\vec{G}$ passes through the center of mass of the rocket, therefore, the moment of this force relative to center of mass is equal to zero.

9.3. Aerodynamic Forces

Aerodynamic forces appear during motion of the flight vehicle in air medium as a result of interaction between the vehicle surface and medium.

In this case it is assumed that the external surface of the rocket is symmetric relative to planes *Cxy* and *Cxz*, and air vanes, if there are such, occupy a neutral position.

Let us suppose first that the rocket moves forward, i.e., $\vec{\omega} = 0$, and that the air medium, in which the flight is accomplished, is fixed relative to the launching system of coordinates. In these conditions the aerodynamic forces are determined by the shape and size of the rocket, parameters of air and velocity of the rocket (body) relative to air. By disregarding the velocity of center of mass of the rocket relative to its body, we will consider that the velocity of the rocket body relative to air medium is equal to the velocity of its center of mass $\vec{V}_C$.

Resultant of all aerodynamic forces, applied to the rocket, will be called *total aerodynamic force* and designated by $\vec{R}_a$.

If velocity $\vec{V}_C$ lies in the plane of symmetry of the rocket, let us say in plane *Cxy*, then total aerodynamic force will lie in the same plane. By the point of its application (center of pressure) it is conditionally possible to consider the point of intersection of vector $\vec{R}_a$ with axis *Cx*. At small angles of attack in view of the assumed symmetry of external surface of the body the center of pressure will coincide with aerodynamic focal point of the rocket, which, as it is known, is defined as the point of longitudinal axis, possessing the property that at fixed controls the moment of aerodynamic forces relative to the axis passing through this point perpendicular to the plane of symmetry (plane *Cxy*), does not depend on the angle of attack. Thus, in this case the center of pressure and focal point coincide with each other and are located on the longitudinal axis of the rocket.

With the presence on the rocket of a fin assembly (stabilizers and air vanes) velocity vector $\vec{V}_C$ may not lie in the plane of symmetry of the rocket and then, strictly speaking, vector $\vec{R}_a$

cannot lie in one plane with vector $\vec{V}_C$ and longitudinal axis of the rocket. However, in this instance vector $\vec{R}_a$ will depart from the plane only insignificantly, passing through vector $\vec{V}_C$ and axis Cx, especially as control of ballistic rockets is usually constructed so that in dense layers of the atmosphere the angle between vector $\vec{V}_C$ and plane of symmetry of the rocket is not large. Considering this, we will consider that in this instance the center of pressure lie on the longitudinal axis of the rocket and coincides with aerodynamic focal point, and vector of total aerodynamic force lies in the plane passing through axis Cx and vector $\vec{V}_C$.

Let us decompose total aerodynamic force $\vec{R}_a$ into two components: drag force $\vec{X}$, directed along the line of action of velocity vector $\vec{V}_C$, and lifting force $\vec{Y}$ (Fig. 1.8). These components by a model can be represented so:

$$X = c_x \varrho S \frac{V_C^2}{2}, \quad Y = c_y \varrho S \frac{V_C^2}{2},$$

where c_x – drag coefficient; c_y – lift coefficient; ϱ – density of air; S – area of midsection of rocket body.

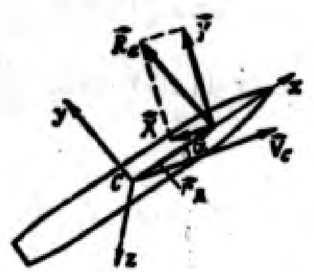

Fig. 1.8.

Having projected aerodynamic force to body axes, we obtain (Figs. 1.8 and 1.9):

$$R_{ax} = -X\cos\alpha + Y\sin\alpha,$$
$$R_{ay} = (X\sin\alpha + Y\cos\alpha)\cos\chi,$$
$$R_{az} = (X\sin\alpha + Y\cos\alpha)\sin\chi.$$

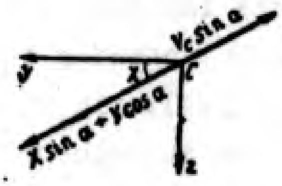

Fig. 1.9.

Here α – angle between velocity vector $\vec{V}_C$ and axis *Cx;* we will call it the *angle of attack;* χ – angle between projections of aerodynamic force $\vec{R}_a$ to plane *Cyz* and axis *Cy*.

From obvious equalities

$$V_{Cy}=-V_C\sin\alpha\cos\chi,\ V_{Cz}=-V_C\sin\alpha\sin\chi$$

we have

$$\cos\chi=-\frac{V_{Cy}}{V_C\sin\alpha},\quad \sin\chi=-\frac{V_{Cz}}{V_C\sin\alpha}.$$

Therefore,

$$R_{ay}=-(X\sin\alpha+Y\cos\alpha)\frac{V_{Cy}}{V_C\sin\alpha},$$
$$R_{az}=-(X\sin\alpha+Y\cos\alpha)\frac{V_{Cz}}{V_C\sin\alpha}.$$

By substituting values of *X* and *Y*, we obtain

$$\begin{aligned}R_{ax}&=-\frac{1}{2}(c_x\cos\alpha-c_y\sin\alpha)\rho SV_C^2,\\ R_{ay}&=-\frac{1}{2}(c_x\sin\alpha+c_y\cos\alpha)\rho S\frac{V_{Cy}V_C}{\sin\alpha},\\ R_{az}&=-\frac{1}{2}(c_x\sin\alpha+c_y\cos\alpha)\rho S\frac{V_{Cz}V_C}{\sin\alpha}.\end{aligned}\qquad(1.9.8)$$

Moment of aerodynamic forces relative to the center of mass of the rocket is equal to vector product of radius vector of center of pressure $\vec{r}_д (x_д, 0, 0)$ by vector of total aerodynamic force:

$$\vec{M}_a = \vec{r}_д \times \vec{R}_a,$$

or, since in the considered case the center of pressure and aerodynamic focal point coincide,

$$\vec{M}_a = \vec{r}_F \times \vec{R}_a.$$

Here $\vec{r}_F (x_F, 0, 0)$ — radius vector of focal point of the vehicle.

By projecting this vector product to axes Cx, Cy, Cz, we obtain

$$M_{ax} = 0;$$
$$M_{ay} = -x_F R_{az} = \frac{1}{2} x_F (c_x \sin\alpha + c_y \cos\alpha) \varrho S \frac{V_{Cz} V_C}{\sin\alpha};$$
$$M_{az} = x_F R_{ay} = -\frac{1}{2} x_F (c_x \sin\alpha + c_y \cos\alpha) \varrho S \frac{V_{Cy} V_C}{\sin\alpha}. \quad (1.9.9)$$

Let us now suppose that air medium, in which the flight of the rocket is accomplished, moves relative to the launching system of coordinates at velocity $\vec{V}_B$. Then the center of mass of the rocket will move relative to air at velocity $\vec{V}_C - \vec{V}_B$ (Fig. 1.10). Aerodynamic forces are determined exactly like velocity of the rocket relative to air medium. Therefore, during consideration of motion of air medium relative to the launching system in formulas (1.9.8) and (1.9.9) velocity V_C must be replaced by modulus of velocity $\vec{V}_C - \vec{V}_B$, and projections of velocity of center of mass to body axes relative to launching axes must be replaced by projections of velocity of center of mass relative to air.

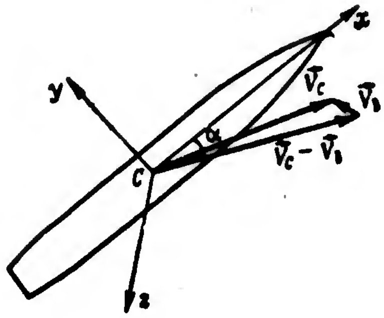

Fig. 1.10.

If the rocket moves not only forward, but also rotates around the center of mass, then additional aerodynamic forces appear. Projections of the resultant of these forces to axes Cx, Cy, Cz let us designate $\Delta R_{a\ x}$, $\Delta R_{a\ y}$, $\Delta R_{a\ z}$, and projections of corresponding (damping) moment to the same axes – $\Delta M_{a\ x}$, $\Delta M_{a\ y}$, $\Delta M_{a\ z}$.

Thus, projections of aerodynamic forces and moments to body axes generally can be represented by formulas:

$$R_{ax}=\Delta R_{ax}-\frac{1}{2}(c_x\cos\alpha-c_y\sin\alpha)\rho S\,|\vec{V}_C-\vec{V}_в|^2;$$

$$R_{ay}=\Delta R_{ay}-\frac{1}{2}(c_x+c_y\operatorname{ctg}\alpha)\rho S(V_{Cy}-V_{вy})\,|\vec{V}_C-\vec{V}_в|; \qquad (1.9.10)$$

$$R_{az}=\Delta R_{az}-\frac{1}{2}(c_x+c_y\operatorname{ctg}\alpha)\rho S(V_{Cz}-V_{вz})\,|\vec{V}_C-\vec{V}_в|;$$

$$M_{ax}=\Delta M_{ax};$$

$$M_{ay}=\Delta M_{ay}+\frac{1}{2}x_F(c_x+c_y\operatorname{ctg}\alpha)\rho S(V_{Cz}-V_{вz})\,|\vec{V}_C-\vec{V}_в|; \qquad (1.9.11)$$

$$M_{az}=\Delta M_{az}-\frac{1}{2}x_F(c_x+c_y\operatorname{ctg}\alpha)\rho S(V_{Cy}-V_{вy})\,|\vec{V}_C-\vec{V}_в|.$$

In these formulas α – angle between vector $\vec{V}_C - \vec{V}_B$. and Cx (see Fig. 1.10). For rockets of the considered class it is characteristic that in flight angle α and angular velocity of rotation ω are small. By using this circumstance, it is possible to considerably simplify formulas (1.9.10) and (1.9.11).

At small α the lift coefficient

$$c_y = c_y^\alpha \alpha, \tag{1.9.12}$$

drag coefficient

$$c_x = c_{x0} + A\alpha^2, \tag{1.9.13}$$

where c_y^α – derivative of lift coefficient with respect to angle α when α = 0; c_{x0} – drag coefficient when α = 0; A – quantity not depending on angle α at small α.

Further, with small ω the projections of resultant of aerodynamic forces, caused by rotation of the vehicle, and corresponding moment to body axes are proportional to projections of angular velocity $\vec{\omega}$ to these axes, namely:

$$\Delta R_{a\,x} = 0,\ \Delta R_{a\,y} = R_{a\,y}^{\omega_z}\omega_z,\ \Delta R_{a\,z} = R_{a\,z}^{\omega_y}\omega_y,$$
$$\Delta M_{a\,x} = M_{a\,x}^{\omega_x}\omega_x,\ \Delta M_{a\,y} = M_{a\,y}^{\omega_y}\omega_y,\ \Delta M_{a\,z} = M_{a\,z}^{\omega_z}\omega_z, \tag{1.9.14}$$

where

$$R_{a\,y}^{\omega_z} = \frac{\partial R_{a\,y}}{\partial \omega_z},\ R_{a\,z}^{\omega_y} = \frac{\partial R_{a\,z}}{\partial \omega_y},\ M_{a\,x}^{\omega_x} = \frac{\partial M_{a\,x}}{\partial \omega_x},$$
$$M_{a\,y}^{\omega_y} = \frac{\partial M_{a\,y}}{\partial \omega_y},\ M_{a\,z}^{\omega_z} = \frac{\partial M_{a\,z}}{\partial \omega_z}.$$

If the point of application of resultant of aerodynamic forces, caused by rotation of the vehicle, is located in front of the center of mass, then $R_{a\,y}^{\omega_z} < 0$, $R_{a\,z}^{\omega_y} > 0$. If, however, this point is located behind the center of mass, then $R_{a\,y}^{\omega_z} > 0$, $R_{a\,z}^{\omega_y} < 0$. In all cases each quantity $M_{a\,x}^{\omega_x}$, $M_{a\,y}^{\omega_y}$, $M_{a\,z}^{\omega_z}$ is smaller than zero.

Let us substitute values of c_x, c_y, $\Delta R_{a\,x}$, $\Delta R_{a\,y}$, $\Delta R_{a\,z}$, $\Delta M_{a\,x}$, $\Delta M_{a\,y}$, $\Delta M_{a\,z}$, determined by formulas (1.9.12)-(1.9.14), in formulas (1.9.10) and (1.9.11) and let us exclude $|\vec{V}_c - \vec{V}_a|$ with the aid of equality

$$|\vec{V}_C-\vec{V}_{\text{в}}|=(V_{Cx}-V_{\text{в}x})\frac{1}{\cos\alpha}. \qquad (1.9.15)$$

Then, by assuming $\sin\alpha\approx\alpha$, $\cos\alpha\approx 1$ and disregarding terms containing α higher than to the first power, we obtain

$$\begin{aligned}
R_{\text{в}x}&=-\frac{1}{2}c_{x0}\varrho S(V_{Cx}-V_{\text{в}x})^2;\\
R_{\text{в}y}&=R_{\text{в}y}^{\omega_z}\omega_z-\frac{1}{2}(c_{x0}+c_y^{\alpha})\varrho S(V_{Cx}-V_{\text{в}x})(V_{Cy}-V_{\text{в}y});\\
R_{\text{в}z}&=R_{\text{в}z}^{\omega_y}\omega_y-\frac{1}{2}(c_{x0}+c_y^{\alpha})\varrho S(V_{Cx}-V_{\text{в}x})(V_{Cz}-V_{\text{в}z});
\end{aligned} \qquad (1.9.16)$$

$$\begin{aligned}
M_{\text{в}x}&=M_{\text{в}x}^{\omega_x}\omega_x;\\
M_{\text{в}y}&=M_{\text{в}y}^{\omega_y}\omega_y+\frac{1}{2}x_F(c_{x0}+c_y^{\alpha})\varrho S(V_{Cx}-V_{\text{в}x})(V_{Cz}-V_{\text{в}z});\\
M_{\text{в}z}&=M_{\text{в}z}^{\omega_z}\omega_z-\frac{1}{2}x_F(c_{x0}+c_y^{\alpha})\varrho S(V_{Cx}-V_{\text{в}x})(V_{Cy}-V_{\text{в}y}),
\end{aligned} \qquad (1.9.17)$$

where $V_{\text{в}x}$, $V_{\text{в}y}$, $V_{\text{в}z}$ – projections of vector $\vec{V}_{\text{в}}$ to body axes.

9.4. Control Forces

For providing rocket flight according to prescribed law it is necessary to have the capability of changing in flight the velocity of center of mass of the rocket and orientation of its axes in space by the necessary means. For this purpose there are used various controls. With deflection of controls from neutral position there appear the forces and moments necessary for control.

According to functions being fulfilled the controls are conveniently broken down into the following 4 groups:

1. Controls of velocity of center of mass of the rocket.

2. Controls of rotation of the rocket around lateral axis Cz.

3. Controls of rotation of rocket around lateral axis Cy.

4. Controls of rotation of rocket around longitudinal axis Cx.

Let us examine the forces and moments which appear with deflection of controls of one group or another.

Controls of velocity of center of mass of the rocket

Change in the velocity of center of mass of the rocket by the required means can be realized with the aid of thrust control of the engine and the application of special braking devices. Usually thrust control is applied in practice. In this instance the control force, directed along axis Cx, is equal to change of engine thrust:

$$\Delta X = p$$

($p > 0$ with increase in thrust in comparison with nominal thrust and $p < 0$ otherwise).

Let us represent control force so:

$$\Delta X = c_{x\delta}\delta_x,$$

where $c_{x\delta}$ – coefficient, equal in this case to nominal engine thrust P; $\delta_x = p/P$ – quantity characterizing relative change of thrust.

Controls of rotation of rocket around lateral axis Cz

It is possible to control rotation of the rocket around axis Cz with the aid of air vanes, jet (gas jet) vanes, auxiliary (vernier) engines; by turning sustainer engines, change of thrust of sustainer engines; with the aid of swiveling nozzles of engines, baffles and so forth.

Despite the variety of means applied for control of rotation of the rocket around lateral axis, in all cases the forces and moments, which appear with deflection of controls, can be recorded identically. Let us examine, for example, certain types of controls.

Air vanes are comparatively small lifting surfaces, installed either directly on the rocket body, or on fixed surfaces, similar to wings, – stabilizers. With deflection of vanes symmetrically to plane *Cxy* (Fig. 1.11) there appears aerodynamic force, lying in plane *Cxy*. The component of aerodynamic force along axis *Cy* can be represented so:

$$\Delta Y = c_y^{\delta} \delta_{\text{в}} q S_{\text{p}},$$

where $\delta_{\text{в}}$ – angle of deflection of vane; c_y^{δ} – derivative of lift coefficient of vane in earth-fixed system of coordinates with respect to angle of deflection of vane; $q=\varrho\frac{V_C^2}{2}$ – dynamic head; S_{p} – area of air vanes.

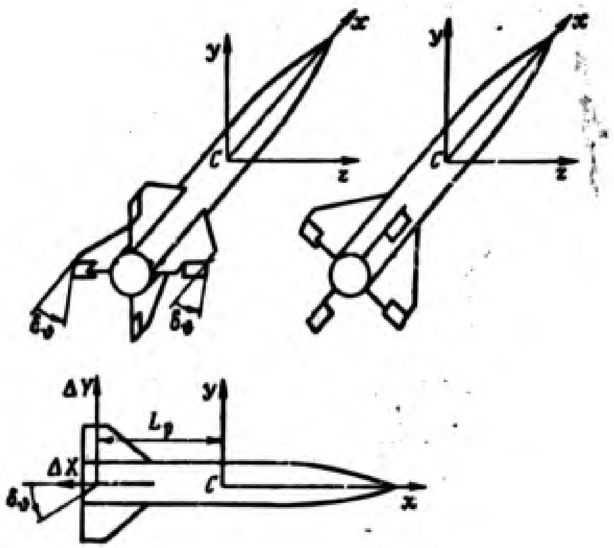

Fig. 1.11.

Another component at small angles of deflection of vane and small angles of attack of the rocket is negligible, and we will subsequently assume $\Delta X = 0$.

Moment of aerodynamic forces relative to axis Cz, appearing with deflection of air vanes, is equal to:

$$\Delta M_z = -L_p \Delta Y,$$

where L_p – distance from center of pressure of vanes to center of mass of the rocket.

Thus, with deflection of air vanes symmetrically to plane Cxy there appear force and moment, projections of which to body axes can be written in the form

$$\Delta X = 0,\ \Delta Y = c_{y\delta} \cdot \delta_\varphi,\ \Delta Z = 0,$$
$$\Delta M_x = \Delta M_y = 0,\ \Delta M_z = c_{\delta\delta} \cdot \delta_\varphi,$$

where

$$c_{y\delta} = c_y^\delta q S_p,\ c_{\delta\delta} = -L_p c_{y\delta}.$$

Jet vanes in many respects are similar to air vanes. The essential difference is only the fact that jet vanes are installed in the flow of gases from engines. Forces and moments, which appear with deflection of jet vanes, can be represented by formulas of the same form as air vanes, with the difference that coefficient c_y^δ must be determined for those conditions, under which the jet vanes are located; by q we should imply dynamic head of gases in the engine nozzle in the area of location of jet vanes.

Rotation of engine around the axis, parallel to axis C_z, is accompanied by the appearance of the following forces and moment:

$$\Delta X = P\cos\delta_\varphi - P = -2P\sin^2\frac{\delta_\varphi}{2},\ \Delta Y = P\sin\delta_\varphi,$$
$$\Delta M_z = -L_p P \sin\delta_\varphi.$$

Usually, angles of rotation of the engine are small. Therefore, it is possible to take

$$\sin\delta_\varphi \approx \delta_\varphi,\ \sin\frac{\delta_\varphi}{2} \approx \frac{\delta_\varphi}{2},\ \cos\delta_\varphi \approx 1.$$

Then, having discarded quantities of the second order of smallness, we will have

$$\Delta X \approx 0,\ \Delta Y = c_{y\delta}\delta_\varphi,\ \Delta M_z = c_{\varphi\delta}\delta_\varphi,$$

where

$$c_{y\delta} = P,\ c_{\varphi\delta} = -LP.$$

If the rocket engine consists of several chambers, then motion around axis Cz can be controlled by *deviation of thrust.* Let us suppose that for this purpose two chambers of the engine are used. If the thrust of one of the chambers is increased, and the other – decreased by p, then total engine thrust will not change, and at the same time there will appear a moment relative to axis Cz, equal to

$$\Delta M_z = -ap = -aP_l\delta_\varphi,$$

where

$$\delta_\varphi = \frac{p}{P_l}.$$

It is clear that in this case we can write

$$\Delta M_z = c_{\varphi\delta}\delta_\varphi,$$

bearing in mind that

$$c_{\varphi\delta} = -aP_l.$$

Analogous expressions take place for other types of controls.

Thus, with deflection of controls of rotation of the rocket around axis Cz there appear forces along axes Cx, Cy and moment along axis Cz, which generally with simultaneous utilization of several types of controls can be represented in the form

$$
\begin{aligned}
&\sum_i \Delta X_i \approx 0,\\
&\sum_i \Delta Y_i = \sum_i c_{y\delta i}\delta_{\theta i},\\
&\sum_i \Delta M_{zi} = \sum_i c_{\theta\delta i}\delta_{\theta i}.
\end{aligned}
\qquad (1.9.18)
$$

Formulas (1.9.18), it is understood, are approximate, because they are obtained without allowing for the rate of change of quantities $\delta_{\theta i}$ and certain other factors, affecting forces and moments created by the controls. However, the effect of neglected factors is small, and for flight vehicle of the considered class expressions (1.9.18) are entirely acceptable.

Controls of rotation of the rocket around lateral axis Cy

For control of rotation around axis Cy there are applied the same controls as for control around axis Cz. In accordance with this, forces and moment, which appear with deflection of controls, can be represented in the following manner:

$$
\begin{aligned}
&\sum_i \Delta X_i \approx 0,\\
&\sum_i \Delta Z_i = \sum_i c_{z\delta i}\delta_{\psi i},\\
&\sum_i \Delta M_{yi} = \sum_i c_{\psi\delta i}\delta_{\psi i}.
\end{aligned}
\qquad (1.9.19)
$$

Coefficients $c_{z\delta i}$, $c_{\psi\delta i}$ are analogous to coefficients $c_{y\delta i}$, $c_{\theta\delta i}$ in formulas (1.9.18), and in each particular case their expressions can be easily found.

In the third equality (1.9.19) in contrast to corresponding equality (1.9.18) coefficients $c_{\psi\delta i}$ are positive, because in this case with positive $\delta_{\psi i}$ there appears positive force ΔZ_i and positive moment ΔM_{yi} (Fig. 1.15).

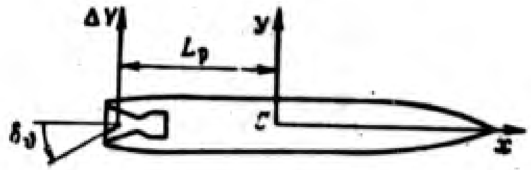

Fig. 1.12.

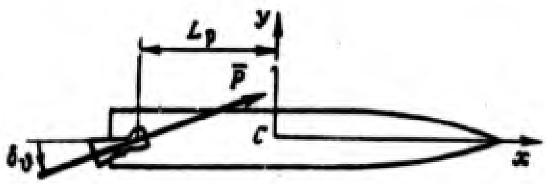

Fig. 1.13.

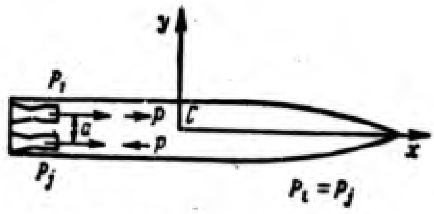

Fig. 1.14.

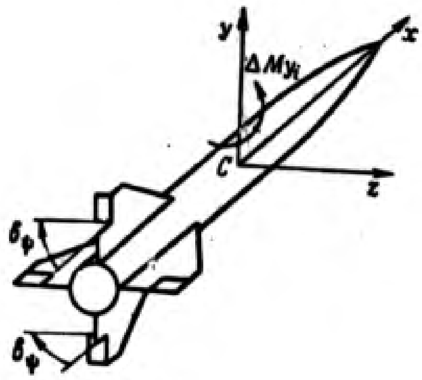

Fig. 1.15.

Controls of rotation of the rocket around longitudinal axis *Cx*

Rotation around axis *Cx* can be controlled with the aid of air and jet vanes, swiveling engines or engines established at certain angles to the longitudinal axis of the rocket, with the aid of auxiliary engines and so forth.

With deflection of controls a moment appears along axis *Cx*, which can be represented in the form

$$\sum_i \Delta M_{xi} = \sum_i c_{\gamma\delta i}\delta_{\gamma i} \quad (c_{\gamma\delta i} < 0), \tag{1.9.20}$$

and force

$$\sum_i \Delta X_i \approx 0. \tag{1.9.21}$$

For example, with deflection of air vanes to angle δ_γ (Fig. 1.16) there appears a moment proportional to angle δ_γ. With positive δ_γ there appears negative moment relative to axis *Cx*. Therefore, $c_{\gamma\delta} < 0$.

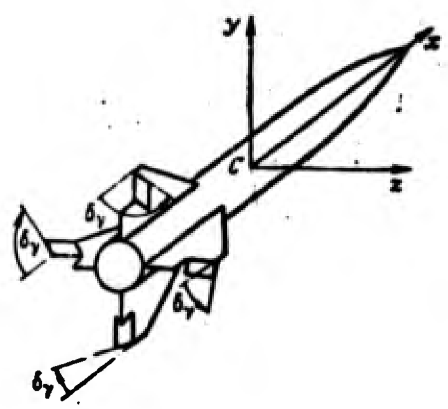

Fig. 1.16.

Forces appearing with deflection of controls from the neutral position are brought to principal vector $\vec{F}_{\text{упр}}$ (applied in the center of mass of the rocket) and to principal moment $\vec{M}_{\text{упр}}$. Projections of force $\vec{F}_{\text{упр}}$ and moment $\vec{M}_{\text{упр}}$ to body axes of coordinates are designated respectively by $F_{\text{упр}\,x}$, $F_{\text{упр}\,y}$, $F_{\text{упр}\,z}$, $M_{\text{упр}\,x}$, $M_{\text{упр}\,y}$, $M_{\text{упр}\,z}$. On the basis of this

$$\begin{aligned}
F_{\text{упр}\,x} &= c_{x\delta}\delta_x,\\
F_{\text{упр}\,y} &= \sum_i \Delta Y_i = \sum_i c_{y\delta i}\delta_{\varphi i},\\
F_{\text{упр}\,z} &= \sum_i \Delta Z_i = \sum_i c_{z\delta i}\delta_{\psi i},\\
M_{\text{упр}\,x} &= \sum_i \Delta M_{xi} = \sum_i c_{\gamma\delta i}\delta_{\gamma i},\\
M_{\text{упр}\,y} &= \sum_i \Delta M_{yi} = \sum_i c_{\psi\delta i}\delta_{\psi i},\\
M_{\text{упр}\,z} &= \sum_i \Delta M_{zi} = \sum_i c_{\varphi\delta i}\delta_{\varphi i}.
\end{aligned} \qquad (1.9.22)$$

If between deflections of controls of each group there exists a proportional relationship, then, by reducing these deflections to deflection of some control of this group, formulas (1.9.22) can be represented in simpler form, namely:

$$\begin{gathered}
F_{\text{упр}\,x} = c_{x\delta}\delta_x,\\
F_{\text{упр}\,y} = c_{y\delta}\delta_\varphi,\quad F_{\text{упр}\,z} = c_{z\delta}\delta_\psi,\\
M_{\text{упр}\,x} = c_{\gamma\delta}\delta_\gamma,\quad M_{\text{упр}\,y} = c_{\psi\delta}\delta_\psi,\quad M_{\text{упр}\,z} = c_{\varphi\delta}\delta_\varphi,
\end{gathered} \qquad (1.9.23)$$

where

$$c_{y\delta} = \sum_i c_{y\delta i}\frac{\delta_{\varphi i}}{\delta_\varphi},\quad c_{z\delta} = \sum_i c_{z\delta i}\frac{\delta_{\psi i}}{\delta_\psi},$$

$$c_{\gamma\delta} = \sum_i c_{\gamma\delta i}\frac{\delta_{\gamma i}}{\delta_\gamma},\quad c_{\psi\delta} = \sum_i c_{\psi\delta i}\frac{\delta_{\psi i}}{\delta_\psi},\quad \text{etc.}$$

9.5. Coriolis Forces

During rotation of the rocket around the center of mass and motion of propellant and gases relative to the body there appear Coriolis forces, principal vector and principal moment of which are represented by formulas:

$$\vec{F}_{\text{к}}=-\int_{\Sigma}\vec{w}_{\text{к}}dm,\quad \vec{M}_{\text{к}}=-\int_{\Sigma}\vec{r}'\times\vec{w}_{\text{к}}dm.$$

Coriolis acceleration

$$\vec{w}_{\text{к}}=2\vec{\omega}\times\vec{v}_r.$$

Therefore,

$$\vec{F}_{\text{к}}=-2\int_{\Sigma}\vec{\omega}\times\vec{v}_r dm,\quad \vec{M}_{\text{к}}=-2\int_{\Sigma}\vec{r}'\times\vec{\omega}\times\vec{v}_r dm. \qquad (1.9.24)$$

By projecting vectors (1.9.24) to body axes, we obtain:

$$F_{\text{к}x}=-2\omega_y\int_{\Sigma}v_{rz}dm+2\omega_z\int_{\Sigma}v_{ry}dm,$$

$$F_{\text{к}y}=-2\omega_z\int_{\Sigma}v_{rx}dm+2\omega_x\int_{\Sigma}v_{rz}dm, \qquad (1.9.25)$$

$$F_{\text{к}z}=-2\omega_x\int_{\Sigma}v_{ry}dm+2\omega_y\int_{\Sigma}v_{rx}dm;$$

$$M_{\text{к}x}=-2\omega_x\int_{\Sigma}y'v_{ry}dm-2\omega_x\int_{\Sigma}z'v_{rz}dm+$$

$$+2\omega_y\int_{\Sigma}y'v_{rx}dm+2\omega_z\int_{\Sigma}z'v_{rx}dm,$$

$$M_{\text{к}y}=-2\omega_y\int_{\Sigma}z'v_{rz}dm-2\omega_y\int_{\Sigma}x'v_{rx}dm+$$

$$+2\omega_z\int_{\Sigma}z'v_{ry}dm+2\omega_x\int_{\Sigma}x'v_{ry}dm, \qquad (1.9.26)$$

$$M_{\text{к}z}=-2\omega_z\int_{\Sigma}x'v_{rx}dm-2\omega_z\int_{\Sigma}y'v_{ry}dm+$$

$$+2\omega_x\int_{\Sigma}x'v_{rz}dm+2\omega_y\int_{\Sigma}y'v_{rz}dm.$$

Here v_{rx}, v_{ry}, v_{rz} and x', y', z' – respectively the projections of relative velocity $\vec{v}_r$ and radius vector $\vec{r}'$ of particle to body axes.

If we consider that the flow of gases and liquids inside the rocket body possesses axial symmetry,

$$\int_S v_{ry} dm = 0, \int_S v_{rz} dm = 0, \int_S y' v_{rx} dm = 0, \int_S z' v_{rx} dm = 0$$

and then

$$F_{\kappa x} = 0,\ F_{\kappa y} = F_{\kappa y}^{\omega_z} \omega_z,\ F_{\kappa z} = F_{\kappa z}^{\omega_y} \omega_y,$$

$$M_{\kappa x} = M_{\kappa x}^{\omega_x} \omega_x,\ M_{\kappa y} = M_{\kappa y}^{\omega_y} \omega_y,\ M_{\kappa z} = M_{\kappa z}^{\omega_z} \omega_z, \tag{1.9.27}$$

where

$$F_{\kappa y}^{\omega_z} = -2 \int_S v_{rx} dm,\ F_{\kappa z}^{\omega_y} = 2 \int_S v_{rx} dm,$$

$$M_{\kappa x}^{\omega_x} = -2 \int_S (y' v_{ry} + z' v_{rz}) dm,\ M_{\kappa y}^{\omega_y} = -2 \int_S (z' v_{rz} + x' v_{rx}) dm,$$

$$M_{\kappa z}^{\omega_z} = -2 \int_S (x' v_{rx} + y' v_{ry}) dm.$$

More detailed research on the question shows that $M_{\kappa x}^{\omega_x}$ is negligible and it is possible to take

$$M_{\kappa x}^{\omega_x} = 0,$$

and coefficients $M_{\kappa y}^{\omega_y}$ and $M_{\kappa z}^{\omega_z}$ are basically determined by the value of integral

$$\int_S x' v_{rx} dm.$$

Coriolis moments can be both buildup, and damping. For particles, which move to center of mass $x' v_{rx} < 0$, and, thus, during motion of these particles there appear buildup moments. For particles, moving from the center of mass $x' v_{rx} > 0$, and, thus, corresponding moments are directed opposite the rotation of the rocket, i.e., are damping. On the whole with location of engines behind the center of mass the damping moments prevail over buildup, and then

$$M_{x_y}^{\omega_y},\ M_{x_z}^{\omega_z} < 0.$$

9.6. Forces, Caused by Displacement of Center of Mass of the Rocket Relative to its Body

Forces caused by displacement of center of mass of the rocket relative to its body are determined by the sum of (see § 5):

$$\vec{F}_r = m\vec{w}_{Cr} + 2m\vec{\omega}\times\vec{V}_{Cr}.$$

The first component is negligible. By rejecting this component, we will approximately have

$$\vec{F}_r = 2m\vec{\omega}\times\vec{V}_{Cr}. \qquad (1.9.28)$$

Let us project equality (1.9.28) to axes Cx, Cy, Cz, we obtain:

$$\begin{aligned} F_{rx} &= (2m\vec{\omega}\times\vec{V}_{Cr})_x = 2m(\omega_y V_{Crz} - \omega_z V_{Cry}),\\ F_{ry} &= (2m\vec{\omega}\times\vec{V}_{Cr})_y = 2m(\omega_z V_{Crx} - \omega_x V_{Crz}),\\ F_{rz} &= (2m\vec{\omega}\times\vec{V}_{Cr})_z = 2m(\omega_x V_{Cry} - \omega_y V_{Crx}), \end{aligned} \qquad (1.9.29)$$

where V_{Crx}, V_{Cry}, V_{Crz} – projections of relative velocity of center of mass to the axes Cx, Cy, Cz.

Considering that the flow of gases and liquids in the rocket body is axially symmetric, we will have

$$F_{rx} = 0,\ F_{ry} = 2mV_{Crx}\omega_z,\ F_{rz} = -2mV_{Crx}\omega_y. \qquad (1.9.30)$$

§10. Dynamic Equations of the Rocket in Expanded Form

Equations of motion of the center of mass of the rocket and equations of rotation around the center of mass together represent a system of dynamic equations of three-dimensional motion of the rocket.

Dynamic equations of the rocket in projections to axes of the earth-fixed system of coordinates, which coincide about principal axes of inertia of the rocket, revolving in space together with the body at identical angular velocity, are represented in the following form:

$$
\begin{aligned}
&m\left(\frac{dV_{Cx}}{dt}+\omega_y V_{Cz}-\omega_z V_{Cy}\right)=G_x+P_x+R_{ax}+F_{\text{к}x}+F_{rx}+F_{\text{упр}\,x},\\
&m\left(\frac{dV_{Cy}}{dt}+\omega_z V_{Cx}-\omega_x V_{Cz}\right)=G_y+P_y+R_{ay}+F_{\text{к}y}+F_{ry}+F_{\text{упр}\,y},\\
&m\left(\frac{dV_{Cz}}{dt}+\omega_x V_{Cy}-\omega_y V_{Cx}\right)=G_z+P_z+R_{az}+F_{\text{к}z}+F_{rz}+F_{\text{упр}\,z},\\
&J_x\frac{d\omega_x}{dt}+(J_z-J_y)\omega_y\omega_z=M_{px}+M_{ax}+M_{\text{к}x}+M_{\text{упр}\,x},\\
&J_y\frac{d\omega_y}{dt}+(J_x-J_z)\omega_z\omega_x=M_{py}+M_{ay}+M_{\text{к}y}+M_{\text{упр}\,y},\\
&J_z\frac{d\omega_z}{dt}+(J_y-J_x)\omega_x\omega_y=M_{pz}+M_{az}+M_{\text{к}z}+M_{\text{упр}\,z}.
\end{aligned}
\qquad (1.10.1)
$$

In the right sides of equations (1.10.1) let us substitute approximate expressions of forces and moments, provided in § 9. We obtain:

$$
\begin{aligned}
&m\left(\frac{dV_{Cx}}{dt}+\omega_y V_{Cz}-\omega_z V_{Cy}\right)=mg_x+P-\\
&\qquad\frac{1}{2}c_{x0}\varrho S(V_{Cx}-V_{\text{в}x})^2+c_{x\delta}\delta_x,\\
&m\left(\frac{dV_{Cy}}{dt}+\omega_z V_{Cx}-\omega_x V_{Cz}\right)=mg_y+\sum_j \eta_j P_j-\\
&\qquad\frac{1}{2}(c_{x0}+c_y^{\alpha})\varrho S(V_{Cx}-V_{\text{в}x})(V_{Cy}-V_{\text{в}y})-\\
&\qquad-\nu_y\omega_z+2mV_{Crx}\omega_z+c_{y\delta}\delta_y,
\end{aligned}
\qquad (1.10.2)
$$

$$m\left(\frac{dV_{Cz}}{dt}+\omega_x V_{Cy}-\omega_y V_{Cx}\right)=mg_z+\sum_j \zeta_j P_j- \quad \text{(Cont'd)}$$

$$\frac{1}{2}(c_{x0}+c_y^{\alpha})\rho S(V_{Cx}-V_{вx})(V_{Cz}-V_{вz})- \quad (1.10.2)$$

$$-\nu_z\omega_y-2mV_{Crx}\omega_y+c_{\psi\delta}\delta_{\psi},$$

where

$$\nu_y=-R^{\omega_z}_{ay}-F^{\omega_z}_{\kappa y},\quad \nu_z=-R^{\omega_y}_{az}-F^{\omega_y}_{\kappa z}$$

and

$$J_x\frac{d\omega_x}{dt}+(J_z-J_y)\omega_y\omega_z=-\mu_x\omega_x+\sum_j(y_j\zeta_j-z_j\eta_j)P_j+c_{\gamma\delta}\delta_{\gamma},$$

$$J_y\frac{d\omega_y}{dt}+(J_x-J_z)\omega_z\omega_x=-\mu_y\omega_y+\sum_j(z_j-x_j\zeta_j)P_j+$$

$$+\frac{1}{2}x_F(c_{x0}+c_y^{z})\rho S(V_{Cx}-V_{вx})(V_{Cz}-V_{вz})+c_{\psi\delta}\delta_{\psi}, \quad (1.10.3)$$

$$J_z\frac{d\omega_z}{dt}+(J_y-J_x)\omega_x\omega_y=-\mu_z\omega_z+\sum_j(x_j\eta_j-y_j)P_j-$$

$$-\frac{1}{2}x_F(c_{x0}+c_y)\rho S(V_{Cx}-V_{вx})(V_{Cy}-V_{вy})+c_{\vartheta\delta}\delta_{\vartheta},$$

where

$$\mu_x=-M^{\omega_x}_{ax}-M^{\omega_x}_{\kappa x},\quad \mu_y=-M^{\omega_y}_{ay}-M^{\omega_y}_{\kappa y},\qquad \mu_z=-M^{\omega_z}_{az}-M^{\omega_z}_{\kappa z}.$$

§ 11. Undisturbed and Disturbed Motions

Despite simplifications, allowed during formulation of dynamic equations of the rocket, these equations, being nonlinear and with variable coefficients, nevertheless remain quite complex. Therefore in practice, depending on the character of the problem being solved, dynamic equations undergo additional simplifications.

Thus, when determining trajectories of the center of mass we consider that parameters of the rocket and control systems have nominal values and flights proceeds in the atmosphere, parameters of which exactly correspond to so-called "standard atmosphere." Besides this, often there are disregarded oscillations of the rocket around the center of mass, excluding equations of rotation of the rocket around center of mass from examination, i.e., there are rejected all the factors which either little affect the trajectory of center of mass, or by its nature carry a random character and cannot be taken into account with construction of the trajectory of center of mass. As a result we obtain some new, simplified system of equations, by integration of which we find the trajectory of center of mass corresponding to accepted idealizations.

The trajectory of center of mass obtained in this manner, of course, will differ from real, realizable in a particular situation, but under certain conditions the theoretical and real trajectory will be close to one another.

Motion of the rocket, described by simplified system of equations, at nominal values of parameters of the vehicle and control system, standard values of parameters of the atmosphere and prescribed initial values of parameters of motion is accepted to call *undisturbed motion*, and corresponding trajectory of center of mass – *undisturbed trajectory*. In contrast to this the motion of the rocket, described by initial equations at real values of parameters of the vehicle and control system, parameters of the atmosphere, etc., is called *disturbed motion;* trajectories in disturbed motion are called *disturbed trajectories*.

During investigation of the stability of motion of flight vehicles usually motion equations preliminarily undergo conversions, essence of which is examined in the following example.

Let us suppose that disturbed and undisturbed motions are described respectively by systems of differential equations:

$$\begin{aligned} \frac{dy'_1}{dt} &= f'_1(y'_1, y'_2, \ldots, y'_n;\ z'_1, z'_2, \ldots, z'_m), \\ &\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots \\ \frac{dy'_n}{dt} &= f'_n(y'_1, y'_2, \ldots, y'_n;\ z'_1, z'_2, \ldots, z'_m) \end{aligned} \quad (1.11.1)$$

and

$$\begin{aligned} \frac{dy_1}{dt} &= f_1(y_1, y_2, \ldots, y_n;\ z_1, z_2, \ldots, z_m), \\ &\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots \\ \frac{dy_n}{dt} &= f_n(y_1, y_2, \ldots, y_n;\ z_1, z_2, \ldots, z_m), \end{aligned} \quad (1.11.2)$$

where $y_1, y_2, \ldots, y_n$ — parameters of motion, and $z_1, z_2, \ldots, z_m$ — some constant or variable parameters (aerodynamic, geometric, weight and other parameters of the vehicle, parameters of the atmosphere and so forth), which in undisturbed motion are considered known. Here and subsequently quantities in disturbed motion are noted by primes.

For convenience of discussion further we will apply compact writing of systems of equations. Thus, let us represent systems (1.11.1) and (1.11.2) in the form

$$\frac{dy'_i}{dt} = f'_i(y'_1, \ldots, y'_n;\ z'_1, \ldots, z'_m) \quad (i = 1, \ldots, n)$$

and

$$\frac{dy_i}{dt} = f_i(y_1, \ldots, y_n;\ z_1, \ldots, z_m) \quad (i = 1, \ldots, n).$$

Let us assume we know undisturbed motion of the considered object corresponding to certain prescribed initial conditions, to which corresponds particular solution $y_i^0 = y_i^0(t)$ $(i=1,\dots,n)$ of equations (1.11.2).

With substitution of particular solution $y_i^0(t)$ $(i=1,\dots,n)$ in equations (1.11.2) the latter become identical:

$$\frac{dy_i^0}{dt} \equiv f_i(y_1^0,\dots,y_n^0;\ z_1,\dots,z_m) \quad (i=1,\dots,n). \tag{1.11.3}$$

Let us assume

$$\begin{aligned} y_i' &= y_i^0 + \Delta y_i \quad (i=1,\dots,n), \\ z_j' &= z_j + \Delta z_j \quad (j=1,\dots,m). \end{aligned} \tag{1.11.4}$$

By subtracting identities (1.11.3) from (1.11.1) and considering (1.11.4), we obtain

$$\frac{d\Delta y_i}{dt} = f_i'(y_1^0+\Delta y_1,\dots,y_n^0+\Delta y_n;\ z_1+\Delta z_1,\dots,z_m+\Delta z_m) - \\ - f_i(y_1^0,\dots,y_n^0;\ z_1,\dots,z_m) \quad (i=1,\dots,n).$$

Hence, by expanding functions f'_i into Taylor series in the right sides of the last equalities, we will have

$$\frac{d\Delta y_i}{dt} = \sum_{j=1}^{n} p_{ij}(t)\Delta y_j + \sum_{j=1}^{m} \pi_{ij}(t)\Delta z_j + \Delta f_i(t) + \\ + R_i(t,\Delta y_1,\dots,\Delta y_n;\ \Delta z_1,\dots,\Delta z_m) \quad (i=1,\dots,n), \tag{1.11.5}$$

where

$$p_{ij} = \left.\frac{\partial f_i'}{\partial y_j}\right|_{\Delta y_1,\dots,\Delta y_n,\Delta z_1,\dots,\Delta z_m=0},$$

$$\pi_{ij} = \left.\frac{\partial f_i'}{\partial z_j}\right|_{\Delta y_1,\dots,\Delta y_n,\Delta z_1,\dots,\Delta z_m=0},$$

$$\Delta f_i = f_i'(y_1^0,\dots,y_n^0;\ z_1,\dots,z_m) - f_i(y_1^0,\dots,y_n^0;\ z_1,\dots,z_m).$$

and R_i – totality of terms, having order higher than the first relatively to disturbances $\Delta y_1, \ldots, \Delta y_n, \Delta z_1, \ldots, \Delta z_m$.

In converted equations of disturbed motion (1.11.5) the sought functions are disturbances Δy_i $(i=1, \ldots, n)$. To each motion of the given mechanical system corresponds a certain particular solution of equations (1.11.5).

In problems of stability of motion there are usually considered solutions of system (1.11.5) at small initial values of disturbances. In this case it is natural to expect that the character of solutions is basically determined by the linear part of equations (1.11.5). Actually, frequently it turns out to be sufficient to consider linear system

$$\frac{d\Delta y_i}{dt}=\sum_{j=1}^{n} p_{ij}(t)\Delta y_j+\sum_{j=1}^{m} \pi_{ij}(t)\Delta z_j+\Delta f_i(t) \quad (i=1,\ldots,n), \tag{1.11.6}$$

so-called *system of equations of disturbed motion of the first approximation*. Disregarding the nonlinear terms considerably simplifies investigation of the equations.

§12. Linearization of Equations of Disturbed Motion

12.1. Basic Prerequisites

With derivation of differential equations of disturbed motion of the rocket as undisturbed motion we consider it programmed motion, being determined in the course of ballistic calculation, when it is considered that parameters of the rocket and control system have nominal values, and parameters of the atmosphere correspond to SA (standard atmosphere).

All the factors, which are not considered in the course of ballistic calculation and cause deviation of real motion from programmed, are considered as disturbances. Disturbances can be broken down into the following groups.

1. *Weight disturbances.* In the course of ballistic calculation it is considered that the mass of rocket and its moments of inertia are changed with the passage of time in a definite known manner. True values of rocket mass and its moments of inertia for different reasons differ from calculated values of these parameters. Let us assume $m'(t)$, $J'_x(t)$, $J'_y(t)$, $J'_z(t)$ — true values of mass and moments of inertia, and $m(t)$, $J_x(t)$, $J_y(t)$, $J_z(t)$ — programmed values. Then the disturbances of these quantities are equal to:

$$\Delta m(t)=m'(t)-m(t), \quad \Delta J_x(t)=J'_x(t)-J_x(t),$$
$$\Delta J_y(t)=J'_y(t)-J_y(t), \quad \Delta J_z(t)=J'_z(t)-J_z(t).$$

2. *Aerodynamic disturbances.* The basic aerodynamic disturbing factor is wind. Of vital importance is also the position of aerodynamic focal point. The computed value of coordinate of focal point $x_F(t)$, because of inaccuracies in determining the position of center of mass and position of focal point of the rocket differ from true value $x'_F(t)$ by

$$\Delta x_F(t)=x'_F(t)-x_F(t).$$

Divergence of aerodynamic coefficients c_x, c_y^a from article to article has no substantial value. However, these coefficients are determined with considerable errors. Disturbances of aerodynamic coefficients are equal to:

$$\Delta c_x=c'_x-c_x, \quad \Delta c_y^a=c_y^{a\prime}-c_y^a.$$

The true value of air density differs from the calculated value, being taken according to SA, by

$$\Delta\rho = \rho' - \rho.$$

Deviations of the area of midsection S also lead to errors in calculations of aerodynamic forces. However, these deviations are insignificant during manufacture of the article, and they can be disregarded.

3. *Disturbances connected with engine thrust.* When determining programmed motion it is considered that engine thrust is directed along the longitudinal axis of the rocket and that the center of mass lies on this axis. Therefore, components of engine thrust on lateral axes Cy, Cz, and also the moment relative to center of mass of the rocket are equal to zero. In actuality, because of distinctive features of the layout and various technological errors there take place misalignment and eccentricity of thrust relative to the center of mass of the rocket. Therefore, components of thrust along lateral axes and the moment of thrust relative to the center of mass, generally speaking, are not equal to zero. Furthermore, the true value of thrust P'_j will differ from calculated P_j by certain quantity

$$\Delta P_j(t) = P'_j(t) - P_j(t).$$

4. *Disturbances of basic parameters of motion.* Basic parameters of motion include angles of pitch ϑ, yaw ψ, roll γ, velocity of center of mass $\vec{V}_C$. The disturbed values of these quantities, as above, we noted by a prime. Disturbances of basic parameters are equal to:

$$\Delta\vartheta(t) = \vartheta'(t) - \vartheta(t), \quad \Delta\psi(t) = \psi'(t) - \psi(t),$$

$$\Delta\gamma(t) = \gamma'(t) - \gamma(t), \quad \Delta\vec{V}_C(t) = \vec{V}'_C(t) - \vec{V}_C(t).$$

Let us assume at moment of time t according to ballistic calculation the earth-fixed system of coordinates occupies a certain position in space. The true position of these axes will be different (Fig. 1.17). Projections of vector $\Delta\vec{V}_C$ to undisturbed axes of earth-fixed system are designated so:

$$(\Delta\vec{V}_C)_x=\Delta V_x,\quad (\Delta\vec{V}_C)_y=\Delta V_y,\quad (\Delta\vec{V}_C)_z=\Delta V_z.$$

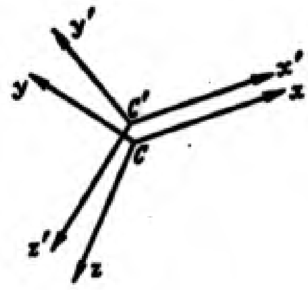

Fig. 1.17.

5. *Disturbances of coefficients* $\nu_y, \nu_z, \mu_x, \mu_y, \mu_z, c_{x\delta}, c_{\vartheta\delta}, c_{\psi\delta}, c_{\gamma\delta}$. These coefficients depend on the geometric parameters of the rocket, relative particle motion inside the body, motion parameters of the rocket. Disturbances of the enumerated coefficients are designated by $\Delta\nu_y, \Delta\nu_z, \Delta\mu_x, \ldots$ These disturbances are determined by errors (disturbances) of corresponding geometric, kinematic and other parameters.

With derivation of differential equations of disturbed motion of the rocket let us agree to consider as small:

– disturbances of basic motion parameters $\Delta\vartheta, \Delta\psi, \Delta\gamma, \Delta V_x, \Delta V_y, \Delta V_z$ and their time derivatives;

– disturbances of mass Δm and moments of inertia $\Delta J_x, \Delta J_y, \Delta J_z$;

– wind velocity V_B, disturbance of air density $\Delta\varrho$, and also disturbances of aerodynamic coefficients $\Delta c_{x0}, \Delta c_y^\alpha$;

– disturbances connected with engine operation, namely: ΔP_j, projections of engine thrust to axes Cy and Cz and thrust relative to center of mass;

– time derivatives of angles of pitch, yaw, roll;

– angle of attack α and its time derivative;

– deflections of controls;

– disturbances of coefficients $\Delta v_y, \Delta v_z, \ldots, \Delta c_{\gamma z}$.

From smallness α and $\frac{d\alpha}{dt}$ follows smallness V_{Cy}, V_{Cz}, $\frac{dV_{Cy}}{dt}$, $\frac{dV_{Cz}}{dt}$. This is easy to see from equalities

$$V_{Cy}=-V_C \sin\alpha\cos\chi, \quad V_{Cz}=-V_C \sin\alpha\sin\chi,$$

cited in § 9.

Usually in programmed motion the angles of yaw and roll are equal to zero. We will also consider that in undisturbed motion

$$\psi, \gamma=0. \tag{1.12.1}$$

Finally, in view of condition (1.12.1), projections of acceleration of the earth's attraction $\vec{g}$ to undisturbed axis Cz at moment of launch is equal to zero, and generally – is a small quantity.

12.2. Auxiliary Relationships

Let us prepare some auxiliary relationships, which determine the connections between quantities in undisturbed and disturbed motions with accuracy to terms of the second order of smallness.

By using equalities (1.8.2), we will have:

$$\omega'_{x'} = \frac{d(\gamma + \Delta\gamma)}{dt} - \frac{d(\vartheta + \Delta\vartheta)}{dt} \sin(\psi + \Delta\psi),$$

$$\omega'_{y'} = \frac{d(\psi + \Delta\psi)}{dt} \cos(\gamma + \Delta\gamma) + \frac{d(\vartheta + \Delta\vartheta)}{dt} \cos(\psi + \Delta\psi) \sin(\gamma + \Delta\gamma),$$

$$\omega'_{z'} = \frac{d(\vartheta + \Delta\vartheta)}{dt} \cos(\psi + \Delta\psi) \cos(\gamma + \Delta\gamma) - \frac{d(\psi + \Delta\psi)}{dt} \sin(\gamma + \Delta\gamma),$$

or, considering (1.12.1),

$$\omega'_{x'} = \frac{d\Delta\gamma}{dt} - \frac{d(\vartheta + \Delta\vartheta)}{dt} \sin\Delta\psi,$$

$$\omega'_{y'} = \frac{d\Delta\psi}{dt} \cos\Delta\gamma + \frac{d(\vartheta + \Delta\vartheta)}{dt} \cos\Delta\psi \sin\Delta\gamma,$$

$$\omega'_{z'} = \frac{d(\vartheta + \Delta\vartheta)}{dt} \cos\Delta\psi \cos\Delta\gamma - \frac{d\Delta\psi}{dt} \sin\Delta\gamma.$$

Hence, with accuracy to quantities of the second order of smallness

$$\begin{aligned} \omega'_{x'} &= \frac{d\Delta\gamma}{dt}, \\ \omega'_{y'} &= \frac{d\Delta\psi}{dt}, \\ \omega'_{z'} &= \frac{d\vartheta}{dt} + \frac{d\Delta\vartheta}{dt}. \end{aligned} \tag{1.12.2}$$

By differentiating equalities (1.12.2), we obtain:

$$\begin{aligned} \frac{d\omega'_{x'}}{dt} &= \frac{d^2\Delta\gamma}{dt^2}, \\ \frac{d\omega'_{y'}}{dt} &= \frac{d^2\Delta\psi}{dt^2}, \\ \frac{d\omega'_{z'}}{dt} &= \frac{d^2\vartheta}{dt^2} + \frac{d^2\Delta\vartheta}{dt^2}. \end{aligned} \tag{1.12.3}$$

From equalities (1.12.2) and (1.12.3) it is evident, incidently, that projections of angular velocity to body axes and their derivatives are small quantities.

Between projections of arbitrary vector $\vec{a}$ to axes of two different systems of coordinates there are the following relationships:

$$
\begin{aligned}
a_{x'} &= a_x \cos(x',x) + a_y \cos(x',y) + a_z \cos(x',z),\\
a_{y'} &= a_x \cos(y',x) + a_y \cos(y',y) + a_z \cos(y',z),\\
a_{z'} &= a_x \cos(z',x) + a_y \cos(z',y) + a_z \cos(z',z).
\end{aligned}
\tag{1.12.4}
$$

As we see, for transition from one system of axes to another it is sufficient to know the matrix (table) of transition

$$
M = \begin{Vmatrix} \cos(x',x) & \cos(x',y) & \cos(x',z) \\ \cos(y',x) & \cos(y',y) & \cos(y',z) \\ \cos(z',x) & \cos(z',y) & \cos(z',z) \end{Vmatrix}.
$$

Values of cosines of angles between axes can be determined as scalar products of unit vectors, directed along corresponding axes.

Let us assume $\vec{i}'$, $\vec{j}'$, $\vec{k}'$ – unit direction vectors of axes $C'x'$, $C'y'$, $C'z'$, and $\vec{i}$, $\vec{j}$, $\vec{k}$ – unit direction vectors of axes Cx, Cy, Cz. Then

$$
M = \begin{Vmatrix} \vec{i}'\cdot\vec{i} & \vec{i}'\cdot\vec{j} & \vec{i}'\cdot\vec{k} \\ \vec{j}'\cdot\vec{i} & \vec{j}'\cdot\vec{j} & \vec{j}'\cdot\vec{k} \\ \vec{k}'\cdot\vec{i} & \vec{k}'\cdot\vec{j} & \vec{k}'\cdot\vec{k} \end{Vmatrix}.
$$

As is known, the scalar product of two vectors is equivalent to the sum of products of their corresponding components. For example,

$$
\vec{i}'\cdot\vec{i} = i_{x_0} i_{x_0} + i_{y_0} i_{y_0} + i_{z_0} i_{z_0}.
$$

By compiling scalar products, using Table 1.1 for this, it is easy to compute matrix elements M:

$$\vec{i}'\cdot\vec{i}=\cos\Delta\psi\cos(\vartheta+\Delta\vartheta)\cos\vartheta+\cos\Delta\psi\sin(\vartheta+\Delta\vartheta)\sin\vartheta=$$
$$=\cos\Delta\psi\cos\Delta\vartheta,$$
$$\vec{i}'\cdot\vec{j}=-\cos\Delta\psi\cos(\vartheta+\Delta\vartheta)\sin\vartheta+\cos\Delta\psi\sin(\vartheta+\Delta\vartheta)\cos\vartheta=$$
$$=\cos\Delta\psi\sin\Delta\vartheta,$$
$$\vec{i}'\cdot\vec{k}=-\sin\Delta\psi,$$
$$\vec{j}'\cdot\vec{i}=-\cos\Delta\gamma\sin(\vartheta+\Delta\vartheta)\cos\vartheta+\sin\Delta\gamma\sin\Delta\psi\cos(\vartheta+\Delta\vartheta)\cos\vartheta+$$
$$+\cos\Delta\gamma\cos(\vartheta+\Delta\vartheta)\sin\vartheta+\sin\Delta\gamma\sin\Delta\psi\sin(\vartheta+\Delta\vartheta)\sin\vartheta=$$
$$=\sin\Delta\gamma\sin\Delta\psi\cos\Delta\vartheta-\cos\Delta\gamma\sin\Delta\vartheta,$$
$$\vec{j}'\cdot\vec{j}=\cos\Delta\gamma\sin(\vartheta+\Delta\vartheta)\sin\vartheta-\sin\Delta\gamma\sin\Delta\psi\cos(\vartheta+\Delta\vartheta)\sin\vartheta+$$
$$+\cos\Delta\gamma\cos(\vartheta+\Delta\vartheta)\cos\vartheta+\sin\Delta\gamma\sin\Delta\psi\sin(\vartheta+\Delta\vartheta)\cos\vartheta=$$
$$=\cos\Delta\gamma\cos\Delta\vartheta+\sin\Delta\gamma\sin\Delta\psi\sin\Delta\vartheta,$$
$$\vec{j}'\cdot\vec{k}=\sin\Delta\gamma\cos\Delta\psi,$$
$$\vec{k}'\cdot\vec{i}=\cos\Delta\gamma\sin\Delta\psi\cos(\vartheta+\Delta\vartheta)\cos\vartheta+\sin\Delta\gamma\sin(\vartheta+\Delta\vartheta)\cos\vartheta-$$
$$-\sin\Delta\gamma\cos(\vartheta+\Delta\vartheta)\sin\vartheta+\cos\Delta\gamma\sin\Delta\psi\sin(\vartheta+\Delta\vartheta)\sin\vartheta=$$
$$=\sin\Delta\gamma\sin\Delta\vartheta+\cos\Delta\gamma\sin\Delta\psi\cos\Delta\vartheta,$$
$$\vec{k}'\cdot\vec{j}=-\cos\Delta\gamma\sin\Delta\psi\cos(\vartheta+\Delta\vartheta)\sin\vartheta-\sin\Delta\gamma\sin(\vartheta+\Delta\vartheta)\sin\vartheta-$$
$$-\sin\Delta\gamma\cos(\vartheta+\Delta\vartheta)\cos\vartheta+\cos\Delta\gamma\sin\Delta\psi\sin(\vartheta+\Delta\vartheta)\cos\vartheta=$$
$$=-\sin\Delta\gamma\cos\Delta\vartheta+\cos\Delta\gamma\sin\Delta\psi\sin\Delta\vartheta,$$
$$\vec{k}'\cdot\vec{k}=\cos\Delta\gamma\cos\Delta\psi.$$

Hence, with accuracy to quantities of the second order of smallness

$$\begin{array}{lll}\vec{i}'\cdot\vec{i}=1, & \vec{i}'\cdot\vec{j}=\Delta\vartheta, & \vec{i}'\cdot\vec{k}=-\Delta\psi,\\ \vec{j}'\cdot\vec{i}=-\Delta\vartheta, & \vec{j}'\cdot\vec{j}=1, & \vec{j}'\cdot\vec{k}=\Delta\gamma,\\ \vec{k}'\cdot\vec{i}=\Delta\psi & \vec{k}'\cdot\vec{j}=-\Delta\gamma, & \vec{k}'\cdot\vec{k}=1.\end{array}$$

This means, with this accuracy

$$M=\begin{pmatrix}1 & \Delta\vartheta & -\Delta\psi\\ -\Delta\vartheta & 1 & \Delta\gamma\\ \Delta\psi & -\Delta\gamma & 1\end{pmatrix}. \qquad (1.12.5)$$

On the basis of equality (1.12.4), using expression (1.12.5), we obtain:

$$\begin{aligned} V'_{Cx'} &= V'_{Cx} + V'_{Cy}\Delta\vartheta - V'_{Cz}\Delta\varphi, \\ V'_{Cy'} &= -V'_{Cx}\Delta\vartheta + V'_{Cy} + V'_{Cz}\Delta\gamma, \\ V'_{Cz'} &= V'_{Cx}\Delta\varphi - V'_{Cy}\Delta\gamma + V'_{Cz}. \end{aligned}$$

Taking into account that

$$V'_{Cx} = V_{Cx} + \Delta V_x, \quad V'_{Cy} = V_{Cy} + \Delta V_y, \quad V'_{Cz} = V_{Cz} + \Delta V_z$$

and retaining quantities of order of smallness not higher than the first, we will have:

$$\begin{aligned} V'_{Cx'} &= V_{Cx} + \Delta V_x, \\ V'_{Cy'} &= -V_{Cx}\Delta\vartheta + V_{Cy} + \Delta V_y, \\ V'_{Cz'} &= V_{Cx}\Delta\varphi + V_{Cz} + \Delta V_z. \end{aligned} \tag{1.12.6}$$

From equalities (1.12.6) by differentiation we find:

$$\begin{aligned} \frac{dV'_{Cx'}}{dt} &= \frac{dV_{Cx}}{dt} + \frac{d\Delta V_x}{dt}, \\ \frac{dV'_{Cy'}}{dt} &= -\frac{dV_{Cx}}{dt}\Delta\vartheta - V_{Cx}\frac{d\Delta\vartheta}{dt} + \frac{dV_{Cy}}{dt} + \frac{d\Delta V_y}{dt}, \\ \frac{dV'_{Cz'}}{dt} &= \frac{dV_{Cx}}{dt}\Delta\varphi + V_{Cx}\frac{d\Delta\varphi}{dt} + \frac{dV_{Cz}}{dt} + \frac{d\Delta V_z}{dt}. \end{aligned} \tag{1.12.7}$$

For projections of acceleration of the earth's attraction and wind velocity we obtain the following relationships:

$$\begin{aligned} g_{x'} &= g_x + g_y\Delta\vartheta - g_z\Delta\varphi, \\ g_{y'} &= -g_x\Delta\vartheta + g_y + g_z\Delta\gamma, \\ g_{z'} &= g_x\Delta\varphi - g_y\Delta\gamma + g_z; \\ V_{\text{в}x'} &= V_{\text{в}x} + V_{\text{в}y}\Delta\vartheta - V_{\text{в}z}\Delta\varphi, \\ V_{\text{в}y'} &= -V_{\text{в}x}\Delta\vartheta + V_{\text{в}y} + V_{\text{в}z}\Delta\gamma, \\ V_{\text{в}z'} &= V_{\text{в}x}\Delta\varphi - V_{\text{в}y}\Delta\gamma + V_{\text{в}z}. \end{aligned}$$

Having discarded terms of the second order of smallness, we find

$$g_{x'}=g_x+g_y\Delta\vartheta,\ g_{y'}=-g_x\Delta\vartheta+g_y,\ g_{z'}=g_x\Delta\psi-g_y\Delta\gamma+g_z, \tag{1.12.8}$$

$$V_{wx'}=V_{wx},\ V_{wy'}=V_{wy},\ V_{wz'}=V_{wz}. \tag{1.12.9}$$

12.3. Linearization of Equations of Disturbed Motion

Let us simplify system of equations (1.10.2) by linearization. Considering these equations as equations of real motion, let us note the quantities entering them by primes, in order to distinguish them from corresponding quantities in undisturbed motion. As indicated above, we will consider programmed motion as undisturbed motion of the rocket.

First equation of forces. We have

$$m'\left(\frac{dV'_{Cx'}}{dt}+\omega'_{y'}V'_{Cz'}-\omega'_{z'}V'_{Cy'}\right)=m'g_{x'}+$$
$$+P'-\frac{1}{2}c'_{x0}\varrho'S\left(V'_{Cx'}-V_{wx'}\right)^2+c'_{x\delta}\delta'_{x'}. \tag{1.12.10}$$

Corresponding equation in programmed motion has the form

$$m\left(\frac{dV_{Cx}}{dt}-\omega_zV_{Cy}\right)=mg_x+P-\frac{1}{2}c_{x0}\varrho SV^2_{Cx}+c_{x\delta}\delta_x. \tag{1.12.11}$$

It is possible to immediately simplify equations (1.12.10) and (1.12.11), having discarded terms of the second order of smallness:

$$m'\frac{dV'_{Cx'}}{dt}=m'g_{x'}+P'-\frac{1}{2}c'_{x0}\varrho'S\left(V'^{2}_{Cx'}-2V'_{Cx'}V_{wx'}\right)+c'_{x\delta}\delta'_{x'}, \tag{1.12.12}$$

$$m\frac{dV_{Cx}}{dt}=mg_x+P-\frac{1}{2}c_{x0}\varrho SV^2_{Cx}+c_{x\delta}\delta_x. \tag{1.12.13}$$

By using the relationships obtain in 12.2, let us write (1.12.2) in the form

$$(m+\Delta m)\left(\frac{dV_{Cx}}{dt}+\frac{d\Delta V_x}{dt}\right)=(m+\Delta m)(g_x+g_y\Delta\vartheta)+P+\Delta P-$$
$$-\frac{1}{2}(c_{x0}+\Delta c_{x0})(\varrho+\Delta\varrho)S\,(V_{Cx}^2+2V_{Cx}\Delta V_x+\Delta V^2-2V_{Cx}V_{wx}-$$
$$-2\Delta V_xV_{wx})+(c_{x\delta}+\Delta c_{x\delta})(\delta_x+\Delta\delta_x).$$

Again let us discard terms of the second and higher order of smallness. We will have

$$m\frac{dV_{Cx}}{dt}+m\frac{d\Delta V_x}{dt}+\Delta m\frac{dV_{Cx}}{dt}=mg_x+mg_y\Delta\vartheta+\Delta mg_x+P+\Delta P-$$
$$-\frac{1}{2}c_{x0}\varrho SV_{Cx}^2-c_{x0}\varrho SV_{Cx}\Delta V_x+c_{x0}\varrho SV_{Cx}V_{wx}-\frac{1}{2}c_{x0}\Delta\varrho SV_{Cx}^2-$$
$$-\frac{1}{2}\Delta c_{x0}\varrho SV_{Cx}^2+c_{x\delta}(\delta_x+\Delta\delta_x).$$

Let us exclude derivative dV_{Cx}/dt with the aid of equation (1.12.13). We obtain

$$m\frac{d\Delta V_x}{dt}+\frac{\Delta m}{m}\left(P-\frac{1}{2}c_{x0}\varrho SV_{Cx}^2\right)=mg_y\Delta\vartheta+\Delta P-c_{x0}\varrho SV_{Cx}\Delta V_x+$$
$$+c_{x0}\varrho SV_{Cx}V_{wx}-\frac{1}{2}c_{x0}SV_{Cx}^2\Delta\varrho-\frac{1}{2}\varrho SV_{Cx}^2\Delta c_{x0}+c_{x\delta}\Delta\delta_x,$$

or finally

$$m\frac{d\Delta V_x}{dt}+c_{x0}\varrho SV_{Cx}\Delta V_x-mg_y\Delta\vartheta=\Delta F_x+c_{x\delta}\Delta\delta_x, \qquad (1.12.14)$$

where $\Delta F_x=-\frac{\Delta m}{m}\left(P-\frac{1}{2}c_{x0}\varrho SV_{Cx}^2\right)+\Delta P+c_{x0}\varrho SV_{Cx}V_{wx}-\frac{1}{2}c_{x0}SV_{Cx}^2\Delta\varrho-\frac{1}{2}\varrho SV_{Cx}^2\Delta c_{x0}$

– axial disturbing force. In the same way let us linearize the remaining equations of system (1.10.2). While not stipulating separately, when writing these or other relationships in case of necessity we will reject terms of the second and higher order of smallness.

The second equation of forces in real and programmed motions with accuracy to terms of the second order of smallness can be accordingly written so:

$$m'\left(\frac{dV'_{cy'}}{dt}+\omega'_{z'}V'_{cx'}\right)=m'g_{y'}+\sum_j \eta_j P'_j-\frac{1}{2}(c'_{x0}+$$

$$+c^{\alpha'}_{y})\varrho' S V'_{cx'}V'_{cy'}+\frac{1}{2}(c'_{x0}+c^{\alpha'}_{y})\varrho' S V'_{cx'}V_{wy'}-\nu'_{y}\omega'_{z'}+c'_{y\delta}\delta'_{\varphi}, \qquad (1.12.15)$$

$$m\left(\frac{dV_{cy}}{dt}+V_{cx}\frac{d\vartheta}{dt}\right)=mg_y-\frac{1}{2}(c_{x0}+c^{\alpha}_{y})\varrho S V_{cx}V_{cy}$$

$$-\nu_y\frac{d\vartheta}{dt}+c_{y\delta}\delta_{\varphi}. \qquad (1.12.16)$$

From (1.12.15) we have

$$(m+\Delta m)\left[\frac{dV_{cy}}{dt}+\frac{d\Delta V_y}{dt}-\frac{dV_{cx}}{dt}\Delta\vartheta-V_{cx}\frac{d\Delta\vartheta}{dt}+\left(\frac{d\vartheta}{dt}+\frac{d\Delta\vartheta}{dt}\right)\times\right.$$

$$\left.\times(V_{cx}+\Delta V_x)\right]=(m+\Delta m)(g_y-g_x\Delta\vartheta)+\sum_j \eta_j P_j+\sum_j \eta_j\Delta P_j-$$

$$-\frac{1}{2}(c_{x0}+\Delta c_x+c^{\alpha}_{y}+\Delta c^{\alpha}_{y})(\varrho+\Delta\varrho)S(V_{cx}+\Delta V_x)(V_{cy}+\Delta V_y-$$

$$-V_{cx}\Delta\vartheta)+\frac{1}{2}(c_{x0}+\Delta c_x+c^{\alpha}_{y}+\Delta c^{\alpha}_{y})(\varrho+\Delta\varrho)S(V_{cx}+\Delta V_x)V_{wy}-$$

$$-(\nu_y+\Delta\nu_y)\left(\frac{d\vartheta}{dt}+\frac{d\Delta\vartheta}{dt}\right)+(c_{y\delta}+\Delta c_{y\delta})(\delta_{\varphi}+\Delta\delta_{\varphi}).$$

Hence

$$m\left(\frac{dV_{cy}}{dt}+\frac{d\Delta V_y}{dt}-\frac{dV_{cx}}{dt}\Delta\vartheta+V_{cx}\frac{d\vartheta}{dt}\right)=m(g_y-g_x\Delta\vartheta)+$$

$$+\Delta m g_y+\sum_j \eta_j P_j-\frac{1}{2}(c_{x0}+c^{\alpha}_{y})\varrho S V_{cx}(V_{cy}+\Delta V_y-V_{cx}\Delta\vartheta)+$$

$$+\frac{1}{2}(c_{x0}+c^{\alpha}_{y})\varrho S V_{cx}V_{wy}-\nu_y\left(\frac{d\vartheta}{dt}+\frac{d\Delta\vartheta}{dt}\right)+c_{y\delta}(\delta_{\varphi}+\Delta\delta_{\varphi}).$$

Let us exclude derivatives $dV_{cx}/dt, dV_{cy}/dt$ with the aid of equalities (1.12.13) and (1.12.16). Then

$$m\frac{d\Delta V_y}{dt}+v_y\frac{d\Delta\theta}{dt}-\left(P-\frac{1}{2}c_{x0}\varrho SV_{cx}^2\right)\Delta\theta=\Delta mg_y+\sum_j\eta_jP_j-$$
$$-\frac{1}{2}(c_{x0}+c_y^a)\varrho SV_{cx}(\Delta V_y-V_{cx}\Delta\theta)+\frac{1}{2}(c_{x0}+c_y^a)\varrho SV_{cx}V_{wy}+$$
$$+c_{y\delta}\Delta\delta_\theta,$$

or

$$m\frac{d\Delta V_y}{dt}+v_y\frac{d\Delta\theta}{dt}+\frac{1}{2}(c_{x0}+c_y^a)\varrho SV_{cx}\Delta V_y-$$
$$-\left(P+\frac{1}{2}c_y^a\varrho SV_{cx}^2\right)\Delta\theta=\Delta F_y+c_{y\delta}\Delta\delta_\theta, \tag{1.12.17}$$

where $\Delta F_y=\Delta mg_y+\sum_j\eta_jP_j+\frac{1}{2}(c_{x0}+c_y^a)\varrho SV_{cx}V_{wy}$ — disturbing force in the direction of axis Cy.

The third equation of forces in real and programmed motions with accuracy to terms of the second order of smallness can be written so:

$$m'\left(\frac{dV'_{cz'}}{dt}-\omega'_{y'}V'_{cx'}\right)=m'g_{z'}+\sum_j\xi_jP'_j-\frac{1}{2}(c'_{x0}+c_y^{a\prime})\varrho'SV'_{cx'}\times$$
$$\times(V'_{cz'}-V_{wz'})-v'_z\omega'_{y'}+c'_{z\delta}\delta'_\psi, \tag{1.12.18}$$

$$m\frac{dV_{cz}}{dt}=mg_z-\frac{1}{2}(c_{x0}+c_y^a)\varrho SV_{cx}V_{cz}+c_{z\delta}\delta_\psi. \tag{1.12.19}$$

As earlier, from (1.12.18) we obtain

$$(m+\Delta m)\left[\frac{dV_{cx}}{dt}\Delta\psi+V_{cx}\frac{d\Delta\psi}{dt}+\frac{dV_{cz}}{dt}+\frac{d\Delta V_z}{dt}-\frac{d\Delta\psi}{dt}(V_{cx}+\right.$$
$$\left.+\Delta V_x)\right]=(m+\Delta m)(g_x\Delta\psi+g_z-g_y\Delta\gamma)+\sum_j\xi_j(P_j+\Delta P_j)-$$
$$-\frac{1}{2}(c_{x0}+\Delta c_{x0}+c_y^a+\Delta c_y^a)(\varrho+\Delta\varrho)S(V_{cx}+\Delta V_x)(V_{cx}\Delta\psi+V_{cz}+$$
$$+\Delta V_z-V_{wz})-(v_z+\Delta v_z)\frac{d\Delta\psi}{dt}+(c_{z\delta}+\Delta c_{z\delta})(\delta_\psi+\Delta\delta_\psi),$$

or

$$m\left(\frac{dV_{Cx}}{dt}\Delta\psi+\frac{dV_{Cz}}{dt}+\frac{d\Delta V_z}{dt}\right)=m(g_x\Delta\psi+g_z-g_y\Delta\gamma)+\sum_j \xi_j P_j-$$
$$-\frac{1}{2}(c_{x0}+c_y^z)\varrho SV_{Cx}(V_{Cx}\Delta\psi+V_{Cz}+\Delta V_z-V_{wz})-\nu_z\frac{d\Delta\psi}{dt}+$$
$$+c_{z\delta}(\delta_\psi+\Delta\delta_\psi).$$

Let us substitute here values of dV_{Cx}/dt, dV_{Cz}/dt. Then

$$\left(P-\frac{1}{2}c_{x0}\varrho SV_{Cx}^2\right)\Delta\psi+m\frac{d\Delta V_z}{dt}=-mg_y\Delta\gamma+\sum_j \xi_j P_j-$$
$$-\frac{1}{2}(c_{x0}+c_y^z)\varrho SV_{Cx}(V_{Cx}\Delta\psi+\Delta V_z-V_{wz})-\nu_z\frac{d\Delta\psi}{dt}+c_{z\delta}\Delta\delta_\psi.$$

Hence

$$m\frac{d\Delta V_z}{dt}+\nu_z\frac{d\Delta\psi}{dt}+\frac{1}{2}(c_{x0}+c_y^z)\varrho SV_{Cx}\Delta V_z+$$
$$+\left(P+\frac{1}{2}c_y^z\varrho SV_{Cx}^2\right)\Delta\psi+mg_y\Delta\gamma=\Delta F_z+c_{z\delta}\Delta\delta_\psi, \qquad (1.12.20)$$

where $\Delta F_z=\frac{1}{2}(c_{x0}+c_y^z)\varrho SV_{Cx}V_{wz}+\sum_j \xi_j P_j$ — disturbing force in direction of axis Cz.

First equation of moments without terms of the second order of smallness has the form

$$J_x'\frac{d\omega_{x'}'}{dt}=-\mu_x'\omega_{x'}'+\sum_j(y_j\xi_j-z_j\eta_j)P_j'+c_{\gamma\delta}\delta_\gamma'. \qquad (1.12.21)$$

Hence

$$(J_x+\Delta J_x)\frac{d^2\Delta\gamma}{dt^2}=-(\mu_x+\Delta\mu_x)\frac{d\Delta\gamma}{dt}+$$
$$+\sum_j(y_j\xi_j-z_j\eta_j)(P_j+\Delta P_j)+(c_{\gamma\delta}+\Delta c_{\gamma\delta})(\delta_\gamma+\Delta\delta_\gamma).$$

or, considering that in programmed motion $\delta_\gamma = 0$,

$$J_x \frac{d^2\Delta\gamma}{dt^2} + \mu_x \frac{d\Delta\gamma}{dt} = \Delta M_x + c_{\gamma\delta}\Delta\delta_\gamma, \qquad (1.12.22)$$

where $\Delta M_x = \sum_j (y_j\xi_j - z_j\eta_j) P_j$ — moment of disturbing forces in projection to axis Cx.

By converting *the second equation of moments*

$$J'_y \frac{d\omega'_{y'}}{dt} = -\mu'_y \omega'_{y'} + \sum_j (z_j - x_j\xi_j) P'_j +$$
$$+ \frac{1}{2} x'_P (c_{x0} + c_y^{\alpha'}) \varrho' S V'_{Cx'} (V'_{Cx'} - V_{wx'}) + c'_{\psi\delta}\delta'_\psi,$$

we obtain

$$J_y \frac{d^2\Delta\psi}{dt^2} + \mu_y \frac{d\Delta\psi}{dt} = \sum_j (z_j - x_j\xi_j) P_j +$$
$$+ \frac{1}{2} x_P (c_{x0} + c_y^\alpha) \varrho S V_{Cx} (V_{Cx}\Delta\psi + V_{Cz} + \Delta V_z - V_{wz}) + c_{\psi\delta}(\delta_\psi + \Delta\delta_\psi).$$

In programmed motion

$$\frac{1}{2} x_P (c_{x0} + c_y^\alpha) \varrho S V_{Cx} V_{Cz} + c_{\psi\delta}\delta_\psi = 0.$$

Therefore

$$J_y \frac{d^2\Delta\psi}{dt^2} + \mu_y \frac{d\Delta\psi}{dt} - \frac{1}{2} x_P (c_{x0} + c_y^\alpha) \varrho S V_{Cx} (\Delta V_z + V_{Cx}\Delta\psi) =$$
$$= \Delta M_y + c_{\psi\delta}\Delta\delta_\psi, \qquad (1.12.23)$$

where $\Delta M_y = -\frac{1}{2} x_P (c_{x0} + c_y^\alpha) \varrho S V_{Cx} V_{wz} + \sum_j (z_j - x_j\xi_j) P_j$ — disturbing moment in projection to axis Cy.

Finally, *the third equation of moments*

$$J'_z \frac{d\omega'_{z'}}{dt} = -\mu'_z \omega'_{z'} + \sum_j (x_j \eta_j - y_j) P'_j -$$
$$-\frac{1}{2} x'_P (c'_{x0} + c^{\alpha'}_y) \varrho' S V'_{Cx'} (V'_{Cy'} - V_{wy'}) + c'_{\mu\delta} \delta'_\varphi$$

is reduced to form

$$J_z\left(\frac{d^2\vartheta}{dt^2} + \frac{d^2\Delta\vartheta}{dt^2}\right) + \mu_z\left(\frac{d\vartheta}{dt} + \frac{d\Delta\vartheta}{dt}\right) = \sum_j (x_j \eta_j - y_j) P_j - \frac{1}{2} x_P (c_{x0} + c^{\alpha}_y) \times$$
$$\times \varrho S V_{Cx} (-V_{Cx}\Delta\vartheta + V_{Cy} + \Delta V_y - V_{wy}) + c_{\mu\delta}(\delta_\varphi + \Delta\delta_\varphi).$$

In programmed motion

$$-\frac{1}{2} x_P (c_{x0} + c^{\alpha}_y) \varrho S V_{Cx} V_{Cy} + c_{\mu\delta}\delta_\varphi = 0.$$

Considering this, we will have

$$J_z \frac{d^2\Delta\vartheta}{dt^2} + \mu_z \frac{d\Delta\vartheta}{dt} + \frac{1}{2} x_P (c_{x0} + c^{\alpha}_y) \varrho S V_{Cx} (\Delta V_y - V_{Cx}\Delta\vartheta) =$$
$$= \Delta M_z + c_{\mu\delta}\Delta\delta_\varphi, \qquad (1.12.24)$$

where $\Delta M_z = \frac{1}{2} x_P (c_{x0} + c^{\alpha}_y) \varrho S V_{Cx} V_{wy} + \sum_j (x_j \eta_j - y_j) P_j - J_z \frac{d^2\vartheta}{dt^2} - \mu_z \frac{d\vartheta}{dt}$ — disturbing moment in projection to axis Cz.

Thus, the system of linearized equations of disturbed motion has the form

$$m \frac{d\Delta V_x}{dt} + c_{x0} \varrho S V_{Cx} \Delta V_x - m g_y \Delta\vartheta = c_{x\delta}\Delta\delta_x + \Delta F_x,$$
$$m \frac{d\Delta V_y}{dt} + \nu_y \frac{d\Delta\vartheta}{dt} + \frac{1}{2}(c_{x0} + c^{\alpha}_y) \varrho S V_{Cx} \Delta V_y -$$
$$-\left(P + \frac{1}{2} c^{\alpha}_y \varrho S V^2_{Cx}\right)\Delta\vartheta = c_{y\delta}\Delta\delta_\varphi + \Delta F_y, \qquad (1.12.25)$$

$$m\frac{d\Delta V_z}{dt}+v_z\frac{d\Delta\psi}{dt}+\frac{1}{2}(c_{x0}+c_y^a)\varrho SV_{Cx}\Delta V_z+$$

$$+\left(P+\frac{1}{2}c_y^a\varrho SV_{Cx}^2\right)\Delta\psi+mg_y\Delta\gamma=c_{z\delta}\Delta\delta_\psi+\Delta F_z,$$

$$J_x\frac{d^2\Delta\gamma}{dt^2}+\mu_x\frac{d\Delta\gamma}{dt}=c_{\gamma\delta}\Delta\delta_\gamma+\Delta M_x,$$

$$J_y\frac{d^2\Delta\psi}{dt^2}+\mu_y\frac{d\Delta\psi}{dt}-\frac{1}{2}x_F(c_{x0}+c_y^a)\varrho SV_{Cx}\Delta V_z-$$

$$-\frac{1}{2}x_F(c_{x0}+c_y^a)\varrho SV_{Cx}^2\Delta\psi=c_{\psi\delta}\Delta\delta_\psi+\Delta M_y,$$

$$J_z\frac{d^2\Delta\vartheta}{dt^2}+\mu_z\frac{d\Delta\vartheta}{dt}+\frac{1}{2}x_F(c_{x0}+c_y^a)\varrho SV_{Cx}\Delta V_y-$$

$$-\frac{1}{2}x_F(c_{x0}+c_y^a)\varrho SV_{Cx}^2\Delta\vartheta=c_{\vartheta\delta}\Delta\delta_\vartheta+\Delta M_z. \qquad \text{(Cont'd)} \quad (1.12.25)$$

In equations (1.12.25) the unknowns are functions ΔV_x, ΔV_y, ΔV_z, $\Delta\vartheta$, $\Delta\psi$, $\Delta\gamma$, $\Delta\delta_x$, $\Delta\delta_\vartheta$, $\Delta\delta_\psi$, $\Delta\delta_\gamma$. Coefficients with these functions and their derivatives, so-called *dynamic coefficients*, are determined by design and geometric parameters of the rocket, its aerodynamic characteristics and various parameters of programmed motion, which are found in the course of ballistic calculation; during analysis of system (1.12.25) the parameters of undisturbed (programmed) motion are considered as known functions of time. Functions ΔF_x, ΔF_y, ΔF_z, ΔM_x, ΔM_y, ΔM_z depend, furthermore, on various disturbances (weight, aerodynamic, etc.), relationship of which to unknown functions ΔV_x, ΔV_y,... is very slight, therefore, by disregarding the effect of unknown functions, the disturbing forces and moments ΔF_x, ΔF_y, ΔF_z, ΔM_x, ΔM_y, ΔM_z are considered as functions of time, not depending upon ΔV_x, ΔV_y, ΔV_z, $\Delta\vartheta$, $\Delta\psi$, $\Delta\gamma$, $\Delta\delta_x$, $\Delta\delta$, $\Delta\delta_\vartheta$, $\Delta\delta_\vartheta$.

§ 13. Splitting of Linearized Equations of Disturbed Motion

It is easy to see that the system of linearized equations of disturbed motion (1.12.25) is decomposed into two independent groups of equations. One of them describes change of parameters ΔV_x, ΔV_y, $\Delta\vartheta$:

$$m\frac{d\Delta V_x}{dt}+c_{xx}\Delta V_x-mg_y\Delta\vartheta=c_{x\delta}\Delta\delta_x+\Delta F_x,$$

$$m\frac{d\Delta V_y}{dt}+\nu_y\frac{d\Delta\vartheta}{dt}+c_{yy}\Delta V_y+c_{y\vartheta}\Delta\vartheta-c_{y\delta}\Delta\delta_\vartheta+\Delta F_y, \qquad (1.13.1)$$

$$J_z\frac{d^2\Delta\vartheta}{dt^2}+\mu_z\frac{d\Delta\vartheta}{dt}+c_{\vartheta y}\Delta V_y+c_{\vartheta\vartheta}\Delta\vartheta=c_{\vartheta\delta}\Delta\delta_\vartheta+\Delta M_z.$$

Here

$$c_{xx}=c_{x0}\varrho SV_{Cx},\ c_{yy}=\frac{1}{2}(c_{x0}+c_y^\alpha)\varrho SV_{Cx},$$

$$c_{y\vartheta}=-\left(P+\frac{1}{2}c_y^\alpha\varrho SV_{Cx}^2\right),\ c_{\vartheta y}=\frac{1}{2}x_P(c_{x0}+c_y^\alpha)\varrho SV_{Cx}, \qquad (1.13.2)$$

$$c_{\vartheta\vartheta}=-\frac{1}{2}x_P(c_{x0}+c_y^\alpha)\varrho SV_{Cx}^2.$$

The other group describes change of $\Delta V_z, \Delta\psi, \Delta\gamma$:

$$m\frac{d\Delta V_z}{dt}+\nu_z\frac{d\Delta\psi}{dt}+c_{zz}\Delta V_z+c_{z\psi}\Delta\psi+mg_y\Delta\gamma=c_{z\delta_\psi}\Delta\delta_\psi+\Delta F_z,$$

$$J_x\frac{d^2\Delta\gamma}{dt^2}+\mu_x\frac{d\Delta\gamma}{dt}=c_{\gamma\delta}\Delta\delta_\gamma+\Delta M_x, \qquad (1.13.3)$$

$$J_y\frac{d^2\Delta\psi}{dt^2}+\mu_y\frac{d\Delta\psi}{dt}+c_{\psi z}\Delta V_z+c_{\psi\psi}\Delta\psi=c_{\psi\delta}\Delta\delta_\psi+\Delta M_y,$$

where

$$c_{zz}=\frac{1}{2}(c_{x0}+c_y^\alpha)\varrho SV_{Cx},\ c_{z\psi}=P+\frac{1}{2}c_y^\alpha\varrho SV_{Cx}^2,$$

$$c_{\psi z}=-\frac{1}{2}x_P(C_{x0}+c_y^\alpha)\varrho SV_{Cx},\ c_{\psi\psi}=-\frac{1}{2}x_P(c_{x0}+c_y^\alpha)\varrho SV_{Cx}^2 \qquad (1.13.4)$$

With change of parameters $\Delta V_x, \Delta V_y, \Delta\vartheta$ parameters $\Delta V_z, \Delta\psi, \Delta\gamma$, as is evident from equations (1.13.1) and (1.13.3), do not change, and vice versa. This means, under conditions when disturbed motion of the rocket is described quite correctly by linearized equations, i.e., when discarded terms containing products and higher degrees of disturbances, do not substantially affect the character of disturbed motion, this motion can be considered as the superposition of two motions, represented by two independent systems (1.13.1) and (1.13.3), investigation of which can be carried out separately. Splitting of system (1.12.25) into independent subsystems considerably simplifies solution of the problems, since in this case are considered systems, order of which is considerably less than the order of the original system.

Usually the linearized equations of disturbed motion are broken down into smaller groups, considering the control channels, which determine deflections of $\Delta\delta_x, \Delta\delta_\vartheta, \Delta\delta_\psi$ and $\Delta\delta_\gamma$, operate independently. In system (1.13.1) the second and third equations can be considered separately from the first equation, i.e., from system (1.13.1) there can be separated independent subsystem

$$\begin{aligned} &m\frac{d\Delta V_y}{dt}+v_y\frac{d\Delta\vartheta}{dt}+c_{yy}\Delta V_y+c_{y\vartheta}\Delta\vartheta=c_{y\delta}\Delta\delta_\vartheta+\Delta F_y,\\ &J_z\frac{d^2\Delta\vartheta}{dt^2}+\mu_z\frac{d\Delta\vartheta}{dt}+c_{\vartheta y}\Delta V_y+c_{\vartheta\vartheta}\Delta\vartheta=c_{\vartheta\delta}\Delta\delta_\vartheta+\Delta M_z, \end{aligned} \quad (1.13.5)$$

which describes disturbed motion of the rocket in terms of pitch. Concerning the first equation of system (1.13.1), it can also be considered independently, if from system (1.13.5) $\Delta\vartheta$ has already been determined as a function of time. Component $mg_y\Delta\vartheta$ in this instance is expedient to include in the composition of disturbing force ΔF_x. Then we will have

$$m\frac{d\Delta V_x}{dt}+c_{xx}\Delta V_x=c_{x\delta}\Delta\delta_x+\Delta F_x, \quad (1.13.6)$$

where

$$\begin{aligned} \Delta F_x=&-\frac{\Delta m}{m}\left(P-\frac{1}{2}c_{x0}\varrho SV_{cx}^2\right)+\Delta P+c_{x0}\varrho SV_{cx}V_{wx}-\\ &-\frac{1}{2}c_{x0}SV_{cx}^2\Delta\varrho-\frac{1}{2}\varrho SV_{cx}^2\Delta c_{x0}+mg_y\Delta\vartheta. \end{aligned} \quad (1.13.7)$$

Equation (1.13.6) describes longitudinal disturbed motion of the rocket.

From system (1.13.3) there can be isolated and examined separately equation

$$J_x \frac{d^2\Delta\gamma}{dt^2} + \mu_x \frac{d\Delta\gamma}{dt} = c_{\gamma\delta}\Delta\delta_\gamma + \Delta M_x, \tag{1.13.8}$$

describing the disturbed motion of the rocket in terms of roll.

The remaining two equations of system (1.13.3) can also be analyzed separately from the remaining groups of equations, if we discard component $mg_y\Delta\gamma$, little affecting the character of disturbed motion of the rocket, or here consider $\Delta\gamma$ as known function of time, determined from equation (1.13.8); in the latter case it is convenient to include component $mg_y\Delta\gamma$ in the composition of disturbing force. Then the last group of equations will appear in the form

$$\begin{aligned} m \frac{d\Delta V_z}{dt} + v_z \frac{d\Delta\psi}{dt} + c_{zz}\Delta V_z + c_{z\psi}\Delta\psi &= c_{z\delta}\Delta\delta_\psi + \Delta F_z, \\ J_y \frac{d^2\Delta\psi}{dt^2} + \mu_y \frac{d\Delta\psi}{dt} + c_{\psi z}\Delta V_z + c_{\psi\psi}\Delta\psi &= c_{\psi\delta}\Delta\delta_\psi + \Delta M_y. \end{aligned} \tag{1.13.9}$$

Equations (1.13.9) describe disturbed motion of the rocket in terms of yaw.

§ 14. Laws of Control

During investigation of disturbed motion of the rocket as the controlled object to equations, which describe motion of the rocket as the object of control, it is necessary to add equations defining the law of forming of controlling pressure on the object.

Input pressure to the regulator (automatic stabilization control), further called control signal of the regulator, is formed usually as a linear combination of controllable quantities and their derivatives. Let us assume the controllable quantities are q_1, $q_2, \ldots, q_n$ – some generalized coordinates of the considered mechanical system (for example, angles of pitch, yaw, etc.). Then the control signal of the regulator

$$v=\sum_{j=1}^{n}(b_{1j}\dot{q}_j+b_{2j}q_j). \qquad (1.14.1)$$

Coefficients b_{1j}, b_{2j} can be both constant, and functions of time t.

Sometimes the second derivatives of generalized coordinates $\ddot{q}_j$ take part in forming the control signal.

The output signal of the regulator can be considered deflection $\Delta\delta$ of control (vane).

Usually a nonlinear connection takes place between control and output signals of the regulator in real systems.

For the sake of simplicity of analysis, frequently, by idealizing the regulator and placing certain limitations on its operation, we consider it as a linear system.

The connection between control and output signals of a linear regulator with parameters, which depend upon time, is represented in the form (see [23]).

$$\Delta\delta=\int_{-\infty}^{t} g(t-t', t')v(t')dt',$$

where $g(t - t', t')$ – pulse transient function of the regulator, which represents the reaction of a preliminarily unexcitable system to control signal in the form of delta function.

In particular, if parameters of the regulator in the considered time interval are invariable, then

$$\Delta\delta = \int_{-\infty}^{t} g(t-t')v(t')\,dt'.$$

Linear laws of control can be represented in the form of linear differential equation, for example by equation

$$T_2^2 \frac{d^2\Delta\delta_\phi}{dt^2} + T_1 \frac{d\Delta\delta_\phi}{dt} + \Delta\delta_\phi = k\Delta\vartheta + k_1 \frac{d\Delta\vartheta}{dt}.$$

In symbolic writing generally these differential equations can be represented in the form

$$P(D)\Delta\delta = \sum_j Q_j(D) q_j,$$

where $P(\lambda)$, $Q_j(\lambda)$ — polynomials of λ, coefficients of which, generally speaking, are functions of t; $D \equiv \frac{d}{dt}$ — symbol of differentiation.

§ 15. General Form of Equations of Disturbed Motion

With derivation of equations (1.12.25) the terms containing products of disturbances or their powers higher than the first were dropped. If this is not done, than after transition to new variable ΔV_x, ..., $\Delta\gamma$ (to disturbances of basic parameters) in the right sides of the equations will appear functions, nonlinear relative to these variable and the remaining disturbances.

For convenience of representation of the system of equations of disturbed motion in such form, let us turn to other marks.

Let us assume x_C', y_C', z_C' – coordinates of center of inertia of the rocket in disturbed motion in system of coordinates $Cxyz$. Then, obviously,

$$\Delta V_x=\frac{dx_{C'}}{dt},\quad \Delta V_y=\frac{dy_{C'}}{dt},\quad \Delta V_z=\frac{dz_{C'}}{dt}.$$

Motion parameters $x_{C'}$, $y_{C'}$, $z_{C'}$, $\Delta\vartheta$, $\Delta\psi$, $\Delta\gamma$, if position of body axes in undisturbed motion is known, determine the position of rocket body in space and can be considered as generalized coordinates of the rocket. Equations of disturbed motion, written relative to these generalized coordinates, are represented in the form of system of nonlinear differential second order equations. By designating generalized coordinates through q_j, through μ_κ – other disturbances and dropping symbol Δ with δ for convenience, equations of disturbed motion of the rocket can be represented in the following general form:

$$\sum_{j=1}^{n}\left[m_{ij}(t)\frac{d^2q_j}{dt^2}+r_{ij}(t)\frac{dq_j}{dt}+n_{ij}(t)q_j\right]=$$
$$=a_i(t)\delta_i+h_i(t,\mu_k)+f_i\left(t,q_j,\frac{dq_j}{dt},\mu_k\right)\quad (i=1,\ldots,n). \qquad (1.15.1)$$

Here by q_1, q_2, ..., q_n there are implied coordinates $x_{C'}$, $y_{C'}$, $z_{C'}$, $\Delta\vartheta$, $\Delta\psi$, $\Delta\gamma$ or part of them depending on whether there is considered the total system of equations or some separated subsystem (for example, a subsystem describing disturbed motion of the rocket in terms of pitch); by symbols m_{ij}, r_{ij}, n_{ij}, a_i there are replaced the designations for coefficients of equations of disturbed motion accepted in § 13 (m, J_x, J_y, J_z, c_{xx}, c_{yy}, c_{zz}, $c_{y\vartheta}$, etc.); functions $h_i(t,\mu_\kappa)$ represent the disturbing forces and moments; finally, by f_i there are designated the nonlinear terms which were dropped in the process of derivation of equation (1.12.25).

The linearized system of equations is obtained from (1.15.1), if we drop the terms that are nonlinear relative to q_j, dq_j/dt, μ_k:

$$\sum_{j=1}^{n}\left[m_{ij}(t)\frac{d^2q_j}{dt^2}+r_{ij}(t)\frac{dq_j}{dt}+n_{ij}(t)q_j\right]=$$
$$=a_i(t)\delta_i+h_i(t,\ \mu_k)\quad (i=1,\ldots,\ n). \qquad (1.15.2)$$

Disturbances μ_k (for example, Δc_{x0}, ΔQ, Δc^{*}_{y},...) insignificantly depend on disturbances of basic parameters, so that this dependence can be disregarded. By considering disturbances μ_k as functions only of time t, we can present equations of disturbed motion (1.15.1) in the form

$$\sum_{j=1}^{n}\left[m_{ij}(t)\frac{d^2q_j}{dt^2}+r_{ij}(t)\frac{dq_j}{dt}+n_{ij}(t)q_j\right]=$$
$$=a_i(t)\delta_i+h_i(t)+f_i\left(t,\ q_j,\ \frac{dq_j}{dt}\right)\quad (i=1,\ldots,n). \qquad (1.15.3)$$

Accordingly, linearized system of equations of disturbed motion will be written so:

$$\sum_{j=1}^{n}\left[m_{ij}(t)\frac{d^2q_j}{dt^2}+r_{ij}(t)\frac{dq_j}{dt}+n_{ij}(t)q_j\right]=a_i(t)\delta_i+h_i(t)\ (i=1,\ldots,\ n). \qquad (1.15.4)$$

Each of systems (1.15.3), (1.15.4) together with equations of the control system forms a closed system of equations relative to variables q_j, δ_i describing the disturbed motion of the rocket as systems of automatic control.

Observation. The form of writing equations of disturbed motion of the rocket, accepted in this paragraph, carries a general character in the sense that equations of disturbed motion of the rocket taking into account mobility of liquids in tanks and elasticity of the construction with some simplifications can also be represented in the form of a system of ordinary linear differential equations of form (1.15.3), (1.15.4).

CHAPTER II

ROCKET STABILITY AND CONTROLLABILITY

§ 1. Concepts of Stability and Controllability

With deflection of controls or the action of external disturbances the rocket, as any other flight vehicle, accomplishes some disturbed motion, the character of which depends on the dynamic characteristics of the vehicle - controllability and stability.

Controllability of the flight vehicle is characterized by its reaction to deflection of controls by change of parameters of motion (velocity, angle of pitch, angle of roll, etc.). The rocket controls must possess sufficient effectiveness for parrying the different type of disturbing factors (wind, eccentricity and misaligment of engine thrust, etc.).

Selection of parameters and determination of the effectiveness of controls are connected with integration of equations of motion of the rocket. Usually it is not possible to obtain accurate analytical solution of these equations, therefore there are widely applied methods of approximate integration of equations, electromodeling and integration of equations with the aid of high-speed discrete computers.

The concept of stability of motion is one of most important concepts in rocket dynamics.

It can be said that the rocket, as a mechanical system, is stable if small actions lead to small effects, and unstable if this does not always take place.

Such definition of stability bears an intuitive character and, it is understood, is unsuitable for solution of concrete problems. It must be replaced by a mathematically strict definition, suitable for establishing quantitative criteria of stability and instability of motion of a mechanical system. The definition of stability, on one hand, must as complete as possible characterize stability as an objective quality of the system, and on the other hand - allow the possibility of construction of a convenient working apparatus for investigation of the stability of motion of particular objects.

At present in literature many various definitions of stability can be encountered. This can be explained by the fact that it is difficult, and perhaps impossible to formulate such a universal concept of stability, which would always completely satisfy the needs of life and would be accepted by all as solely true. Below are given some definitions of stability, having gained popularity and acknowledgement with solution of various problems of mathematical physics, mechanics and technology.

1.1. Static Stability

Some presentation of the stability of motion of a rocket can be obtained by investigating the motion of a rocket with fixed controls (i.e., when $\delta \equiv 0$) with small deviations of motion parameters from their values in equilibrium state, when the sum of motion parameters from their values in equilibrium state, when the sum of moments of all external and reactive forces, affecting the rocket, is equal to zero.

Let us consider, for example, motion of the rocket in the pitching, plane considering that the longitudinal axis of the rocket and velocity of center of mass lie in one vertical plane. The sum of moments relative to axes can be represented so:

$$\sum M_z = M_{a\,z} + M_{\text{пр}\,z}, \tag{2.1.1}$$

where $M_{a\,z}$ - moment of aerodynamic forces (without aerodynamic moment, created by deflection of air vanes, if there are such); $M_{\text{пр}\,z}$ - moment of all other forces (external and reactive), affecting the rocket.

According to (1.9.9)

$$\sum M_z = x_p R_{a\,y} + M_{\text{пр}\,z}, \tag{2.1.2}$$

where $R_{a\,y}$ - component of aerodynamic force along axis Cy, determined by formula (1.9.8). Since in this case

$$V_{Cy} = -V_C \sin\alpha,$$

then

$$R_{a\,y} = (c_x \sin\alpha + c_y \cos\alpha) S\varrho \frac{V_C^2}{2}. \tag{2.1.3}$$

By considering small angles of attack, when it is possible to take $\sin\alpha \approx \alpha$, $\cos\alpha \approx 1$ and $c_y = c_y^{\alpha}\alpha$, instead of (2.1.3) we will have

$$R_{a\,y} = (c_x + c_y^{\alpha})\alpha S\varrho \frac{V_C^2}{2}. \tag{2.1.4}$$

With change of angle of attack in expression (2.1.2) there are changes only $R_{a\,y}$. If moment is not too great, then for certain value of $\alpha = \alpha_{\text{бал}}$ there sets in balancing of the flight vehicle, when the sum of all moments affecting the vehicle is equal to zero, i.e.,

$$x_p R_{a\,y}|_{\alpha=\alpha_{\text{бал}}} + M_{\text{пр}\,z} = 0. \tag{2.1.5}$$

By subtracting equality (2.1.5) from (2.1.2), we obtain

$$\sum M_z = x_F (R_{zy} - R_{zy}|_{\alpha=\alpha_{бал}}), \qquad (2.1.6)$$

or, in view of equality (2.1.4),

$$\sum M_z = x_F (c_y + c_y^{\circ}) S \varrho \frac{V_C^2}{2} (\alpha - \alpha_{бал}). \qquad (2.1.7)$$

The next three cases are possible.

1. The $x_F < 0$ (aerodynamic focal point is located behind the center of mass of the vehicle).

If $\alpha > \alpha_{бал}$, in this instance $\Sigma M_z < 0$, i.e., the rocket is affected by a moment which is directed toward decrease of the difference of $\alpha - \alpha_{бал}$ (Fig. 2.1a).

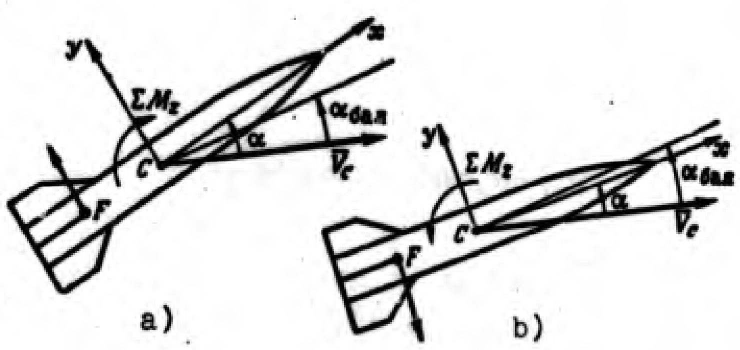

Fig. 2.1.

If $\alpha < \alpha_{бал}$, then $\Sigma M_z > 0$ and again the moment affecting the rocket, tries to bring α closer to $\alpha_{бал}$ (Fig. 2.1b).

Thus, if $x_F < 0$, then with disturbance of balance of the flight vehicle the moments affecting the vehicle are directed toward decrease of $|\alpha - \alpha_{бал}|$. In this instance we say taht the flight vehicle possesses *static stability*.

2. The $x_F = 0$ (aerodynamic focal point coincides with the center of mass of the vehicle).

As can be seen from expression (2.1.7), with any small change of angle of attack no moment appears. The flight vehicle is in a state of indifferent equilibrium.

3. The $x_F > 0$ (aerodynamic focal point is located in front of the center of mass of the vehicle).

In this case the sign of the moment coincides with the sign of difference of $\alpha - \alpha_{бал}$. This means that with deviation in any direction from the position of equilibrium there appears a moment, directed toward incraase of $|\alpha - \alpha_{бал}|$ (Fig. 2.2a, b). We say *static stability* takes place.

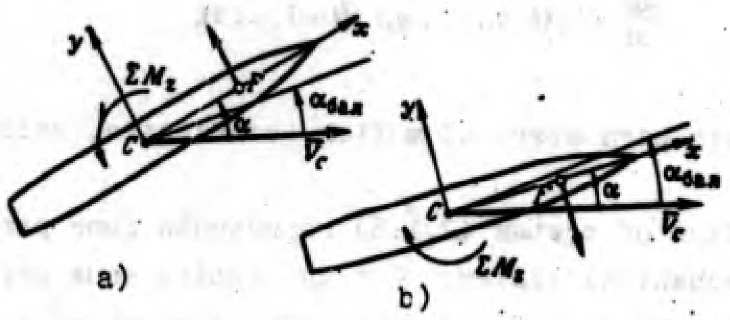

Fig. 2.2.

The concept of static stability cannot be identified with the concept of stability of motion. The presence of static stability only attests to the fact that at a given moment of time on the flight vehicle act forces, which are trying to return it to position of equilibirum. But this does not exclude such a possibility, at which the vehicle in its approach to the position of equilibrium will oscillate relative to the position of equilibrium and in this case its maximum deviation from equilibrium position will be increased with the passage of time. Further, static stability is a characteristic of a flight vehicle with fixed controls. Steering by means of deflection of controls radically changes the dynamic properties of the vehicle. Let us assume the rocket is even statically unstable and because of the control system it can accomplish motion, differing little from desired. And vice versa, if the control system is desired erroneously, the rocket, even possessing static stability, will be inadmissibly highly deflected from the predetermined trajectory.

The concept of stability of motion of the rocket, as any other mechanical system, must reflect the character of its actual motion in time.

1.2. Lyapunov Stability

Let us assum there is given a mechanical system, motion of which is described by a system of differential equations, reduced to standard from

$$\frac{dy_i}{dt}=Y_i(t, y_1, \ldots, y_n) \quad (i=1, \ldots n), \tag{2.1.8}$$

where y_i - certain parameters of motion (coordinate, velocity, etc.).

Every solution of system (2.1.8) represents some particular motion of our mechanical system. Let us examine some particular motion of the system, to which corresponds solution $y_i^0 = f_i(t)$ $(i = 1, \ldots, n)$ of equations (2.1.8). Lyapunov calls this particular motion undisturbed in contrast to other, disturbed motions of the given mechanical system.

In equations (2.1.8) let us replace variables

$$x_i=y_i-f_i(t) \quad (i=1, 2, \ldots, n). \tag{2.1.9}$$

We obtain the following system of differential equations relative to disturbances x_i:

$$\frac{dx_i}{dt}=X_i(t, x_1, x_2, \ldots, x_n) \quad (i=1, 2, \ldots, n), \tag{2.1.10}$$

where

$$X_i(t, x_1, x_2, \ldots, x_n) = Y_i(t, x_1+f_1, x_2+f_2, \ldots, x_n+f_n) - \\ -Y_i(t, f_1, \ldots, f_n).$$

To every motion of the considered mechanical system there corresponds, in view of (2.1.9), a particular solution of equations (2.1.10). In particular trivial (zero) solutions $x_1 = x_2 = \ldots = x_n = 0$ of equations (2.1.10) corresponds to undisturbed motion of the system.

Lyapunov gives the following definition of stability of undisturbed motion (trivial solution of equations (2.1.10)).[12].

Let us assume $L_1, \ldots, L_n$ in essence are randomly assigned positive numbers. If at every L_i, no matter how small, there can be selected positive numbers $E_1, \ldots, E_n$ so that at every real x_{i0}, satisfying conditions

$$|x_{i0}| \leq E_i \quad (i=1, 2, \ldots, n), \tag{2.1.11}$$

and at every t, exceeding t_0, there were fulfilled inequalities

$$|x_i| < L_i \quad (i=1, 2, \ldots, n), \tag{2.1.12}$$

then undisturbed motion with respect to quantities x_i is stable; otherewise - it is unstable.

Determination of Lyapunov stability can be given by the following geometric interpretation.

In n-dimensional space of quantities $x_1, x_2, \ldots, x_n$ there is assigned a parallelepiped with the center at the origin of coordinates and with sides parallel to the coordinate planes. The quantity of sides is determined by numbers $2L_1, \ldots, 2L_n$. These numbers are assigned arbitarily and can be as small as desired (but not equal to zero). If for the given parallelepiped it is possible to construct another parallelepiped with sides determined by positive numbers $2E_1$, ..., $2E_n$, and namely such that beginning from moment of time t_0 functions $x_i(t)$ remain inside the first parallelepiped, if their initial values, i.e., x_{i0}, were located inside the second parallelepiped, then undisturbed motion with respect to quantities x_i are stable (in Fig. 2.3a there are shown sections of these parallelepiped with plane of two axes of coordinates x_i and x_j and change of these coordinates in time in the case of stable zero solution). In other words, undistrubed motion is stable, if all disturbances x_i (i = 1, ..., n), values of which at initial moment of time t_0 are rather small, at which t, larger than t_0, do not exceed limits determined by assigned, as small as desired numbers L_i (Fig. 2.3b).

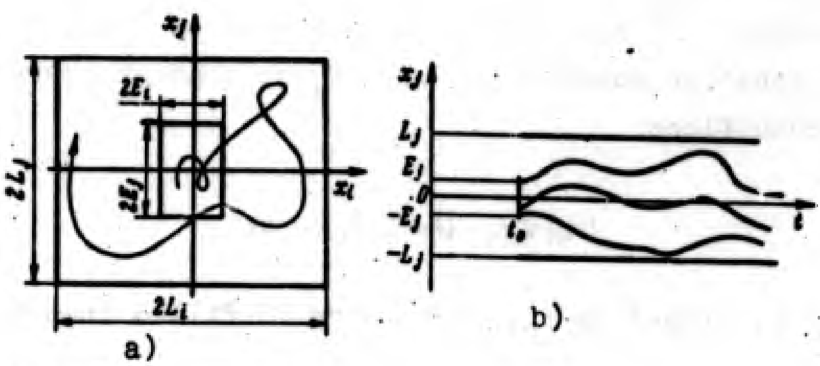

Fig. 2.3.

Determination of Lyapunov stability does not maintain any concrete quantitative limitations on quantities x_i. Stability of motion according to Lyapunov is a certain characteristic of strength and inflexibility of motion to the actions of disturbances. This involves the mechanical sense of the concept of stability, which Lyapunov puts into it.

1.3. Stability in Finite Interval of Time. Technical Stability

An essential moment in determining Lyapunov stability is limitation of deviations of x_i in an infinite interval of time by condition (2.1.12). If we convert to limitations in finite time interval, even as large as desired, any meaning in Lyapunov determination is lost, inasmuch as in any finite time interval any motion, including that not having mechanical stability, satisfies conditions (2.1.11) and (2.1.12). Meanwhile, real objects are of interest during a finite interval of time and therefore the introduction of the concept of motion acquires meaning and value for a finite interval of time is very expedient. The concept of stability of motion acquires meaning and value for a finite time interval if we introduce a relationship between regions, limiting x_i ($i = 1, 2, \ldots, n$) when $t = t_0$ and $t > t_0$.

Let us give two different definitions of stability of motion in finite time interval, proposed by G. V. Kamenkov and N. D. Moiseyev.

G. V. Kamenkov, retaining the mechanical sense of the concept of stability, which Lyapunov puts into it, poses problems of stability of motion in finite time interval in the following manner [6].

If differential equations of disturbed motion (2.1.10) are such that at rather small positive number A quantities x_s, considered as time functions, satisfy condition

$$\sum_{s=1}^{n} (a_{s1}x_1 + a_{s2}x_2 + \ldots + a_{sn}x_n)^2 \leqslant A \tag{2.1.13}$$

in finite time interval $[t_0, t_0 + \Delta t]$, if only the initial values of these functions x_{i0} satisfy condition

$$\sum_{s=1}^{n} (a_{s1}x_{10} + a_{s2}x_{20} + \ldots + a_{sn}x_{n0})^2 \leqslant 0, \tag{2.1.14}$$

where numbers $a_{s1}, a_{s2}, \ldots$ are such that $\begin{vmatrix} a_{11}a_{12}\ldots a_{1n} \\ a_{21}a_{22}\ldots a_{2n} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{n1}a_{n2}\ldots a_{nn} \end{vmatrix} \neq 0,$ then undisturbed motion will be stable in time interval Δt; otherwise - it is unstable, i.e., $\Delta t = 0$.

The given definition of stability allows the following geometric interpretation.

Let us assume at certain moment of time $t = t_0$ the system obtained some nonzero, randomly small deviations $x_{10}, \ldots, x_{n0}$ and these deviations were located inside or on the surface of n-dimensional ellipsoid:

$$\sum_{s=1}^{n} (a_{s1}x_1 + a_{s2}x_2 + \ldots + a_{sn}x_n)^2 = A. \qquad (2.1.15)$$

Then if deviations $x_s(t)$ ($s = 1, 2, \ldots, n$) remained inside or on the surface of this ellipsoid at least to moment of time $t = t_1 > t_0$, then motion is stable in interval $[t_0, t_1]$: otherwise - it is unstable.

Number A, by means of which limitation is introduced on quantities x_s, is not assumed assigned in advance: for stabiltiy of the system there is required only fulfilling of inequalities (2.1.13), (2.1.14) for rather small value A. Therefore it is comparatively easy to establish conditions of stability and instability of the system and evaluate time interval $\Delta t = t_1 - t_0$, within which these conditions are observed.

Concept of stability in finite time interval, proposed by G. V. Kamenkov, bears a local character. Usually it is require to investigate the behavior of the mechanical system within a certain prescribed time interval. In order to have the possibility of judging the stability of motion within the prescribed time interval, it is required to conduct a certain number of the same type of investigations at various "initial" values of t_0 from the considered interval.

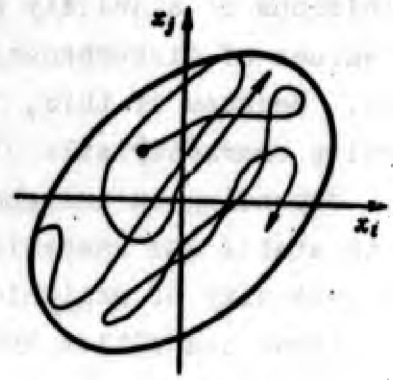

Fig. 2.4.

By examining mechanical systems, distrubed motion of which is described by equations

$$\frac{dx_s}{dt}=X_s(t, x_1, x_2, \ldots, x_n)+p_s(t) \quad (s=1, 2, \ldots, n), \tag{2.1.16}$$

where $p_s(t)$ - disturbing forces, acting on the system, N. D. Moiseyev in the following manner defines the so-called *technical stability* of trivial solution of equations:

$$\frac{dx_s}{dt}=X_s(t, x_1, x_2, \ldots, x_n) \quad (s=1, 2, \ldots, n). \tag{2.1.17}$$

Zero solution $x_1 = x_2 = \ldots = x_n = 0$ of equations (2.1.17) is called possessing technical stability relative to assigned upper limits of initial deviations of $\bar{x}_{s0} > 0$, assigned upper limits of disturbing forces $\bar{p}_s > 0$ and assigned upper limits of subsequent deviations of $\bar{x}_s > 0$ on assigned segment of values $0 \leq t \leq \bar{t}$ in only the case where any solutions $x_s = x_s(t)$ of system of equations (2.1.16) at any initial values of x_{s0}, satisfying condition $|x_{s0}| \leq \bar{x}_{s0}$, and at arbitrary disturbing forces, limited by unique condition $|p_s(t)| \leq \bar{p}_s$ when $0 \leq t \leq \bar{t}$, will satisfy the condition

$$|x_s(t)|<\bar{x}_s \quad (s=1, 2, \ldots, n)$$

for all values of t, not exceeding $\bar{t}$.

Here, in contrast to definitions of stability by Lyapunov and Kamenkov, maximum permissible values of disturbances are assumed finite and assigned in advance. Because of this, "technical stability" is largely a subjective characteristic of motion of a mechanical system: any motion, depending on assigned values of limiting deviations, can be both stable and unstable. Certain inconveniences result from the necessity of assigning limiting deviations: rational values of these quantities are usually unknown and difficult to determine.

Nevertheless, formulation of the problem of stability, proposed by N. D. Moiseyev, allowing in each concrete case in the most acceptable manner to evaluate the mechanical system itself, on the strength of its utilized qualities is very attractive for technical applications, and specifically for problems of rocket dynamics.

It is possible to become acquainted with methods of investigation of the technical stability of mechanical systems in book [7].

1.4. Local Stability

Disturbed motion of the rocket is described by a system of equations, consisting of equations of disturbed motion of the vehicle as the object of control (1.15.3) or linearized equations (1.15.4), and equations of the control system. In flight on the rocket constantly act various disturbances (wind, weight and others). Because of this, in equations of disturbed motion there figure disturbing forces. Thus, in linearized equations of disturbed motion (1.15.4) these disturbing forces are represented by functions $h_i(t)$.

To each particular solution of equations of disturbed motion there corresponds a certain distrubed motion of the rocket. To undisturbed motion of the rocket there corresponds trivial (zero) solution of equations, which are obtained from equations of disturbed motion if all disturbing forces are discarded in them. Stability and instability of undisturbed motion are equivalent to stability and instability of the shown trivial solution, and the stability of this trivial solution is determined by the behavior of particular solutions of equations of disturbed motion, in which there figure constantly acting disturbances.

Thus, as in the formulation of N. D. Moiseyev, solution of the problem of stability of undisturbed motion of the rocket is connected with investigation of solutions of the system of equations with the presence of disturbing forces, whereas the motion being investigaed is represented by a particular solution of another system of equations, not containing disturbing forces. Analysis of stability of motion under these conditions is a difficult problem, and its examination exceeds the bounds of educational literature.

In this book we will limit ourselves to simplified formulation of problem, which, incidently, is widely used in practice.

Let us assume that during some finite time interval the flight vehicle is affected by disturbing forces and their action at moment of time t_0, which we will call *initial moment*, ceases. As a consequence of the action of disturbing forces, kinematic parameters of motion of the flight vehicle when $t = t_0$ will differ from their values in undisturbed motion: $\vec{V}'(t_0) = \vec{V}(t_0) + \Delta\vec{V}(t_0)$ $\vartheta'(t_0) = \vartheta(t_0) + \Delta\vartheta(t_0)$, etc.

Solution of the problem of stability of undisturbed motion when $t > t_0$ is connected with the character of change of disturbances $\Delta\vartheta(t)$, $\Delta\psi(t)$, ... The rocket is considered stable or unstable when $t > t_0$ in certain interval $[t_0, t_1]$ depending on how disturbances of kinematic parameters are changed in this interval.

Inasmuch as change of distrubances $\Delta\bar{V}(t)$, $\Delta\vartheta(t)$, $\Delta\psi(t)$, ... when $t > t_0$ proceeds under conditions when there are no disturbing forces, disturbances of kinematic parameters are represented as particular solution of equations of disturbed motion of the rocket corresponding to initial values of these parameters ($\Delta V(t_0)$, $\Delta\theta(t_0)$, $\Delta\psi(t_0)$, ...) in which all the disturbing forces have been dropped. Thus, in the given formulation the subject of investigations becomes simpler equations, because of which the problem of stability is made substantially easier.

Disturbing forces, applied to the rocket, in actuality and when $t > t_0$ continue to affect the disturbance of kinematic parameters of motion. But if we consider rather small time intervals $[t_0, t_1]$, then, most likely, the effect from the action of disturbing forces during this interval in comparison with the effect from the action of these forces for a long time, preceding moment t_0, will be small.

The stability with the shown schematization of disturbed motion of the rocket we will call *local stability*.

1.5. Methods of Solution of Problems of Stability

All methods of solution of the problems of stability of motion can be broken down into two categories. The first category includes those methods that are based on determination of general or particular solution of equations of disturbed motion. Lyapunov called the totality of all methods of the first category the first method. During investigation of stability by first method the question about integration of equations of disturbed motion is central. If it is possible to integrate these equations in closed form, then investigation of stability no longer further presents serious difficulties.

The second category includes those methods that do not require finding particular or general solutions of equations of disturbed motion. Lyapunov called the totality of all methods of the second category the second method.

During solution of the problem of stability both by first and second methods we usually resort to various methods, which simplify the investigation of equations of disturbed motion. Thus, the problem will be considerably simplified if we are limited by consideration of linearized equations of disturbed motion - equations of the first approximation.

Equations of disturbed motion of the rocket of the first approximation are represented in the form of linear equations, coefficients of which depend on the kinematic parameters of undisturbed motion (velocity, flight altitude, etc.) and design parameters of the vehicle (mass, moments of inertia, etc.). Since undisturbed motion occurs under conditions when the velocity and altitude of flight, mass and moments of inertia are rapidly changed, it carries a clearly expressed unsteady character. Because of this, coefficients of equations of disturbed motion are functions of time t. Investigation of system with various coefficients, let us assume even linear, nevertheless is a very complex problem. Therefore, in practice further simplifying the problem, we reduce it to investigation of systems of linear equations with constant coefficients by means of the so-called method of quenching of coefficients, having received wide use at present.

§2. Method of Quenched Coefficients

Method of quenched coefficients - this is a method of investigation of equations of disturbed motion, based upon the method of *quenching* of coefficients of equations. The essence of the method of quenching of coefficients consists of the following.

Let us assume the equations of disturbed motion are represented in the form

$$\sum_{j=1}^{n}\left[m_{ij}(t)\frac{d^2q_j}{dt^2}+r_{ij}(t)\frac{dq_j}{dt}+n_{ij}(t)q_j\right]=a_i(t)\delta_i+h_i(t)$$

$$(i=1, 2,\ldots, n). \qquad (2.2.1)$$

Coefficients of equations $m_{ij}(t)$, $r_{ij}(t)$, $n_{ij}(t)$, $a_i(t)$, $h_i(t)$ depend on kinematic parameters of undisturbed motion and flight vehicle parameters and are known functions of time. On an undisturbed trajectory we select a certain number of characteristic points $t_1, \ldots, t_k, \ldots, t_m$ so that from the behavior of the vehicle in the vicinities of these points it would be possible with sufficient certainty to judge its behavior on the entire trajectory. For studying the motion of a rocket in the vicinities of point t_k instead of system of equations (2.2.1) with variable coefficients we consider the system of equations with constant coefficients, equal to values of these coefficients at fixed moment of time t_k, i.e., system

$$\sum_{j=1}^{n}\left[m_{ij}(t_k)\frac{d^2q_j}{dt^2}+r_{ij}(t_k)\frac{dq_j}{dt}+n_{ij}(t_k)q_j\right]=a_i(t_k)\delta_i+h_i(t)$$

$$(i=1, 2, \ldots, n). \qquad (2.2.2)$$

A new system of equations, as the system with constant coefficients, can be easily integrated. The behavior of solution of the original system (2.2.1) is judged by the solution of system (2.2.2).

It is clear that the solutions of these two systems will not coincide, but if coefficients of the equations are changed rather slowly, then with some reliability it is possible to assert that in small vicinity of point t_k the solution of the original system will little differ from solution of the system with quenched coefficients. As a measure of "slowness" of change of coefficients of equations there is sometimes taken their change during the transient process of the flight vehicle as the controllable object. Coefficients of equations are considered slowly changing, if their change during the transient process does not exceed 10-15% - accuracy of determination of dynamic coefficients.

It must be said that it is possible to judge the validity of application of the method of quenched coefficients only by the behavior of coefficients of equations on a certain segment of time only with known precaution. This method can be permissible with rapid change of coefficients, and, conversely, inadmissible with their slighter change. The validity of application of the method of quenching of coefficients in every concrete case requires substantiation. Utilization of the method of quenching of coefficients is entirely justified at the initial stage of designing of a flight vehicle for preliminary selection of parameters of the vehicle and control system. Final check of the stability and controllability of the flight vehicle must be performed on the basis of stricter methods with allowance for the variability of coefficients of equations of disturbed motion.

In practice of the design offices and scientific research organizations there is widely applied the method of quenching of coefficients, since this gives the possibility of using well worked-out engineering methods of investigation of linear stationary systems of automatic control.

§ 3. Criteria of Stability of Motion According to the Method of Quenched Coefficients

Let us assume that disturbed motion is described by the following system of linear differential equations:

$$\sum_{j=1}^{n}\left[m_{ij}(t)\frac{d^2q_j}{dt^2}+r_{ij}(t)\frac{dq_j}{dt}+n_{ij}(t)q_j\right]=f_i(t)\quad (i=1,\ 2,\dots,\ n). \qquad (2.3.1)$$

Linearized equations of disturbed motion of a rocket have such a form when deflections controls are prescribed as functions of time or when the connection between control and output signals of the regulator is likewise prescribed by linear differential equations. In the latter case the deflections of controls δ_ν are considered, al along with q_j, as generalized coordinates of the given mechanical system.

To undisturbed motion there corresponds trivial solution $q_j=0,\ \frac{dq_j}{dt}=0\ (j=1,2,\ldots,n)$ of equations

$$\sum_{j=1}^{n}\left[m_{ij}(t)\frac{d^2q_j}{dt^2}+r_{ij}(t)\frac{dq_j}{dt}+n_{ij}(t)\,q_j\right]=0\quad (i=1,2,\ldots,n). \qquad (2.3.2)$$

Let us fix moment of time $t = t_k$ and let us invetigate the local stability of trivial solution of system (2.3.2) with values of t close to t_k. For this it is necessary to study the behavior of particular solutions of equations of disturbed motion when $t \geq t_k$ on the assumption that the action of disturbing forces at moment of time t_k is ceased, i.e., solutions of homogeneous system (2.3.2) must be investigated.

By using the method of quenched coefficients, further instead of equations (2.3.2) let us introduce into consideration the equations with quenched coefficients

$$\sum_{j=1}^{n}\left[m_{ij}\frac{d^2q_j}{dt^2}+r_{ij}\frac{dq_j}{dt}+n_{ij}q_j\right]=0\quad (i=1,2,\ldots,n), \qquad (2.3.3)$$

where

$$m_{ij}=m_{ij}(t_k),\quad r_{ij}=r_{ij}(t_k),\quad n_{ij}=n_{ij}(t_k),$$

and let us investigate from the beginning the behavior of solutions of these equations.

We will seek the particular solution of system of linear differential equations with constant coefficients (2.3.3) in the form

$$q_j=k_je^{\lambda t}\quad (j=1,\ldots,n), \qquad (2.3.4)$$

where λ, k_j - constants, which are determined in the following manner.

Let us substitute (2.3.4) in equations (2.3.3). After reduction by $e^{\gamma t}$ we obtain

$$\sum_{j=1}^{n}[m_{ij}\lambda^2+r_{ij}\lambda+n_{ij}]k_j=0 \quad (i=1, 2, \dots, n). \tag{2.3.5}$$

System of algebraic equations (2.3.5) possesses trivial solution $k_1 = k_2 = \dots = k_n = 0$, which, as evident from (2.3.4), determines the trivial solution of system of equations (2.3.3). System of linear homogeneous equations (2.3.5) has a nonzero solution, if only the determinant of system

$$D(\lambda)=\begin{vmatrix} m_{11}\lambda^2+r_{11}\lambda+n_{11}\dots m_{1n}\lambda^2+r_{1n}\lambda+n_{1n} \\ m_{21}\lambda^2+r_{21}\lambda+n_{21}\dots m_{2n}\lambda^2+r_{2n}\lambda+n_{2n} \\ \dots\dots\dots\dots\dots\dots \\ m_{n1}\lambda^2+r_{n1}\lambda+n_{n1}\dots m_{nn}\lambda^2+r_{nn}\lambda+n_{nn} \end{vmatrix} \tag{2.3.6}$$

is equal to zero.

Having expanded this determinant, we obtain polynomial

$$D(\lambda)=a_0\lambda^{2n}+a_1\lambda^{2n-1}+\dots+a_{2n-1}\lambda+a_{2n}, \tag{2.3.7}$$

called characteristic polynomial of the system. The coefficient at λ^{2n} is equal to determinant, consisting of coefficients of equations (2.3.5) at λ^2:

$$a_0=\begin{vmatrix} m_{11} & m_{12} & \dots & m_{1n} \\ m_{21} & m_{22} & \dots & m_{2n} \\ \dots & \dots & \dots & \dots \\ m_{n1} & m_{n2} & \dots & m_{nn} \end{vmatrix}.$$

We will consider that $a_0 \neq 0$, and therefore polynomial (2.3.7) is a polynomial of $2n$ power relative to λ.

From higher algebra it is known that polynomial of $2n$ power with real coefficients has $2n$ roots, real or complex. Among these roots, generally speaking, they can be equal. For simplicity we will assume that all roots of the characteristic polynomial are different.

Let us assume λ_i - one of roots of characteristic polynomial. Since

$$D(\lambda_i)=0,$$

system (2.3.5) when $\lambda = \lambda_i$ has nonzero solution

$$k_j=k_{ji} \quad (j=1,\ldots, n). \tag{2.3.8}$$

This solution with accuracy to within arbitrary constant coefficient, nonzero, is a unique solution of system (2.3.5), since λ_i is a simple root of the characteristic polynomial.

Thus, to root λ_i corresponds particular solution of system of differential equations (2.3.3)

$$q_{ji}=k_{ji}e^{\lambda_i t} \quad (j=1, 2, \ldots, n), \tag{2.3.9}$$

or in expanded form

$$\begin{aligned} q_{1i}&=k_{1i}e^{\lambda_i t},\\ q_{2i}&=k_{2i}e^{\lambda_i t},\\ &\cdots\cdots\\ q_{ni}&=k_{ni}e^{\lambda_i t}. \end{aligned} \tag{2.3.10}$$

By introducing into examination vector $q^{(i)}$ with components $q_{1i}, q_{2i}, \ldots, q_{ni}$ and vector $k^{(i)}$ with components $k_{1i}, k_{2i}, \ldots, k_{ni}$

let us represent the particular solution, corresponding to root λ_i in the following compact form:

$$q^{(i)}=k^{(i)}e^{\lambda_i t}. \tag{2.3.11}$$

To each root of characteristic polynomial there corresponds a particular solution of form (2.3.11). The considered system of differential equations is linear, therefore, the superposition principle is valid, according to which any linear combination of particular solutions is also a solution of this system. General solution of system of equations (2.3.3) is represented by linear combination

$$q=\sum_{i=1}^{2n} c_i q^{(i)}=\sum_{i=1}^{2n} c_i k^{(i)}e^{\lambda_i t}, \tag{2.3.12}$$

containing $2n$ arbitrary constants $c_1, c_2, \ldots, c_{2n}$. Arbitrary constants c_i are determined from initial conditions (by initial values of disturbances $q(t_k)$ and $\left.\frac{dq}{dt}\right|_{t=t_k}$).

If λ - complex root of characteristic polynomial, i.e., $\lambda_i = \alpha_i + i\beta_i$ ($\beta_i \neq 0$), then components of vector $k^{(i)}$, as the solution of algebraic equations (2.3.5) with complex coefficients, will also be complex numbers. Therefore $k^{(i)}$ is a vector of form $p^{(i)} + iq^{(i)}$ where $p^{(i)}$ and $q^{(i)}$ - vectors with real components. Among the remaining roots there is certainly root λ_k, complex conjugate to root λ_i. Corresponding vector $k^{(k)}$ is a vector, complex conjugate to vector $k^{(i)}$, i.e.,

$$k^{(k)}=\bar{k}^{(i)}=p^{(i)}-iq^{(i)}.$$

Thus, to the pair of complex conjugate roots λ_i and $\lambda_k = \bar{\lambda}_i$ corresponds particular solution of system (2.3.3)

$$q^{(i,k)}=k^{(i)}e^{\lambda_i t}c_i+\bar{k}^{(i)}e^{\bar{\lambda}_i t}c_k. \tag{2.3.13}$$

In order that this particular solution would be real, it is necessary that c_i and c_k be complex conjugate numbers.

Considering that $c_k = \bar{c}_i$ and representing complex number c_i in the form

$$c_i = c_{i0} e^{i\varphi_i},$$

it is possible to write this particular solution so:

$$q^{(i,k)} = (p^{(i)} + iq^{(i)}) e^{(\alpha_i + i\beta_i)t} c_{i0} e^{i\varphi_i} + (p^{(i)} - iq^{(i)}) e^{(\alpha_i - i\beta_i)t} c_{i0} e^{-i\varphi_i}.$$

Hence, using relationship

$$e^{\pm ix} = \cos x \pm i \sin x,$$

we obtain

$$q^{(i,k)} = 2c_{i0} e^{\alpha_i t} \left[p^{(i)} \cos(\beta_i t + \varphi_i) - q^{(i)} \sin(\beta_i t + \varphi_i)\right]. \qquad (2.3.14)$$

Values of c_{i0} and ϕ_i are determined by initial conditions.

The provided relationships permit easily obtaining the conditions of stability and instability of trivial solution of the equations with constant coefficients (2.3.3).

Let us compute the modulus of vector $q^{(i)}$ representing particular solution (2.3.10). By using (2.3.11), we find

$$|q^{(i)}| = \sqrt{\bar{q}^{(i)} \cdot q^{(i)}} = \sqrt{\bar{k}^{(i)} \cdot k^{(i)} e^{\bar{\lambda}_i t} e^{\lambda_i t}} = |k^{(i)}| e^{\mathrm{Re}\,\lambda_i t} = |k^{(i)}| e^{\alpha_i t}, \qquad (2.3.15)$$

Hence it is apparent that the modulus of vector $q^{(i)}$ when $t \to \infty$ grows unlimitedly if $\alpha_i > 0$, it is decreased, asymptotically approaching zero if $\alpha_i < 0$, and remains constant if $\alpha_i = 0$ (see Fig. 2.5).

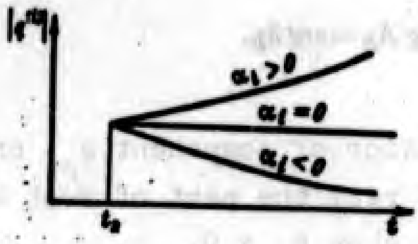

Fig. 2.5.

Moduli of components of vector $q^{(i)}$ behave analogously. We have

$$q_{ji} = k_{ji} e^{(\alpha_i + i\beta_i)t}.$$

Hence, since

$$|e^{i\beta_i t}| = 1,$$

then

$$|q_{ji}| = |k_{ji}| e^{\alpha_i t}, \tag{2.3.16}$$

and, this means the modulus of component q_{ji} of vector $q^{(i)}$ when $t \to \infty$ grows unlimitedly when $\alpha_i > 0$, asymptotically approaching zero when $\alpha_i < 0$, and remains constant if $\alpha_i = 0$.

Let us assume

$$A_{ji} = k_{ji} e^{\alpha_i t}.$$

Then

$$q_{ji} = A_{ji} e^{i\beta_i t}. \tag{2.3.17}$$

Factor A_{ji} can be considered as complex amplitude with modulus

$$|A_{ji}| = |k_{ji}| e^{\alpha_i t}$$

and argument

$$\arg A_{jl} = \arg k_{jl}.$$

Figure 2.6 shows the behavior of component q_{ji} on the complex plane depending on the sign of real the part of root λ_i. The given figures correspond to the case when $\beta_i > 0$.

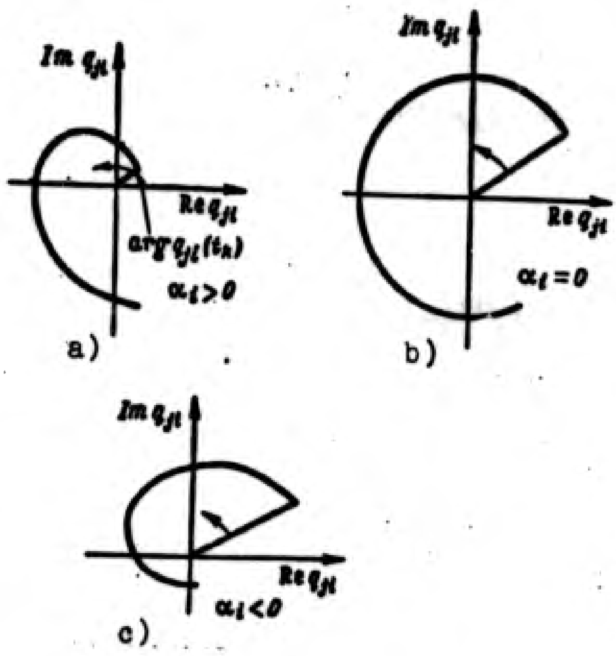

Fig. 2.6.

Character of change of the real part of coordinate q_{ji} (when $\beta_i \neq 0$) when $\alpha_i > 0$, $\alpha_i < 0$ and $\alpha_i = 0$ is shown in Fig. 2.7.

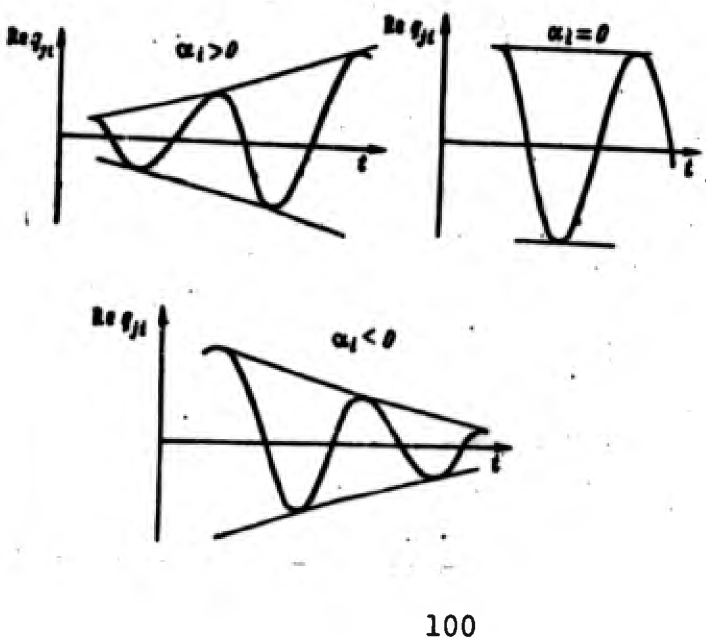

Fig. 2.7.

If λ_i is the real root, in this instance q_{ji} is changed with respect to the exponent (Fig. 2.8).

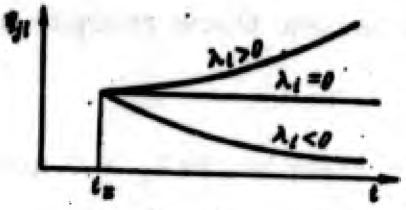

Fig. 2.8

By differentiating (2.3.10), we obtain

$$\frac{dq_{11}}{dt} = k_{11}\lambda_1 e^{\lambda_1 t},$$
$$\cdots\cdots\cdots$$
$$\frac{dq_{n1}}{dt} = k_{n1}\lambda_1 e^{\lambda_1 t}.$$

Hence it is clear that behavior of $\frac{dq_{ji}}{dt}$ is analogous to the behavior of q_{ji}: if $\mathrm{Re}\lambda_i > 0$, then $|\frac{dq_{ji}}{dt}|$ - ascending function; if $\mathrm{Re}\lambda_i < 0$, then $|\frac{dq_{ji}}{dt}|$ - descending function, and if $\mathrm{Re}\lambda_i = 0$, then $|\frac{dq_{ji}}{dt}|$ = const. Modulus of vector $\frac{dq^{(i)}}{dt}$ in exactly the same manner.

Let us suppose that all roots of characteristic polynomial have negative real parts. In this instance all particular solutions of $q^{(i)}$ and their derivatives in time with unlimited growth of t approach zero. Whatever constants c_i are, general solution (2.3.12) and its derivative in time also approach zero when $t \to \infty$. Consequently, trivial solution of system (2.3.3) is stable according to Lyapunov, and at any initial disturbances.

If the investigated particular solution is not only stable, but also possesses the property that all disturbances, initial values of which are rather small, with unlimited growth of t asymptotically approach zero, then we say that the given particular solution is asymptotically stable. In the considered case, as we see, trivial solution of equations (2.3.3) is not only stable, but is asymptotically stable.

If there is at least one root with a positive real part, then from any small region of initial values there emerges a solution, unlimited when $t \to \infty$.

Actually, let us assume, for example, $\mathrm{Re}\lambda_j > 0$ and

$$|q(t_k)| \leqslant \delta, \qquad (2.3.18)$$

where δ - as small a positive quantity as desired. Let us assume

$$c_i = 0 \quad (i \neq j), \quad c_j \neq 0.$$

In this case we will have

$$q(t) = k^{(j)} e^{\lambda_j t} c_j. \qquad (2.3.19)$$

When $t = t_k$ $q(t_k) = k^{(j)} e^{\lambda_j t_k} c_j$.

It is clear that c_j can always be selected so that condition (2.3.18) is fulfilled. But, regardless of this, solution (2.3.19) when $t \to \infty$ is not limited; conditions (2.1.11) and (2.1.12) are not observed, and, this means, trivial solution of equations (2.3.3) is unstable according to Lyapunov.

Finally, let us assume that along with roots with negative real parts there are roots with zero real parts and there are no roots with positive real parts. In this instance all solutions of system (2.3.3) and their derivatives in time when $t > t_k$ are limited, but there are solutions, moduli of which remain constant. Actually let us assume, for example, $\lambda_j = i\beta_j$. Then such solution is

$$q^{(j)} = k^{(j)} e^{i\beta_j t} c_j. \qquad (2.3.20)$$

Solutions of type (2.3.20) do not disturb the stability of trivial solution of equations (2.3.3), but the stability of this solution is no longer asymptotic.

Thus, in order that trivial solution of the system of equations with constant coefficients (2.3.3) would be asymptotically stable, it is necessary and sufficient that all roots of characteristic polynomial have negative real parts. If among the roots of characteristic polynomial there is at least one with positive real part, then the trivial solution is unstable. This remains valid even when there are multiple roots.

If among roots of characteristic polynomial there are no roots with positive real parts, but there are simple roots with zero real parts, then the trivial solution is stable, but not asymptotically. Stability of the trivial solution is retained even in the presence of multiple roots with zero real parts, if only the number of groups of solutions, corresponding to these roots, is equal to their multiplicity.

Thus, the question of stability of trivial solution of equations (2.3.3) is solved by signs of real parts of roots of characteristic polynomial (2.3.7).

Thus, we established conditions of stability and instability of the trivial solution of equations (2.3.3) in infinite time interval. But we are interested in the behavior of solutions not when $t \to \infty$, but within finite time interval, starting from moment t_k.

Inasmuch as (2.3.3) is a system of differential equations with constant coefficients, then the behavior of its particular solutions in infinite time interval, with the exception of cases of multiple roots of characteristic polynomial, fully determines its behavior in any finite time interval, and vice versa. Actually, if the trivial solution of equations (2.3.3) is stable according to Lyapunov, then all particular solutions of these equations of type

(2.3.11) with respect to modulus are nonincreasing functions when $t \geq t_k$. If, however, the trivial solution of equations (2.3.3) is unstable according to Lyapunov, then there is certainly at least one particular solution of form (2.3.11), which when $t \geq t_k$ at small time interval is a function increasing in modulus. To these considerations let us add the following. It would be possible to show that if at moment of time t_k the real parts of all roots of characteristic polynomial are negative, then stability takes place in finite time interval according to G. V. Kamenkov, i.e., starting from moment of time t_k there is finite time interval, during which all disturbances, included in initial moment t_k inside a certain ellipsoid, do not exceed the limit of this ellipsoid. If, however, there is at least one root with positive real part, then the investigated motion does not possess stability in finite time interval in the meaning of G. V. Kamenkov.

Considering the above and excluding the case of multiple roots of characteristic polynomial from examination, it is possible to consider the above-formulated conditions of stability of trivial solution of equations (2.3.3) according to Lyapunov as conditions of stability of this solution in finite time interval with origin at fixed point t_k.[1] Thus, the question of stability of trivial solution of equations with quenched coefficients (2.3.3) is solved bolth in infinite and finite time interval.

If solutions of system (2.3.2) and systems with quenched coefficients (2.3.3) coincided, then the formulated conditions of stability of trivial solution of system (2.3.3) could have been directly converted to trivial solution of equations (2.3.2), which represents the undisturbed motion being investigated. In fact, systems (2.3.2) and (2.3.3) are different and their solutions can be considered as close, if we are limited by small vicinities of

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[1]Let us note that the concept "asymptotic stability" makes sense only for infinite time interval, therefore, during examination of finite time interval in formulations of conditions of stability the term "asymptotic stability" should be replaced by the term "stability."

point t_k. Therefore, signs of real parts of roots of characteristic polynomial can be considered only as approximate criteria of stability of undisturbed motion [trivial solution of equations (2.3.2)] within a small time interval.

Inasmuch as disturbed motion when $t \geq t_k$ was studied on the assumption that the action of disturbing forces when $t = t_k$ was ceased, these criteria are in ssence the approximate criteria of local stability.

To investigate equations of disturbed motion it is not complusory to determine roots of characteristic polynomial. It is sufficient to obtain the characteristic polynomial of the system, and then use some criterion, permitting by coefficients of characteristic polynomial to judge its roots, for example, by Hurwitz criterion.

For characteristic polynomial

$$D(\lambda) = a_0\lambda^m + a_1\lambda^{m-1} + \ldots + a_{m-1}\lambda + a_m$$

the condition of negativity of real parts of all roots of characteristic polynomial is reduced to the fact that when $a_0 > 0$ are all m Hurwitz determinants, being obtained from the determinant consisting of coefficients of the following characteristic polynomial, should be greater than

$$\begin{vmatrix} a_1 & a_3 & a_5 & \ldots & 0 & 0 \\ a_0 & a_2 & a_4 & \ldots & 0 & 0 \\ 0 & a_1 & a_3 & \ldots & 0 & 0 \\ 0 & a_0 & a_2 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & a_{m-1} & 0 \\ 0 & 0 & 0 & \ldots & m_{m-2} & a_m \end{vmatrix},$$

namely:

$$\begin{aligned}
&\Delta_1 = a_1 > 0,\\
&\Delta_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} > 0,\\
&\Delta_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} > 0,\\
&\dots\dots\dots\dots\\
&\Delta_m = a_m \Delta_{m-1} > 0.
\end{aligned} \qquad (2.3.21)$$

During analysis of disturbed motion of rockets Nyquist criterion is widely applied. Formulation of Nyquist is given below (§ 5), where this criterion is used for establishing maximum permissible values of frequency characteristics of automatic stabilization control from condition of stability of the automatic control system of the rocket.

The basis of criterion of Hurwitz, Nyquist and other criteria of stability of stationary systems is given in courses of automatic control theory.

§4. The Rocket as a Linear Object of Automatic Control

Disturbed motion of the rocket as a closed system of automatic control is described by a system of equations consisting of equations of disturbed motion of the rocket as the object of control, and equations describing operation of control systems. The unknown functions in this system of equations are disturbances of parameters of motion ΔV_x, ΔV_y, ΔV_z, $\Delta\vartheta$, $\Delta\psi$, $\Delta\gamma$ and deflections of controls $\Delta\delta_x$, $\Delta\delta_\vartheta$, $\Delta\delta_\psi$, $\Delta\delta_\gamma$.

To investigate the closed system of automatic control it is necessary to know the dynamic properties of its separate elements and, specifically, the dynamic properties of the rocket as the object of control. Knowledge of dynamic properties of the vehicle we separately consider equations of disturbed motion of the rocket as the object of control, assuming in them deflections of controls as known function of time.

The system "flight vehicle - regulator" generally speaking, is nonlinear: both the vehicle and regulator are described by nonlinear equations. In problems of dynamics the nonlinear terms of equations of disturbed motion of the rocket as an object of control are usually disregarded, being limited by consideration of linearized equations of disturbed motion (1.13.5), (1.13.6), (1.13.8), (1.13.9). In most cases linear approximation is sufficient with respect to the regulator. However, often there appears the necessity of investigation of the system with allowance for nonlinearity of the regulator, for example during investigation of self-oscillations in system "flight vehicle - regulator."

Linear systems (1.13.5), (1.13.6), (1.13.8) and (1.13.9) consist of nonhomogeneous differential equations, the right sides of which at assigned values of deflections $\Delta\delta_x$, $\Delta\delta_\vartheta$, $\Delta\delta_\psi$, $\Delta\delta_\gamma$ are certain functions of time, not depending on unknown functions ΔV_x, ΔV_y, ΔV_z, $\Delta\vartheta$, $\Delta\psi$, $\Delta\gamma$.

General solution of nonhomogenous linear system, as is known, is made up of the general solution of homogeneous system and particular solution of nonhomogenous system.

General solution of homogeneous system describes free, or proper, motion of flight vehicle,[1] and particular solution of nonhomogenous system - its forced motion. In accordance with this, disturbed motion, which appears with deflection of controls or action of external disturbing forces on the vehicle, is made up of free and forced motions.

Dynamic properties of the rocket as an object of control are determined by the character of disturbed motion, appearing with deflection of controls or the action of disturbing forces. Usually there are investigated reactions of the vehicle to gradual deflection

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[1]Proper motion corresponds to fixed controls.

of controls and deflection of controls according to harmonic law.

Reaction of the vehicle to gradual deflection of controls is described by transient functions of the flight vehicle, which are solutions of differential equations under zero initial conditions and on the assumption that there are no disturbing forces, and deflections of controls ($\Delta\delta_x$, $\Delta\delta_\vartheta$, $\Delta\delta_\psi$, $\Delta\delta_\gamma$) up to fixed moment of time t_0 (initial moment) are identically equal to zero, and when $t \geq t_0$ have constant nonzero values.

Reaction of the vehicle to deflection of controls according to harmonic law is described by frequency characteristics of the flight vehicle. These characteristics are components of the solution of differential equations of disturbed motion with the absence of disturbing forces and deflection of controls according to harmonic law.

Investigation of disturbed motion of the vehicle, which appears with the action of external disturbing forces, for example atmospheric disturbances, is often conducted according to the following scheme.

We assume that during some time interval right up to moment of time t_0 disturbing force acted on the vehicle, due to which the kinematic parameters of motion of the flight vehicle when $t = t_0$ differ from their values in undisturbed motion, i.e., $\Delta V_x(t_0) \neq 0$, $\Delta V_y(t_0) \neq 0$, $\Delta V_z(t_0) \neq 0$, etc. Further behavior of the vehicle is studied on the assumption that deflections of controls $\Delta\delta_i$ are equal to zero and disturbing forces are absent. Thus, the problem is reduced to the study of proper motion of the vehicle, caused by initial disturbances of parameters of motion (disturbances when $t = t_0$).

During analysis of disturbed motion below there is used the method of quenching of coefficients. Consequently, this analysis must be considered as approximate, since actually there are studied dynamic properties of the flight vehicle under artificial conditions, when coefficients of equations of disturbed motion, beginning from moment of time t_0, remain constant. In actuality the coefficients of equations are variables, and the variability of coefficients can sometimes be the cause of such dynamic effects, which cannot be revealed within the framework of the method of quenched coefficients.

4.1. The Character of Free Disturbed Pitch and Yaw Motion

Free disturbed pitch motion of the flight vehicle is described by system of equations (1.13.5) with zero right sides:

$$\begin{aligned} m\frac{d\Delta V_y}{dt} + v_y\frac{d\Delta\vartheta}{dt} + c_{yy}\Delta V_y + c_{y\vartheta}\Delta\vartheta &= 0, \\ J_z\frac{d^2\Delta\vartheta}{dt^2} + \mu_z\frac{d\Delta\vartheta}{dt} + c_{\vartheta y}\Delta V_y + c_{\vartheta\vartheta}\Delta\vartheta &= 0. \end{aligned} \qquad (2.4.1)$$

Let us assume

$$\begin{aligned} \Delta V_y(t_0) = \Delta V_{y0}, \quad \Delta\vartheta(t_0) = \Delta\vartheta_0, \\ \Delta\dot{\vartheta}(t_0) = \Delta\dot{\vartheta}_0. \end{aligned} \qquad (2.4.2)$$

It is required to construct the solution of system (2.4.1) at initial conditions (2.4.2).

By considering the coefficients of equations as quenched, we will seek the particular solution of system (2.4.1) in the form

$$\Delta V_y = Ae^{\lambda t}, \quad \Delta\vartheta = Be^{\lambda t}. \qquad (2.4.3)$$

After substitution of (2.4.3) in (2.4.1) and reductions by common factor $e^{\lambda t}$ we obtain a system of algebraic equations

$$\begin{aligned} (m\lambda + c_{yy})A + (v_y\lambda + c_{y\vartheta})B &= 0, \\ c_{\vartheta y}A + (J_z\lambda^2 + \mu_z\lambda + c_{\vartheta\vartheta})B &= 0. \end{aligned} \qquad (2.4.4)$$

Homogeneous system of linear algebraic equations (2.4.4), except trivial solution $A = B = 0$, has nonzero solutions, if only the determinant of the system is equal to zero, i.e.,

$$\begin{vmatrix} m\lambda + c_{vv} & \nu_y\lambda + c_{v\vartheta} \\ c_{\vartheta v} & J_z\lambda^2 + \mu_z\lambda + c_{\vartheta\vartheta} \end{vmatrix} = 0. \qquad (2.4.5)$$

Having expanded the determinant, let us represent characteristic equation (2.4.5) in the form

$$D(\lambda) \equiv (m\lambda + c_{vv})(J_z\lambda^2 + \mu_z\lambda + c_{\vartheta\vartheta}) - \nu_y c_{\vartheta v}\lambda - c_{\vartheta v}c_{v\vartheta} = 0. \qquad (2.4.6)$$

Let us assume λ_1, λ_2, λ_3 - roots of characteristic equation. Usually all three roots are obtained different. In this case to each root λ_i ($i = 1, 2, 3$) there corresponds a particular solution of system (2.4.1):

$$\Delta V_{vi} = A_i e^{\lambda_i t}, \quad \Delta\vartheta_i = B_i e^{\lambda_i t}, \qquad (2.4.7)$$

where A_i, B_i - solution of algebraic system (2.4.4) when $\lambda = \lambda_i$.

General solution of system (2.4.1) is represented in the form of a linear combination of particular solutions:

$$\begin{aligned} \Delta V_v &= c_1 A_1 e^{\lambda_1 t} + c_2 A_2 e^{\lambda_2 t} + c_3 A_3 e^{\lambda_3 t}; \\ \Delta\vartheta &= c_1 B_1 e^{\lambda_1 t} + c_2 B_2 e^{\lambda_2 t} + c_3 B_3 e^{\lambda_3 t}. \end{aligned} \qquad (2.4.8)$$

Here c_1, c_2, c_3 - arbitrary constants, which are determined by initial conditions (2.4.2).

Coefficients of the characteristic equation are real quantitites. Therefore, during analysis of free disturbed motion the following two cases can be presented:

1) all three roots real;

2) one real root and two complex conjugate roots.

In the first case each particular solution (2.4.7) has an aperiodic character, functions ΔV_{yi}, ΔV_i in modulus increase in time if the corresponding root is positive, and are decreased if this root is negative. Free motion of the flight vehicle is made up of three aperiodic motions. If among the roots of characteristic polynomial there will be at least one positive root, then disturbance ΔV_y, $\Delta\vartheta$ at any nonzero initial conditions, beginning with a certain moment of time $t_1 (t_0 \le t_1 < \infty)$, will be ascending functions.

In the second case to the pair of complex conjugate roots $\lambda_2 = \alpha + i\beta$, $\lambda_3 = \alpha - i\beta$ corresponds particular solutions

$$\Delta\vartheta_{2,3} = c_2 B_2 e^{(\alpha+i\beta)t} + c_3 B_3 e^{(\alpha-i\beta)t}. \qquad (2.4.9)$$

The expression for $\Delta V_{y2,\,3}$ has analogous form [see (2.4.8)].

Since initial disturbances ΔV_{y0} and $\Delta\vartheta_0$ are represented by real numbers, then $c_2 B_2$ and $c_3 B_3$ will be complex conjugate values, i.e., if

$$c_2 B_2 = a - ib,$$

then

$$c_3 B_3 = a + ib.$$

Considering this and bearing in mind that $e^{\pm i\beta t} = \cos\beta t \pm$ $\pm \sin\beta t$, we will have

$$\begin{aligned}\Delta\vartheta_{2,3} &= (a - ib) e^{(\alpha+i\beta)t} + (a + ib) e^{(\alpha-i\beta)t} = \\ &= e^{\alpha t}[(a - ib)(\cos\beta t + i\sin\beta t) + (a + ib)(\cos\beta t - i\sin\beta t)] = \\ &= 2e^{\alpha t}(a\cos\beta t + b\sin\beta t).\end{aligned}$$

Hence

$$\Delta\vartheta_{3,4} = Be^{\alpha t}\sin(\beta t + \varphi), \tag{2.4.10}$$

where

$$B = 2\sqrt{a^2 + b^2}, \quad \varphi = \operatorname{arctg}\frac{a}{b}.$$

As we can see, to the pair of complex conjugate roots corresponds oscillatory motion with frequency β, phase ϕ and variable amplitude $Be^{\alpha t}$ ascending in exponent when $\alpha > 0$ and descending when $\alpha < 0$.

Roots of characteristic equation (2.4.6) are easily found in the case when flight is performed outside the atmosphere ($\rho = 0$). In this case $c_{\vartheta\vartheta} = c_{\vartheta y} = c_{yy} = 0$, and therefore

$$D(\lambda) \equiv m\lambda(J_z\lambda^2 + \mu_z\lambda).$$

Hence we find: $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = \frac{\mu_z}{J_z}$.

Roots of characteristic equation are determined simply when $\rho \neq 0$, but the aerodynamic focal point of the vehicle coincides with the center of mass ($x_F = 0$). In this case $c_{\vartheta\vartheta} = c_{\vartheta y} = 0$, and therefore

$$D(\lambda) \equiv (m\lambda + c_{yy})(J_z\lambda^2 + \mu_z\lambda).$$

Accordingly $\lambda_1 = -\frac{c_{yy}}{m}$, $\lambda_2 = 0$, $\lambda_3 = -\frac{\mu_z}{J_z}$.

Generally the roots of characteristic polynomial can be approximately determined by means of expansion of polynomial into factors in the following manner.

Let us introduce numbers z_1, z_2, z_3 into consideration and determine them so that the following equality would be fulfilled:

$$(m\lambda + c_{yy})(J_z\lambda^2 + \mu_z\lambda + c_{\vartheta\vartheta}) - v_y c_{\vartheta y}\lambda - c_{i\vartheta}c_{\vartheta y} =$$
$$= (m\lambda + c_{yy} + z_1)[J_z\lambda^2 + (\mu_z + z_2)\lambda + c_{\vartheta\vartheta} + z_3]. \tag{2.4.11}$$

By equating in this equality the coefficients with identical degrees of λ, we obtain system of equations

$$\begin{gathered}
J_z z_1 + m z_2 = 0, \\
\mu_z z_1 + c_{yy} z_2 + m z_3 + z_1 z_2 = -\nu_y c_{\vartheta y}, \\
c_{\vartheta\vartheta} z_1 + c_{yy} z_3 + z_1 z_3 = -c_{\vartheta y} c_{z\vartheta}
\end{gathered}$$

relative to z_1, z_2, z_3.

Let us rewrite the obtained system of nonlinear algebraic equations so:

$$\begin{gathered}
J_z z_1 + m z_2 = 0, \\
\mu_z z_1 + (c_{yy} + z_1) z_2 + m z_3 = -\nu_y c_{\vartheta y}, \\
c_{\vartheta\vartheta} z_1 + (c_{yy} + z_1) z_3 = -c_{\vartheta y} c_{z\vartheta}.
\end{gathered} \tag{2.4.12}$$

By solving (2.4.12) formally as a system of linear equations, coefficients of which depend on parameter $c_{yy} + z_1$, we will have

$$\begin{gathered}
z_1 = \frac{-m^2 c_{\vartheta y} c_{z\vartheta} + \nu_y m c_{\vartheta y} (c_{yy} + z_1)}{J_z (c_{yy} + z_1)^2 + m^2 c_{\vartheta\vartheta} - \mu_z m (c_{yy} + z_1)}, \\
z_2 = -\frac{J_z}{m} z_1, \\
z_3 = -\frac{1}{m}\left[\mu_z - \frac{J_z}{m}(c_{yy} + z_1)\right] z_1 - \frac{1}{m}\nu_y c_{\vartheta y}.
\end{gathered} \tag{2.4.13}$$

Thus, the problem is reduced to determination of z_1 from the first equation of system (2.4.13). Determination of z_1 is conveniently performed by the method of successive approximations. Let us assume

$$z_1^{[0]} = -c_{yy}. \tag{2.4.14}$$

By substituting the value of z_1 in the first equation (2.4.13), we obtain refined values of z_1:

$$z_1^{[1]} = -\frac{c_{\vartheta y} c_{y\vartheta}}{c_{\vartheta\vartheta}}. \tag{2.4.15}$$

By taking into account that $c_{\vartheta y} = x_F c_{yy}$, $c_{\vartheta\vartheta} = -x_F V_{cx} c_{yy}$, and $c_{y\vartheta} = -\left(P + \frac{1}{2} c_y^\alpha \rho S \frac{V_{cx}^2}{2}\right)$, we will have

$$z_1^{[1]} = \frac{c_{y\vartheta}}{V_{cx}} = -\left(c_{yy} + \frac{P - X_0}{V_{cx}}\right), \tag{2.4.16}$$

where $X_0 = \frac{1}{2} c_{x0} \rho S V_{cx}^2$ - drag force at angle of attack $\alpha = 0$.

The following approximation gives

$$z_1^{[2]} = \frac{-c_{\vartheta y} c_{y\vartheta} - \nu_y c_{\vartheta y} \frac{P - X_0}{m V_{cx}}}{J_z \left(\frac{P - X_0}{m V_{cx}}\right)^2 + \mu_z \frac{P - X_0}{m V_{cx}} + c_{\vartheta\vartheta}}. \tag{2.4.17}$$

With the exception of those sections of trajectory where the product of $\rho x_F V_{cx}$ is close to zero, usually, as numerical calculations show,

$$|c_{\vartheta\vartheta}| \gg J_z \left(\frac{P - X_0}{m V_{cx}}\right)^2 + \mu_z \frac{P - X_0}{m V_{cx}}. \tag{2.4.18}$$

Futhermore,

$$|c_{y\vartheta}| \gg \left|\nu_y \frac{P - X_0}{m V_{cx}}\right|.$$

Therefore, at condition (2.4.18) $z_1^{[2]} \approx z_1^{[1]}$ so that it is possible to be limited by approximation (2.4.16).

Finally, it is possible to disregard quantity z_3 in comparison with $c_{\vartheta\vartheta}$. Actually, if $x_F = 0$, then $c_{\vartheta y} = 0$ and, thus, $z_1 = 0$, $z_2 = 0$, $z_3 = 0$ (see (2.4.13)). If however, $x_F \neq 0$, then, as calculations show, usually $|c_{\vartheta\vartheta}| \gg |z_3|$.

Thus, the characterhistic polynomial can be approximately presented in the form

$$D(\lambda) \cong (m\lambda + c_{yy} + z_1)\left[J_z\lambda^2 + \left(\mu_z - \frac{J_z}{m} z_1\right)\lambda + c_{\theta\theta}\right], \quad (2.4.19)$$

where z_1 is determined by approximate formulas (2.4.16) or (2.4.17).

Polynomial (2.4.19) has such roots:

$$\lambda_1 = -\frac{1}{m}(c_{yy} + z_1), \quad (2.4.20)$$

$$\lambda_{2,3} = -\frac{1}{2}\left(\frac{\mu_z}{J_z} - \frac{z_1}{m}\right) \pm \sqrt{\frac{1}{4}\left(\frac{\mu_z}{J_z} - \frac{z_1}{m}\right)^2 - \frac{c_{\theta\theta}}{J_z}}.$$

If $\rho x_F V_{cx} = 0$, then $z_1 = 0$ and $\lambda_1 = -\frac{c_{yy}}{m} < 0$. If $\rho x_F V_{cx} \neq 0$ and condition (2.4.18) is fulfilled, then

$$c_{yy} + z_1 \approx -\frac{P - X_0}{V_{Cx}} < 0$$

and thus, $\lambda_1 \approx \frac{P - X_0}{mV_{cx}} > 0$. In any case, real root λ_1, as calculations show, is small in absolute value, so that slow aperiodic motion of the rocket corresponds to this root.

The two others roots (λ_2 and λ_3) depending on the sign of $c_{\theta\theta}$ can be both real and complex conjugate.

1. If $c_{\theta\theta} < 0$ ($x_F > 0$, i.e., the rocket is statically unstable), then roots λ_2 and λ_3 are real, one of them positive, and the other negative. To the positive root corresponds aperiodic motion, in the process of which the corresponding components of disturbances $\Delta\theta$, ΔV_y increase in modulus; to the negative root corresponds aperiodic motion, in the process of which the corresponding components of disturbances $\Delta\theta$, ΔV_y are decreased in modulus.

2. If $c_{\theta\theta} > 0$ ($x_F < 0$, the rocket is statically stable), then $\frac{\mu_z}{J_z} - \frac{z_1}{m} > 0$, since $z_1 < 0$. Therefore, when

$$\frac{1}{4}\left(\frac{\mu_z}{J_z}-\frac{z_1}{m}\right)^2-\frac{c_{\theta\theta}}{J_z}\geqslant 0, \tag{2.4.21}$$

then roots λ_2 and λ_3 are real and negative; corresponding motions of the vehicle carry a subsidence character.

Let us now assume

$$\frac{1}{4}\left(\frac{\mu_z}{J_z}-\frac{z_1}{m}\right)^2-\frac{c_{\theta\theta}}{J_z}<0. \tag{2.4.22}$$

In this case roots λ_2, λ_3 are conveniently presented in the form

$$\lambda_{2,3}=\frac{1}{T}\left(-\xi\pm i\sqrt{1-\xi^2}\right), \tag{2.4.23}$$

where

$$T=\sqrt{\frac{J_z}{c_{\theta\theta}}}, \tag{2.4.24}$$

$$\xi=\frac{1}{2\sqrt{J_z c_{\theta\theta}}}\left(\mu_z-\frac{J_z}{m}z_1\right). \tag{2.4.25}$$

In view of condition (2.4.22) $1-\xi^2>0$ and, thus λ_2, λ_3 - complex conjugate roots. Motion corresponding to these roots carries an oscillatory character. Since the real parts of roots λ_2, λ_3 are always negative, this oscillatory process is damped.

Relationship

$$\frac{\xi}{T}=\frac{1}{2}\left(\frac{\mu_z}{J_z}-\frac{z_1}{m}\right). \tag{2.4.26}$$

determines the rate of attenuation and is called *the damping* (or *attenuation*) *coefficient*, and ξ called *logarithmic decrement.*

The angular frequency of free oscillations is determined by the imaginary part of roots $\lambda_{2,\ 3}$:

$$\omega = \frac{\sqrt{1-\xi^2}}{T} = \sqrt{\frac{c_{\psi\psi}}{J_z} - \frac{1}{4}\left(\frac{\mu_z}{J_z} - \frac{z_1}{m}\right)^2}\ 1/\text{s}. \qquad (2.4.27)$$

The frequency of free oscillations is most affected by the degree of static stability, the measure of which is the coordinate of aerodynamic focal point x_F. The larger $|x_F|$ ($x_F < 0$), the greater is $c_{\psi\psi}$ and, thus, the greater is ω.

In the absence of damping $\left(\frac{\mu_z}{J_z} - \frac{z_1}{m} = 0\right)$ we have

$$\omega_c = \sqrt{\frac{c_{\psi\psi}}{J_z}} = \frac{1}{T}\ 1/\text{s}. \qquad (2.4.28)$$

The frequency of free oscillations in the absence of damping ω_c is called *the frequency of natural oscillations* of the vehicle. The frequency of natural oscillations in Hz is equal to:

$$f = \frac{\omega_c}{2\pi}. \qquad (2.4.29)$$

The period of natural oscillations is determined by formula

$$T_c = \frac{2\pi}{\omega_c} = 2\pi T. \qquad (2.4.30)$$

Free disturbed yawing motion of the rocket is described by equations:

$$\begin{aligned} &m\frac{d\Delta V_z}{dt} + v_z\frac{d\Delta\psi}{dt} + c_{zz}\Delta V_z + c_{z\psi}\Delta\psi = 0, \\ &J_y\frac{d^2\Delta\psi}{dt^2} + \mu_y\frac{d\Delta\psi}{dt} + c_{\psi z}\Delta V_z + c_{\psi\psi}\Delta\psi = 0 \end{aligned} \qquad (2.4.31)$$

at initial conditions:

$$\Delta V_z(t_0) = \Delta V_{z0}, \quad \Delta\psi(t_0) = \Delta\psi_0, \quad \Delta\dot{\psi}(t_0) = \Delta\dot{\psi}_0.$$

The system of equations (2.4.31) with accuracy to within designations does not differ from system (2.4.1), which describes free disturbed pitching motion of the rocket, so that the entire preceding analysis of equations (2.4.1) can be directly transfered to equations (2.4.31).

4.2. Transfer Functions of the Vehicle During Pitch and Yaw

The method of quenching of coefficients makes possible the utilization of convenient methods of investigation of steady-state systems of automatic control when designing and investigating the control systems of rockets. The application of these methods is based on knowledge of transfer functions or frequency characteristics of the rocket as the object of control.

As is known, the transfer function of linear element of the automatic control system is the ratio of Laplace transform of output quantity to Laplace transform of input quantity under zero initial conditions [24]. For the rocket as the element of automatic control system, the input quantities are deflections of controls, and also disturbing forces and moments; the output quantities are increments of motion parameters ($\Delta\vartheta$, $\Delta\gamma$, ΔV_y and so forth).

Transfer functions can be obtained from the equations of disturbed motion of the rocket by means of their Laplace transformation with quenched coefficients.

For Laplace transformation of equations of disturbed pitching motion of the rocket under initial conditions.

$$\Delta V_y(t_0)=\Delta\vartheta(t_0)=\Delta\dot{\vartheta}(t_0)=0 \qquad (2.4.32)$$

let us multiply the left and right sides of equations (1.13.5) by e^{-pt} and integrate them with respect to t_0 within limits from t_0 to ∞. Then, by designating through $V_y(p)$ $\vartheta(p)$, $\delta_\vartheta(p)$, $F_y(p)$, $M_z(p)$, respectively the Laplace transforms of function ΔV_y, $\Delta\vartheta$, $\Delta\delta_\vartheta$, ΔF_y, ΔM_z:

$$V_y(p)=\int_0^\infty \Delta V_y e^{-p\tau}d\tau,\quad \vartheta(p)=\int_0^\infty \Delta\vartheta e^{-p\tau}d\tau,\quad \delta_\theta(p)=\int_0^\infty \Delta\delta_\theta e^{-p\tau}d\tau,$$

$$F_y(p)=\int_0^\infty \Delta F_y e^{-p\tau}d\tau,\quad M_z(p)=\int_0^\infty \Delta M_z e^{-p\tau}d\tau \quad (\tau=t-t_0)$$

and considering that at condition (2.4.32)

$$\int_{t_0}^\infty \Delta\vartheta_y e^{-pt}dt=e^{-pt_0}\int_0^\infty \Delta\vartheta_y e^{-p\tau}d\tau=e^{-pt_0}\vartheta(p),$$

$$\int_{t_0}^\infty \frac{d\Delta\vartheta}{dt}e^{-pt}dt=\left[\Delta\vartheta e^{-pt}\right]_{t_0}^\infty+p\int_{t_0}^\infty \Delta\vartheta_y e^{-pt}dt=e^{-pt_0}p\vartheta(p),$$

$$\int_{t_0}^\infty \frac{d^2\Delta\vartheta}{dt^2}e^{-pt}dt=\left[\frac{d\Delta\vartheta}{dt}e^{-pt}\right]_{t_0}^\infty+p\int_{t_0}^\infty \frac{d\Delta\vartheta}{dt}e^{-pt}dt=e^{-pt_0}p^2\vartheta(p)$$

and in exactly the same manner

$$\int_{t_0}^\infty \Delta V_y e^{-pt}dt=e^{-pt_0}V_y(p),\quad \int_{t_0}^\infty \frac{d\Delta V_y}{dt}e^{-pt}dt=e^{-pt_0}pV_y(p),$$

$$\int_{t_0}^\infty \Delta\delta_\theta e^{-pt}dt=e^{-pt_0}\delta_\theta(p),\quad \int_{t_0}^\infty \Delta F_y e^{-pt}dt=e^{-pt_0}F_y(p),$$

$$\int_{t_0}^\infty \Delta M_z e^{-pt}dt=e^{-pt_0}M_z(p),$$

we obtain

$$\begin{aligned}&(mp+c_{yy})V_y(p)+(v_y p+c_{y\vartheta})\vartheta(p)=c_{y\delta}\delta_\theta(p)+F_y(p),\\&c_{\vartheta y}V_y(p)+(J_z p^2+\mu_z p+c_{\vartheta\vartheta})\vartheta(p)=c_{\vartheta\delta}\delta_\theta(p)+M_z(p).\end{aligned}\qquad(2.4.33)$$

By solving system of algebraic equations (2.4.33) relative to $\vartheta(p)$ and $V_y(p)$, we obtain

$$\vartheta(p)=\frac{\begin{vmatrix}mp+c_{yy} & c_{y\delta}\delta_\theta(p)+F_y(p)\\ c_{\vartheta y} & c_{\vartheta\delta}\delta_\theta(p)+M_z(p)\end{vmatrix}}{(mp+c_{yy})(J_z p^2+\mu_z p+c_{\vartheta\vartheta})-v_y c_{\vartheta y}p-c_{y\vartheta}c_{\vartheta y}},\qquad(2.4.34)$$

$$V_y(p)=\frac{\begin{vmatrix} c_{y\delta}\delta_\theta(p)+F_y(p) & \nu_y p + c_{y\theta} \\ c_{\theta\delta}\delta_\theta(p)+M_z(p) & J_z p^2+\mu_z p + c_{\theta\theta} \end{vmatrix}}{(mp+c_{yy})(J_z p^2+\mu_z p+c_{\theta\theta})-\nu_y c_{\theta y} p - c_{y\theta}c_{\theta y}}. \qquad (2.4.35)$$

Having expanded the determinants, standing in the numerators, we will have

$$\theta(p)=W_\theta^\delta(p)\delta_\theta(p)+W_\theta^F(p)F_y(p)+W_\theta^M(p)M_z(p), \qquad (2.4.36)$$

$$V_y(p)=W_V^\delta(p)\delta_\theta(p)+W_V^F(p)F_y(p)+W_V^M(p)M_z(p). \qquad (2.4.37)$$

Here $W_\theta^\delta(p)$, $W_V^\delta(p)$, $W_\theta^F(p)$, $W_V^F(p)$, $W_\theta^M(p)$, $W_V^M(p)$ - transfer functions of the rocket, which correspond to different pairs of input and output signals.

In expanded form, specifically,

$$W_\theta^\delta(p)=\frac{(mp+c_{yy})c_{\theta\delta}-c_{\theta y}c_{y\delta}}{(mp+c_{yy})(J_z p^2+\mu_z p+c_{\theta\theta})-\nu_y c_{\theta y} p - c_{y\theta}c_{\theta y}}. \qquad (2.4.38)$$

In the denominator of transfer functions stands characteristic polynomial of system of equations (1.13.5). The polynomial, as was shown above, can be approximately represented in the form (2.4.19). By replacing the characteristic polynomial by its approximate expression, we will have

$$W_\theta^\delta(p)=\frac{(mp+c_{yy})c_{\theta\delta}-c_{\theta y}c_{y\delta}}{(mp+c_{yy}+z_1)\left[J_z p^2+\left(\mu_z-\frac{J_z}{m}z_1\right)p+c_{\theta\theta}\right]}. \qquad (2.4.39)$$

Transfer function (2.4.39) is conveniently presented so:

$$W_\theta^\delta(p)=\frac{k(\tau_1 p+1)}{(\tau_2 p+1)(T^2p^2+2T\xi p+1)}. \qquad (2.4.40)$$

Here $k = \frac{c_{yy}c_{\theta\delta}-c_{\theta y}c_{y\delta}}{(c_{yy}+z_1)c_{\theta\theta}}$ - transfor coefficient of the vehicle; τ_1, τ_2, T - time constants, where

$$\tau_1=\frac{mc_{\vartheta\delta}}{c_{y\vartheta}c_{\vartheta\delta}-c_{\vartheta y}c_{y\delta}},\quad \tau_2=\frac{m}{c_{yy}+x_1},$$

and T is determined by formula (2.4.24).

Coefficient ξ (logarithmic decrement), as above, is represented by formula (2.4.25).

By using the transfer function (2.4.40), it is possible to construct the transient function, representing the reaction of the vehicle for angle $\Delta\vartheta$ to deflections of controls.

With gradual deflection of controls

$$\delta_\vartheta(p)=\frac{\Delta\delta_\vartheta}{p}.$$

Considering this, from (2.4.40) we find

$$\frac{\vartheta(p)}{\Delta\delta_\vartheta}=\frac{k(\tau_1 p+1)}{p(\tau_2 p+1)(T^2p^2+2T\xi p+1)}. \qquad (2.4.41)$$

Hence, by Laplace inverse transformation one can determine transient function $\Delta\vartheta(t)/\Delta\delta_\vartheta$, which represents the law of change of angle $\Delta\vartheta$ with gradual deflection of controls and initial conditions (2.4.32).

By the same way it is possible to construct transient function $\Delta V_y(t)/\Delta\delta_\vartheta$, representing the law of change of disturbance of velocity ΔV_y with gradual deflection of controls and initial conditions (2.4.32).

Evident expressions of transient functions will be listed in the next point after some simplifications of equations of disturbed motion.

Transfer functions of the rocket in terms of yawing are analogous in form to transform functions of the rocket in terms of pitching, Thus, for instance,

$$W_{\psi}^{\delta}(p)=\frac{\psi(p)}{\delta_{\psi}(p)}=\frac{k(\tau_1 p+1)}{(\tau_2 p+1)(T^2p^2+2T\xi p+1)},$$

only now

$$k=\frac{c_{zz}c_{\psi\delta}-c_{\psi z}c_{z\delta}}{(c_{zz}+z_1)\,c_{\psi\psi}},\qquad \tau_1=\frac{mc_{\psi\delta}}{c_{zz}c_{\psi\delta}-c_{\psi z}c_{z\delta}},$$

$$\tau_2=\frac{m}{c_{zz}+z_1},\quad T=\sqrt{\frac{J_y}{c_{\psi\psi}}},\quad \xi=\frac{1}{2\sqrt{J_y c_{\psi\psi}}}\left(\mu_y-\frac{J_y}{m}z_1\right),$$

and z_1 is determined by approximate formulas

$$z_1^{[1]}=-\left(c_{zz}+\frac{P-X_0}{V_{Cx}}\right)$$

and

$$z_1^{[2]}=-\frac{c_{\psi z}c_{z\psi}+\nu_z c_{\psi z}\dfrac{P-X}{mV_{Cx}}}{J_y\left(\dfrac{P-X_0}{mV_{Cx}}\right)^2+\mu_y\dfrac{P-X_0}{mV_{Cx}}+c_{\psi\psi}}.$$

4.3. Simplified Equations of Disturbed Motion and Transfer Functions in Terms of Pitch and Yaw

Transient process, which begins with deflection of controls, has the following characteristic.

In the first seconds after deflection of controls the variation of velocity ΔV_y, which was equal to zero at initial moment t_0, changes very slowly, remaining small in magnitude. Therefore, at the first stage of disturbed motion the effect of variation of velocity of the center of mass of the vehicle on its rotation around the center of mass is little. Considering this, it is possible to simplify equations of disturbed motion (1.13.5), assuming in the second equation of system (1.13.5) that $\Delta V_y = 0$ and rejecting the first equation. Then the disturbed motion of the rocket at the first stage will be approximately represented by differential equation

$$J_z \frac{d^2\Delta\vartheta}{dt^2} + \mu_z \frac{d\Delta\vartheta}{dt} + c_{\vartheta\vartheta}\Delta\vartheta = c_{\vartheta\delta}\Delta\delta_\vartheta + \Delta M_z. \tag{2.4.42}$$

The expediency of such simplification of system (1.13.5) is dictated still by the fact that inasmuch as we use the method of quenching of coefficients, we can count on the reliability of obtained conclusions applicable only to small time segments.

With the shown simplification of equations transfer functions $W_\vartheta^\delta(p)$ and W_ϑ^M are represented in the form

$$W_\vartheta^\delta(p) = \frac{c_{\vartheta\delta}}{J_z p^2 + \mu_z p + c_{\vartheta\vartheta}}, \tag{2.4.43}$$

$$W_\vartheta^M(p) = \frac{1}{J_z p^2 + \mu_z p + c_{\vartheta\vartheta}}. \tag{2.4.44}$$

Assuming

$$T = \sqrt{\frac{J_z}{c_{\vartheta\vartheta}}}, \quad \xi = \frac{\mu_z}{2\sqrt{J_z c_{\vartheta\vartheta}}}, \quad k = \frac{c_{\vartheta\delta}}{c_{\vartheta\vartheta}}, \quad k_M = \frac{1}{c_{\vartheta\vartheta}}, \tag{2.4.45}$$

we will have

$$W_\vartheta^\delta(p) = \frac{k}{T^2p^2 + 2T\xi p + 1}, \tag{2.4.46}$$

$$W_\vartheta^M(p) = \frac{k_M}{T^2p^2 + 2T\xi p + 1}. \tag{2.4.47}$$

After analogous simplification of equations of disturbed yawing motion we will have

$$J_y \frac{d^2\Delta\psi}{dt^2} + \mu_y \frac{d\Delta\psi}{dt} + c_{\psi\psi}\Delta\psi = c_{\psi\delta}\Delta\delta + \Delta M_y. \tag{2.4.48}$$

Hence we obtain the following expressions for transfer functions:

$$W_{\psi}^{\delta}(p)=\frac{c_{\psi\delta}}{J_{y}p^{2}+\mu_{y}p+c_{\psi\psi}}, \qquad (2.4.49)$$

$$W_{\psi}^{M}(p)=\frac{1}{J_{y}p^{2}+\mu_{y}p+c_{\psi\psi}}. \qquad (2.4.50)$$

Let us investigate the reaction of the vehicle in terms of angle of pitch to gradual deflection of controls.

With gradual deflection of controls

$$\vartheta(p)=W_{\vartheta}^{\delta}(p)\delta_{\vartheta}(p)=\frac{c_{\vartheta\delta}\Delta\delta_{\vartheta}}{p\,(J_{z}p^{2}+\mu_{z}p+c_{\vartheta\vartheta})}. \qquad (2.4.51)$$

Hence, by factoring trinomial

$$D(p)=J_{z}p^{2}+\mu_{z}p+c_{\vartheta\vartheta}$$

we obtain

$$\frac{\vartheta(p)}{\Delta\delta_{\vartheta}}=\frac{c_{\vartheta\delta}}{J_{z}}\,\frac{1}{p\,(p-p_{1})\,(p-p_{2})}. \qquad (2.4.52)$$

Here p_1, p_2 - roots of trinomial $D(p)$:

$$p_{1}=-\frac{1}{2}\,\frac{\mu_{z}}{J_{z}}+\sqrt{\frac{\mu_{z}^{2}}{4J_{z}^{2}}-\frac{c_{\vartheta\vartheta}}{J_{z}}},\quad p_{2}=-\frac{1}{2}\,\frac{\mu_{z}}{J_{z}}-\sqrt{\frac{\mu_{z}^{2}}{4J_{z}^{2}}-\frac{c_{\vartheta\vartheta}}{J_{z}}}. \qquad (2.4.53)$$

Let us apply Laplace inverse transformation to (2.4.52); we obtain transient function

$$\frac{\Delta\vartheta(t)}{\Delta\delta_{\text{в}}}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{c_{\delta\text{в}}}{J_z}\frac{e^{pt}}{p(p-p_1)(p-p_2)}dp. \qquad (2.4.54)$$

Here c - arbitrary real number, exceeding the real parts of roots (2.4.53).

The right side of relationship (2.4.54) is equal to the sum of residues of integrands. Residue of function $f(p)$, corresponding to pole p_i[1] of multiplicity κ, as is known, is determined so:

$$R_i=\operatorname*{Res}_{p=p_i}f(p)=\frac{1}{(\kappa-1)!}\lim_{p\to p_i}\frac{d^{\kappa-1}[f(p)(p-p_i)^{\kappa}]}{dp^{\kappa-1}}. \qquad (2.4.55)$$

Specifically, if p_i is a simple pole of function $f(p)$, then

$$R_i\equiv\operatorname*{Res}_{p=p_i}f(p)=\lim_{p\to p_i}[f(p)(p-p_i)].$$

Let us examine the following possible cases.

Statically unstable flight vehicle ($x_F > 0$). In this instance $c_{\alpha\text{в}} < 0$, therefore $p_1 > 0$, and $p_2 < 0$. The integrand in expression (2.4.54) has three simple poles: p_1, p_2, 0. Corresponding resudies of integrand are equal to:

$$R_1=\frac{c_{\delta\text{в}}}{J_z}\frac{e^{p_1t}}{p_1(p_1-p_2)},\quad R_2=\frac{c_{\delta\text{в}}}{J_z}\frac{e^{p_2t}}{p_2(p_2-p_1)},\quad R_3=\frac{c_{\delta\text{в}}}{J_z}\frac{1}{p_1p_2}.$$

---

[1]Point p^i is κ-multiple pole of function $f(p)$, if it is κ-multiple zero of function $\frac{1}{f(p)}\equiv\phi(p)$. In turn, κ-multiple zero of function $\phi(p)$ is characterized by relationships

$$\varphi(p_i)=0,\quad \frac{d\varphi(p_i)}{dp}=0,\ldots,\quad \frac{d^{\kappa-1}\varphi(p_i)}{dp^{\kappa-1}}=0,\quad \frac{d^{\kappa}\varphi(p_i)}{dp^{\kappa}}\neq 0.$$

Since $p_1 p_2 = \frac{c_{\theta\theta}}{J_z}$, residue R_3 is exactly equivalent to transfer coefficient k (see (2.4.45)).

Considering this, we obtain

$$\frac{\Delta\vartheta(t)}{\Delta\delta_0} = \sum_i R_i = \frac{c_{\theta\delta}}{J_z} \frac{e^{p_1 t}}{p_1(p_1 - p_2)} + \frac{c_{\theta\delta}}{J_z} \frac{e^{p_2 t}}{p_2(p_2 - p_1)} + k. \qquad (2.4.56)$$

Transient function (2.4.56) determines the law of change of disturbance $\Delta\vartheta(t)$ with gradual deflection of controls and zero initial values of disturbances $\Delta\vartheta$ and $\Delta\dot{\vartheta}$. Since $p_1 > 0$, and $p_2 < 0$, the first component of transient function increases with passage of time, and the second component - diminishes, approaching zero.

Statically neutral flight vehicle ($x_F = 0$). In this instance $c_{\theta\theta} = 0$, therefore $p_1 = 0$, and $p_2 = -\frac{\mu_z}{J_z}$.

The integrand has double pole 0 and simple pole p_2. Corresponding residues are equal to:

$$R_1 = -\frac{c_{\theta\delta}}{J_z} \frac{1 + tp_2}{p_2^2}, \qquad R_2 = \frac{c_{\theta\delta}}{J_z} \frac{e^{p_2 t}}{p_2^2}.$$

This means

$$\frac{\Delta\vartheta(t)}{\Delta\delta_0} = \frac{c_{\theta\delta}}{J_z} \frac{e^{p_2 t} - 1}{p_2^2} - \frac{c_{\theta\delta}}{J_z} \frac{t}{p_2}. \qquad (2.4.57)$$

As we can see, in this instance the transient function has a component, increasing in modulus with passage of time, however, now this rise proceeds according to linear law.

Statically stable flight vehicle ($x_F < 0$). In this instance $c_{\theta\theta} > 0$, and the following variants are possible.

a) $\frac{\mu_z^2}{J_z^2} > \frac{c_{\theta\theta}}{J_z}$. In this case roots p_1 and p_2 are negative and different. Transient function $\frac{\Delta\theta(t)}{\Delta\delta_0}$ is again represented by formula (2.4.56), but now both the first and second of its components are descending functions with respect to modulus; with increase of t disturbance $\Delta\theta(t)$ approaches a steady value, equal to $k\Delta\delta_0$;

b) $\frac{\mu_z^2}{4J_z^2} > \frac{c_{\theta\theta}}{J_z}$. From (2.4.53) we have $p_1 = p_2 = -\frac{1}{2}\frac{\mu_z}{J_z}$. For transient function we obtain the following expression:

$$\frac{\Delta\theta(t)}{\Delta\delta_0} = \frac{c_{\theta\delta}}{J_z}\left[\frac{1}{p_2^2} - \frac{1 - tp_2}{p_2^2} e^{p_2 t}\right]. \qquad (2.4.58)$$

With increase of $t\ \Delta\theta(t) \to \frac{c_{\theta\delta}\Delta\delta_0}{J_z p_2^2} = \frac{4J_z c_{\theta\delta}}{\mu_z^2}\Delta\delta_0$;

c) $\frac{\mu_z^2}{4J_z^2} < \frac{c_{\theta\theta}}{J_z}$. In this instance roots p_1, p_2 - complex conjugate:

$$p_1 = -\frac{1}{2}\frac{\mu_z}{J_z} + i\sqrt{\frac{c_{\theta\theta}}{J_z} - \frac{\mu_z^2}{4J_z^2}}, \quad p_2 = -\frac{1}{2}\frac{\mu_z}{J_z} - i\sqrt{\frac{c_{\theta\theta}}{J_z} - \frac{\mu_z^2}{4J_z^2}}.$$

In notations (2.4.45) we will have

$$p_1 = -\frac{\xi}{T} + i\frac{\sqrt{1-\xi^2}}{T}, \quad p_2 = -\frac{\xi}{T} - i\frac{\sqrt{1-\xi^2}}{T}. \qquad (2.4.59)$$

Since $p_1 \neq p_2$, $p_1 \neq 0$, $p_2 \neq 0$ and, thus, poles of integrand are simple, the transient function is determined by formula (2.4.56) at values p_1 and p_2, determined by formulas (2.4.59).

By substituting values of p_1 and p_2 in (2.4.56), after some conversions we obtain

$$\frac{\Delta\vartheta(t)}{\Delta\delta_\text{в}}=k\left[1-e^{-\frac{\xi}{T}t}\left(\cos\frac{\sqrt{1-\xi^2}}{T}t+\frac{\xi}{\sqrt{1-\xi^2}}\sin\frac{\sqrt{1-\xi^2}}{T}t\right)\right]. \qquad (2.4.60)$$

The last relationship can be reduced to the form

$$\frac{\Delta\vartheta(t)}{\Delta\delta_\text{в}}=k\left[1-\frac{e^{-\frac{\xi}{T}}}{\sqrt{1-\xi^2}}\cos\left(\frac{\sqrt{1-\xi^2}}{T}\,t-\varphi\right)\right], \qquad (2.4.61)$$

where

$$\operatorname{tg}\varphi=\frac{\xi}{\sqrt{1-\xi^2}}.$$

From (2.4.61) it is evident that the transient process is oscillatory, moreover oscillations are damped, since $-\frac{\xi}{T} = -\frac{1}{2}\frac{\mu_z}{J_z} < 0$. The greater the damping coefficient ξ/T the faster the oscillatory process attenuates. With the passage of time disturbance $\Delta\vartheta(t)$ approaches a steady value, equal to $k\Delta\delta_\text{в}$. Transfer coefficient k in this case can be treated as the relationship of steady value of output quantity $\Delta\vartheta$ the quantity of input signal $\Delta\delta_\text{в}$.

4.4. Transfer Functions of the Vehicle in Terms of Roll

By converting equations of disturbed rolling motion (1.13.8) according to Laplace under zero initial conditions, we obtain

$$(J_x p^2+\mu_x p)\gamma(p)=c_{\gamma\delta}\delta_\gamma(p)+M_x(p). \qquad (2.4.62)$$

where

$$\gamma(p)=\int_0^\infty \Delta\gamma e^{-pt}dt, \quad \delta_\gamma(p)=\int_0^\infty \Delta\delta_\gamma e^{-pt}dt, \quad M_x(p)=\int_0^\infty \Delta M_x e^{-pt}dt.$$

Hence

$$\gamma(p)=\frac{c_{\gamma\delta}}{p(J_x p+\mu_x)}\delta_\gamma(p)+\frac{1}{p(J_x p+\mu_x)}M_x'(p). \qquad (2.4.63)$$

Coefficients at $\delta_\gamma(p)$ and $M_x(p)$ are transfer functions of the vehicle in terms of roll W_γ^δ and W_γ^M. These transfer functions can be represented in the form

$$W_\gamma^\delta(p)=\frac{k}{p(Tp+1)}, \quad W_\gamma^M(p)=\frac{k_M}{p(Tp+1)}. \qquad (2.4.64)$$

Here $k = \frac{c_{\gamma\delta}}{\mu_x}$, $k_M = \frac{1}{\mu_x}$ - transfer coefficients, $T = \frac{J_x}{\mu_x}$ - time constant.

Reaction of the vehicle to gradual deflection of controls with respect to roll is described by transient function $\Delta\gamma(t)/\Delta\delta_\gamma$ which can be obtained by using transfer function $W_\gamma^\delta(p)$, by means of Laplace inverse transformation. With gradual deflection of controls

$$\frac{\gamma(p)}{\Delta\delta_\gamma}=\frac{1}{p}W_\gamma^\delta(p)=\frac{k}{p^2(Tp+1)}.$$

Hence

$$\frac{\Delta\gamma(t)}{\Delta\delta_\gamma}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{k}{p^2(Tp+1)}e^{pt}dp \qquad (2.4.65)$$

and after calculation of integral

$$\frac{\Delta\gamma(t)}{\Delta\delta_\gamma}=k\left[t-T\left(1-e^{-\frac{t}{T}}\right)\right]. \qquad (2.4.66)$$

By differentiating (2.4.66), we obtain the expression for angular velocity of the vehicle with respect to roll as a function of time:

$$\frac{\Delta\dot{\gamma}(t)}{\Delta\delta_\gamma}=k\left(1-e^{-\frac{t}{T}}\right). \qquad (2.4.67)$$

With increase of t the angular velocity $\Delta\dot{\gamma}$, changing aperiodically approaches the value of this velocity in steady state, equal to $k\Delta\delta_\gamma$. The rate of passage of this aperiodic transient process depends on the value of time constant T: the greater T is, the slower this process occurs, and vice versa.

Transfer coefficient k can be treated as the ratio of steady value of output quantity $\Delta\gamma$ to the quantity of output signal $\Delta\delta_\gamma$.

4.5. Transfer Functions of the Rocket in Longitudinal Motion

Laplace transformation of equation of disturbed longitudinal motion of the rocket (1.13.6) under zero initial conditions leads to equality

$$(mp+c_{xx})V_x(p)=c_{x\delta}\delta_x(p)+F_x(p), \qquad (2.4.68)$$

where $V_x(p)$, $\delta_x(p)$, $F_x(p)$ - Laplace transforms of functions ΔV_x, $\Delta\delta_x$, ΔF_x.

From (2.4.68) we find transfer functions of the rocket in longitudinal motion:

$$W_V^\delta(p)=\frac{c_{x\delta}}{mp+c_{xx}}, \quad W_V^F(p)=\frac{1}{mp+c_{xx}}. \qquad (2.4.69)$$

In more convenient form

$$W_V^{\delta}(p) = \frac{k}{Tp+1}; \quad W_V^{F}(p) = \frac{k_F}{Tp+1}. \qquad (2.4.70)$$

Here $k = \frac{a_{x\delta}}{a_{xx}}$, $k_F = \frac{1}{a_{xx}}$ - transform coefficients; $T = \frac{m}{a_{xx}}$ - time constant of the vehicle.

With gradual deflection of input signal the transient conditions process is started, which is described by transient function

$$\frac{\Delta V_x(t)}{\Delta \delta_x} = k\left(1 - e^{-\frac{t}{T}}\right). \qquad (2.4.71)$$

As we can see, the dynamic properties of the vehicle with respect to ΔV_x coincide with properties of aperiodic link. With increase of t output signal ΔV_x approaches in exponent to the steady value of disturbance ΔV_x. The rate of passage of this aperiodic process depends on the quantity of time constant T: the greater T is, the slower this process occurs.

From (2.4.71) it is evident that transfer coefficients k can be treated as the ratio of steady value of output quantity ΔV_x to the value of input quantity $\Delta \delta_x$.

The transfer coefficient k_F has analogous physical meaning: it is equal to the ratio of steady value of output quantity ΔV_x to the value of input quantity ΔF_x.

4.6. Frequency Characteristics of the Rocket

Processes in the control system of the rocket are frequently difficult to describe by simple differential equations, rather convenient for analysis. Therefore, in practice there are widely applied methods of investigations which use frequency characteristics of the system. Frequency characteristics of elements of the system, equations of which are unknown, can be obtained experimentally. The application of frequency methods is especially expedient for systems of high order, since in this instance the solutions of differential equations and algebraic criteria of stability become very bulky.

For application of frequency methods it is necessary to know the frequency characteristics of the rocket as an object of control and frequency characteristics of the control system. Frequency characteristics of the rocket as an object of control can be obtained from expressions for transfer functions of the rocket by means of replacement of parameter p by $i\omega$. However, for greater clarity we obtain frequency characteristics directly from equations of disturbed motion. Concerning the frequency characteristics of the control system, they are usually determined experimentally.

Frequency characteristics of the rocket represent its forced motion with deflection of controls according to harmonic law and the absence of disturbing forces.

Let us examine *pitching motion*. For construction of frequency characteristics in equations (1.13.5) let us assume

$$\Delta F_y = 0, \quad \Delta M_z = 0, \quad \Delta \delta_\varphi = e^{i\omega t} \tag{2.4.72}$$

and considering that coefficients of equations are quenched, we will seek unknown functions ΔV_x, $\Delta\theta$ in the form

$$\Delta V_y = V_y e^{i\omega t}, \quad \Delta\vartheta = \vartheta e^{i\omega t}. \tag{2.4.73}$$

After substitution of (2.4.72) and (2.4.73) in equations (1.13.5) we obtain system of algebraic equations

$$\begin{aligned} &(c_{yy} + i\omega m) V_y + (c_{y\vartheta} + i\omega \nu_y)\vartheta = c_{y\delta}, \\ &c_{\vartheta y} V_y + (c_{\vartheta\vartheta} + i\mu_z\omega - \omega^2 J_z)\vartheta = c_{\vartheta\delta}. \end{aligned} \tag{2.4.74}$$

Let us solve system (2.4.74) relative to θ. We obtain frequency transfer function

$$\theta=\frac{(c_{yy}c_{\vartheta\delta}-c_{\vartheta y}c_{y\delta})+i\omega m c_{\vartheta\delta}}{c_{yy}(c_{\vartheta\vartheta}-\omega^2 J_z)-\omega^2\mu_z m-c_{y\vartheta}c_{\vartheta y}+i\left[\omega m(c_{\vartheta\vartheta}-\omega^2 J_z)+\mu_z\omega c_{yy}-c_{\vartheta y}\omega\nu_y\right]}. \qquad (2.4.75)$$

By comparing it with (2.4.38), we can see that

$$\theta(i\omega)=W_{\vartheta}^{\delta}(p)\big|_{p=i\omega}.$$

Let us write θ in the form

$$\theta=\theta_1(\omega)+i\theta_2(\omega), \qquad (2.4.76)$$

where θ_1, θ_2 - real and imagir.ary components of frequency transfer function r pectively, determined by formulas:

$$\theta_1=\frac{c_1(c_{yy}c_{\vartheta\delta}-c_{\vartheta y}c_{y\delta})+c_2\omega m c_{\vartheta\delta}}{c_1^2+c_2^2}, \qquad (2.4.77)$$

$$\theta_2=\frac{c_1\omega m c_{\vartheta\delta}-c_2(c_{yy}c_{\vartheta\delta}-c_{\vartheta y}c_{y\delta})}{c_1^2+c_2^2}.$$

Here

$$c_1=c_{yy}(c_{\vartheta\vartheta}-\omega^2 J_z)-\omega^2\mu_z m-c_{y\vartheta}c_{\vartheta y},$$
$$c_2=\omega m(c_{\vartheta\vartheta}-\omega^2 J_z)+\omega\mu_z c_{yy}-\nu_y\omega c_{\vartheta y}. \qquad (2.4.78)$$

Frequency transfer function can be presented thus:

$$\theta=k(\omega)e^{i\varphi(\omega)} \qquad (2.4.79)$$

where

$$k(\omega)=|\theta(i\omega)|=\sqrt{\theta_1^2(\omega)+\theta_2^2(\omega)} \qquad (2.4.80)$$

- amplitude frequency characteristics of the object (amplitification factor);

$$\varphi=\arg\theta(i\omega) \qquad (2.4.81)$$

- phase frequency characteristic of the object (phase shift).

Accordingly, function $\Delta\vartheta$, being the output quantity for the object as the linear link of automatic control system (and input quantity for automatic stabilization control), is represented by formula

$$\Delta\vartheta = k(\omega)e^{i[\omega t+\varphi(\omega)]} \tag{2.4.82}$$

As we can see, harmonic oscillations of controls cause harmonic oscillations of output quantity of the object with the same frequency, but with changed amplitude and phase. These changes are characterized by amplification factor $k(\omega)$ and by phase shift $\phi(\omega)$.

Having substituted (2.4.77) in (2.4.80), we will have

$$k(\omega) = \frac{|c_{\vartheta\delta}|}{\sqrt{c_1^2+c_2^2}}\sqrt{\left(1-x_p\frac{c_{y\delta}}{c_{\vartheta\delta}}\right)^2 c_{yy}^2+\omega^2 m}. \tag{2.4.83}$$

If control is carried out by means of thrust misalignment, the $c_{y\delta} = 0$, and in this instance

$$k(\omega) = \frac{|c_{\vartheta\delta}|}{\sqrt{c_1^2+c_2^2}}\sqrt{c_{yy}^2+\omega^2 m^2}. \tag{2.4.84}$$

During control with the aid of air or jet vanes or by means of turning of sustainer engines $c_{\vartheta\delta} = -L_p c_{y\delta}$. In this case formula (2.4.83) assumes the form

$$k(\omega) = \frac{|c_{\vartheta\delta}|}{\sqrt{c_1^2+c_2^2}}\sqrt{\left(1+\frac{x_p}{L_p}\right)^2 c_{yy}^2+\omega^2 m}. \tag{2.4.85}$$

Phase shift $\phi(\omega)$ is represented as the difference of arguments of the numerator and denominator of frequency transfer function. Let us turn to examination of rolling *motion of the rocket*. In equation of disturbed motion (1.13.8) let us assume

$$\Delta M_x = 0, \quad \Delta\delta_1 = e^{i\omega t}. \qquad (2.4.86)$$

Again considering the coefficients of equation as quenched, we will seek unknown function $\Delta\gamma$ in the form

$$\Delta\gamma = \Gamma e^{i\omega t}. \qquad (2.4.87)$$

After substitution of (2.4.86) and (2.4.87) in equation (1.13.8) we obtain

$$(-\omega^2 J_x + i\omega\mu_x)\Gamma = c_{\gamma\delta}.$$

Hence

$$\Gamma = \frac{c_{\gamma\delta}}{-J_x\omega^2 + i\mu_x\omega}, \qquad (2.4.88)$$

or

$$\Gamma = \Gamma_1(\omega) + i\Gamma_2(\omega), \qquad (2.4.89)$$

where

$$\Gamma_1(\omega) = \frac{-c_{\gamma\delta}J_x}{J_x^2\omega^2 + \mu_x^2}, \quad \Gamma_2(\omega) = \frac{-c_{\gamma\delta}\mu_x}{(J_x^2\omega^2 + \mu_x^2)\omega}. \qquad (2.4.90)$$

Here $\Gamma_1(\omega)$, $\Gamma_2(\omega)$ - real and imaginary components of frequency transfer function respectively.

Frequency transfer function can be presented in the form

$$\Gamma = k(\omega)e^{i\varphi(\omega)}, \qquad (2.4.91)$$

where

$$k(\omega) = |\Gamma(i\omega)| = \frac{|c_{\gamma\delta}|}{\sqrt{J_x^2\omega^4 + \mu_x^2\omega^2}}, \qquad (2.4.92)$$

$$\varphi(\omega)=\arg\Gamma(i\omega)=\arg\left(-\frac{c_{\gamma\delta}}{J_x\omega^2}\right)-\arg\left(1-i\frac{\mu_x}{J_x\omega}\right).$$

Since $c_{\gamma\delta} < 0$,

$$\varphi=-\arg\left(1-i\frac{\mu_x}{J_x\omega}\right)=-\arg\operatorname{tg}\left(-\frac{\mu_x}{J_x\omega}\right)=\arg\operatorname{tg}\frac{\mu_x}{J_x\omega}. \quad (2.4.93)$$

Functions '.(ω) and $\phi(\omega)$ represent the amplitude frequency and phase frequency characteristics respectively of the rocket as an object of control.

§ 5. Requirements for Frequency Characteristics of Automatic Stabilization Control from Condition of Stability of the Automatic Control System of the Rocket

If the frequency characteristics of the rocket as an object of control and frequency characteristics of automatic stabilization control are known, then we can determine the frequency characteristics of an open automatic control system.

Let us assume $k_o(\omega)$ and $\phi_o(\omega)$ - respectively the amplification factor and phase shift of the object, and $k_a(\omega)$ and $\phi_a(\omega)$ - amplification factor and phase shift of automatic stabilization control. Then, as is known, the amplification factor and phase shift of an open automatic control system are determined by relationships:

$$k(\omega)=k_0(\omega)k_a(\omega),$$
$$\varphi(\omega)=\varphi_0(\omega)+\varphi_a(\omega). \quad (2.5.1)$$

If θ_{o1}, θ_{o2} - respectively the real and imaginary components of frequency transfer function of the object, and θ_{a1}, θ_{a2} - real and imaginary components of frequency transfer function of the automatic stabilization control then real and imaginary components of the frequency transfer function of an open system are determined by equalities:

$$\theta_1(\omega)=\theta_{o1}(\omega)\theta_{a1}(\omega)-\theta_{o2}(\omega)\theta_{a2}(\omega),$$
$$\theta_2(\omega)=\theta_{o1}(\omega)\theta_{a2}(\omega)+\theta_{o2}(\omega)\theta_{a1}(\omega). \qquad (2.5.2)$$

By relationships (2.5.1) or (2.5.2) it is possible to construct amplitude-phase-frequency characteristics (hodographs) of the open system. By the shape of this hodograph, adhering to Nyquist stability criterion, it is possible to judge the stability of motion of the rocket.

Nyquist criterion of stability is formulated in the following manner.

Let us assume $W_o(p)$ - transfer function of the rocket as an object of control, and $W_a(p)$ - transfer function of automatic stabilization control. Then the transfer function of the open automatic control system, as is known, is equal to:

$$W(p)=W_o(p)W_a(p)=\frac{R(p)}{Q(p)}, \qquad (2.5.3)$$

where $R(p)$ and $Q(p)$ - polynomial of p; degree of polynomial $R(p)$ is not higher than the degree of polynomial $Q(p)$.

From (2.5.3), incidently, with substitution of $p = i(\omega)$ there is obtained frequency transfer function of the open system:

$$W(i\omega)\equiv\theta(i\omega)=k(\omega)e^{i\varphi(\omega)}=\theta_1(\omega)+i\theta_2(\omega). \qquad (2.5.4)$$

<u>Nyquist stability criterion</u>. Let us assume the denominator of transfer function of the open automatic control system (2.5.3) contains l roots in the right half-plane of the complex plane and $n-l$ roots - in the left. Then with change of frequency ω from $-\infty$ to $+\infty$ for a system stable in closed state the resulting turn of the end of the vector of frequency transfer function $\theta(i\omega)$ of the open system relative to point (+1, 0) counterclockwise should form an angle, equal to $2\pi l$, i.e., amplitude-phase-frequency characteristic

should encompass point (+1, 0) as many times as the denominator of transfer function of the open system contains roots in the right half-plane.

<u>Note</u>: The provided formulation of Nyquist stability criterion is valid on the assumption that the output quantity of the object with the same sign is the input quantity or part of the input (control) quantity of the regulator. In theory of automatic control usually during forming of control quantity of the regulator quantity of the object as a link of automatic control system is taken with opposite sign. In this instance the characteristic point, relative to which there is considered rotation of the end of vector $\theta(i\omega)$, becomes point (-1, 0).

At the initial stage of designing the rocket, when parameters of automatic stabilization control have not yet been determined, but frequency characteristics of the rocket as an object of control are already known, with the aid of Nyquist stability criterion it is possible to establish the necessary limitations for frequency characteristics of automatic stabilization control from the condition of stability of the system, which will permit refining the range of possible values of parameters of the automatic stabilization control and, thus, will simplify the problem of selection of its rational values.

Below we will limit ourselves to examination of pitching and rolling motion of the rocket, bearing in mind that corresponding analysis of other control channels is performed analogously.

5.1. Requirements for Frequency Characteristics of Automatic Stabilization Control with Respect to Pitch Channel

For simplicity and clarity of analysis let us use equation (2.4.42), which approximately describes disturbed pitching motion of rocket. In this instance for transfer function of the rocket as an object of control we have the following expression:

$$W_0(p) \equiv W_\delta^\theta(p) = \frac{c_{\theta\delta}}{J_z p^2 + \mu_z p + c_{\theta\theta}}. \qquad (2.5.5)$$

By substituting $i\omega$ instead of p in (2.5.5), we obtain frequency transfer function of the object

$$W_0(i\omega) \equiv \theta_0(i\omega) = \frac{-c_{\theta\delta}}{J_z\omega^2 - c_{\theta\theta} - i\mu_z\omega}. \qquad (2.5.6)$$

By dividing the real and imaginary parts of frequency transfer function, we will have

$$\theta_0(i\omega) = \theta_{01}(\omega) + i\theta_{02}(\omega), \qquad (2.5.7)$$

where

$$\theta_{01}(\omega) = \frac{-c_{\theta\delta}(J_z\omega^2 - c_{\theta\theta})}{(J_z\omega^2 - c_{\theta\theta})^2 + \mu_z^2\omega^2},$$
$$\theta_{02}(\omega) = \frac{-c_{\theta\delta}\mu_z\omega}{(J_z\omega^2 - c_{\theta\theta})^2 + \mu_z^2\omega^2}. \qquad (2.5.8)$$

Formulas (2.5.8) are conveniently written so:

$$\theta_{01}(\omega) = \frac{-\bar{c}_{\theta\delta}(\omega^2 - \bar{c}_{\theta\theta})}{(\omega^2 - \bar{c}_{\theta\theta})^2 + \bar{\mu}_z^2\omega^2},$$
$$\theta_{02}(\omega) = \frac{-\bar{c}_{\theta\delta}\bar{\mu}_z\omega}{(\omega^2 - \bar{c}_{\theta\theta})^2 + \bar{\mu}_z^2\omega^2}. \qquad (2.5.9)$$

Here

$$\bar{c}_{\psi\psi}=\frac{c_{\psi\psi}}{J_z},\quad \bar{\mu}_z=\frac{\mu_z}{J_z},\quad \bar{c}_{\psi\delta}=\frac{c_{\psi\delta}}{J_z}. \tag{2.5.10}$$

The following three cases are possible.

1. *Rocket is statically stable, i.e.,* $x_F < 0$. In this case

$$\bar{c}_{\psi\psi}>0 \tag{2.5.11}$$

and therefore the denominator of transfer function (2.5.5) does not have roots in the right half-plane.

With change of ω from -∞ to +∞ the end of the vector of frequency transfer function $\theta_0(i\omega)$ describes the curve shown in Fig. 2.9, as follows from analysis of formulas (2.5.9) taking into account the fact that $c_{\psi\delta} < 0$, $\mu_z > 0$. From condition of Nyquist stability since the denominator of transfer function does not have roots in the right half-plane, it follows that the hodograph of the open system should not encompass point (+1, 0).[1] The desirable shape of the hodograph of an open system is shown in Fig. 2.9 by a broken line. In this instance there will take place a stability margin with respect to phase, being measured by magnitude of angle $\phi_{\text{зап}}$ between the axis of abscissa and ray, drawn from the origin of coordinates through the point of intersection of hodograph of the open system with the circumference of unit radius. For creation of such margin of stability it is necessary that the automatic stabilization control at frequencies, close to frequency ω^1, at which the amplification factor of the open system is equal to one, had phase lead. Phase shift of automatic stabilization control must satisfy condition

$$2k\pi\leqslant\varphi_a+\varphi_0\leqslant(2k+1)\pi \quad (k=0,1,2,\ldots), \tag{2.5.12}$$

[1]Here and further it is assumed that the denominator of transfer function of automatic stabilization control does not have roots in the right half-plane.

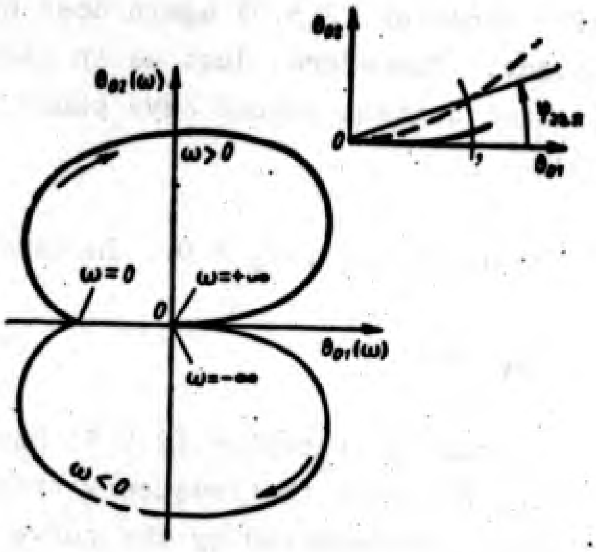

Fig. 2.9.

2. *Rocket is statically netural, i.e.,* $x_F = 0$. In this case

$$\bar{c}_{34} = 0 \tag{2.5.13}$$

and therefore

$$\theta_{01}^{\cdot} = \frac{-\bar{c}_{34}}{\omega^2 + \bar{\mu}_z^2}, \quad \theta_{02}^{\cdot}(\omega) = \frac{-\bar{c}_{34}\bar{\mu}_z}{(\omega^2 + \bar{\mu}_z^2)\,\omega}. \tag{2.5.14}$$

The end of vector $\theta_0(i\omega)$ describes the curve shown in Fig. 2.10.

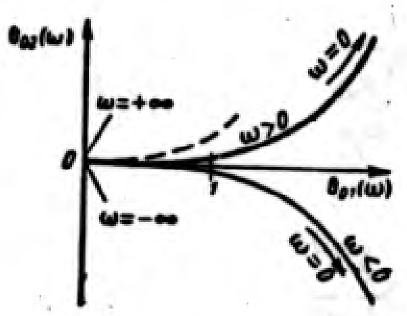

Fig. 2.10.

The denominator of transfer function (2.5.5) again does not have roots in the right half-plane. Therefore, just as in case $x_F < 0$, the automatic stabilization control should have phase lead according to condition (2.5.12).

3. *Rocket is statically unstable, i.e.,* $x_F > 0$. In this case

$$\bar{c}_{\theta\theta} < 0 \tag{2.5.15}$$

and therefore the denominator of transfer function (2.5.5) has one root in the right half-plane. In this case the frequency transfer function of the object of control is represented by the curve shown in Fig. 2.11. Accordingly to Nyquist criterion, for stability of the system in closed state the end of the vector of frequency transfer function of the open system with change of ω from $-\infty$ to $+\infty$ must be turned to angle 2π. From Fig. 2.11 it is evident that to provide stability with some phase margin it is necessary that phase ϕ_a of automatic stabilization control satisfies condition (2.5.12). But this is insufficient in the considered case. It is still necessary that the hodograph of the open system intersect the axis of abscissa when $\omega = 0$ to the right of point (+1, 0). Therefore it is necessary to fulfill one more condition:

$$k_0(0)k_a(0) > 1. \tag{2.5.16}$$

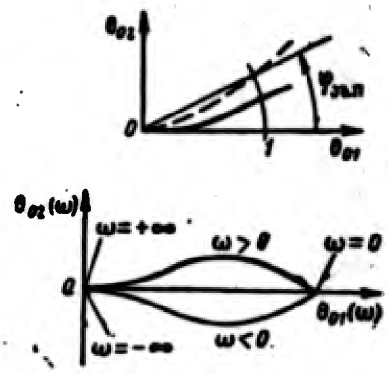

Fig. 2.11.

Since amplification factor of the object

$$k_o(\omega) = \sqrt{\theta_{o1}^2(\omega) + \theta_{o2}^2(\omega)} = \frac{|c_{\delta\delta}|}{\sqrt{(\omega^2 - \bar{c}_{\delta\delta})^2 + \bar{\mu}_z^2\omega^2}}, \tag{2.5.17}$$

when ω = 0 we have

$$k_o(0) = \frac{\bar{c}_{\delta\delta}}{\bar{c}_{\delta\delta}}. \tag{2.5.18}$$

Considering (2.5.18), from (2.5.16) we find the lower limit of permissible values of amplification factor of the automatic stabilization control:

$$k_a(0) > \frac{\bar{c}_{\delta\delta}}{\bar{c}_{\delta\delta}}. \tag{2.5.19}$$

Thus, in all the considered cases there is required phase lead of automatic stabilization control at frequencies close to frequency ω*, and if the rocket is statically unstable, furthermore, the amplification factor of automatic stabilization control when ω = 0 must satisfy condition (2.5.19).

Frequency ω*, close to which condition (2.5.12) must be fulfilled, is determined by equality

$$k_o(\omega^*) k_a(\omega^*) = 1. \tag{2.5.20}$$

This frequency can be found so. Let us substitute value $k_o(\omega^*)$ in equality (2.5.20) according to (2.5.17). We obtain

$$\frac{|c_{\delta\delta}|}{\sqrt{(\omega^{*2} - \bar{c}_{\delta\delta})^2 + \bar{\mu}_z^2\omega^{*2}}} k_a(\omega^*) = 1.$$

In view of the smallness of $\bar{\mu}_z$, hence follows approximate equality

$$\omega^{*2} \approx \bar{c}_{\vartheta\vartheta} + |\bar{c}_{\vartheta\delta}| k_a(\omega^*) \qquad (2.5.21)$$

If the relationship of amplification factor of the automatic stabilization control k_a to frequency ω is known, then from (2.5.21) it is simple to determine the desired frequency ω^* graphically or by means of successive approximations

Since usually $\phi_o(\omega^*) \stackrel{\sim}{\sim} 0$ (and when $\mu_z = 0$ $\phi_o(\omega^*) = 0$), the condition of phase lead of automatic stabilization control can be simplified, having written it so:

$$2k\pi < \varphi_a < (2k+1)\pi \quad (k=0, 1, 2, \ldots). \qquad (2.5.22)$$

Finally, let us estimate the maximum permissible value of amplification factor of the automatic stabilization control at large ω.

Let us assume ω_o - frequency at which the automatic stabilization control does not give phase shift. Approximately at this frequency the hodograph of open automatic control system intersects the axis of abscissa, since phase shift, created by the object, is very small. By disregarding value $\phi_o(\omega_o)$, for coordinate of the point of intersection of hodograph of open automatic control system with the axis of abscissa we will have the following expression:

$$\begin{aligned}\theta_1(\omega_0) = k(\omega_0) = \theta_{01}(\omega_0) k_a(\omega_0) = \\ = \frac{-\bar{c}_{\vartheta\delta}(\omega_0^2 - \bar{c}_{\vartheta\vartheta})}{(\omega_0^2 - \bar{c}_{\vartheta\vartheta})^2 + \mu_z \omega_0^2} k_a(\omega_0).\end{aligned} \qquad (2.5.23)$$

From condition of stability of the system the hodograph of open system should intersect the axis of abscissa (at frequencies close to ω_o) to the left of point (+1, 0) and, thus, the amplification factor of automatic stabilization control must satisfy inequality

$$\frac{-\bar{c}_{\psi\delta}(\omega_0^2-\bar{c}_{\psi\psi})}{(\omega_0^2-\bar{c}_{\psi\psi})^2+\bar{\mu}_z^2\omega_0^2}k_s(\omega_0)<1. \quad (2.5.24)$$

Hence, assuming that $\omega_0^2 > \bar{c}_{\psi\psi}$, and taking into account that $\bar{c}_{\psi\delta} < 0$, we obtain

$$k_s(\omega_0)<\frac{(\omega_0^2-\bar{c}_{\psi\psi})^2+\bar{\mu}_z^2\omega_0^2}{-\bar{c}_{\psi\delta}(\omega_0^2-\bar{c}_{\psi\psi})}. \quad (2.5.25)$$

In view of the smallness of coefficient $\bar{\mu}_z$ relationship (2.5.25) can be simplified, having dropped component $\bar{\mu}_z^2\ \omega_0^2$. Then we will have

$$k_s(\omega_0)<\frac{\omega_0^2-\bar{c}_{\psi\psi}}{-\bar{c}_{\psi\delta}}. \quad (2.5.26)$$

5.2. Requirements for Frequency Characteristics of Automatic Stabilization Control with Respect to Roll Channel

Transfer function of the rocket as the object of control for the pair of input and output signals $\Delta\delta_\gamma$-$\Delta\gamma$ is represented in the form of [see (2.4.64)]

$$W_o(p)=W_\gamma^\delta(p)=\frac{c_{\gamma\delta}}{J_x}\frac{1}{p\left(p+\frac{\mu_x}{J_x}\right)}. \quad (2.5.27)$$

Real and imaginary components of frequency transfer function are represented by formula (2.4.90), which can be written so:

$$\Gamma_{o1}=\frac{-\bar{c}_{\gamma\delta}}{\omega^2+\bar{\mu}_x^2},\quad \Gamma_{o2}=\frac{-\bar{c}_{\gamma\delta}\bar{\mu}_x}{(\omega^2+\bar{\mu}_x^2)\,\omega}.$$

Here

$$\bar{c}_{\gamma\delta}=\frac{c_{\gamma\delta}}{J_x},\quad \bar{\mu}_x=\frac{\mu_x}{J_x}.$$

With change of ω from $-\infty$ to $+\infty$ the end of the vector of frequency transfer function $\Gamma_o(i\omega)$ describes the curve shown in Fig. 2.12. Since the denominator of transfer function (2.5.27) does not have roots in the right half-plane, then from the condition of Nyquist stability the hodograph of open system should not encompass point (+1, 0), and for this it is necessary that the automatic stabilization control at frequencies close to frequency ω^*, at which the amplification factor of open system is equal to one, had phase lead; more accurately, that at these frequencies there would be fulfilled condition

$$2k\pi \leqslant \varphi_a + \varphi_o \leqslant (2k+1)\pi \quad (k=0, 1, 2, \ldots). \tag{2.5.28}$$

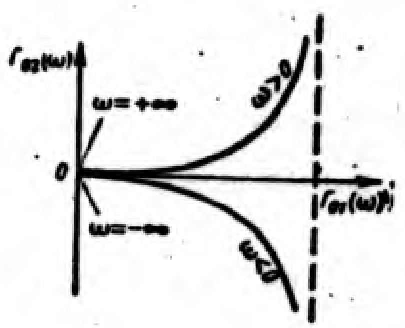

Fig. 2.12.

Frequency ω^* can be determined in the following manner.

Let us assume $k_o(\omega)$ and $k_a(\omega)$ - amplification factor of the object and automatic stabilization control respectively. Then, according to condition,

$$k_o(\omega^*)k_a(\omega^*)=1. \tag{2.5.29}$$

Taking into account that

$$k_o(\omega)=\sqrt{\Gamma_{o1}^2+\Gamma_{o2}^2}=\frac{|c_{74}|}{\sqrt{(\omega^2+\bar{\mu}_x^2)\,\omega^2}}, \tag{2.5.30}$$

from (2.5.29) we obtain

$$|c_{\gamma\delta}|k_a(\omega^*)=\sqrt{(\omega^{*2}+\mu_x^2)\omega^{*2}}.$$

In view of the smallness of μ_x hence follows approximate equality

$$\omega^{*2}=|\bar{c}_{\gamma\delta}|k_a(\omega^*). \tag{2.5.31}$$

If the amplification factor of automatic stabilization control as a function of ω is known, then from (2.5.31) it is possible, at least approximately, to determine the value of ω*.

§ 6. Stability Regions. D-Division

During calculation and designing of the system of control of the rocket there is usually investigated the effect of various structural parameters of the rocket and parameters of the regulator on the stability of the system. For this purpose we construct regions of stability in the plane of one parameter or in the plane o two parameters at fixed values of all other parameters. The plane of the investigated parameters is subjected to so-called D-division by means of constructions of a line, which divides the regions with definite distribution of roots of the characteristic polynomial of a closed system (with the same number of roots with negative real parts and with positive real parts). Of practical interest is the part of curves of D-division, that is the boundary of the region with maximum number of roots with negative real parts. For isolation of this region, which can be the region of stability, there is introduced shading of lines of D-division according to the rule given below. In order to establish whether in fact the selected region is the region of stability or not, it is necessary for one (any) point of this region to check the stability with the aid of some criterion.

Below as an example there is shown the construction of the region of stability in a plane of two parameters. There is considered disturbed pitching motion of the rocket. Construction of the regions of stability with respect to yaw and roll is performed completely analogously.

Disturbed pitching motion of the rocket is described by differential equations:

$$\begin{aligned} &m(t)\frac{d\Delta V_y}{dt}+\nu_y(t)\frac{d\Delta\vartheta}{dt}+c_{yy}(t)\Delta V_y+c_{y\vartheta}(t)\Delta\vartheta= \\ &\qquad =c_{y\delta}(t)\Delta\delta+\Delta F_y. \\ &J_z(t)\frac{d^2\Delta\vartheta}{dt^2}+\mu_z(t)\frac{d\Delta\vartheta}{dt}+c_{\vartheta\vartheta}(t)\Delta\vartheta+c_{\vartheta y}(t)\Delta V_y= \\ &\qquad =c_{\vartheta\delta}(t)+\Delta M_z. \end{aligned} \tag{2.6.1}$$

To equations (2.6.1) there must be added another equation of the connection between $\Delta\delta$ and the controllable quantities. Let us take the following law of control:

$$T_2^2\frac{d^2\Delta\delta}{dt^2}+T_1\frac{d\Delta\delta}{dt}+\Delta\delta=k\Delta\vartheta+k_1\frac{d\Delta\vartheta}{dt}. \tag{2.6.2}$$

For the other law of control the corresponding working formulas are simply obtained by repeating the operations being given further.

By following the method of quenched coefficients, let us record time t and examine the system of homogeneous equations with constant coefficients

$$\begin{aligned} &m\frac{d\Delta V_y}{dt}+\nu_y\frac{d\Delta\vartheta}{dt}+c_{yy}\Delta V_y+c_{y\vartheta}\Delta\vartheta-c_{y\delta}\Delta\delta=0, \\ &J_z\frac{d^2\Delta\vartheta}{dt^2}+\mu_z\frac{d\Delta\vartheta}{dt}+c_{\vartheta\vartheta}\Delta\vartheta+c_{\vartheta y}\Delta V_y-c_{\vartheta\delta}\Delta\delta=0, \\ &T_2^2\frac{d^2\Delta\delta}{dt^2}+T_1\frac{d\Delta\delta}{dt}+\Delta\delta-k_1\frac{d\Delta\vartheta}{dt}-k\Delta\vartheta=0. \end{aligned} \tag{2.6.3}$$

Let us construct lines of D-division in the plane of two parameters k and k_1, assuming that T_1 and T_2^2 - assigned quantities.

Characteristic equations of system (2.6.3) has the form

$$D(\lambda) = a_0 T_2^2 \lambda^5 + (a_1 T_2^2 + a_0 T_1)\lambda^4 + (a_0 + a_2 T_2^2 + a_1 T_1)\lambda^3 +$$
$$+ (a_1 + a_3 T_2^2 + a_2 T_1 + b_1 k_1)\lambda^2 + (a_2 + a_3 T_1 + b_1 k + b_2 k_1)\lambda +$$
$$+ (a_3 + b_2 k) = 0, \qquad (2.6.4)$$

where

$$\begin{aligned} a_0 &= mJ_z, \\ a_1 &= m\mu_z + c_{yy} J_z, \\ a_2 &= mc_{\vartheta\vartheta} + \mu_z c_{yy} - c_{\vartheta y} v_y \\ a_3 &= c_{yy} c_{\vartheta\vartheta} - c_{y\vartheta} c_{\vartheta y}, \\ b_1 &= -mc_{\vartheta\vartheta}, \\ b_2 &= c_{y\vartheta} c_{\vartheta y} - c_{yy} c_{\vartheta\vartheta}. \end{aligned} \qquad (2.6.5)$$

Equation (2.6.4) can be rewritten so:

$$kP(\lambda) + k_1 Q(\lambda) + S(\lambda) = 0, \qquad (2.6.6)$$

where

$$P(\lambda) = b_1 \lambda + b_2,$$
$$Q(\lambda) = (b_1 \lambda + b_2)\lambda,$$
$$S(\lambda) = a_0 T_2^2 \lambda^5 + (a_1 T_2^2 + a_0 T_1)\lambda^4 + (a_0 + a_2 T_2^2 + a_1 T_1)\lambda^3 +$$
$$+ (a_1 + a_3 T_2^2 + a_2 T_1)\lambda^2 + (a_2 + a_3 T_1)\lambda + a_3.$$

Let us introduce substitution $\lambda = i\omega$. We will have

$$P(i\omega) = P_1(\omega) + iP_2(\omega),$$
$$Q(i\omega) = Q_1(\omega) + iQ_2(\omega),$$
$$S(i\omega) = S_1(\omega) + iS_2(\omega).$$

Here

$$P_1(\omega) = b_2, \qquad P_2(\omega) = b_1\omega,$$
$$Q_1(\omega) = -b_1\omega^2, \quad Q_2(\omega) = b_2\omega,$$

$$S_1(\omega)=(a_1T_2^2+a_0T_1)\omega^4-(a_1+a_3T_2^2+a_2T_1)\omega^2+a_3,$$
$$S_2(\omega)=a_0T_2^2\omega^5-(a_0+a_2T_2^2+a_1T_1)\omega^3+(a_2+a_3T_1)\omega.$$

By equating separately the real and imaginary parts to zero in equation (2.6.6), we obtain parametric equations of lines of D-division of the plane of parameters k and k_1:

$$\left.\begin{aligned}kP_1(\omega)+k_1Q_1(\omega)+S_1(\omega)=0,\\ kP_2(\omega)+k_1Q_2(\omega)+S_2(\omega)=0.\end{aligned}\right\} \quad (2.6.7)$$

Let us solve equalities (2.6.7) relative to k and k_1. Then

$$k=\frac{\begin{vmatrix}-S_1(\omega) & Q_1(\omega)\\ -S_2(\omega) & Q_2(\omega)\end{vmatrix}}{\begin{vmatrix}P_1(\omega) & Q_1(\omega)\\ P_2(\omega) & Q_2(\omega)\end{vmatrix}},\quad k_1=\frac{\begin{vmatrix}P_1(\omega) & -S_1(\omega)\\ P_2(\omega) & -S_2(\omega)\end{vmatrix}}{\begin{vmatrix}P_1(\omega) & Q_1(\omega)\\ P_2(\omega) & Q_2(\omega)\end{vmatrix}},$$

or, by expanding the determinants,

$$\begin{aligned}k&=\frac{1}{b_1^2\omega^2+b_2^2}(A_{16}\omega^6+A_{14}\omega^4+A_{12}\omega^2+A_{10}),\\ k_1&=\frac{1}{b_1^2\omega^2+b_2^2}(A_{24}\omega^4+A_{22}\omega^2+A_{20}),\end{aligned} \quad (2.6.8)$$

where

$$\begin{aligned}&A_{16}=-a_0b_1T_2^2,\\ &A_{14}=b_1(a_0+a_2T_2^2+a_1T_1)-b_2(a_1T_2^2+a_0T_1),\\ &A_{12}=b_2(a_1+a_3T_2^2+a_2T_1)-b_1(a_2+a_3T_1),\\ &A_{10}=-a_3b_2,\\ &A_{24}=b_1(a_1T_2^2+a_0T_1)-a_0b_2T_2^2,\\ &A_{22}=b_2(a_0+a_2T_2^2+a_1T_1)-b_1(a_1+a_3T_2^2+a_2T_1),\\ &A_{20}=a_3b_1-b_2(a_2+a_3T_1).\end{aligned} \quad (2.6.9)$$

To every value of ω there correspond definite values of k and k_1, i.e., the point in plane kk_1. By adding different values of ω (from $-\infty$ to ∞), it is possible in plane kk_1 to construct the curve of D-division. In this case k should be plotted along the axis of abscissas, and k_1 - along the axis of ordinates.

Further there is performed shading of curve of D-division according to the following rule. If during motion along this curve toward growth of ω (i.e., from ω = -∞ to ω = ∞) the principal determinant of system (2.6.7) is positive, then the curve is shaded from the left. If in this case the principal determinant is negative, the the curve is shaded from the right. In our case the principal determinant is equal to:

$$\Delta = \begin{vmatrix} P_1(\omega) & Q_1(\omega) \\ P_2(\omega) & Q_2(\omega) \end{vmatrix} = (b_1^2\omega^2 + b_2^2)\omega.$$

Therefore, with growth of ω from 0 to ∞ Δ > 0, and, thus, the curve should be shaded from the left.

Inasmuch as in formulas (2.6.8) ω enters the fourth power, each point of the curve corresponds to two values of ω: ±|ω|, i.e., with growth of ω from -∞ to ∞ our curve is passed twice: from -∞ to 0 and from 0 to ∞. With growth of ω from -∞ to 0 Δ < 0, therefore the curve should be shaded from the right. As a result the curve is shaded twice from the same side.

When ω = 0

$$\left.\begin{aligned} k(0) &= -\frac{a_4}{b_2}, \\ k_1(0) &= \frac{a_3 b_1 - b_1(a_2 + a_3 T_1)}{b_2^2}. \end{aligned}\right\} \quad (2.6.10)$$

The point with coordinates $k(0)$, $k_1(0)$ is specific, since when ω = 0 both the principal determinant of system (2.6.7) and the following determinants become zero:

$$\begin{vmatrix} -S_1(\omega) & Q_1(\omega) \\ -S_2(\omega) & Q_2(\omega) \end{vmatrix}, \quad \begin{vmatrix} P_1(\omega) & -S_1(\omega) \\ P_2(\omega) & -S_2(\omega) \end{vmatrix}.$$

When $\omega = 0$ system (2.6.7) is reduced to one equation

$$k = -\frac{a_2}{b_2}. \qquad (2.6.11)$$

This equation is a straight line, passing through a specific point (2.6.10). The straight line should be shaded so that near the specific point the shading of straight line and curve would be directed to the same side. The region, covered by the maximum number of shadings, will be the range of stability, if there generally is such at the considered moment of time.

Figure 2.13 shows line of D-division and region of stability for a rocket with law of control (2.6.2). A similar construction of regions of stability is provided for various fixed moments of time with 10-20 s interval. The overall part of all stability regions obtained in this manner is the region of those values of parameters k and k_1, at which stability is provided on the entire trajectory.

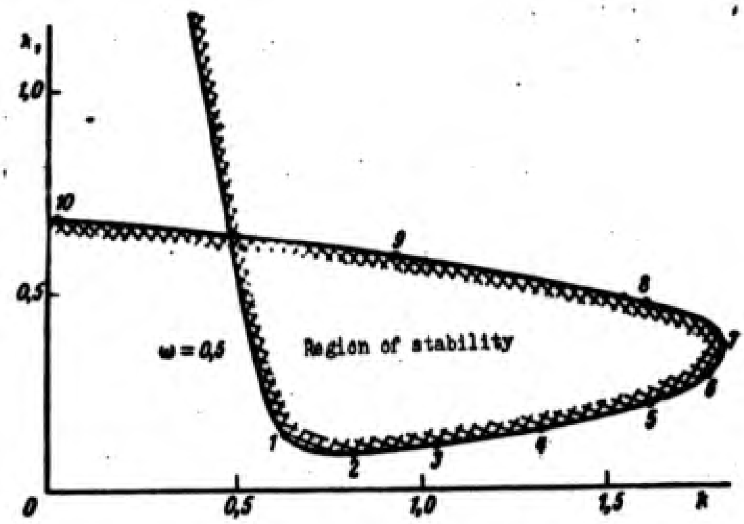

Fig. 2.13.

§ 7. Refinement of the Method of Quenched Coefficients. Accounts of Variability of Coefficients of Equations

In § 3 we formulated the approximate criterion of local stability of undisturbed motion, to which corresponds trivial solution of equations (2.3.2), based on the method of "quenching" of coefficients. In this section there are derived criteria of local stability of undisturbed motion [trivial solution of equations (2.3.2)], considering the variability of coefficients of equations.

For convenience of further account let us present the system of differential equations (2.3.2) in standard form. For this purpose, considering generalized velocities as additional generalized coordinates, let us introduce new variables

$$x_i = \begin{cases} \dfrac{dq_i}{dt} (i = 1, 2, \ldots n), \\ q_{i-n} (i = n + 1, \ldots, 2n). \end{cases} \tag{2.7.1}$$

In new variables system (2.3.2) assumes the form

$$\begin{gathered} \sum_{j=1}^{n} m_{ij}(t) \frac{dx_j}{dt} = -\sum_{j=1}^{n} r_{ij}(t) x_j - \sum_{j=1}^{n} n_{ij}(t) x_{n+j}, \\ \frac{dx_{n+i}}{dt} = x_i \quad (i = 1, 2, \ldots, n). \end{gathered} \tag{2.7.2}$$

Determinant, composed of coefficients m_{ij} with leading derivatives in system (2.3.2), can be shown nonzero, therefore the following determinant is not equal to zero

$$\begin{vmatrix} m_{11} m_{12} \ldots m_{1n} 0 \ldots 0 \\ m_{21} m_{22} \ldots m_{2n} 0 \ldots 0 \\ \ldots\ldots\ldots\ldots \\ m_{n1} m_{n2} \ldots m_{nn} 0 \ldots 0 \\ 0 \quad 0 \qquad 0 \quad 1 \ldots 0 \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ 0 \quad 0 \ldots 0 \quad 0 \ldots 1 \end{vmatrix}$$

consisting of coefficients with derivatives in equations (2.7.2). This means, system (2.7.2) is solvable relative to derivatives dx_i/dt. By solving it as a nonhomogeneous algebraic system relative to derivatives dx_i/dt, we obtain a system of differential equations in standard form

$$\frac{dx_i}{dt} = \sum_{j=1}^{m} u_{ij}(t)\,x_j \quad (i = 1, 2, \ldots, m;\ m = 2n). \tag{2.7.3}$$

Further it is assumed that $u_{ij}(t)$ - any necessary number of once differentiable functions.

7.1. Asymptotic Integration of Equations

If coefficients of equations (2.7.3) are constants, then their solution is constructed just as the solution of system (2.3.3). So, if

$$\frac{dx_i}{dt} = \sum_{j=1}^{m} u_{ij}x_j \quad (i = 1, 2, \ldots, m), \tag{2.7.4}$$

where u_{ij} = const, then the particular solution is sought in the form

$$x_i = k_i e^{\lambda t} \quad (i = 1, 2, \ldots, m). \tag{2.7.5}$$

Substitution of (2.7.5) in equations (2.7.4) leads to the system of algebraic equations

$$\sum_{j=1}^{m} u_{ij}k_j - k_i\lambda = 0 \quad (i = 1, 2, \ldots, m). \tag{2.7.6}$$

System of algebraic equations (2.7.6) has nonzero solutions only at values of λ at which the determinant of the system becomes zero. The determinant of the system is polynomial (characteristic polynomial) of m power relative to λ. This polynomial has m roots λ_1,

$\lambda_2, \ldots, \lambda_m$. To simple root λ_σ with accuracy to arbitrary nonzero coefficient corresponds unique nonzero solution of algebraic system (2.7.6): $k_{1\sigma}, k_{2\sigma}, \ldots, k_{m\sigma}$. To root λ_σ and to nonzero solution of algebraic equations (2.7.6) when $\lambda = \lambda_\sigma$ corresponds the particular solution of differential equations (2.7.4).

$$x_{i\sigma} = k_{i\sigma} e^{\lambda_\sigma t} \quad (i = 1, 2, \ldots, m). \tag{2.7.7}$$

If all the roots of characteristic polynomial are simple, then the general solution of equations (2.7.4) is represented so:

$$x_i = \sum_{\sigma=1}^{m} c_\sigma k_{i\sigma} e^{\lambda_\sigma t} \quad (i = 1, 2, \ldots, m), \tag{2.7.8}$$

where c_σ ($\sigma = 1, 2, \ldots, m$) - arbitrary constants.

The problem of stability of trivial solution of equations (2.7.4) is determined depending on signs of real parts of roots of characteristic polynomial. According to the method of quenched coefficients the signs of real parts of roots are considered as criterion of stability when coefficients of equations are variables.

In order to obtain more accurate stability criteria of trivial solution of equations (2.7.3), adhering to the first method of Lyapunov, it is necessary to solve equations (2.7.3) more accurately than by the method of quenched coefficents.

Below is indicated the method of construction of approximate solution of equations (2.7.3) with allowance for variability of coefficients of equations. This method is based on ideas of asymptotic integration of differential equations using the concept "slow time," introduced into mathematical physics by N. M. Krylov and N. N. Bogolyubov [9].

Instead of system (2.7.3) let us introduce into examination the system

$$\frac{dx_i}{dt} = \sum_{j=1}^{m} a_{ij}(\tau) x_j \quad (i = 1, 2, \ldots, m), \tag{2.7.9}$$

where $\tau = \varepsilon t$ - so-called "slow time"; ε - parameter.

When $\varepsilon = 1$ systems (2.7.3) and (2.7.9) coinciue, so that if we can construct the solution of equations (2.7.9), then, by assuming in this solution $\varepsilon = 1$, we obtain the solution of system (2.7.3) of interest to us.

We will seek the particular solution of equations (2.7.9) in the form

$$x_i(t, \varepsilon) = \tilde{k}_i(\tau, \varepsilon) e^{\int \tilde{\lambda}(\tau, \varepsilon) dt} \quad (i = 1, 2, \ldots, m). \tag{2.7.10}$$

By substituting (2.7.10) in (2.7.9), after reduction by common multiple $\exp \int \tilde{\lambda}(\tau, \varepsilon) dt$

$$\sum_j a_{ij}(\tau) \tilde{k}_j(\tau, \varepsilon) - \tilde{k}_i(\tau, \varepsilon) \tilde{\lambda}(\tau, \varepsilon) = \varepsilon \frac{d\tilde{k}_i(\tau, \varepsilon)}{d\tau} \quad (i = 1, 2, \ldots, m). \tag{2.7.11}$$

Let us determine functions $\tilde{k}_j(\tau, \varepsilon)$ and $\tilde{\lambda}(\tau, \varepsilon)$ so that equalities (2.7.11) are fulfilled indentically relative to parameter ε. We will construct these functions in the form of series according to powers ε:

$$\begin{aligned} \tilde{k}_i(\tau, \varepsilon) &= k_i(\tau) + \varepsilon k_i^{[1]}(\tau) + \varepsilon^2 k_i^{[2]}(\tau) + \ldots \quad (i = 1, 2, \ldots, m), \\ \tilde{\lambda}(\tau, \varepsilon) &= \lambda(\tau) + \varepsilon \lambda^{[1]}(\tau) + \varepsilon^2 \lambda^{[2]}(\tau) + \ldots \end{aligned} \tag{2.7.12}$$

Let us substitute (2.7.12) in (2.7.11):

$$\begin{aligned} &\sum_j a_{ij}(k_j + \varepsilon k_j^{[1]} + \varepsilon^2 k_j^{[2]} + \ldots) - (k_i + \varepsilon k_i^{[1]} + \ldots)(\lambda + \varepsilon \lambda^{[1]} + \ldots) = \\ &= \varepsilon \left(\frac{dk_i}{d\tau} + \varepsilon \frac{dk_i^{[1]}}{d\tau} + \varepsilon^2 \frac{dk_i^{[1]}}{d\tau} + \ldots \right) \quad (i = 1, 2, \ldots, m). \end{aligned} \tag{2.7.13}$$

In equalities (2.7.13) let us equate the terms containing ε in equal powers. As a reuslt we obtain the following systems of equalities:

$$\sum_j u_{ij}(\tau) k_j(\tau) - k_i(\tau)\lambda(\tau) = 0 \qquad (i=1, 2, \ldots, m); \tag{2.7.14}$$

$$\sum_j u_{ij}(\tau) k_j^{[1]}(\tau) - k_i^{[1]}(\tau)\lambda(\tau) = k_i(\tau)\lambda^{[1]}(\tau) + \frac{dk_i(\tau)}{d\tau} \qquad (i=1,2,\ldots,m);$$

$$\sum_j u_{ij}(\tau) k_j^{[2]}(\tau) - k_i^{[2]}(\tau)\lambda(\tau) = k_i(\tau)\lambda^{[2]}(\tau) + \tag{2.7.15}$$

$$+k_i^{[1]}(\tau)\lambda^{[1]}(\tau) + \frac{dk_i^{[1]}(\tau)}{d\tau} \qquad (i=1,2,\ldots m); \tag{2.7.15a}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

which the terms of series (2.7.12) should satisfy.

Before proceeding to the solution of obtained algebraic systems, let us examine some general features of systems of linear algebraic equations.

7.1.1. Some information from the theory of systems of linear algebraic equations

Let us examine the algebraic system of linear nonhomogenous equations

$$\begin{array}{l} (u_{11}-\lambda)k_1 + u_{12}k_2 + \ldots u_{1m}k_m = f_1, \\ u_{21}k_1 + (u_{22}-\lambda)k_2 + \ldots + u_{2m}k_m = f_2, \\ \cdots\cdots\cdots\cdots\cdots \\ u_{m1}k_1 + u_{m2}k_2 + \ldots + (u_{mm}k_m - \lambda) = f_m \end{array} \tag{2.7.16}$$

and homogeneous system corresponding to it

$$\begin{array}{l} (u_{11}-\lambda)k_1 + u_{12}k_2 + \ldots + u_{1m}k_m = 0, \\ u_{21}k_1 + (u_{22}-\lambda)k_2 + \ldots + u_{2m}k_m = 0, \\ \cdots\cdots\cdots\cdots\cdots \\ u_{m1}k_1 + u_{m2}k_2 + \ldots + (u_{mm}-\lambda)k_m = 0. \end{array} \tag{2.7.17}$$

In order that homogeneous system (2.7.17) would have nonzero solution, it is necessary and sufficient that the determinant of system

$$\begin{vmatrix} u_{11}-\lambda & u_{12} & \dots & u_{1m} \\ u_{21} & u_{22}-\lambda & \dots & u_{2m} \\ \dots & \dots & \dots & \dots \\ u_{m1} & u_{m2} & \dots & u_{mm}-\lambda \end{vmatrix} \quad (2.7.18)$$

be equal to zero. Determinant (2.7.18) in opened form is a polynomial (characteristic polynomial) of power m, which has m roots: $\lambda_1, \lambda_2, \dots, \lambda_m$ (u_{ij} - real quantities). We will further assume that all these roots are different.

Let us assume λ_σ - one of the roots of characteristic polynomial. When $\lambda = \lambda_\sigma$, inasmuch as λ_σ - simple root of characteristic polynomial, the rank of determinant (2.8.18) is equal to m - 1 and therefore system (2.7.17) with accuracy to arbitrary nonzero coefficient has unique nonzero solution. This means that if

$$k_{1\sigma}, k_{2\sigma}, \dots, k_{m\sigma} \quad (2.7.19)$$

- nonzero solution of system (2.7.17), then all other solutions of this system have the form

$$ck_{1\sigma}, ck_{2\sigma}, \dots, ck_{m\sigma}.$$

Solution of (2.7.19) can be represented in vector K_σ with components $k_{1\sigma}, k_{2\sigma}, \dots, k_{m\sigma}$. The available arbitrariness permits selecting the solution of (2.7.19) in such a way that the following condition would be fulfilled

$$|K_\sigma| = 1. \quad (2.7.20)$$

Let us remember that the modulus of vector A with components $a_1, \dots, a_m$ is defined as the square root of the scalar product of vector A by itself:

$$|A| = \sqrt{(A, A)},$$

and the scalar product of vector A by vector B with components b_1, b_2, ..., b_m in the case when components of vectors are numbers from the field of complex numbers, is represented by equality

$$(A, B) = \sum_{j=1}^{m} a_j \bar{b}_j.$$

Conjugate homogeneous system

$$\begin{aligned} &(u_{11} - \lambda)\mu_1 + u_{21}\mu_2 + \ldots + u_{m1}\mu_m = 0, \\ &u_{12}\mu_1 + (u_{22} - \lambda)\mu_2 + \ldots + u_{m2}\mu_m = 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &u_{1m}\mu_1 + u_{2m}\mu_2 + \ldots + (u_{mm} - \lambda)\mu_m = 0 \end{aligned} \qquad (2.7.21)$$

when $\lambda = \lambda_s$ (λ_s - root characteristic polynomial) with accuracy to within arbitrary nonzero coefficient also has unique nonzero solution, which is written so:

$$\mu_{s1}, \mu_{s2}, \ldots, \mu_{sm}. \qquad (2.7.22)$$

Between the solutions of homogeneous system (2.7.17) and conjugate system (2.7.21) there exist relationships

$$\sum_{j=1}^{m} \mu_{sj} k_{je} \begin{cases} \neq 0 & (s = e), \\ = 0 & (s \neq e). \end{cases} \qquad (2.7.23)$$

This can be shown by the following means.

We have

$$\begin{aligned} &(u_{11} - \lambda_e) k_{1e} + u_{12} k_{2e} + \ldots + u_{1m} k_{me} = 0, \\ &u_{21} k_{1e} + (u_{22} - \lambda_e) k_{2e} + \ldots + u_{2m} k_{me} = 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &u_{m1} k_{1e} + u_{m2} k_{2e} + \ldots + (u_{mm} - \lambda_e) k_{me} = 0 \end{aligned} \qquad (2.7.24)$$

and

$$
\begin{aligned}
&(u_{11}-\lambda_s)\mu_{s1}+u_{21}\mu_{s2}+\ldots+u_{m1}\mu_{sm}=0,\\
&u_{12}\mu_{s1}+(u_{22}-\lambda_s)\mu_{s2}+\ldots+u_{m2}\mu_{sm}=0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&u_{1m}\mu_{s1}+u_{2m}\mu_{s2}+\ldots+(u_{mm}-\lambda_s)\mu_{sm}=0.
\end{aligned}
\qquad (2.7.25)
$$

Let us multiply each i-th line in system (2.7.24) by μ_{si}, and in system (2.7.25) - by $-k_{j\sigma}$ and sum up the results. We obtain

$$(\lambda_s-\lambda_\sigma)(\mu_{s1}k_{1\sigma}+\mu_{s2}k_{2\sigma}+\ldots+\mu_{sm}k_{m\sigma})=0. \qquad (2.7.26)$$

Hence there directly follow relationships (2.7.23).

Nonzero solution (2.7.22) of conjugate system (2.7.21) is also determined with accuracy to arbitrary nonzero coefficient, therefore it can always be selected so that the first equality (2.7.23) would be fulfilled in the following manner:

$$\sum_{j=1}^{m}\mu_{\sigma j}k_{j\sigma}=1. \qquad (2.7.27)$$

Subsequently we will assume that solutions of system (2.7.17) and conjugate system (2.7.21) are selected in accordance with conditions (2.7.20) and (2.7.27).

Finally, let us note one condition of solvability of nonhomogenous system (2.7.16).

In order that the system of linear nonhomogenous equations (2.7.16) when $\lambda = \lambda_\sigma$ would be solvable, it is necessary and sufficient that the following equality be fulfilled

$$\sum_{j=1}^{m}\mu_{\sigma j}\cdot f_j=0, \qquad (2.7.28)$$

where $\mu_{\sigma 1}, \mu_{\sigma 2}, \ldots, \mu_{\sigma m}$ - any solution of conjugate system (2.7.21).

Let us prove this necessity. Let us assume that system (2.7.16) is solvable (commonly) and $\mu_{\sigma 1}$, $\mu_{\sigma 2}$, ..., $\mu_{\sigma m}$ - some solution of conjugate system. Let us sum up equalities (2.7.16) together, having preliminarily multiplied each i-th equality by $\mu_{\sigma i}$. We obtain

$$[(u_{11}-\lambda)\mu_{\sigma 1}+u_{21}\mu_{\sigma 2}+\ldots+u_{m1}\mu_{\sigma m}]k_1+$$
$$+[u_{12}\mu_{\sigma 1}+(u_{22}-\lambda)\mu_{\sigma 2}+\ldots+u_{m2}\mu_{\sigma m}]k_2+$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+[u_{1m}\mu_{\sigma 1}+u_{2m}\mu_{\sigma 2}+\ldots+(u_{mm}-\lambda)\mu_{\sigma m}]k_m=\sum_{j=1}^{m}\mu_{\sigma j}f_j$$

When $\lambda = \lambda_\sigma$ all the square brackets become zero, since $\mu_{\sigma 1}$, $\mu_{\sigma 2}$, ..., $\mu_{\sigma m}$ is the solution of conjugate homogeneous system corresponding to root λ_σ, and, consequently,

$$\sum_{j=1}^{m}\mu_{\sigma j}f_j=0.$$

Adequacy. Let us assume condition (2.7.28) is observed. Let us prove then that system (2.7.16) is common.

According to theorem of Kronecker-Capella the system of linear equations (2.7.16) when $\lambda = \lambda_\sigma$ then and only then is common, when the rank of matrix

$$U=\begin{vmatrix} u_{11}-\lambda_\sigma & u_{12} & \ldots & u_{1m} \\ u_{21} & u_{22}-\lambda_\sigma & \ldots & u_{2m} \\ u_{m1} & u_{m2} & \ldots & u_{mm}-\lambda_\sigma \end{vmatrix}$$

is equal to the rank of expanded matrix

$$U_1=\begin{vmatrix} u_{11}-\lambda_\sigma & u_{12} & \ldots & u_{1m} f_1 \\ u_{21} & u_{22}-\lambda_\sigma & \ldots & u_{2m} f_2 \\ \cdots & \cdots & & \cdots \\ u_{m1} & u_{m2} & \ldots & u_{mm}\ \lambda_\sigma f_m \end{vmatrix}.$$

In the considered case the rank of matrix U is equal to $m - 1$, since λ_σ is the simple root of characteristic polynomial. Let us show that the rank of expanded matrix U_1 is also equal to $m - 1$.

Without detriment for generality let us assume that to nonzero minor of matrix U occupies the left upper corner. For this minor in matrix U_1 there are two bordering minors: one of them - determinant of matrix U, which is equal to zero, and the other - determinant

$$\begin{vmatrix} u_{11}-\lambda_\sigma & u_{12} & u_{1m-1} & f_1 \\ u_{21} & u_{22}-\lambda_\sigma & u_{2m-1} & f_2 \\ u_{m1} & u_{m2} & u_{mm-1} & f_m \end{vmatrix}. \qquad (2.7.29)$$

Determinant (2.7.29) is also equal to zero. Actually, if we multiply lines of this determinant respectively by $\mu_{\sigma 1}, \mu_{\sigma 2}, \ldots, \mu_{\sigma m}$ and then sum them up, then, in view of condition (2.7.28), as a result we obtain zero, which designates the linear dependence of determinant. Hence it follows that determinant of (2.7.29) is equal to zero.

Thus, minors of matrix U_1, bordering the minor of rank $m - 1$ not equal to zero, are equal to zero. Therefore, the rank of matrix U_1 is equal to $m - 1$, i.e., is equal to the rank of matrix U. This means system (2.7.16) when $\lambda = \lambda_\sigma$ is solvable.

7.1.2. Recurrence formulas

Let us return to systems of equations (2.7.14), (2.7.15) and so forth. Let us first consider system (2.7.14). In expanded form this system has the form of equalities (2.7.17). To each root λ_σ of characteristic polynomial there corresponds a solution of system $k_{1\sigma}, k_{2\sigma}, \ldots, k_{m\sigma}$, which, we will consider, satisfies condition (2.7.20). Since coefficients U_{ij} are functions of time, roots $\lambda_1, \lambda_2, \ldots, \lambda_m$ are also functions of time t. Further we will assume that in the considered time interval all roots remain simple, i.e.,

$$|\lambda_\sigma(t) - \lambda_s(t)| > \varepsilon > 0 \quad (s \neq \sigma). \tag{2.7.30}$$

When $\lambda = \lambda_\sigma$ and $k_i = k_{i\sigma}$ equality (2.7.14) is fulfilled identically, and remaining algebraic systems (2.7.15), (2.7.15a), ..., can be written so:

$$\sum_j a_{lj}(\tau) k_{j\sigma}^{[\nu]}(\tau) - k_{l\sigma}^{[\nu]}(\tau) \lambda_\sigma(\tau) = k_{l\sigma}^{(\tau)} \lambda_\sigma^{[\nu]}(\tau) + g_l^{[\nu-1]}(\tau)$$

$$(l = 1, 2, \ldots, m; \quad \nu = 1, 2, \ldots). \tag{2.7.31}$$

Here

$$g_l^{[0]} = \frac{dk_{l\sigma}}{d\tau},$$
$$g_l^{[1]} = k_{l\sigma}^{[1]} \lambda_\sigma^{[1]} + \frac{dk_{l\sigma}^{[1]}}{d\tau}, \ldots \tag{2.7.32}$$

and so forth.

Let us suppose that $k_{j\sigma}^{[1]}$, $\lambda_\sigma^{[1]}$, ..., $k_{j\sigma}^{[\nu-1]}$, $\lambda_\sigma^{[\nu-1]}$ are already determined. Then $g_i^{[\nu-1]}$ is a known function, and the quantities of the next approximation $k_{j\sigma}^{[\nu]}$ $\lambda_\sigma^{[\nu]}$ can be determined from ν-th system (2.7.31) in the following manner.

Let us assume

$$\mu_{s1}, \mu_{s2}, \ldots, \mu_{sm} \tag{2.7.33}$$

is the solution of homogeneous system, conjugate to system (2.7.17), when $\lambda = \lambda_s$, i.e., solution of system

$$\sum_j a_{jl}\mu_j - \mu_l \lambda_s = 0 \quad (l = 1, 2, \ldots, m). \tag{2.7.34}$$

We will assume that all solutions of form (2.7.33), corresponding to roots λ_1, λ_2, ..., λ_m and satisfying, of course, equalities (2.7.23), are selected from condition (2.7.27).

Condition of solvability of ν-th linear system (2.7.31) will be written down so:

$$\sum_{l=1}^{m} \mu_{\sigma l}\left(k_{l\sigma}\lambda_{\sigma}^{[\nu]}+g_{l}^{[\nu-1]}\right)=0.$$

Hence, considering (2.7.27), we find

$$\lambda_{\sigma}^{[\nu]}=-\sum_{l=1}^{m} \mu_{\sigma l} g_{l}^{[\nu-1]}. \qquad (2.7.35)$$

For determination of $k_{j\sigma}^{[\nu]}$ let us multiply each i-th equality of ν-th system (2.7.31) by μ_{si} and sum up the results. We obtain

$$\sum_{i} \mu_{si}\left(\sum_{j} u_{ij} k_{j\sigma}^{[\nu]}-k_{i\sigma}^{[\nu]}\lambda_{\sigma}\right)=\sum_{i} \mu_{si}\left(k_{i\sigma}\lambda_{\sigma}^{[\nu]}+g_{i}^{[\nu-1]}\right),$$

or, by changing the order of summation,

$$\sum_{j} k_{j\sigma}^{[\nu]} \sum_{i} u_{ij}\mu_{si}-\sum_{i} \mu_{si} k_{i\sigma}^{[\nu]}\lambda_{\sigma}=\sum_{i} \mu_{si} k_{i\sigma}\lambda_{\sigma}^{[\nu]}+\sum_{i} \mu_{si} g_{i}^{[\nu-1]}.$$

Hence, taking into account that

$$\sum_{i} u_{ij}\mu_{si}=\lambda_{s}\mu_{sj},$$

[see (2.7.34)], we obtain

$$\sum_{i} \mu_{si} k_{i\sigma}^{[\nu]}\left(\lambda_{s}-\lambda_{\sigma}\right)=\sum_{i} \mu_{si} k_{i\sigma}\lambda_{\sigma}^{[\nu]}+\sum_{i} \mu_{si} g_{i}^{[\nu-1]}. \qquad (2.7.36)$$

When $s \neq \sigma$ $\sum_{i} \mu_{si} k_{is}=0$, $\lambda_{s} \neq \lambda_{\sigma}$ and from (2.7.36) we have

$$\sum_{l=1}^{m} \mu_{sl} k_{l\sigma}^{[\nu]}=\frac{\sum_{l=1}^{m} \mu_{sl} g_{l}^{[\nu-1]}}{\lambda_{s}-\lambda_{\sigma}} \quad (s \neq \sigma).$$

When $s = \sigma$ the right side of equality (2.7.36) is equal to zero in view of (2.7.35) and $\lambda_s = \lambda_\sigma$, so that we can take

$$\sum_{l=1}^{m} p_{\sigma l} k_{l\sigma}^{[\nu]} = c_\sigma^{[\nu]},$$

where $c_\sigma^{[\nu]}$ - arbitrary, necessary number of times, differentiable function.

Thus, the sought quantities $k_{j\sigma}^{[\nu]}$ satisfy the following system of algebraic equations:

$$\sum_{l=1}^{m} p_{sl} k_{l\sigma}^{[\nu]} = \begin{cases} b_{s\sigma}^{[\nu-1]} & (s \neq \sigma), \\ c_\sigma^{[\nu]} & (s = \sigma), \end{cases} \quad (2.7.37)$$

where

$$b_{s\sigma}^{[\nu-1]} = \frac{1}{\lambda_\sigma - \lambda_s} \sum_{l=1}^{m} p_{sl} g_l^{[\nu-1]}. \quad (2.7.38)$$

The determinant of system (2.7.37) is equal to:

$$\Delta = \begin{vmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{vmatrix}.$$

As we can see, the lines of this determinant are nonzero solutions of system (2.7.34) when $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_m$. Since all roots λ_i are different, all m nonzero solutions of system (2.7.34) are linearly independent and, thus, determinant Δ is nonzero. Considering this, by Cramer rule from (2.7.37) we find

$$k_{l\sigma}^{[\nu]} = \sum_{s \neq \sigma} \frac{\Delta_{sl}}{\Delta} b_{s\sigma}^{[\nu-1]} + \frac{\Delta_{\sigma l}}{\Delta} c_\sigma^{[\nu]} \quad (l = 1, 2, \ldots, m).$$

Here Δ_{si} - signed minor of element μ_{si} of determinant Δ.

It is possible to show that ratio $\frac{\Delta_{si}}{\Delta} = k_{is}$. Considering this, we will have

$$k_{l\sigma}^{[\nu]} = \sum_{s \neq \sigma} k_{ls} b_{s\sigma}^{[\nu-1]} + k_{l\sigma} c_{\sigma}^{[\nu]} \quad (l = 1, 2, \ldots, m). \tag{2.7.39}$$

Let us use the arbitrariness, available during determination of $k_{i\sigma}^{[\nu]}$ in the following manner.

Let us assume K_σ - vector with components $k_{1\sigma}$, $k_{2\sigma}$, ..., $k_{m\sigma}$; $K_\sigma^{[\nu]}$ - vector with components $k_{1\sigma}^{[\nu]}$, $k_{2\sigma}^{[\nu]}$, ..., $k_{m\sigma}^{[\nu]}$; $\tilde{K}_\sigma$ - vector with components $\tilde{k}_{1\sigma}$, $\tilde{k}_{2\sigma}$, ..., $\tilde{k}_{m\sigma}$. Then m of the first series (2.7.12) can be written so:

$$\tilde{K}_\sigma(\tau, \varepsilon) = K_\sigma(\tau) + \varepsilon K_\sigma^{[1]}(\tau) + \varepsilon^2 K_\sigma^{[2]}(\tau) + \ldots \tag{2.7.40}$$

Terms of this series $K_\sigma^{[\nu]}$ (ν = 1, 2, ...) in accordance with (2.7.39) are determined by formula

$$K_\sigma^{[\nu]} = \sum_{s \neq \sigma} K_s b_{s\sigma}^{[\nu-1]} + K_\sigma c_\sigma^{[\nu]}. \tag{2.7.41}$$

Let us subordinate the selection of arbitrary functions $c_\sigma^{[\nu]}$ (ν = 1, 2, ...) to condition of equality of modulus of vector K_σ to one. The square of modulus of vector K_σ is equal to:

$$\begin{aligned} |\tilde{K}_\sigma|^2 = (\tilde{K}_\sigma, \tilde{K}_\sigma) = (K_\sigma, K_\sigma) + \varepsilon\left[(K_\sigma, K_\sigma^{[1]}) + (K_\sigma^{[1]}, K_\sigma)\right] + \\ + \varepsilon^2\left[(K_\sigma, K_\sigma^{[2]}) + (K_\sigma^{[1]}, K_\sigma^{[1]}) + (K_\sigma^{[2]}, K_\sigma)\right] + \varepsilon^3 \ldots \end{aligned} \tag{2.7.42}$$

In view of (2.7.20)

$$(K_\sigma, K_\sigma) = 1. \tag{2.7.43}$$

Further, considering (2.7.41) and (2.7.43), we find

$$(K_\sigma, K_\sigma^{[1]}) + (K_\sigma^{[1]}, K_\sigma) = \Big(K_\sigma, \sum_{s \neq \sigma} K_s b_{s\sigma}^{[0]} + K_\sigma c_\sigma^{[1]}\Big) +$$

$$+ \Big(\sum_{s \neq \sigma} K_s b_{s\sigma}^{[0]} + K_\sigma c_\sigma^{[1]}, K_\sigma\Big) = \sum_{s \neq \sigma} (K_\sigma, K_s b_{s\sigma}^{[0]}) + \sum_{s \neq \sigma} (K_s b_{s\sigma}^{[0]}, K_\sigma) + 2c_\sigma^{[1]}.$$

Hence it is apparent that if we take

$$c_\sigma^{[1]} = -\frac{1}{2} \sum_{s \neq \sigma} \left[(K_\sigma, K_s b_{s\sigma}^{[0]}) + (K_s, K_\sigma b_{s\sigma}^{[0]})\right].$$

then the component in (2.7.42), containing ε in the first power, will become zero.

By the same method it is possible to select $c_\sigma^{[2]}$ so that in (2.7.42) the component containing ε^2 and so forth would become zero. Thus, during corresponding selection of $c_\sigma^{[1]}$, $c_\sigma^{[2]}$, $c_\sigma^{[3]}$, ... we will have

$$|\bar{K}_\sigma| = 1. \tag{2.7.44}$$

While not stipulating especially, further we will assume that the selection of functions $c_\sigma^{[\nu]}$ is subordinate to condition (2.7.44).

The obtained recurrent formulas (2.7.35) and (2.7.39) permit successively determining the terms of series (2.7.12) [and, consequently, series (2.7.40)] by means of which the particular solution of differential equations (2.7.9), corresponding to root $\lambda_\sigma(\tau)$ of characteristic polynomial, is represented in the form of (2.7.10). To each root $\lambda_s(\tau)$ (s = 1, 2, ..., m) there corresponds a particular solution of such form.

By assuming in (2.7.12) $\varepsilon = 1$, we obtain particular solutions of the original system (2.7.3):

$$x_{i\sigma}(t)=\tilde{x}_{i\sigma}(t)e^{\int \tilde{\lambda}_\sigma(t)dt} \quad (i=1,\ldots,m;\ \sigma=1,\ldots,m). \tag{2.7.45}$$

corresponding to roots $\lambda_1(t)$, $\lambda_2(t)$, ..., $\lambda_m(t)$ of characteristic polynomial. Here

$$\tilde{x}_{i\sigma}(t)=\tilde{x}_{i\sigma}(\tau,\varepsilon)\big|_{\varepsilon=1}, \quad \tilde{\lambda}_\sigma(t)=\tilde{\lambda}_\sigma(\tau,\varepsilon)\big|_{\varepsilon=1}.$$

In vector writing the particular solutions (2.7.45) have the form

$$X_\sigma(t)=\tilde{X}_\sigma(t)e^{\int \tilde{\lambda}_\sigma(t)dt} \quad (\sigma=1,2,\ldots,m), \tag{2.7.46}$$

where X_σ - vector with components $x_{1\sigma}$, $x_{2\sigma}$, ..., $x_{m\sigma}$.

Constructed particular solutions are formed solutions of system (2.7.3), since the convergence of series (2.7.12) is not proven. For practical purposes this is not necessary, since the time consumption of calculations sharply increases with increase of number ν, and for all practical purposes there can be determined only a certain number of the first terms of series (2.7.12).

By retaining in series (2.7.12) a finite number of first terms, we will have approximate solutions of equations (2.7.3). Calculations show that during examination of equations of disturbed motion of a rocket it is sufficient to be limited only by the first two terms of expansion (2.7.12). The account of subsequent terms does not introduce substantial correction into the result.

7.2. Stability Criteria

Let us investigate the behavior of particular solutions in the vicinity of fixed moment of time t_0.

Modulus of particular solution of differential equations (2.7.3) is equal to:

$$|X_\sigma|=\sqrt{(X_\sigma, X_\sigma)}=\sqrt{(\tilde{X}_\sigma, \tilde{X}_\sigma)e^{2\mathrm{Re}\int \tilde{\lambda}_\sigma(t)dt}}$$

or, considering (2.7.44),

$$|X_\sigma| = e^{\int \text{Re}\,\tilde{\lambda}_\sigma(t)\,dt}. \tag{2.7.47}$$

By differentiating (2.7.47), we obtain

$$\frac{d|X_\sigma|}{dt} = \text{Re}\,\tilde{\lambda}_\sigma(t)\,e^{\int \text{Re}\,\tilde{\lambda}_\sigma(t)\,dt}. \tag{2.7.48}$$

Let us assume Re $\tilde{\lambda}_\sigma(t_0) < 0$. Then with respect to continuity inequality $\tilde{\lambda}_\sigma(t) < 0$ takes place within a certain finite interval of time. According to (2.7.48) in this interval the modulus of particular solution X_σ is a diminishing function.

If Re $\tilde{\lambda}_\sigma(t_0) > 0$, then, at least in a rather small vicinity of point t_0, X_σ is a function increasing a modulus.

From this follows the following conditions of stability and instability of undisturbed motion [trivial solution of equations (2.7.3)].

If all functions $\tilde{\lambda}_\sigma(t)$ satisfy inequalities

$$\text{Re}\,\tilde{\lambda}_\sigma(t_0) < 0 \quad (\sigma = 1, 2, \ldots, m), \tag{2.7.49}$$

then undisturbed motion [trivial solution of equations (2.7.3)] possesses stability in finite time interval $[t_0, t_0 + \Delta t]$; if however among functions $\tilde{\lambda}_\sigma$ there is at least one function $\tilde{\lambda}_i(t)$, such that

$$\text{Re}\,\tilde{\lambda}_i(t_0) > 0,$$

then undisturbed motion is unstable ($\Delta t = 0$).

Let us substitute the values of functions $\tilde{\lambda}_\sigma$ in (2.7.49). Then the condition of stability will be written so:

$$\mathrm{Re}\left(\lambda_\sigma + \lambda_\sigma^{[1]} + \lambda_\sigma^{[2]} + \ldots\right) < 0 \quad (\sigma = 1, 2, \ldots, m). \qquad (2.7.50)$$

By retaining in the left sides of inequalities (2.7.50) only the first terms, we obtain stability criterion in very simple form, coinciding with criterion of stability according to the method of quenched coefficients:

$$\mathrm{Re}\,\lambda_\sigma < 0 \quad (\sigma = 1, 2, \ldots, m). \qquad (2.7.51)$$

By retaining the two first terms in decompositions λ_σ, we obtain refined criterion of stability, already considering the variability of coefficients of equations:

$$\mathrm{Re}\left(\lambda_\sigma + \lambda_\sigma^{[1]}\right) < 0 \quad (\sigma = 1, 2, \ldots, m).$$

Having substituted here the values of $\lambda_\sigma^{[1]}$, we will have

$$\mathrm{Re}\left(\lambda_\sigma - \sum_{l=1}^{m} \nu_{\sigma l} \frac{dk_{l\sigma}}{dt}\right) < 0 \quad (\sigma = 1, 2, \ldots, m). \qquad (2.7.52)$$

The following approximation gives

$$\mathrm{Re}\left(\lambda_\sigma + \lambda_\sigma^{[1]} + \lambda_\sigma^{[2]}\right) < 0$$

and so forth.

7.3. The Applicability of Approximate Criteria of Stability

Stability and instability of undisutrbed motion, as was shown above, are determined depending on the signs of functions

$$\mathrm{Re}\left(\lambda_\sigma + \lambda_\sigma^{[1]} + \ldots + \lambda_\sigma^{[l]}\right) \quad (\sigma = 1, 2, \ldots, m). \qquad (2.7.53)$$

The time consumption of calculation of terms of series

$$\lambda_s + \lambda_s^{[1]} + \lambda_s^{[2]} + \ldots \tag{2.7.54}$$

sharply increases with increase in serial number, therefore it is convenient to use the provided stability criterion at small l. Inasmuch as these criteria are not accurate, then when among functions (2.7.53) there are some very close to zero, approximate criteria of stability become doubtful. Therefore, it is important, on the one hand, to correctly determine number l, and on the other - to be able to evaluate the authenticity of approximate criteria of stability with selected l. Strict solution of these questions is very difficult. It is possible to recommend the following practical method of selection of l and evaluation of authenticity of approximate criteria of stability. We will consider that number l is selected correctly, and corresponding stability criterion possesses sufficient authenticity, if as l there has been selected the least whole positive number or 0, at which

$$\frac{\left|\operatorname{Re}\lambda_s^{[l+1]}(t_0)\right|}{\left|\operatorname{Re}\left[\lambda_s(t_0) + \lambda_s^{[1]}(t_0) + \ldots + \lambda_s^{[l]}(t_0)\right]\right|} \ll 1, \quad (s = 1, 2, \ldots, m), \tag{2.7.55}$$

understanding the strengthened inequality in the sense that its left side is at least one order less than the right side.

Condition (2.7.55) can be used, specifically, for determination of the applicability of the method of quenched coefficients. Thus, when fulfilling inequalities

$$\frac{\left|\operatorname{Re}\lambda_s^{[1]}(t_0)\right|}{\left|\operatorname{Re}\lambda_s(t_0)\right|} \ll 1 \quad (s = 1, 2, \ldots, m) \tag{2.7.56}$$

approximate condition of stability will have the form

$$\operatorname{Re}\lambda_s < 0 \quad (s = 1, 2, \ldots m), \tag{2.7.57}$$

which coincides in accuracy with the condition of stability according to the method of quenched coefficients. Nonfulfillment of condition

(2.7.56), at least at one σ, designates the inapplicability of the method of quenched coefficients.

Stability criteria, established above, are valid under the condition that all roots of characteristic polynomial are simple. With the presence of equal roots these criteria are inapplicable, since in this instance the recurrence formulas for $K_\sigma^{[\kappa]}$ and $\lambda_\sigma^{[\kappa]}$ lose meaning.

In conclusion - several words about the influence of variability of coefficients of equations on the stability of motion of the system.

By analyzing the relationships, which determine $\lambda_\sigma^{[1]}$, $\lambda_\sigma^{[2]}$, ..., it can be expected that the influence of variability of coefficients of equations of disturbed motion, generally speaking, will be greater in those sections of the trajectory where derivatives $dk_{j\sigma}/dt$ are great, i.e., where there is a sharp change of components of solution of equations (2.7.14). Calculations of particular systems confirm this assumption. For an example let us give results of calculations, connected with the investigation of free oscillations of a mechanical system, motion of which is described by system of equations of form

$$\sum_j m_{ij}(t)\frac{d^2q_j}{dt^2}+\sum_j n_{ij}(t)q_j=0 \quad (i=1,2,\ldots,m), \tag{2.7.58}$$

where $m_{ij} \equiv m_{ji}$, $n_{ij} \equiv n_{ji}$.

Figure 2.14 shows change of three nonzero roots of characteristic polynomial of differential equations, obtained as a result of conversion of equations (2.7.58) to standard form. These and other nonzero roots are purely imaginary and therefore Re $\lambda_\sigma = 0$. According to the simplest stability criterion the system is located on the boundary of the region of stability. This result coincides with conclusions obtained by the method of quenched coefficients, and expresses the physical fact that if coefficients of equations would be constants, then oscillations of the system would proceed with constant amplitudes. The variability of coefficients not only quantitatively, but also qualitatively changes the picture. On Fig. 2.15 there are presented functions $\lambda_\sigma^{[1]}$. Calculation of subsequent corrections of $\lambda_\sigma^{[2]}$ showed that on the entire considered range of time $\lambda_\sigma^{[2]}$ there are negligible values, so that the influence of variability of coefficients of equations, at least in the given example, is practically completely considered by functions $\lambda_\sigma^{[1]}$.

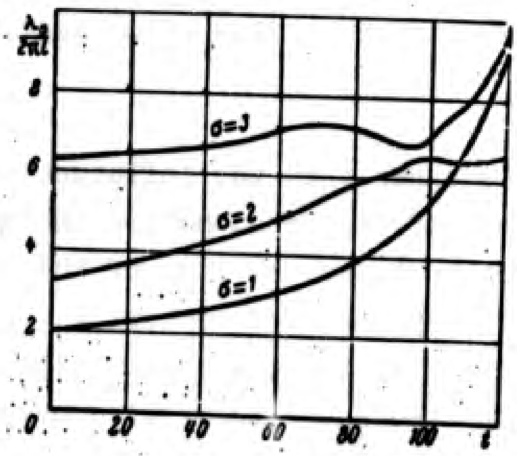

Fig. 2.14.

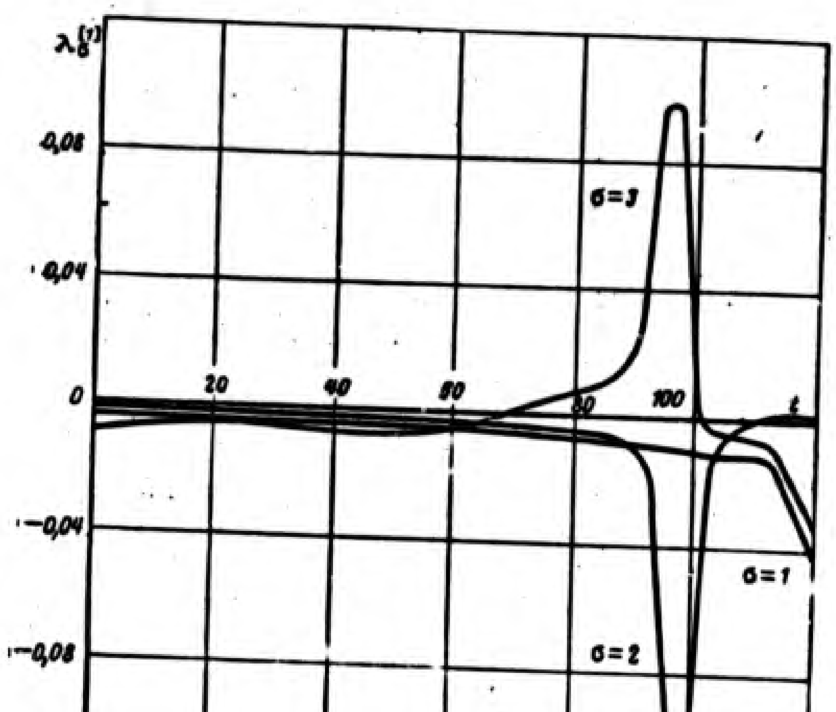

Fig. 2.15.

From Fig. 2.15 it is evident that the variability of coefficients can show both a damping influence, and, conversely, can be the cause for excitation of oscillations in the system. In this case the variability of coefficients leads to damping of oscillations of the second tone and buildup of oscillations of the third tone. These effects are manifested the most strongly in the section where roots λ_2 and λ_3 approach each other. At the same time, the approach and even intersection of roots λ_1 and λ_2 did not noticeably change the character of shape of curves Re $(\lambda_\sigma + \lambda^{[1]})$. This means that convergence of roots of characteristic polynomial will not necessarily involve increase of the influence of variability of coefficients.

More detailed analysis shows that the influence of variability of coefficients of equations of disturbed motion on the stability of undisturbed motion is great in those sections of the trajectory where there is a sharp change of components of vectors K_σ, consisting of solutions of system (2.7.14). In turn, a sharp change of components of these vectors can be expected where there is convergence of roots of characteristic polynomial with each other. On those sections of the trajectory where among the roots of characteristic polynomial there are no close roots, the influence of variability of coefficients on the character of motion of the object is small, and therefore the investigation of stability within the framework of the method of quenched coefficients can be sufficient, if, of course, the time interval, within limits of which motion of the system is considered, is not too great: for a large time interval, even with small effect from the variability of coefficients, the amplitude of oscillations can grow to inadmissibly large values.

§ 8. Effectiveness and Maximum Permissible Deflections of Controls

The effectiveness and maximum permissible deflections are basic indices of controls.

The effectiveness of controls is characterized by the ratio of controlling moment, which appears with unit deflection of controls, to the corresponding inertia moment of the flight vehicle. Thus, the effectiveness of pitch control is determined by dynamic coefficient $c_{\vartheta\delta}/J_z$; the effectiveness of yaw controls is determined by dynamic coefficient $c_{\psi\delta}/J_y$; the effectiveness of roll controls is determined by dyanmic coefficient $c_{\gamma\delta}/J_x$.

The effectiveness and maximum permissible deflections of controls determine the maximum controlling moments, which can be created by the controls. Maximum moments, being created by the controls of pitch, yaw and roll, are referred to corresponding inertia moments of the rocket, are equal to:

$$\frac{c_{\vartheta\delta}}{J_z}\delta_{\vartheta\,max}, \quad \frac{c_{\psi\delta}}{J_y}\delta_{\psi\,max}, \quad \frac{c_{\gamma\delta}}{J_x}\delta_{\gamma\,max},$$

where $\delta_{\vartheta\,max}$, $\delta_{\psi\,max}$, $\delta_{\gamma\,max}$ - maximum permissible deflections of controls of pitch, yaw and roll respectively.

Requirements of the effectiveness and limiting deflections of controls are developed when designing the control system of the rocket depending on the problems being placed on these systems and their operating conditions. These requirements can usually be satisfied at various combinations of effectiveness and limiting deflections of controls, and the assignment of the designer is to determine the optimum ratio between these characteristics. All things being equal, the effectiveness of controls is directly proportional to the area of air and jet vanes, engine thrust, utilized for creation of controlling moment, and so forth. Therefore, the natural tendency of the designer toward economy of weight causes a tendency to decrease of the effectiveness of controls. However, decrease in the size and weight of controls does not necessarily lead to decrease in the weight of the entire rocket. The necessity of increase in the maximum permissible deflection of controls in connection with decrease of its effectiveness in the final analysis can lead to increase in the weight of the entire rocket, since with large deflections of controls there is a decrease in their effectiveness, increase of aerodynamic drag of air vanes and losses of thrust of swiveling engines, utilized for creation of controlling moment. Furthermore, increase in the limiting deflections of controls can be difficult with respect to various structural considerations.

Thus, problems of determination of the required effectiveness and limiting deflections of controls are closely interconnected and must be solved together for finding optimum solutions.

The solution of these problems is conveniently sought by means of comparison of several variants, which are distinguished by different values of limiting deflections of controls. The calculation of each variant is reduced to determination of the required effectiveness of controls at predetermined values of limiting deflections and at the condition that the vehicle possesses the necessary dynamic qualitites, for example with the action of various disturbing factors on the vehicle in flight the disturbances of parameters of motion at the end of the powered-flight phase do not exceed definite limits. In such formulation for determining the required effectiveness of controls it is necessary to know how parameters of disturbed motion of the vehicle are changed with the presence of limitations on deflections of controls, i.e., under the condition that

$$|\delta'(t)| \leqslant \delta_{max}. \qquad (2.8.1)$$

Thus, the selection of basic parameters of controls (effectiveness and limiting deflections) are connected with determination of disturbances of parameters of motion of the rocket in condition (2.8.1). For determination of the disturbances of parameters of motion it is necessary to integrate equations of disturbed motion together with the accepted law of control.

Condition (2.8.1) can be somewhat simplified. Taking into account that

$$\delta'(t)=\delta(t)+\Delta\delta(t),$$

where $\delta(t)$ - deflection of control, required for flight along a programmed trajectory, according to (2.8.1) we have

$$|\delta(t)+\Delta\delta(t)|\leqslant\delta_{max}. \quad (2.8.2)$$

A ballistic rocket is a low-maneuverable flight vehicle, and required deflections of controls during flight along the programmed trajectory are small. Therefore, condition (2.8.2), which must be satisfied in all controlled flight phases, can be replaced by approximate condition

$$|\Delta\delta(t)|\leqslant\delta_{max}. \quad (2.8.3)$$

8.1. Effectiveness of Pitch and Yaw Controls

Disturbed motion of the rocket in pitching and yawing planes is described by similar equations, therefore investigation of the effectivenesses and selection of rational parameters of pitching and yawing controls can be conducted by identical methods. Considering this, we will be limited here by examination only of yawing motion.

Disturbed yawing motion of the rocket is described by system of equations (1.13.9). The law of control in a rather general case has the form

$$T_2^2\frac{d^2\Delta\delta_\psi}{dt^2}+T_1\frac{d\Delta\delta_\psi}{dt}+\Delta\delta_\psi=k\Delta\psi+k_1\frac{d\Delta\psi}{dt}-k_{V_z}\Delta V_z-k_z\Delta z. \quad (2.8.4)$$

Thus, closed system of equations of disturbed yawing motion of the rocket is represented so:

$$\begin{aligned} m\frac{d\Delta V_z}{dt}+v_z\frac{d\Delta\psi}{dt}+c_{zz}\Delta V_z+c_{z\psi}\Delta\psi&=c_{z\delta}\Delta\delta_\psi+\Delta F_z,\\ J_y\frac{d^2\Delta\psi}{dt^2}+\mu_y\frac{d\Delta\psi}{dt}+c_{\psi z}\Delta V_z+c_{\psi\psi}\Delta\psi&=c_{\psi\delta}\Delta\delta_\psi+\Delta M_y. \end{aligned} \quad (2.8.5)$$

$$k_1 \frac{d\Delta\psi}{dt} - k_{V_z}\Delta V_z + k\Delta\psi - k_z\Delta z = T_2^2 \frac{d^2\Delta\delta_\psi}{dt^2} + T_1 \frac{d\Delta\delta_\psi}{dt} + \Delta\delta_\psi,$$

$$\frac{d\Delta z}{dt} = \Delta V_z. \qquad (2.8.5 \text{ Cont'd.})$$

Unknown functions are $\Delta z(t)$, $\Delta V_z(t)$, $\Delta\psi(t)$ and $\Delta\delta(t)$. In order to judge the effectiveness of controls of the rocket, it is necessary to know how these functions are changed on the trajectory with predetermined initial values and the action of all types of factory on the rocket.

Thus, the problem of investigation of controllability of a rocket in yawing leads to the necessity of integration of system of equations (2.8.5) with assigned initial conditions:

$$\Delta z(t_0) = \Delta z_0, \quad \left.\frac{d\Delta z}{dt}\right|_{t=t_0} = \Delta V_{z0}, \quad \Delta\psi(t_0) = \Delta\psi_0,$$
$$\left.\frac{d\Delta\psi}{dt}\right|_{t=t_0} = \Delta\dot{\psi}_0, \quad \Delta\delta_\psi(t_0) = \Delta\delta_{\psi_0}, \quad \left.\frac{d\Delta\delta_\psi}{dt}\right|_{t=t_0} = \Delta\dot{\delta}_0 \qquad (2.8.6)$$

and limitation

$$|\Delta\delta_\psi(t)| \leqslant \delta_{\psi\,max}. \qquad (2.8.7)$$

Limitation of deflection controls introduces nonlinearity into the problem. This difficulty can be overcome by the following means.

Let us suppose that $\Delta\delta_\psi = \Delta\delta_\psi(t)$ is the solution of system (2.8.5) with initial conditions (2.8.6) without allowing for condition (2.8.7). This solution, beginning from initial moment t_0, remains valid even with condition (2.8.7) while

$$|\Delta\delta_\psi(t)| < \delta_{\psi\,max}$$

or

$$|\Delta\delta_\psi(t)| = \delta_{\psi\,max},$$

but in the latter case - if $|\Delta\delta_\psi(t)|$ - nonincreasing function.

Let us assume at some moment of time t_1 deflection of control reaches a limiting value and in this case $|\Delta\delta_\psi(t)|$ - increasing function.

Then starting from this moment the disturbed motion of the rocket will be described by other equations, namely: equations

$$m\frac{d\Delta V_z}{dt}+v_z\frac{d\Delta\psi}{dt}+c_{zz}\Delta V_z+c_{z\psi}\Delta\psi=\pm c_{z\delta}\delta_{\psi\,max}+\Delta F_z,$$
$$J_y\frac{d^2\Delta\psi}{dt^2}+\mu_y\frac{d\Delta\psi}{dt}+c_{\psi z}\Delta V_z+c_{\psi\psi}\Delta\psi=\pm c_{\psi\delta}\delta_{\psi\,max}+\Delta M_y,$$
$$\frac{d\Delta z}{dt}=\Delta V_z \qquad (2.8.8)$$

with initial conditions:

$$\Delta z(t_1)=\Delta z_1,\ \left.\frac{d\Delta z}{dt}\right|_{t=t_1}=\Delta V_{z1},$$
$$\Delta\psi(t_1)=\Delta\psi_1,\ \left.\frac{d\Delta\psi}{dt}\right|_{t=t_1}=\Delta\dot{\psi}_1,$$

where Δz_1, ΔV_{z1}, $\Delta\psi_1$, $\Delta\psi_1$ - values at moment of time t_1 of corresponding functions, representing the solution of system (2.8.5) at initial conditions (2.8.6). Before coefficients $c_{z\delta}$ and $c_{\psi\delta}$ one should accept sign "+", if $\Delta\delta_\psi(t_1) > 0$, and "-" if $\Delta\delta(t_1) < 0$.

System (2.8.8) describes disturbed motion of the rocket right up to moment of time t_2, when

$$|k\Delta\psi+k_1\Delta\dot{\psi}-k_{V_z}\Delta V_z-k_z\Delta z|=\delta_{\psi\,max}, \qquad (2.8.9)$$

the left side of equality (2.8.9) at moment t_2 - diminishing function.

Beginning wit moment of time t_2 again the disturbed motion is described by system (2.8.5) and so forth.

Thus, sought quantities $\Delta z(t)$, $\Delta V_z(t)$, $\Delta\psi(t)$ can be obtained as solutions of systems (2.8.5) and (2.8.8), "joined" with each other at boundary points t_1, t_2, ...

Let us suppose that from those or other considerations there are prescribed maximum permissible values of disturbances Δz, ΔV_z, $\Delta\psi$ $\Delta\dot{\psi}$ (either on the entire trajectory or on its separate phases). Then, by solving systems of equations of disturbed motion at various fixed values of parameters of controls, it is possible to select rational values of parameters of controls under the condition that limitations, placed on parameters of motion, are not disturbed.

Integration of equations should be performed separately, assuming each time that the vehicle is affected by disturbing forces and moment, connected to one of the disturbing factors. The action of all types of disturbances can be summarized by root-mean-square law, if we assume that these disturbances are independent and each of them is subordinate to a standard law of distribution. Let us assume, for example, x_1, ..., x_n - values of parameter of motion x with action n independent disturbances. Then the selection of parameters of the controls must be subordinate to condition

$$\sqrt{\sum_i x_i^2} \leqslant \bar{x},$$

where $\bar{x}$ - maximum permissible value of parameter of motion x.

Basic disturbances, influencing the selection of effectiveness of controls, are wind and disturbances from the propulsion system.

Systems of equations (2.8.5) and (2.8.8) can generally be integrated by numerical method. With some additional simplifications the construction of solutions of these systems in quadratures is possible.

Let us suppose that beginning with moment of time t_0 the vehicle is affected by constant disturbing forces (ΔF_z = const, ΔM_y = const) and that disturbed motion is described by differential equations, coefficients of which are also constant. At moment of time t_0 the transient process is started, in which at first there is a rapid change of angular parameters of motion. If the rocket is stable, then soon the angular parameters assume values, equal to, or more accurately, close to their values in steady state. Subsequently these parameters remain constants or, by slowly changing, asymptotically approach steady values of quantities, so that their derivatives are close to zero. Therefore, disturbed motion of the vehicle after termination of the stage of rapid change of angular parameters can approximately be described by equations, which are obtained from initial equations of disturbed motion, if in the latter we disregard derivatives of angular quantities.

If disturbing forces and coefficients of equations change slowly, then after the first stage of disturbed motion, when there is a rapid change of angular parameters of motion, a certain quasi-static mode of flight sets in, in which the angular quantities are changed, but slowly, so that in this instance the derivatives of angular parameters remain small quantities. Considering this, equations of disturbed motion can be simplified, having dropped derivatives of angular parameters of motion and derivatives of angles of deflection of controls.

Concerning the first stage of disturbed motion, here the equations of disturbed motion can be considerably simplified. If the rocket is stable, then the first stage of disturbed motion is finished relatively rapidly and during this time the dynamic coefficients of equations are changed little. Therefore, during analysis of the effectiveness of controls during the first stage of disturbed motion, which begins as a result of the action of disturbing forces according to step law, or brief action of disturbances, as, for example, during separation of stages, etc., the variability of coefficients can be disregarded. Then the problem is reduced to integration of linear differential equations with constant coefficients, whi[illegible] no longer presents difficulty.

At the first stage of disturbed motion the angular parameters undergo rapid change, and velocity ΔV_z and coordinate Δz are changed insignificantly. Therefore, it is possible to further simplify equations (2.8.5), having dropped the first and fourth equations and disregarding the effect of disturbances ΔV_z, Δz on the rotatory motion of the rocket around the center of mass.

Thus, when the considered disturbed motion of a stable rocket has a quasi-static character, the equations of disturbed motion can be simplified, having dropped the terms containing derivatives of angular quantities; during analysis of processes at the first stage of disturbed motion, which sets in as a result of the action of various disturbances, it is possible to use the method of quenching of coefficients, and also to disregard the effect of disturbances ΔV_z, Δz on rotatory motion of the rocket.

8.1.1. Integration of equations in the case of slow change of disturbing action with the absence of control of motion of center of mass ($k_z = k_{V_z} = 0$)

By dropping derivatives of angular quantities in equations (2.8.5) and taking into account that in this case $k_z = k_{V_z} = 0$, we obtain the following system of equations, which describe approximately the disturbed motion of the rocket in quasi-static mode:

$$
\begin{aligned}
&m\frac{d\Delta V_z}{dt}+c_{zz}\Delta V_z+c_{z\psi}\Delta\psi=c_{z\delta}\Delta\delta_\psi+\Delta F_z,\\
&c_{\psi z}\Delta V_z+c_{\psi\psi}\Delta\psi=c_{\psi\delta}\Delta\delta_\psi+\Delta M_y,\\
&k\Delta\psi=\Delta\delta_\psi.
\end{aligned}
\tag{2.8.10}
$$

In the case when the vanes occupy limiting positions, we have

$$
\begin{aligned}
&m\frac{d\Delta V_z}{dt}+c_{zz}\Delta V_z+c_{z\psi}\Delta\psi=\pm c_{z\delta}\delta_{\psi\,\max}+\Delta F_z,\\
&c_{\psi z}\Delta V_z+c_{\psi\psi}\Delta\psi=\pm c_{\psi\delta}\delta_{\psi\,\max}+\Delta M_y.
\end{aligned}
\tag{2.8.11}
$$

System (2.8.11) acts up to the moment of time when the first (after the vanes reached stops) relationships are simultanelusly fulfilled:

$$|k\Delta\psi| = \delta_{\psi\,max}, \quad \frac{d^l}{dt^l}|k\Delta\psi| < 0. \tag{2.8.12}$$

Here l designates the smallest derivative of $|k\Delta\psi|$ of order not equal to 0.

Let us first examine system (2.8.10).

Assuming that $c_{\psi\psi} - kc_{\psi\delta} \neq 0$, from the second and third equations we find

$$\Delta\psi = \frac{1}{c_{\psi\psi} - kc_{\psi\delta}}(\Delta M_y - c_{\psi z}\Delta V_z), \tag{2.8.13}$$

$$\Delta\delta = \frac{k}{c_{\psi\psi} - kc_{\psi\delta}}(\Delta M_y - c_{\psi z}\Delta V'_z). \tag{2.8.14}$$

Having substituted the values of $\Delta\psi$ and $\Delta\delta$ in the first equation of system (2.8.10), we obtain

$$\frac{d\Delta V_z}{dt} + P(t)\Delta V_z = Q(t), \tag{2.8.15}$$

where

$$P = \frac{1}{m}\left(c_{zz} + \frac{kc_{\psi z}c_{z\delta} - c_{z\psi}c_{\psi z}}{c_{\psi\psi} - kc_{\psi\delta}}\right),$$

$$Q = \frac{1}{m}\left(\Delta F_z + \frac{kc_{z\delta} - c_{z\psi}}{c_{\psi\psi} - kc_{\psi\delta}}\right). \tag{2.8.16}$$

General solution of linear differential equation (2.8.15) has the form

$$\Delta V_z = e^{-\int\limits_{t_0}^{t} P\,dt}\left(c + \int\limits_{t_0}^{t} Q e^{\int\limits_{t_0}^{t'} P\,dt} dt'\right). \qquad (2.8.17)$$

Arbitrary constant c is determined by initial value of ΔV_z. By assuming in (2.8.17) $t = t_0$, we obtain

$$\Delta V_z(t_0) = \Delta V_{z0} = c.$$

Thus,

$$\Delta V_z = e^{-\int\limits_{t_0}^{t} P\,dt}\left(\Delta V_{z0} + \int\limits_{t_0}^{t} Q e^{\int\limits_{t_0}^{t'} P\,dt} dt'\right). \qquad (2.8.18)$$

Formulas (2.8.13) and (2.8.14) together with expression (2.8.18) represent the solution of system (2.8.10).

By the same means we can determine parameters of motion ΔV_z and $\Delta\psi$ when the controls occupy limiting positions.

From the second equation of system (2.8.11) we have

$$\Delta\psi = \frac{1}{c_{\psi\psi}}(\Delta M_y \pm c_{\psi\delta}\delta_{\psi\,max} - c_{\psi z}\Delta V_z). \qquad (2.8.19)$$

By substituting (2.8.19) in the first equation of system (2.8.11), we again obtain linear differential equation of form (2.8.15), only now

$$P = \frac{1}{m}\left(c_{zz} - \frac{c_{z\psi}c_{\psi z}}{c_{\psi\psi}}\right),$$

$$Q = \frac{1}{m}\left[\Delta F_z \pm \left(c_{z\delta} - \frac{c_{z\psi}}{c_{\psi\psi}}\right)\delta_{\psi\,max} - \frac{c_{z\psi}}{c_{\psi\psi}}\Delta M_y\right]. \qquad (2.8.20)$$

8.1.2. Account of control of velocity of center of mass ($k_z = 0$, $k_{V_z} \neq 0$)

From (2.8.5) in this instance ensue approximate equations, describing rocket motion in quasi-static mode:

$$
\begin{aligned}
&m\frac{d\Delta V_z}{dt}+c_{zz}\Delta V_z+c_{z\psi}\Delta\psi=c_{z\delta}\Delta\delta_\psi+\Delta F_z,\\
&c_{\psi z}\Delta V_z+c_{\psi\psi}\Delta\psi=c_{\psi\delta}\Delta\delta_\psi+\Delta M_y,\\
&-k_{V_z}\Delta V_z+k\Delta\psi=\Delta\delta_\psi.
\end{aligned}
\qquad (2.8.21)
$$

This system also leads to linear differential equation of the first order

$$\frac{d\Delta V_z}{dt}+P\Delta V_z=Q. \qquad (2.8.22)$$

Here

$$
\begin{aligned}
&P=\frac{1}{m}\left[c_{zz}+\frac{kc_{\psi z}c_{z\delta}-k_{V_z}(c_{z\psi}c_{\psi\delta}+c_{\psi\psi}c_{z\delta})-c_{z\psi}c_{\psi z}}{c_{\psi\psi}-kc_{\psi\delta}}\right],\\
&Q=\frac{1}{m}\left(\Delta F_z+\frac{kc_{z\delta}-c_{z\psi}}{c_{\psi\psi}-kc_{\psi\delta}}\Delta M_y\right).
\end{aligned}
\qquad (2.8.23)
$$

Deflection $\Delta\psi$ and $\Delta\delta_\psi$ are determined by formulas:

$$
\begin{aligned}
&\Delta\psi=\frac{1}{c_{\psi\psi}-kc_{\psi\delta}}[\Delta M_y-(c_{\psi z}+k_{V_z}c_{\psi\delta})\Delta V_z],\\
&\Delta\delta=\frac{1}{c_{\psi\psi}-kc_{\psi\delta}}[k\Delta M_y-(kc_{\psi z}+k_{V_z}c_{\psi\psi})\Delta V_z],
\end{aligned}
$$

which follow from the second and third equations of system (2.8.21).

8.1.3. Account of control of velocity and lateral displacement of center of mass

Quasi-static mode of disturbed motion in this instance is described by equations:

$$m\frac{d\Delta V_z}{dt}+c_{zz}\Delta V_z+c_{z\psi}\Delta\psi=c_{z\delta}\Delta\delta_\psi+\Delta F_z,$$
$$c_{\psi z}\Delta V_z+c_{\psi\psi}\Delta\psi=c_{\psi\delta}\Delta\delta_\psi+\Delta M_y, \qquad (2.8.26)$$
$$-k_z\Delta z-k_{V_z}\Delta V_z+k\Delta\psi=\Delta\delta,$$
$$\frac{d\Delta z}{dt}=\Delta V_z.$$

From the second and third equations we have

$$\Delta\psi=\frac{1}{c_{\psi\psi}-kc_{\psi\delta}}[\Delta M_y-(c_{\psi z}+c_{\psi\delta}k_{V_z})\Delta V_z-k_z c_{\psi\delta}\Delta z], \qquad (2.8.27)$$

$$\Delta\delta=\frac{1}{c_{\psi\psi}-kc_{\psi\delta}}[k\Delta M_y-(kc_{\psi z}+k_{V_z}c_{\psi\psi})\Delta V_z-k_z c_{\psi\psi}\Delta z]. \qquad (2.8.28)$$

Having substituted the values of $\Delta\psi$ and $\Delta\delta$ in the first equation of system (2.8.26), we obtain

$$\frac{d\Delta V_z}{dt}+P(t)\Delta V_z+R(t)\Delta z=Q(t), \qquad (2.8.29)$$

where P and Q are determined by formulas (2.8.23), and

$$R=\frac{k_z c_{\psi\psi} c_{z\delta}}{m(c_{\psi\psi}-kc_{\psi\delta})}. \qquad (2.8.30)$$

Considering the fourth equation of system (2.8.26), we will have

$$\frac{d^2\Delta z}{dt^2}+P(t)\frac{d\Delta z}{dt}+R(t)\Delta z=Q(t). \qquad (2.8.31)$$

Thus, in the considered case the problem is reduced to construction of a solution of equations (2.8.31) at initial conditions

$$\Delta z(t_0)=\Delta z_0, \quad \Delta V_z(t_0)=\Delta V_{z0}. \qquad (2.8.32)$$

This equation can be integrated numerically. Having determined Δz and then ΔV_z, it is possible to even calculate $\Delta\psi$ and $\Delta\delta$ by formulas (2.8.27) and (2.8.28).

8.1.4. Integration of equations at the first stage of disturbed motion

As was already said, during investigation of the effectiveness of controls at the first stage of disturbed motion it is possible to disregard the variability of coefficients of equations, and also the effect of disturbances ΔV_z and Δz on the rotatory motion of the rocket. In this case system of equations (2.8.5) assumes the form

$$\begin{gathered} J_y \frac{d^2\Delta\psi}{dt^2} + \mu_y \frac{d\Delta\psi}{dt} + c_{\psi\psi}\Delta\psi - c_{\psi\delta}\Delta\delta_\psi = \Delta M_y, \\ -k_1 \frac{d\Delta\psi}{dt} - k\Delta\psi + T_2^2 \frac{d^2\Delta\delta_\psi}{dt^2} + T_1 \frac{d\Delta\delta_\psi}{dt} + \Delta\delta_\psi = 0. \end{gathered} \tag{2.8.33}$$

Here J_y, $c_{\psi\psi}$ and $c_{\psi\delta}$ and all the remaining coefficients - constants. Equations (2.8.33) can be used every time, when controllability is investigated with large initial disturbances of parameters of motion or intermittent action of external disturbances. For integration of equations it is convenient to use Laplace transformations.

Initial conditions:

$$\Delta\psi(t_0) = \Delta\psi_0, \quad \left.\frac{d\Delta\psi}{dt}\right|_{t=t_0} = \Delta\dot{\psi}_0, \quad \Delta\delta_\psi(t_0) = \Delta\delta_{\psi 0}, \quad \left.\frac{d\Delta\delta_\psi}{dt}\right|_{t=t_0} = \Delta\dot{\delta}_{\psi 0}. \tag{2.8.34}$$

Having transformed equations (2.8.33) according to Laplace, we obtain

$$\begin{gathered} (J_y p^2 + \mu_y p + c_{\psi\psi})\psi(p) - c_{\psi\delta}\delta_\psi(p) = \\ = M_y(p) + (pJ_y + \mu_y)\Delta\psi_0 + J_y\Delta\dot{\psi}_0, \\ -(k_1 p + k)\psi(p) + (T_2^2 p^2 + T_1 p + 1)\delta_\psi(p) = \\ = T_2^2(p\Delta\delta_{\psi 0} + \Delta\dot{\delta}_{\psi 0}) + T_1\Delta\delta_{\psi 0} - k_1\Delta\psi_0, \end{gathered}$$

where $\psi(p)$, $\delta_\psi(p)$, $M_y(p)$ - Laplace transforms of functions $\Delta\psi$, $\Delta\delta_\psi$, ΔM_y respectively.

By solving the obtained system, for example relative to $\delta_\psi(p)$, we will have

$$\delta_\psi(p)=\frac{1}{(J_y p^2+\mu_y p+c_{\psi\psi})(T_2^2 p^2+T_1 p+1)-c_{\psi\delta}(k_1 p+k)}\{[M_y(p)+$$

$$+(pJ_y+\mu_y)\Delta\psi_0+J_y\Delta\dot{\psi}_0](k_1 p+k)+(J_y p^2+\mu_y p+c_{\psi\psi})[T_2^2(p\Delta\delta_{\psi 0}+$$

$$+\Delta\dot{\delta}_{\psi 0})+T_1\Delta\delta_{\psi 0}-k_1\Delta\psi_0]\}. \quad (2.8.35)$$

If we now perform inverse Laplace transformations, then we find function $\Delta\delta_\psi$:

$$\Delta\delta_\psi(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\delta_\psi(p)e^{pt}dp. \quad (2.8.36)$$

Integral (2.8.36) is calculated with the aid of residues of integrands. Construction of $\Delta\delta_\psi(t)$ is performed in the following sequence.

1. For prescribed disturbances $\Delta M_y(t)$ there is determined transform $M_y(p)$:

$$M_y(p)=\int_0^\infty \Delta M_y(t)e^{-pt}dt.$$

For most types of disturbances encountered in practice the corresponding transforms can be found in tables, which are listed in textbooks according on the theory of automatic control.

2. Expression $\delta_\psi(p)$ is represented as the relationship of two polynomials $M(p)$ and $N(p)$:

$$\delta_\psi(p)=\frac{M(p)}{N(p)}.$$

3. Zeros of polynomial $N(p)$ are determined, i.e., those values of p_i at which $N(p_i) = 0$.

4. Residues of function $\delta_\psi(p)$ are determined. Residue of the function, corresponding to simple pole p_i (pole of function $\delta_\psi(p)$ is zero of polynomial $N(p)$, if p_i is not also zero, and function $M(p)$), in this case is equal to:

$$R_i = \lim_{p \to p_i} \frac{M(p)\, e^{pt}(p - p_i)}{N(p)} = \frac{M(p_i)}{\left(\frac{dN}{dp}\right)_{p=p_i}} e^{p_i t}.$$

5. The right side of expression (2.8.36) is equivalent to the sum of all residues of integrands. Thus,

$$\Delta\delta_\psi(t) = \sum_i \frac{M(p_i)}{\left(\frac{dN}{dp}\right)_{p=p_i}} e^{p_i t}. \tag{2.8.37}$$

By this means it is possible to also construct function $\Delta\psi(t)$.

By using expression (2.8.37), it is simple to determine maximum values of $|\Delta\delta_\psi(t)|$. For this let us equate derivative $\frac{d\Delta\delta_\psi}{dt}$ to zero We obtain, by designating through t^* the values of the argument, at which this derivative is equal to zero,

$$\sum_i \frac{p_i M(p_i)\, e^{p_i t^*}}{\left(\frac{dN}{dp}\right)_{p=p_i}} = 0. \tag{2.8.38}$$

By determining t^* from this equality and substituting it in (2.8.37), we will have

$$\Delta\delta_{\psi\,max,\,min}(t^*)=\sum_i \frac{M(p_i)}{\left(\frac{dN}{dp}\right)_{p=p_i}} e^{p_i t^*}. \qquad (2.8.39)$$

Inasmuch as the process is oscillatory, equality (2.8.38), generally speaking, will be fulfilled at several values of the argument. However, by knowing that the vehicle is stable, it should be expected that maximum deflection of the vane will take place with minimum of t^*, but it is necessary to bear in mind that other possibilities are not excluded.

Analogously it is possible to construct function $\Delta\psi$ under conditions when the vanes occupy limiting positions.

In this instance from (2.8.8) we have approximate relationship

$$J_y \frac{d^2\Delta\psi}{dt^2}+\mu_y \frac{d\Delta\psi}{dt}+c_{\psi\psi}\Delta\psi=\pm c_{\psi\delta}\delta_{\psi\,max}+\Delta M_y. \qquad (2.8.40)$$

As a result of Laplace transformation we obtain

$$(J_y p^2+\mu_y p+c_{\psi\psi})\psi(p)=\pm\frac{c_{\psi\delta}\delta_{\psi\,max}}{p}+M_y(p)+ \\ +(J_y p+\mu_y)\Delta\psi_0+J_y\Delta\dot{\psi}_0. \qquad (2.8.41)$$

Hence

$$\psi(p)=\frac{1}{p(J_y p^2+\mu_y p+c_{\psi\psi})}[\pm c_{\psi\delta}\delta_{\psi\,max}+pM_y(p)+ \\ +(J_y p+\mu_y)\Delta\psi_0+J_y\Delta\dot{\psi}_0]. \qquad (2.8.42)$$

Inverse Laplace transformation of function $\psi(p)$ gives

$$\Delta\psi(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\psi(p)e^{pt}dp. \qquad (2.8.43)$$

Further there is calculated the right side of relationship (2.8.43) according to the scheme given above.

8.2. Effectiveness of Controls on Rolling

Disturbed rolling motion of the rocket is described by equation

$$J_x \frac{d^2\Delta\gamma}{dt^2} + \mu_x \frac{d\Delta\gamma}{dt} = c_{\gamma\delta}\Delta\delta_\gamma + \Delta M_x. \qquad (2.8.44)$$

Let us take the law of control, for example, in the form

$$T_2^2 \frac{d^2\Delta\delta_\gamma}{dt^2} + T_1 \frac{d\Delta\delta_\gamma}{dt} + \Delta\delta_\gamma = k\Delta\gamma + k_1 \frac{d\Delta\gamma}{dt}. \qquad (2.8.45)$$

Quasi-static mode of rolling motion of the rocket is described by equations, which are obtained from (2.8.44) and (2.8.45), if we drop the derivatives of angular quantities:

$$\begin{gathered} c_{\gamma\delta}\Delta\delta_\gamma + \Delta M_x = 0, \\ \Delta\delta_\gamma = k\Delta\gamma. \end{gathered} \qquad (2.8.46)$$

Hence

$$\Delta\delta_\gamma = -\frac{\Delta M_x}{c_{\gamma\delta}} = -\frac{\Delta M_x/J_x}{c_{\gamma\delta}/J_x}, \qquad (2.8.47)$$

i.e., angle of deflection of vane is directly proportional to the effectiveness of the vane.

Steady value of the angular of roll is equal to:

$$\Delta\gamma = \frac{\Delta\delta_\gamma}{k} = -\frac{\Delta M_x/J_x}{kc_{\gamma\delta}/J_x}. \qquad (2.8.48)$$

During investigation of the effectiveness of controls on roll in transient conditions the coefficients of equation (2.8.44) can be considered as constant. Then the solutions of equations (2.8.44) and (2.8.45) under initial conditions

$$\Delta\gamma(t_0)=\Delta\gamma_0, \qquad \Delta\delta_\gamma(t_0)=\Delta\delta_{\gamma 0}$$
$$\left.\frac{d\Delta\gamma}{dt}\right|_{t=t_0}=\Delta\dot{\gamma}_0 \quad \left.\frac{d\Delta\delta_\gamma}{dt}\right|_{t=t_0}=\Delta\dot{\delta}_{\gamma 0} \tag{2.8.49}$$

can be obtained so. Having performed Laplace transformation of these equations, we obtain

$$(J_x p^2+\mu_x p)\gamma(p)-c_\gamma\delta(p)=M_x p+(pJ_y+\mu_x)\Delta\gamma_0+J_y\Delta\dot{\gamma}_0,$$
$$-(k_1 p+k)\gamma(p)+(T_2^2p^2+T_1p+1)\delta_\gamma(p)=$$
$$=(T_2^2p+T_1)\Delta\delta_{\gamma 0}+T_2^2\Delta\dot{\delta}_{\gamma 0}-k_1\Delta\gamma_0. \tag{2.8.50}$$

Here $\gamma(p)$, $\delta_\gamma(p)$, $M_x(p)$ - transforms of function $\Delta\gamma$, $\Delta\delta_\gamma$, ΔM_x according to Laplace.

By solving the system of algebraic equations (2.8.50) relative to $\delta_\gamma(p)$ and $\gamma(p)$ it is possible to then determine the sought functions $\Delta\delta_\gamma(t)$ and $\Delta\gamma(t)$ in accordance with expressions:

$$\Delta\delta_\gamma(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\delta_\gamma(p)e^{pt}\,dp, \quad \Delta\gamma(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\gamma(p)e^{pt}\,dp. \tag{2.8.51}$$

CHAPTER III

STABILIZATION OF ROCKET MOTION WITH ACCOUNT OF MOBILITY OF FLUIDS IN FUEL TANKS

§ 1. Basic Statements About the Physical Properties of Liquid Propellant. Velocity Potential

With derivation of differential motion equations of a rocket, considering the mobility of fluids in fuel tanks, we will consider the liquid components of propellant as *perfect and incompressible fluids*. The question of the account of viscosity of fluids in equations of motion of a rocket will be illuminated below, in § 22. If motions, being accomplished by fluids in the initial launch system of coordinates x_0, y_0, z_0, at some moment of time will be vortex-free, then, as is known from the course of hydrodynamics (see [8]), they will remain vortex-free subsequently in accordance with the perfectness of these fluids being assumed by us. We will consider that at some initial moment of time the fluids are in a state of rest with respect to the initial launch coordinates system x_0, y_0, z_0. In this case all subsequent motions of fluids, being accomplished by them in system of coordinates x_0, y_0, z_0, will be *vortex-free*, since the state of rest is a particular case of vortex-free motion.

Having established specific numbering for fuel tanks of the rocket, by $\vec{v}_j$ let us designate the vector, which determines velocities, with which particles of fluid, located in the tank with number j, move in initial launch system of coordinates. Since motions, accomplished by fluids in initial launch system of coordinates, are assumed vortex-free, the rotor of vector $\vec{v}_j$ at any moment of time should satisfy condition

$$\operatorname{rot} \vec{v}_j = 0 \tag{3.1.1}$$

in the region occupied by fluid in the tank with number j. According to this, the velocity field of fluid, being determined by vector $\vec{v}_j$, must be *potential*, in other words vector $\vec{v}_j$ should be expressed by the gradient of its potential. Thus, there must take place equality

$$\vec{v}_j = \operatorname{grad} \Phi_j, \tag{3.1.2}$$

where Φ_j – velocity potential for fluid, which is located in the tank with number j.

Divergence of vector $\vec{v}_j$ at any moment of time t must satisfy condition

$$\operatorname{div} \vec{v}_j = 0 \tag{3.1.3}$$

in the region occupied by fluid in j-th tank, in accordance with the assumed incompressibility of liquid components of propellant.

According to (3.1.2) and (3.1.3) the velocity potential must satisfy Laplace equation

$$\operatorname{div} \operatorname{grad} \Phi_j = \nabla^2 \Phi_j = 0 \tag{3.1.4}$$

in the region occupied by fluid in the tank with number j. In the following paragraph there are examined boundary conditions, which must be fulfilled on the boundary of this region.

§ 2. Boundary Conditions for Velocity Potentials

Velocity $\vec{v}_j$, at which the particle of fluid moves in initial launch system of coordinates x_0, y_0, z_0, can be represented in the form of the sum of velocity of following, being generated by motion of fixed system of coordinates x, y, z, and relative velocity $\vec{v}_{j\ \text{отн}}$, at the given particle of fluid moves in coordinate system x, y, z. By using the formula for velocity of following known from theoretical mechanics, we obtain relationship

$$\vec{v}_j=\vec{v}_0+\vec{\omega}\times\vec{r}+\vec{v}_{j\ \text{отн}}, \tag{3.2.1}$$

where $\vec{v}_0$ – velocity at which the origin of coordinate system x, y, z moves; $\vec{\omega}$ – angular velocity of rotation of fixed system of coordinates; $\vec{r}$ – radius vector of the examined particle of fluid in movable coordinate system x, y, z.[1] According to (3.1.2) and (3.2.1) there should take place equality

$$\operatorname{grad}\Phi_j=\vec{v}_0+\vec{\omega}\times\vec{r}+\vec{v}_{j\ \text{отн}}. \tag{3.2.2}$$

In accordance with formula (3.2.2), having established boundary conditions for relative velocity $\vec{v}_{j\ \text{отн}}$, the desired boundary conditions for velocity potential Φ_j. Let us examine the conditions, which must be satisfied by relative velocity $\vec{v}_{j\ \text{отн}}$ on the surface of the tank being wetted and on the free surface of fluid.

---

[1]We will subsequently consider that the origin of movable system of coordinates coincides with the center of mass of the rocket when the fluids in the fuel tanks are limited by flat free surfaces, standard for the longitudinal axis of the rocket. In connection with this in contrast to Chapter I the origin of movable system of coordinates will be designated by the letter O. With oscillations of free surfaces of fluids the center of mass C will be deflected from point O and velocity $\vec{v}_O$ will differ from velocity $\vec{v}_C$.

Let us designate by S_j the internal surface of j-th tank and by σ_j – the plane section of this tank, normal to axis x and constructed thus so that the volume of region V_j, limited by surfaces S_j and σ_j, would be equal to the volume of fluid located in the tank (Fig. 3.1). If $x = x_j(t)$ – equation of plane σ_j, then the equation of free surface of fluid, which we will designate subsequently by σ_j^*, can be presented in the form

$$x = x_j(t) + f_j(y, z, t), \tag{3.2.3}$$

where $f_j(y, z, t)$ – function, determining deflections of the points of free surface of fluid σ_j^* from plane σ_j.

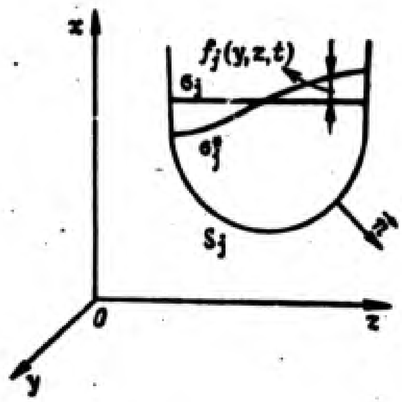

Fig. 3.1.

Let us examine first the particle motion of fluid, flowing around the wetted surface S_j. Vector of relative velocity of motion of this particle $\vec{v}_{j\,\text{отн}}$ will always lie in the plane tangent to surface S_j. By $\vec{n}$ let us designate the unit vector of the external normal to surface S_j (see Fig. 3.1). In the considered case vectors $\vec{v}_{j\,\text{отн}}$ and $\vec{n}$ must be mutually orthogonal, their scalar product must be equal to zero and, thus, on the wetted surface there must be fulfilled condition

$$\vec{v}_{j\,\text{отн}} \cdot \vec{n} = 0 \text{ on } S_j. \qquad (3.2.4)$$

Let us examine further the relative particle motion of fluid, remaining in the process of this motion on free surface σ_j^*. By A let us designate the point, at which this particle of fluid is located at moment of time t, and by A_1 – the point, to which it is moved at moment of time $t + dt$ (Fig. 3.2). Movement of $\vec{AA}_1$, accomplished by the particle of fluid in fixed system of coordinates for element of time dt, will be equal to $\vec{v}_{j\,\text{отн}} dt$. Point A_1 will lie on the free surface of fluid, corresponding to moment of time $t + dt$ and shown in Fig. 3.2 by dotted line. In accordance with the equation of free surface (3.2.3) the point of this surface with coordinates y, z at the moment of time $t + dt$ will have coordinate $x + dx$, determined by relationship

$$x + dx = x_j(t + dt) + f_j(y, z, t + dt). \qquad (3.2.5)$$

According to (3.2.3) and (3.2.5) there will take place equality

$$dx = \frac{dx_j}{dt} dt + \frac{\partial f_j}{\partial t} dt. \qquad (3.2.6)$$

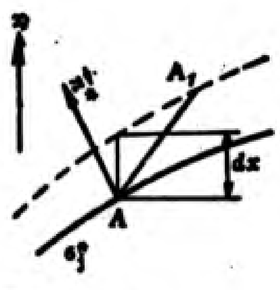

Fig. 3.2.

As can be seen from Fig. 3.2, projection of movement $\vec{AA}_1$ in the direction of external normal n^* to surface σ_j^* will be equal to $dx \cos(n^*, x)$, or according to (3.2.6)

$$\left(\frac{dx_j}{dt}+\frac{\partial f_j}{dt}\right)\cos(n^*, x)\,dt. \qquad (3.2.7a)$$

At the same time, this projection can be found in terms of scalar by multiplying movement $\vec{AA}_1$ by unit vector of the external normal to surface σ_j^*, which we designate by $\vec{n}^*$. In this instance for the considered projection we obtain expression

$$\vec{v}_{j\,\text{отн}}\cdot\vec{n}^*\,dt. \qquad (3.2.7b)$$

By comparing expressions (3.2.7a) and (3.2.7b), we arrive at boundary condition

$$\vec{v}_{j\,\text{отн}}\cdot\vec{n}^*=\left(\frac{dx_j}{dt}+\frac{\partial f_j}{\partial t}\right)\cos(n^*, x) \text{ on } \sigma_j^*. \qquad (3.2.8)$$

According to (3.2.2), (3.2.4) and (3.2.8) velocity potential must satisfy boundary conditions

$$\begin{aligned}
&\operatorname{grad}\Phi_j\cdot\vec{n}=\vec{v}_0\cdot\vec{n}+(\vec{\omega}\times r)\cdot\vec{n} \text{ on } S_j,\\
&\operatorname{grad}\Phi_j\cdot\vec{n}^*=\vec{v}_0\cdot\vec{n}^*+(\vec{\omega}\times\vec{r})\cdot\vec{n}^*+\\
&+\left(\frac{dx_j}{dt}+\frac{\partial f_j}{\partial t}\right)\cos(n^*, x) \text{ on } \sigma_j^*.
\end{aligned} \qquad (3.2.9)$$

By using equalities:

$$\begin{aligned}
&\operatorname{grad}\Phi_j\cdot\vec{n}=\frac{\partial\Phi_j}{\partial x}\cos(n, x)+\frac{\partial\Phi_j}{\partial y}\cos(n, y)+\\
&+\frac{\partial\Phi_j}{\partial z}\cos(n, z)=\frac{\partial\Phi_j}{\partial n}\\
&\operatorname{grad}\Phi_j\cdot\vec{n}^*=\frac{\partial\Phi_j}{\partial x}\cos(n^*, x)+\frac{\partial\Phi_j}{\partial y}\cos(n^*, y)+\\
&+\frac{\partial\Phi_j}{\partial z}\cos(n^*, z)=\frac{\partial\Phi_j}{\partial n^*}
\end{aligned} \qquad (3.2.10)$$

and equalities:

$$(\vec{\omega}\times\vec{r})\cdot\vec{n}=(\vec{r}\times\vec{n})\cdot\vec{\omega},\quad (\vec{\omega}\times\vec{r})\cdot\vec{n}^{*}=(\vec{r}\times\vec{n}^{*})\cdot\vec{\omega}, \tag{3.2.11}$$

ensuing from the possibility of cyclic permutation of coefficients in mixed product of vectors, it is possible to reduce boundary conditions (3.2.9) to the form

$$\begin{aligned}&\frac{\partial\Phi_j}{\partial n}=\vec{v}_0\cdot\vec{n}+\vec{\omega}\cdot(\vec{r}\times\vec{n}) \text{ on } S_j,\\ &\frac{\partial\Phi_j}{\partial n^{*}}=\vec{v}_0\cdot\vec{n}^{*}+\vec{\omega}\cdot(\vec{r}\times\vec{n}^{*})+\\ &\qquad+\left(\frac{dx_j}{dt}+\frac{\partial f_j}{\partial t}\right)\cos(n^{*},x) \text{ on } \sigma_j^{*}.\end{aligned} \tag{3.2.12}$$

If on the investigated section of rocket flight the tank with number j is not emptied, plane σ_j, shown in Fig. 3.1, does not change its position in fixed system of coordinates x, y, z and function $x_j(t)$ keeps a constant value. In this instance boundary conditions (3.2.12) take the form

$$\begin{aligned}&\frac{\partial\Phi_j}{\partial n}=\vec{v}_0\cdot\vec{n}+\vec{\omega}\cdot(\vec{r}\times\vec{n}) \text{ on } S_j,\\ &\frac{\partial\Phi_j}{\partial n^{*}}=\vec{v}_0\cdot\vec{n}^{*}+\vec{\omega}\cdot(\vec{r}\times\vec{n}^{*})+\\ &\qquad+\frac{\partial f_j}{\partial t}\cos(n^{*},x) \text{ on } \sigma_j^{*}.\end{aligned} \tag{3.2.13}$$

For the emptying tanks boundary conditions (3.2.12) must be augmented by the boundary condition, which should be fulfilled in the intake sections of expended fuel line . In this instance the velocity potential will be made up of potential Φ_j, which satisfies boundary conditions (3.2.13), and additional velocity potential, determining the process of drain of fluid from an immobile tank, in the course of which the free surface of fluid remains flat and perpendicular to xis x. Since drain of fluid for all practical purposes is not reflected on oscillations, accomplished by the fluid in the fuel tank, we will limit ourselves to account of velocity potential Φ_j, determined by boundary conditions (3.2.13), and in connection with this, in all further considerations we will proceed from differential equation (3.1.4) with boundary conditions (3.2.13).

§ 3. Small Oscillations of Fluids in Fuel Tanks

As was already shown in the preface, during examination of the question about stabilization of rocket motion with allowance for mobility of fluids in fuel tanks we will be limited to investigation of small oscillations of fluids, in the process of which the free surfaces σ_j^* remain close to plane surfaces σ_j (see Fig. 3.1). In this instance boundary value problem for velocity potential Φ_j, obtained in the previous paragraph, can be simplified, while subsequently will significantly facilitate motion stability analysis.

Let us introduce into consideration function Ψ_j, connected to velocity potential Φ_j by relationship

$$\Psi_j=\Phi_j-\vec{v}_0\cdot\vec{r}=\Phi_j-v_{0x}x-v_{0y}y-v_{0z}z. \tag{3.3.1}$$

By differentiating both parts of relationship (3.3.1), we obtain equalities:

$$\begin{gathered}\nabla^2\Psi_j=\nabla^2\Phi_j,\\ \frac{\partial\Psi_j}{\partial n}=\frac{\partial\Phi_j}{\partial n}-v_{0x}\cos(n,x)-v_{0y}\cos(n,y)-v_{0z}\cos(n,z)=\\ =\frac{\partial\Phi_j}{\partial n}-\vec{v}_0\cdot\vec{n}\text{ on }S_j,\\ \frac{\partial\Psi_j}{\partial n^*}=\frac{\partial\Phi_j}{\partial n^*}-v_{0x}\cos(n^*,x)-v_{0y}\cos(n^*,y)-\\ -v_{0z}\cos(n^*,z)=\frac{\partial\Phi_j}{\partial n^*}-\vec{v}_0\cdot\vec{n}^*\text{ on }\sigma_j^*.\end{gathered} \tag{3.3.2}$$

In accordance with formulas (3.1.4), (3.2.13) and (3.3.2) function Ψ_j must satisfy Laplace equation

$$\nabla^2 \Psi_j = 0 \tag{3.3.3}$$

and boundary conditions:

$$\begin{aligned} &\frac{\partial \Psi_j}{\partial n} = \vec{\omega} \cdot (\vec{r} \times \vec{n}) \text{ on } S_j, \\ &\frac{\partial \Psi_j}{\partial n^*} = \vec{\omega} \cdot (\vec{r} \times \vec{n}^*) + \frac{\partial f_j}{\partial t} \cos(n^*, x) \text{ on } \sigma_j^* \end{aligned} \tag{3.3.4}$$

According to (3.1.2) and (3.3.1) there will take place relationship

$$\vec{v}_j = \vec{v}_0 + \operatorname{grad} \Psi_j. \tag{3.3.5}$$

From formulas (3.2.1) and (3.3.5) ensues equality

$$\vec{\omega} \times \vec{r} + \vec{v}_{j\,\text{отн}} = \operatorname{grad} \Psi_j. \tag{3.3.6}$$

In accordance with the assumption about the slowness of rotation of the rocket and about the smallness of oscillations of fluids in the fuel tanks, velocities $\vec{\omega} \times \vec{r} + \vec{v}_{j\ \text{OTH}}$ will always be small and allowing relatively small errors when determining their potential Ψ_j, we obtain errors of the second order of smallness when seeking absolute velocities of motion of particles of fluid $\vec{v}_j$. At small oscillations of fluid in the tank with number j the free surface of liquid σ_j^* will remain close to plane σ_j. Thus, by disregarding small quantities of the second order of smallness during investigation of motion of fluid, it is possible to transfer the second of boundary conditions (3.3.4) from surface σ_j^* to close surface σ_j, substituting in this case the direction of normal n^* to surface σ_j^* by direction of normal n to surface σ_j. Boundary conditions (3.3.4) in this instance will take the form

$$\frac{\partial \Psi_j}{\partial n} = \vec{\omega}\cdot(\vec{r}\times\vec{n}) \text{ on } S_j,$$
$$\frac{\partial \Psi_j}{\partial n} = \vec{\omega}\cdot(\vec{r}\times\vec{n}) + \frac{\partial f_j}{\partial t} \text{ on } \sigma_j \qquad (3.3.7)$$

(on surface σ_j cos $(n, x) = 1$, since the direction of external normal n to plane σ_j coincides with direction of axis x).

The question about construction of the solution of Laplace equation (3.3.3), satisfying boundary conditions (3.3.7), is comprehensively examined by us in the following paragraphs of this chapter.

§ 4. Zhukovskiy Potentials

Let us first examine the case when the surface of fluid, located in the j-th tank, is closed by a cap, located in section of tank σ_j. As can be seen from Fig. 3.1, in this instance function $f_j(y, z, t)$ in the process of motion will remain identically equal to zero and in accordance with this, boundary conditions (3.3.7) will take the form

$$\frac{\partial \Psi_j}{\partial n} = \vec{\omega}\cdot(\vec{r}\times\vec{n}) \text{ on } S_j \text{ and } \sigma_j. \qquad (3.4.1)$$

Let us introduce into consideration vector function $\vec{\varphi}_j$, satisfying Laplace equation

$$\nabla^2\vec{\varphi}_j = 0 \qquad (3.4.2)$$

and boundary condition

$$\frac{\partial \vec{\varphi}_j}{\partial n} = \vec{r}\times\vec{n} \text{ on } S_j \text{ and } \sigma_j. \qquad (3.4.3)$$

According to (3.4.2) and (3.4.3) function

$$\Psi_j = \vec{\omega}\cdot\vec{\varphi}_j \qquad (3.4.4)$$

will satisfy differential equation (3.3.3) and boundary condition (3.4.1). Thus, in the examined case the problem about motion of fluid, located in the tank with number j, is reduced to solution of differential equation (3.4.2) with boundary condition (3.4.3).

Problem of finding function ϕ, satisfying Laplace equation

$$\nabla^2\varphi=0 \tag{3.4.5}$$

in the region limited by surface S, and satisfying boundary condition

$$\frac{\partial\varphi}{\partial n}=F \text{ on } S, \tag{3.4.6}$$

where F – function assigned on boundary surface S, carries the name Neumann problem for Laplace equation. For solvability of Neumann problem it is necessary and sufficient that function F satisfy condition

$$\int_S F\,ds=0 \tag{3.4.7}$$

(see [21]). Thus, necessary and sufficient conditions of solvability of differential equation (3.4.2) with boundary condition (3.4.3) in vector form can be expressed by equality

$$\int_{S_j+\sigma_j} \vec{r}\times\vec{n}\,ds=0. \tag{3.4.8}$$

Let us show that condition (3.4.8) is always fulfilled. Having converted surface integrals into volumetric by means of Gauss divergence formulas we find:

$$\begin{aligned}
&\int_{S_j+\sigma_j} [y\cos(n,z)-z\cos(n,y)]\,ds=\int_{V_j}\left(\frac{\partial y}{\partial z}-\frac{\partial z}{\partial y}\right)dv=0,\\
&\int_{S_j+\sigma_j} [z\cos(n,x)-x\cos(n,z)]\,ds=\int_{V_j}\left(\frac{\partial z}{\partial x}-\frac{\partial x}{\partial z}\right)dv=0,
\end{aligned} \tag{3.4.9}$$

$$\int_{s_j+\sigma_j} [x\cos(n,y)-y\cos(n,x)]ds=\int_{V_j}\left(\frac{\partial x}{\partial y}-\frac{\partial y}{\partial x}\right)dv=0 \quad (3.4.9 \text{ cont'd})$$

(V_j - region lying inside closed surface $S_j + \sigma_j$).

Scalar equalities (3.4.9) are equivalent to vector equality (3.4.8). Thus, boundary value problem, formed by differential equation (3.4.2) with boundary condition (3.4.3), is always solvable, which it was required to prove.

According to (3.3.5) and (3.4.4) the velocity of motion of particles of fluid in the considered case will be determined by formula

$$\vec{v}_j=\vec{v}_0+\text{grad}(\vec{\omega}\cdot\vec{\varphi}_j). \qquad (3.4.10)$$

Differential equation (3.4.5) with boundary condition (3.4.6) determines the desired function ϕ with accuracy to arbitrary constant component. In accordance with this equation (3.4.2) with boundary condition (3.4.3) determines the vector function $\vec{\phi}_j$, with accuracy to arbitrary constant vector component. According to (3.4.10) the form of vector function $\vec{v}_j$, determining the velocity field, does not depend on the selection of this constant component and can be taken as any.

When $\omega_x = 1$, $\omega_y = 0$, $\omega_z = 0$ formula (3.4.10) assumes the form

$$\vec{v}_j=\vec{v}_0+\text{grad}\,\varphi_{jx}, \qquad (3.4.11)$$

to oase $\omega_x = 0$, $\omega_y = 1$, $\omega_z = 0$ there corresponds equality

$$\vec{v}_j=\vec{v}_0+\text{grad}\,\varphi_{jy} \qquad (3.4.12)$$

and in case $\omega_x = 0$, $\omega_y = 0$, $\omega_z = 1$ velocity $\vec{v}_j$ will be determined by formula

$$\vec{v}_j=\vec{v}_0+\text{grad}\,\varphi_{jz}. \qquad (3.4.13)$$

Thus, projections ϕ_{jx}, ϕ_{jy}, ϕ_{jz} of vector $\vec{\phi}_j$ are velocity potentials, appearing with rotation of the rocket body with unit angular velocity around axes x, y, z respectively. Velocity potentials ϕ_{jx}, ϕ_{jy}, ϕ_{jz} were introduced into consideration for the first time by N. Ye. Zhukovskiy, therefore functions ϕ_{jx}, ϕ_{jy} and ϕ_{jz} we will subsequently call *Zhukovskiy potentials*.

§ 5. Small Oscillations of Fluid in an Immobile Tank. Motion Potential.

Let us now examine small oscillations of fluids, which can take place in the fuel tanks of an immobile rocket, occupying a vertical position (case of the rocket standing for launch). Assuming in (3.3.5) and (3.3.7) $\vec{v}_0 = 0$, $\vec{\omega} = 0$, for vector $\vec{v}_j$ we obtain formula

$$\vec{v}_j = \operatorname{grad} \Psi_j, \tag{3.5.1}$$

where Ψ_j – solution of Laplace equation (3.3.3), which satisfies boundary conditions:

$$\frac{\partial \Psi_j}{\partial n} = 0 \text{ on } S_j, \quad \frac{\partial \Psi_j}{\partial n} = \frac{\partial f_j}{\partial t} \text{ on } \sigma_j. \tag{3.5.2}$$

Let us introduce into examination function ψ_j, satisfying Laplace equation

$$\nabla^2 \psi_j = 0 \tag{3.5.3}$$

and boundary conditions

$$\frac{\partial \psi_j}{\partial n} = 0 \text{ on } S_j, \quad \frac{\partial \psi_j}{\partial n} = f_j \text{ on } \sigma_j. \tag{3.5.4}$$

According to (3.5.3) and (3.5.4) function

$$\Psi_j = \frac{\partial \psi_j}{\partial t} \qquad (3.5.5)$$

will satisfy differential equation (3.3.3) and boundary condition (3.5.2) in accordance with equalities:

$$\nabla^2 \frac{\partial \psi_j}{\partial t} = \frac{\partial \nabla^2 \psi_j}{\partial t}, \quad \frac{\partial}{\partial n}\left(\frac{\partial \psi_j}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial \psi_j}{\partial n}\right), \qquad (3.5.6)$$

ensuing from the possibilities of change of order of differentiation of the function, depending upon several independent variables. Thus, in the considered case the problem about motion of fluid, located in the tank with number j, can be reduced to solution of differential equation (3.5.3) with boundary conditions (3.5.4). For solvability of this Neumann problem it is necessary and sufficient that function f_j satisfy condition

$$\int_{\sigma_j} f_j ds = 0. \qquad (3.5.7)$$

As can be seen from Fig. 3.1, with small oscillations of fluid in j-th fuel tank the integral, which figures in the left side of equality (3.5.7), determines the difference between the volume of region limited by surfaces S_j and σ_j^*, and the volume of region V_j, limited by surface S_j and σ_j. Section of tank σ_j was so constructed above that these two volumes would be equal to each other. Thus, condition (3.5.7) will be fulfilled.

Let us now introduce into examination vector $\vec{u}_j(x, y, z, t)$, determining movements which the particle of fluid, having coordinates x, y, z in the state of rest, accomplishes in j-th tank. Velocity of motion of the particle of fluid, having coordinates x, y, z at the moment of time t, is determined by vector $\vec{v}_j(x, y, z, t)$. Thus, the velocity of motion of the particle, having coordinates x, y, z in the state of rest, will be determined by vector $\vec{v}_j(x + u_{jx}, y + u_{jy}, z + u_{jz}, t)$, since at the moment of time t this particle of fluid will have coordinates $x + u_{jx}$, $y + u_{jx}$, $z + u_{jz}$, where u_{jx}, u_{jy}, u_{jz} – projections of vector of movements $\vec{u}_j$. From this follows equality

$$\frac{\partial \vec{u}_j}{\partial t} = \vec{v}_j(x + u_{jx},\ y + u_{jy},\ z + u_{jz},\ t). \tag{3.5.8}$$

With small oscillations of fluid, allowing errors of the second order of smallness, it is possible to substitute equality (3.5.8) by approximate relationship

$$\frac{\partial \vec{u}_j}{\partial t} = \vec{v}_j(x, y, z, t), \tag{3.5.9}$$

or according to (3.5.1) and (3.5.5)

$$\frac{\partial \vec{u}_j}{\partial t} = \operatorname{grad} \frac{\partial \psi_j}{\partial t}. \tag{3.5.10}$$

By changing the order of differentiation of function ψ_j, hence we obtain equality

$$\frac{\partial}{\partial t}\left(\vec{u}_j - \operatorname{grad} \psi_j\right) = 0. \tag{3.5.11}$$

Let us suppose now that at a certain moment of time $t = t_0$ the particles of fluid, located in j-th tank, occupy positions corresponding to the state of rest, i.e.,

$$\vec{u}_j = 0 \text{ in } V_j \text{ when } t = t_0,\ f_j(y, z, t_0) \equiv 0 \tag{3.5.12}$$

(see Fig. 3.1). According to (3.5.12) with $t = t_0$ the boundary value problem, being formed by differential equation (3.5.3) and boundary conditions (3.5.4), will be homogeneous Neumann problem and there will exist relationship

$$\psi_j \equiv c_j \text{ when } t = t_0, \qquad (3.5.13)$$

where c_j – some constant. In accordance with equality (3.5.11) the difference of $\vec{u}_j$ – grad ψ_j should not depend on time t, while according to (3.5.12) and (3.5.13) when $t = t_0$ this difference should be identically equal to zero. Thus, there should exist identical equality

$$\vec{u}_j = \text{grad}\,\psi_j \text{ in } V_j. \qquad (3.5.14)$$

As we see, in case of small oscillations of fluid in an immobile tank the movements accomplished by particles of fluid possess potential ψ_j, determined by differential equation (3.5.3) with boundary conditions (3.5.4). In accordance with this the solution of Laplace equation (3.5.3), which satisfies boundary conditions (3.5.4), we will subsequently call *motion potential*, bearing in mind the movements caused by oscillations of free surface of fluid in the tank with number j.

By knowing Zhukovskiy potentials ϕ_{jx}, ϕ_{jy}, ϕ_{jz} and motion potential ψ_j, it is possible to construct a general expression for velocity potential Ψ_j, figuring in formula (3.3.5) and determined by differential equation (3.3.3) with boundary conditions (3.3.7). According to (3.4.2), (3.4.3), (3.5.3), (3.5.4) and (3.5.6) function

$$\Psi_j = \vec{\omega} \cdot \vec{\varphi}_j + \frac{\partial \psi_j}{\partial t} \qquad (3.5.15)$$

will satisfy differential Laplace equation (3.3.3) and boundary conditions (3.3.7). Thus, according to (3.3.5) and (3.5.15) generally the velocities, at which particles of fluid move in the launch system of coordinates, which is located in j-th tank, will be determined by relationship

$$\vec{v}_j = \vec{v}_0 + \text{grad}\left(\vec{\omega} \cdot \vec{\varphi}_j + \frac{\partial \psi_j}{\partial t}\right). \qquad (3.5.16)$$

Motion potential ψ_j can be found from differential equation (3.5.3) with boundary conditions (3.5.4), by knowing function f_j, determining the oscillations which the free surface of fluid accomplishes in the tank with number j. The question of seeking functions f_j is considered by us in the following paragraph.

§ 6. Oscillations of Free Surfaces of Fluids in Fuel Tanks

Pressure in the region occupied by fluid in the tank with number j, which we will subsequently designate by $p_j(x, y, z, t)$, on free surface σ_j^* must coincide with the pressure of gas located above the free surface, in other words with boost pressure. Proceeding from condition

$$p_j = p_j^{(0)} \text{ on } \sigma_j^*, \tag{3.6.1}$$

where $p_j^{[0]}$ – boost pressure in j-th fuel tank, below we obtain the boundary value problem, which determines the oscillations of free surface of fluid in the tank with number j according to assigned vector functions of time $\vec{v}_0(t)$ and $\vec{\omega}(t)$, characterizing motion, accomplished by the rocket body in the launch system of coordinates x_0, y_0, z_0.

In accordance with d'Alembert principle pressure p_j will satisfy equation of equilibrium

$$\text{grad}\, p_j = \varrho_j \vec{Q}_j, \tag{3.6.2}$$

where ϱ_j – density of fluid in the tank with number j; $\vec{Q}_j$ – mass force, acting on this fluid (see [8]), if we include mass force of inertia in force $\vec{Q}_j$. Thus, pressure p_j can be determined, having assumed in (3.6.2)

$$\vec{Q}_j = \vec{g} - \vec{w}_j, \tag{3.6.3}$$

where $\vec{g}$ – acceleration due to gravity, acting on the fluid; $\vec{w}_j$ – vector, determining the accelerations, at which particles of fluid, located in j-th tank, move in launch system of coordinates x_0, y_0, z_0.

According to (3.3.5) there should exist equality

$$\vec{w}_j = \vec{w}_0 + \frac{d(\text{grad}\,\Psi_j)}{dt}, \tag{3.6.4}$$

where $\vec{w}_0 = \frac{d\vec{v}_0}{dt}$ – acceleration, at which the origin of fixed system of coordinates x, y, z moves.

We will assume velocity potential Ψ_j found in the form of time function t and three-dimensional coordinates x, y, z. By calculating the acceleration of some particles of fluid by formula (3.6.4), it is necessary to consider the dependence of coordinates of this particle x, y, z on the time, furthermore, with differentiation of vector grad Ψ_j there should be considered rotation of coordinate system x, y, z, adding vector product $\vec{\omega} \times$ grad Ψ_j to the local time derivative t of vector grad Ψ_j. By designating through $\vec{e}_x$, $\vec{e}_y$, $\vec{e}_z$ the unit vectors, directions of which correspond to positive positions of coordinate axes x, y, z, we find

$$\begin{aligned}
\frac{d(\text{grad}\,\Psi_j)}{dt} &= \frac{d}{dt}\left(\frac{\partial\Psi_j}{\partial x}\vec{e}_x + \frac{\partial\Psi_j}{\partial y}\vec{e}_y + \frac{\partial\Psi_j}{\partial z}\vec{e}_z\right) = \\
&= \left(\frac{\partial^2\Psi_j}{\partial x\,\partial t} + \frac{\partial^2\Psi_j}{\partial x^2}\frac{dx}{dt} + \frac{\partial^2\Psi_j}{\partial x\,\partial y}\frac{dy}{dt} + \frac{\partial^2\Psi_j}{\partial x\,\partial z}\frac{dz}{dt}\right)\vec{e}_x + \\
&+ \left(\frac{\partial^2\Psi_j}{\partial y\,\partial t} + \frac{\partial^2\Psi_j}{\partial x\,\partial y}\frac{dx}{dt} + \frac{\partial^2\Psi_j}{\partial y^2}\frac{dy}{dt} + \frac{\partial^2\Psi_j}{\partial y\,\partial z}\frac{dz}{dt}\right)\vec{e}_y + \\
&+ \left(\frac{\partial^2\Psi_j}{\partial z\,\partial t} + \frac{\partial^2\Psi_j}{\partial x\,\partial z}\frac{dx}{dt} + \frac{\partial^2\Psi_j}{\partial y\,\partial z}\frac{dy}{dt} + \frac{\partial^2\Psi_j}{\partial z^2}\frac{dz}{dt}\right)\vec{e}_z + \\
&+ \vec{\omega}\times\text{grad}\,\Psi_j = \text{grad}\left(\frac{\partial\Psi_j}{\partial t} + \frac{\partial\Psi_j}{\partial x}\frac{dx}{dt} + \frac{\partial\Psi_j}{\partial y}\frac{dy}{dt} + \right. \\
&\left. + \frac{\partial\Psi_j}{\partial z}\frac{dz}{dt}\right) + \vec{\omega}\times\text{grad}\,\Psi_j,
\end{aligned}$$

or

$$\frac{d(\mathrm{grad}\,\Psi_j)}{dt}=\mathrm{grad}\left(\frac{\partial\Psi_j}{\partial t}+\vec{v}_{j\,\text{отн}}\cdot\mathrm{grad}\,\Psi_j\right)+\vec{\omega}\times\mathrm{grad}\,\Psi_j, \qquad (3.6.5)$$

where $\vec{v}_{j\ \text{OTH}}$ – velocity, at which the considered particle of fluid moves on fixed system of coordinates x, y, z. In view of the smallness of velocities $\vec{v}_{j\ \text{OTH}}$, $\vec{\omega}$ and grad Ψ_j assumed by us, formula (3.6.5) can be substituted by approximate formula

$$\frac{d(\mathrm{grad}\,\Psi_j)}{dt}=\mathrm{grad}\,\frac{\partial\Psi_j}{\partial t}, \qquad (3.6.6)$$

disregarding small quantities of the second order of smallness. According to (3.6.3), (3.6.4), and (3.6.6) equation (3.6.2) can be reduced to form

$$\mathrm{grad}\,p_j=\varrho_j\left(\vec{g}-\vec{w}_0-\mathrm{grad}\,\frac{\partial\Psi_j}{\partial t}\right). \qquad (3.6.7)$$

In accordance with equality

$$\vec{w}_0-\vec{g}=\mathrm{grad}\,[(w_{0x}-g_x)x+(w_{0y}-g_y)y+(w_{0z}-g_z)z]=$$
$$=\mathrm{grad}\left[\left(\vec{w}_0-\vec{g}\right)\cdot\vec{r}\right] \qquad (3.6.8)$$

equation (3.6.7) can be given the form

$$\mathrm{grad}\left\{p_j+\varrho_j\left[\left(\vec{w}_0-\vec{g}\right)\cdot\vec{r}+\frac{\partial\Psi_j}{\partial t}\right]\right\}=0. \qquad (3.6.9)$$

According to (3.6.9) there should exist dependence

$$p_j=-\varrho_j\left[\left(\vec{w}_0-\vec{g}\right)\cdot\vec{r}+\frac{\partial\Psi_j}{\partial t}\right]+c_j(t), \qquad (3.6.10)$$

where $c_j(t)$ — a certain function of time t, which will subsequently be excluded from examination.

By using formula (3.5.15), we find

$$\frac{\partial \Psi_j}{\partial t} = \frac{d\vec{\omega}}{dt} \cdot \vec{\varphi}_j + \vec{\omega} \cdot \frac{\partial \vec{\varphi}_j}{\partial t} + \frac{\partial^2 \psi_j}{\partial t^2}. \qquad (3.6.11)$$

If on the investigated section of rocket flight the tank with number j is not emptied, vector function $\vec{\phi}_j$, being determined by equation (3.4.2) with boundary condition (3.4.3), does not depend on time t and equality (3.6.11) in this case assumes the form

$$\frac{\partial \Psi_j}{\partial t} = \frac{d\vec{\omega}}{dt} \cdot \vec{\varphi}_j + \frac{\partial^2 \psi_j}{\partial t^2}. \qquad (3.6.12)$$

In the case of emptying of j-th tank derivative $\frac{\partial \vec{\varphi}_j}{\partial t}$ for all practical purposes can be referred to the number of small quantities and in conformity with the assumed smallness of angular velocity $\vec{\omega}$ it is possible to substitute relationship (3.6.11) by approximate relationship (3.6.12), allowing errors of the second order of smallness.

According to (3.6.12) formula (3.6.10) can be given the form

$$p_j = -\rho_j \left[(\vec{w}_0 - \vec{g}) \cdot \vec{r} + \frac{d\vec{\omega}}{dt} \cdot \vec{\varphi}_j + \frac{\partial^2 \psi_j}{\partial t^2} \right] + c_j(t). \qquad (3.6.13)$$

In accordance with the equation of free surface (3.2.3) condition (3.6.1) can be presented in the form

$$p_j(x_j + f_j, y, z, t) = p_j^{(0)},$$

or

$$p_j(x_j, y_j, z, t)+\left(\frac{\partial p_j}{\partial x}\right)_{x=x_j} f_j+\frac{1}{2}\left(\frac{\partial^2 p_j}{\partial x^2}\right)_{x=x_j} (f_j)^2+$$
$$+\ldots=p_j^{(0)}. \qquad (3.6.14)$$

With small oscillations of free surface condition (3.6.14) can be substituted with approximate condition

$$p_j(x_j, y, z, t)+\left(\frac{\partial p_j}{\partial x}\right)_{x=x_j} f_j=p_j^{(0)}, \qquad (3.6.15)$$

allowing errors of highest orders of smallness. By placing p_j from (3.6.13) into (3.6.15), we obtain equality

$$-\varrho_j\left[(w_{0x}-g_x)x_j+(w_{0y}-g_y)y+(w_{0z}-g_z)z+\frac{d\vec{\omega}}{dt}\cdot(\vec{\varphi}_j)_{x=x_j}+\right.$$
$$\left.+\left(\frac{\partial^2\psi_j}{\partial t^2}\right)_{x=x_j}\right]+c_j(t)-\varrho_j\left[w_{0x}-g_x+\right.$$
$$\left.+\frac{d\vec{\omega}}{dt}\left(\frac{\partial\vec{\varphi}_j}{\partial x}\right)_{x=x_j}+\left(\frac{\partial^3\psi_j}{\partial t^2\,\partial x}\right)_{x=x_j}\right]f_j=p_j^{(0)}. \qquad (3.6.16)$$

In accordance with the assumption about smallness of oscillations of free surface and about smallness of angular acceleration $d\vec{\omega}/dt$ equality (3.6.16) can be substituted by approximate equality

$$-\varrho_j\left[(w_{0x}-g_x)x_j+(w_{0y}-g_y)y+(w_{0z}-g_z)z+\right.$$
$$\left.+\frac{d\vec{\omega}}{dt}\cdot(\vec{\varphi}_j)_{x=x_j}+\left(\frac{\partial^2\psi_j}{\partial t^2}\right)_{x=x_j}\right]+c_j(t)-\varrho_j(w_{0x}-g_x)f_j=p_j^{(0)}, \qquad (3.6.17)$$

allowing errors of the second order of smallness.

Having introduced meaning

$$C_j(t)=\frac{c_j(t)-p_j^{(0)}}{\varrho_j}-(w_{0x}-g_x)x_j, \qquad (3.6.18)$$

it is possible to give relationship (3.6.17) the form

$$(w_{0x}-g_x)f_j+(w_{0y}-g_y)y+(w_{0z}-g_z)z+ \\ +\frac{d\vec{\omega}}{dt}\cdot(\vec{\varphi}_j)_{x=x_j}+\left(\frac{\partial^2\psi_j}{\partial t^2}\right)_{x=x_j}=C_j(t),$$

or

$$(w_{0x}-g_x)f_j+(w_{0y}-g_y)y+(w_{0z}-g_z)z+\frac{d\vec{\omega}}{dt}\cdot\vec{\varphi}_j+ \\ +\frac{\partial^2\psi_j}{\partial t^2}=C_j(t) \text{ on } \sigma_j, \tag{3.6.19}$$

since equation of plane σ_j has the form $x = x_j$.

By placing f_j from (3.6.19) into (3.5.4), for motion potential Ψ_j we obtain boundary conditions

$$\left.\begin{aligned} &\frac{\partial\psi_j}{\partial n}=0 \text{ on } S_j, \\ &\frac{\partial^2\psi_j}{\partial t^2}+(w_{0x}-g_x)\frac{\partial\psi_j}{\partial n}+(w_{0y}-g_y)y+(w_{0z}-g_z)z+ \\ &\quad+\frac{d\vec{\omega}}{dt}\cdot\vec{\varphi}_j=C_j(t) \text{ on } \sigma_j. \end{aligned}\right\} \tag{3.6.20}$$

According to (3.5.4), by knowing motion potential ψ_j, we can determine function $f_j(y, z, t)$ by formula

$$f_j=\left(\frac{\partial\psi_j}{\partial x}\right)_{x=x_j} \tag{3.6.21}$$

(on plane σ_j the direction of external normal n coincides with positive direction of axis x). Thus, function $f_j(y, z, t)$, determining oscillations of free surface of fluid in the tank with number j, can be found, having solved Laplace equation (3.5.3) with boundary conditions (3.6.20).

In the conclusion of this paragraph let us show that the solution of the problem about oscillations of free surface of fluid, located in j-th tank, does not depend on the form of function $C_j(t)$, figuring in boundary conditions (3.6.20). Let us consider function of time $\gamma(t)$, which satisfies differential equation

$$\frac{d^2\gamma}{dt^2}=C_j(t)-C_j^*(t). \tag{3.6.22}$$

By assuming in equation (3.5.3) and in boundary conditions (3.6.20)

$$\psi_j=\psi_j^*+\gamma(t), \tag{3.6.23}$$

according to (3.6.22) we obtain differential equation

$$\nabla^2\psi_j^*=0 \tag{3.6.24}$$

with boundary conditions:

$$\begin{gathered}\frac{\partial\psi_j^*}{\partial n}=0 \text{ on } S_j,\\ \frac{\partial^2\psi_j^*}{\partial t^2}+(w_{0x}-g_x)\frac{\partial\psi_j^*}{\partial n}+(w_{0y}-g_y)y+(w_{0z}-g_z)z+\\ +\frac{d\vec{\omega}}{dt}\cdot\vec{\varphi}_j=C_j^*(t) \text{ on } \sigma_j.\end{gathered} \tag{3.6.25}$$

Thus, substitution of (3.6.23) transfers boundary problem for function ψ_j, formed by differential equation (3.5.3) with boundary conditions (3.6.20), into analogous boundary value problem for function ψ_j^*, containing instead of function $C_j(t)$ another function $C^*_j(t)$ In this case according to (3.6.23) there will exist equality

$$\left(\frac{\partial\psi_j^*}{\partial x}\right)_{x=x_j}=\left(\frac{\partial\psi_j}{\partial x}\right)_{x=x_j}$$

and in accordance with formula (3.6.21), carrying out the transition from function $C_j(t)$ to function $C_j^*(t)$, we will finally obtain the same function $f_j(y, z, t)$, determining oscillations of free surface in the tank with number j. Further we will select functions $C_j(t)$, $j = 1, 2, \ldots$ proceeding from convenience of conducting the necessary calculations.

§ 7. Oscillations of Center of Mass of the Rocket, Equation of Forces

Let us now turn to investigation of the influence, which mobility of fluids in fuel tanks has on motion, accomplished by the rocket body.

According to assumption, the origin of movable coordinate system O coincides with the center of mass of the rocket C, if fluids located in the fuel tanks are limited by flat free surfaces, normal to the axis of the rocket. The small oscillations of free surfaces of fluids examined above cause small oscillations of the center of mass of the rocket relative to the origin of fixed system of coordinates x, y, z and thus affect the motion being accomplished by this coordinate system.

The position, being occupied by the center of mass of the rocket in fixed system of coordinates, will be determined by vector formula

$$\vec{r}_C = \frac{1}{m}\left(\int\limits_{V_0} \vec{r}\varrho\, dv + \sum_{j=1}^{N} \varrho_j \int\limits_{V_j^*} \vec{r}\, dv\right), \tag{3.7.1}$$

where $\vec{r}_C$ – radius vector of center of mass; m – overall mass of rocket; N – number of fuel tanks; V_0 – region occupied by solid elements of the construction; V_j^* – region occupied by fluid in j-th tank; $\vec{r}$ – radius vector of element of volume dv; ϱ – density of solid elements of the construction; $\varrho_1, \varrho_2, \ldots \varrho_N$ – density of fluids, located in the fuel tanks.

Region V_j^* is limited by the wetted surface of the tank S_j and by free surface of fluid σ_j^*. If $F(x, y, z)$ – some function of space variables x, y, z, then, as can be seen from Fig. 3.1, at small deviations of free surface σ_j^* from plane σ_j it is possible to assume

$$\int_{V_j^*} F\,dv = \int_{V_j} F\,dv + \int_{\sigma_j} F f_j\,d\sigma, \qquad (3.7.2)$$

where V_j – the region limited by surfaces S_j and σ_j.[1] In accordance with equality (3.7.2) formula (3.7.1) can be converted to form

$$\vec{r}C = \frac{1}{m}\left(\int_{V_0} \vec{r}\varrho\,dv + \sum_{j=1}^{N} \varrho_j \int_{V_j} \vec{r}\,dv + \sum_{j=1}^{N} \varrho_j \int_{\sigma_j} \vec{r} f_j\,d\sigma\right). \qquad (3.7.3)$$

In the absence of deviations f_j of free surfaces σ_j^* from planes σ_j the center of mass of the rocket should coincide with the origin of fixed system of coordinates x, y, z. Thus, according to (3.7.3) there should exist equality

$$\int_{V_0} \vec{r}\varrho\,dv + \sum_{j=1}^{N} \varrho_j \int_{V_j} \vec{r}\,dv = 0. \qquad (3.7.4)$$

By using dependence (3.7.4), it is possible to convert formula (3.7.3) to form

$$\vec{r}_c = \frac{1}{m}\sum_{j=1}^{N} \varrho_j \int_{\sigma_j} \vec{r} f_j\,d\sigma. \qquad (3.7.5)$$

Formula (3.7.5) determines the oscillations of center of mass of the rocket, being caused by oscillations of free surfaces of fluids in the fuel tanks.

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[1]Formula (3.7.2) is accurate only for a cylindrical fuel tank.

As was already indicated in Chapter I, equations of motion of the rocket can be formulated as equations of motion of a material system of constant composition, having connected the reactive forces to external forces acting on the rocket. In accordance with the theorem about motion of center of mass of a material system there should take place equality

$$m\vec{w}_C=\vec{F}+m\vec{g}, \tag{3.7.6}$$

where $\vec{w}_C$ – absolute acceleration of center of mass of the rocket, found without allowing for variability of its composition; $\vec{F}$ – principal vector of aerodynamic and reactive forces, acting on the rocket; $\vec{g}$ – acceleration due to gravity.[1]

By representing absolute accordance $\vec{w}_C$ in the form of the sum of migratory, relative and Coriolis accelerations, we find

$$\vec{w}_C=\vec{w}_0+\frac{d\vec{\omega}}{dt}\times\vec{r}_C+\vec{\omega}\times(\vec{\omega}\times\vec{r}_C)+\frac{\tilde{d}^2\vec{r}_C}{dt^2}+2\vec{\omega}\times\frac{\tilde{d}\vec{r}_C}{dt}. \tag{3.7.7}$$

By allowing errors of the highest orders of smallness, it is possible to substitute equality (3.7.7) by approximate equality

$$\vec{w}_C=\vec{w}_0+\frac{\tilde{d}^2\vec{r}_C}{dt^2}, \tag{3.7.8}$$

since rotation of the rocket is assumed slow by us and oscillations, being accomplished by the center of mass of the rocket, must be small in accordance with the smallness of oscillations of free surfaces of fluids assumed by us.

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[1]In Chapter 1 for the principal vector and principal moment of forces acting on the rocket, we applied designations $\Sigma\vec{F}$ and $\Sigma\vec{M}$. By sign Σ we emphasized the addition of reactive and Coriolis forces to external forces. Subsequently, for brevity, sign Σ will be dropped by us.

According to (3.7.5) acceleration $\delta^2 \vec{r}_C/dt^2$, found without allowing for variability of composition of the rocket, will be determined by relationship

$$\frac{\delta^2 \vec{r}_C}{dt^2} = -\frac{1}{m} \sum_{j=1}^{N} \varrho_j \int_{\tau_j} \vec{r} \frac{\partial^2 f_j}{\partial t^2} d\tau. \quad (3.7.9)$$

In accordance with formulas (3.7.8) and (3.7.9) equation (3.7.6) can be given the form

$$m(\vec{w}_0 - \vec{g}) + \sum_{j=1}^{N} \varrho_j \int_{\tau_j} \vec{r} \frac{\partial^2 f_j}{\partial t^2} d\tau = \vec{F}. \quad (3.7.10)$$

Equation (3.7.10) is the *equation of forces*, constructed with allowance for the mobility of fluids, located in the fuel tanks of the rocket.

§ 8. Moment of Momentum of the Rocket

By changing to construction of *equation of moments*, let us first derive the formula, determining the moment of momentum of a rocket relative to its center of mass, considering the mobility of liquid components of propellant during the derivation.

Taking into account the mobility of fluids, located in the fuel tanks of the rocket, the desired moment of momentum $\vec{L}_C$ should be found from relationship

$$\vec{L}_C = \int_{V_0} (\vec{r} - \vec{r}_C) \times (\vec{v}_0 + \vec{\omega} \times \vec{r}) \varrho \, dv + \sum_{j=1}^{N} \varrho_j \int_{V_j^*} (\vec{r} - \vec{r}_C) \times \times \vec{v}_j \, dv, \quad (3.8.1)$$

where $\vec{v}_j$ – vector determining the velocities, at which particles of fluid, located in the tank with number j, move in launch system of coordinates. By substituting $\vec{v}_j$ from (3.3.5) into (3.8.1), we obtain equality

$$\vec{L}_C=\int_{V_0}(\vec{r}-\vec{r}_C)\times(\vec{v}_0+\vec{\omega}\times\vec{r})\varrho dv+\sum_{j=1}^{N}\varrho_j\int_{V_j^*}(\vec{r}-\vec{r}_C)\times$$
$$\times(\vec{v}_0+\operatorname{grad}\Psi_j)dv,$$

or

$$\vec{L}_C=\left[\int_{V_0}(\vec{r}-\vec{r}_C)\varrho dv+\sum_{j=1}^{N}\varrho_j\int_{V_j^*}(\vec{r}-\vec{r}_C)dv\right]\times\vec{v}_0+$$
$$+\int_{V_0}(\vec{r}-\vec{r}_C)(\vec{\omega}\times\vec{r})\varrho dv+\sum_{j=1}^{N}\varrho_j\int_{V_j^*}(\vec{r}-\vec{r}_C)\times\operatorname{grad}\Psi_j dv. \tag{3.8.2}$$

By using formula (3.7.1) and equality

$$m=\int_{V_0}\varrho dv+\sum_{j=1}^{N}\varrho_j\int_{V_j^*}dv, \tag{3.8.3}$$

determining the overall mass of rocket m, we find

$$\int_{V_0}(\vec{r}-\vec{r}_C)\varrho dv+\sum_{j=1}^{N}\varrho_j\int_{V_j^*}(\vec{r}-\vec{r}_C)dv=\int_{V_0}\vec{r}\varrho dv+$$
$$+\sum_{j=1}^{N}\varrho_j\int_{V_j^*}\vec{r}dv-\left(\int_{V_0}\varrho dv+\sum_{j=1}^{N}\varrho_j\int dv\right)\vec{r}_C=0. \tag{3.8.4}$$

According to (3.8.4) formula (3.8.2) can be given the form

$$\vec{L}_C=\int_{V_0}(\vec{r}-\vec{r}_C)\times(\vec{\omega}\times\vec{r})\rho\, dv+$$

$$+\sum_{j=1}^{N}\rho_j\int_{V_j^*}(\vec{r}-\vec{r}_C)\times\operatorname{grad}\Psi_j\, dv. \tag{3.8.5}$$

By considering the smallness of vectors $\vec{r}_C$, $\vec{\omega}$ and grad Ψ_j and the proximity of regions V_j^* and V_j, it is possible with errors of the highest orders of smallness to replace formula (3.8.5) by approximate formula

$$\vec{L}_C=\int_{V_0}\vec{r}\times(\vec{\omega}\times\vec{r})\rho\, dv+\sum_{j=1}^{N}\rho_j\int_{V_j}\vec{r}\times\operatorname{grad}\Psi_j\, dv. \tag{3.8.6}$$

By carrying out conversion of volumetric integral into surface by means of Gauss divergence formulas, we find:

$$\int_{V_j}\vec{r}\times\operatorname{grad}\Psi_j\, dv=\int_{V_j}\Big[\Big(y\frac{\partial\Psi_j}{\partial z}-z\frac{\partial\Psi_j}{\partial y}\Big)\vec{e}_x+\Big(z\frac{\partial\Psi_j}{\partial x}-x\frac{\partial\Psi_j}{\partial z}\Big)\vec{e}_y+$$

$$+\Big(x\frac{\partial\Psi_j}{\partial y}-y\frac{\partial\Psi_j}{\partial x}\Big)\vec{e}_z\Big]dv=\int_{V_j}\Big\{\frac{\partial}{\partial x}\big[(z\vec{e}_y-y\vec{e}_z)\Psi_j\big]+$$

$$+\frac{\partial}{\partial y}\big[(x\vec{e}_z-z\vec{e}_x)\Psi_j\big]+\frac{\partial}{\partial z}\big[(y\vec{e}_x-x\vec{e}_y)\Psi_j\big]\Big\}dv=$$

$$=\int_{S_j+\sigma_j}\Psi_j\big[(z\vec{e}_y-y\vec{e}_z)\cos(n,x)+(x\vec{e}_z-z\vec{e}_x)\cos(n,y)+$$

$$+(y\vec{e}_x-x\vec{e}_y)\cos(n,z)\big]d\sigma=\int_{S_j+\sigma_j}\Psi_j\Big\{[y\cos(n,z)-$$

$$-z\cos(n,y)]\vec{e}_x+[z\cos(n,x)-x\cos(n,z)]\vec{e}_y+$$

$$+[x\cos(n,y)-y\cos(n,x)]\vec{e}_z\Big\}d\sigma=\int_{S_j+\sigma_j}\Psi_j\vec{r}\times\vec{n}\, d\sigma,$$

or according to (3.4.3)

$$\int_{V_j}\vec{r}\times\operatorname{grad}\Psi_j\, dv=\int_{S_j+\sigma_j}\Psi_j\frac{\partial\vec{\varphi}}{\partial n}j\, d\sigma. \tag{3.8.7}$$

Functions Ψ_j and $\vec{\Phi}_j$ are solutions of equations (3.3.3) and (3.4.2), and, thus, in accordance with Green's theorem there should take place equality

$$\int_{S_j+\sigma_j} \Psi_j \frac{\partial \vec{\varphi}_j}{\partial n} d\sigma = \int_{S_j+\sigma_j} \vec{\varphi}_j \frac{\partial \Psi_j}{\partial n} d\sigma. \quad (3.8.8)$$

According to (3.3.7) and (3.8.8) formula (3.8.7) can be given the form

$$\int_{V_j} \vec{r} \times \operatorname{grad} \Psi_j dv = \int_{S_j+\sigma_j} \vec{\varphi}_j \vec{\omega} \cdot (\vec{r} \times \vec{n}) d\sigma + \int_{\sigma_j} \vec{\varphi}_j \frac{\partial f_j}{\partial t} d\sigma,$$

or, if we use dependence (3.4.3)

$$\int_{V_j} \vec{r} \times \operatorname{grad} \Psi_j dv = \int_{S_j+\sigma_j} \vec{\varphi}_j \vec{\omega} \cdot \frac{\partial \vec{\varphi}_j}{\partial n} d\sigma + \int_{\sigma_j} \vec{\varphi}_j \frac{\partial f_j}{\partial t} d\sigma. \quad (3.8.9)$$

By substituting (3.8.9) in (3.8.6), for moment of momentum $\vec{L}_C$ we obtain formula

$$\vec{L}_C = \int_{V_0} \vec{r} \times (\vec{\omega} \times \vec{r}) \varrho dv + \sum_{j=1}^{N} \varrho_j \int_{S_j+\sigma_j} \vec{\varphi}_j \vec{\omega} \cdot \frac{\partial \vec{\varphi}_j}{\partial n} d\sigma + \sum_{j=1}^{N} \varrho_j \int_{\sigma_j} \vec{\varphi}_j \frac{\partial f_j}{\partial t} d\sigma,$$

or

$$\vec{L}_C = \vec{L}_C^{(0)} + \sum_{j=1}^{N} \varrho_j \int_{\sigma_j} \vec{\varphi}_j \frac{\partial f_j}{\partial t} d\sigma, \quad (3.8.10)$$

[Translator's Note: The vector $\vec{L}_C^0$ is shown exactly as it appears in the original and as it should appear according to errata changes however incorrectly it may be written.]

where

$$\vec{L}_C^0 = \int_{V_0} \vec{r} \times (\vec{\omega} \times \vec{r}) \varrho \, dv + \sum_{j=1}^{N} \varrho_j \int_{S_j+\sigma_j} \vec{\varphi}_{j\omega} \cdot \frac{\partial \vec{\varphi}_j}{\partial n} \, d\sigma. \qquad (3.8.11)$$

In formula (3.8.10) vector $\vec{L}_C^{(0)}$ determines the moment of momentum of the rocket relative to its center of mass, corresponding to the case when in the process of motion the free surfaces of fluids do not accomplish oscillations and remain plane and normal to the axis of the rocket. The second component in the right side of formula (3.8.10) determines the additional moment of momentum, being generated by oscillations of free surfaces of fluids, located in the fuel tanks.

§ 9. Account of the Mobility of Liquid Components of Propellant During Calculation of Moments of Inertia of Operation

Let us examine in more detail the moment of momentum $\vec{L}_C^0$, being determined by formula (3.8.11). In vector formula (3.8.11) by projecting vectors to the axes of fixed system of coordinates x, y, z, we obtain formulas:

$$\begin{aligned}
L_{Cx}^{(0)} &= \int_{V_0} [y(\omega_x y - \omega_y x) - z(\omega_z x - \omega_x z)] \varrho \, dv + \\
&+ \sum_{j=1}^{N} \varrho_j \int_{S_j+\sigma_j} \varphi_{jx} \left(\omega_x \frac{\partial \varphi_{jx}}{\partial n} + \omega_y \frac{\partial \varphi_{jy}}{\partial n} + \right. \\
&\left. + \omega_z \frac{\partial \varphi_{jz}}{\partial n} \right) d\sigma, \\
L_{Cy}^{(0)} &= \int_{V_0} [z(\omega_y z - \omega_z y) - x(\omega_x y - \omega_y x)] \varrho \, dv + \\
&+ \sum_{j=1}^{N} \varrho_j \int_{S_j+\sigma_j} \varphi_{jy} \left(\omega_x \frac{\partial \varphi_{jx}}{\partial n} + \omega_y \frac{\partial \varphi_{jy}}{\partial n} + \omega_z \frac{\partial \varphi_{jz}}{\partial n} \right) d\sigma, \\
L_{Cz}^{(0)} &= \int_{V_0} [x(\omega_z x - \omega_x z) - y(\omega_y z - \omega_z y)] \varrho \, dv + \\
&+ \sum_{j=1}^{N} \varrho_j \int_{S_j+\sigma_j} \varphi_{jz} \left(\omega_x \frac{\partial \varphi_{jx}}{\partial n} + \omega_y \frac{\partial \varphi_{jy}}{\partial n} + \omega_z \frac{\partial \varphi_{jz}}{\partial n} \right) d\sigma.
\end{aligned} \qquad (3.9.1)$$

Formulas (3.9.1) can be given the form

$$L_{Cx}^{(0)} = J_{xx}\omega_x + J_{xy}\omega_y + J_{xz}\omega_z,$$
$$L_{Cy}^{(0)} = J_{yx}\omega_x + J_{yy}\omega_y + J_{yz}\omega_z, \qquad (3.9.2)$$
$$L_{Cz}^{(0)} = J_{zx}\omega_x + J_{zy}\omega_y + J_{zz}\omega_z,$$

where

$$J_{xx} = \int\limits_{V_0} (y^2+z^2)\rho\, dv + \sum_{j=1}^{N} \rho_j \int\limits_{S_j+\sigma_j} \varphi_{jx} \frac{\partial \varphi_{jx}}{\partial n}\, d\sigma,$$
$$J_{xy} = -\int\limits_{V_0} xy\rho\, dv + \sum_{j=1}^{N} \rho_j \int\limits_{S_j+\sigma_j} \varphi_{jx} \frac{\partial \varphi_{jy}}{\partial n}\, d\sigma,$$
$$J_{xz} = -\int\limits_{V_0} xz\rho\, dv + \sum_{j=1}^{N} \rho_j \int\limits_{S_j+\sigma_j} \varphi_{jx} \frac{\partial \varphi_{jz}}{\partial n}\, d\sigma,$$
$$J_{yx} = -\int\limits_{V_0} xy\rho\, dv + \sum_{j=1}^{N} \rho_j \int\limits_{S_j+\sigma_j} \varphi_{jy} \frac{\partial \varphi_{jx}}{\partial n}\, d\sigma,$$
$$J_{yy} = \int\limits_{V_0} (x^2+z^2)\rho\, dv + \sum_{j=1}^{N} \rho_j \int\limits_{S_j+\sigma_j} \varphi_{jy} \frac{\partial \varphi_{jy}}{\partial n}\, d\sigma, \qquad (3.9.3)$$
$$J_{yz} = -\int\limits_{V_0} yz\rho\, dv + \sum_{j=1}^{N} \rho_j \int\limits_{S_j+\sigma_j} \varphi_{jy} \frac{\partial \varphi_{jz}}{\partial n}\, d\sigma,$$
$$J_{zx} = -\int\limits_{V_0} xz\rho\, dv + \sum_{j=1}^{N} \rho_j \int\limits_{S_j+\sigma_j} \varphi_{jz} \frac{\partial \varphi_{jx}}{\partial n}\, d\sigma,$$
$$J_{zy} = -\int\limits_{V_0} yz\rho\, dv + \sum_{j=1}^{N} \rho_j \int\limits_{S_j+\sigma_j} \varphi_{jz} \frac{\partial \varphi_{jy}}{\partial n}\, d\sigma,$$
$$J_{zz} = \int\limits_{V_0} (x^2+y^2)\rho\, dv + \sum_{j=1}^{N} \rho_j \int\limits_{S_j+\sigma_j} \varphi_{jz} \frac{\partial \varphi_{jz}}{\partial n}\, d\sigma.$$

According to (3.4.2) functions ϕ_{jx}, ϕ_{jy}, and ϕ_{jz} are solutions of Laplace equations, and, thus, in accordance with Green's theorem there should take place equalities:

$$\int\limits_{S_j+\sigma_j} \varphi_{jy} \frac{\partial \varphi_{jz}}{\partial n} d\sigma = \int\limits_{S_j+\sigma_j} \varphi_{jz} \frac{\partial \varphi_{jy}}{\partial n} d\sigma,$$

$$\int\limits_{S_j+\sigma_j} \varphi_{jx} \frac{\partial \varphi_{jz}}{\partial n} d\sigma = \int\limits_{S_j+\sigma_j} \varphi_{jz} \frac{\partial \varphi_{jx}}{\partial n} d\sigma, \tag{3.9.4}$$

$$\int\limits_{S_j+\sigma_j} \varphi_{jx} \frac{\partial \varphi_{jy}}{\partial n} d\sigma = \int\limits_{S_j+\sigma_j} \varphi_{jy} \frac{\partial \varphi_{jx}}{\partial n} d\sigma.$$

From formulas (3.9.3) and (3.9.4) ensue relationships:

$$J_{yx} = J_{xy},\ J_{zx} = J_{xz},\ J_{zy} = J_{yz}. \tag{3.9.5}$$

In accordance with equalities (3.9.2) and (3.9.5) the moment of momentum $\vec{L}_C^{(0)}$ is equal to the moment of momentum of a solid, for which moments of inertia relative to axes x, y, z – J_{xx}, J_{yy}, J_{zz} and centrifugal inertia moments J_{xy}, J_{xz}, J_{yz} have values determined by formulas (3.9.3).

By using relationships (3.9.2) and (3.9.5), it is possible to give equality (3.8.10) the form

$$\vec{L}_C = (J_{xx}\omega_x + J_{xy}\omega_y + J_{xz}\omega_z)\vec{e}_x + (J_{xy}\omega_x + J_{yy}\omega_y + J_{yz}\omega_z)\vec{e}_y + \\ +(J_{xz}\omega_x + J_{yz}\omega_y + J_{zz}\omega_z)\vec{e}_z + \sum_{j=1}^{N} \vec{\varrho}_j \int\limits_{\sigma_j} \vec{\varphi}_j \frac{\partial f_j}{\partial t} d\sigma. \tag{3.9.6}$$

According to (3.9.6) in the absence of oscillations of free surfaces of fluids, i.e., when $f_j \equiv 0$, $j = 1, 2, \ldots, N$, the moment of momentum of the rocket relative to its center of mass can be determined as the moment of momentum of a solid, having calculated the inertia moments of the rocket relative to axes of fixed system of coordinates x, y, z – J_{xx}, J_{yy}, J_{zz} and centrifugal inertia moments of the rocket J_{xy}, J_{xz}, J_{yz} by formulas (3.9.3)

§ 10. Equation of Moments. Zhukovskiy Theorem

In § 7 we obtained equation of forces (3.7.10). Let us now turn to derivation of equation of moments. In accordance with the theorem about change of moment of momentum, at any moment of time there should exist equality

$$\frac{d\vec{L}_C}{dt} = \vec{M}_C. \qquad (3.10.1)$$

where $\frac{d\vec{L}_C}{dt}$ – time derivative from moment of momentum $\vec{L}_C$, calculated without allowing for variability of the composition of the rocket; $\vec{M}_C$ – main moment of aerodynamic and reactive forces affecting the rocket (main moment of forces of gravity relative to center of mass of the rocket will always be equal to zero).

Equation (3.10.1) can be given the form

$$\frac{\delta\vec{L}_C}{dt} + \vec{\omega}\times\vec{L}_C = \vec{M}_C. \qquad (3.10.2)$$

According to (3.9.6) vector product $\vec{\omega} \times \vec{L}_C$ is a small quantity of the second order of smallness in accordance with the smallness of oscillations of free surfaces of fluids and smallness of angular velocity $\vec{\omega}$, assumed by us. Thus, disregarding small quantities of the second order of smallness, it is possible to substitute equation (3.10.2) by approximate equation

$$\frac{\delta\vec{L}_C}{dt} = \vec{M}_C. \qquad (3.10.3)$$

Calculating the local time derivative $\delta\vec{L}_C/dt$ by formula (3.9.6) without allowing for variability of the composition of the rocket, we obtain relationship

$$
\begin{aligned}
\frac{\delta \vec{L}_C}{dt} = & \left(J_{xx}\frac{d\omega_x}{dt} + J_{xy}\frac{d\omega_y}{dt} + J_{xz}\frac{d\omega_z}{dt}\right)\vec{e}_x + \\
& + \left(J_{xy}\frac{d\omega_x}{dt} + J_{yy}\frac{d\omega_y}{dt} + J_{yz}\frac{d\omega_z}{dt}\right)\vec{e}_y + \\
& + \left(J_{xz}\frac{d\omega_x}{dt} + J_{yz}\frac{d\omega_y}{dt} + J_{zz}\frac{d\omega_z}{dt}\right)\vec{e}_z + \\
& + \sum_{j=1}^{N} \rho_j \iint_{\sigma_j} \vec{\varphi}_j \frac{\partial^2 f_j}{\partial t^2} d\sigma.
\end{aligned} \tag{3.10.4}
$$

Let us further designate through $\vec{M}$ the main moment of aerodynamic and reactive forces relative to the origin of fixed system of coordinates x, y, z. Moments $\vec{M}$ and $\vec{M}_C$ will be connected together by relationship

$$\vec{M}_C = \vec{M} - \vec{r}_C \times \vec{F}, \tag{3.10.5}$$

where $\vec{r}_C$ – radius vector of center of mass of the rocket; $\vec{F}$ – main vector of aerodynamic and reactive forces, acting on the rocket.

According to (3.7.6) and (3.7.8) formula (3.10.5) can be converted to the form

$$\vec{M}_C = \vec{M} - m\vec{r}_C \times (\vec{w}_C - \vec{g}) = \vec{M} - m\vec{r}_C \times (\vec{w}_0 - \vec{g}) - m\vec{r}_C \times \frac{\delta^2 \vec{r}_C}{dt^2}. \tag{3.10.6}$$

The smallness of oscillations of free surfaces of fluids being assumed by us involves the smallness of oscillations being accomplished by the center mass of the rocket in fixed system of coordinates x, y, z. Thus, disregarding small quantities of the second order of smallness, it is possible to replace equality (3.10.6) by approximate equality

$$\vec{M}_C = \vec{M} - m\vec{r}_C \times (\vec{w}_0 - \vec{g}),$$

or according to (3.7.5)

$$\vec{M}_C=\vec{M}-\sum_{j=1}^{N}\varrho_j\int_{\sigma_j}\vec{r}f_j d\sigma\times(\vec{w}_0-\vec{g}). \qquad (3.10.7)$$

By substituting (3.10.4) and (3.10.7) in (3.10.3), we obtain equation

$$\left(J_{xx}\frac{d\omega_x}{dt}+J_{xy}\frac{d\omega_y}{dt}+J_{xz}\frac{d\omega_z}{dt}\right)\vec{e}_x+\left(J_{xy}\frac{d\omega_x}{dt}+J_{yy}\frac{d\omega_y}{dt}+J_{yz}\frac{d\omega_z}{dt}\right)\vec{e}_y+\left(J_{xz}\frac{d\omega_x}{dt}+J_{yz}\frac{d\omega_y}{dt}+J_{zz}\frac{d\omega_z}{dt}\right)\vec{e}_z+\sum_{j=1}^{N}\varrho_j\iint_{\sigma_j}\vec{\varphi}_j\frac{\partial^2 f_j}{\partial t^2}d\sigma+\sum_{j=1}^{N}\varrho_j\iint_{\sigma_j}\vec{r}f_j d\sigma\times(\vec{w}_0-\vec{g})=\vec{M}. \qquad (3.10.8)$$

In the absence of oscillations of free surfaces of fluids, i.e., when $f_j = 0$, $j = 1, 2, \ldots, N$, equation of forces (3.7.10) and equation of moments (3.10.8) change into equations of motion of a solid, which we proceeded from in Chapters I and II, with the only difference that during consideration of relative motions, being accomplished by fluids in the fuel tanks, the moments of inertia of the rocket must be calculated not by formulas of dynamics of solid, but by formulas (3.9.3).

In [5] N. Ye. Zhukovskiy examined the problem about motion of a solid with cavities, completely filled with perfect fluid, and showed that such a material system moves under the action of a predetermined system of forces, as an equivalent solid of the same mass, possessing moments of inertia, determined by formulas (3.9.3) (in this instance surfaces σ_j, $j = 1, 2, \ldots, N$, figuring in formulas (3.9.3), are absent). Thus, in the absence of oscillations of free surfaces of fluids, the equations of forces and moments (3.7.10) and (3.10.8) directly ensue from the works of Zhukovskiy indicated by us. Formulas (3.9.3) will be subsequently called *Zhukovskiy formulas*.

§ 11. Equations of Motion of the Rocket, Considering the Mobility of Fluids in Fuel Tanks

Equations of forces and moments (3.7.10) and (3.10.8) contain system of function $f_j(y, z, t)$, $j = 1, 2, \ldots, N$, characterizing oscillations of free surfaces of fluids in fuel tanks. At the same time, the boundary value problem, being formed by differential equation (3.5.3) with boundary conditions (3.6.20) and determining function $f_j(y, z, t)$ in accordance with formula (3.6.21), contains accelerations w_{0x}, w_{0y}, w_{0z}, $d\omega_x/dt$, $d\omega_y/dt$ and $d\omega_z/dt$, characterizing motion accomplished by the rocket body. Thus, the problem about motion of the body and the problem about oscillations of fluids in fuel tanks can be solved only as a result of their joint examination. In order to obtain a complete system of equations, determining motion of the rocket body and motions accomplished by fluids located in fuel tanks, one should exclude functions $f_j(y, z, t)$ from equations of forces and moments (3.7.10) and (3.10.8) by means of relationship (3.6.21) and connect them to obtained dependences of equation (3.5.3) with boundary conditions (3.6.30). We obtain system of equations

$$
\begin{aligned}
& m(\vec{w}_0 - \vec{g}) + \sum_{j=1}^{N} \varrho_j \int_{\sigma_j} \vec{r} \left(\frac{\partial^3 \psi_j}{\partial x\, \partial t^2} \right)_{x=x_j} d\sigma = \vec{F}, \\
& \left(J_{xx} \frac{d\omega_x}{dt} + J_{xy} \frac{d\omega_y}{dt} + J_{xz} \frac{d\omega_z}{dt} \right) \vec{e}_x + \left(J_{xy} \frac{d\omega_x}{dt} + J_{yy} \frac{d\omega_y}{dt} + \right. \\
& \left. + J_{yz} \frac{d\omega_z}{dt} \right) \vec{e}_y + \left(J_{xz} \frac{d\omega_x}{dt} + J_{yz} \frac{d\omega_y}{dt} + J_{zz} \frac{d\omega_z}{dt} \right) \vec{e}_z + \\
& + \sum_{j=1}^{N} \varrho_j \int_{\sigma_j} \vec{\varphi}_j \left(\frac{\partial^3 \psi_j}{\partial x\, \partial t^2} \right)_{x=x_j} d\sigma + \sum_{j=1}^{N} \varrho_j \int_{\sigma_j} \vec{r} \left(\frac{\partial \psi_j}{\partial x} \right)_{x=x_j} d\sigma \times \\
& \qquad \times (\vec{w}_0 - \vec{g}) = \vec{M}, \\
& \nabla^2 \psi_j = 0, \quad j = 1, 2, \ldots, N, \\
& \frac{\partial \psi_j}{\partial n} = 0 \text{ on } S_j, \quad j = 1, 2, \ldots, N, \\
& \frac{\partial^2 \psi_j}{\partial t^2} + (w_{0x} - g_x) \frac{\partial \psi_j}{\partial n} + (w_{0y} - g_y) y + (w_{0z} - g_z) z + \\
& \qquad + \frac{d\vec{\omega}}{dt} \cdot \vec{\varphi}_j = C_j(t) \text{ on } \sigma_j, \quad j = 1, 2, \ldots, N.
\end{aligned}
\tag{3.11.1}
$$

Equations (3.11.1) determine motion accomplished by the body of the rocket, and system of functions $\psi_j(x, y, z, t)$, $j = 1, 2, \ldots N$. By using formula (3.6.21), by functions ψ_j it is possible to find the system of functions $f_j(y, z, t)$ $j = 1, 2, \ldots, N$, determining the oscillations of free surfaces of fluids in fuel tanks. By knowing vector functions of time $\vec{v}_0(t)$ and $\vec{\omega}(t)$ and functions $\psi_j(x, y, z, t)$, $j = 1, 2, \ldots, N$, by formula (3.5.16) it is possible to construct the system of vector functions $\vec{v}_j(x, y, z, t)$, $j = 1, 2, \ldots, N$, determining the velocity field of fluids located in fuel tanks. Thus, equations (3.11.1) completely determine motions accomplished by the rocket body and fluids located in its fuel tanks, and these equations can be considered as equations of motion of the rocket, considering the mobility of fluids in fuel tanks.

For construction of equations of motion (3.11.1) it is necessary to preliminarily find Zhukovskiy potentials ϕ_{jx}, ϕ_{jy}, ϕ_{jz}, $j = 1, 2, \ldots, N$, clearly entering equations (3.11.1) and figuring in Zhukovskiy formulas (3.9.3). The question about calculation of Zhukovskiy potentials is considered in the following paragraphs of this chapter.

§ 12. Calculation of Zhukovskiy Potentials

During examination of the question about calculation of Zhukovskiy potentials ϕ_{jx}, ϕ_{jy}, ϕ_{jz} we will assume that all wetted surfaces S_j, $j = 1, 2, \ldots, N$, are surfaces of rotation. With the presence of intertank equipment, disturbing this property of the wetted surface, calculation determination of Zhukovskiy potentials is very complicated, investigation of dynamic effects, which appear with installation of such intertank equipment, is conducted usually by experimental methods.

Zhukovskiy potentials ϕ_{jx}, ϕ_{jy}, ϕ_{jz} are projections to coordinate axes x, y, z of vector $\vec{\Phi}_j$, determined by equation (3.4.2) and boundary condition (3.4.3). By $\vec{r}_j$ let us designate the radius vector of center of flat free surface of fluid σ_j in fixed system of coordinates x, y, z (Fig. 3.3) and instead of vector function $\vec{\phi}_j$ let us introduce into examination vector function $\vec{\phi}_j^0$, determined by relationship

$$\vec{\varphi}_j^{(0)}=\vec{\varphi}_j-\vec{r}_j\times\vec{r}. \qquad (3.12.1)$$

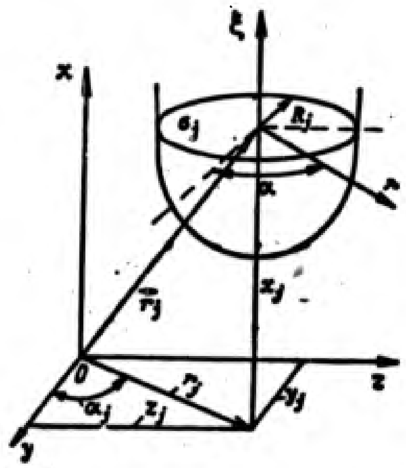

Fig. 3.3.

According to (3.12.1) there will exist equality:

$$\nabla^2\vec{\varphi}_j^{(0)}=\nabla^2\vec{\varphi}_j-\vec{r}_j\times\nabla^2\vec{r},\quad \frac{\partial\vec{\varphi}_j^{(0)}}{\partial n}=\frac{\partial\vec{\varphi}_j}{\partial n}-\vec{r}_j\times\frac{\partial\vec{r}}{\partial n}. \qquad (3.12.2)$$

By using differential equation (3.4.2), boundary condition (3.4.3) and relationships:

$$\nabla^2\vec{r}=\nabla^2(x\vec{e}_x+y\vec{e}_y+z\vec{e}_z)=0,$$

$$\frac{\partial\vec{r}}{\partial n}=\frac{\partial}{\partial n}(x\vec{e}_x+y\vec{e}_y+z\vec{e}_z)=\cos(n,x)\vec{e}_x+\cos(n,y)\vec{e}_y+ \qquad (3.12.3)$$

$$+\cos(n,z)\vec{e}_z=\vec{n},$$

from (3.12.2) we obtain differential equation

$$\nabla^2 \vec{\varphi}_j^{(0)} = 0 \tag{3.12.4}$$

and boundary condition

$$\frac{\partial \vec{\varphi}_j^{(0)}}{\partial n} = (\vec{r} - \vec{r}_j) \times \vec{n} \quad \text{on } S_j \text{ and } \sigma_j. \tag{3.12.5}$$

Let us further construct system of cylindrical coordinates ξ, r, α, shown in Fig. 3.3. The connection between rectangular coordinates x, y, z and cylindrical coordinates ξ, r, α will be determined by formulas:

$$\begin{aligned} x &= x_j + \xi, \\ y &= y_j + r\cos\alpha, \\ z &= z_j + r\sin\alpha, \end{aligned} \tag{3.12.6}$$

where x_j, y_j and z_j – projections of vector $\vec{r}_j$ to coordinate axes x, y, z (see Fig. 3.3). Differential Laplace equation (3.12.4) in cylindrical coordinates ξ, r, α will have the form

$$\frac{\partial^2 \vec{\varphi}_j^{(0)}}{\partial \xi^2} + \frac{\partial^2 \vec{\varphi}_j^{(0)}}{\partial r^2} + \frac{1}{r} \frac{\partial \vec{\varphi}_j^{(0)}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \vec{\varphi}_j^{(0)}}{\partial \alpha^2} = 0 \tag{3.12.7}$$

(see [21]). According to (3.12.5) and (3.12.7) the projections of vector $\vec{\phi}_j^0$ to coordinate axes x, y, z – $\phi_{jx}^{(0)}$, $\phi_{jy}^{(0)}$, $\phi_{jz}^{(0)}$ must satisfy differential equations:

$$\begin{aligned} &\frac{\partial^2 \varphi_{jx}^{(0)}}{\partial \xi^2} + \frac{\partial^2 \varphi_{jx}^{(0)}}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi_{jx}^{(0)}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \varphi_{jx}^{(0)}}{\partial \alpha^2} = 0, \\ &\frac{\partial^2 \varphi_{jy}^{(0)}}{\partial \xi^2} + \frac{\partial^2 \varphi_{jy}^{(0)}}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi_{jy}^{(0)}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \varphi_{jy}^{(0)}}{\partial \alpha^2} = 0, \\ &\frac{\partial^2 \varphi_{jz}^{(0)}}{\partial \xi^2} + \frac{\partial^2 \varphi_{jz}^{(0)}}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi_{jz}^{(0)}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \varphi_{jz}^{(0)}}{\partial \alpha^2} = 0 \end{aligned} \tag{3.12.8}$$

and boundary conditions:

$$\left.\begin{aligned}\frac{\partial\varphi_{jx}^{(0)}}{\partial n}&=(y-y_j)n_z-(z-z_j)n_y\\ \frac{\partial\varphi_{jy}^{(0)}}{\partial n}&=(z-z_j)n_x-(x-x_j)n_z\\ \frac{\partial\varphi_{jz}^{(0)}}{\partial n}&=(x-x_j)n_y-(y-y_j)n_x\end{aligned}\right\}\ \text{on } S_j \text{ and } \sigma_j. \qquad (3.12.9)$$

Let us further designate through χ the angle between direction of external normal n to surface S_j and σ_j and direction of negative semiaxis ξ (Fig. 3.4). As can be seen from the figure, projections n_x, n_y and n_z of unit vector of external normal $\vec{n}$ will be determined by formulas:

$$\begin{aligned}n_x&=-\cos\chi,\\ n_y&=\sin\chi\cos\alpha,\\ n_z&=\sin\chi\sin\alpha.\end{aligned} \qquad (3.12.10)$$

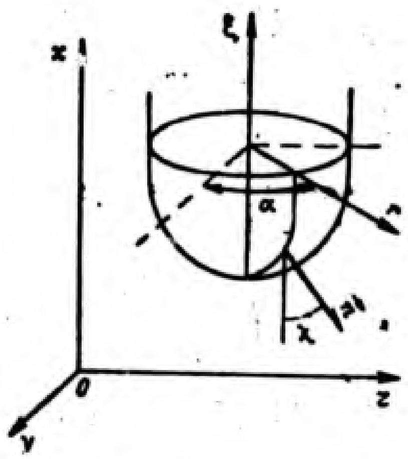

Fig. 3.4.

According to (3.12.6) and (3.12.10) boundary conditions (3.12.9) can be given the form

$$\left.\begin{aligned}&\frac{\partial\varphi_{jx}^{(0)}}{\partial n}=0\\&\frac{\partial\varphi_{jy}^{(0)}}{\partial n}=-(r\cos\chi+\xi\sin\chi)\sin\alpha\\&\frac{\partial\varphi_{jz}^{(0)}}{\partial n}=(r\cos\chi+\xi\sin\chi)\cos\alpha\end{aligned}\right\}\text{ on } S_j \text{ and } \sigma_j. \qquad (3.12.11)$$

We will satisfy the first of differential equations (3.12.8) and the first of boundary conditions (3.12.11), having assumed

$$\varphi_{jx}^{(0)}=0. \qquad (3.12.12)$$

For determination of functions $\phi_{jy}^{(0)}$ and $\phi_{jz}^{(0)}$ let us introduce into examination the solution of two-dimensional differential equation

$$\frac{\partial^2\theta_j}{\partial\xi^2}+\frac{\partial^2\theta_j}{\partial r^2}+\frac{1}{r}\frac{\partial\theta_j}{\partial r}-\frac{\theta_j}{r^2}=0, \qquad (3.12.13)$$

regular on axis $r = 0$, in other words unlimitedly differentiable at the points of this axis, and satisfying boundary condition

$$\frac{\partial\theta_j}{\partial n}=r\cos\chi+\xi\sin\chi \text{ on } l_j, \qquad (3.12.14)$$

where l_j – generatrix of closed surface of rotation $S_j + \sigma_j$. According to (3.12.13) and (3.12.14) we will satisfy the second and third of differential equations (3.12.8) and the second and third of boundary conditions (3.12.11), having assumed

$$\varphi_{jy}^{(0)}=-\theta_j\sin\alpha,\quad \varphi_{jz}^{(0)}=\theta_j\cos\alpha \qquad (3.12.15)$$

(on external normal n polar angle α keeps constant value, and, thus $\partial\alpha/\partial n = 0$).

By carrying out in equality (3.12.1) projection of vectors to coordinate axes x, y, z, we obtain formulas:

$$\varphi_{jx}=\varphi_{jx}^{(0)}+y_jz-z_jy,\quad \varphi_{jy}=\varphi_{jy}^{(0)}+z_jx-x_jz,\quad \varphi_{jz}=\varphi_{jz}^{(0)}+x_jy-y_jx,$$

or according to (3.12.12) and (3.12.15)

$$\left.\begin{aligned}\varphi_{jx}&=y_jz-z_jy,\\ \varphi_{jy}&=-\theta_j\sin\alpha+z_jx-x_jz,\\ \varphi_{jz}&=\theta_j\cos\alpha+x_jy-y_jx.\end{aligned}\right\} \qquad (3.12.16)$$

Thus, having solved two-dimensional boundary problem, being formed by differential equation (3.12.13) and boundary condition (3.12.14), it is possible to then find Zhukovskiy potentials ϕ_{jx}, ϕ_{jy} and ϕ_{jz} by formulas (3.12.16). For solution of differential equation (3.12.13) with boundary condition (3.12.14) there are known many calculation methods; it is possible to use, specifically, the method discussed in book [22]. Sometimes function θ_j can be found in the form of decomposition by these or other specific functions. Two such cases are comprehensively examined by us in the following paragraph.

§ 13. The Simplest Examples of Calculation of Zhukovskiy Potentials

13.1. Cylindrical Fuel Tank

In this instance generatrix l_j of closed surface $S_j + \sigma_j$ is a broken line, shown in Fig. 3.5. Angle χ, figuring in boundary condition (3.12.14), on contour l_j takes values determined by formulas:

$$\left.\begin{aligned}&\chi=0 \text{ when } \xi=-H, && 0<r<R,\\ &\chi=\frac{\pi}{2} \text{ when } r=R, && -H<\xi<0,\\ &\chi=\pi \text{ when } \xi=0, && 0<r<R,\end{aligned}\right\} \qquad (3.13.1)$$

where R — radius of tank; H — its filling depth.

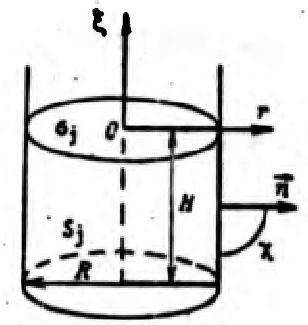

Fig. 3.5.

According to (3.13.1) boundary condition (3.12.14) in this case will have the form

$$\begin{aligned}
&\frac{\partial\theta_j}{\partial\xi} = -r \text{ when } \xi = -H, \quad 0 < r < R, \\
&\frac{\partial\theta_j}{\partial r} = \xi \quad \text{when } r = R, \quad -H < \xi < 0, \\
&\frac{\partial\theta_j}{\partial\xi} = -r \text{ when } \xi = 0, \quad 0 < r < R.
\end{aligned} \tag{3.13.2}$$

Assuming in (3.12.13) and (3.13.2)

$$\theta_j = \xi r + \theta_j^*. \tag{3.13.3}$$

for function θ_j^* we obtain differential equation

$$\frac{\partial^2\theta_j^*}{\partial\xi^2} + \frac{\partial^2\theta_j^*}{\partial r^2} + \frac{1}{r}\frac{\partial\theta_j^*}{\partial r} - \frac{\theta_j^*}{r^2} = 0 \tag{3.13.4}$$

with boundary conditions:

$$\begin{aligned}
&\frac{\partial\theta_j^*}{\partial\xi} = -2r \text{ when } \xi = -H, \quad 0 < r < R, \\
&\frac{\partial\theta_j^*}{\partial r} = 0 \quad \text{when } r = R, \quad -H < \xi < 0, \\
&\frac{\partial\theta_j^*}{\partial\xi} = -2r \text{ when } \xi = 0, \quad 0 < r < R.
\end{aligned} \tag{3.13.5}$$

Let us first construct particular solution of differential equation (3.13.4) of form

$$\theta_j^* = f(\xi) g(r), \tag{3.13.6}$$

regular on axis $r = 0$ and satisfying the second of boundary conditions (3.13.5). By substituting (3.13.6) in (3.13.4), we obtain equality

$$\frac{d^2 f}{d\xi^2} g + f\left(\frac{d^2 g}{dr^2} + \frac{1}{r}\frac{dg}{dr} - \frac{g}{r^2}\right) = 0,$$

or

$$\frac{\frac{d^2 f}{d\xi^2}}{f} = -\frac{\frac{d^2 g}{dr^2} + \frac{1}{r}\frac{dg}{dr} - \frac{g}{r^2}}{g}. \tag{3.13.7}$$

The left side of equality (3.13.7) does not depend on variable r, and the right side of this equality does not depend on variable ξ. Thus, the ratios, figuring in equality (3.13.7), must represent a certain constant. By designating this constant through λ^2, for function $f(\xi)$ we obtain differential equation

$$\frac{d^2 f}{d\xi^2} - \lambda^2 f = 0 \tag{3.13.8}$$

and for function $g(r)$ – differential equation

$$\frac{d^2 g}{dr^2} + \frac{1}{r}\frac{dg}{dr} + \left(\lambda^2 - \frac{1}{r^2}\right) g = 0. \tag{3.13.9}$$

Assuming in (3.13.9)

$$r = \frac{\varrho}{\lambda}. \tag{3.13.10}$$

for function g we obtain Bessel differential equation

$$\frac{d^2g}{d\varrho^2}+\frac{1}{\varrho}\frac{dg}{d\varrho}+\left(1-\frac{1}{\varrho^2}\right)g=0. \qquad (3.13.11)$$

Solution of differential equation (3.13.11), regular at point $\varrho=0$, is determined by Bessel function of the first type and first order – $J_1(\varrho)$. Thus, differential equation (3.13.9) according to (3.13.10) possesses solution

$$g=J_1(\lambda r), \qquad (3.13.12)$$

regular at point $r = 0$.

General solution of differential equation (3.13.8) has the form

$$f=Ae^{\lambda\xi}+Be^{-\lambda\xi}, \qquad (3.13.13)$$

where A and B – arbitrary constants. According to (3.13.6), (3.13.12) and (3.13.13) differential equation (3.13.4) has solution

$$\theta_j^*=(Ae^{\lambda\xi}+Be^{-\lambda\xi})J_1(\lambda r), \qquad (3.13.14)$$

regular on axis $r = 0$ and containing three arbitrary constants A, B and λ.

By substituting (3.13.14) in the second of boundary conditions (3.13.5), for parameter λ we obtain equation

$$J_1'(\lambda R)=0. \qquad (3.13.15)$$

According to (3.13.15) function θ_j^*, determined by formula (3.13.14), will satisfy the second of boundary conditions (3.13.5) when

$$\lambda=\frac{\nu_k}{R} \quad (k=1,2,\ldots). \qquad (3.13.16)$$

where ν_1, ν_2, ... – sequence of solutions of transcendental equation

$$J_1'(\nu)=0. \tag{3.13 17}$$

We will now seek the solution of boundary value problem, formed by differential equation (3.13.4) and boundary conditions (3.13.5), in the form of series

$$\Phi_j^* = \sum_{k=1}^{\infty}\left(A_k e^{\frac{\nu_k \xi}{R}} + B_k e^{-\frac{\nu_k \xi}{R}}\right) J_1\left(\frac{\nu_k r}{R}\right). \tag{3.13.18}$$

We showed above that each of the terms of series (3.13.18) represents the solution of differential equation (3.13.4), regular on axis $r = 0$ and satisfying the second of boundary conditions (3.13.5) Thus, it remains to find the values of coefficients A_k and B_k ($k = 1, 2, \ldots$), at which series (3.13.18) will satisfy the first and third of these boundary conditions. By substituting series (3.13.18) in these two boundary conditions, we obtain equalities:

$$\left.\begin{aligned} &\sum_{k=1}^{\infty}\frac{\nu_k}{R}\left(A_k e^{-\frac{\nu_k H}{R}} - B_k e^{\frac{\nu_k H}{R}}\right) J_1\left(\frac{\nu_k r}{R}\right) = -2r \\ &\sum_{k=1}^{\infty}\frac{\nu_k}{R}(A_k - B_k) J_1\left(\frac{\nu_k r}{R}\right) = -2r \end{aligned}\right\} \; 0 < r < R. \tag{3.13.19}$$

For Bessel function of the first order there take place integral relationships:

$$\begin{aligned} &\int_0^R r J_1(\alpha r) J_1(\beta r)\, dr = \frac{R\left[\beta J_1(\alpha R) J_1'(\beta R) - \alpha J_1(\beta R) J_1'(\alpha R)\right]}{\alpha^2-\beta^2}, \quad \beta \neq \alpha \\ &\int_0^R r J_1^2(\alpha r)\, dr = \frac{R^2}{2}\left\{\left[J_1'(\alpha R)\right]^2 + \left(1 - \frac{1}{\alpha^2 R^2}\right) J_1^2(\alpha R)\right\}, \\ &\int_0^R r^2 J_1(\alpha r)\, dr = \frac{R^2}{\alpha}\left[\frac{J_1(\alpha R)}{\alpha R} - J_1'(\alpha R)\right] \end{aligned} \tag{3.13.20}$$

(see [27]).

By assuming in (3.13.20) $\alpha = \nu_l/R$, $\beta = \nu_k/R$, we obtain equalities:

$$\int_0^R rJ_1\left(\frac{\nu_l r}{R}\right)J_1\left(\frac{\nu_k r}{R}\right)dr = 0, \quad k \neq l$$
$$\int_0^R rJ_1^2\left(\frac{\nu_l r}{R}\right)dr = \frac{R^2}{2}\left(1-\frac{1}{\nu_l^2}\right)J_1^2(\nu_l), \qquad (3.13.21)$$
$$\int_0^R r^2J_1\left(\frac{\nu_l r}{R}\right)dr = \frac{R^3}{\nu_l^2}J_1(\nu_l),$$

since numbers ν_1, ν_2, ... are solutions of equations (3.13.17). Having multiplied both sides of relationships (3.13.19) by $rJ_1\left(\frac{\nu_l r}{R}\right)$ and integrated them with respect to r from 0 to R, we obtain according to (3.13.21) equations:

$$A_l e^{-\frac{\nu_l H}{R}} - B_l e^{\frac{\nu_l H}{R}} = -\frac{4R^2}{\nu_l(\nu_l^2-1)J_1(\nu_l)},$$
$$A_l - B_l = -\frac{4R^2}{\nu_l(\nu_l^2-1)J_1(\nu_l)}. \qquad (3.13.22)$$

From equations (3.12.22) we find

$$A_l = \frac{4R^2\left(1-e^{\frac{\nu_l H}{R}}\right)}{\nu_l(\nu_l^2-1)J_1(\nu_l)\left(e^{\frac{\nu_l H}{R}}-e^{-\frac{\nu_l H}{R}}\right)},$$
$$B_l = \frac{4R^2\left(1-e^{-\frac{\nu_l H}{R}}\right)}{\nu_l(\nu_l^2-1)J_1(\nu_l)\left(e^{\frac{\nu_l H}{R}}-e^{-\frac{\nu_l H}{R}}\right)}. \qquad (3.13.23)$$

By substituting (3.13.23) in (3.13.18), we obtain formula

$$\theta_j^\circ = 4R^2 \sum_{k=1}^{\infty} \frac{\left[e^{\frac{\nu_k\xi}{R}} + e^{-\frac{\nu_k\xi}{R}} - e^{\frac{\nu_k(H+\xi)}{R}} - e^{-\frac{\nu_k(H+\xi)}{R}}\right) J_1\left(\frac{\nu_k r}{R}\right)}{\nu_k(\nu_k^2-1)\left(e^{\frac{\nu_k H}{R}} - e^{-\frac{\nu_k H}{R}}\right) J_1(\nu_k)},$$

or

$$\theta_j^\circ = 4R^2 \sum_{k=1}^{\infty} \frac{\left[\operatorname{ch}\frac{\nu_k\xi}{R} - \operatorname{ch}\frac{\nu_k(H+\xi)}{R}\right] J_1\left(\frac{\nu_k r}{R}\right)}{\nu_k(\nu_k^2-1)\operatorname{sh}\frac{\nu_k H}{R} J_1(\nu_k)}. \qquad (3.13.24)$$

According to (3.12.16), (3.13.3) and (3.13.24) in the considered example the Zhukovskiy potentials will be determined by relationships:

$$\varphi_{jx} = y_j z - z_j y,$$

$$\varphi_{jy} = -\left\{\xi r + 4R^2 \sum_{k=1}^{\infty} \frac{\left[\operatorname{ch}\frac{\nu_k\xi}{R} - \operatorname{ch}\frac{\nu_k(H+\xi)}{R}\right] J_1\left(\frac{\nu_k r}{R}\right)}{\nu_k(\nu_k^2-1)\operatorname{sh}\frac{\nu_k H}{R} J_1(\nu_k)}\right\}\sin\alpha + z_j x - x_j z, \qquad (3.13.2$$

$$\varphi_{jz} = \left\{\xi r + 4R^2 \sum_{k=1}^{\infty} \frac{\left[\operatorname{ch}\frac{\nu_k\xi}{R} - \operatorname{ch}\frac{\nu_k(H+\xi)}{R}\right] J_1\left(\frac{\nu_k r}{R}\right)}{\nu_k(\nu_k^2-1)\operatorname{sh}\frac{\nu_k H}{R} J_1(\nu_k)}\right\}\cos\alpha + x_j y - y_j x.$$

13.2. Half-Filled Spherical Tank

In this instance the generatrix l_j of closed surface $S_j + \sigma_j$ the contour, shown in Fig. 3.6. In differential equation (3.12.13) let us change from rectangular coordinates ξ, r to polar coordinates ϱ, β, having assumed

$$\xi = -\varrho\cos\beta, \quad r = \varrho\sin\beta. \qquad (3.13.26)$$

In accordance with transition formulas:

$$
\begin{aligned}
&\frac{\partial\theta_j}{\partial\xi} = -\frac{\partial\theta_j}{\partial\varrho}\cos\beta + \frac{1}{\varrho}\frac{\partial\theta_j}{\partial\beta}\sin\beta,\\
&\frac{\partial\theta_j}{\partial r} = \frac{\partial\theta_j}{\partial\varrho}\sin\beta + \frac{1}{\varrho}\frac{\partial\theta_j}{\partial\beta}\cos\beta,\\
&\frac{\partial^2\theta_j}{\partial\xi^2} + \frac{\partial^2\theta_j}{\partial r^2} = \frac{\partial^2\theta_j}{\partial\varrho^2} + \frac{1}{\varrho}\frac{\partial\theta_j}{\partial\varrho} + \frac{1}{\varrho^2}\frac{\partial^2\theta_j}{\partial\beta^2}
\end{aligned}
\qquad (3.13.27)
$$

differential equation (3.12.13) takes the form

$$
\frac{\partial^2\theta_j}{\partial\varrho^2} + \frac{2}{\varrho}\frac{\partial\theta_j}{\partial\varrho} + \frac{1}{\varrho^2}\frac{\partial^2\theta_j}{\partial\beta^2} + \frac{\operatorname{ctg}\beta}{\varrho^2}\frac{\partial\theta_j}{\partial\beta} - \frac{\theta_j}{\varrho^2\sin^2\beta} = 0. \qquad (3.13.28)
$$

Contour l_j in the considered case is made up of the arc of circumference $\varrho = R$ and segment of straight line $\beta = \pi/2$ (Fig. 3.6). Angle l_j, figuring in boundary condition (3.12.14), on contour χ will be determined by equalities:

$$
\begin{aligned}
&\chi = \beta \ \text{when} \ \varrho = R,\ 0 < \beta < \frac{\pi}{2},\\
&\chi = \pi \ \text{when} \ \beta = \frac{\pi}{2},\ 0 < \varrho < R.
\end{aligned}
\qquad (3.13.29)
$$

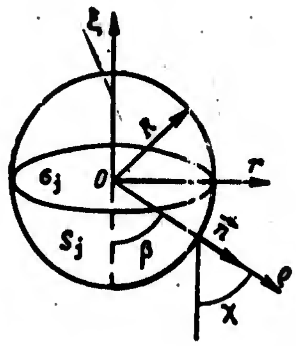

Fig. 3.6.

According to (3.13.27) the derivative of $\partial\theta_j/\partial n$ will be determined on contour l_j by formulas:

$$\frac{\partial\theta_j}{\partial n}=\frac{\partial\theta_j}{\partial\varrho} \quad \text{when} \quad \varrho=R,\ 0<\beta<\frac{\pi}{2}, \tag{3.13.30}$$

$$\frac{\partial\theta_j}{\partial n}=\frac{\partial\theta_j}{\partial\xi}=-\frac{\partial\theta_j}{\partial\varrho}\cos\beta+\frac{1}{\varrho}\frac{\partial\theta_j}{\partial\beta}\sin\beta=\frac{1}{\varrho}\frac{\partial\theta_j}{\partial\beta} \quad \text{when} \quad \beta=\frac{\pi}{2},\ 0<\varrho<R.$$

In accordance with equalities (3.13.26), (3.13.29) and (3.13.30) boundary condition (3.12.14) in the considered example will have the form

$$\frac{\partial\theta_j}{\partial\varrho}=0 \quad \text{when} \quad \varrho=R,\ 0<\beta<\frac{\pi}{2},$$
$$\frac{\partial\theta_j}{\partial\beta}=-\varrho^2 \quad \text{when} \quad \beta=\frac{\pi}{2},\ 0<\varrho<R. \tag{3.13.31}$$

Assuming in (3.13.28) and (3.13.31)

$$\theta_j=\frac{\varrho^2}{2}\sin 2\beta+\theta_j^*, \tag{3.13.32}$$

for function θ_j^* we obtain differential equation

$$\frac{\partial^2\theta_j^*}{\partial\varrho^2}+\frac{2}{\varrho}\frac{\partial\theta_j^*}{\partial\varrho}+\frac{1}{\varrho^2}\frac{\partial^2\theta_j^*}{\partial\beta^2}+\frac{\operatorname{ctg}\beta}{\varrho^2}\frac{\partial\theta_j^*}{\partial\beta}-\frac{\theta_j^*}{\varrho^2\sin^2\beta}=0 \tag{3.13.33}$$

with boundary conditions:

$$\frac{\partial\theta_j^*}{\partial\varrho}=-R\sin 2\beta \quad \text{when} \quad \varrho=R,\ 0<\beta<\frac{\pi}{2},$$
$$\frac{\partial\theta_j^*}{\partial\beta}=0 \quad \text{when} \quad \beta=\frac{\pi}{2},\ 0<\varrho<R. \tag{3.13.34}$$

Let us first construct particular solution of differential equation (3.13.33) of form

$$\theta_j^* = f(\varrho)\, g(\beta). \tag{3.13.35}$$

regular on segment $\beta = 0$, $0 \le \varrho \le R$ and satisfying the second of conditions (3.13.34). By substituting (3.13.35) in (3.13.33), we obtain equality

$$\left(\frac{d^2 f}{d\varrho^2} + \frac{2}{\varrho}\frac{df}{d\varrho}\right) g + \frac{f}{\varrho^2}\left(\frac{d^2 g}{d\beta^2} + \operatorname{ctg}\beta \frac{dg}{d\beta} - \frac{g}{\sin^2\beta}\right) = 0.$$

or

$$\frac{\dfrac{d^2 f}{d\varrho^2} + \dfrac{2}{\varrho}\dfrac{df}{d\varrho}}{\dfrac{f}{\varrho^2}} = -\frac{\dfrac{d^2 g}{d\beta^2} + \operatorname{ctg}\beta \dfrac{dg}{d\beta} - \dfrac{g}{\sin^2\beta}}{g}. \tag{3.13.36}$$

The left side of equality (3.13.36) does not depend on variable β, and the right side of this equality does not depend on variable ϱ. Thus, rations which figure in equality (3.13.36), must represent a certain constant. By designating this constant through λ, for function $f(\varrho)$ we obtain differential equation

$$\frac{d^2 f}{d\varrho^2} + \frac{2}{\varrho}\frac{df}{d\varrho} - \lambda \frac{f}{\varrho^2} = 0 \tag{3.13.37}$$

and for function $g(\beta)$ – differential equation

$$\frac{d^2 g}{d\beta^2} + \operatorname{ctg}\beta \frac{dg}{d\beta} + \left(\lambda - \frac{1}{\sin^2\beta}\right) g = 0. \tag{3.13.38}$$

Assuming in (3.13.38)

$$\cos\beta = \eta. \tag{3.13.39}$$

for function g we obtain Legendre differential

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{dg}{d\eta}\right]+\left(\lambda-\frac{1}{1-\eta^2}\right)g=0. \qquad (3.13.40)$$

In order that function θ_j^*, determined by formula (3.13.35), would be regular when $\beta = 0$ and would satisfy the second of boundary conditions (3.13.34), function g according to (3.13.39) should be regular when $\eta = 1$ and must satisfy condition

$$\frac{dg}{d\eta}=0 \text{ when } \eta=0. \qquad (3.13.41)$$

Differential equation (3.13.40) has a solution, which satisfies condition (3.13.41) and is regular at point $\eta = 1$; when

$$\lambda=2k(2k-1) \qquad (3.13.42)$$

(k = 1, 2, ...) this solution is determined by associated Legendre function of the first type $P_{2k-1}^{(1)}(\eta)$. Thus, according to (3.13.39) initial differential equation (3.13.38) when $\lambda = 2k\ (2k - 1)$ has solution

$$g=P_{2k-1}^{(1)}(\cos\beta). \qquad (3.13.43)$$

regular at point $\beta = 0$ and satisfying condition

$$\frac{dg}{d\beta}=0 \text{ when } \beta=\frac{\pi}{2}. \qquad (3.13.44)$$

By substituting (3.13.42) in (3.13.37), for function $f(\varrho)$ we obtain Euler differential equation

$$\frac{d^2f}{d\varrho^2}+\frac{2}{\varrho}\frac{df}{d\varrho}-2k(2k-1)\frac{f}{\varrho^2}=0. \qquad (3.13.45)$$

Solution of differential equation (3.13.45), regular at point $\varrho = 0$, has the form

$$f = \varrho^{2k-1}. \qquad (3.13.46)$$

According to (3.13.35), (3.13.43) and (3.13.46) differential equation (3.13.33) has infinite sequence of particular solutions

$$\theta_j^* = \varrho^{2k-1} P_{2k-1}^{(1)}(\cos\beta) \qquad (k = 1, 2, \ldots), \qquad (3.13.47)$$

regular on segment $\beta = 0$, $0 \leq \varrho \leq R$ and satisfying the second of conditions (3.13.34).

We will now seek the solution of boundary value problem, formed by differential equation (3.13.33) and boundary conditions (3.13.34), in the form of series

$$\theta_j^* = \sum_{k=1}^{\infty} c_k \varrho^{2k-1} P_{2k-1}^{(1)}(\cos\beta). \qquad (3.13.48)$$

Each term of series (3.13.48) satisfies differential equation (3.13.33) and the second of boundary conditions (3.13.34). Thus, it remains to find the values of coefficients c_1, c_2, ..., at which series (3.13.48) will satisfy the first of these boundary conditions. By substituting series (3.13.48) in the first of conditions (3.13.34), we obtain equality

$$\sum_{k=1}^{\infty} (2k-1) c_k R^{2k-3} P_{2k-1}^{(1)}(\cos\beta) = -\sin 2\beta \left(0 < \beta < \frac{\pi}{2}\right).$$

or

$$\sum_{k=1}^{\infty} (2k-1) c_k R^{2k-3} P_{2k-1}^{(1)}(\eta) = -2\eta\sqrt{1-\eta^2} \; (0 < \eta < 1). \qquad (3.13.49)$$

By using formula

$$P_2^{(1)}(\eta) = 3\eta\sqrt{1-\eta^2}, \qquad (3.13.50)$$

it is possible to convert equality (3.13.50) thus:

$$\sum_{k=1}^{\infty}(2k-1)\,c_k R^{2k-3} P_{2k-1}^{(1)}(\eta) = -\frac{2}{3}P_2^{(1)}(\eta)\ (0<\eta<1). \qquad (3.13.51)$$

For associated Legendre functions, figuring in (3.13.51), there exist integral relationships (see [27]):

$$\int_0^1 P_{2k-1}^{(1)}(\eta)\,P_{2l-1}^{(1)}(\eta)\,d\eta = 0 \text{ when } k \neq l,$$

$$\int_0^1 \left[P_{2l-1}^{(1)}(\eta)\right]^2 d\eta = \frac{(2l-1)\,2l}{4l-1}, \qquad (3.13.52)$$

$$\int_0^1 P_2^{(1)}(\eta)\,P_{2l-1}^{(1)}(\eta)\,d\eta = \frac{3(-1)^l l}{(l+1)(2l-3)}\,\frac{1.3\ldots(2l-1)}{2.4\ldots 2l}.$$

Having multiplied (3.13.51) by $P_{2l-1}^{(1)}(\eta)$ and integrated with respect to η from 0 to 1, according to (3.13.52) we obtain equality

$$\frac{(2l-1)^2 2l}{4l-1}\,c_l R^{2l-3} = \frac{2(-1)^{l-1} l}{(l+1)(2l-3)}\,\frac{1.3\ldots(2l-1)}{2.4\ldots 2l},$$

in accordance with which coefficients c_1, c_2, ..., should have values determined by formula

$$c_l = \frac{(-1)^{l-1}(4l-1)}{(l+1)(2l-3)(2l-1)^2}\,\frac{1.3\ldots(2l-1)}{2.4\ldots 2l}\,R^{3-2l}. \qquad (3.13.53)$$

By substituting (3.13.53) in (3.13.48), for function θ_j^* expansion

$$\varphi_j^{*}=R^2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(4k-1)}{(k+1)(2k-3)(2k-1)^2}\,\frac{1.3\ldots(2k-1)}{2.4\ldots 2k}\left(\frac{\varrho}{R}\right)^{2k-1}P_{2k-1}^{(1)}(\cos\beta). \qquad (3.13.54)$$

According to (3.12.16), (3.13.32) and (3.13.54) in the given example the Zhukovskiy potentials will be determined by relationships:

$$\begin{aligned}
\varphi_{jx}&=yj_z-zj_y,\\
\varphi_{jy}&=-\left[\frac{\varrho^2}{2}\sin 2\beta+R^2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(4k-1)}{(k+1)(2k-3)(2k-1)^2}\,\frac{1.3\ldots(2k-1)}{2.4\ldots 2k}\times\right.\\
&\qquad\left.\times\left(\frac{\varrho}{R}\right)^{2k-1}P_{2k-1}^{(1)}(\cos\beta)\right]\sin\alpha+zj_x-xj_z,\\
\varphi_{jz}&=\left[\frac{\varrho^2}{2}\sin 2\beta+R^2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(4k-1)}{(k+1)(2k-3)(2k-1)^2}\,\frac{1.3\ldots(2k-1)}{2.4\ldots 2k}\times\right.\\
&\qquad\left.\times\left(\frac{\varrho}{R}\right)^{2k-1}P_{2k-1}^{1}(\cos\beta)\right]\cos\alpha+xj_y-yj_x.
\end{aligned} \qquad (3.13.55)$$

§ 14. Calculation of Moments of Inertia of the Rocket

According to (3.9.3), with allowance for mobility of liquid components of propellant, the moments of inertia of the rocket are expressed by relationships:

$$\begin{aligned}
&J_{xx}=J_{xx}^{(0)}+\sum_{j=1}^{N}J_{xx}^{(j)},\ J_{yy}=J_{yy}^{(0)}+\sum_{j=1}^{N}J_{yy}^{(j)},\ J_{zz}=J_{zz}^{(0)}+\sum_{j=1}^{N}J_{zz}^{(j)},\\
&J_{xy}=J_{xy}^{(0)}+\sum_{j=1}^{N}J_{xy}^{(j)},\ J_{xz}=J_{xz}^{(0)}+\sum_{j=1}^{N}J_{xz}^{(j)},\ J_{yz}=J_{yz}^{(0)}+\\
&+\sum_{j=1}^{N}J_{yz}^{(j)},
\end{aligned} \qquad (3.14.1)$$

where $J_{xx}^{(0)}$, $J_{yy}^{(0)}$, $J_{zz}^{(0)}$, $J_{xy}^{(0)}$, $J_{xz}^{(0)}$, $J_{yz}^{(0)}$ – moments of inertia of a rocket not filled with fuel; $J_{xx}^{(j)}$, $J_{yy}^{(j)}$, $J_{zz}^{(j)}$, $J_{xy}^{(j)}$, $J_{xz}^{(j)}$, J_{yz}^{j} $(1 \leq j \leq N)$ – moments of inertia of liquid mass, located in the tank with number j, which can be found by formula:

$$J_{xx}^{(j)}=\varrho_j\int\limits_{S_j+\sigma_j}\varphi_{jx}\frac{\partial\varphi_{jx}}{\partial n}d\sigma,\quad J_{yy}^{(j)}=\varrho_j\int\limits_{S_j+\sigma_j}\varphi_{jy}\frac{\partial\varphi_{jy}}{\partial n}d\sigma,$$

$$J_{zz}^{(j)}=\varrho_j\int\limits_{S_j+\sigma_j}\varphi_{jz}\frac{\partial\varphi_{jz}}{\partial n}d\sigma,\quad J_{xy}^{(j)}=\varrho_j\int\limits_{S_j+\sigma_j}\varphi_{jx}\frac{\partial\varphi_{jy}}{\partial n}d\sigma,\qquad (3.14.2)$$

$$J_{xz}^{(j)}=\varrho_j\int\limits_{S_j+\sigma_j}\varphi_{jx}\frac{\partial\varphi_{jz}}{\partial n}d\sigma,\quad J_{yz}^{(j)}=\varrho_j\int\limits_{S_j+\sigma_j}\varphi_{jy}\frac{\partial\varphi_{jz}}{\partial n}.$$

In accordance with equality (3.12.1) vector function $\vec{\phi}$ can be represented in the form

$$\vec{\varphi}_j=\vec{\varphi}_j^{(0)}+\vec{\varphi}_j^{(1)},\qquad (3.14.3)$$

where $\vec{\phi}_j^{(0)}$ – vector function, determined by differential equation (3.12.4) with boundary condition (3.12.5);

$$\vec{\varphi}_j^{(1)}=\vec{r}_j\times\vec{r}\qquad (3.14.4)$$

– vector function, which according to (3.12.3) satisfies differential equation

$$\nabla^2\vec{\varphi}_j^{(1)}=0\qquad (3.14.5)$$

with boundary condition

$$\frac{\partial\vec{\varphi}_j^{(1)}}{\partial n}=\vec{r}_j\times\vec{n}\quad\text{on } S_j \text{ and } \sigma_j.\qquad (3.14.6)$$

By using relationships (3.12.12) and (3.14.3), it is possible to convert formulas (3.14.2) thus:

$$J_{xx}^{(j)}=\varrho_j\int\limits_{S_j+\sigma_j}\varphi_{jx}^{(1)}\frac{\partial\varphi_{jx}^{(1)}}{\partial n}d\sigma,$$

$$J_{xy}^{(j)}=\varrho_j\int\limits_{S_j+\sigma_j}\varphi_{jx}^{(1)}\frac{\partial\left(\varphi_{jy}^{(0)}+\varphi_{jy}^{(1)}\right)}{\partial n}d\sigma,\qquad (3.14.7)$$

$$J_{xz}^{(j)}=\varrho_j\int\limits_{S_j+\sigma_j}\varphi_{jx}^{(1)}\frac{\partial\left(\varphi_{jz}^{(0)}+\varphi_{jz}^{(1)}\right)}{\partial n}d\sigma,$$

$$J_{yy}^{(j)} = \varrho_j \int\limits_{S_j+\epsilon_j} \left[\varphi_{jy}^{(1)} \frac{\partial(\varphi_{jy}^{(0)} + \varphi_{jy}^{(1)})}{\partial n} + \varphi_{jy}^{(0)} \frac{\partial \varphi_{jy}^{(1)}}{\partial n} \right] d\sigma +$$

$$+ \varrho_j \int\limits_{S_j+\epsilon_j} \varphi_{jy}^{(0)} \frac{\partial \varphi_{jy}^{(0)}}{\partial n} d\sigma,$$

$$J_{yz}^{(j)} = \varrho_j \int\limits_{S_j+\epsilon_j} \left[\varphi_{jy}^{(1)} \frac{\partial(\varphi_{jz}^{(0)} + \varphi_{jz}^{(1)})}{\partial n} + \varphi_{jy}^{(0)} \frac{\partial \varphi_{jz}^{(1)}}{\partial n} \right] d\sigma +$$

$$+ \varrho_j \int\limits_{S_j+\epsilon_j} \varphi_{jy}^{(0)} \frac{\partial \varphi_{jz}^{(0)}}{\partial n} d\sigma, \qquad (3.14.8)$$

$$J_{zz}^{(j)} = \varrho_j \int\limits_{S_j+\epsilon_j} \left[\varphi_{jz}^{(1)} \frac{\partial(\varphi_{jz}^{(0)} + \varphi_{jz}^{(1)})}{\partial n} + \varphi_{jz}^{(0)} \frac{\partial \varphi_{jz}^{(1)}}{\partial n} \right] d\sigma +$$

$$+ \varrho_j \int\limits_{S_j+\epsilon_j} \varphi_{jz}^{(0)} \frac{\partial \varphi_{jz}^{(0)}}{\partial n} d\sigma.$$

According to (3.12.4) and (3.14.5) functions $\phi_{jy}^{(0)}$, $\phi_{jz}^{(0)}$, $\phi_{jy}^{(1)}$, $\phi_{jz}^{(1)}$ satisfy Laplace equation and in accordance with Green's theorem there should take place equalities:

$$\int\limits_{S_j+\epsilon_j} \varphi_{jy}^{(0)} \frac{\partial \varphi_{jy}^{(1)}}{\partial n} d\sigma = \int\limits_{S_j+\epsilon_j} \varphi_{jy}^{(1)} \frac{\partial \varphi_{jy}^{(0)}}{\partial n} d\sigma,$$

$$\int\limits_{S_j+\epsilon_j} \varphi_{jy}^{(0)} \frac{\partial \varphi_{jz}^{(1)}}{\partial n} d\sigma = \int\limits_{S_j+\epsilon_j} \varphi_{jz}^{(1)} \frac{\partial \varphi_{jy}^{(0)}}{\partial n} d\sigma, \qquad (3.14.9)$$

$$\int\limits_{S_j+\epsilon_j} \varphi_{jz}^{(0)} \frac{\partial \varphi_{jz}^{(1)}}{\partial n} d\sigma = \int\limits_{S_j+\epsilon_j} \varphi_{jz}^{(1)} \frac{\partial \varphi_{jz}^{(0)}}{\partial n} d\sigma.$$

Thus, formulas (3.14.8) can be given the form

$$J_{yy}^{(j)} = \varrho_j \int\limits_{S_j+\epsilon_j} \varphi_{jy}^{(1)} \frac{\partial(2\varphi_{jy}^{(0)} + \varphi_{jy}^{(1)})}{\partial n} d\sigma + \varrho_j \int\limits_{S_j+\epsilon_j} \varphi_{jy}^{(0)} \frac{\partial \varphi_{jy}^{(0)}}{\partial n} d\sigma, \qquad (3.14.10)$$

$$J_{yz}^{(j)} = \varrho_j \int\limits_{S_j+\sigma_j} \varphi_{jy}^{(1)} \frac{\partial(\varphi_{jz}^{(0)} + \varphi_{jz}^{(1)})}{\partial n} d\sigma + \varrho_j \int\limits_{S_j+\sigma_j} \varphi_{jz}^{(1)} \frac{\partial \varphi_{jy}^{(0)}}{\partial n} d\sigma +$$

$$+ \varrho_j \int\limits_{S_j+\sigma_j} \varphi_{jy}^{(0)} \frac{\partial \varphi_{jz}^{(0)}}{\partial n} d\sigma, \qquad (3.14.10 \text{ cont'd})$$

$$J_{zz}^{(j)} = \varrho \int\limits_{S_j+\sigma_j} \varphi_{jz}^{(1)} \frac{\partial(2\varphi_{jz}^{(0)} + \varphi_{jz}^{(1)})}{\partial n} d\sigma + \varrho_j \int\limits_{S_j+\sigma_j} \varphi_{jz}^{(0)} \frac{\partial \varphi_{jz}^{(0)}}{\partial n} d\sigma.$$

By using relationships (3.12.5), (3.14.4) and (3.14.6), it is possible to convert formulas (3.14.7) and (3.14.10):

$$J_{xx}^{(j)} = \varrho_j \int\limits_{S_j+\sigma_j} (y_j z - z_j y)[y_j \cos(n,z) - z_j \cos(n,y)] d\sigma,$$

$$J_{xy}^{(j)} = \varrho_j \int\limits_{S_j+\sigma_j} (y_j z - z_j y)[z \cos(n,x) - x \cos(n,z)] d\sigma,$$

$$J_{xz}^{(j)} = \varrho_j \int\limits_{S_j+\sigma_j} (y_j z - z_j y)[x \cos(n,y) - y \cos(n,x)] d\sigma,$$

$$J_{yy}^{(j)} = \varrho_j \int\limits_{S_j+\sigma_j} (z_j x - x_j z)[(2z - z_j)\cos(n,x) -$$

$$-(2x - x_j)\cos(n,z)] d\sigma + \varrho_j \int\limits_{S_j+\sigma_j} \varphi_{jy}^{(0)} \frac{\partial \varphi_{jy}^{(0)}}{\partial n} d\sigma,$$

$$J_{yz}^{(j)} = \varrho_j \int\limits_{S_j+\sigma_j} \{(z_j x - x_j z)[x \cos(n,y) - y \cos(n,x)] + \qquad (3.14.11)$$

$$+ (x_j y - y_j x)[(z - z_j)\cos(n,x) -$$

$$-(x - x_j)\cos(n,z)]\} d\sigma + \varrho_j \int\limits_{S_j+\sigma_j} \varphi_{jy}^{(0)} \frac{\partial \varphi_{jz}^{(0)}}{\partial n} d\sigma,$$

$$J_{zz}^{(j)} = \varrho_j \int\limits_{S_j+\sigma_j} (x_j y - y_j x)[(2x - x_j)\cos(n,y) -$$

$$-(2y - y_j)\cos(n,x)] d\sigma + \varrho_j \int\limits_{S_j+\sigma_j} \varphi_{jz}^{(0)} \frac{\partial \varphi_{jz}^{(0)}}{\partial n} d\sigma.$$

By converting surface integrals into volumetric in formulas (3.14.11) by means of Gauss divergence formulas, we obtain relation ships:

$$
\begin{aligned}
J_{xx}^{(j)} &= \rho_j \int_{V_j} (y_j^2 + z_j^2)\, dv, \\
J_{xy}^{(j)} &= -\rho_j \int_{V_j} x y_j\, dv, \\
J_{xz}^{(j)} &= -\rho_j \int_{V_j} x z_j\, dv, \\
J_{yy}^{(j)} &= \rho_j \int_{V_j} [(2x - x_j) x_j + (2z - z_j) z_j]\, dv + \\
&+ \rho_j \int_{S_j + \sigma_j} \varphi_{jy}^{(0)} \frac{\partial \varphi_{jy}^{(0)}}{\partial n}\, d\sigma, \\
J_{yz}^{(j)} &= -\rho_j \int_{V_j} [y z_j + (z - z_j) y_j]\, dv + \\
&+ \rho_j \int_{S_j + \sigma_j} \varphi_{jy}^{(0)} \frac{\partial \varphi_{jz}^{(0)}}{\partial n}\, d\sigma, \\
J_{zz}^{(j)} &= \rho_j \int_{V_j} [(2x - x_j) x_j + (2y - y_j) y_j]\, dv + \\
&+ \rho_j \int_{S_j + \sigma_j} \varphi_{jz}^{(0)} \frac{\partial \varphi_{jz}^{(0)}}{\partial n}\, d\sigma.
\end{aligned}
\qquad (3.14.12)
$$

The mass of fluid, located in the tank with number j – m_j and coordinates of the center of mass of this fluid $x_C^{(j)}$, $y_C^{(j)}$, $z_C^{(j)}$ will be determined by formulas:

$$
\begin{aligned}
& m_j = \rho_j \int_{V_j} dv, \quad x_C^{(j)} = \frac{\rho_j}{m_j} \int_{V_j} x\, dv, \quad y_C^{(j)} = \frac{\rho_j}{m_j} \int_{V_j} y\, dv, \\
& z_C^{(j)} = \frac{\rho_j}{m_j} \int_{V_j} z\, dv.
\end{aligned}
\qquad (3.14.13)
$$

The center of mass of fluid, located in j-th tank, should lie on the axis of this tank in accordance with the assumption about axial symmetry of fuel tanks. Thus, there should take place equalities:

$$
y_C^{(j)} = y_j, \quad z_C^{(j)} = z_j. \qquad (3.14.14)
$$

According to (3.14.13) and (3.14.14) formulas (3.14.12) can be given the form

$$\begin{gathered}
J_{xx}^{(j)}=m_j(y_j^2+z_j^2),\quad J_{xy}^{(j)}=-m_j x_C^{(j)} y_j,\quad J_{xz}^{(j)}=-m_j x_C^{(j)} z_j,\\
J_{yy}^{(j)}=m_j\left[(2x_C^{(j)}-x_j)x_j+z_j^2\right]+\varrho_j\int\limits_{S_j+\sigma_j}\varphi_{jy}^{(0)}\frac{\partial\varphi_{jy}^{(0)}}{\partial n}\,d\sigma,\\
J_{yz}^{(j)}=-m_j y_j z_j+\varrho_j\int\limits_{S_j+\sigma_j}\varphi_{jy}^{(0)}\frac{\partial\varphi_{jz}^{(0)}}{\partial n}\,d\sigma,\\
J_{zz}^{(j)}=m_j\left[(2x_C^{(j)}-x_j)x_j+y_j^2\right]+\varrho_j\int\limits_{S_j+\sigma_j}\varphi_{jz}^{(0)}\frac{\partial\varphi_{jz}^{(0)}}{\partial n}\,d\sigma.
\end{gathered}\tag{3.14.15}$$

The integral from some function f on axisymmetrical surface $S_j + \sigma_j$ can be found by formula

$$\int\limits_{S_j+\sigma_j} f\,d\sigma=\int\limits_{l_j}\int\limits_0^{2\pi} f r\,ds\,d\alpha,\tag{3.14.16}$$

where l_j – generatrix of surface of integration $S_j + \sigma_j$. By using formulas (3.12.15) and (3.14.16), we find

$$\begin{gathered}
\int\limits_{S_j+\sigma_j}\varphi_{jy}^{(0)}\frac{\partial\varphi_{jy}^{(0)}}{\partial n}\,d\sigma=\int\limits_{S_j+\sigma_j}\theta_j\frac{\partial\theta_j}{\partial n}\sin^2\alpha\,d\sigma=\int\limits_{l_j}\int\limits_0^{2\pi}\theta_j\frac{\partial\theta_j}{\partial n}\times\\
\times r\sin^2\alpha\,ds\,d\alpha=\int\limits_0^{2\pi}\sin^2\alpha\,d\alpha\int\limits_{l_j}\theta_j\frac{\partial\theta_j}{\partial n}r\,ds=\pi\int\limits_{l_j}\theta_j\frac{\partial\theta_j}{\partial n}r\,ds,\\
\int\limits_{S_j+\sigma_j}\varphi_{jy}^{(0)}\frac{\partial\varphi_{jz}^{(0)}}{\partial n}\,d\sigma=-\int\limits_{S_j+\sigma_j}\theta_j\frac{\partial\theta_j}{\partial n}\sin\alpha\cdot\cos\alpha\,d\sigma=\\
=-\int\limits_{l_j}\int\limits_0^{2\pi}\theta_j\frac{\partial\theta_j}{\partial n}r\sin\alpha\cos\alpha\,ds\,d\alpha=-\int\limits_0^{2\pi}\sin\alpha\cos\alpha\times\\
\times d\alpha\int\limits_{l_j}\theta_j\frac{\partial\theta_j}{\partial n}r\,ds=0,
\end{gathered}\tag{3.14.17}$$

$$\int_{S_j+\sigma_j} \varphi_{jz}^{(0)} \frac{\partial \varphi_{jz}^{(0)}}{\partial n} d\sigma = \int_{S_j+\sigma_j} \theta_j \frac{\partial \theta_j}{\partial n} \cos^2 \alpha d\sigma = \int_{l_j}\int_0^{2\pi} \theta_j \frac{\partial \theta_j}{\partial n} \times$$

$$\times r \cos^2 \alpha ds d\alpha = \int_0^{2\pi} \cos^2 \alpha d\alpha \int_{l_j} \theta_j \frac{\partial \theta_j}{\partial n} r ds = \qquad (3.14.17 \text{ cont'd})$$

$$= \pi \int_{l_j} \theta_j \frac{\partial \theta_j}{\partial n} r ds.$$

By substituting (3.14.17) in (3.14.15), for moments of inertia $J_{xx}^{(j)}$, $J_{yy}^{(j)}$, $J_{zz}^{(j)}$ we obtain formulas:

$$J_{xx}^{(j)} = m_j (y_j^2 + z_j^2),$$
$$J_{yy}^{(j)} = m_j [(2x_C^{(j)} - x_j) x_j + z_j^2] + \pi \varrho_j \int_{l_j} \theta_j \frac{\partial \theta_j}{\partial n} r ds, \qquad (3.14.18)$$
$$J_{zz}^{(j)} = m_j [(2x_C^{(j)} - x_j) x_j + y_j^2] + \pi \varrho_j \int_{l_j} \theta_j \frac{\partial \theta_j}{\partial n} r ds$$

and for centrifugal inertia moments $J_{xy}^{(j)}$, $J_{xz}^{(j)}$, $J_{yz}^{(j)}$ – formulas:

$$J_{xy}^{(j)} = -m_j x_C^{(j)} y_j, \quad J_{xz}^{(j)} = -m_j x_C^{(j)} z_j, \quad J_{yz}^{(j)} = -m_j y_j z_j. \qquad (3.14.19)$$

According to (3.14.1) and (3.14.19) the centrifugal moments of inertia of the rocket are determined by relationships:

$$J_{xy} = J_{xy}^{(0)} - \sum_{j=1}^{N} m_j x_C^{(j)} y_j,$$
$$J_{xz} = J_{xz}^{(0)} - \sum_{j=1}^{N} m_j x_C^{(j)} z_j, \qquad (3.14.20)$$
$$J_{yz} = J_{yz}^{(0)} - \sum_{j=1}^{N} m_j y_j z_j.$$

Just as in Chapter I, we will assume below that planes $y = 0$ and $z = 0$ are planes of symmetry of the rocket. In this instance the principal axes of inertia of a rocket not filled with fuel coincide with axes x, y, z, in other words

$$J_{xy}^{(0)} = J_{xz}^{(0)} = J_{yz}^{(0)} = 0. \tag{3.14.21}$$

When $y_j \neq 0$ to the mass of fluid m_j with center of mass at point $x = x_C^{(j)}$, $y = y_j$, $z = z_j$ in view of the symmetry there will always correspond the same mass of fluid m_j with center of mass at point $x = x_C^{(j)}$, $y = -y_j$, $z = z_j$. Analogously when $z_j \neq 0$ to the mass of fluid m_j with center of mass at point $x = x_C^{(j)}$, $y = y_j$, $z = z_j$ in view of symmetry there will correspond the same mass of fluid with center of mass at point $x = x_C^{(j)}$, $y = y_j$, $z = -z_j$. Thus, there will always exist equalities

$$\sum_{j=1}^{N} m_j x_C^{(j)} y_j = \sum_{j=1}^{N} m_j x_C^{(j)} z_j = \sum_{j=1}^{N} m_j y_j z_j = 0. \tag{3.14.22}$$

According to (3.14.20), (3.14.21) and (3.14.22) during consideration of mobility of liquid components of propellant, equalities

$$J_{xy} = J_{xz} = J_{yz} = 0 \tag{3.14.23}$$

are retained and axes of fixed system of coordinates x, y, z remain principal axes of inertia of the rocket. Moments of inertia of the rocket J_{xx}, J_{yy}, J_{zz} retain the role of principal moments of inertia J_x, J_y, J_z:

$$J_{xx} = J_x,\ J_{yy} = J_y,\ J_{zz} = J_z. \tag{3.14.24}$$

Let us turn to examination of inertia moments $J_{xx}^{(j)}$, $J_{yy}^{(j)}$, $J_{zz}^{(j)}$, determined by formulas (3.14.18). By using identity

$$(2x_C^{(j)}-x_j)\,x_j=x_C^{(j)2}-(x_j-x_C^{(j)})^2, \qquad (3.14.25)$$

it is possible to give formulas (3.14.18) the form

$$\begin{aligned} J_{xx}^{(j)}&=m_j(y_j^2+z_j^2),\\ J_{yy}^{(j)}&=m_j(x_C^{(j)2}+z_j^2)+J_j,\\ J_{zz}^{(j)}&=m_j(x_C^{(j)2}+y_j^2)+J_j, \end{aligned} \qquad (3.14.26)$$

where

$$J_j=\pi\varrho_j\int_{l_j}\theta_j\frac{\partial\theta_j}{\partial n}r\,ds-m_j(x_j-x_C^{(j)})^2. \qquad (3.14.27)$$

The difference $x_j - x_C^{(j)}$ determines the depth, at which the center of mass of fluid, which is located in the tank with number j, is located; this difference does not depend on the arrangement of j-th tank. Function θ_j, determined by differential equation (3.12.13) with boundary condition (3.12.14), also does not depend on the arrangement of this tank. Thus, quantity J_j, figuring in formulas (3.14.26), depends only on the configuration of the tank with number j, on the level of filling of this tank with fluid and on the density of fluid located in the tank; quantity J_j does not depend on the arrangement of the tank. When $x_C^{(j)} = 0$, $y_j = 0$, $z_j = 0$, i.e., when the center of gravity of liquid mass, which is located in the tank with number j, coincides with origin of fixed system of coordinates x, y, z, formulas (3.14.26) assume the form

$$J_{xx}^{(j)}=0,\quad J_{yy}^{(j)}=J_j,\quad J_{zz}^{(j)}=J_j. \qquad (3.14.28)$$

According to (3.14.28) the quantity of J_j determines the moment of inertia of fluid located in j-th tank, relative to the lateral axis, passing through the center of mass of this fluid. Moment of inertia of liquid relative to longitudinal axis of the tank according to (3.14.28) is equal to zero, which corresponds to the original assumptions about the axial symmetry of fuel tanks and about the perfectness of liquid components of propellant.

In its physical sense quantity J_j is nonnegative, and, by introducing the radius of inertia into examination h_j, it is possible to assume

$$J_j = m_j h_j^2. \qquad (3.14.29)$$

In accordance with formulas (3.14.27) and (3.14.29) the radius of inertia h_j can be found by formula

$$h_j = \left[\frac{\pi}{v_j}\int_{l_j} \theta_j \frac{\partial \theta_j}{\partial n} r\,ds - (x_j - x_C^{(j)})^2\right]^{\frac{1}{2}}, \qquad (3.14.30)$$

where $v_j = \frac{m_j}{\varrho_j}$ - volume of fluid located in j-th tank.

According to (3.14.1), (3.14.24), (3.14.26) and (3.14.29) the moments of inertia J_x, J_y, J_z can be calculated by formulas:

$$\begin{aligned} J_x &= J_{xx}^{(0)} + \sum_{j=1}^{N} m_j (y_j^2 + z_j^2), \\ J_y &= J_{yy}^{(0)} + \sum_{j=1}^{N} m_j (x_C^{(j)2} + z_j^2 + h_j^2), \\ J_z &= J_{zz}^{(0)} + \sum_{j=1}^{N} m_j (x_C^{(j)2} + y_j^2 + h_j^2). \end{aligned} \qquad (3.14.31)$$

By substituting in (3.14.31) the moments of inertia of a rocket not filled with fuel $J_{xx}^{(0)}$, $J_{yy}^{(0)}$, $J_{zz}^{(0)}$ by their expanded expressions, for principal inertia moments of the rocket J_x, J_y, J_z we obtain working formulas:

$$
\begin{aligned}
J_x &= \int\limits_{V_0} (y^2+z^2)\varrho dv + \sum_{j=1}^{N} m_j (y_j^2 + z_j^2),\\
J_y &= \int\limits_{V_0} (x^2+z^2)\varrho dv + \sum_{j=1}^{N} m_j (x_C^{(j)2} + z_j^2 + h_j^2),\\
J_z &= \int\limits_{V_0} (x^2+y^2)\varrho dv + \sum_{j=1}^{N} m_j (x_C^{(j)2} + y_j^2 + h_j^2).
\end{aligned}
\qquad (3.14.32)
$$

According to (3.14.30) the determination of radius of inertia h_j is reduced to solution of boundary value problem, formed by differential equation (3.12.13) and boundary condition (3.12.14). As an example let us calculate the radius of inertia of liquid mass, located in a cylindrical fuel tank. In this case contour l_j is made up of three rectilinear segments: $\xi = -H$, $0 < r < R$; $r = R$, $-H < \xi < 0$; $\xi = 0$, $0 < r < R$ (see Fig. 3.5). According to (3.12.14) and (3.13.1) in the considered case derivative of $\partial\theta_j/\partial n$ will be determined on contour l_j by equalities:

$$
\begin{aligned}
\frac{\partial\theta_j}{\partial n} &= r \text{ when } \xi = -H, \quad 0 < r < R,\\
\frac{\partial\theta_j}{\partial n} &= \xi \text{ when } r = R, \quad -H < \xi < 0,\\
\frac{\partial\theta_j}{\partial n} &= -r \text{ when } \xi = 0, \quad 0 < r < R.
\end{aligned}
\qquad (3.14.33)
$$

In accordance with equalities (3.14.33) the contour integral, which figures in formula (3.14.30), can be represented in the form

$$
\int\limits_{l_j} \theta_j \frac{\partial\theta_j}{\partial n} r ds = \int\limits_0^R (\theta_j)_{\xi=-H} r^2 dr + \int\limits_{-H}^{0} (\theta_j)_{r=R} \xi R d\xi - \int\limits_0^R (\theta_j)_{\xi=0} r^2 dr. \qquad (3.14.34)
$$

By using formulas (3.13.3) and (3.13.24), we find

$$(\theta_j)_{\xi=-H}=-Hr+4R^2\sum_{k=1}^{\infty}\frac{\left(\operatorname{ch}\frac{\nu_k H}{R}-1\right)J_1\left(\frac{\nu_k r}{R}\right)}{\nu_k\left(\nu_k^2-1\right)\operatorname{sh}\frac{\nu_k H}{R}\,J_1(\nu_k)},$$

$$(\theta_j)_{r=R}=R\xi+4R^2\sum_{k=1}^{\infty}\frac{\left[\operatorname{ch}\frac{\nu_k\xi}{R}-\operatorname{ch}\frac{\nu_k(H+\xi)}{R}\right]}{\nu_k\left(\nu_k^2-1\right)\operatorname{sh}\frac{\nu_k H}{R}}, \qquad (3.14.35)$$

$$(\theta_j)_{\xi=0}=4R^2\sum_{k=1}^{\infty}\frac{\left(1-\operatorname{ch}\frac{\nu_k H}{R}\right)J_1\left(\frac{\nu_k r}{R}\right)}{\nu_k\left(\nu_k^2-1\right)\operatorname{sh}\frac{\nu_k H}{R}\,J_1(\nu_k)}.$$

By substituting (3.14.35) in (3.14.34), we obtain expansion

$$\int_{l_j}\theta_j\frac{\partial\theta_j}{\partial n}\,r\,ds=-H\int_0^R r^3dr+8R^2\sum_{k=1}^{\infty}\frac{\operatorname{ch}\frac{\nu_k H}{R}-1}{\nu_k\left(\nu_k^2-1\right)\operatorname{sh}\frac{\nu_k H}{R}\,J_1(\nu_k)}\times$$

$$\times\int_0^R J_1\left(\frac{\nu_k r}{R}\right)r^2dr+R^2\int_{-H}^0\xi^2d\xi+4R^3\sum_{k=1}^{\infty}\frac{1}{\nu_k\left(\nu_k^2-1\right)\operatorname{sh}\frac{\nu_k H}{R}}\times$$

$$\times\int_{-H}^0\left[\operatorname{ch}\frac{\nu_k\xi}{R}-\operatorname{ch}\frac{\nu_k(H+\xi)}{R}\right]\xi\,d\xi. \qquad (3.14.36)$$

By calculating definite integrals, figuring in formula (3.14.36), we find

$$\int_0^R r^3dr=\frac{R^4}{4},\quad \int_0^R J_1\left(\frac{\nu r}{R}\right)r^2dr=\frac{R^3}{\nu_k^3}J_1(\nu_k),$$

$$\int_{-H}^0\xi^2d\xi=\frac{H^3}{3},\quad \int_{-H}^0\left[\operatorname{ch}\frac{\nu_k\xi}{R}-\operatorname{ch}\frac{\nu_k(H+\xi)}{R}\right]\xi\,d\xi=$$

$$=\frac{2R^2}{\nu_k^2}\left(\operatorname{ch}\frac{\nu_k H}{R}-1\right)-\frac{HR}{\nu_k}\operatorname{sh}\frac{\nu_k H}{R} \qquad (3.14.37)$$

[see (3.13.21)]. By substituting (3.14.37) in (3.14.36) and applying identity

$$\frac{\operatorname{ch} u-1}{\operatorname{sh} u}=\operatorname{th}\frac{u}{2}. \qquad (3.14.38)$$

we obtain expansion

$$\int_{l_j}\theta_j\frac{\partial\theta_j}{\partial n}r\,ds=\frac{H^3R^2}{3}-\frac{HR^4}{4}\left[1+16\sum_{k=1}^{\infty}\frac{1}{\nu_k^2(\nu_k^2-1)}\right]+$$
$$+16R^5\sum_{k=1}^{\infty}\frac{\operatorname{th}\frac{\nu_k H}{2R}}{\nu_k^3(\nu_k^2-1)}. \qquad (3.14.39)$$

By using tables of Bessel functions, we find

$$\sum_{k=1}^{\infty}\frac{1}{\nu_k^2(\nu_k^2-1)}=\frac{1}{8} \qquad (3.14.40)$$

[let us recall that ν_1, ν_2, ... – sequence of solutions of equations (3.13.17)]. According to (3.14.40) formula (3.14.39) can be given the form

$$\int_{l_j}\theta_j\frac{\partial\theta_j}{\partial n}r\,ds=\frac{H^3R^2}{3}-\frac{3HR^4}{4}+16R^5\sum_{k=1}^{\infty}\frac{\operatorname{th}\frac{\nu_k H}{2R}}{\nu_k^3(\nu_k^2-1)}. \qquad (3.14.41)$$

In the considered case

$$v_j=\pi HR^2,\quad x_j-x_C^{(j)}=\frac{H}{2}. \qquad (3.14.42)$$

By substituting (3.14.41) and (3.14.42) in (3.14.30), for radius of inertia h_j we obtain expression

$$h_j=\left[\frac{H^2}{12}-\frac{3R^2}{4}+\frac{16R^3}{H}\sum_{k=1}^{\infty}\frac{\operatorname{th}\frac{\nu_k H}{2R}}{\nu_k^3(\nu_k^2-1)}\right]^{\frac{1}{2}}. \qquad (3.14.43)$$

According to (3.14.40) in the limit, when ratio H/R approaches zero, equality (3.14.43) is degenerated into limiting equality

$$h_j=\frac{R}{2} \qquad (3.14.44)$$

$\left(\lim_{u\to 0}\frac{\operatorname{th}u}{u}=1\right)$. When $\frac{R}{H}\to 0$ from (3.14.43) ensues limiting relationship

$$h_j=\frac{H}{2\sqrt{3}}. \qquad (3.14.45)$$

In the considered example the radius of inertial of "frozen" liquid mass is determined by formula

$$h_j=\sqrt{\frac{H^2}{12}+\frac{R^2}{4}}. \qquad (3.14.46)$$

According to (3.14.44), (3.14.45) and (3.14.46) during calculation of radius of inertia of liquid mass, located in a cylindrical fuel tank, the effect, which the account of mobility of fluid gives, disappears when $H/R\to 0$ and when $H/R\to\infty$. On Fig. 3.7 a solid line shows the relationship between ratios h_j/R and H/R, found by formula (3.14.43), i.e., with allowance for mobility of fluid. For comparison in the same figure the broken line shows the relationship between ratios h_j/R and H/R, found without allowing for mobility of fluid by formula (3.14.46).

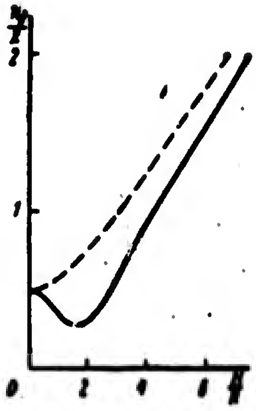

Fig. 3.7.

§ 15. Oscillations of Free Surface of Fluid in Axisymmetrical Fuel Tank

The problem about oscillations of free surface of fluid, located in the fuel tank with number j, was reduced by us in § 6 to finding function ψ_j, satisfying Laplace equation (3.5.3) and boundary conditions (3.6.20). In the case of axial symmetry of the tank the boundary problem, formed by differential equation (3.5.3) and boundary conditions (3.6.20), can be substantially simplified.

By carrying out the transition from rectangular coordinates x, y, z to cylindrical coordinates ξ, r, α, in § 6 we obtained formulas (3.12.16) for Zhukovskiy potentials ϕ_{jx}, ϕ_{jy}, ϕ_{jz}. In accordance with transition formulas (3.12.6) relationships (3.12.16) can be given the form

$$\left.\begin{aligned}\varphi_{jx}&=r(y_j\sin\alpha-z_j\cos\alpha),\\ \varphi_{jy}&=z_j\xi-(x_jr+\theta_j)\sin\alpha,\\ \varphi_{jz}&=-y_j\xi+(x_jr+\theta_j)\cos\alpha.\end{aligned}\right\}\qquad(3.15.1)$$

On surface σ_j coordinate ξ becomes zero (see Fig. 3.3). Thus, according to (3.15.1) there should exist equalities:

$$\left.\begin{aligned}\varphi_{jx}&=r(y_j\sin\alpha-z_j\cos\alpha)\\ \varphi_{jy}&=-[x_jr+(\theta_j)_{\xi-0}]\sin\alpha\\ \varphi_{jz}&=[x_jr+(\theta_j)_{\xi-0}]\cos\alpha\end{aligned}\right\}\text{ on } \sigma_j. \qquad (3.15.2)$$

Coordinates y_j, z_j of the center of flat free surface can be presented in the form

$$y_j=r_j\cos\alpha_j,\ z_j=r_j\sin\alpha_j, \qquad (3.15.3)$$

where r_j – distance from axis of j-th tank to axis x; α_j – angle between plane $z = 0$ and the plane passing through the axis of this tank and axis x (see Fig. 3.3).

In accordance with equalities (3.15.3) the first of formulas (3.15.2) can be converted so:

$$\varphi_{jx}=r_jr\cdot\sin(\alpha-\alpha_j) \text{ on } \sigma_j. \qquad (3.15.4)$$

Function θ_j represents the solution of differential equation (3.12.13), regular on axis ξ. According to (3.12.13) when $r \to 0$ ratio θ_j/r approaches a finite limit, determined by equality

$$\lim_{r\to 0}\frac{\theta_j}{r}=\left(\frac{\partial\theta_j}{\partial r}\right)_{r=0}. \qquad (3.15.5)$$

Hence, specifically, finiteness ensues at point $r = 0$ of relationship

$$k_j=-\frac{(\theta_j)_{\xi=0}}{r}. \qquad (3.15.6)$$

By introducing into examination function k_j, determined by relationship (3.15.6), it is possible to convert the second and third of formulas (3.15.2):

$$\left.\begin{aligned}\varphi_{jy}&=-(x_j-k_j)r\sin\alpha\\ \varphi_{jz}&=(x_j-k_j)r\cos\alpha\end{aligned}\right\}\text{ on } \sigma_j. \qquad (3.15.7)$$

According to (3.12.6), (3.15.4) and (3.15.7) boundary conditions (3.6.20) can be given the form

$$\frac{\partial\psi_j}{\partial n}=0 \quad \text{on } S_j,$$

$$\frac{\partial^2\psi_j}{\partial t^2}+(w_{0x}-g_x)\frac{\partial\psi_j}{\partial n}+(w_{0y}-g_y)(y_j+r\cos\alpha)+$$
$$+(w_{0z}-g_z)(z_j+r\sin\alpha)+\frac{d\omega_x}{dt}r_j r\sin(\alpha-\alpha_j)- \qquad (3.15.8)$$
$$-\frac{d\omega_y}{dt}(x_j-k_j)\,r\sin\alpha+\frac{d\omega_z}{dt}(x_j-k_j)\,r\cos\alpha=C_j(t)\,\text{on } \sigma_j.$$

Having represented Laplace equation (3.5.3) in cylindrical coordinates ξ, r, α, for function ψ_j we obtain boundary value problem, formed by differential equation

$$\frac{\partial^2\psi_j}{\partial\xi^2}+\frac{\partial^2\psi_j}{\partial r^2}+\frac{1}{r}\frac{\partial\psi_j}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi_j}{\partial\alpha^2}=0$$

and boundary conditions (3.15.8).

In § 6 we showed that the solution of the problem about oscillations of free surface of fluid, located in j-th tank, depends on the form of function $C_j(t)$, entering the boundary conditions for motion potential ψ_j. For simplification of boundary conditions (3.15.8) let us assume

$$C_j(t)=y_j(w_{0y}-g_y)+z_j(w_{0z}-g_z). \qquad (3.15.10)$$

In this instance boundary conditions (3.15.8) will take the form

$$\frac{\partial\psi_j}{\partial n}=0 \quad \text{on } S_j,$$

$$\frac{\partial^2\psi_j}{\partial t^2}+(w_{0x}-g_x)\frac{\partial\psi_j}{\partial n}+r_j\frac{d\omega_x}{dt}r\sin(\alpha-\alpha_j)+$$
$$+\left[w_{0y}-g_y+(x_j-k_j)\frac{d\omega_z}{dt}\right]r\cos\alpha+ \qquad (3.15.11)$$
$$+\left[w_{0z}-g_z-(x_j-k_j)\frac{d\omega_y}{dt}\right]r\sin\alpha=0 \quad \text{on } \sigma_j.$$

Generatrix l_j of surface of revolution $S_j + \sigma_j$ is made up of the generatrix of wetted surface S_j (we will subsequently designate this generatrix by l_j^*) and of rectilinear segment $\xi = 0$, $0 < r < R_j$, where R_j – radius of flat free surface σ_j (see Fig. 3.3). Let us introduce into examination the solution ψ_{jx} of two-dimensional differential equation

$$\frac{\partial^2\psi_{jx}}{\partial\xi^2}+\frac{\partial^2\psi_{jx}}{\partial r^2}+\frac{1}{r}\frac{\partial\psi_{jx}}{\partial r}-\frac{\psi_{jx}}{r^2}=0, \tag{3.15.12}$$

regular on axis $r = 0$ and satisfying boundary conditions

$$\begin{gathered}\frac{\partial\psi_{jx}}{\partial n}=0 \text{ on } l_j^*,\\ \frac{\partial^2\psi_{jx}}{\partial t^2}+(w_{0x}-g_x)\frac{\partial\psi_{jx}}{\partial\xi}+rr_j\frac{d\omega_x}{dt}=0\\ \text{when } \xi=0,\ 0<r<R_j,\end{gathered} \tag{3.15.13}$$

function ψ_{jy}, regular on axis ξ and satisfying differential equation

$$\frac{\partial^2\psi_{jy}}{\partial\xi^2}+\frac{\partial^2\psi_{jy}}{\partial r^2}+\frac{1}{r}\frac{\partial\psi_{jy}}{\partial r}-\frac{\psi_{jy}}{r^2}=0 \tag{3.15.14}$$

and boundary conditions

$$\begin{gathered}\frac{\partial\psi_{jy}}{\partial n}=0 \text{ on } l_j^*,\\ \frac{\partial^2\psi_{jy}}{\partial t^2}+(w_{0x}-g_x)\frac{\partial\psi_{jy}}{\partial\xi}+r\left[w_{0y}-g_y+(x_j-k_j)\frac{d\omega_z}{dt}\right]=0\\ \text{when } \xi=0,\ 0<r<R_j,\end{gathered} \tag{3.15.15}$$

and solution ψ_{jz} of differential equation

$$\frac{\partial^2\psi_{jz}}{\partial\xi^2}+\frac{\partial^2\psi_{jz}}{\partial r^2}+\frac{1}{r}\frac{\partial\psi_{jz}}{\partial r}-\frac{\psi_{jz}}{r^2}=0, \tag{3.15.16}$$

regular on axis ξ and satisfying boundary conditions

$$\frac{\partial\psi_{jz}}{\partial n}=0 \text{ on } l_j,$$

$$\frac{\partial^2\psi_{jz}}{\partial t^2}+(w_{0x}-g_x)\frac{\partial\psi_{jz}}{\partial\xi}+r\left[w_{0z}-g_z-(x_j-k_j)\frac{d\omega_y}{dt}\right]=0 \tag{3.15.17}$$

$$\text{when } \xi=0,\ 0<r<R_j.$$

According to (3.15.12), (3.15.13), (3.15.14), (3.15.15), (3.15.16) and (3.15.17) function

$$\psi_j=\psi_{jx}\sin(\alpha-\alpha_j)+\psi_{jy}\cos\alpha+\psi_{jz}\sin\alpha \tag{3.15.18}$$

will represent the solution of Laplace equation (3.15.9), regular inside closed surface $S_j + \sigma_j$ and satisfying boundary conditions (3.15.11) (on surface σ_j the direction of normal n coincides with the direction of positive semiaxis ξ).

Thus, in case of axial symmetry of j-th tank the three-dimensional problem, formed by differential equation (3.15.9) and boundary conditions (3.15.11), can be reduced to two-dimensional boundary value problems for functions ψ_{jx}, ψ_{jy}, ψ_{jz}.

In accordance with formulas (3.6.21), (3.12.6) and (3.15.18) function $f_j(r, \alpha, t)$, which determines oscillations of free surface of fluid in j-th tank, will have the form

$$f_j=\left(\frac{\partial\psi_{jx}}{\partial\xi}\right)_{\xi=0}\sin(\alpha-\alpha_j)+\left(\frac{\partial\psi_{jy}}{\partial\xi}\right)_{\xi=0}\cos\alpha+\left(\frac{\partial\psi_{jz}}{\partial\xi}\right)_{\xi=0}\sin\alpha. \tag{3.15.19}$$

Apparent axial acceleration $w_{0x} - g_x$ figures in boundary conditions (3.15.13), (3.15.15) and (3.15.17) as the coefficient with partial derivative with respect to ξ of the sought function. Of the lateral apparent accelerations $w_{0y} - g_y$, $w_{0z} - g_z$ and angular accelerations $d\omega_x/dt$, $d\omega_y/dt$, $d\omega_z/dt$ the boundary conditions (3.15.13) contain only acceleration $d\omega_x/dt$ alone. Boundary conditions

(3.15.15) include accelerations $w_{0y} - g_y$ and $d\omega_z/dt$, and boundary conditions (3.15.17) – accelerations $w_{0z} - g_z$ and $d\omega_y/dt$. Thus, function ψ_{jz} determines the oscillations of free surface, caused by oscillations of the rocket body in the rolling plane, function ψ_{jx} determines oscillations of free surface, generated by oscillations of the body in pitching plane, function ψ_{jz} determines oscillations of free surface, caused by oscillations of the rocket in yawing plane.

In accordance with formula (3.15.19) the oscillations of free surface of fluid in j-th tank, caused by revolution of the rocket around its longitudinal axis, are characterized by nodal line, lying in plane $\alpha = \alpha_j$; along this line function f_j always remains equal to zero. As can be seen from Fig. 3.3, plane $\alpha = \alpha_j$ always passes through axis x. Oscillations of free surface of fluid, being generated by oscillations of the rocket body in the pitching plane, according to (3.15.19) are characterized by nodal lines, lying in plane $\alpha = \pi/2$, i.e., in plane $y = y_j$ (see Fig. 3.3). Oscillations of free surface, caused by oscillations of the rocket in yawing plane, are characterized by nodal line, lying in plane $z = z_j$.

According to (3.15.6) quantity k_j, figuring in boundary conditions (3.15.15) and (3.15.17), can be found, having solved the boundary value problem formed by differential equation (3.12.13) and boundary condition (3.12.14). While the tank with number j is not emptied, quantity k_j does not depend on time t and is a function of only variable r alone. In the process of emptying of j-th tank there appears the relationship of quantity k_j to time t, since the form of function θ_j depends on the filling level of the tank.

As an example let us determine quantity k_j for fluid, located in a cylindrical fuel tank. In the considered case according to (3.13.3) and (3.13.24) there will exist equality

$$(k_j)_{\xi=0} = -4R^2 \sum_{k=1}^{\infty} \frac{\left(\mathrm{ch}\,\frac{\nu_k H}{R} - 1\right) J_1\left(\frac{\nu_k r}{R}\right)}{\nu_k\left(\nu_k^2 - 1\right)\mathrm{sh}\,\frac{\nu_k H}{R} J_1(\nu_k)}. \quad (3.15.20)$$

By substituting (3.15.20) in (3.15.6) and by using identity (3.14.38), for quantity k_j we obtain expansion

$$k_j = 4R \sum_{k=1}^{\infty} \frac{\operatorname{th}\frac{\nu_k H}{2R} J_1\left(\frac{\nu_k r}{R}\right) \frac{R}{\nu_k r}}{(\nu_k^2 - 1) J_1(\nu_k)}. \qquad (3.15.21)$$

In the limit, when ratio H/R approaches infinity, equality (3.15.21) is degenerated into limiting equality

$$k_j = 4R \sum_{k=1}^{\infty} \frac{J_1\left(\frac{\nu_k r}{R}\right) \frac{R}{\nu_k r}}{(\nu_k^2 - 1) J_1(\nu_k)}. \qquad (3.15.22)$$

When $H/R \to 0$ function k_j degenerates into identical zero. Figure 3.8 for differnt values of ratio H/R shows the relationship between ratios k_j/R and r/R, found by formula (3.15.21).

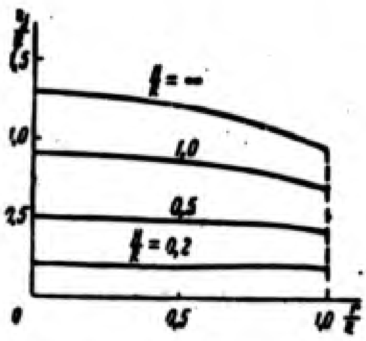

Fig. 3.8.

For a spherical tank, half filled with fluid, according to (3.13.32) and (3.13.54) there will exist equality

$$(\theta_j)_{k=0} = R^2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(4k-1)}{(k+1)(2k-3)(2k-1)^2} \frac{1.3 \ldots (2k-1)}{2 \cdot 4 \ldots 2k} \left(\frac{r}{R}\right)^{2k-1} \times$$
$$\times P_{2k-1}^{(1)}(0) \qquad (3.15.23)$$

[in accordance with transition formulas (3.13.26) $\beta = \frac{\pi}{2}$ and $\varrho = r$ when $\xi = 0$]. By substituting (3.15.23) in (3.15.6) and by using relationship

$$P^{(1)}_{2k-1}(0) = (-1)^{k-1}\, 2k \frac{1.3 \ldots (2k-1)}{2.4 \ldots 2k} \tag{3.15.24}$$

(see [27]), for quantity k_j we obtain expansion

$$k_j = R\left\{\frac{3}{4} - \sum_{k=2}^{\infty} \frac{4k-1}{2k(k+1)(2k-3)} \times \right.$$
$$\left. \times \left[\frac{1\cdot 3 \ldots (2k-3)}{2\cdot 4 \ldots (2k-2)}\right]^2 \left(\frac{r}{R}\right)^{2k-2}\right\}.$$

or

$$k_j = -R \sum_{k=1}^{\infty} \frac{2k(4k-1)}{(k+1)(2k-3)(2k-1)^2} \times$$
$$\times \left[\frac{1.3 \ldots (2k-1)}{2.4 \ldots 2k}\right]^2 \left(\frac{r}{R}\right)^{2k-2}. \tag{3.15.25}$$

Figure 3.9 shows the relationship between ratios k_j/R and r/R, found by formula (3.15.25).

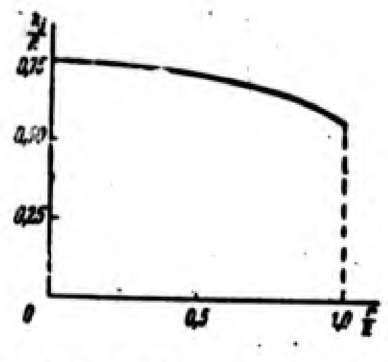

Fig. 3.9.

§ 16. Equation of Longitudinal Motion, Equations of Motion in Pitching, Yawing and Rolling Planes

Let us now return to examination of equations of motion (3.11.1). In the first two equations (3.11.1) by projecting vectors to coordinate axes x, y, z and using in this case transition formulas (3.12.6) and relationships (3.14.23) and (3.14.24), we obtain three equations of forces:

$$
\begin{aligned}
&m(w_{0x}-g_x)+\sum_{j=1}^{N}\varrho_j\int_{\sigma_j}x_j\left(\frac{\partial^3\psi_j}{\partial\xi\,\partial t^2}\right)_{\xi=0}d\sigma=F_x,\\
&m(w_{0y}-g_y)+\sum_{j=1}^{N}\varrho_j\int_{\sigma_j}(y_j+r\cos\alpha)\left(\frac{\partial^3\psi_j}{\partial\xi\,\partial t^2}\right)_{\xi=0}d\sigma=F_y,\\
&m(w_{0z}-g_z)+\sum_{j=1}^{N}\varrho_j\int_{\sigma_j}(z_j+r\sin\alpha)\left(\frac{\partial^3\psi_j}{\partial\xi\,\partial t^2}\right)_{\xi=0}d\sigma=F_z
\end{aligned}
\tag{3.16.1}
$$

and three equations of moments:

$$
\begin{aligned}
&J_x\frac{d\omega_x}{dt}+\sum_{j=1}^{N}\varrho_j\int_{\sigma_j}\varphi_{jx}\left(\frac{\partial^3\psi_j}{\partial\xi\,\partial t^2}\right)_{\xi=0}d\sigma+(w_{0z}-g_z)\times\\
&\times\sum_{j=1}^{N}\varrho_j\int_{\sigma_j}(y_j+r\cos\alpha)\left(\frac{\partial\psi_j}{\partial\xi}\right)_{\xi=0}d\sigma-(w_{0y}-g_y)\times\\
&\times\sum_{j=1}^{N}\varrho_j\int_{\sigma_j}(z_j+r\sin\alpha)\left(\frac{\partial\psi_j}{\partial\xi}\right)_{\xi=0}d\sigma=M_x,\\
&J_y\frac{d\omega_y}{dt}+\sum_{j=1}^{N}\varrho_j\int_{\sigma_j}\varphi_{jy}\left(\frac{\partial^3\psi_j}{\partial\xi\,\partial t^2}\right)_{\xi=0}d\sigma+(w_{0x}-g_x)\times\\
&\times\sum_{j=1}^{N}\varrho_j\int_{\sigma_j}(z_j+r\sin\alpha)\left(\frac{\partial\psi_j}{\partial\xi}\right)_{\xi=0}d\sigma-(w_{0z}-g_z)\times\\
&\times\sum_{j=1}^{N}\varrho_j\int_{\sigma_j}x_j\left(\frac{\partial\psi_j}{\partial\xi}\right)_{\xi=0}d\sigma=M_y,
\end{aligned}
\tag{3.16.2}
$$

$$J_z \frac{d\omega_z}{dt}+\sum_{j=1}^{N} c_j \int_{\sigma_j} \varphi_{jz} \left(\frac{\partial^3 \psi_j}{\partial \xi \, \partial t^2}\right)_{\xi=0} d\sigma + (w_{0y}-g_y) \times$$

$$\times \sum_{j=1}^{N} c_j \int_{\sigma_j} x_j \left(\frac{\partial \psi_j}{\partial \xi}\right)_{\xi=0} d\sigma - (w_{0x}-g_x) \times \qquad \text{(3.16.2 cont'd)}$$

$$\times \sum_{j=1}^{N} c_j \int_{\sigma_j} (y_j + r\cos\alpha) \left(\frac{\partial \psi_j}{\partial \xi}\right)_{\xi=0} d\sigma = M_z.$$

In § 15 we obtained equalities (3.15.4) and (3.15.7) for Zhukovskiy potentials ϕ_{jx}, ϕ_{jy}, ϕ_{jz} and formula (3.15.18) for motion potential ψ_j. By using these formulas, let us transform the surface integrals, figuring in equations (3.16.1) and (3.16.2).

The integral from some function f with respect to flat free surface of fluid σ_j can be represented in the form

$$\int_{\sigma_j} f d\sigma = \int_0^{R_j} \int_0^{2\pi} f r \, dr \, d\alpha, \qquad (3.16.3)$$

where R_j – radius of free surface σ_j.

Having converted formula (3.15.18):

$$\psi_j = (\psi_{jy} - \psi_{jx}\sin\alpha_j)\cos\alpha + (\psi_{jz} + \psi_{jx}\cos\alpha_j)\sin\alpha, \qquad (3.16.4)$$

we find

$$\int_{\sigma_j} x_j \left(\frac{\partial^3 \psi_j}{\partial \xi \, \partial t^2}\right)_{\xi=0} d\sigma = x_j \int_{\sigma_j} \left\{ \left[\frac{\partial^3 (\psi_{jy} - \psi_{jx}\sin\alpha_j)}{\partial \xi \, \partial t^2}\right]_{\xi=0} \cos\alpha + \right.$$

$$\left. + \left[\frac{\partial^3 (\psi_{jz} + \psi_{jx}\cos\alpha_j)}{\partial \xi \, \partial t^2}\right]_{\xi=0} \sin\alpha \right\} d\sigma = x_j \left\{ \int_0^{2\pi} \cos\alpha \, d\alpha \times \right.$$

$$\times \int_0^{R_j} \left[\frac{\partial^3 (\psi_{jy} - \psi_{jx}\sin\alpha_j)}{\partial \xi \, \partial t^2}\right]_{\xi=0} r dr + \int_0^{2\pi} \sin\alpha \, d\alpha \times$$

$$\left. \times \int_0^{R_j} \left[\frac{\partial^3 (\psi_{jz} + \psi_{jx}\cos\alpha_j)}{\partial \xi \, \partial t^2}\right]_{\xi=0} r dr \right\} = 0, \qquad (3.16.5)$$

$$\iint_{\sigma_j} (y_j + r\cos\alpha)\left(\frac{\partial^3\psi_j}{\partial\xi\,\partial t^2}\right)_{\xi=0} d\sigma = \iint_{\sigma_j} (y_j + r\cos\alpha) \times$$

$$\times \Big\{\left[\frac{\partial^3(\psi_{jy} - \psi_{jx}\sin\alpha_j)}{\partial\xi\,\partial t^2}\right]_{\xi=0} \cos\alpha + \left[\frac{\partial^3(\psi_{jz} + \psi_{jx}\cos\alpha_j)}{\partial\xi\,\partial t^2}\right]_{\xi=0} \sin\alpha\Big\}\, d\sigma =$$

$$= y_j \Big\{\int_0^{2\pi} \cos\alpha\, d\alpha \int_0^{R_j} \left[\frac{\partial^3(\psi_{jy} - \psi_{jx}\sin\alpha_j)}{\partial\xi\,\partial t^2}\right]_{\xi=0} r\,dr + \int_0^{2\pi} \sin\alpha\, d\alpha \times$$

$$\times \int_0^{R_j} \left[\frac{\partial^3(\psi_{jz} + \psi_{jx}\cos\alpha_j)}{\partial\xi\,\partial t^2}\right]_{\xi=0} r\,dr\Big\} + \int_0^{2\pi} \sin\alpha\cos\alpha\, d\alpha \times$$

$$\times \int_0^{R_j} \left[\frac{\partial^3(\psi_{jz} + \psi_{jx}\cos\alpha_j)}{\partial\xi\,\partial t^2}\right]_{\xi=0} r^2 dr + \int_0^{2\pi} \cos^2\alpha\, d\alpha \times$$

$$+ \int_0^{R_j} \left[\frac{\partial^3(\psi_{jy} - \psi_{jx}\sin\alpha_j)}{\partial\xi\,\partial t^2}\right]_{\xi=0} r^2 dr = \pi \int_0^{R_j} \left[\frac{\partial^3(\psi_{jy} - \psi_{jx}\sin\alpha_j)}{\partial\xi\,\partial t^2}\right]_{\xi=0} r^2 dr =$$

$$= \pi \int_0^{R_j} \left(\frac{\partial^3\psi_{jy}}{\partial\xi\,\partial t^2}\right)_{\xi=0} r^2 dr - \pi\sin\alpha_j \int_0^{R_j} \left(\frac{\partial^3\psi_{jx}}{\partial\xi\,\partial t^2}\right)_{\xi=0} r^2 dr, \qquad (3.16.6)$$

$$\int_{\sigma_j} (z_j + r\sin\alpha)\left(\frac{\partial^3\psi_j}{\partial\xi\,\partial t^2}\right)_{\xi=0} d\sigma = \int_{\sigma_j} (z_j + r\sin\alpha)\Big\{\left[\frac{\partial^3(\psi_{jy} - \psi_{jx}\sin\alpha_j)}{\partial\xi\,\partial t^2}\right]_{\xi=0} \times$$

$$\times \cos\alpha + \left[\frac{\partial^3(\psi_{jz} + \psi_{jx}\cos\alpha_j)}{\partial\xi\,\partial t^2}\right]_{\xi=0} \sin\alpha\Big\}\, d\sigma = z_j \Big\{\int_0^{2\pi} \cos\alpha\, d\alpha \times$$

$$\times \int_0^{R_j} \left[\frac{\partial^3(\psi_{jy} - \psi_{jx}\sin\alpha_j)}{\partial\xi\,\partial t^2}\right]_{\xi=0} r\,dr + \int_0^{2\pi} \sin\alpha\, d\alpha \times$$

$$\times \int_0^{R_j} \left[\frac{\partial^3(\psi_{jz} + \psi_{jx}\cos\alpha_j)}{\partial\xi\,\partial t^2}\right]_{\xi=0} r\,dr\Big\} + \int_0^{2\pi} \sin\alpha\cos\alpha\, d\alpha \times$$

$$\times \int_0^{R_j} \left[\frac{\partial^3(\psi_{jy} - \psi_{jx}\sin\alpha_j)}{\partial\xi\,\partial t^2}\right]_{\xi=0} r^2 dr + \int_0^{2\pi} \sin^2\alpha\, d\alpha \times$$

$$\times \int_0^{R_j} \left[\frac{\partial^3(\psi_{jz} + \psi_{jx}\cos\alpha_j)}{\partial\xi\,\partial t^2}\right]_{\xi=0} r^2 dr = \pi \int_0^{R_j} \left[\frac{\partial^3(\psi_{jz} + \psi_{jx}\cos\alpha_j)}{\partial\xi\,\partial t^2}\right]_{\xi=0} r^2 dr =$$

$$= \pi \int_0^{R_j} \left(\frac{\partial^3\psi_{jz}}{\partial\xi\,\partial t^2}\right)_{\xi=0} r^2 dr + \pi\cos\alpha_j \int_0^{R_j} \left(\frac{\partial^3\psi_{jx}}{\partial\xi\,\partial t^2}\right)_{\xi=0} r^2 dr. \qquad (3.16.7)$$

According to (3.16.5) the first of equations (3.16.1) can be given the form

$$m(w_{0x}-g_x)=F_x. \tag{3.16.8}$$

Equation of longitudinal motion (3.16.8) is no different from the equation placed by us in the basis of investigation of longitudinal motion of the rocket in Chapter 1. Thus, oscillations of fluids in the fuel tanks do not affect longitudinal motion of the rocket.

According to (3.16.6) and (3.16.7) the second and third of equations (3.16.1) can be reduced to the form

$$\begin{gathered}
m(w_{0y}-g_y)+\pi\sum_{j=1}^{N}\rho_j\int_0^{R_j}\left(\frac{\partial^3\psi_{jy}}{\partial\xi\,\partial r^2}\right)_{\xi=0}r^2dr-\\
-\pi\sum_{j=1}^{N}\rho_j\sin\alpha_j\int_0^{R_j}\left(\frac{\partial^3\psi_{jz}}{\partial\xi\,\partial r^2}\right)_{\xi=0}r^2dr=F_y,\\
m(w_{0z}-g_z)+\pi\sum_{j=1}^{N}\rho_j\int_0^{R_j}\left(\frac{\partial^3\psi_{jz}}{\partial\xi\,\partial r^2}\right)_{\xi=0}r^2dr+\\
+\pi\sum_{j=1}^{N}\rho_j\cos\alpha_j\int_0^{R_j}\left(\frac{\partial^3\psi_{jz}}{\partial\xi\,\partial r^2}\right)_{\xi=0}r^2dr=F_z.
\end{gathered} \tag{3.16.9}$$

Let us turn to conversion of equations of moments (3.16.2). By using formulas (3.15.4), (3.15.7), (3.15.3) and (3.16.4), we find

$$\int_{\sigma_j} \varphi_{jx} \left(\frac{\partial^3 \psi_j}{\partial \xi\, \partial t^2}\right)_{\xi=0} d\sigma = r_j \int_{\sigma_j} (\sin\alpha \cos\alpha_j - \cos\alpha \sin\alpha_j) \times$$

$$\times \left\{ \left[\frac{\partial^3(\psi_{jy} - \psi_{jx}\sin\alpha_j)}{\partial \xi\, \partial t^2}\right]_{\xi=0} \cos\alpha + \left[\frac{\partial^3(\psi_{jz} + \psi_{jx}\cos\alpha_j)}{\partial \xi\, \partial t^2}\right]_{\xi=0} \sin\alpha \right\} r\, d\sigma =$$

$$= r_j \left\{ \int_0^{2\pi} \sin\alpha \cos\alpha\, d\alpha \int_0^{R_j} \left\{ \cos\alpha_j \left[\frac{\partial^3(\psi_{jy} - \psi_{jx}\sin\alpha_j)}{\partial \xi\, \partial t^2}\right]_{\xi=0} - \sin\alpha_j \times \right.\right.$$

$$\left. \times \left[\frac{\partial^3(\psi_{jz} + \psi_{jx}\cos\alpha_j)}{\partial \xi\, \partial t^2}\right]_{\xi=0} \right\} r^2 dr + \cos\alpha_j \int_0^{2\pi} \sin^2\alpha\, d\alpha \times$$

$$\times \int_0^{R_j} \left[\frac{\partial^3(\psi_{jz} + \psi_{jx}\cos\alpha_j)}{\partial \xi\, \partial t^2}\right]_{\xi=0} r^2 dr - \sin\alpha_j \int_0^{2\pi} \cos^2\alpha\, d\alpha \times$$

$$\left. \times \int_0^{R_j} \left[\frac{\partial^3(\psi_{jy} - \psi_{jx}\sin\alpha_j)}{\partial \xi\, \partial t^2}\right]_{\xi=0} r^2 dr \right\} = \pi r_j \left\{ \cos\alpha_j \times \right.$$

$$\left. \times \int_0^{R_j} \left[\frac{\partial^3(\psi_{jz} + \psi_{jx}\cos\alpha_j)}{\partial \xi\, \partial t^2}\right]_{\xi=0} r^2 dr - \sin\alpha_j \int_0^{R_j} \left[\frac{\partial^3(\psi_{jy} - \psi_{jx}\sin\alpha_j)}{\partial \xi\, \partial t^2}\right]_{\xi=0} r^2 dr \right\} =$$

$$= \pi r_j \int_0^{R_j} \left(\frac{\partial^3 \psi_{jx}}{\partial \xi\, \partial t^2}\right)_{\xi=0} r^2 dr - \pi r_j \sin\alpha_j \int_0^{R_j} \left(\frac{\partial^3 \psi_{jy}}{\partial \xi\, \partial t^2}\right)_{\xi=0} r^2 dr +$$

$$+ \pi r_j \cos\alpha_j \int_0^{R_j} \left(\frac{\partial^3 \psi_{jz}}{\partial \xi\, \partial t^2}\right)_{\xi=0} r^2 dr, \qquad (3.16.10)$$

$$\int_{\sigma_j} \varphi_{jy} \left(\frac{\partial^3 \psi_j}{\partial \xi\, \partial t^2}\right)_{\xi=0} d\sigma = - \int_{\sigma_j} (x_j - k_j) \sin\alpha \left\{ \left[\frac{\partial^3(\psi_{jy} - \psi_{jx}\sin\alpha_j)}{\partial \xi\, \partial t^2}\right]_{\xi=0} \cos\alpha + \right.$$

$$\left. + \left[\frac{\partial^3(\psi_{jz} + \psi_{jx}\cos\alpha_j)}{\partial \xi\, \partial t^2}\right]_{\xi=0} \sin\alpha \right\} r\, d\sigma = - \int_0^{2\pi} \sin\alpha \cos\alpha\, d\alpha \times$$

$$\times \int_0^{R_j} \left[\frac{\partial^3(\psi_{jy} - \psi_{jx}\sin\alpha_j)}{\partial \xi\, \partial t^2}\right]_{\xi=0} (x_j - k_j) r^2 dr - \int_0^{2\pi} \sin^2\alpha\, d\alpha \times$$

$$\times \int_0^{R_j} \left[\frac{\partial^3(\psi_{jz} + \psi_{jx}\cos\alpha_j)}{\partial \xi\, \partial t^2}\right]_{\xi=0} (x_j - k_j) r^2 dr =$$

$$= -\pi \int_0^{R_j} \left[\frac{\partial^3(\psi_{jz} + \psi_{jx}\cos\alpha_j)}{\partial \xi\, \partial t^2}\right]_{\xi=0} (x_j - k_j) r^2 dr =$$

$$= -\pi \int_0^{R_j} \left(\frac{\partial^3 \psi_{jz}}{\partial \xi\, \partial t^2}\right)_{\xi=0} (x_j - k_j) r^2 dr - \pi \cos\alpha_j \int_0^{R_j} \left(\frac{\partial^3 \psi_{jx}}{\partial \xi\, \partial t^2}\right)_{\xi=0} (x_j - k_j) r^2 dr, \qquad (3.16.11)$$

$$\int_{\sigma_j} \varphi_{jx}\left(\frac{\partial^3\psi_j}{\partial\xi\,\partial t^2}\right)_{\xi=0} d\sigma = \int_{\sigma_j} (x_j-k_j)\cos\alpha\left\{\left[\frac{\partial^3(\psi_{jy}-\psi_{jx}\sin\alpha_j)}{\partial\xi\,\partial t^2}\right]_{\xi=0}\cos\alpha+\right.$$

$$\left.+\left[\frac{\partial^3(\psi_{jz}+\psi_{jx}\cos\alpha_j)}{\partial\xi\,\partial t^2}\right]_{\xi=0}\sin\alpha\right\} r\,d\sigma = \int_0^{2\pi}\sin\alpha\cos\alpha\,d\alpha\,\times$$

$$\times\int_0^{R_j}\left[\frac{\partial^3(\psi_{jz}+\psi_{jx}\cos\alpha_j)}{\partial\xi\,\partial t^2}\right]_{\xi=0}(x_j-k_j)\,r^2dr+\int_0^{2\pi}\cos^2\alpha\,d\alpha\,\times$$

$$\times\int_0^{R_j}\left[\frac{\partial^3(\psi_{jy}-\psi_{jx}\sin\alpha_j)}{\partial\xi\,\partial t^2}\right]_{\xi=0}(x_j-k_j)\,r^2dr=$$

$$=\pi\int_0^{R_j}\left[\frac{\partial^3(\psi_{jy}-\psi_{jx}\sin\alpha_j)}{\partial\xi\,\partial t^2}\right]_{\xi=0}(x_j-k_j)\,r^2dr=$$

$$=\pi\int_0^{R_j}\left(\frac{\partial^3\psi_{jy}}{\partial\xi\,\partial t^2}\right)_{\xi=0}(x_j-k_j)\,r^2dr-\pi\sin\alpha_j\int_0^{R_j}\left(\frac{\partial^3\psi_{jx}}{\partial\xi\,\partial t^2}\right)_{\xi=0}(x_j-k_j)\,r^2dr. \qquad (3.16.12)$$

By substituting derivative $\partial^3\psi_j/\partial\xi\partial t^2$ in the left sides of formulas (3.16.5), (3.16.6) and (3.16.7) by derivative $\partial\psi_j/\partial\xi$, we obtain relationships:

$$\begin{aligned}
&\int_{\sigma_j} x_j\left(\frac{\partial\psi_j}{\partial\xi}\right)_{\xi=0} d\sigma=0,\\
&\int_{\sigma_j}(y_j+r\cos\alpha)\left(\frac{\partial\psi_j}{\partial\xi}\right)_{\xi=0} d\sigma=\pi\int_0^{R_j}\left(\frac{\partial\psi_{jy}}{\partial\xi}\right)_{\xi=0} r^2dr-\\
&\qquad-\pi\sin\alpha_j\int_0^{R_j}\left(\frac{\partial\psi_{jx}}{\partial\xi}\right)_{\xi=0} r^2dr,\\
&\int_{\sigma}(z_j+r\sin\alpha)\left(\frac{\partial\psi_j}{\partial\xi}\right)_{\xi=0} d\sigma=\pi\int_0^{R_j}\left(\frac{\partial\psi_{jz}}{\partial\xi}\right)_{\xi=0} r^2dr+\\
&\qquad+\pi\cos\alpha_j\int_0^{R_j}\left(\frac{\partial\psi_{jx}}{\partial\xi}\right)_{\xi=0} r^2dr.
\end{aligned} \qquad (3.16.13)$$

According to (3.16.10), (3.16.11), (3.16.12) and (3.16.13) the equations of moments (3.16.2) can be given the form

$$
\left.
\begin{aligned}
& J_x \frac{d\omega_x}{dt} + \pi \sum_{j=1}^{N} \varrho_j r_j \int_0^{R_j} \left(\frac{\partial^3 \psi_{jz}}{\partial \xi \, \partial t^2} \right)_{\xi=0} r^2 dr - \pi \sum_{j=1}^{N} \varrho_j r_j \sin \alpha_j \times \\
& \times \int_0^{R_j} \left(\frac{\partial^3 \psi_{jy}}{\partial \xi \, \partial t^2} \right)_{\xi=0} r^2 dr + \pi \sum_{j=1}^{N} \varrho_j r_j \cos \alpha_j \int_0^{R_j} \left(\frac{\partial^3 \psi_{jz}}{\partial \xi \, \partial t^2} \right)_{\xi=0} r^2 dr + \\
& + \pi (w_{0z} - g_z) \left[\sum_{j=1}^{N} \varrho_j \int_0^{R_j} \left(\frac{\partial \psi_{jy}}{\partial \xi} \right)_{\xi=0} r^2 dr - \sum_{j=1}^{N} \varrho_j \sin \alpha_j \times \right. \\
& \left. \times \int_0^{R_j} \left(\frac{\partial \psi_{jz}}{\partial \xi} \right)_{\xi=0} r^2 dr \right] - \pi (w_{0y} - g_y) \left[\sum_{j=1}^{N} \varrho_j \int_0^{R_j} \left(\frac{\partial \psi_{jz}}{\partial \xi} \right)_{\xi=0} r^2 dr + \right. \\
& \left. + \sum_{j=1}^{N} \varrho_j \cos \alpha_j \int_0^{R_j} \left(\frac{\partial \psi_{jz}}{\partial \xi} \right)_{\xi=0} r^2 dr \right] = M_x, \\
& J_y \frac{d\omega_y}{dt} - \pi \sum_{j=1}^{N} \varrho_j \int_0^{R_j} \left(\frac{\partial^3 \psi_{jz}}{\partial \xi \, \partial t^2} \right)_{\xi=0} (x_j - k_j) r^2 dr - \\
& - \pi \sum_{j=1}^{N} \varrho_j \cos \alpha_j \int_0^{R_j} \left(\frac{\partial^3 \psi_{jz}}{\partial \xi \, \partial t^2} \right)_{\xi=0} (x_j - k_j) r^2 dr + \\
& + \pi (w_{0x} - g_x) \left[\sum_{j=1}^{N} \varrho_j \int_0^{R_j} \left(\frac{\partial \psi_{jz}}{\partial \xi} \right)_{\xi=0} r^2 dr + \sum_{j=1}^{N} \varrho_j \cos \alpha_j \times \right. \\
& \left. \times \int_0^{R_j} \left(\frac{\partial \psi_{jz}}{\partial \xi} \right)_{\xi=0} r^2 dr \right] = M_y, \\
& J_z \frac{d\omega_z}{dt} + \pi \sum_{j=1}^{N} \varrho_j \int_0^{R_j} \left(\frac{\partial^3 \psi_{jy}}{\partial \xi \, \partial t^2} \right)_{\xi=0} (x_j - k_j) r^2 dr - \\
& - \pi \sum_{j=1}^{N} \varrho_j \sin \alpha_j \int_0^{R_j} \left(\frac{\partial^3 \psi_{jz}}{\partial \xi \, \partial t^2} \right)_{\xi=0} (x_j - k_j) r^2 dr - \\
& - \pi (w_{0x} - g_x) \left[\sum_{j=1}^{N} \varrho_j \int_0^{R_j} \left(\frac{\partial \psi_{jy}}{\partial \xi} \right)_{\xi=0} r^2 dr - \right. \\
& \left. - \sum_{j=1}^{N} \varrho_j \sin \alpha_j \int_0^{R_j} \left(\frac{\partial \psi_{jz}}{\partial \xi} \right)_{\xi=0} r^2 dr \right] = M_z.
\end{aligned}
\right\}
\quad (3.16.14)
$$

When cos $\alpha_j \neq 0$ in view of symmetry of the rocket, to the fuel tank with number j there will always correspond exactly the same fuel tank with a certain number k, symmetrically arranged with respect to plane $y = 0$ (see Fig. 3.3), and there will take place equalities:

$$\psi_{kx} = \psi_{jx},\ \psi_{kz} = \psi_{jz},\ \varrho_k = \varrho_j,\ r_k = r_j,\ R_k = R_j,\ x_k = x_j,$$
$$k_k = k_j,\ \cos\alpha_k = -\cos\alpha_j. \tag{3.16.15}$$

From this follows relationships:

$$\sum_{j=1}^{N} \varrho_j \cos\alpha_j \int_0^{R_j} \left(\frac{\partial^3 \psi_{jz}}{\partial\xi\, \partial t^2}\right)_{\xi=0} r^2 dr = 0,$$
$$\sum_{j=1}^{N} \varrho_j r_j \cos\alpha_j \int_0^{R_j} \left(\frac{\partial^3 \psi_{jz}}{\partial\xi\, \partial t^2}\right)_{\xi=0} r^2 dr = 0,$$
$$\sum_{j=1}^{N} \varrho_j \cos\alpha_j \int_0^{R_j} \left(\frac{\partial \psi_{jz}}{\partial\xi}\right)_{\xi=0} r^2 dr = 0,$$
$$\sum_{j=1}^{N} \varrho_j \cos\alpha_j \int_0^{R_j} \left(\frac{\partial^3 \psi_{Lz}}{\partial\xi\, \partial t^2}\right)_{\xi=0} (x_j - k_j) r^2 dr = 0. \tag{3.16.16}$$

Analogously when sin $\alpha_j \neq 0$ to the fuel tank with number j there will always correspond exactly the same fuel tank with certain number k, symmetrically arranged with respect to plane $z = 0$, there will exist relationships:

$$\psi_{kx} = \psi_{jx},\ \psi_{ky} = \psi_{jy},\ \varrho_k = \varrho_j,\ r_k = r_j,\ R_k = R_j,\ x_k = x_j,$$
$$k_k = k_j,\ \sin\alpha_k = -\sin\alpha_j. \tag{3.16.17}$$

Equalities (3.16.17) involve relationships:

$$\sum_{j=1}^{N} \varrho_j \sin\alpha_j \int_0^{R_j} \left(\frac{\partial^3 \psi_{jz}}{\partial\xi\, \partial t^2}\right)_{\xi=0} r^2 dr = 0,$$
$$\sum_{j=1}^{N} \varrho_j r_j \sin\alpha_j \int_0^{R_j} \left(\frac{\partial^3 \psi_{jy}}{\partial\xi\, \partial t^2}\right)_{\xi=0} r^2 dr = 0, \tag{3.16.18}$$

$$\sum_{j=1}^{N} Q_j \sin\alpha_j \int_0^{R_j} \left(\frac{\partial\psi_{jz}}{\partial\xi}\right)_{\xi=0} r^2 dr = 0,$$

$$\sum_{j=1}^{N} C_j \sin\alpha_j \int_0^{R_j} \left(\frac{\partial^3\psi_{jz}}{\partial\xi\,\partial t^2}\right)_{\xi=0} (x_j - k_j) r^2 dr = 0. \qquad (3.16.18 \text{ cont'd})$$

According to (3.16.16) and (3.16.18) equations of forces (3.16.9) can be converted so:

$$m(w_{0y} - g_y) + \pi \sum_{j=1}^{N} Q_j \int_0^{R_j} \left(\frac{\partial^3\psi_{jy}}{\partial\xi\,\partial t^2}\right)_{\xi=0} r^2 dr = F_y,$$

$$m(w_{0z} - g_z) + \pi \sum_{j=1}^{N} Q_j \int_0^{R_j} \left(\frac{\partial^3\psi_{jz}}{\partial\xi\,\partial t^2}\right)_{\xi=0} r^2 dr = F_z, \qquad (3.16.19)$$

and equations of moments (3.16.14) can be given the form

$$J_x \frac{d\omega_x}{dt} + \pi \sum_{j=1}^{N} Q_j r_j \int_0^{R_j} \left(\frac{\partial^3\psi_{jz}}{\partial\xi\,\partial t^2}\right)_{\xi=0} r^2 dr + \pi(w_{0z} - g_z) \times$$

$$\times \sum_{j=1}^{N} C_j \int_0^{R_j} \left(\frac{\partial\psi_{jy}}{\partial\xi}\right)_{\xi=0} r^2 dr - \pi(w_{0y} - g_y) \times$$

$$\times \sum_{j=1}^{N} Q_j \int_0^{R_j} \left(\frac{\partial\psi_{jz}}{\partial\xi}\right)_{\xi=0} r^2 dr = M_x,$$

$$J_y \frac{d\omega_y}{dt} - \pi \sum_{j=1}^{N} C_j \int_0^{R_j} \left(\frac{\partial^3\psi_{jz}}{\partial\xi\,\partial t^2}\right)_{\xi=0} (x_j - k_j) r^2 dr + \qquad (3.16.20)$$

$$+ \pi(w_{0x} - g_x) \sum_{j=1}^{N} C_j \int_0^{R_j} \left(\frac{\partial\psi_{jz}}{\partial\xi}\right)_{\xi=0} r^2 dr = M_y,$$

$$J_z \frac{d\omega_z}{dt} + \pi \sum_{j=1}^{N} Q_j \int_0^{R_j} \left(\frac{\partial^3\psi_{jy}}{\partial\xi\,\partial t^2}\right)_{\xi=0} (x_j - k_j) r^2 dr -$$

$$- \pi(w_{0x} - g_x) \sum_{j=1}^{N} C_j \int_0^{R_j} \left(\frac{\partial\psi_{jy}}{\partial\xi}\right)_{\xi=0} r^2 dr = M_z.$$

By uniting the first of equations (3.16.19) with the third of equations (3.16.20) and adding to this system of equations the differential equations (3.15.14) with boundary conditions (3.15.15), we obtain equations:

$$
\begin{aligned}
&m(w_{0y}-g_y)+\pi\sum_{j=1}^{N}\varrho_j\int_0^{R_j}\left(\frac{\partial^3\psi_{jy}}{\partial\xi\,\partial t^2}\right)_{\xi=0} r^2dr=F_y,\\
&J_z\frac{d\omega_z}{dt}+\pi\sum_{j=1}^{N}\varrho_j\int_0^{R_j}\left(\frac{\partial^3\psi_{jy}}{\partial\xi\,\partial t^2}\right)_{\xi=0}(x_j-k_j)r^2dr-\\
&-\pi(w_{0x}-g_x)\sum_{j=1}^{N}\varrho_j\int_0^{R_j}\left(\frac{\partial\psi_{jy}}{\partial\xi}\right)_{\xi=0} r^2dr=M_z,\\
&\frac{\partial^2\psi_{jy}}{\partial\xi^2}+\frac{\partial^2\psi_{jy}}{\partial r^2}+\frac{1}{r}\frac{\partial\psi_{jy}}{\partial r}-\frac{\psi_{jy}}{r^2}=0,\ j=1,2,\ldots,N,\\
&\frac{\partial\psi_{jy}}{\partial n}=0 \text{ on } l_j^*,\ j=1,2,\ldots,N,\\
&\frac{\partial^2\psi_{jy}}{\partial t^2}+(w_{0x}-g_x)\frac{\partial\psi_{jy}}{\partial\xi}+r\left[w_{0y}-g_y+(x_j-k_j)\frac{d\omega_z}{dt}\right]=\\
&=0 \text{ when } \xi=0,\ 0<r<R_j,\ j=1,2,\ldots,N.
\end{aligned}
\tag{3.16.21}
$$

Equations (3.16.21) determine the motion being accomplished by the rocket body in the pitching plane, and oscillations of free surfaces of fluids in the fuel tanks appearing during this motion, which according to (3.15.19) are characterized by functions

$$
f_j=\left(\frac{\partial\psi_{jy}}{\partial\xi}\right)_{\xi=0}\cos\alpha,\ j=1,2,\ldots,N. \tag{3.16.22}
$$

Thus, equations (3.16.21) are *motion equations of the rocket in the pitching plane*, considering the mobility of fluids, located in the fuel tanks of the rocket.

Analogously, by uniting the second of equations (3.16.19) with the second of equations (3.16.20) and adding to this system of equations the differential equations (3.15.16) with boundary conditions (3.15.17), we obtain *motion equations of the rocket in yawing plane:*

$$
\begin{gathered}
m(w_{0z}-g_z)+\pi\sum_{j=1}^{N}\varrho_j\int_0^{R_j}\left(\frac{\partial^3\psi_{jz}}{\partial\xi\,\partial t^2}\right)_{\xi=0}r^2dr=F_z,\\
J_y\frac{d\omega_y}{dt}-\pi\sum_{j=1}^{N}\varrho_j\int_0^{R_j}\left(\frac{\partial^3\psi_{jz}}{\partial\xi\,\partial t^2}\right)_{\xi=0}(x_j-k_j)r^2dr+\\
+\pi(w_{0x}-g_x)\sum_{j=1}^{N}\varrho_j\int_0^{R_j}\left(\frac{\partial\psi_{jz}}{\partial\xi}\right)_{\xi=0}r^2dr=M_y,\\
\frac{\partial^2\psi_{jz}}{\partial\xi^2}+\frac{\partial^2\psi_{jz}}{\partial r^2}+\frac{1}{r}\frac{\partial\psi_{jz}}{\partial r}-\frac{\psi_{jz}}{r^2}=0,\ j=1,2,\ldots,N,\\
\frac{\partial\psi_{jz}}{\partial n}=0\ \text{on}\ l_j,\ j=1,2,\ldots,N,\\
\frac{\partial^2\psi_{jz}}{\partial t^2}+(w_{0x}-g_x)\frac{\partial\psi_{jz}}{\partial\xi}+r\left[w_{0z}-g_z-(x_j-k_j)\frac{d\omega_y}{dt}\right]=\\
=0\ \text{when}\ \xi=0,\ 0<r<R_j,\ j=1,2,\ldots,N.
\end{gathered}
\tag{3.16.23}
$$

Motion equations in the yawing plane (3.16.23) can be obtained from equations of motion in the pitching plane (3.16.21), replacing quantities $w_{0y} - g_y$, $\frac{d\omega_z}{dt}$, F_y, M_z, J_z and ψ_{jy} respectively by quantities $w_{0z} - g_z$, $-\frac{d\omega_y}{dt}$, F_z, $-M_y$, J_y and ψ_{jz}. Thus, motion equations in the pitching plane and equations of motion in yawing plane have completely identical structure and just as in Chapter I, we will be limited subsequently to examination of motion equations in the pitching plane, bearing in mind the applicability of all the obtained results to equations of motion in yawing plane.

By adding to the first of equations of moments (3.16.20) the differential equations (3.15.12) with boundary conditions (3.15.13), we obtain equations:

$$
\begin{gathered}
J_x\frac{d\omega_x}{dx}+\pi\sum_{j=1}^{N}\varrho_j r_j\int_0^{R_j}\left(\frac{\partial^3\psi_{jz}}{\partial\xi\,\partial t^2}\right)_{\xi=0}r^2dr+\\
+\pi(w_{0z}-g_z)\sum_{j=1}^{N}\varrho_j\int_0^{R_j}\left(\frac{\partial\psi_{jy}}{\partial\xi}\right)_{\xi=0}r^2dr-\\
-\pi(w_{0y}-g_y)\sum_{j=1}^{N}\varrho_j\int_0^{R_j}\left(\frac{\partial\psi_{jz}}{\partial\xi}\right)_{\xi=0}r^2dr=M_x,
\end{gathered}
\tag{3.16.24}
$$

$$\frac{\partial^2\psi_{jx}}{\partial\xi^2}+\frac{\partial^2\psi_{jx}}{\partial r^2}+\frac{1}{r}\frac{\partial\psi_{jx}}{\partial r}-\frac{\psi_{jx}}{r^2}=0, \quad j=1, 2, \ldots, N,$$

$$\frac{\partial\psi_{jx}}{\partial n}=0 \quad \text{on} \quad \Gamma_j, \quad j=1, 2, \ldots, N,$$

$$\frac{\partial^2\psi_{jx}}{\partial t^2}+(w_{0x}-g_x)\frac{\partial\psi_{jx}}{\partial\xi}+rr_j\frac{d\omega_x}{dt}=0 \quad \text{when} \quad \xi=0, \tag{3.16.24 cont'd}$$

$$0<r<R_j, \quad j=1, 2, \ldots, N.$$

Equations (3.16.24) determine rotation of the rocket body around its longitudinal axis and oscillations of free surfaces of fluids in the fuel tanks appearing during this rotation, which according to (3.15.19) are characterized by functions

$$f_j=\left(\frac{\partial\psi_{jx}}{\partial\xi}\right)_{\xi=0}\sin(\alpha-\alpha_j), \quad j=1, 2, \ldots, N. \tag{3.16.25}$$

Thus, equations (3.16.24) are *motion equations of the rocket in rolling plane*, considering the mobility of fluids located in the fuel tanks of the rocket.

Equations (2.16.24) maintain quantities $w_{0y} - g_y$, $w_{0z} - g_z$, ψ_{jy} and ψ_{jz}, characterizing motion, being accomplished by the rocket in pitching and yawing planes. Components, containing these quantities, determine the effect which oscillations of the rocket in pitching and yawing planes have on rotation of the rocket around its longitudinal axis. This effect appears due to the fact that oscillations of the free surfaces of fluids, caused by oscillations of the center of gravity of the rocket relative to its axis and thus affect the axial component of the principal moment of external forces relative to the center of gravity of the rocket. In view of the relative smallness of lateral apparent accelerations $w_{0y} - g_y$ and $w_{0z} - g_z$, the indicated effect is practically very slight and usually is disregarded. By dropping the terms in (3.16.24) characterizing the given effect, we obtain equations:

$$
\begin{gathered}
J_x \frac{d\omega_x}{dt} + \pi \sum_{j=1}^{N} \varrho_j r_j \int_0^{R_j} \left(\frac{\partial^3 \psi_{jx}}{\partial \xi \, \partial t^2} \right)_{\xi=0} r^2 dr = M_x, \\
\frac{\partial^2 \psi_{jx}}{\partial \xi^2} + \frac{\partial^2 \psi_{jx}}{\partial r^2} + \frac{1}{r} \frac{\partial \psi_{jx}}{\partial r} - \frac{\psi_{jx}}{r^2} = 0, \ j = 1, 2, \ldots, N, \\
\frac{\partial \psi_{jx}}{\partial n} = 0 \ \text{on} \ \mathring{l}_j, \ j = 1, 2, \ldots, N, \\
\frac{\partial^2 \psi_{jx}}{\partial t^2} + (w_{0x} - g_x) \frac{\partial \psi_{jx}}{\partial \xi} + r r_j \frac{d\omega_x}{dt} = 0 \ \text{when} \ \xi = 0, \\
0 < r < R_j, \ j = 1, 2, \ldots, N.
\end{gathered}
\tag{3.16.26}
$$

In equations (3.16.26) r_j – distance from axis of j-th fuel tank to the longitudinal axis of the rocket x. When $r_1 = r_2 = \ldots = r_N = 0$, i.e., when the axes of all fuel tanks coincide with the longitudinal axis of the rocket, equations (3.16.26) are decomposed into equation

$$
J_x \frac{d\omega_x}{dt} = M_x \tag{3.16.27}
$$

and homogeneous differential equations for functions ψ_{jx} with homogeneous boundary conditions. According to (3.14.32) in this case

$$
J_x = \int_{V_0} (y^2 + z^2) \varrho \, dv \tag{3.16.28}
$$

$(y_j^2 + z_j^2 = r_j^2)$. Thus, when $r_1 = r_2 = \ldots = r_N = 0$ the presence of fluids in fuel tanks does not affect the rotation of the rocket around its longitudinal axis, which agrees with the original assumptions about axial symmetry of the fuel tanks and about the perfectness of liquid components of the propellant.

Direct investigation of stability of motion of the rocket on the basis of motion equations, constructed in this paragraph, is very difficult because of the presence of differential equations in partial derivatives in these equations of motion. In connection with this, motion equations of the rocket, considering the mobility of liquid components of propellant, will be transformed below into systems of ordinary differential equations, with respect to which there are applicable usual methods of investigation of the stability of motion. This conversion is based on the concept of natural oscillations of free surfaces of fluids, for examination of which we will transfer to the following paragraph of this chapter.

§ 17. Natural Oscillations of Free Surfaces of Fluids in Fuel Tanks

In § 15 for functions ψ_{jx}, ψ_{jy} and ψ_{jz} we obtained differential equations (3.15.12), (3.15.14) and (3.15.16) with boundary conditions (3.15.13), (3.15.15) and (3.15.17). For these three boundary value problems it is possible to establish a single writing introducing into examination function ϑ_j, which satisfies differential equation

$$\frac{\partial^2 \vartheta_j}{\partial \xi^2}+\frac{\partial^2 \vartheta_j}{\partial r^2}+\frac{1}{r}\,\frac{\partial \vartheta_j}{\partial r}-\frac{\vartheta_j}{r^2}=0, \qquad (3.17.1)$$

regular on axis $r = 0$ and satisfying boundary conditions:

$$\frac{\partial \vartheta_j}{\partial n}=0 \text{ on } f_j,$$
$$\frac{\partial^2 \vartheta_j}{\partial t^2}+(w_{0x}-g_x)\,\frac{\partial \vartheta_j}{\partial \xi}+G_j(r,t)=0 \text{ when } \xi=0,\ 0<r<R_j, \qquad (3.17.2)$$

where $G_j(r, t)$ – a certain assigned function. When

$$G_j=r r_j \frac{d\omega_z}{dt} \qquad (3.17.3)$$

the solution of boundary value problem formed by differential equation (3.17.1) and boundary conditions (3.17.2), will determine function ψ_{jx}, when

$$G_j = r\left[w_{0y} - g_y + (x_j - k_j)\frac{d\omega_z}{dt}\right] - \text{ function } \psi_{jy} \text{ and when} \quad (3.17.4)$$

$$G_j = r\left[w_{0z} - g_z - (x_j - k_j)\frac{d\omega_y}{dt}\right] - \text{ function } \psi_{jz}. \quad (3.17.5)$$

According to (3.17.3), (3.17.4) and (3.17.5) with the absence of lateral apparent accelerations $w_{0y} - g_y$, $w_{0z} - g_z$ and angular accelerations $\frac{d\omega_x}{dt}$, $\frac{d\omega_y}{dt}$, $\frac{d\omega_z}{dt}$ in all three cases function $G_j(r, t)$ will be identically equal to zero and boundary conditions (3.17.2) will degenerate into homogeneous boundary conditions:

$$\frac{\partial \Phi_j}{\partial n} = 0 \text{ on } f_j,$$

$$\frac{\partial^2 \Phi}{\partial t^2} + (w_{0x} - g_x)\frac{\partial \Phi_j}{\partial \xi} = 0 \text{ when } \xi = 0,\ 0 < r < R_j. \quad (3.17.6)$$

Differential equation (3.17.1) with boundary conditions (3.17.6) has obvious or, as it is accepted to say in such cases, trivial solution $\Phi_j = 0$. To this trivial solution corresponds identical equality of functions ψ_{jx}, ψ_{jy} and ψ_{jz} to zero, and consequently, function f_j, determining oscillations of free surface of fluid in j-th tank in accordance with formula (3.15.19). Thus, in the absence of lateral apparent accelerations and angular accelerations one of the possible motions of fluids, located in fuel tanks, is such motion at which the free surfaces of fluids do not oscillate, remaining flat and normal to the longitudinal axis of rocket x in the process of motion. However, as will be seen further, along with trivial solution $\Phi_j = 0$ differential equation (3.17.1) with homogeneous boundary conditions (3.17.6) has infinite sequence of linearly independent nontrivial solutions, different from solution $\Phi_j = 0$. Oscillations of free surface of fluid in j-th tank, corresponding to these nontrivial solutions of differential equation (3.17.1) with boundary conditions (3.17.6), are called *natural oscillations* of this free surface.

The problem about natural oscillations of the free surface of fluid can be solved by the method of separation of variables, which in mathematical physics it is accepted to call the Fourier method. We will seek nontrivial solution of differential equation (3.17.1), regular on axis $r = 0$ and satisfying boundary conditions (3.17.6), in the form of the product of two functions

$$\vartheta_j(\xi, r, t) = \Theta_j(\xi, r)\beta_j(t). \tag{3.17.7}$$

By substituting (3.17.7) in (3.17.1) and (3.17.6), we obtain differential equation

$$\frac{\partial^2\Theta_j}{\partial\xi^2}+\frac{\partial^2\Theta_j}{\partial r^2}+\frac{1}{r}\frac{\partial\Theta_j}{\partial r}-\frac{\Theta_j}{r^2}=0, \tag{3.17.8}$$

boundary condition

$$\frac{\partial\Theta_j}{\partial n}=0 \text{ on } l_j^{*} \tag{3.17.9}$$

and boundary condition

$$\Theta_j\frac{d^2\beta_j}{dt^2}+(w_{0x}-g_x)\frac{\partial\Theta_j}{\partial\xi}\beta_j=0 \text{ when } \xi=0,\ 0<r<R_j. \tag{3.17.10}$$

Condition (3.17.10) can be converted so:

$$\left[\frac{\frac{\partial\Theta_j}{\partial\xi}}{\Theta_j}\right]_{\xi=0}=-\frac{\frac{d^2\beta}{dt^2}}{(w_{0x}-g_x)\beta_j}. \tag{3.17.11}$$

The left side of equality (3.7.11) does not depend on variable t, and the right side of this equality does not depend on variable r. Thus, the ratios, which figure in equality (3.17.11), must represent some constant. By designating the constant through λ_j, for function Θ_j we obtain boundary condition

$$\frac{\partial\Theta_j}{\partial\xi}=\lambda_j\Theta_j \text{ when } \xi=0,\ 0<r<R_j \qquad (3.17.12)$$

and for function β_j differential equation

$$\frac{d^2\beta_j}{dt^2}+\lambda_j(w_{0x}-g_x)\beta_j=0. \qquad (3.17.13)$$

Thus, the problem about natural oscillations of free surface of fluid, located in j-th fuel tank, is reduced to finding those values of numerical parameter λ_j, at which differential equation (3.17.8) has nontrivial solutions, regular on axis ξ and satisfying boundary conditions (3.17.9) and (3.17.12). These values of parameter λ_j are called *eigenvalues* of boundary value problem, formed by differential equation (3.17.8) and boundary conditions (3.17.9) and (3.17.12); nontrivial solutions of this boundary value problem are called its *eigenfunctions*. Basic properties of eigenfunctions and eigenvalues of differential equation (3.17.8) with boundary conditions (3.17.9) and (3.17.12) are found by us in the following paragraph.

§ 18. Forms and Frequencies of Natural Oscillations of Free Surface

In order to investigate the eigenfunctions and eigenvalues of the boundary value problem, formed by differential equation (3.17.8) and boundary conditions (3.17.9) and (3.17.12), let us introduce into consideration function $U_j(\xi, r, \alpha)$, determined by equality

$$U_j=\Theta_j\cos\alpha. \qquad (3.18.1)$$

According to (3.17.8), (3.17.9) and (3.17.12) function U_j must satisfy Laplace equation

$$\frac{\partial^2 U_j}{\partial\xi^2}+\frac{\partial^2 U_j}{\partial r^2}+\frac{1}{r}\frac{\partial U_j}{\partial r}+\frac{1}{r^2}\frac{\partial^2 U_j}{\partial\alpha^2}=0 \qquad (3.18.2)$$

and boundary conditions:

$$\frac{\partial U_j}{\partial n}=0 \text{ on } S_j,$$
$$\frac{\partial U_j}{\partial n}=\lambda_j U_j \text{ on } \sigma_j \qquad (3.18.3)$$

(let us recall that contour l_j^* is the generatrix of wetted surface of revolution S_j, and rectilinear segment $\xi = 0$, $0 < r < R_j$ – generatrix of flat free surface σ_j).

Let us prove first that differential equation (3.17.8) can have nontrivial solutions, which satisfy boundary conditions (3.17.9) and (3.17.12) only with real and moreover positive values of parameter λ_j. Let us assume that a certain complex value of parameter λ_j is the eigenvalue of the boundary value problem, formed by differential equation (3.17.8) and boundary conditions (3.17.9) and (3.17.12). The appropriate eigenfunction Θ_j will be complex, function U_j, found by formula (3.18.1), will also be complex. According to (3.18.3) there will exist equality

$$\int_{S_j+\sigma_j}\frac{\partial U_j}{\partial n}\bar{U}_j d\sigma=\lambda_j\int_{\sigma_j}U_j\bar{U}_j d\sigma=\lambda_j\int_{\sigma_j}|U_j|^2 d\sigma. \qquad (3.18.4)$$

By converting the surface integral into volumetric by means of Gauss divergence formulas, we find

$$\int_{S_j+\sigma_j}\frac{\partial U_j}{\partial n}\bar{U}_j d\sigma=\int_{S_j+\sigma_j}\bar{U}\left[\frac{\partial U_j}{\partial x}\cos(n,x)+\frac{\partial U_j}{\partial y}\cos(n,y)+\right.$$
$$\left.+\frac{\partial U_j}{\partial z}\cos(n,z)\right]d\sigma=\int_{V_j}\left[\frac{\partial}{\partial x}\left(\bar{U}_j\frac{\partial U_j}{\partial x}\right)+\frac{\partial}{\partial y}\left(\bar{U}_j\frac{\partial U_j}{\partial y}\right)+\right.$$
$$\left.+\frac{\partial}{\partial z}\left(\bar{U}_j\frac{\partial U_j}{\partial z}\right)\right]dv=\int_{V_j}\left(\bar{U}_j\nabla^2U_j+\frac{\partial U_j}{\partial x}\frac{\partial \bar{U}_j}{\partial x}+\frac{\partial U_j}{\partial y}\frac{\partial \bar{U}_j}{\partial y}+\frac{\partial U_j}{\partial z}\frac{\partial \bar{U}_j}{\partial z}\right)dv=$$
$$=\int_{V_j}\left(\left|\frac{\partial U_j}{\partial x}\right|^2+\left|\frac{\partial U_j}{\partial y}\right|^2+\left|\frac{\partial U_j}{\partial z}\right|^2\right)dv, \qquad (3.18.5)$$

since according to (3.18.2) $\nabla^2 U_j = 0$.

Function Θ_j is nontrivial solution of differential equation (3.17.8) and, thus, in accordance with formula (3.18.1), function U_j can keep constant value in region V_j. From this follows inequality

$$\left|\frac{\partial U_j}{\partial x}\right|^2+\left|\frac{\partial U_j}{\partial y}\right|^2+\left|\frac{\partial U_j}{\partial z}\right|^2 \not\equiv 0 \quad \text{in } V_j \tag{3.18.6}$$

and inequality

$$U_j \not\equiv 0 \quad \text{on } \sigma_j \tag{3.18.7}$$

(when $U_j \equiv 0$ on σ_j the boundary value problem, formed by differential Laplace equation (3.18.2) and boundary conditions (3.18.3), degenerates into homogeneous Neumann problem, general solution of which has form U_j = const). According to (3.18.4) and (3.18.5) there should take place equality

$$\lambda_j = \frac{\int\limits_{V_j}\left(\left|\frac{\partial U_j}{\partial x}\right|^2+\left|\frac{\partial U_j}{\partial y}\right|^2+\left|\frac{\partial U_j}{\partial z}\right|^2\right)dv}{\int\limits_{\sigma_j}|U_j|^2\,d\sigma}. \tag{3.18.8}$$

In accordance with inequalities (3.18.6) and (3.18.7) the integrals, which figure in formula (3.18.8), is real and positive. Thus, having assumed that the boundary value problem, formed by differential equation (3.17.8) and boundary conditions (3.17.9) and (3.17.12), has complex eigenvalue, we arrived at a contradiction, thereby having proven the realness of eigenvalues of this boundary value problem. Incidentally, having established equality (3.18.8), we also proved the positivity of these eigenvalues.

Considering the positivity of eigenvalues λ_j and positivity of axial apparent acceleration $w_{0x} - g_x$, it is possible to give differential equation (3.17.13) the form

$$\frac{d^2\beta_j}{dt^2}+\omega_j^2\beta_j=0, \tag{3.18.9}$$

where

$$\omega_j=\sqrt{\lambda_j(w_{0x}-g_x)}. \tag{3.18.10}$$

By solving the problem about natural oscillations of free surface of fluid by Fourier method, we established the possibility of appearance of natural oscillations, in the process of which functions ψ_{jx}, ψ_{jy} and ψ_{jz} take values, determined by formula (3.17.7). As an example let us consider natural oscillations of free surface of fluid in the pitching plane. Assuming in accordance with formula (3.17.7)

$$\psi_{jy}=\Theta_j(\xi, r)\beta_j(t), \tag{3.18.11}$$

let us investigate function $f_j(r, \alpha, t)$, determining deflections of points of the free surface of fluid from plane σ_j. According to (3.16.22) and (3.18.11) in the considered case there will exist equality

$$f_j=\beta_j(t)F_j(r)\cos\alpha, \tag{3.18.12}$$

where

$$F_j(r)=\left(\frac{\partial\Theta_j}{\partial\xi}\right)_{\xi=0}. \tag{3.18.13}$$

In accordance with equality (3.18.12) in the cross section of free surface with arbitrary plane, passing through the axis of the tank, deflections of points of the free surface from plane σ_j, determined by function f_j, at any moment of time t are proportional to values taken by function $F_j(r)$. The proportionality factor depends on the position of intersecting plane and on the values being taken by time factor $\beta_j(t)$. According to (3.18.12) the greatest deflections of points of the free surface from plane σ_j appear in half-planes $\alpha = 0$ and $\alpha = \pi$, in half-planes $\alpha = \pm\frac{\pi}{2}$ movements f_j are absent, through these half-planes pass the nodal line. Time factor $\beta_j(t)$ is determined by differential equation (3.18.9), quantity ω_j, entering this equation, characterizes the frequency of oscillations of free surface considered by us. Thus, function $F_j(r)$ determines *the form of natural oscillations* of the free surface, quantity ω_j determines the *frequency of oscillations*.

According to (3.18.10) the frequency of natural oscillations is proportional to the square root of apparent axial acceleration $w_{0x} - g_x$, in other words, is proprotional to the square root of axial overload. Changes, undergone by axial overload in the process of rocket flight, involve changes of frequency of natural oscillations ω_j. Until j-th fuel tank is emptied, numerical factor λ_j and function $F_j(r)$ do not undergo changes. In the process of emptying of j-th tank there changes contour l_j^*, figuring in boundary condition (3.17.9), and radius R_j of flat free surface σ_j, entering the boundary condition (3.17.12). In connection with this, there appears the relationship of eigenvalue λ_j and eigenfunction Θ_j to time t in the boundary value problem, formed by differential equation (3.17.8) and boundary conditions (3.17.9) and (3.17.12). According to (3.18.13) the relationship of function Θ_j to time involves the relationship of function F_j to time, determining the form of natural oscillations of the free surface of fluid.

In case of relationship of function Θ_j to time t solution (3.17.7) of differential equation (3.17.1) with boundary conditions (3.17.6) becomes already approximate, since with substitution of function Θ_j from (3.17.7) in the second of boundary conditions (3.17.6), not only function β_j, but also function Θ_j should be differentiated with respect to time. However, the theoretical and experimental investigations conducted on this question attest to the insignificance of errors appearing in this case.

Forms of natural oscillations of free surfaces of fluids possess an integral property, which it is accepted to call *the property of orthogonality*. In order to establish this property, let us examine two eigenfunctions θ_j and θ_j^*, corresponding to two different eigenvalues λ_j and λ_j^* of the boundary value problem, formed by differential equation (3.17.8) and boundary conditions (3.17.9) and (3.17.12). Along with function (3.18.1), which satifies differential equation (3.18.2) and boundary conditions (3.18.3), we will consider function

$$U_j^* = \theta_j^* \cos\alpha, \qquad (3.18.14)$$

satisfying Laplace equation

$$\frac{\partial^2 U_j^*}{\partial \xi^2} + \frac{\partial^2 U_j^*}{\partial r^2} + \frac{1}{r}\frac{\partial U_j^*}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U_j^*}{\partial \alpha^2} = 0 \qquad (3.18.15)$$

and boundary conditions:

$$\begin{aligned} &\frac{\partial U_j^*}{\partial n} = 0 \quad \text{on } S_j, \\ &\frac{\partial U_j^*}{\partial n} = \lambda_j^* U_j^* \quad \text{on } \sigma_j. \end{aligned} \qquad (3.18.16)$$

In accordance with Green's theorem there will take place equality

$$\int_{S_j+\sigma_j} U_j \frac{\partial U_j^*}{\partial n} d\sigma = \int_{S_j+\sigma_j} U_j^* \frac{\partial U_j}{\partial n} d\sigma. \qquad (3.18.17)$$

By using boundary conditions (3.18.3) and (3.18.16), it is possible to reduce equality (3.18.17) to the form

$$\lambda_j^* \int_{\sigma_j} U_j U_j^* d\sigma = \lambda_j \int_{\sigma_j} U_j U_j^* d\sigma, \qquad (3.18.18)$$

or

$$(\lambda_j^* - \lambda_j) \int_{\sigma_j} U_j U_j^* d\sigma = 0.$$

From this follows relationship

$$\int_{\sigma_j} U_j U_j^* d\sigma = 0, \qquad (3.18.19)$$

since according to condition $\lambda_j^* \neq \lambda_j$.

According to (3.16.3), (3.18.1) and (3.18.14) equality (3.18.19) can be reduced to form

$$\int_0^{R_j} \int_0^{2\pi} (\Theta_j \Theta_j^*)_{t=0}\, r \cos^2 \alpha dr\, d\alpha = 0,$$

or

$$\int_0^{2\pi} \cos^2 \alpha d\alpha \int_0^{R_j} (\Theta_j \Theta_j^*)_{t=0}\, r dr = 0. \qquad (3.18.20)$$

Relationship (3.18.20) can be given the form

$$\int_0^{R_j} (\Theta_j \Theta_j^*)_{t=0}\, r dr = 0, \qquad (3.18.21)$$

since

$$\int_0^{2\pi} \cos \alpha d\alpha = \pi \neq 0.$$

In accordance with boundary condition (3.17.12) there must exist equalities:

$$(\Theta_j)_{\xi=0}=\frac{1}{\lambda_j}\left(\frac{\partial\Theta_j}{\partial\xi}\right)_{\xi=0},\quad (\Theta_j^*)_{\xi=0}=\frac{1}{\lambda_j^*}\left(\frac{\partial\Theta_j^*}{\partial\xi}\right)_{\xi=0} \qquad (3.18.22)$$

when $0 < r < R_j$.

By placing (3.18.22) in (3.18.21), we obtain relationship

$$\int_0^{R_j}\left(\frac{\partial\Theta_j}{\partial\xi}\frac{\partial\Theta_j^*}{\partial\xi}\right)_{\xi=0} rdr=0,$$

or according to (3.18.13)

$$\int_0^{R_j} F_j(r)F_j^*(r)rdr=0, \qquad (3.18.23)$$

where

$$F_j^*(r)=\left(\frac{\partial\Theta_j^*}{\partial\xi}\right)_{\xi=0} \qquad (3.18.24)$$

– function, which determines the form of natural oscillations, corresponding to eigenvalue λ_j^* of the boundary value problem, formed by differential equation (3.17.8) and boundary conditions (3.17.9) and (3.17.12).

Thus, any two forms of natural oscillations of the free surface, corresponding to different eigenvalues λ_j and λ_j^*, are connected together by integral relationship (3.18.23), expressing the so-called property of orthogonality of forms of natural oscillations.

In this paragraph we showed that the forms and frequencies of natural oscillations of the free surface of fluid, located in j-th tank, can be determined, having found the eigenvalues and eigenfunctions of the boundary value problem, formed by differential equation (3.17.8) and boundary conditions (3.17.9) and (3.17.12). As the simplest example we will indicate in the following paragraph the solution of this boundary value problem for the case of cylindrical configuration of a fuel tank.

§ 19. Natural Oscillations of Free Surface of Fluid in a Cylindrical Fuel Tank

In the considered case contour l_j^* is made up of two rectilinear segments $\xi = -H$, $0 < r < R$ and $r = R$, $-H < \xi < 0$ (see Fig. 3.5). Thus, boundary conditions (3.17.9) and (3.17.12) in this case will have the form

$$\begin{aligned} &\frac{\partial\theta_j}{\partial\xi}=0 \text{ when } \xi=-H,\ 0<r<R,\\ &\frac{\partial\theta_j}{\partial r}=0 \text{ when } r=R,\ -H<\xi<0,\\ &\frac{\partial\theta_j}{\partial\xi}=\lambda_j\theta_j \text{ when } \xi=0,\ 0<r<R \end{aligned} \quad (3.19.1)$$

(radius R_j of flat free surface σ_j in the considered example is always equal to the radius of cylindrical tank R). In § 13 we showed that differential equation (3.13.4) has particular solution (3.13.14), regular on axis ξ and containing three arbitrary constants A, B and λ. In accordance with this we will seek the solution of differential equation (3.17.8), regular on axis ξ and satisfying boundary conditions (3.19.1) in the form

$$\theta_j=(Ae^{\lambda\xi}+Be^{-\lambda\xi})J_1(\lambda r). \quad (3.19.2)$$

In order that function θ_j, determined by formula (3.19.2), would satisfy the first and third of boundary conditions (3.19.1), constants A and B must satisfy equations:

$$\left.\begin{array}{l} Ae^{-\lambda H} - Be^{\lambda H} = 0, \\ (\lambda - \lambda_j) A - (\lambda + \lambda_j) B = 0. \end{array}\right\} \quad (3.19.3)$$

By equating the determinant of homogeneous system of equations (3.19.3) to zero, we obtain equation

$$(\lambda + \lambda_j) e^{-\lambda H} - (\lambda - \lambda_j) e^{\lambda H} = 0, \quad (3.19.4)$$

which should be satisfied by parameter λ_j so that equations (3.19.3) would have nontrivial solution. By determining λ_j from equation (3.19.4), we find

$$\lambda_j = \lambda \frac{e^{\lambda H} - e^{-\lambda H}}{e^{\lambda H} + e^{-\lambda H}} = \lambda \operatorname{th} \lambda H. \quad (3.19.5)$$

By substituting (3.19.2) in the second of boundary conditions (3.19.1), we obtain equation

$$J_1'(\lambda R) = 0. \quad (3.19.6)$$

By solving equation (3.19.6), for constant λ we obtain infinite sequence of values

$$\lambda = \frac{\nu_k}{R}, \quad k = 1, 2, \ldots, \quad (3.19.7)$$

where ν_1, ν_2 ... – sequence of solutions of transcendental equation

$$J_1'(\nu) = 0. \quad (3.19.8)$$

By substituting (3.19.7) in (3.19.5), we find infinite sequence of eigenvalues of the considered boundary value problem

$$\lambda_j = \frac{\nu_k}{R} \operatorname{th} \frac{\nu_k H}{R}, \quad k = 1, 2, \ldots . \tag{3.19.9}$$

We will satisfy equations (3.19.3), having assumed

$$A = \frac{C}{2} e^{\lambda H}, \quad B = \frac{C}{2} e^{-\lambda H}, \tag{3.19.10}$$

where C – arbitrary constant. By substituting (3.19.10) in (3.19.2), for function Θ_j we obtain expression

$$\Theta_j = C \operatorname{ch} \lambda (H + \xi) J_1 (\lambda r). \tag{3.19.11}$$

According to (3.19.7) and (3.19.11) the eigenfunctions of the considered boundary value problem should have the form

$$\Theta_j = C_k \operatorname{ch} \frac{\nu_k (H + \xi)}{R} J_1 \left(\frac{\nu_k r}{R} \right), \quad k = 1, 2, \ldots . \tag{3.19.12}$$

where C_1, C_2, ... – arbitrary constants.

By substituting (3.19.9) in (3.18.10) and (3.19.12) in (3.18.13), for frequencies of natural oscillations of the free surface we obtain values

$$\omega_j = \sqrt{\frac{\nu_k}{R} \operatorname{th} \frac{\nu_k H}{R} (w_{0x} - g_x)}, \quad k = 1, 2, \ldots . \tag{3.19.13}$$

and for forms of natural oscillations expression

$$F_j(r) = C_k \frac{\nu_k}{R} \operatorname{sh} \frac{\nu_k H}{R} J_1 \left(\frac{\nu_k r}{R} \right), \quad k = 1, 2, \ldots \tag{3.19.14}$$

By giving to constants C_1, C_2, ... values determined by relationship

$$C_k = \frac{R}{\nu_k \operatorname{sh} \frac{\nu_k H}{R}}. \tag{3.19.15}$$

for the forms of natural oscillations of free surface we obtain expressions

$$F_j(r) = J_1\left(\frac{\nu_k r}{R}\right), \quad k = 1, 2, \ldots \tag{3.19.16}$$

According to (3.19.13) when $H/R \to \infty$ natural frequencies ω_j approach finite limits $\omega_j^{(0)}$, determined by formula

$$\omega_j^{(0)} = \sqrt{\frac{\nu_k}{R}(w_{0x} - g_x)}, \quad k = 1, 2, \ldots \tag{3.19.17}$$

Figure 3.10 shows the relationship between ratios $\omega_j/\omega_j^{(0)}$ and H/R for the first three frequencies of natural oscillations (k = 1, 2, 3).

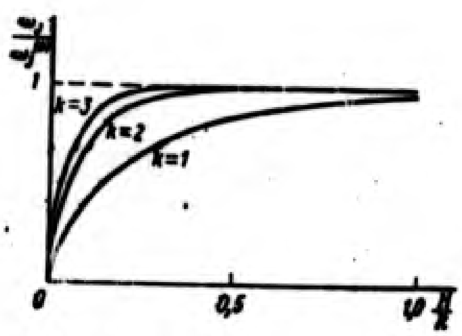

Fig. 3.10.

In accordance with formula (3.19.16) in the considered case the forms of natural oscillations do not depend on the filling depth of the fuel tank. Figure 3.11 shows the form of cross section of free surface of fluid with plane passing through the axis of the tank for the first three tones of oscillations (k = 1, 2, 3).

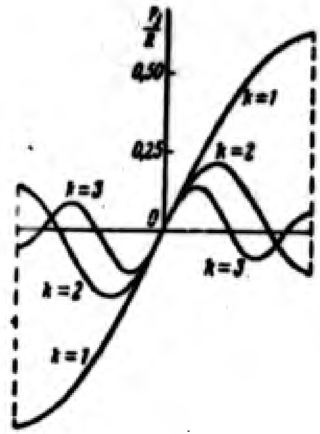

Fig. 3.11.

§ 20. Forced Oscillations of Free Surfaces of Fluids in Fuel Tanks

Let us now turn to examination of forced oscillations of free surfaces of fluids, i.e., those oscillations which are caused by lateral apparent accelerations $w_{0y} - g_y$, $w_{0z} - g_z$ and by angular accelerations $d\omega_x/dt$, $d\omega_y/dt$, $d\omega_z/dt$, appearing in the process of rocket flight. Functions ψ_{jx}, ψ_{jy} and ψ_{jz}, characterizing these oscillations, are determined by differential equations (3.17.1) with boundary conditions (3.17.2), in which $G_j(r, t)$ – function, the possible form of which is shown in formulas (3.17.3), (3.17.4) and (3.17.5). The boundary value problem, formed by differential equation (3.17.1) and boundary conditions (3.17.2), can be solved, having constructed for function $G_j(r, t)$ expansion into series with respect to forms of natural oscillations of fluid $F_j^k(r)$. Let us assume $\omega_j^{(1)}$, $\omega_j^{(2)}$, ... – frequencies of natural oscillations of fluid, located in j-th fuel tank, and $F_j^{(1)}(r)$, $F_j^{(2)}(r)$, ... – functions determining the corresponding forms of natural oscillations (we will consider natural frequencies renumbered in there ascending order). In accordance with properties of orthogonality of the forms of natural oscillations, which we established above in § 18, there will exist equalities

$$\int_0^{R_j} F_j^{(k)}(r) F_j^{(l)}(r) r dr = 0 \text{ when } l \neq k. \tag{3.20.1}$$

Coefficients of expansion of function $G_j(r, t)$ into series with respect to forms of natural oscillations $F_j(r)$ will depend on time t. Thus, the desired expansion will have the form

$$G_j(r, t) = \sum_{l=1}^{\infty} c_j^{(l)}(t) F_j^{(l)}(r). \tag{3.20.2}$$

For finding the coefficients of series (3.20.2) in both sides of equality (3.20.2) let us multiply by $F_j^{(k)}(r)r$ and integrate with respect to r from zero to R_j. According to (3.20.1) we obtain relationship

$$\int_0^{R_j} G_j(r, t) F_j^{(k)}(r) r dr = c_j^{(k)}(t) \int_0^{R_j} [F_j^{(k)}(r)]^2 r dr. \tag{3.20.3}$$

In accordance with equality (3.20.3) expansion (3.20.2) can be given the form

$$G_j(r, t) = \sum_{k=1}^{\infty} \frac{\int_0^{R_j} G_j(r, t) F_j^{(k)}(r) r dr \, F_j^{(k)}(r)}{\int_0^{R_j} [F_j^{(k)}(r)]^2 r dr}. \tag{3.20.4}$$

We will now seek the solution of differential equation (3.17.1) with boundary conditions (3.17.2) in the form of series

$$\vartheta_j = \sum_{k=1}^{\infty} \beta_j^{(k)}(t) \Theta_j^{(k)}(\xi, r), \tag{3.20.5}$$

where $\theta_j^{(1)}(\xi, r)$, $\theta_j^{2}(\xi, r)$, ... – eigenfunctions of boundary value problem, formed by differential equation (3.17.8) and boundary conditions (3.17.9) and (3.17.12), i.e., functions satisfying differential equations

$$\frac{\partial^2\theta_j^{(k)}}{\partial\xi^2}+\frac{\partial^2\theta_j^{(k)}}{\partial r^2}+\frac{1}{r}\frac{\partial\theta_j^{(k)}}{\partial r}-\frac{\theta_j^{(k)}}{r^2}=0,\ k=1,\ 2,\ldots \tag{3.20.7}$$

and boundary conditions

$$\left.\begin{array}{c}\dfrac{\partial\theta_j^{(k)}}{\partial n}=0 \text{ on } l_j\\ \dfrac{\partial\theta_j^{(k)}}{\partial\xi}=\lambda_j^{(k)}\theta_j^{(k)} \text{ when } \xi=0,\ 0<r<R_j\end{array}\right\} k=1,\ 2,\ldots, \tag{3.20.7}$$

in which $\lambda_j^{(1)}$, $\lambda_j^{(2)}$ – eigenvalues of the given boundary value problem. According to (3.20.6) and (3.20.7) series (3.20.5) will satisfy differential equation (3.17.1) and the first of boundary conditions (3.17.2), the second of boundary conditions (3.20.7) will be fulfilled if there will take place identical equality

$$\sum_{k=1}^{\infty}\left[\frac{d^2\beta_j^{(k)}}{dt^2}(\theta_j^{(k)})_{\xi=0}+(w_{0x}-g_x)\beta_j^{(k)}\left(\frac{\partial\theta_j^{(k)}}{\partial\xi}\right)_{\xi=0}\right]+$$
$$+G_j(r,\ t)=0 \text{ when } 0<r<R_j. \tag{3.20.8}$$

In accordance with formulas (3.18.13) and (3.20.7) relationship (3.20.8) can be given the form

$$\sum_{k=1}^{\infty}\left[\frac{1}{\lambda_j^{(k)}}\frac{d^2\beta_j^{(k)}}{dt^2}+(w_{0x}-g_x)\beta_j^{(k)}\right]F_j^{(k)}(r)+G_j(r,\ t)=0$$
$$\text{when } 0<r<R_j, \tag{3.20.9}$$

where $F_j^{(1)}(r)$, $F_j^{(2)}(r)$ – functions, which determine the forms of natural oscillations of free surface of fluid, located in j-th tank. By substituting function $G_j(r, t)$ in (3.20.9) by its expansion (3.20.4), we obtain equality

$$\sum_{k=1}^{\infty}\left\{\frac{1}{\lambda_j^{(k)}}\frac{d^2\beta_j^{(k)}}{dt^2}+(w_{0x}-g_x)\beta_j^{(k)}+\frac{\int_0^{R_j}G_j(r,t)F_j^{(k)}(r)rdr}{\int_0^{R_j}[F_j^{(k)}(r)]^2rdr}\right\}F_j^{(k)}(r)=$$

$$=0 \text{ when } 0<r<R_j, \qquad (3.20.10)$$

which will take place if functions $\beta_j^{(k)}(t)$, $k = 1, 2, \ldots$ will satisfy differential equations

$$\frac{d^2\beta_j^{(k)}}{dt^2}+\lambda_j^{(k)}(w_{0x}-g_x)\beta_j^{(k)}+\lambda_j^{(k)}\frac{\int_0^{R_j}G_j(r,t)F_j^{(k)}(r)rdr}{\int_0^{R_j}[F_j^{(k)}(r)]^2rdr}=0,\ k=1,2,\ldots \qquad (3.20.11)$$

According to (3.18.10) equations (3.20.11) can be given the form

$$\frac{d^2\beta_j^{(k)}}{dt^2}+\omega_j^{(k)2}\beta_j^{(k)}+\lambda_j^{(k)}\frac{\int_0^{R_j}G_j(r,t)F_j^{(k)}(r)rdr}{\int_0^{R_j}[F_j^{(k)}(r)]^2rdr}=0,\ k=1,2,\ldots \qquad (3.20.12)$$

By using expansion of function $G_j(r, t)$ into infinite series (3.20.4) as a basis, for solution of differential equation (3.17.1) with boundary conditions (3.17.2) we obtained infinite series (3.20.5), in which $\beta_j^{(1)}(t)$, $\beta_j^{(2)}(t)$, ... – functions determined by differential equations (3.20.12). For all practical purposes the assigned function $G_j(r, t)$ with sufficient accuracy can always represent finite sum of series (3.20.4). In this instance the solution of differential equation (3.17.1) with boundary conditions (3.17.2) will be determined by the sum of the appropriate number of terms of series (3.20.5).

Let us first construct function ψ_{jx}, figuring in equations of motion in the rolling plane (3.16.24). In the considered case function $G_j(r, t)$ is determined by formula (3.17.3). According to (3.17.3) in this case there will take place equality

$$\int_0^{R_j} G_j(r,t) F_j^{(k)}(r) r dr = r_j \frac{d\omega_x}{dt} \int_0^{R_j} F_j^{(k)}(r) r^2 dr \qquad (3.20.13)$$

and expansion (3.20.4) will have the form

$$G_j(r,t) = r_j \frac{d\omega_x}{dt} \sum_{k=1}^{\infty} \frac{\int_0^{R_j} F_j^{(k)}(r) r^2 dr}{\int_0^{R_j} \left[F_j^{(k)}(r)\right]^2 r dr} F_j^{(k)}(r). \qquad (3.20.14)$$

As can be seen from Fig. 3.11, for the first tone of natural oscillations of free surface of fluid, located in cylindrical fuel tank, the section of free surface with plane passing through the axis of the tank little differs from rectilinear, in other words

$$F_j^{(1)}(r) \approx C_j r. \qquad (3.20.15)$$

According to (3.12.6) and (3.18.12) function f_j, determining deflections of free surface of fluid from plane σ_j can be approximately expressed in this instance by relationship

$$f_j = C_j \beta_j(t) r \cos\alpha = C_j \beta_j(t)(y - y_j). \qquad (3.20.16)$$

In accordance with equality (3.20.16) the first tone of oscillations of the free surface of fluid in a cylindrical fuel tank is characterized by the fact that in the process of oscillations the free surface retains a configuration close to flat. Numerous calculations, and also experimental investigations showed that the first tone of natural oscillations of free surface possesses this feature regardless of the configuration of the fuel tank. By using approximate relationship (3.20.15), we find

$$\int_0^{R_j} F_j^{(k)}(r)\,r^2dr \approx \frac{1}{C_j}\int_0^{R_j} \dot{F}_j^{(k)}(r)\,F_j^{(1)}(r)\,rdr. \tag{3.20.17}$$

According to (3.20.1)

$$\int_0^{R_j} F_j^{(k)}(r)\,F_j^{(1)}(r)\,rdr = 0 \text{ when } k \geqslant 2. \tag{3.20.18}$$

In accordance with equalities (3.20.17) and (3.20.18) in the expansion, figuring in the right side of relationship (3.20.14), for all practical purposes we can be limited by the first term. In this instance according to (3.20.5), (3.20.12) and (3.20.13) the desired function ψ_{jx} will be expressed by the product of

$$\psi_{jx} = \beta_j^{(1)}(t)\,\theta_j^{(1)}(\xi, r), \tag{3.20.19}$$

where $\beta_j^{(1)}(t)$ – function determined by differential equation

$$\frac{d^2\beta_j^{(1)}}{dt^2} + \omega_j^{(1)2}\beta_j^{(1)} + \lambda_j^{(1)} r_j \frac{d\omega_z}{dt} \frac{\int_0^{R_j} F_j^{(1)}(r)\,r^2dr}{\int_0^{R_j} [F_j^{(1)}(r)]^2\, rdr} = 0. \tag{3.20.20}$$

Let us now turn to construction of function ψ_{jy}, figuring in equations of motion in the pitching plane (3.16.21). In this case function $G_j(r, t)$ is determined by formula (3.17.4). According to (3.17.4) in the considered case there will exist equality

$$\int_0^{R_j} G_j(r, t)\,F_j^{(k)}(r)\,rdr = \left(\dot{w}_{0y} - g_y + x_j\frac{d\omega_z}{dt}\right)\int_0^{R_j} F_j^{(k)}(r)\,r^2dr -$$

$$-\frac{d\omega_z}{dt}\int_0^{R_j} F_j^{(k)}(r)\,k_j r^2dr \tag{3.20.21}$$

and expansion (3.20.4) will have the form

$$G_j(r,t)=\left(w_{0y}-g_y+x_j\frac{d\omega_z}{dt}\right)\sum_{k=1}^{\infty}\frac{\int_0^{R_j}F_j^{(k)}(r)\,r^2dr}{\int_0^{R_j}\left[F_j^{(k)}(r)\right]^2 r\,dr}F_j^{(k)}(r)-$$

$$-\frac{d\omega_z}{dt}\sum_{k=1}^{\infty}\frac{\int_0^{R_j}F_j^{(k)}(r)\,k_j r^2dr}{\int_0^{R_j}\left[F_j^{(k)}(r)\right]^2 r\,dr}F_j^{(k)}(r). \qquad (3.20.22)$$

We showed above that in the first of the series, figuring in the right side of relationship (3.20.22), it is practically possible to be limited by the first term of this series. Let us examine the second of the expansions entering formula (3.20.22). Let us return to the example of cylindrical fuel tank. Figure 3.12 shows for this case the relationship of ratio $F_j^{(1)}/k_j$ to variable r, calculated by formulas (3.15.21) and (3.19.16) at various values of ratio H/R. In the considered example ratio $F_j^{(1)}/k_j$ represents the function of variable r, close to linear, in other words

$$\frac{F_j^{(1)}(r)}{k_j(r)}\approx C_j^{*}r. \qquad (3.20.23)$$

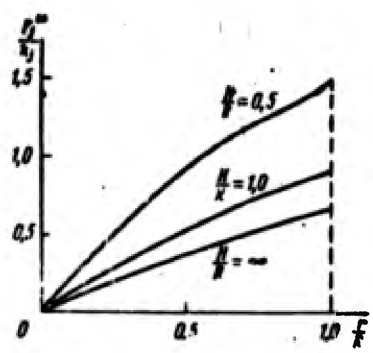

Fig. 3.12.

Calculations, being conducted for tanks of very diverse configurations, attest to the fact that approximate equality (3.20.23) is observed regardless of the shape of the fuel tank. By using approximate relationship (3.20.23), we find

$$\int_0^{R_j} F_j^{(k)}(r)\,k_j r^2 dr \approx \frac{1}{C_j^{\circ}} \int_0^{R_j} F_j^{(k)}(r) F_j^{(1)}(r)\,r dr. \qquad (3.20.24)$$

In accordance with equalities (3.20.18) and (3.20.24) in the second of series, figuring in the right side of relationship (3.20.22), just as in the first of these series, it is practically possible to be limited by the first term of expansion. In this instance according to (3.20.5), (3.20.12) and (3.20.21) the desired function ψ_{jy} will be expressed by the product of

$$\psi_{jy} = \beta_j^{(1)}(t)\,\Theta_j^{(1)}(\xi, r), \qquad (3.20.25)$$

where $\beta_j^{(1)}(t)$ – function determined by differential equation

$$\frac{d^2\beta_j^{(1)}}{dt^2} + \omega_j^{(1)2}\beta_j^{(1)} + \lambda_j^{(1)}\left[w_{0y} - g_y + (x_j - \delta_j)\frac{d\omega_z}{dt}\right]\frac{\int_0^{R_j} F_j^{(1)}(r)\,r^2 dr}{\int_0^{R_j}\left[F_j^{(1)}(r)\right]^2 r dr} = 0, \qquad (3.20.26)$$

in which

$$\delta_j = \frac{\int_0^{R_j} F_j^{(1)}(r)\,k_j r^2 dr}{\int_0^{R_j} F_j^{(1)}(r)\,r^2 dr}. \qquad (3.20.27)$$

By introducing various designations for time functions, figuring in formulas for ψ_{jx} and ψ_{jy}, and having dropped the upper indices in relationships (3.20.19), (3.20.20), (3.20.25), (3.20.26), for convenience of subsequent computations to formulas determining functions

ψ_{jx} and ψ_{jy}, let us give the form

$$\left.\begin{aligned} \psi_{jy} &= \beta_j(t)\,\Theta_j(\xi, r), \\ \psi_{jz} &= \alpha_j(t)\,\Theta_j(\xi, r), \end{aligned}\right\} \tag{3.20.28}$$

where $\beta_j(t)$ and $\alpha_j(t)$ – functions determined by differential equations

$$\frac{d^2\beta_j}{dt^2} + \omega_j^2\beta_j + \lambda_j\left[w_{0y} - g_y + (x_j - \delta_j)\frac{d\omega_z}{dt}\right]\frac{\int_0^{R_j} F_j r^2 dr}{\int_0^{R_j} F_j^2 r\,dr} = 0, \tag{3.20.29}$$

$$\frac{d^2\alpha_j}{dt^2} + \omega_j^2\alpha_j + \lambda_j r_j \frac{d\omega_x}{dt}\,\frac{\int_0^{R_j} F_j r^2 dr}{\int_0^{R_j} F_j^2 r\,d^2 r} = 0,$$

δ_j – quantity determined by relationship

$$\delta_j = \frac{\int_0^{R_j} F_j k_j r^2 dr}{\int_0^{R_j} F_j r^2 dr}. \tag{3.20.30}$$

By subsequently using formulas (3.20.28), (3.20.29) and (3.20.30), we will bear in mind that in these formulas λ_j and Θ_j – first eigenvalue and first eigenfunction of boundary value problem, formed by differential equation (3.17.8) and boundary conditions (3.17.9) and (3.17.12), and ω_j and F_j – first frequency and first form respectively of natural oscillations of free surface of fluid in j-th tank.

Until the fuel tank with number j is emptied, quantity δ_j, being determined by formula (3.20.30), remains constant; in the process of emptying of j-th tank there appears relationship of quantity δ_j to time t. Figure 3.13 shows the relationship between ratios δ_j/R and H/R, calculated for cylindrical fuel tank by formulas (3.15.21), (3.19.16) and (3.20.30).

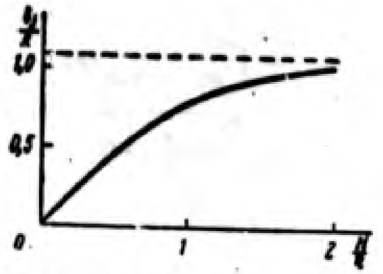

Fig. 3.13.

§ 21. Calculation of Forms and Frequencies of Natural Oscillations of Free Surfaces by the Method of Successive Approximations

In the previous paragraph we reduced finding of functions ψ_{jy}, ψ_{jx}, figuring in equations of motion of the rocket in pitching and rolling planes, to integration of ordinary differential equations (3.20.29). Construction of these differential equations requires calculation of the first form and first frequency of natural oscillations of the free surface of fluid, located in j-th fuel tank, in other words, calculation of the first eigenvalue and first eigenfunction of the boundary value problem, formed by differential equation (3.17.8) and boundary conditions (3.17.9) and (3.17.12) [forms and frequencies of natural oscillations will be determined by relationships (3.18.10) and (3.18.13)]. In this paragraph we will give the process of successive approximations, making it possible to relatively simply calculate the first eigenvalue and first eigenfunction of this boundary value problem for a fuel tank of arbitrary axisymmetric configuration.

By knowing the eigenfunction of boundary value problem Θ_j, we can determine the corresponding eigenvalue λ_j, having used boundary condition (3.17.12). According to (3.17.12) there must exist equality

$$\int_0^{R_j}\left(\frac{\partial\Theta_j}{\partial\xi}\Theta_j\right)_{\xi=0} r\,dr=\lambda_j\int_0^{R_j}(\Theta_j)^2_{\xi=0}\,r\,dr, \qquad (3.21.1)$$

in accordance with which eigenvalue λ_j can be found by formula

$$\lambda_j = \frac{\int_0^{R_j} \left(\frac{\partial \theta_j}{\partial \xi} \theta_j\right)_{\xi=0} r dr}{\int_0^{R_j} (\theta_j)^2_{\xi=0} r dr}. \tag{3.21.2}$$

In accordance with differential equation (3.17.8) and boundary conditions (3.17.9) and (3.17.12) we will determine the $k + 1$ approximation for eigen function $\theta_{j,\ k+1}$ with respect to j-th approximation θ_{jk}, by solving differential equation

$$\frac{\partial^2 \theta_{j,\ k+1}}{\partial \xi^2} + \frac{\partial^2 \theta_{j,\ k+1}}{\partial r^2} + \frac{1}{r}\frac{\partial \theta_{j,\ k+1}}{\partial r} - \frac{\theta_{j,\ k+1}}{r^2} = 0 \tag{3.21.3}$$

with boundary conditions:

$$\frac{\partial \theta_{j,\ k+1}}{\partial n} = 0 \quad \text{on} \quad l_j,$$

$$\frac{\partial \theta_{j,\ k+1}}{\partial \xi} = \lambda_{jk}\theta_{jk} \quad \text{when} \quad \xi = 0,\ 0 < r < R_j. \tag{3.21.4}$$

The k-th approximation of λ_{jk} for sought eigenvalue λ_j, figuring in boundary conditions (3.21.4), according to (3.21.2) we will calculate by formula

$$\lambda_{jk} = \frac{\int_0^{R_j} \left(\frac{\partial \theta_{jk}}{\partial \xi} \theta_{jk}\right)_{\xi=0} r dr}{\int_0^{R} (\theta_{jk})^2_{\xi=0} r dr}. \tag{3.21.5}$$

For construction of initial approximation θ_{j1} it is possible to use approximate relationship (3.20.15). According to (3.18.13) and (3.20.15) for the first eigenfunction θ_j there should take place approximate relationship

$$\left(\frac{\partial\theta_j}{\partial\xi}\right)_{\xi=0}\approx C_j r. \qquad (3.21.6)$$

Since eigenfunctions are determined with accuracy to arbitrary constant factors, for the first approximation θ_{j1} it is possible to establish relationship

$$\left(\frac{\partial\theta_{j1}}{\partial\xi}\right)_{\xi=0}=r. \qquad (3.21.7)$$

According to (3.17.8), (3.17.9) and (3.21.7) the first approximation θ_{j1} can be found from differential equation

$$\frac{\partial^2\theta_{j1}}{\partial\xi^2}+\frac{\partial^2\theta_{j1}}{\partial r^2}+\frac{1}{r}\frac{\partial\theta_{j1}}{\partial r}-\frac{\theta_{j1}}{r^2}=0 \qquad (3.21.8)$$

with boundary conditions:

$$\frac{\partial\theta_{j1}}{\partial n}=0 \quad \text{on } l_j^*,$$
$$\frac{\partial\theta_{j1}}{\partial\xi}=r \text{ when } \xi=0,\ 0<r<R_j. \qquad (3.21.9)$$

In accordance with formulas (3.18.10) and (3.18.13) the k-th approximation ω_{jk} for frequency of natural oscillations ω_j will be determined by relationship

$$\omega_{jk}=\sqrt{\lambda_{jk}(w_{0x}-g_x)} \qquad (3.21.10)$$

and k-th approximation $F_{jk}(r)$ for function $F_j(r)$, determining the form of natural oscillations of free surface, – by relationship

$$F_{jk}(r)=\left(\frac{\partial\theta_{jk}}{\partial\xi}\right)_{\xi=0}. \qquad (3.21.11)$$

According to (3.21.4) and (3.21.9) formula (3.21.11) can be replaced by formulas

$$F_{j1}(r)=r,\ F_{jk}(r)=\lambda_{j,k-1}(\theta_{j,k-1})_{\xi=0}\ \text{when}\ k\geq 2. \qquad (3.21.12)$$

Calculation of each of successive approximations θ_{jk} is reduced to solution of differential equation

$$\frac{\partial^2\theta_j}{\partial\xi^2}+\frac{\partial^2\theta_j}{\partial r^2}+\frac{1}{r}\frac{\partial\theta_j}{\partial r}-\frac{\theta_j}{r^2}=0 \qquad (3.21.13)$$

with boundary conditions:

$$\frac{\partial\theta_j}{\partial n}=0\ \text{on}\ l_j^*,$$
$$\frac{\partial\theta_j}{\partial\xi}=F_j(r)\ \text{when}\ \xi=0,\ 0<r<R_j, \qquad (3.12.14)$$

where $F_j(r)$ – assigned function. Boundary value problem of such type is comprehensively examined in [22]. In this book it is shown that the given boundary value problem is always solvable and has a unique solution, there is indicated one of the possible calculation methods for solution of this boundary value problem.

In order to illustrate the method of successive approximations given in this paragraph, let us consider two examples.

21.1. Cylindrical Fuel Tank

In the considered case contour l_j^* is made up of two rectilinear segments $\xi = -H$, $0 < r < R$ and $r = R$, $-H < \xi < 0$ and in accordance with these boundary conditions (3.21.9) assume the form

$$\frac{\partial\theta_{j1}}{\partial\xi}=0\ \text{when}\ \xi=-H,\ 0<r<R,$$
$$\frac{\partial\theta_{j1}}{\partial r}=0\ \text{when}\ r=R,\ -H<\xi<0, \qquad (3.21.15)$$
$$\frac{\partial\theta_{j1}}{\partial\xi}=r\ \text{when}\ \xi=0,\ 0<r<R.$$

In § 13 we showed that the solution of differential equation (3.13.4), satisfying the second of boundary conditions (3.13.5), can be represented in the form of series (3.13.18). Thus, having assumed

$$\theta_{j1} = \sum_{k=1}^{\infty}\left(A_k e^{\frac{\nu_k t}{R}} + B_k e^{-\frac{\nu_k t}{R}}\right) J_1\left(\frac{\nu_k r}{R}\right). \tag{3.21.16}$$

we will satisfy differential equation (3.21.8) and the second of boundary conditions (3.21.15). In order that series (3.21.16) would satisfy the first and third of boundary conditions (3.21.15), there must take place equalities:

$$\sum_{k=1}^{\infty} \nu_k \left(A_k e^{-\frac{\nu_k H}{R}} - B_k e^{\frac{\nu_k H}{R}}\right) J_1\left(\frac{\nu_k r}{R}\right) = 0, \quad 0 < r < R \tag{3.21.17}$$

and

$$\sum_{k=1}^{\infty} \frac{\nu_k}{R}(A_k - B_k) J_1\left(\frac{\nu_k r}{R}\right) = r, \quad 0 < r < R. \tag{3.21.18}$$

Having multiplied both sides of relationship (3.21.18) by $rJ_1\left(\frac{\nu_l r}{R}\right)$ and integrated them with respect to r from 0 to R, according to (3.13.21) we obtain equation

$$A_l - B_l = \frac{2R^2}{\nu_l(\nu_l^2 - 1) J_1(\nu_l)}. \tag{3.21.19}$$

Relationship (3.21.17) will take place if at any whole positive l coefficients A_l and B_l will satisfy equation

$$A_l e^{-\frac{\nu_l H}{R}} - B_l e^{\frac{\nu_l H}{R}} = 0. \tag{3.21.20}$$

By solving the system of equations (3.21.19) and (3.21.20), for coefficients A_l and B_l we obtain expressions:

$$A_l = \frac{2R^2}{\nu_l(\nu_l^2-1)J_1(\nu_l)\left(1-e^{-\frac{2\nu_l H}{R}}\right)},$$

$$B_l = \frac{2R^2 e^{-\frac{2\nu_l H}{R}}}{\nu_l(\nu_l^2-1)J_1(\nu_l)\left(1-e^{-\frac{2\nu_l H}{R}}\right)}. \quad (3.21.21)$$

According to (3.21.15) and (3.21.21) in the considered case function Θ_{j1} can be represented by series

$$\Theta_{j1} = \sum_{k=1}^{\infty} \frac{2R^2\left[e^{\frac{\nu_k(H+\xi)}{R}} + e^{-\frac{\nu_k(H+\xi)}{R}}\right] J_1\left(\frac{\nu_k r}{R}\right)}{\nu_k(\nu_k^2-1)J_1(\nu_k)\left(e^{\frac{\nu_k H}{R}} - e^{-\frac{\nu_k H}{R}}\right)}$$

or

$$\Theta_{j1} = \sum_{k=1}^{\infty} \frac{2R^2 \operatorname{ch}\frac{\nu_k(H+\xi)}{R} J_1\left(\frac{\nu_k r}{R}\right)}{\nu_k(\nu_k^2-1)J_1(\nu_k)\operatorname{sh}\frac{\nu_k H}{R}}. \quad (3.21.22)$$

By using expansion (3.21.22) and the third of boundary conditions (3.21.15), we obtain relationship

$$\int_0^R \left(\frac{\partial\Theta_{j1}}{\partial\xi} - \Theta_{j1}\right)_{\xi=0} r\,dr = \sum_{k=1}^{\infty} \frac{2R^2 \operatorname{cth}\frac{\nu_k H}{R}}{\nu_k(\nu_k^2-1)J_1(\nu_k)} \int_0^R J_1\left(\frac{\nu_k r}{R}\right) r^2\,dr,$$

or according to (3.13.21)

$$\int_0^R \left(\frac{\partial\Theta_{j1}}{\partial\xi}\Theta_{j1}\right)_{\xi=0} r\,dr = 2R^3 \sum_{k=1}^{\infty} \frac{\operatorname{cth}\frac{\nu_k H}{R}}{\nu_k^3(\nu_k^2-1)}. \tag{3.21.23}$$

In accordance with formulas (3.13.21) and (3.21.22) in this case

$$\int_0^R (\Theta_{j1})^2_{\xi=0} r\,dr = 4R^4 \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{\operatorname{cth}\frac{\nu_k H}{R}\operatorname{cth}\frac{\nu_l H}{R}}{\nu_k(\nu_k^2-1)\nu_l(\nu_l^2-1)J_1(\nu_k)J_1(\nu_l)} \times$$
$$\times \int_0^R J_1\left(\frac{\nu_k r}{R}\right) J_1\left(\frac{\nu_l r}{R}\right) r\,dr = 4R^4 \sum_{k=1}^{\infty} \frac{\operatorname{cth}^2\frac{\nu_k H}{R}}{\nu_k^2(\nu_k^2-1)^2 J_1^2(\nu_k)} \times$$
$$\times \int_0^R J_1^2\left(\frac{\nu_k r}{R}\right) r\,dr = 2R^6 \sum_{k=1}^{\infty} \frac{\operatorname{cth}^2\frac{\nu_k H}{R}}{\nu_k^4(\nu_k^2-1)}. \tag{3.21.24}$$

According to (3.21.5), (3.21.23) and (3.21.24) in the considered case the first approximation λ_{j1} for the sought eigenvalue λ_j can be found by formula

$$\lambda_{j1} = \frac{\displaystyle\sum_{k=1}^{\infty} \frac{\operatorname{cth}\frac{\nu_k H}{R}}{\nu_k^3(\nu_k^2-1)}}{\displaystyle R\sum_{k=1}^{\infty} \frac{\operatorname{cth}^2\frac{\nu_k H}{R}}{\nu_k^4(\nu_k^2-1)}}. \tag{3.21.25}$$

In accordance with formulas (3.21.12) and (3.21.22) function $F_{j2}(r)$, approximately determining the sought form of natural oscillations, in this case will be determined by series

$$F_{j2}(r) = 2\lambda_{j1}R^2 \sum_{k=1}^{\infty} \frac{\text{cth}\frac{\nu_k H}{R}}{\nu_k(\nu_k^2-1)} \cdot \frac{J_1\left(\frac{\nu_k r}{R}\right)}{J_1(\nu_k)}. \quad (3.21.26)$$

approximation ω_{j1} for sought frequency of natural oscillations according to (3.21.10) will be determined by relationship

$$\omega_{j1} = \sqrt{\lambda_{j1}(w_{0x} - g_x)}. \quad (3.21.27)$$

In § 19 we obtained an accurate solution of the problem about natural oscillations of the free surface of fluid in a cylindrical fuel tank, in accordance with this accurate solution the function $F_j(r)$, characterizing the form of natural oscillations, and frequency ω_j for the first tone of natural oscillations of free surface are determined by formulas [see (3.19.13) and (3.19.14)]:

$$F_j(r) = CJ_1\left(\frac{\nu_1 r}{R}\right), \quad \omega_j = \sqrt{\frac{\nu_1}{R}\,\text{th}\,\frac{\nu_1 H}{R}(w_{0x} - g_x)}, \quad (3.21.28)$$

where C – arbitrary factor, not depending on variable r.

In the table given below for several values of ratio H/R there are shown particular values of function $F_{j2}(r)$, calculated according to formulas (3.21.25), (3.21.26) and (3.21.27), approximately determining the form of natural oscillations, and approximate values ω_{j1} for frequency of natural oscillations of the free surface of fluid ω_j. For comparison in the same table there are shown particular values of function $F_j(r)$, calculated according to formulas (3.21.28), accurately determining the form of natural oscillations, and accurate values of frequency of natural oscillations ω_j (factor C was determined so that mean values of functions $F_j(r)$ and $F_{j2}(r)$ would coincide in interval $0 < r < R$).

r/R	$H/R=0{,}2$		$H/R=0{,}5$		$H/R=1{,}0$		$H/R=\infty$	
	$\frac{F_{j2}(r)}{R}$	$\frac{F_j(r)}{R}$	$\frac{F_{j2}(r)}{R}$	$\frac{F_j(r)}{R}$	$\frac{F_{j2}(r)}{R}$	$\frac{F_j(r)}{R}$	$\frac{F_{j2}(r)}{R}$	$\frac{F_j(r)}{R}$
0	0	0	0	0	0	0	0	0
0,1	0,127	0,132	0,123	0,130	0,121	0,129	0,120	0,129
0,2	0,250	0,258	0,242	0,255	0,237	0,254	0,236	0,254
0,3	0,369	0,379	0,357	0,375	0,350	0,374	0,349	0,374
0,4	0,480	0,490	0,466	0,484	0,460	0,483	0,458	0,483
0,5	0,582	0,589	0,569	0,583	0,563	0,581	0,561	0,581
0,6	0,670	0,674	0,660	0,666	0,655	0,664	0,654	0,664
0,7	0,742	0,741	0,736	0,734	0,735	0,731	0,735	0,730
0,8	0,801	0,790	0,800	0,783	0,803	0,780	0,804	0,779
0,9	0,839	0,819	0,846	0,812	0,852	0,809	0,853	0,808
1,0	0,855	0,831	0,867	0,823	0,878	0,820	0,880	0,819

Continued

$H/R=0{,}2$		$H/R=0{,}5$	
$\frac{\omega_{j1}\sqrt{R}}{\sqrt{w_{0x}-g_x}}$	$\frac{\omega_j\sqrt{R}}{\sqrt{w_{0x}-g_x}}$	$\frac{\omega_{j1}\sqrt{R}}{\sqrt{w_{0x}-g_x}}$	$\frac{\omega_j\sqrt{R}}{\sqrt{w_{0x}-g_x}}$
0,807	0,806	1,157	1,156

Continued

$H/R=1{,}0$		$H/R=\infty$	
$\frac{\omega_{j1}\sqrt{R}}{\sqrt{w_{0x}-g_x}}$	$\frac{\omega_j\sqrt{R}}{\sqrt{w_{0x}-g_x}}$	$\frac{\omega_{j1}\sqrt{R}}{\sqrt{w_{0x}-g_x}}$	$\frac{\omega_j\sqrt{R}}{\sqrt{w_{0x}-g_x}}$
1,323	1,321	1,359	1,357

From the provided table it is evident that for achievement of accuracy, acceptable during engineering calculations, in the considered case it is sufficient to calculate the first approximations for eigenvalue λ_j and eigenfunctions θ_j.

21.2. Spherical Fuel Tank, Half-Filled with Fluid

During consideration of this example let us change from rectangular coordinates ξ, r to polar coordinates ϱ, β in accordance with transition formulas (3.13.26). In § 13 we showed that differential equation (3.12.13) in this instance assumes the form (3.13.28), for differentiation of function with respect to normal n to contour l_j we obtained formulas (3.13.30). Thus, by changing to polar coordinates in formulas (3.21.8) and (3.21.9), we obtain differential equation

$$\frac{\partial^2\theta_{j1}}{\partial\varrho^2}+\frac{2}{\varrho}\frac{\partial\theta_{j1}}{\partial\varrho}+\frac{1}{\varrho^2}\frac{\partial^2\theta_{j1}}{\partial\beta^2}+\frac{\operatorname{ctg}\beta}{\varrho^2}\frac{\partial\theta_{j1}}{\partial\beta}-\frac{\theta_{j1}}{\varrho^2\sin^2\beta}=0 \qquad (3.21.29)$$

with boundary conditions

$$\begin{aligned}&\frac{\partial\theta_{j1}}{\partial\varrho}=0 \text{ when } \varrho=R,\ 0<\beta<\frac{\pi}{2},\\&\frac{\partial\theta_{j1}}{\partial\beta}=\varrho^2 \text{ when } \beta=\frac{\pi}{2},\ 0<\varrho<R.\end{aligned} \qquad (3.21.30)$$

In § 13 for differential equation (3.13.28) with boundary conditions (3.13.31) we obtained the solution, determined by formulas (3.13.32) and (3.13.54). In accordance with these formulas the solution of differential equation (3.21.29) with boundary conditions (3.21.30) should have the form

$$\theta_{j1} = -\frac{\varrho^2}{2}\sin 2\xi + R^2 \sum_{k=1}^{\infty} \frac{(-1)^k (4k-1)}{(k+1)(2k-3)(2k-1)^2} \times$$

$$\times \frac{1.3\ldots(2k-1)}{2.4\ldots 2k} \times \left(\frac{\varrho}{R}\right)^{2k-1} P^{(1)}_{2k-1}(\cos\xi). \qquad (3.21.31)$$

According to (3.21.31) there will exist equality

$$(\theta_{j1})_{\xi=0} = R^2 \sum_{k=1}^{\infty} \frac{(-1)^k (4k-1)}{(k+1)(2k-3)(2k-1)^2} \times$$

$$\times \frac{1.3\ldots(2k-1)}{2.4\ldots 2k} \left(\frac{r}{R}\right)^{2k-1} P^{(1)}_{2k-1}(0) \qquad (3.21.32)$$

(in accordance with transition formulas (3.13.26) $\varrho = r$, $\beta = \frac{\pi}{2}$ when $\xi = 0$). By using formula

$$P^{(1)}_{2k-1}(0) = (-1)^{k-1}\, 2k \frac{1.3\ldots(2k-1)}{2.4\ldots 2k} \qquad (3.21.33)$$

(see [27]), it is possible to convert equality (3.21.32) to the form

$$(\theta_{j1})_{\xi=0} = -R^2 \sum_{k=1}^{\infty} \frac{2k(4k-1)}{(k+1)(2k-3)(2k-1)^2} \times$$

$$\times \left[\frac{1.3\ldots(2k-1)}{2.4\ldots 2k}\right]^2 \left(\frac{r}{R}\right)^{2k-1}. \qquad (3.21.34)$$

By using expansion (3.21.34) and the second of boundary conditions (3.21.9), we obtain relationships:

$$\left.\begin{aligned} &\int_0^R \left(\frac{\partial\theta_{j1}}{\partial\xi}\theta_{j1}\right)_{\xi=0} r\,dr = -R^2 \sum_{k=1}^{\infty} \frac{2k(4k-1)}{(k+1)(2k-3)(2k-1)^2} \times \\ &\times \left[\frac{1.3\ldots(2k-1)}{2.4\ldots 2k}\right]^2 \int_0^R \left(\frac{r}{R}\right)^{2k-1} r^2\,dr = \\ &= -R^5 \sum_{k=1}^{\infty} \frac{k(4k-1)}{(k+1)^2(2k-3)(2k-1)^2} \left[\frac{1.3\ldots(2k-1)}{2.4\ldots 2k}\right]^2. \end{aligned}\right\} \qquad (3.21.35)$$

$$
\left.\begin{aligned}
&\int_0^R (\Theta_{j1})^2_{\xi=0}\, r\,dr = \\
&= R^4 \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{4k(4k-1)\,l\,(4l-1)}{(k+1)(2k-3)(2k-1)^2(l+1)(2l-3)(2l-1)^2}\times \\
&\times\left[\frac{1.3\ldots(2k-1)}{2.4\ldots2k}\right]^2\left[\frac{1.3\ldots(2l-1)}{2.4\ldots2l}\right]^2\int_0^R\left(\frac{r}{R}\right)^{2k+2l-2} r\,dr = \\
&= R^6 \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{2k(4k-1)\,l\,(4l-1)}{(k+1)(2k-3)(2k-1)^2(l+1)(2l-3)(2l-1)^2(k+l)}\times \\
&\times\left[\frac{1.3\ldots(2k-1)}{2.4\ldots2k}\right]^2\left[\frac{1.3\ldots(2l-1)}{2.4\ldots2l}\right]^2.
\end{aligned}\right\} \quad (3.21.35 \text{ cont'd})
$$

By summation of the numerical series, figuring in formulas (3.21.35), we find

$$
\int_0^R \left(\frac{\partial\Theta_{j1}}{\partial\xi}\Theta_{j1}\right)_{\xi=0} r\,dr = 0{,}1592R^5, \quad \int_0^R (\Theta_{j1})^2_{\xi=0}\, r\,dr = 0{,}1007R^6. \qquad (3.21.36)
$$

According to (3.21.5) and (3.21.36) in the considered case

$$
\lambda_{j1} = \frac{1{,}581}{R}. \qquad (3.21.37)
$$

In accordance with formulas (3.21.12), (3.21.34) and (3.21.37) function $F_{j2}(r)$, approximately determining the sought form of natural oscillations, in this case will be determined by series

$$
F_{j2}(r) = -1{,}581R \sum_{k=1}^{\infty} \frac{2k(4k-1)}{(k+1)(2k-3)(2k-1)^2}\times \left[\frac{1.3\ldots(2k-1)}{2.4\ldots2k}\right]^2\left(\frac{r}{R}\right)^{2k-1}. \qquad (3.21.38)
$$

approximation of ω_{j1} for sought frequency of natural oscillations ω_j according to (3.21.10) will be determined by relationship

$$\omega_{j1}=1,257\sqrt{\frac{w_{0x}-g_x}{R}}. \tag{3.21.39}$$

Calculation of further successive approximations in the considered example leads to function $F_j(r)$, graph of which is shown by the solid line in Fig. 3.14, and to formula

$$\omega_j=1,253\sqrt{\frac{w_{0x}-g_x}{R}} \tag{3.21.40}$$

for frequency of natural oscillations of free surface of fluid. The broken line in Fig. 3.14 shows the graph of function $F_{j2}(r)$, constructed in accordance with formula (3.21.38). As can be seen, in the given example for achievement of accuracy acceptable during engineering calculations, it is sufficient to determine the first approximations for eigenvalue λ_j and eigenfunction Θ_j.

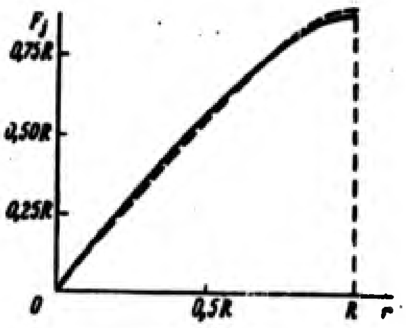

Fig. 3.14.

§ 22. Account of Energy Dissipation in Equation of Oscillations of Free Surface of Fluid

While investigating in § 18 the natural oscillations of free surface of fluid, for function of time $\beta_j(t)$ we obtained differential equation (3.18.9). If axial overload retains a constant value and the level of fluid in the fuel tank does not change, the frequency of natural oscillations ω_j according to (3.18.10) remains constant and the general solution of differential equation (3.18.9) in this instance can be represented in the form

$$\beta_j = C\cos(\omega_j t + \varphi), \tag{3.22.1}$$

where C and ϕ – arbitrary constants.

According to (3.22.1) in the considered case the free surface of the fluid will accomplish harmonic oscillations, which is found in accordance with the original assumption about the perfectness of fluid, on the basis of which all reasonings were conducted above. In actual conditions, because of internal friction and friction of fluid against the wetted surface of the tank, natural oscillations of the free surface of fluid will always be damped. Numerous experimental investigations show that oscillation energy dissipation can be taken into account, by substituting differential equation (3.18.9) by differential equation

$$\frac{d^2\beta_j}{dt^2} + \varepsilon_j \frac{d\beta_j}{dt} + \omega_j^2 \beta_j = 0, \tag{3.22.2}$$

where ε_j – positive coefficient, which determines the rapidity of damping of natural oscillations of the free surface of fluid.

Determination of damping factor ε_j by calculation is a very complex problem, and, as a rule, for finding it we use experimental methods. The question about experimental determination of the damping factor is comprehensively examined in book [14].

The damping factor depends on the amplitude of oscillations of the free surface of fluid, the damping factor rises as the amplitude of oscillations increases. In the question about stabilization of oscillations of free surfaces of fluids this circumstance plays an important role. For a tank with smooth walls the relationship of damping factor to the amplitude of oscillations is manifested weakly, however, with the presence of special intertank damping equipment this relationship becomes very essential.

In the process of experimental determination of the damping factor there is simultaneously established the experimental value of the frequency of natural oscillations of fluid.

§ 23. Conversion of Equations of Motion into Systems of Ordinary Differential Equations

We reduced the determination of functions $\psi_{jx}(\xi, r, t)$ and $\psi_{jy}(\xi, r, t)$ in § 20 to integration of ordinary differential equations (3.20.29). Having sought functions $\alpha_j(t)$ and $\beta_j(t)$ from these equations, we can further determine functions $\psi_{jx}(\xi, r, t)$ and $\psi_{jy}(\xi, r, t)$ by formulas (3.20.28). By using formulas (3.20.28), below we will convert equations of motion of the rocket in pitching and rolling planes, constructed in § 16, to systems of ordinary differential equations.

According to (3.20.28) the first two equations of (3.16.21) can be given the form

$$\begin{gathered} m(w_{0y}-g_y)+\pi\sum_{j=1}^{N}\varrho_j\frac{d^2\zeta_j}{dt^2}\int_0^{R_j}\left(\frac{\partial\theta_j}{\partial\xi}\right)_{\xi=0}r^2dr=F_y, \\ J_z\frac{d\omega_z}{dt}+\pi\sum_{j=1}^{N}\varrho_j\frac{d^2\zeta_j}{dt^2}\int_0^{R_j}\left(\frac{\partial\theta_j}{\partial\xi}\right)_{\xi=0}(x_j-k_j)r^2dr- \\ -\pi(w_{0x}-g_x)\sum_{j=1}^{N}\varrho_j\zeta_j\int_0^{R_j}\left(\frac{\partial\theta_j}{\partial\xi}\right)_{\xi=0}r^2dr=M_z \end{gathered} \quad (3.23.1)$$

(in the process of emptying the tank with number t there appears the relationship of eigenfunction Θ_j to time, however, with differentiation of function ψ_{jx} and ψ_{jy} this relationship can be disregarded, since the rate of change of function Θ_j in this instance is very small as compared to the rate of change of functions α_j and β_j).

By introducing meanings:

$$a_{jy} = \pi\rho_j \int_0^{R_j} \left(\frac{\partial\theta_j}{\partial\xi}\right)_{\xi=0} r^2 dr,$$

$$a_{j\theta} = \pi\rho_j \int_0^{R_j} \left(\frac{\partial\theta_j}{\partial\xi}\right)_{\xi=0} (x_j - k_j) r^2 dr, \tag{3.23.2}$$

it is possible to reduce equations (3.23.1) to the form

$$m(w_{0y} - g_y) + \sum_{j=1}^{N} a_{jy} \frac{d^2\beta_j}{dt^2} = F_y,$$

$$J_z \frac{dw_z}{dt} + \sum_{j=1}^{N} a_{j\theta} \frac{d^2\beta_j}{dt^2} - (w_{0x} - g_x) \sum_{j=1}^{N} a_{jy}\beta_j = M_z. \tag{3.23.3}$$

According to (3.18.13) formulas (3.23.2) can be given the form

$$a_{jy} = \pi\rho_j \int_0^{R_j} F_j(r) r^2 dr,$$

$$a_{j\theta} = \pi\rho_j \int_0^{R_j} F_j(r)(x_j - k_j) r^2 dr = a_{jy}\left(x_j - \frac{\int_0^{R} F_j(r) k_j r^2 dr}{\int_0^{R} F_j(r) r^2 dr}\right),$$

or, if we use meaning (3.20.30),

$$a_{jy} = \pi\rho_j \int_0^{R_j} F_j(r) r^2 dr, \quad a_{j\theta} = (x_j - \delta_j) a_{jy}. \tag{3.23.4}$$

Taking into account damping, the first of equations (3.20.29) assumes the form

$$\frac{d^2\beta_j}{dt^2}+\varepsilon_j\frac{d\beta_j}{dt}+\omega_j^2\beta_j+\lambda_j\left[w_{0y}-g_y+(x_j-\delta_j)\frac{d\omega_z}{dt}\right]\frac{\int_0^{R_j}F_j r^2 dr}{\int_0^{R_j}F_j^2 r dr}=0,$$

or

$$\frac{\pi\varrho_j}{\lambda_j}\int_0^{R_j}F_j^2 r dr\left(\frac{d^2\beta_j}{dt^2}+\varepsilon_j\frac{d\beta_j}{dt}+\omega_j^2\beta_j\right)+\pi\varrho_j\int_0^{R_j}F_j r^2 dr\times$$
$$\times\left[w_{0y}-g_y+(x_j-\delta_j)\frac{d\omega_z}{dt}\right]=0. \qquad (3.23.5)$$

By introducing meaning

$$\mu_j=\frac{\pi\varrho_j}{\lambda_j}\int_0^{R_j}F_j^2 r dr \qquad (3.23.6)$$

and by using meanings (3.23.4), it is possible to reduce equation (3.23.5) to the form

$$\mu_j\left(\frac{d^2\beta_j}{dt^2}+\varepsilon_j\frac{d\beta_j}{dt}+\omega_j^2\beta_j\right)+a_{jy}(w_{0y}-g_y)+a_{j\omega}\frac{d\omega_z}{dt}=0. \qquad (3.23.7)$$

According to (3.23.3) and (3.23.7) the equations of motion of the rocket in pitching plane can be represented in the form

$$\begin{aligned}
&m(w_{0y}-g_y)+\sum_{j=1}^{N}a_{jy}\frac{d^2\beta_j}{dt^2}=F_y,\\
&J_z\frac{d\omega_z}{dt}+\sum_{j=1}^{N}a_{j\omega}\frac{d^2\beta_j}{dt^2}-(w_{0x}-g_x)\sum_{j=1}^{N}a_{jy}\beta_j=M_z,\\
&\mu_j\left(\frac{d^2\beta_j}{dt^2}+\varepsilon_j\frac{d\beta_j}{dt}+\omega_j^2\beta_j\right)+a_{jy}(w_{0y}-g_y)+\\
&+a_{j\omega}\frac{d\omega_z}{dt}=0,\quad j=1,2,\ldots,N.
\end{aligned} \qquad (3.23.8)$$

The system of differential equations (3.23.8) determines the motion, accomplished by the rocket body in the pitching plane, and time functions $\beta_1(t)$, $\beta_2(t)$, ..., $\beta_N(t)$, characterizing oscillations of free surfaces of fluids in the fuel tanks.

Functions $F_1(r)$, $F_2(r)$, ..., $F_N(r)$, figuring in formulas (3.23.4) and (3.23.6), in the process of calculation of forms of natural oscillations of fluids are determined to within arbitrary factors, which do not depend on variable r. In order to give uniqueness to coefficients of differential equations (3.23.8), for functions $F_j(r)$, $j = 1, 2, ..., N$, let us establish additional condition

$$\frac{dF_j}{dr}=1 \text{ when } r=0, \qquad (3.23.9)$$

in the presence of which the system of functions $F_1(r)$, $F_2(r)$, ..., $F_N(r)$ will already be determined uniquely. According to (3.18.12) in this instance functions $f_j(r, \alpha, t)$, determining the deflections of points of free surfaces of fluids from planes σ_j, will always satisfy condition

$$\left(\frac{\partial f_j}{\partial r}\right)_{r=0}=\beta_j(t)\cos\alpha. \qquad (3.23.10)$$

Specifically in half-plane $\alpha = 0$, normal to nodal line, there will exist equality

$$\beta_j(t)=\left(\frac{\partial f_j}{\partial r}\right)_{r=0}. \qquad (3.23.11)$$

With small oscillations of free surface of fluid in j-th tank the right side of equality (3.23.11) represents the angle between the plane tangent to free surface σ_j^* at point $r = 0$, and plane σ_j. Thus, when fulfilling condition (3.23.9) function $\beta_j(t)$ will determine the angle of rotation of free surface of fluid in j-th tank around its nodal line in the section of free surface normal to this line (Fig. 3.15).

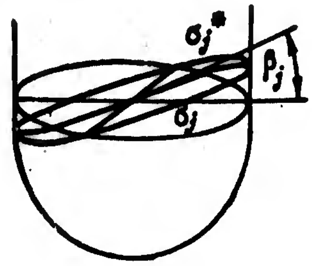

Fig. 3.15.

Let us turn to conversion of equations of motion of the rocket in rolling plane. According to (3.20.28) the first of equations (3.16.24) can be given the form

$$J_x \frac{d\omega_x}{dt} + \pi \sum_{j=1}^{N} \varrho_j r_j \frac{d^2 a_j}{dt^2} \int_0^{R_j} \left(\frac{\partial \Theta_j}{\partial \xi} \right)_{\xi=0} r^2 dr = M_x,$$

or

$$J_x \frac{d\omega_x}{dt} + \sum_{j=1}^{N} a_{j\gamma} \frac{d^2 a_j}{dt^2} = M_x, \qquad (3.23.12)$$

where

$$a_{j\gamma} = \pi \varrho_j r_j \int_0^{R_j} \left(\frac{\partial \Theta_j}{\partial \xi} \right)_{\xi=0} r^2 dr. \qquad (3.23.13)$$

In accordance with formulas (3.23.2) and (3.23.13) coefficients $\alpha_{j\gamma}$ and α_{jy} are connected by relationship

$$a_{j\gamma} = r_j a_{jy}. \qquad (3.23.14)$$

Taking damping into account the second of equations (3.20.29) takes the form

$$\frac{d^2a_j}{dt^2}+\varepsilon_j\frac{da_j}{dt}+\omega_j^2a_j+\lambda_jr_j\frac{d\omega_x}{dt}\frac{\int_0^{R_j}F_jr^2dr}{\int_0^{R_j}F_j^2rdr}=0,$$

or

$$\frac{\pi Q_j}{\lambda_j}\int_0^{R_j}F_j^2rdr\left(\frac{d^2a_j}{dt^2}+\varepsilon_j\frac{da_j}{dt}+\omega_j^2a_j\right)+\pi Q_jr_j\frac{d\omega_x}{dt}\int_0^{R_j}F_jr^2dr=0. \quad (3.23.15)$$

According to (3.23.4) and (3.23.14) there should exist equality

$$a_{j1}=\pi Q_jr_j\int_0^{R_j}F_j(r)r^2dr. \quad (3.23.16)$$

In accordance with formulas (3.23.6) and (3.23.16) equation (3.23.15) can be given the form

$$\mu_j\left(\frac{d^2a_j}{dt^2}+\varepsilon_j\frac{da_j}{dt}+\omega_j^2a_j\right)+a_{j1}\frac{d\omega_x}{dt}=0. \quad (3.23.17)$$

According to (3.23.12) and (3.23.17) the equations of motion of the rocket in rolling plane can be represented in the form

$$\begin{gathered}J_x\frac{d\omega_x}{dt}+\sum_{j=1}^{N}a_{j1}\frac{d^2a_j}{dt^2}=M_x,\\ \mu_j\left(\frac{d^2a_j}{dt^2}+\varepsilon_j\frac{da_j}{dt}+\omega_j^2a_j\right)+a_{j1}\frac{d\omega_x}{dt}=0,\\ j=1,2,\ldots,N.\end{gathered} \quad (3.23.18)$$

System of differential equations (3.23.18) determines rotation of the rocket body around its longitudinal axis and time functions $\alpha_1(t)$, $\alpha_2(t)$, ..., $\alpha_N(t)$, characterizing the oscillations of free surfaces of fluids, caused by this rotation. With fulfilling of condition (3.23.9), function $\alpha_j(t)$ determines the angle of rotation of free surface of fluid in j-th tank around its nodal line in the section of free surface normal to this line.

§ 24. Equations of Disturbed Motion, Considering the Mobility of Liquid Components of Propellant

Let us turn now to construction of differential equations of disturbed motion of the rocket, considering the mobility of liquid components of propellant. Let us first examine the motion of the rocket in pitching plane.

Let us assume $\beta_1(t), \beta_2(t), \ldots, \beta_N(t)$ – functions characterizing oscillations of free surfaces of fluids, accomplished by them during undisturbed motion of the rocket in the pitching plane, and $\beta'_1(t), \beta'_2(t), \ldots, \beta'_N(t)$ – functions characterizing oscillations of free surfaces, appearing with disturbed motion of the rocket. According to (3.23.8) functions $\beta_1(t), \beta_2(t), \ldots, \beta_N(t)$ must satisfy differential equations

$$\mu_j\left(\frac{d^2\beta_j}{dt^2}+\varepsilon_j\frac{d\beta_j}{dt}+\omega_j^2\beta_j\right)+a_{jy}(w_{0y}-g_y)+a_{jz}\frac{d\omega_z}{dt}=0,$$
$$j=1, 2, \ldots, N, \tag{3.24.1}$$

and functions $\beta'_1(t), \beta'_2(t), \ldots, \beta'_N(t)$ – equations

$$\mu_j\left(\frac{d^2\beta'_j}{dt^2}+\varepsilon_j\frac{d\beta'_j}{dt}+\omega_j^2\beta'_j\right)+a_{jy}(w'_{0y'}-g_{y'})+a_{jz}\frac{d\omega'_z}{dt}=0,$$
$$j=1, 2, \ldots, N, \tag{3.24.2}$$

where $w'_{0y'} - g_{y'}$ – apparent lateral acceleration in the pitching plane, appearing in the process of disturbed motion of the rocket; ω'_z – angular velocity of rotation of the rocket around axis z, corresponding to disturbed motion [in view of the smallness of oscillations of free surfaces of fluids assumed by us, in equations (3.24.2) it is possible to disregard small changes of coefficients of these differential equations, which can appear during transition from undisturbed to disturbed motion of the rocket]. In accordance with equations (3.24.1) and (3.24.2) the differences of

$$\Delta\beta_j=\beta'_j-\beta_j, \quad j=1, 2, \ldots, N, \tag{3.24.3}$$

must satisfy differential equations

$$\mu_j\left(\frac{d^2\Delta\beta_j}{dt^2}+\varepsilon_j\frac{d\Delta\beta_j}{dt}+\omega_j^2\Delta\beta_j\right)+a_{jy}(w'_{0y'}-g_{y'}-w_{0y}+g_y)+$$
$$+a_{jz}\frac{d(\omega'_z-\omega_z)}{dt}=0,\ j=1,2,\ldots,N. \qquad (3.24.4)$$

By using transition matrix (1.12.5), we find:

$$w'_{0y'}-g'_{y'}=w'_{0y}-g_y-(w_{0x}-g_x)\Delta\vartheta=w_{0y}-g_y-$$
$$-(w_{0x}-g_x)\Delta\vartheta+\frac{d\Delta V_y}{dt},$$
$$\omega'_z=\frac{d\vartheta'}{dt}=\frac{d\vartheta}{dt}+\frac{d\Delta\vartheta}{dt}=\omega_z+\frac{d\Delta\vartheta}{dt} \qquad (3.24.5)$$

(with error of the second order of smallness $\vec{v}'_0\cong\vec{V}'_C=\vec{V}_C+\Delta\vec{V}_C$).

According to (3.24.5) differential equations (3.24.4) can be given the form

$$\mu_j\left(\frac{d^2\Delta\beta_j}{dt^2}+\varepsilon_j\frac{d\Delta\beta_j}{dt}+\omega_j^2\Delta\beta_j\right)+a_{jy}\frac{d\Delta V_y}{dt}+a_{jz}\frac{d^2\Delta\vartheta}{dt^2}-$$
$$-a_{jy}(w_{0x}-g_x)\Delta\vartheta=0,\quad j=1,2,\ldots,N. \qquad (3.24.6)$$

The first two equations of (3.23.8) can be represented in the form

$$m(w_{0y}-g_y)=F_y+F_{\varkappa y},$$
$$J_z\frac{d\omega_z}{dt}=M_z+M_{\varkappa z}, \qquad (3.24.7)$$

where

$$F_{\varkappa y}=-\sum_{j=1}^{N}a_{jy}\frac{d^2\beta_j}{dt^2}\quad\text{and}\quad M_{\varkappa z}=-\sum_{j=1}^{N}a_{jz}\frac{d^2\beta_j}{dt^2}+$$
$$+(w_{0x}-g_x)\sum_{j=1}^{N}a_{jy}\beta_j \qquad (3.24.8)$$

– the force and moment respectively, appearing as a result of mobility of fluids.

In Chapter I, not considering force $F_{ж\,y}$ and moment $M_{ж\,z}$ we obtained equations of disturbed motion:

$$m\frac{d\Delta V_y}{dt}+v_y\frac{d\Delta\vartheta}{dt}+c_{yy}\Delta V_y+c_{y\vartheta}\Delta\vartheta=c_{y\delta}\Delta\delta_\varphi+\Delta F_y,$$
$$J_z\frac{d^2\Delta\vartheta}{dt^2}+\mu_z\frac{d\Delta\vartheta}{dt}+c_{\vartheta y}\Delta V_y+c_{\vartheta\vartheta}\Delta\vartheta=c_{\vartheta\delta}\Delta\delta_\varphi+\Delta M_z. \qquad (3.24.9)$$

With allowance for force $F_{ж\,y}$ and moment $M_{ж\,y}$ appearing during the transition from

$$m\frac{d\Delta V_y}{dt}+v_y\frac{d\Delta\vartheta}{dt}+c_{yy}\Delta V_y+c_{y\vartheta}\Delta\vartheta=c_{y\delta}\Delta\delta_\varphi+\Delta F_y+\Delta F_{жy},$$
$$J_z\frac{d^2\Delta\vartheta}{dt^2}+\mu_z\frac{d\Delta\vartheta}{dt}+c_{\vartheta y}\Delta V_y+c_{\vartheta\vartheta}\Delta\vartheta=c_{\vartheta\delta}\Delta\delta_\varphi+\Delta M_z+\Delta M_{жz}, \qquad (3.24.10)$$

where $\Delta F_{ж\,y}$ and $\Delta M_{ж\,z}$ – changes of force $F_{ж\,y}$ and moment $M_{ж\,z}$, appearing during the transition from undisturbed motion of the rocket to its disturbed motion.

According to (3.24.8) there will exist equalities:

$$\Delta F_{жy}=-\sum_{j=1}^{N}a_{jy}\frac{d^2\Delta\beta_j}{dt^2},$$
$$\Delta M_{жz}=-\sum_{j=1}^{N}a_{j\vartheta}\frac{d^2\Delta\beta_j}{dt^2}+(w_{0x}-g_x)\sum_{j=1}^{N}a_{jy}\Delta\beta_j \qquad (3.24.11)$$

[in view of the assumed smallness of oscillations of free surfaces of fluids, it is possible to disregard small changes of coefficients of differential expressions (3.24.8), which can appear during the transition from undisturbed to disturbed motion of the rocket].

By substituting (3.24.11) in (3.24.10) and to the obtained equations adding differential equations (3.24.6) we obtain system of equations

$$
\begin{aligned}
& m\frac{d\Delta V_y}{dt}+v_y\frac{d\Delta\vartheta}{dt}+c_{yy}\Delta V_y+c_{y\vartheta}\Delta\vartheta+\sum_{j=1}^{N}a_{jy}\frac{d^2\Delta\beta_j}{dt^2}= \\
& =c_{y\delta}\Delta\delta_\varphi+\Delta F_y, \\
& J_z\frac{d^2\Delta\vartheta}{dt^2}+\mu_z\frac{d\Delta\vartheta}{dt}+c_{\vartheta y}\Delta V_y+c_{\vartheta\vartheta}\Delta\vartheta+\sum_{j=1}^{N}a_{j\vartheta}\frac{d^2\Delta\beta_j}{dt^2}- \\
& -(w_{0x}-g_x)\sum_{j=1}^{N}a_{jy}\Delta\beta_j=c_{\vartheta\delta}\Delta\delta_\varphi+\Delta M_z, \\
& \mu_j\left(\frac{d^2\Delta\beta_j}{dt^2}+\varepsilon_j\frac{d\Delta\beta_j}{dt}+\omega_j^2\Delta\beta_j\right)+a_{jy}\frac{d\Delta V_y}{dt}+a_{j\vartheta}\frac{d^2\Delta\vartheta}{dt^2}- \\
& -a_{jy}(w_{0x}-g_x)\Delta\vartheta=0,\quad j=1,\ 2,\ldots,N.
\end{aligned}
\qquad (3.24.12)
$$

Equations (3.24.12) are differential equations of disturbed motion of the rocket in the pitching plane, considering the mobility of liquid components of propellant.

Let us turn to examination of rotation of the rocket around its longitudinal axis. According to (3.23.18) disturbances $\Delta\alpha_j$, appearing with transition from undisturbed to disturbed motion of the rocket, must satisfy differential equations:

$$\mu_j\left(\frac{d^2\Delta\alpha_j}{dt^2}+\varepsilon_j\frac{d\Delta\alpha_j}{dt}+\omega_j^2\Delta\alpha_j\right)+a_{j\gamma}\frac{d\Delta\omega_x}{dt}=0,\quad j=1,\ 2,\ldots,N,$$

or

$$\mu_j\left(\frac{d^2\Delta\alpha_j}{dt^2}+\varepsilon_j\frac{d\Delta\alpha_j}{dt}+\omega_j^2\Delta\alpha_j\right)+a_{j\gamma}\frac{d^2\Delta\gamma}{dt^2}=0,\quad j=1,\ 2,\ldots,N, \qquad (3.24.13)$$

since $\omega_x=\frac{d\gamma}{dt}$.

The first of equations (3.23.18) can be given the form

$$J_x\frac{d\omega_x}{dt}=M_x+M_{x\,r}, \qquad (3.24.14)$$

where

$$M_{\text{ж} x} = -\sum_{j=1}^{N} a_{j\gamma} \frac{d^2 a_j}{dt^2}. \qquad (3.24.15)$$

In Chapter I, while not considering moment $M_{\text{ж} x}$ in equation (3.24.14), we obtained equation of disturbed motion

$$J_x \frac{d^2 \Delta\gamma}{dt^2} + \mu_x \frac{d\Delta\gamma}{dt} = c_{\gamma\delta} \Delta\delta_\gamma + \Delta M_x. \qquad (3.24.16)$$

With allowance for moment $M_{\text{ж} x}$ equation (3.24.16) takes the form

$$J_x \frac{d^2 \Delta\gamma}{dt^2} + \mu_x \frac{d\Delta\gamma}{dt} = c_{\gamma\delta} \Delta\delta_\gamma + \Delta M_x + \Delta M_{\text{ж} x}, \qquad (3.24.17)$$

where

$$\Delta M_{\text{ж} x} = -\sum_{j=1}^{N} a_{j\gamma} \frac{d^2 \Delta a_j}{dt^2} \qquad (3.24.18)$$

according to (3.24.15). By substituting (3.24.18) in (3.24.17) and adding differential equations (3.24.13) to the obtained equation, we obtain system of equations

$$\begin{gathered} J_x \frac{d^2 \Delta\gamma}{dt^2} + \mu_x \frac{d\Delta\gamma}{dt} + \sum_{j=1}^{N} a_{j\gamma} \frac{d^2 \Delta a_j}{dt^2} = c_{\gamma\delta} \Delta\delta_\gamma + \Delta M_x, \\ \mu_j \left(\frac{d^2 \Delta a_j}{dt^2} + \varepsilon_j \frac{d\Delta a_j}{dt} + \omega_j^2 \Delta a_j \right) + a_{j\gamma} \frac{d^2 \Delta\gamma}{dt^2} = 0, \\ j = 1, 2, \ldots, N. \end{gathered} \qquad (3.24.19)$$

Equations (3.24.19) represent differential equations of disturbed motion of the rocket in rolling plane, considering the mobility of liquid components of propellant.

§ 25. Calculation of Coefficients of Differential Equations of Disturbed Motion

The question about calculation of coefficients $c_{yy}, c_{y\vartheta}, c_{y1}, c_{\vartheta y}, c_{\vartheta\vartheta}, c_{\vartheta 1}, c_{\gamma 1}, \nu_y, \mu_x, \mu_z$, figuring in equations of disturbed motion (3.24.12) and (3.24.19), was examined by us in Chapter I. Below we will provide a summary of working formulas for calculation of the remaining coefficients of these differential equations — $a_{jy}, a_{j\vartheta}, a_{j\gamma}, \mu_j, \omega_j, J_x, J_z$ (calculation of moments of inertia J_x and J_z in Chapter I was considered without allowing for mobility of liquid components of propellant).

The basis of the necessary calculations is solution of differential equations

$$\frac{\partial^2\theta_j}{\partial\xi^2}+\frac{\partial^2\theta_j}{\partial r^2}+\frac{1}{r}\frac{\partial\theta_j}{\partial r}-\frac{\theta_j}{r^2}=0,\quad j=1,\ 2,\ldots,N \qquad (3.25.1)$$

with boundary conditions

$$\frac{\partial\theta_j}{\partial n}=r\cos\chi+\xi\sin\chi \text{ на } l_j,\quad j=1,\ 2,\ldots,N \qquad (3.25.2)$$

and finding the first eigenvalues and the first eigenfunctions of boundary value problems, formed by differential equations

$$\frac{\partial^2\Theta_j}{\partial\xi^2}+\frac{\partial^2\Theta_j}{\partial r^2}+\frac{1}{r}\frac{\partial\Theta_j}{\partial r}-\frac{\Theta_j}{r^2}=0,\quad j=1,\ 2,\ldots,N \qquad (3.25.3)$$

and boundary conditions:

$$\left.\begin{aligned}&\frac{\partial\Theta_j}{\partial n}=0 \quad \text{on } \Gamma_j\\ &\frac{\partial\Theta_j}{\partial n}=\lambda_j\Theta_j \text{ when } \xi=0,\ 0<r<R_j\end{aligned}\right\}\ j=1,\ 2,\ldots,N \qquad (3.25.4)$$

(see §§ 12 and 17). By applying the method of successive approximations, given in § 21, for calculation of eigenvalues λ_j and eigenfunctions Θ_j to solution of differential equations (3.25.1) with boundary conditions of form

$$\frac{\partial\theta_j}{\partial n} = f_j(s) \text{ on } l_j, \; j = 1, 2, \ldots, N, \qquad (3.25.5)$$

where $f_j(s)$ – assigned functions of length of arc s of contour l_j.

By knowing functions θ_j and Θ_j, $j = 1, 2, \ldots, N$, and numbers $\lambda_1, \lambda_2, \ldots, \lambda_N$, it is possible to calculate the coefficients of differential equations of disturbed motion a_{jy}, μ_j, ω_j, J_x, J_z by formulas:

$$\begin{aligned}
a_{jy} &= \pi\varrho_j \int_0^{R_j} F_j(r) r^2 dr, \\
\mu_j &= \frac{\pi\varrho_j}{\lambda_j} \int_0^{R_j} F_j^2(r) r dr, \\
\omega_j &= \sqrt{\lambda_j (w_{0x} - g_x)}, \\
J_x &= \int_{V_0} (y^2 + z^2) \varrho dv + \sum_{j=1}^{N} m_j (y_j^2 + z_j^2), \\
J_z &= \int_{V_0} (x^2 + y^2) \varrho dv + \sum_{j=1}^{N} m_j (x_C^{(j)2} + y_j^2 + h_j^2),
\end{aligned} \qquad (3.25.6)$$

where

$$\begin{aligned}
F_j(r) &= \left(\frac{\partial\Theta_j}{\partial\xi}\right)_{\xi=0}, \\
h_j &= \left[\frac{\pi}{v_j} \int_{l_j} \theta_j \frac{\partial\theta_j}{\partial n} r ds - (x_j - x_C^{(j)})^2\right]^{\frac{1}{2}}
\end{aligned} \qquad (3.25.7)$$

(see §§ 14, 18 and 23).

By knowing coefficients a_{jy}, $j = 1, 2, \ldots, N$, it is possible to calculate coefficients $a_{j\vartheta}$ and $a_{j\psi}$, $j = 1, 2, \ldots, N$ by formulas:

$$a_{j\vartheta} = (x_j - \delta_j) a_{jy}, \; a_{j\psi} = r_j a_{jy}, \qquad (3.25.8)$$

where

$$\delta_j = \frac{\int_0^{R_j} F_j k_j r^2 dr}{\int_0^{R_j} F_j r^2 dr}, \qquad k_j = -\frac{(\theta_j)_{\xi=0}}{r} \qquad (3.25.9)$$

(see §§ 15, 20 and 23),

In all the reasonings, conducted in this chapter, for simplicity we assumed that the axis of fuel tank always intersects the liquid mass, which is located in this tank. If the tank with number j is doughnut-shaped in the second of boundary conditions (3.25.4) interval $0 < r < R_j$ must be replaced by interval $R_j^{(0)} < r < R_j$ where $R_j^{(0)}$ - internal radius of flat figure σ_j, in this case having the shape of a ring. In this case there is dropped the requirement of regularity of solutions of differential equations (3.25.1) and (3.25.3) on the axis of fuel tank $r = 0$. In formulas (3.25.6) and (3.25.9) for a_{jy}, μ_j and δ_j the interval of integration $(0, R_j)$ in the considered case should be replaced by interval $(R_j^{(0)}, R_j)$.

Until the tank with number j is emptied, coefficients $a_{jy}, a_{j\theta}, a_{j\tau}$ and μ_j retain constant values. In the process of emptying of j-th tank there appears relationship of these coefficients to time t.

As was shown above in § 22, in the process of experimental determination of damping facotrs ε_j there is simultaneously achieved experimental determination of the frequencies of natural oscillations of free surfaces of fluids ω_j. By knowing frequencies $\omega_1, \omega_2, \ldots, \omega_N$, it is possible to approximately calculate the remaining coefficients of differential equations of disturbed motion, while not resorting to solution of differential equations (3.25.1) and (3.25.3) with boundary conditions (3.25.2) and (3.25.4). According to (3.20.15) for the first tone of natural oscillations of free surface it is possible to approximately assume

$$F_j(r) = C_j r. \tag{3.25.10}$$

In order that condition (3.23.9) would be fulfilled, constant C_j must be equal to one. Assuming in formulas (3.25.6) for coefficients a_{jy} and μ_j

$$F_j(r) = r, \tag{3.25.11}$$

we obtain approximate relationships:

$$a_{jy}=\pi\varrho_j\int_0^{R_j} r^3dr=\frac{\pi}{4}\varrho_jR_j^4,$$

$$\mu_j=\frac{\pi\varrho_j}{\lambda_j}\int_0^{R_j} r^3dr=\frac{\pi\varrho_j}{4\lambda_j}R_j^4. \qquad (3.25.12)$$

According to (3.25.6) there must exist equality

$$\lambda_j=\frac{\omega_j^2}{w_{0x}-g_x}. \qquad (3.25.13)$$

By using relationship (3.25.13), it is possible to reduce formulas (3.25.12) to the form

$$a_{jy}=\frac{\pi\varrho_jR_j^4}{4},\quad \mu_j=\frac{\pi\varrho_jR_j^4}{4\omega_j^2}(w_{0x}-g_x). \qquad (3.25.14)$$

Calculation of moment of inertia J_x without allowing for mobility of liquid components of propellant leads to overstated results. Thus,

$$J_x<\int_{V_0}(x^2+y^2)+\varrho dv+\sum_{j=1}^{N}\varrho_j\int_{V_j}(x^2+y^2)dv. \qquad (3.25.15)$$

On the other hand, according to (3.25.6)

$$J_x>\int_{V_0}(x^2+y^2)\varrho dv+\sum_{j=1}^{N}m_j(x_C^{(j)2}+y_j^2). \qquad (3.25.15a)$$

In accordance with inequalities (3.25.15) and (3.25.15a) approximate formula

$$J_x=\int_{V_0}(x^2+y^2)\varrho dv+\frac{1}{2}\sum_{j=1}^{N}\left[\varrho_j\int_{V_j}(x^2+y^2)dv+m_j(x_C^{(j)2}+y_j^2)\right] \qquad (3.25.16)$$

gives error with absolute value not exceeding the sum of

$$\frac{1}{2}\sum_{j=1}^{N}\left[\varrho_j\int_{V_j}(x^2+y^2)dv-m_j(x_C^{(j)2}+y_j^2)\right]. \qquad (3.25.17)$$

The sum of (3.25.17) usually comprises only several percent of the approximate value of moment of inertia J_z, determined by formula (3.25.16) By substituting in (3.25.6) the accurate formula for moment of inertia J_z by approximate formula (3.25.16), we obtain working formulas for moments of inertia:

$$J_x=\int_{V_0}(y^2+z^2)\varrho dv+\sum_{j=1}^{N}m_j(y_j^2+z_j^2),$$
$$J_z=\int_{V_0}(x^2+y^2)\varrho dv+\frac{1}{2}\sum_{j=1}^{N}\left[\varrho_j\int_{V_j}(x^2+y^2)dv+m_j(x_j^{(j)2}+y_j^2)\right]. \qquad (3.25.18)$$

With substantial withdrawal of the free surface of fluid in j-th tank from plane $x = 0$, passing through origin of fixed system of coordinates, quantity δ_j is usually small in comparison with the absolute quantity of coordinate x_j. If, however, the free surface of fluid in j-th tank is close to plane $x = 0$, coefficient $a_{j\varphi}$ in view of its smallness does not play a substantial role during investigation of the stability of motion of the rocket. By disregarding in (3.25.8) quantity δ_j we obtain working formulas:

$$a_{j\varphi}=x_j a_{jy}, \quad a_{j\gamma}=r_j a_{jy}. \qquad (3.25.19)$$

Thus, having obtained experimental values of damping factors ε_j and frequencies of natural oscillations ω_j, it is possible to orientate the investigation of stability of motion of the rocket, by using simple working formulas (3.25.14), (3.25.18) and (3.25.19) during calculation of remaining coefficients of differential equations of disturbed motion.

§ 26. Frequency Characteristics of the Rocket as the Object of Automatic Control

In Chapter II during investigation of the stability of motion of the rocket we described the connection between input signal of automatic stabilization control and its output signal by the simplest linear differential equations. If we consider the mobility of liquid components of propellant in the process of investigation of stability of motion of the rocket it is necessary to consider oscillations of its body in a substantially greater range of frequencies; for such a wide frequency range the description of "law of control" by differential equations becomes very complex. In connection with this, to investigate the stability of motion of the rocket with allowance for mobility of liquid components of propellant we usually use the frequency methods of the theory of automatic control.

In this paragraph we consider the question about calculation of these frequency characteristics, in the two subsequent paragraphs we indicated possible means of providing stability of motion of the rocket with allowance for mobility of liquid components of propellant.

Just as in Chapter II, during investigation of the stability of motion of the rocket we will "quench" the coefficients of differential equations of disturbed motion, considering that in the examined time interval these coefficients keep values corresponding to a certain fixed initial moment of time $t = \tau$.

Let us first examine motion of the rocket in pitching plane. For construction of the sought frequency characteristics in equations of disturbed motion (3.24.12) let us assume

$$\Delta F_y=0, \quad \Delta M_z=0, \quad \Delta\delta_\varphi=e^{i\omega t} \tag{3.26.1}$$

and, considering the coefficients of these differential equations constant, we will seek unknown functions ΔV_y, $\Delta\vartheta$, $\Delta\beta_j$, $j=1, 2, \ldots, N$, in the form

$$\Delta V_y=Ve^{i\omega t}, \quad \Delta\vartheta=\theta e^{i\omega t}, \quad \Delta\beta_j=B_je^{i\omega t}, \quad j=1, 2, \ldots N. \tag{3.26.2}$$

By substituting (3.26.1) and (3.26.2) in (3.24.12), for factors V, θ and B_j, $j = 1, 2, \ldots, N$ we obtain system of algebraic equations

$$
\begin{aligned}
&(c_{yy}+i\omega m)V+(c_{y\theta}+i\nu_y\omega)\theta-\omega^2\sum_{j=1}^{N}a_{jy}B_j=c_{yz},\\
&c_{\theta y}V+(c_{\theta\theta}+i\mu_z\omega-\omega^2 J_z)\theta-\sum_{j=1}^{N}[(w_{0x}-g_x)a_{jy}+\\
&+\omega^2 a_{j\theta}]B_j=c_{\theta z},\\
&\mu_j(\omega_j^2-\omega^2+i\varepsilon_j\omega)B_j+ia_{jy}\omega V-[(w_{0x}-g_x)a_{jy}+\\
&+\omega^2 a_{j\theta}]\theta=0,\quad j=1,2,\ldots,N.
\end{aligned}
\qquad (3.26.3)
$$

According to (3.26.3) there must exist equalities

$$
B_j=\frac{-ia_{jy}\omega V+[(w_{0x}-g_x)a_{jy}+\omega^2 a_{j\theta}]\theta}{\mu_j(\omega_j^2-\omega^2+i\varepsilon_j\omega)},\quad j=1,2,\ldots,N. \qquad (3.26.4)
$$

By substituting (3.26.4) in the first two equations of (3.26.3) for unknown V and θ we obtain a system of two algebraic equations

$$
\begin{aligned}
&\left[c_{yy}+i\omega m+\sum_{j=1}^{N}\frac{ia_{jy}^2\omega^3}{\mu_j(\omega_j^2-\omega^2+i\varepsilon_j\omega)}\right]V+\Big\{c_{y\theta}+i\nu_y\omega-\\
&-\sum_{j=1}^{N}\frac{[(w_{0x}-g_x)a_{jy}+\omega^2 a_{j\theta}]\omega^2 a_{jy}}{\mu_j(\omega_j^2-\omega^2+i\varepsilon_j\omega)}\Big\}\theta=c_{yz},\\
&\Big\{c_{\theta y}+\sum_{j=1}^{N}\frac{[(w_{0x}-g_x)a_{jy}+\omega^2 a_{j\theta}]ia_{jy}\omega}{\mu_j(\omega_j^2-\omega^2+i\varepsilon_j\omega)}\Big\}V+\Big\{c_{\theta\theta}+i\mu_z\omega-\\
&-\omega^2 J_z-\sum_{j=1}^{N}\frac{[(w_{0x}-g_x)a_{jy}+\omega^2 a_{j\theta}]^2}{\mu_j(\omega_j^2-\omega^2+i\varepsilon_j\omega)}\Big\}\theta=c_{\theta z}.
\end{aligned}
\qquad (3.26.5)
$$

By introducing meanings:

$$
\begin{aligned}
&c_{yy}+i\omega m+\sum_{j=1}^{N}\frac{ia_{jy}^2\omega^3}{\mu_j(\omega_j^2-\omega^2+i\varepsilon_j\omega)}=a_{11}(\omega)+ib_{11}(\omega),\\
&c_{y\theta}+i\nu_y\omega-\sum_{j=1}^{N}\frac{[(w_{0x}-g_x)a_{jy}+\omega^2 a_{j\theta}]\omega^2 a_{jy}}{\mu_j(\omega_j^2-\omega^2+i\varepsilon_j\omega)}=a_{12}(\omega)+ib_{12}(\omega),\\
&c_{\theta y}+\sum_{j=1}^{N}\frac{[(w_{0x}-g_x)a_{jy}+\omega^2 a_{j\theta}]ia_{jy}\omega}{\mu_j(\omega_j^2-\omega^2+i\varepsilon_j\omega)}=a_{21}(\omega)+ib_{21}(\omega),\\
&c_{\theta\theta}+i\mu_z\omega-\omega^2 J_z-\sum_{j=1}^{N}\frac{[(w_{0x}-g_x)a_{jy}+\omega^2 a_{j\theta}]^2}{\mu_j(\omega_j^2-\omega^2+i\varepsilon_j\omega)}=\\
&=a_{22}(\omega)+ib_{22}(\omega),
\end{aligned}
\qquad (3.26.6)
$$

it is possible to reduce equations (3.26.5) o form

$$(a_{11}+ib_{11})V+(a_{12}+ib_{12})\theta=c_{y\theta},$$
$$(a_{21}+ib_{21})V+(a_{22}+ib_{22})\theta=c_{\theta\theta}. \qquad (3.26.7)$$

By separating the real and imaginary pa: ts in the left sides of relationships (3.26.6), for functions $a_{jk}(\omega)$ and $b_{jk}(\omega)$, $j=1,2, k=1,2,$ we obtain working formulas:

$$a_{11}(\omega)=c_{yy}+\sum_{j=1}^{N}\frac{\varepsilon_j\omega^4a_{jy}^2}{\mu_j\left[(\omega_j^2-\omega^2)^2+\varepsilon_j^2\omega^2\right]},$$
$$b_{11}(\omega)=\omega m+\sum_{j=1}^{N}\frac{(\omega_j^2-\omega^2)\,\omega^3a_{jy}^2}{\mu_j\left[(\omega_j^2-\omega^2)^2+\varepsilon_j^2\omega^2\right]},$$
$$a_{12}(\omega)=c_{y\theta}-\sum_{j=1}^{N}\frac{(\omega_j^2-\omega^2)\,\omega^2a_{jy}\left[(w_{0x}-g_x)\,a_{jy}+\omega^2a_{j\theta}\right]}{\mu_j\left[(\omega_j^2-\omega^2)^2+\varepsilon_j^2\omega^2\right]},$$
$$b_{12}(\omega)=\nu_y\omega+\sum_{j=1}^{N}\frac{\varepsilon_j\omega^3a_{jy}\left[(w_{0x}-g_x)\,a_{jy}+\omega^2a_{j\theta}\right]}{\mu_j\left[(\omega_j^2-\omega^2)^2+\varepsilon_j^2\omega^2\right]},$$
$$a_{21}(\omega)=c_{\theta y}+\sum_{j=1}^{N}\frac{\varepsilon_j\omega^2a_{jy}\left[(w_{0x}-g_x)\,a_{jy}+\omega^2a_{j\theta}\right]}{\mu_j\left[(\omega_j^2-\omega^2)^2+\varepsilon_j^2\omega^2\right]}, \qquad (3.26.8)$$
$$b_{21}(\omega)=\sum_{j=1}^{N}\frac{(\omega_j^2-\omega^2)\,\omega a_{jy}\left[(w_{0x}-g_x)\,a_{jy}+\omega^2a_{j\theta}\right]}{\mu_j\left[(\omega_j^2-\omega^2)^2+\varepsilon_j^2\omega^2\right]},$$
$$a_{22}(\omega)=c_{\theta\theta}-\omega^2J_x-\sum_{j=1}^{N}\frac{(\omega_j^2-\omega^2)\left[(w_{0x}-g_x)\,a_{jy}+\omega^2a_{j\theta}\right]^2}{\mu_j\left[(\omega_j^2-\omega^2)^2+\varepsilon_j^2\omega^2\right]},$$
$$b_{22}(\omega)=\mu_x\omega+\sum_{j=1}^{N}\frac{\varepsilon_j\omega\left[(w_{0x}-g_x)\,a_{jy}+\omega^2a_{j\theta}\right]^2}{\mu_j\left[(\omega_j^2-\omega^2)^2+\varepsilon_j^2\omega^2\right]}.$$

By determining θ from equations (3.26.7), we find

$$\theta=\frac{c_{\theta\theta}(a_{11}+ib_{11})-c_{y\theta}(a_{21}+ib_{21})}{(a_{11}+ib_{11})(a_{22}+ib_{22})-(a_{12}+ib_{12})(a_{21}+ib_{21})}=$$
$$=\frac{c_{\theta\theta}a_{11}-c_{y\theta}a_{21}+i(c_{\theta\theta}b_{11}-c_{y\theta}b_{21})}{a_{11}a_{22}-a_{12}a_{21}-b_{11}b_{22}+b_{12}b_{21}+i(a_{11}b_{22}-a_{12}b_{21}+b_{11}a_{22}-b_{12}a_{21})},$$

or

$$\theta=\theta_1(\omega)+i\theta_2(\omega), \qquad (3.26.9)$$

where $\theta_1(\omega)$ and $\theta_2(\omega)$ – functions of frequency ω, determined by formulas:

$$
\begin{aligned}
\theta_1 &= \frac{c_1(c_{\vartheta\delta}a_{11} - c_{y\delta}a_{21}) + c_2(c_{\vartheta\delta}b_{11} - c_{y\delta}b_{21})}{c_1^2 + c_2^2},\\
\theta_2 &= \frac{c_1(c_{\vartheta\delta}b_{11} - c_{y\delta}b_{21}) - c_2(c_{\vartheta\delta}a_{11} - c_{y\delta}a_{21})}{c_1^2 + c_2^2},\\
c_1 &= a_{11}a_{22} - a_{12}a_{21} - b_{11}b_{22} + b_{12}b_{21},\\
c_2 &= a_{11}b_{22} - a_{12}b_{21} + b_{11}a_{22} - b_{12}a_{21}.
\end{aligned}
\qquad (3.26.10)
$$

Relationship (3.26.9) can be reduced to form

$$\theta = k_0^{\cdot}(\omega)e^{i\varphi_0(\omega)}, \qquad (3.26.11)$$

where

$$k_0^{\cdot}(\omega) = \sqrt{\theta_1^2(\omega) + \theta_2^2(\omega)}, \quad \varphi_0^{\cdot}(\omega) = \operatorname{arctg}\frac{\theta_2(\omega)}{\theta_1(\omega)}. \qquad (3.26.12)$$

According to (3.26.2) and (3.26.11) in the considered case disturbance $\Delta\vartheta$, being the input signal for automatic stabilization control, will be determined by relationship

$$\Delta\vartheta = k_0^{\cdot}(\omega)e^{i[\omega t + \varphi_0(\omega)]}. \qquad (3.26.13)$$

Thus, in the absence of disturbing forces and moments, harmonic oscillations of actuating elements of the control system cause harmonic oscillations of the input signal of automatic stabilization control, characterized by *amplification factor* $k_0(\omega)$ and $\varphi_0(\omega)$. Functions $k_0(\omega)$ and $\varphi_0(\omega)$, for which we established calculation formulas (3.26.8), (3.26.10) and (3.26.12), determine *amplitude-frequency* and *phase-frequency characteristic* of the rocket as the object of automatic control for the case of oscillations of the rocket in pitching plane.

Let us turn to examination of the rotation of the rocket around its longitudinal axis. Having assumed in equations of disturbed motion (3.24.19) that

$$\Delta M_x = 0, \ \Delta\delta_1 = e^{i\omega t} \qquad (3.26.14)$$

and considering the coefficients of these differential equations "quenched" at a certain moment of time $t = \tau$, we will seek unknown functions $\Delta\gamma$ and $\Delta\alpha_j$, $j = 1, 2, \ldots, N$ in the form

$$\Delta\gamma = \Gamma e^{i\omega t}, \quad \Delta\alpha_j = A_j e^{i\omega t}, \quad j = 1, 2, \ldots N. \tag{3.26.15}$$

By substituting (3.26.14) and (3.26.15) in (3.24.19), for coefficients Γ and A_j, $j = 1, 2, \ldots, N$ we obtain system of algebraic equations

$$\begin{aligned} &(i\omega\mu_x - \omega^2 J_x)\Gamma - \omega^2 \sum_{j=1}^{N} a_{j1} A_j = c_{\gamma\delta}, \\ &\mu_j(\omega_j^2 - \omega^2 + i\varepsilon_j\omega) A_j - \omega^2 a_{j1}\Gamma = 0, \\ &j = 1, 2, \ldots, N. \end{aligned} \tag{3.26.16}$$

By excluding unknowns A_j, $j = 1, 2, \ldots, N$, from equations (3.26.16) we obtain equation

$$-\left[\omega^2 J_x - i\omega\mu_x + \sum_{j=1}^{N} \frac{a_{j1}^2 \omega^4}{\mu_j(\omega_j^2 - \omega^2 + i\varepsilon_j\omega)}\right]\Gamma = c_{\gamma\delta},$$

or

$$[C_1(\omega) + iC_2(\omega)]\Gamma = c_{\gamma\delta}, \tag{3.26.17}$$

where

$$\begin{aligned} C_1(\omega) &= -\omega^2 J_x - \sum_{j=1}^{N} \frac{(\omega_j^2 - \omega^2)\omega^4 a_{j1}^2}{\mu_j[(\omega_j^2 - \omega^2)^2 + \varepsilon_j^2\omega^2]}, \\ C_2(\omega) &= \omega\mu_x + \sum_{j=1}^{N} \frac{\varepsilon_j\omega^5 a_{j1}^2}{\mu_j[(\omega_j^2 - \omega^2)^2 + \varepsilon_j^2\omega^2]}. \end{aligned} \tag{3.26.18}$$

According to (3.26.17) there will exist equality

$$\Gamma = \frac{e_{\gamma\delta}}{C_1 + iC_2},$$

or

$$\Gamma = \Gamma_1(\omega) + i\Gamma_2(\omega), \tag{3.26.19}$$

where

$$\Gamma_1(\omega) = \frac{e_{\gamma\delta} C_1(\omega)}{C_1^2(\omega) + C_2^2(\omega)}, \quad \Gamma_2(\omega) = -\frac{e_{\gamma\delta} C_2(\omega)}{C_1^2(\omega) + C_2^2(\omega)}. \tag{3.26.20}$$

Having given formula (3.26.19) the form

$$\Gamma = k_0(\omega) e^{i\varphi_0(\omega)}, \tag{3.26.21}$$

where

$$k_0(\omega) = \sqrt{\Gamma_1^2(\omega) + \Gamma_2^2(\omega)}, \quad \varphi_0(\omega) = \operatorname{arctg} \frac{\Gamma_2(\omega)}{\Gamma_1(\omega)}, \tag{3.26.22}$$

according to (3.26.15) and (3.6.21) we obtain relationship

$$\Delta\gamma = k_0(\omega) e^{i[(\omega t + \varphi_0(\omega)]}. \tag{3.26.23}$$

In accordance with equality (3.6.23), functions $k_0(\omega)$ and $\varphi_0(\omega)$. for which we obtained working formulas (3.26.18), (3.26.20) and (3.6.22), determine the amplitude-frequency and phase-frequency characteristics of the rocket as the object of automatic control for the case of rotation of the rocket around its longitudinal axis.

Having determined the frequency characteristics of the rocket as the object of control for some of the control channels and having assigned the frequency characteristics of automatic stabilization control, it is possible to construct a hodograph of open system of automatic control, and, by using frequency criterion of stability, to make a conclusion about the stability or instability of motion of the rocket with assigned frequency characteristics of automatic stabilization control. If the amplification factor and phase shift of automatic stabilization control are determined by functions $k_a(\omega)$ and $\varphi_a(\omega)$, then amplification factor k and phase shift φ of the open system of automatic control will be determined by relationship:

$$\left.\begin{aligned} k &= k_0(\omega)k_a(\omega), \\ \varphi &= \varphi_0(\omega)+\varphi_a(\omega). \end{aligned}\right\} \quad (3.26.24)$$

Relationships (3.26.24) determine the equation of hodograph in polar coordinates k and φ. In rectangular coordinates X and Y the equation of hodograph will have the form

$$\left.\begin{aligned} X &= k_0(\omega)k_a(\omega)\cos[\varphi_0(\omega)+\varphi_a(\omega)], \\ Y &= k_0(\omega)k_a(\omega)\sin[\varphi_0(\omega)+\varphi_a(\omega)]. \end{aligned}\right\} \quad (3.26.25)$$

Having constructed the hodograph of the open system in accordance with the calculation scheme, discussed in this paragraph the conclusion can be made concerning local stability, or instability of motion of the rocket at moment of time $t = \tau$, having used Nyquist criterion.

§ 27. Phase Stabilization of Oscillations of the Rocket Body and Free Surfaces of Fluids in Fuel Tanks

During consideration of the question about finding the frequency characteristics of automatic stabilization control, providing stability of motion of the rocket, we will proceed from approximate calculation of frequency characteristics of the rocket as the object of automatic control, using as a basis the following simplifications of equations (3.26.3) and (3.26.16).

1. Harmonic oscillations of actuating elements of control system with frequency ω excite oscillations of the free surface of fluid in the fuel tank with number k for all practical purposes only at values of frequency ω very close to the frequency of free oscillations of this free surface ω_k. If among the remaining frequencies

of free oscillations $\omega_1, \omega_2, \ldots, \omega_{k-1}, \omega_{k+1}, \ldots, \omega_r$ there are no frequencies close to ω_k, then during calculation of the frequency characteristics of the rocket as a controllable object for frequencies ω, close to frequencies ω_k, for all practical purposes it is possible to consider oscillations of free surfaces of fluids in the fuel tanks with numbers $1, 2, \ldots, k-1, k+1, \ldots, N$. In this instance in the sums, which figure in equations (3.26.3), there are retained only the components containing unknown quantity B_k, and in the sum, which figures in equations (3.26.16), there will be retained only one component, containing unknown quantity A_k.

2. When determining unknowns V and θ from the first two equations of system (3.26.3) for frequencies ω, close to the frequencies of natural oscillations of free surfaces of fluids, the basic role among components containing these unknowns is played by components $i\omega mV$ and $\omega^2 J_z \theta$. In connection with this, during approximate calculation of frequency characteristics of the rocket as a controllable object for frequencies ω, close to frequencies of natural oscillations of fluids in fuel tanks, in equations (3.26.3) and (3.26.16) it is possible to disregard the components containing coefficients $c_{yy}, c_{y\theta}, c_{\theta y}, c_{\theta\theta}, \nu_y, \mu_z$ and μ_x as factors.

Thus, approximate calculation of frequency characteristics of the rocket as the object of automatic control of frequencies ω, close to frequency ω_k, can be carried out by substituting system of equations (3.26.3) by equations:

$$\begin{gathered}
i\omega mV - \omega^2 a_{ky} B_k = c_{y\delta}, \\
-\omega^2 J_z \theta - [(w_{0x} - g_x) a_{ky} + \omega^2 a_{k\theta}] B_k = c_{\theta\delta}, \\
\mu_k (\omega_k^2 - \omega^2 + i\varepsilon_k \omega) B_k + i a_{ky} \omega V - \\
- [(w_{0x} - g_x) a_{ky} + \omega^2 a_{k\theta}] \theta = 0
\end{gathered} \qquad (3.27.1)$$

and system of equations (3.26.16) - by equations:

$$\begin{gathered}
-\omega^2 J_x \Gamma - \omega^2 a_{k\gamma} A_k = c_{\gamma\delta}, \\
\mu_k (\omega_k^2 - \omega^2 + i\varepsilon_k \omega) A_k - \omega^2 a_{k\gamma} \Gamma = 0.
\end{gathered} \qquad (3.27.2)$$

Let us first examine the question about stabilization of motion of the rocket in the pitching plane. By excluding unknowns V and θ from equations (3.27.1), we obtain equation

$$\mu_k(\omega_k^2-\omega^2+i\varepsilon_k\omega)B_k+\frac{a_{ky}}{m}(c_{y\delta}+\omega^2a_{ky}B_k)+$$
$$+\frac{1}{\omega^2J_z}[(w_{0x}-g_x)a_{ky}+\omega^2a_{k\vartheta}]\{c_{\vartheta\delta}+$$
$$+[(w_{0x}-g_x)a_{ky}+\omega^2a_{k\vartheta}]B_k\}=0,$$

or

$$\{(ma_{k\vartheta}^2+J_za_{ky}^2-\mu_kmJ_z)\omega^4+[\mu_kmJ_z\omega_k^2+2m(w_{0x}-$$
$$-g_x)a_{ky}a_{k\vartheta}]\omega^2+m(w_{0x}-g_x)^2a_{ky}^2+i\mu_kmJ_z\varepsilon_k\omega^3\}B_k=$$
$$=-(ma_{k\vartheta}c_{\vartheta\delta}+J_za_{ky}c_{y\delta})\omega^2-m(w_{0x}-g_x)a_{ky}c_{\vartheta\delta}. \quad (3.27.3)$$

Biquadratic equation

$$(ma_{k\vartheta}^2+J_za_{ky}^2-\mu_kmJ_z)\omega^4+[\mu_kmJ_z\omega_k^2+$$
$$+2m(w_{0x}-g_x)a_{ky}a_{k\vartheta}]\omega^2+m(w_{0x}-g_x)^2a_{ky}^2=0. \quad (3.27.4)$$

always has positive real solution ω_k^*, close to frequency ω_k. For its approximate detection let us assume in (3.27.4).

$$\omega=\omega_k+\eta. \quad (3.27.5)$$

By disregarding in this case the highest degrees of small quantity η, we obtain equation

$$\{4(ma_{k\vartheta}^2+J_za_{ky}^2-\mu_kmJ_z)\omega_k^3+2[\mu_kmJ_z\omega_k^2+$$
$$+2m(w_{0x}-g_x)a_{ky}a_{k\vartheta}]\omega_k\}\eta+(ma_{k\vartheta}^2+J_za_{ky}^2)\omega_k^4+$$
$$+2m(w_{0x}-g_x)a_{ky}a_{k\vartheta}\omega_k^2+m(w_{0x}-g_x)^2a_{ky}^2=0. \quad (3.27.6)$$

According to (3.18.10) and (3.23.4) equation (3.27.6) can be reduced to form

$$\left\{-2\mu_k m J_z + a_{ky}^2\left[4m(x_k-\delta_k)^2+4J_z+\frac{4m(x_k-\delta_k)}{\lambda_k}\right]\right\}\eta+$$
$$+\omega_k a_{ky}^2\left[J_z+m(x_k-\delta_k)^2+\frac{2m(x_k-\delta_k)}{\lambda_k}+\frac{m}{\lambda_k^2}\right]=0,$$

or

$$\left\{1-\frac{2a_{ky}^2}{\mu_k m}\left[1+\frac{m(x_k-\delta_k)}{\lambda_k J_z}+\frac{m(x_k-\delta_k)^2}{J_z}\right]\right\}\eta=$$
$$=\frac{\omega_k a_{ky}^2}{2\mu_k m}\left[1+\frac{m}{J_z}\left(x_k-\delta_k+\frac{1}{\lambda_k}\right)^2\right]. \quad (3.27.7)$$

Ratio $\frac{a_{ky}^2}{\mu_k m}$ is always small in comparison with one and in accordance with equality (3.27.7) it is possible to approximately assume

$$\eta=\frac{\omega_k a_{ky}^2}{2\mu_k m}\left(1+\frac{m x_k^{**}}{J_z}\right), \quad (3.27.8)$$

where

$$x_k^*=x_k-\delta_k+\frac{1}{\lambda_k}. \quad (3.27.9)$$

According to (3.27.5) and (3.27.8) solution ω_k^* of biquadratic equation (3.27.4) can be approximately calculated by formula

$$\omega_k^*=\omega_k\left[1+\frac{a_{ky}^2}{2\mu_k m}\left(1+\frac{m x^{**}}{J_z}\right)\right]. \quad (3.27.10)$$

With values of frequency ω, close to frequency ω_k^*, quantity can be approximately determined from equation (3.27.3), assuming in equation (3.27.3)

$$\omega=\omega_k^*+\delta \quad (3.27.11)$$

and in the left side of this equation disregarding small quantities of order of smallness δ^2 and $\varepsilon_k\delta$ and in the right side of the equation - small quantities of order of smallness δ. Equation (3.27.3) in this instance takes the form

$$\{\{4(ma_{k\theta}^2+J_z a_{ky}^2-\mu_k m J_z)\omega_k^{*4}+2[\mu_k m J_z \omega_k^{*2}+2m(w_{0x}-g_x)a_{ky}a_{k\theta}]\omega_k^{*}\}\delta+i\mu_k m J_z \varepsilon_k \omega_k^{*4}\}B_k= =-(ma_{k\theta}c_{\theta\delta}+J_z a_{ky}c_{y\delta})\omega_k^{*2}-m(w_{0x}-g_x)c_{ky}c_{\theta\delta} \tag{3.27.12}$$

[let us recall that ω_k^* - solution of equation (3.27.4)]. By disregarding small quantities of order of smallness $\delta\eta$ and $\varepsilon_k\eta\,(\eta=\omega_k^*-\omega_k)$ in the left side of equation (3.27.12) and in the right side of this equation - small quantities of order of smallness η, in equation (3.27.12) it is possible to substitute frequency ω_k^* by frequency ω_k. According to (3.18.10) and (3.23.4) in this instance equation (3.27.12) could be given the form

$$\omega_k\mu_k m J_z\left\{2\left\{1-\frac{2a_{ky}^2}{\mu_k m}\left[1+\frac{m(x_k-\delta_k)}{\lambda_k J_z}+\frac{m(x_k-\delta_k)^2}{J_z}\right]\right\}\delta-i\varepsilon_k\right\}B_k= =a_{ky}\left[J_z c_{y\delta}+m\left(x_k-\delta_k+\frac{1}{\lambda_k}\right)c_{\theta\delta}\right]. \tag{3.27.13}$$

Since ratio $\frac{a_{ky}^2}{\mu_k m}$ is small in comparison with one, equation (3.27.13) can be substituted by approximate equation

$$(2\delta-i\varepsilon_k)B_k=\frac{a_{ky}}{\omega_k\mu_k m J_z}\left[J_z c_{y\delta}+m\left(x_k-\delta_k+\frac{1}{\lambda_k}\right)c_{\theta\delta}\right],$$

or according to (3.27.9)

$$(2\delta-i\varepsilon_k)B_k=\frac{a_{ky}(J_z c_{y\delta}+m x_k^* c_{\theta\delta})}{\omega_k\mu_k m J_z}. \tag{3.27.14}$$

In accordance with the second of equations (3.27.1) function $\theta(\omega)$ can be found by formula

$$\theta=-\frac{[(w_{0x}-g_x)a_{ky}+\omega^2 a_{k\theta}]B_k+c_{\theta\delta}}{\omega^2 J_z}. \tag{3.27.15}$$

By substituting B_k from (3.27.14) in (3.27.15) we obtain relationship

$$\theta = -\frac{1}{\omega^2 \omega_k \mu_k m J_z^2 (2\delta - i\varepsilon_k)} \{ [(w_{0x} - g_x) a_{ky} + \omega^2 a_{k\theta}] \times$$

$$\times a_{ky} (J_z c_{y\delta} + m x_k^{\circ} c_{\theta\delta}) + \omega_k \mu_k m J_z c_{\theta\delta} (2\delta - i\varepsilon_k) \}. \tag{3.27.16}$$

By considering the proximity of frequency ω to frequency ω_k and smallness of quantities δ and ε_k, assumed by us, it is possible to substitute equality (3.27.16) by approximate equality

$$\theta = -\frac{[(w_{0x} - g_x) a_{ky} + \omega_k^2 a_{k\theta}] a_{ky} (J_z c_{y\delta} + m x_k^{\circ} c_{\theta\delta})}{\omega_k^3 \mu_k m J_z^2 (2\delta - i\varepsilon_k)}. \tag{3.27.17}$$

In accordance with formulas (3.18.10) and (3.23.4) relationship (3.27.17) can be given the form

$$\theta = \frac{\left(x_k - \delta_k + \frac{1}{\lambda_k}\right) a_{ky}^2 (J_z c_{y\delta} + m x_k^{\circ} c_{\theta\delta})}{\omega_k \mu_k m J_z^2 (2\delta - i\varepsilon_k)},$$

or, if we use meaning (3.27.9),

$$\theta = -\frac{x_k^{\circ} (J_z c_{y\delta} + m x_k^{\circ} c_{\theta\delta}) a_{ky}^2}{\omega_k \mu_k m J_z^2 (2\delta - i\varepsilon_k)}. \tag{3.27.18}$$

According to (3.26.11) and (3.27.18) there will exist equalities:

$$\begin{aligned} k_0(\omega) \cos \varphi_0(\omega) &= \operatorname{Re} \theta = -\frac{2 x_k^{\circ} (J_z c_{y\delta} + m x_k^{\circ} c_{\theta\delta}) a_{ky}^2 \delta}{\omega_k \mu_k m J_z^2 (4\delta^2 + \varepsilon_k^2)}, \\ k_0(\omega) \sin \varphi_0(\omega) &= \operatorname{Im} \theta = -\frac{x_k^{\circ} (J_z c_{y\delta} + m x_k^{\circ} c_{\theta\delta}) a_{ky}^2 \varepsilon_k}{\omega_k \mu_k m J_z^2 (4\delta^2 + \varepsilon_k^2)}. \end{aligned} \tag{3.27.19}$$

In accordance with formulas (3.26.25) and (3.27.19) in the onsidered case the equation of hodograph of the open system of automat c control will have the form

$$X=-\frac{x_k^\circ(J_z c_{yk}+m x_k^\circ c_{bk})a_{ky}^2}{\omega_k \mu_k m J_z^2(4\delta^2+\varepsilon_k^2)}k_a(\omega)\times$$
$$\times[2\delta\cos\varphi_a(\omega)-\varepsilon_k\sin\varphi_a(\omega)],\qquad (3.27.20)$$
$$Y=-\frac{x_k^\circ(J_z c_{yk}+m x_k^\circ c_{bk})a_{ky}^2}{\omega_k \mu_k m J_z^2(4\delta^2+\varepsilon_k^2)}k_a(\omega)\times$$
$$\times[2\delta\sin\varphi_a(\omega)+\varepsilon_k\cos\varphi_a(\omega)].$$

By considering the proximity of frequency ω to frequency ω_k, assumed by us, it is possible to substitute equalities (3.27.20) by approximate equalities:

$$X=-\frac{x_k^\circ(J_z c_{yk}+m x_k^\circ c_{bk})a_{ky}^2}{\omega_k \mu_k m J_z^2(4\delta^2+\varepsilon_k^2)}k_a(\omega_k)[2\delta\cos\varphi_a(\omega_k)-\varepsilon_k\sin\varphi_a(\omega_k)],$$
$$Y=-\frac{x_k^\circ(J_z c_{yk}+m x_k^\circ c_{bk})a_{ky}^2}{\omega_k \mu_k m J_z^2(4\delta^2+\varepsilon_k^2)}k_a(\omega_k)[2\delta\sin\varphi_a(\omega_k)+\varepsilon_k\cos\varphi_a(\omega_k)],$$

or

$$X=\frac{2c_k\varepsilon_k[2\delta\cos\varphi_a(\omega_k)-\varepsilon_k\sin\varphi_a(\omega_k)]}{4\delta^2+\varepsilon_k^2},$$
$$Y=\frac{2c_k\varepsilon_k[2\delta\sin\varphi_a(\omega_k)+\varepsilon_k\cos\varphi_a(\omega_k)]}{4\delta^2+\varepsilon_k^2},\qquad (3.27.21)$$

where

$$c_k=-\frac{k_a(\omega_k)x_k^\circ(J_z c_{yk}+m x_k^\circ c_{bk})a_{ky}^2}{2\varepsilon_k\omega_k\mu_k m J_z^2}.\qquad (3.27.22)$$

According to (3.27.21) there should take place equalities:

$$X\cos\varphi_a(\omega_k)+Y\sin\varphi_a(\omega_k)=\frac{4c_k\varepsilon_k\delta}{4\delta^2+\varepsilon_k^2},$$
$$X\sin\varphi_a(\omega_k)-Y\cos\varphi_a(\omega_k)=-\frac{2c_k\varepsilon_k^2}{4\delta^2+\varepsilon_k^2}.\qquad (3.27.23)$$

in accordance with which parameter δ can be expressed through coordinates of hodograph X, Y by relationship

$$\delta = -\frac{c_k[X\cos\varphi_a(\omega_k) + Y\sin\varphi_a(\omega_k)]}{2[X\sin\varphi_a(\omega_k) - Y\cos\varphi_a(\omega_k)]}. \qquad (3.27.24)$$

By substituting (3.27.24) in the second of equalities (3.27.23), it is possible to reduce the equation of hodograph to form

$$-2c_k = [X\sin\varphi_a(\omega_k) - Y\cos\varphi_a(\omega_k)]\times$$
$$\times\left\{1 + \frac{[X\cos\varphi_a(\omega_k) + Y\sin\varphi_a(\omega)]^2}{[X\sin\varphi_a(\omega_k) - Y\cos\varphi_a(\omega_k)]^2}\right\} =$$
$$= \frac{X^2 + Y^2}{X\sin\varphi_a(\omega_k) - Y\cos\varphi_a(\omega_k)},$$

or

$$X^2 + Y^2 + 2c_k[X\sin\varphi_a(\omega_k) - Y\cos\varphi_a(\omega_k)] = 0. \qquad (3.27.25)$$

Equation (3.27.25) represents an equation of circumference. Reduction of this equation to canonical form

$$[X + c_k\sin\varphi_a(\omega_k)]^2 + [Y - c_k\cos\varphi_a(\omega_k)]^2 = c_k^2 \qquad (3.27.26)$$

shows that coordinates of the center of the given circumference X_0, Y_0 are respectively equal to:

$$X_0 = -c_k\sin\varphi_a(\omega_k), \quad Y_0 = c_k\cos\varphi_a(\omega_k), \qquad (3.27.27)$$

radius of circumference is equal to $|c_k|$, the circumference passes through the origin of coordinates.

When $X_0 < 0$ the hodograph of the open system with values of frequency ω, close to frequency ω_k, will have the form shown in Fig. 3.16 by a solid line for case $Y_0 > 0$ and broken line – for case

$Y_0 < 0$. In both cases regardless of the radius of circumference with frequencies ω, close to frequency ω_k, the hodograph of the open system can encompass point $X = 1$, $Y = 0$. According to (3.27.27) with our introduced simplifications the conditions

$$c_k \sin\varphi_a(\omega_k) > 0 \tag{3.27.28}$$

are sufficient conditions of stability of oscillations of the rocket body and free surfaces of fluids in the fuel tanks.[1] Condition (3.27.28) is called the condition of *phase stabilization,* since it is fulfilled by the creation of phase lead in the automatic stabilization control at frequency $\omega = \omega_k$ when $c_k > 0$ or phase lag at this frequency when $c_k < 0$; the value, taken at frequency $\omega = \omega_k$ by the amplification factor of automatic stabilization control, does not play a role in this case. In accordance with formula (3.27.22) the condition of phase stabilization (3.27.28) can be given the form

$$x_k^\circ (J_z c_{\mu k} + m x_k^\circ c_{\delta k}) \sin\varphi_a(\omega_k) < 0. \tag{3.27.29}$$

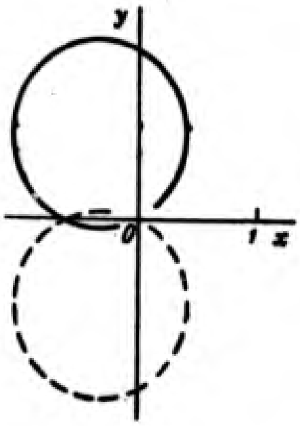

Fig. 3.16.

If the control system does not create controlling force and its actuating elements shape only the controlling moment (control of thrust misalignment), condition (3.27.29) assumes the form

[1]After simplifications, made in the beginning of this paragraph, the transfer function cannot have poles in the right half-plane.

$$\sin\varphi_a(\omega_k) > 0, \tag{3.27.30}$$

since in this instance $c_{yk} = 0$, $c_{\delta k} < 0$. Thus, in the considered case the phase stabilization of oscillations can be provided by realizing phase lead in the automatic stabilization control in the range of frequencies encompassing all the frequencies of free oscillations of free surfaces of fluids ω_1, ω_2, ..., ω_N.

If actuating elements of the control system create controlling force, then

$$c_{\delta k} = -l c_{yk}, \tag{3.27.31}$$

where l – distance from the line of action of controlling force to plane $x = 0$. In this instance condition (3.27.29) assumes the form

$$x_k^\circ \left(1 - \frac{m x_k^\circ l}{J_z}\right) \sin\varphi_a(\omega_k) < 0 \tag{3.27.32}$$

(coefficient $c_y\delta$ is always positive). According to (3.27.32) for phase stabilization of oscillations there is required phase lead at frequency ω_k when

$$x_k^\circ < 0 \text{ and } x_k^\circ > \frac{J_z}{ml} \tag{3.27.33}$$

and phase lag – when

$$0 < x_k^\circ < \frac{J_z}{ml}. \tag{3.27.34}$$

Coordinate x_k° usually little differs from coordinate x_k. Thus, by realizing phase lead in the automatic stabilization control in the range of frequencies encompassing the frequencies of natural oscillations of free surfaces of fluids ω_1, ω_2, ..., ω_N, we accomplish phase stabilization of the oscillations for fuel tanks, whose free surfaces of fluids lie lower than the center of mass of the rocket. For the remaining fuel tanks the conditions of phase stabilization, as a rule, will be disturbed in this case. Realization of phase lead in the automatic stabilization control for some fuel tanks and phase lag for other tanks is practically impossible, since frequencies of natural oscillations $\omega_1, \omega_2, \ldots, \omega_N$ usually lie in the small frequency range and, furthermore, are changed in the process of rocket flight. In connection with this, in the automatic stabilization control for the entire range of frequencies of natural oscillations of free surfaces of fluids there is usually realized phase lead and in the upper fuel tanks, for which in the first stage of flight the condition of phase stabilization is disturbed, there are installed perforated caps, which for all practical purposes prevent the possibility of oscillations of free surfaces of fluids in these tanks until these tanks begin to empty. When the propellant in these tanks begins to be consumed, the surfaces of fluids in them, as a rule, are already lower than the center of mass of the rocket and the stability of motion is not disturbed. The places for installation of caps are selected with allowance for minimum permissible level of servicing of the tanks.

By the means indicated above it is usually possible to achieve stabilization of oscillations for the larger part of powered flight sections the conditions of stability are continuously disturbed. This case is considered by us in the following paragraph of this chapter.

Let us turn to examination of rotation of the rocket around its longitudinal axis. By excluding unknown quantity Γ from equations (3.27.2) we obtain equation

$$\left[J_{x}\mu_{k}(\omega_k^2-\omega^2+i\varepsilon_k\omega)+\omega^2 a_{k\gamma}^2\right] A_k+a_{k\gamma}c_{\gamma k}=0. \qquad (3.27.35)$$

Assuming in (3.27.35)

$$\omega_k^2=\left(1-\frac{a_{k\gamma}^2}{J_x\mu_k}\right)\omega_k^{*2}, \qquad (3.27.36)$$

it is possible to convert this equation:

$$\left[\left(1-\frac{a_{k\gamma}^2}{J_x\mu_k}\right)(\omega_k^{*2}-\omega^2)+i\varepsilon_k\omega\right]A_k=-\frac{a_{k\gamma}c_{\gamma k}}{J_x\mu_k}. \quad (3.27.37)$$

With values of frequency ω close to frequency ω_k^*, where

$$\omega_k^*=\frac{\omega_k}{\sqrt{1-\frac{a_{k\gamma}^2}{J_x\mu_k}}} \quad (3.27.38)$$

in accordance with equality (3.27.36), quantity A_k can be approximately determined from equation (3.27.37), assuming in this equation

$$\omega=\omega_k^*+\delta \quad (3.27.39)$$

and disregarding small quantities of order of smallness δ^2 and $\varepsilon_k\delta$ in the left side of the equation. Equation (3.27.37) in this instance takes the form

$$\left[2\left(1-\frac{a_{k\gamma}^2}{J_x\mu_k}\right)\delta-i\varepsilon_k\right]A_k=\frac{a_{k\gamma}c_{\gamma k}}{\omega_k^*J_x\mu_k}. \quad (3.27.40)$$

Ratio $\frac{a_{k\gamma}^2}{J_x\mu_k}$ is always small in comparison with one, and according to (3.27.38) frequency ω_k^* will always be close to frequency ω_k. In accordance with this equation (3.27.40) can be substituted by approximate equation

$$(2\delta-i\varepsilon_k)A_k=\frac{a_{k\gamma}c_{\gamma k}}{\omega_kJ_x\mu_k}. \quad (3.27.41)$$

By substituting A_k from (3.27.41) into the first of equations (3.27.2), for quantity Γ we obtain expression

$$\Gamma=-\frac{\omega^2a_{k\gamma}^2+\omega_kJ_x\mu_k(2\delta-i\varepsilon_k)}{\omega^2\omega_kJ_x^2\mu_k(2\delta-i\varepsilon_k)}c_{\gamma k}. \quad (3.27.42)$$

Considering the smallness of δ and ε_k, it is possible to replace formula (3.27.42) by approximate formula

$$\Gamma = -\frac{a_{k\gamma}^2 e_{\gamma\delta}}{\omega_k J_x^2 \mu_k (2\delta - i\varepsilon_k)}. \qquad (3.27.43)$$

According to (3.26.21) and (3.27.43) there will exist equalities:

$$\begin{aligned} k_0(\omega)\cos\varphi_0(\omega) &= \operatorname{Re}\Gamma = -\frac{2a_{k\gamma}^2 e_{\gamma\delta}\delta}{\omega_k J_x^2 \mu_k (4\delta^2+\varepsilon_k^2)}, \\ k_0(\omega)\sin\varphi_0(\omega) &= \operatorname{Im}\Gamma = -\frac{a_{k\gamma}^2 e_{\gamma\delta}\varepsilon_k}{\omega_k J_x^2 \mu_k (4\delta^2+\varepsilon_k^2)}. \end{aligned} \qquad (3.27.44)$$

In accordance with formulas (3.26.25) and (3.27.44) in the considered case the equation of hodograph of the open system of automatic control will have the form

$$\begin{aligned} X &= -\frac{a_{k\gamma}^2 e_{\gamma\delta}}{\omega_k J_x^2 \mu_k (4\delta^2+\varepsilon_k^2)} k_a(\omega)[2\delta\cos\varphi_a(\omega) - \varepsilon_k \sin\varphi_a(\omega)], \\ Y &= -\frac{a_{k\gamma}^2 e_{\gamma\delta}}{\omega_k J_x^2 \mu_k (4\delta^2+\varepsilon_k^2)} k_a(\omega)[2\delta\sin\varphi_a(\omega) + \varepsilon_k \cos\varphi_a(\omega)]. \end{aligned} \qquad (3.27.45)$$

By considering the proximity of frequency ω to frequency ω_k, assumed by us, it is possible to substitute equalities (3.27.45) by approximate equalities:

$$\begin{aligned} X &= -\frac{a_{k\gamma}^2 e_{\gamma\delta}}{\omega_k J_x^2 \mu_k (4\delta^2+\varepsilon_k^2)} k_a(\omega_k)[2\delta\cos\varphi_a(\omega_k) - \\ &\quad - \varepsilon_k \sin\varphi_a(\omega_k)], \\ Y &= -\frac{a_{k\gamma}^2 e_{\gamma\delta}}{\omega_k J_x^2 \mu_k (4\delta^2+\varepsilon_k^2)} k_a(\omega_k)[2\delta\sin\varphi_a(\omega_k) + \\ &\quad + \varepsilon_k \cos\varphi_a(\omega_k)]. \end{aligned} \qquad (3.27.46)$$

Assuming in this case

$$c_k = -\frac{h_k(\omega_k)\, a_{k\gamma}^2\, c_{\gamma\delta}}{2\kappa_k \omega_k \mu_k l_k^2}, \tag{3.27.47}$$

it is possible to reduce the equation of hodograph of open system (3.27.46) to form (3.27.26), established by us earlier in examining the motion of rocket in the pitching plane. In the considered case there will always take place inequality

$$c_k > 0, \tag{3.27.48}$$

since coefficient $c_\gamma \delta$ is always negative. In accordance with inequality (3.27.48) the conditions of phase stabilization (3.27.28) for the case of rotation of the rocket around its longitudinal axis will have the form

$$\sin \varphi_k(\omega_k) > 0, \quad k = 1, 2, \ldots N. \tag{3.27.49}$$

Thus, in the considered case phase stabilization of oscillations can be provided by realizing phase lead in the automatic stabilization control in the range of frequencies encompassing all frequencies of free oscillations of free surfaces of fluids $\omega_1, \omega_2, \ldots, \omega_N$.

In conclusion of this paragraph let us note that the criteria of stability indicated in it are based on simplifications of the calculation scheme made in the beginning of the paragraph, and sometimes can be incorrect. A strict conclusion about the stability of motion can be made only on the basis of an accurate construction of hodograph.

§ 28. Self-Oscillations of the Rocket in Pitching Plane

Let us now examine the case when for the fuel tank with number k the condition of phase stabilization of oscillations in pitching plane (3.27.28) cannot be provided. According to (3.27.27) when

$c_k \sin\varphi_k(\omega_k)<0$ coordinate X_0 of the center of circumference, approximately determining the type of hodograph of the open system of frequencies ω close to frequency ω_k, will be positive. In this case, this circumference will have the shape shown in Fig. 3.17 by solid line for case $Y_0 > 0$ and by broken line - for case $Y_0 < 0$. In accordance with the equation of the considered circumference (3.27.26) the points of its intersection with straight line $Y = 0$ will be determined by quadratic equation

$$[X+c_k \sin\varphi_k(\omega_k)]^2+c_k^2 \cos^2\varphi_k(\omega_k)=c_k^2,$$

or

$$X[X+2c_k \sin\varphi_k(\omega_k)]=0. \qquad (3.28.1)$$

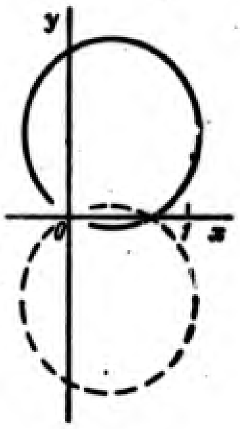

Fig. 3.17.

Thus, the circumference will intersect axis X at points with coordinates

$$X=0 \text{ and } X=-2c_k \sin\varphi_k(\omega_k)$$

and the condition of stability of oscillations of k-th fuel tank will have the form

$$-2c_k \sin\varphi_k(\omega_k)<1. \qquad (3.28.2)$$

In accordance with formula (3.27.22) condition (3.28.2) can be given the form

$$\frac{k_a(\omega_k)\sin\varphi_a(\omega_k)\,x_k^{\bullet}\left(J_z c_{y\delta}+m x_k^{\bullet} c_{\delta\delta}\right)a_{k\mu}^2}{\varepsilon_k\omega_k\mu_k m J_z^2}<1,$$

or

$$\varepsilon_k>\varepsilon_k^{(0)}, \tag{3.28.3}$$

where

$$\varepsilon_k^{(0)}=\frac{k_a(\omega_k)\sin\varphi_a(\omega_k)\,x_k^{\bullet}\left(J_z c_{y\delta}+m x_k^{\bullet} c_{\delta\delta}\right)a_{k\mu}^2}{\omega_k\mu_k m J_z^2}. \tag{3.28.4}$$

Quantity $\varepsilon_k^{(0)}$ determines the minimum value of damping factor ε_k, at which oscillations still remain stable. Specifically, when $\varepsilon_k=\varepsilon_k^{(0)}$ the hodograph of open system of automatic control at certain frequency $\omega_k^{(0)}$, close to frequency ω_k, passes through point $X = 1$, $Y = 0$, and in this case the oscillations acquire a harmonic character.

As was already noted above, in § 22, the camping factor depends on the amplitude of oscillations of free surface of fluid in the fuel tank and increases with growth of amplitude of these oscillations. The limit which the damping factor ε_k, approaches when the amplitude of oscillations of the free surface of fluid in k-th tank approaches zero, as a rule, is inadequate for fulfilling the condition of stability (3.28.3). In this instance equality $\varepsilon_k=\varepsilon_k^{(0)}$ is fulfilled only for some particular value of amplitude of oscillations of the free surface of fluid in k-th fuel tank, which can be calculated, having available for this tank experimental data about the relationship of damping factor to the amplitude of oscillations of the free surface. The value of amplitude of oscillations of the free surface of fluid in k-th fuel tank, at which damping factor ε_k, assumes value $\varepsilon_k^{(0)}$, determined by formula (3.28.4), we will designate below through $\beta_k^{(0)}$.[1] In the considered case one of the possible motions of the rocket will be its motion in the pitching plane, at which the free surface of fluid in k-th fuel tank accomplishes harmonic oscillations with amplitude $\beta_k^{(0)}$. With values of amplitude of oscillations of the free surface of fluid in k-th fuel tank, which do not reach critical value $\beta_k^{(0)}$, there will exist inequality $\varepsilon_k<\varepsilon_k^{(0)}$, oscillations will be unstable and their amplitude will increase. With values of amplitude of oscillations of the free surface exceeding $\beta_k^{(0)}$, there will exist inequality $\varepsilon_k>\varepsilon_k^{(0)}$ and oscillations will be damped. Thus, the above-indicated motion of the rocket in the pitching plane, at which the free surface of fluid in k-th fuel tank accomplishes harmonic oscillations with amplitude $\beta_k^{(0)}$, is *stable self-oscillating motion,* which will inevitably appear in the conditions analyzed by us.

In § 26 we showed that when $\Delta\delta_\vartheta=e^{i\omega t}, \Delta F_y=0, \Delta M_z=0$ the differential equation of disturbed motion (3.24.12) have a solution, determined by formulas (3.26.2). In accordance with this when

$$\Delta\delta_\vartheta=\delta_0 e^{i\omega t},\quad \Delta F_y=0,\quad \Delta M_z=0 \tag{3.28.5}$$

differential equations of distributed motion will have solutions

$$\Delta V_y=\delta_0 V e^{i\omega t},\quad \Delta\vartheta=\delta_0\theta e^{i\omega t},\quad \Delta\beta_j=\delta_0 B_j e^{i\omega t},\quad j=1,2,\ldots,N, \tag{3.28.6}$$

since these differential equations are linear. Thus, oscillations of actuating elements of the control system, having amplitude δ_0, cause oscillations of the free surface of fluid in k-th fuel tank, having amplitude $\delta_0|B_k|$. According to (3.27.11) and (3.27.14) there should exist equality

$$\sqrt{4(\omega-\omega_k^*)+\varepsilon_k^2}\,|B_k|=\frac{a_{ky}\,|J_z c_{y\varepsilon}+m\,x_{2k}c_{\vartheta\varepsilon}|}{\omega_k\mu_k m J_z}. \tag{3.28.7}$$

[1]During calculation of amplitude $\beta_k^{(0)}$ we should consider the relationship of frequency characteristics of the automatic stabilization control, which figure in formula (3.28.4), to the amplitude of input signal of the automatic stabilization control. The account of nonlinearity of the automatic stabilization control is examined in detail at the end of the paragraph.

In accordance with this equality the amplitude of oscillations of the free surface of fluid in k-th fuel tank will be determined by expression

$$\frac{a_{k\nu}\left|J_z c_{yk} + m x_k^{\circ} c_{bk}\right| \delta_0}{\omega_k \mu_k m J_z \sqrt{4(\omega - \omega_k^{\circ})^2 + \varepsilon_k^2}}, \tag{3.28.8}$$

where ω – frequency of oscillations of actuating elements of the control system; δ_0 – amplitude of these oscillations. Assuming in this expression $\omega = \omega_k^{(0)}$, where $\omega_k^{(0)}$ -- frequency of investigated self-oscillations, and equating the obtained result to the amplitude of self-oscillation of free surface of fluid in k-th fuel tank $\beta_k^{(0)}$, for the amplitude of self-oscillations of actuating elements of the control system we obtain formula

$$\delta_0 = \frac{\omega_k \mu_k m J_z \sqrt{4(\omega_k^{(0)} - \omega_k^{\circ})^2 + \varepsilon_k^{(0)2}}}{a_{k\nu}\left|J_z c_{yk} + m x_k^{\circ} c_{bk}\right|} \beta_k^{(0)}, \tag{3.28.9}$$

where $\varepsilon_k^{(0)}$ – value of damping factor ε_k, corresponding to the self-oscillating mode of oscillations considered by us.

The point of the hodograph of open system, corresponding to the frequency of self-oscillation $\omega_k^{(0)}$, has coordinates $X = 1$, $Y = 0$. In accordance with formula (3.27.11) and the second of formulas (3.27.21) there should take place equality

$$2(\omega_k^{(0)} - \omega_k^{\circ}) \sin \varphi_a(\omega_k) + \varepsilon_k^{(0)} \cos \varphi_a(\omega_k) = 0. \tag{3.28.10}$$

According to (3.28.10) formula (3.28.9) can be converted so:

$$\delta_0 = \frac{\omega_k \mu_k m J_z \varepsilon_k^{(0)}}{a_{k\nu}\left|J_z c_{yk} + m x_k^{\circ} c_{bk}\right| \left|\sin \varphi_a(\omega_k)\right|} \beta_k^{(0)}. \tag{3.28.11}$$

In the considered case the quantity, figuring in the right side of equality (3.28.4), is positive and formula (3.28.4) can be given the form

$$\beta_k^{(0)}=\frac{k_a(\omega_k)\,|\sin\varphi_a(\omega_k)|\,|x_k^{*}|\,|J_z c_{y3}+m x_k^{*} c_{\beta 3}|\,a_{ky}^2}{\omega_k \mu_k m J_z^2}. \qquad (3.28.12)$$

By substituting (3.28.12) in (3.28.11), for the amplitude of self-oscillation of actuating elements of the control system we obtain formula

$$\delta_0=\frac{k_a(\omega_k)\,|x_k^{*}|\,a_{ky}}{J_z}\beta_k^{(0)}. \qquad (3.28.13)$$

In self-oscillating mode the amplitude of oscillations of input signal of the automatic stabilization control ϑ_0 will be connected with the amplitude of oscillations of actuating elements of the control system δ_0 by relationship

$$\delta_0=k_a(\omega_k^{(0)})\,\vartheta_0. \qquad (3.28.14)$$

By considering the proximity of frequency of self-oscillation $\omega_k^{(0)}$ to frequency ω_k, it is possible to substitute equality (3.28.14) by approximate equality

$$\delta_0=k_a(\omega_k)\,\vartheta_0. \qquad (3.28.15)$$

According to (3.28.13) and (3.28.15) the amplitude of self-oscillations of the input signal of automatic stabilization control can be determined by formula

$$\vartheta_0=\frac{|x_k^{*}|\,a_{ky}}{J_z}\beta_k^{(0)}. \qquad (3.28.16)$$

By means of formulas (3.28.4) and (3.28.16) we can determine the amplitude of self-oscillations of the free surface of fluid in k-th fuel tank $\beta_k^{(0)}$ and the amplitude of self-oscillations of the input signal of automatic stabilization control ϑ_0, having available experimental data about the relationship of damping factor to the amplitude of of oscillations of the free surface in k-th tank.

Large amplitudes of self-oscillation of free surfaces of fluids in fuel tanks are inadmissible, since they cause rapid drop of boost pressure.[1] Large amplitudes of self-oscillations of the input signal of automatic stabilization control are also inadmissible, since because of these oscillations the interferences at the input of the automatic stabilization control can exceed their permissible level. Reduction of the amplitudes of self-oscillations to their permissible level is attained by selection of rather effective inter-tank equipment, raising the damping factors. Raising the damping factor can be achieved, specifically, by installation of longitudinal perforated ribs in the fuel tank. The cross section of the tank, equipped with such longitudinal damping ribs, is shown in Fig. 3.18. The number of damping ribs and their profile are selected so that the required damping factor would be attained in the given tank at a permissible amplitude of oscillations of the free surface of fluid.

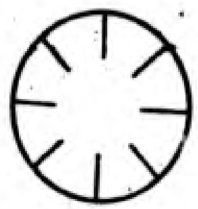

Fig. 3.18.

With the presence of the longitudinal ribs, shown in Fig. 3.18, the wetted surface of the tank will cease being a surface of revolution, which was proposed by us in the greater part of this chapter. In this instance the moment of inertia of fluid, located in the fuel tank, becomes nonzero relative to the axis of this tank, since in this instance in the process of rotation of the tank around its longitudinal axis the fluid is drawn into rotation by the ribs seizing

[1]The effect of oscillations of the free surface of fluid in the fuel tank on the boost pressure is comprehensively examined usually in the course of rocket design.

it. Furthermore, with the presence of ribs the oscillations of the tank around its longitudinal axis cause oscillations of the free surface of the fluid located in it, which were not considered by us during examination of the stability of motion of the rocket in rolling plane. Calculation investigation of the above-indicated effects is connected with large calculating difficulties; for account of these effects in the question about stabilization of rotation of the rocket around its longitudinal axis there are usually applied experimental methods. Description of these experimental methods is contained in book [14].

As was noted above, the amplification factor of automatic stabilization control $k_a(\omega_k)$ and phase shift $\varphi_a(\omega_k)$ depend on the amplitudes of oscillations of input signal ϑ_0. During calculation if the amplitudes of self-oscillations this dependence is considered in the following manner. In the first approximation when determining the required damping $\varepsilon_k^{(0)}$ in formula (3.28.4) we place limiting values, which quantities $k_a(\omega_k)$ and $\varphi_a(\omega_k)$ approach when $\vartheta_0 \to 0$. Having found further the amplitude of oscillations of free surface of fluid $\beta_k^{(0)}$, by formula (3.28.16) we determine the first approximation for amplitude of oscillations of the input signal of automatic stabilization control ϑ_0. In the next approximation for determination of required damping $\varepsilon_k^{(0)}$ in formula (3.28.4) we place values of amplification factor $k_a(\omega_k)$ and phase shift $\varphi_a(\omega_k)$, corresponding to the first approximation for amplitude of oscillations of the input signal of automatic stabilization control ϑ_0, further we calculate the second approximation for the amplitude of oscillations of free surface of fluid $\beta_k^{(0)}$ and for amplitude of oscillations of the input signal of automatic stabilization control ϑ_0 and so forth. This process of successive approximations is usually well convergent and rapidly leads to the target.

The methods of investigation discussed in this chapter are applied for preliminary selection of the frequency characteristics of automatic stabilization controls and for development of intertank equipment, necessary for stabilization of oscillations of liquid components of propellant. Further check of stability of motion and final selection of frequency characteristics of automatic stabilization controls are performed usually by the method of electromodeling with actual equipment of the control system.

CHAPTER IV

STABILIZATION OF ROCKET MOTION WITH ACCOUNT OF ELASTICITY OF ITS CONSTRUCTION

§ 1. The Simplest Statement of the Problems About Flexural Vibrations of the Rocket Body

During investigation of elastic oscillations of the rocket, which appear in the process of its flight, we will restrict ourselves to examination of flexural vibrations of the rocket body in the pitching plane. In the question about stabilization of flexural vibrations of the rocket all calculation schemes for yawing plane are practically no different from corresponding calculation schemes for the pitching plane. By replacing the bending equation in calculation schemes, constructed for pitching plane, by equation of torsion, calculation schemes can be obtained for solution of the problem about stabilization of torsional vibrations of the rocket.

Calculation of deformations of various structural elements of the rocket, appearing under action of predetermined loads, is a very complex problem of structural mechanics and in connection with this, during formation of motion equations of the rocket, considering the elasticity of its construction, we always proceed from replacement of actual construction of the rocket by some simplified mechanical model of it. The question about stabilization of flexural vibrations of the rocket body will be examined proceeding from the simplest original assumptions, which we will formulate in this paragraph.

With derivation of equations of rocket motion in pitching plane, considering the elasticity of its construction, we will proceed below from the following assumptions.

1. We will assume that the rocket body is deformed as a hollow elastic rod, and during calculation of its flexural deformations we will proceed from corresponding equations of strength of materials.

2. We will disregard the mutual affect of compressive (stretching) shear, bending and twisting strains and in accordance with this when examining flexural vibrations of the rocket we will use equations of pure bending.

3. We will assume that particles of the rocket, which at initial moment of time lie in some flat cross section of it do not change their mutual arrangement in the process of motion of the rocket. In reference to the rocket body this assumption coincides with the hypothesis of flat cross sections, forming the basis of strength of materials. For liquid propellant components the given assumption excludes the possibility of wave formation on the free surface of liquids. This assumption also excludes the possibility of account of real deformations of structural elements, fastened to the supporting body, and deformations of attachment points.

4. We will assume that the extreme cross sections of the rocket are free from loads. In accordance with this, during examination of bending of the rocket we will consider that shearing force and bending moment at the ends of the rocket become zero.

In previous chapters we assumed that the axes of fixed system of coordinates x, y, z coincide with principal central axes of inertia when free surfaces of fluids have the shape of planes, normal to the longitudinal axis of the rocket. Subsequently, by considering the elasticity of rocket construction during derivation of equations of motion, as before we will assume that axes of mobile system of coordinates x, y, z always coincide with principal central axes of inertia of the rocket.

Bending of the rocket body in the pitching plane affects the aerodynamic and reactive forces, acting on the rocket. In each cross section of the rocket body we will consider so-called "local" angle of attack, made up of angle α, formed by positive direction of axis x and direction of velocity vector of the center of mass of the rocket, and angle of rotation of the given section, appearing in the process of bending of the body. Lateral aerodynamic load per unit of length will be considered proportional to this local angle of attack.[1] The effect of bending of the body on the vector of total thrust force of engines $\vec{P}$ will involve rotation of this vector, as a result of which in mobile system of coordinates x, y, z at vector $\vec{P}$ there will appear a lateral component. We will consider this lateral force as concentrated force, applied to the rocket body at the place of attachment of the power plant to it. Furthermore, bending of the rocket body will cause displacement of the line of action of longitudinal component of vector $\vec{P}$, as a result of which there will appear additional moment of reactive forces relative to axis z. We will disregard the effect of bending of rocket body on controlling forces.

In the following paragraph, being based on the above-formulated original prerequisites, we will derive differential equation of flexural vibrations of the rocket body in the pitching plane, by using which we further investigate the stability of motion of the rocket in this plane with allowance for elasticity of the rocket construction.

[1]Such type of account of the effect of bending of rocket body on aerodynamic forces is very approximate, however, we used it in dynamic calculations, since oscillations of aerodynamic forces, caused by flexural vibrations of the body, insignificantly affect the motion of the rocket.

§ 2. Differential Equation of Flexural Vibrations

In accordance with the assumptions formulated in the previous paragraph, differential equation of flexural vibrations of the rocket body in the pitching plane can be obtained, having constructed the equation of oscillations of an elastic hollow rod, which undergoes pure bending in plane $z = 0$ in the process of oscillations. Proceeding from d'Alembert principle the equation of oscillations of an elastic rod can be formulated from equations of its elastic equilibrium, having added forces of inertia to the external forces acting on the rod. In case of pure bending in plane $z = 0$ equation of equilibrium of elastic rod have the form (see [25])

$$\frac{dQ}{dx}=q_y,\quad \frac{dM}{dx}=Q-m_z,\quad B_y\frac{d^2v}{dx^2}=M, \tag{4.2.1}$$

where q_y - *lateral linear force,* acting on the rod; Q - *shearing force,* appearing in the elastic rod; m_z - *linear moment of external forces,* acting on the rod; M - *bending moment;* v - *deflection of rod;* B_y *-flexural rigidity.*

In order to obtain equations of flexural vibrations of the rod from equations of equilibrium (4.2.1) to lateral lines force q_y it is necessary to add lateral linear force q_{yu}, created by forces of inertia, and to linear moment m_z - linear moment of forces of inertia m_{zu}. Furthermore, in equations (4.2.1) the symbols of ordinary differentiation should be replaced by symbols of partial differentiation, since in the process of oscillations unknown functions v, M and Q will depend on two independent variables - coordinates of section x and time t. As a result we obtain system of equations

$$\frac{\partial Q}{\partial x}=q_y+q_{yu},\quad \frac{\partial M}{\partial x}=Q-m_z-m_{zu},\quad B_y\frac{\partial^2 v}{\partial x^2}=M. \tag{4.2.2}$$

In order to determine linear force q_{yu} and linear moment m_{zu}, let us calculate the forces of inertia appearing in the process of motion, being accomplished by the rocket body in launch system of coordinates $x_0y_0z_0$. As coordinate axes x, y, z we selected in § 1 axes combined with principal central axes of inertia of the rocket. In this instance the relative motion of particles of the rocket in mobile coordinate system x, y, z can appear only as a result of deformations of the rocket, since its forward movement in system of coordinates x, y, z changes the position occupied by the center of mass of the rocket in this coordinate system, and rotation of the rocket relative to mobile coordinate axes disturbs the equality of centrifugal inertia moment J_{xy}, J_{xz}, J_{yz} to zero. In connection with our assumed smallness of elastic oscillations the relative velocities of particles of the rocket $\vec{v}_{\text{OTH}}$ can be considered small. Rotation of the rocket is assumed slow by us and, thus, when determining absolute accelerations of particles of the rocket $\vec{w}$ it is possible to disregard migratory acceleration $\vec{\omega}\times(\vec{\omega}\times\vec{r})$ and Coriolis acceleration $2(\vec{\omega}\times\vec{v}_{\text{отн}})$ ($\vec{\omega}$ - angular velocity of rotation of mobile coordinate system x, y, z, $\vec{r}$ - radius vector of particle of the rocket in this coordinate system). For absolute acceleration of rocket particles $\vec{w}$ we obtain approximate formula

$$\vec{w}=\vec{w}_0+\frac{d\vec{\omega}}{dt}\times\vec{r}+\vec{w}_{\text{отн}}, \tag{4.2.3}$$

where $\vec{w}_0$ - absolute acceleration of the origin of mobile system of coordinates x, y, z; $\vec{w}_{\text{отн}}$ - relative accelerations of rocket particles.

According to (4.2.3) in the case of plane motion of the rocket, i.e., when $\omega_x\equiv\omega_y\equiv v_{0z}\equiv 0$, there will exist equalities:

$$\begin{aligned} w_x &= w_{0x}-y\frac{d\omega_z}{dt}+w_{\text{отн}\,x},\\ w_y &= w_{0y}+x\frac{d\omega_z}{dt}+w_{\text{отн}\,y}. \end{aligned} \tag{4.2.4}$$

Let us determine now the relative accelerations $w_{отн\,x}$ and $w_{отн\,y}$, which appear in the process of flexural vibrations of the rocket in the pitching plane. Let us examine the relative motion of rocket particles, which lie in its plane cross section, having coordinate x in nondeformed condition of the body. In accordance with assumptions made in § 1, in the process of flexural vibrations these particles will not change their mutual arrangement and, thus, the particle having coordinates x, y in nondeformed state of the rocket body in the process of bending will assume positions determined by coordinates x', y', where in accordance with Fig. 4.1.

$$x'=x+u-y\sin\varphi,\quad y'=y+v-y(1-\cos\varphi), \tag{4.2.5}$$

Here u, v - forward movements of the section in the direction of axes x and y; ϕ - angle of rotation of the section around its axis, parallel to axis z. Angle ϕ will be connected with deflection of body v by relationship

$$\mathrm{tg}\,\varphi=\frac{\partial v}{\partial x}. \tag{4.2.6}$$

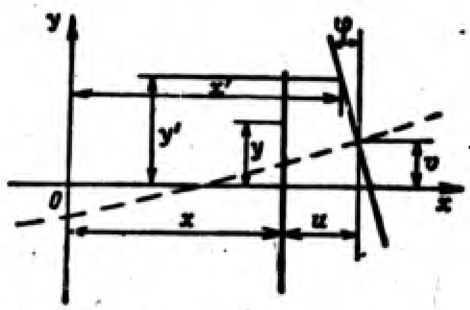

Fig. 4.1.

In accordance with the assumption about smallness of elastic oscillations of the rocket, disregarding small quantities of higher orders of smallness, it is possible to assume

$$\sin\varphi=\frac{\partial v}{\partial x},\quad \cos\varphi=1. \tag{4.2.7}$$

According to (4.2.7) relationships (4.2.5) can be substituted by approximate relationships

$$x' = x + u - y\frac{\partial v}{\partial x}, \quad y' = y + v. \qquad (4.2.8)$$

By using formulas (4.2.8), we find

$$w_{\text{отн}\,x} = \frac{\partial^2 x'}{\partial t^2} = \frac{\partial^2 u}{\partial t^2} - y\frac{\partial^3 v}{\partial x\,\partial t^2},$$
$$w_{\text{отн}\,y} = \frac{\partial^2 y'}{\partial t^2} = \frac{\partial^2 v}{\partial t^2}. \qquad (4.2.9)$$

By substituting (4.2.9) in (4.2.4), for absolute accelerations w_x, w_y formulas:

$$w_x = w_{0x} + \frac{\partial^2 u}{\partial t^2} - y\left(\frac{d\omega_z}{dt} + \frac{\partial^3 v}{\partial x\,\partial t^2}\right),$$
$$w_y = w_{0y} + \frac{\partial^2 v}{\partial t^2} + x\frac{d\omega_z}{dt}. \qquad (4.2.10)$$

Let us now construct two plane cross sections of the rocket S and S', having coordinates x and $x + dx$ in nondeformed state of the body (Fig. 4.2). The elementary cylinder (shaded in the figure) with the area of base ds and height dx is affected by force of inertia, projection of which to axis x is equal to $w_x \rho ds dx$, projection of this force to axis y is equal to $w_y \rho ds dx$ (ρ - density of the considered elementary cylinder). Thus, lateral load $q_{yu} dx$ and moment load $m_{zu} dx$, created by forces of inertia affecting the particles of the rocket, enclosed between sections S and S', will be determined by relationships:

$$q_{yu}\,dx = -\int_S w_y \rho\,ds\,dx, \quad m_{zu}\,dx = \int_S w_x y \rho\,ds\,dx. \qquad (4.2.11)$$

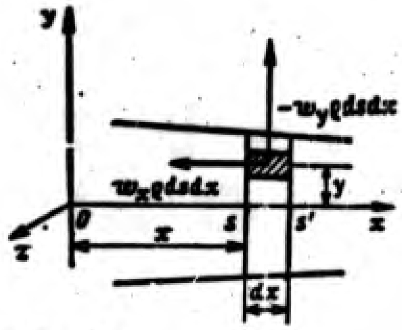

Fig. 4.2.

According to (4.2.11) linear force q_{yu} and linear moment m_{zu} can be found by formulas:

$$q_{yu} = -\int_S w_y \rho \, ds, \quad m_{zu} = \int_S w_x y \rho \, ds. \tag{4.2.12}$$

By substituting (4.2.10) in (4.2.12), we obtain relationships:

$$\begin{gathered} q_{yu} = -\left(w_{0y} + \frac{\partial^2 v}{\partial t^2} + x\frac{d\omega_z}{dt}\right)\int_S \rho \, ds, \\ w_{zu} = \left(w_{0x} + \frac{\partial^2 u}{\partial t^2}\right)\int_S y\rho \, ds - \left(\frac{d\omega_z}{dt} + \frac{\partial^3 v}{\partial x \, \partial t^2}\right)\int_S y^2 \rho \, ds. \end{gathered} \tag{4.2.13}$$

We assume everywhere that planes $y = 0$ and $z = 0$ are planes of symmetry of the rocket. Thus, there should take place equality

$$\int_S y\rho \, ds = 0, \tag{4.2.14}$$

in accordance with which formulas (4.2.13) can be given the form

$$\begin{gathered} q_{yu} = -\mu\left(w_{0y} + \frac{\partial^2 v}{\partial t^2} + x\frac{d\omega_z}{dt}\right), \\ m_{zu} = -J_z\left(\frac{d\omega_z}{dt} + \frac{\partial^3 v}{\partial x \, \partial t^2}\right), \end{gathered} \tag{4.2.15}$$

where

$$\mu = \int_S \rho \, ds, \quad J_z = \int_S y^2 \rho \, ds. \tag{4.2.16}$$

By substituting (4.2.15) in (4.2.2), we obtain equations:

$$\begin{aligned}
&\frac{\partial Q}{\partial x}=q_y-\mu\left(w_{0y}+\frac{\partial^2 v}{\partial t^2}+x\frac{\partial\omega_z}{\partial t}\right),\\
&\frac{\partial M}{\partial x}=Q-m_z+J_z\left(\frac{d\omega_z}{dt}+\frac{\partial^3 v}{\partial x\,\partial t^2}\right),\\
&B_z\frac{\partial^2 v}{\partial x^2}=M.
\end{aligned}\tag{4.2.17}$$

By using the second and third equations from (4.2.17), we find

$$Q=\frac{\partial}{\partial x}\left(B_z\frac{\partial^2 v}{\partial x^2}\right)+m_z-J_z\left(\frac{d\omega_z}{dt}+\frac{\partial^3 v}{\partial x\,\partial t^2}\right).\tag{4.2.18}$$

By substituting shearing force Q from (4.2.18) in the first equation of system (4.2.17), for function $v(x, t)$ we obtain differential equation in partial derivatives

$$\begin{aligned}
&\frac{\partial^2}{\partial x^2}\left(C_z\frac{\partial^2 v}{\partial x^2}\right)-\frac{\partial}{\partial x}\left(J_z\frac{\partial^3 v}{\partial x\,\partial t^2}\right)+\mu\frac{\partial^2 v}{\partial t^2}=q_y-\\
&-\frac{\partial m_z}{\partial x}-\mu w_{0y}-\left(\mu x-\frac{dJ_z}{dx}\right)\frac{d\omega_z}{dt}.
\end{aligned}\tag{4.2.19}$$

In accordance with assumptions formulated in § 1, there should exist equalities

$$M=Q=0 \text{ when } x=a \text{ and } x=b,\tag{4.2.20}$$

where a and b - abscissas of extreme cross sections of the rocket. According to (4.2.17), (4.2.18) and (4.2.20) the solution of differential equation (4.2.19) $v(x, t)$ must satisfy boundary conditions:

$$\left.\begin{aligned}
&B_z\frac{\partial^2 v}{\partial x^2}=0\\
&\frac{\partial}{\partial x}\left(B_z\frac{\partial^2 v}{\partial x^2}\right)-J_z\frac{\partial^3 v}{\partial x\,\partial t^2}=-m_z+J_z\frac{d\omega_z}{dt}
\end{aligned}\right|\ \text{when } x=a \text{ and when } x=b.\tag{4.2.21}$$

Thus, at prescribed loads q_y and m_z and with predetermined motion of mobile system of coordinates x, y, z the determination of deflection of body $v(x, t)$ is reduced to solution of differential equation of flexural vibrations (4.2.19) with boundary conditions (4.2.21). The question about motion, accomplished by mobile coordinate system x, y, z, is considered by us in the following paragraph of this chapter.

§ 3. Equation of Forces and Equation of Moments

Equations of motion being accomplished by mobile system of coordinates x, y, z in initial launch coordinate system x_0, y_0, z_0, can be constructed, having used the theorem of motion of the center of mass of a material system and theorem of change of moment of momentum, since axes of coordinates x, y, z are combined by us with principal central axes of inertia of the rocket. If to external forces, acting on the rocket, we add reactive forces and designate the principal vector of the obtained system of forces by $\vec{Q}$ and the principal moment of these forces relative to origin of mobile coordinate system x, y, z, by $\vec{M}$ there will exist equalities:

$$mw_{0y}=Q_y, \quad J_z\frac{d\omega_z}{dt}=M_z, \qquad (4.3.1)$$

where J_z - moment of inertia of the rocket relative to axis z let us recall that rotation of the rocket is assumed slow by us). In § 2 when examining bending of the rocket body in the pitching plane through q_y we designated linear lateral force, acting on the rocket, and through m_z - linear moment of external forces. Thus, external forces, acting in the pitching plane on the section of the rocket contained between its cross sections, having abscissas x and $x + dx$, will be reduced to force $q_y dx$ and to moment $m_z dz$. The moment of this system of forces relative to axis z will be equal to $(q_y x+m_z)dx$. In accordance with this, the projection of principal vector of external forces Q_y and projection of principal moment M_z can be determined by formulas:

$$Q_y = \int_a^b q_y\,dx, \quad M_z = \int_a^b (q_y x + m_z)\,dx. \tag{4.3.2}$$

According to (4.3.2) equations (4.3.1) can be given the form

$$m w_{0y} = \int_a^b q_y\,dx,$$

$$J_z \frac{d\omega_z}{dt} = \int_a^b (q_y x + m_z)\,dx. \tag{4.3.3}$$

Equations of forces and moments (4.3.3) together with differential equation of flexural vibrations (4.2.19) and boundary conditions (4.2.21) determine the motion of mobile coordinate system in the pitching plane and flexural vibrations of body, appearing in this plane in the process of motion of the rocket.

Let us now show that function $v(x, t)$, found from differential equation (4.2.19) with boundary conditions (4.2.21), will always satisfy integral conditions:

$$\int_a^b v\mu\,dx = 0, \quad \int_a^b \left(xv\mu + \frac{\partial v}{\partial x} j_z\right) dx = 0 \tag{4.3.3}$$

in accordance with equations (4.3.3), which determine accelerations w_{0y} and $\frac{d\omega_z}{dt}$. Integral relationships (4.3.4) play an important role during investigation of flexural vibrations of the rocket body.

By integrating both sides of equality (4.2.19) with respect to variable x from a to b, we obtain relationship

$$\left[\frac{\partial}{\partial x}\left(B_y \frac{\partial^2 v}{\partial x^2}\right) - j_z \frac{\partial^3 v}{\partial x\,\partial t^2}\right]_{x=a}^{x=b} + \int_a^b \frac{\partial^2 v}{\partial t^2} \mu\,dx =$$

$$= \int_a^b q_y\,dx - w_{0y} \int_a^b \mu\,dx + \left(-m_z + j_z \frac{d\omega_z}{dt}\right)_{x=a}^{x=b} - \frac{d\omega_z}{dt} \int_a^b x\mu\,dx,$$

or according to (4.2.21)

$$\int_a^b \frac{\partial^2 v}{\partial t^2}\mu\, dx = \int_a^b q_v\, dx - w_{0v}\int_a^b \mu\, dx - \frac{d\omega_z}{dt}\int_a^b x\mu\, dx. \quad (4.3.5)$$

Having multiplied both sides of equation (4.2.19) by x and integrated with respect to x from a to b, we obtain equality

$$\int_a^b x\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(B_v\frac{\partial^2 v}{\partial x^2}\right) - j_z\frac{\partial^3 v}{\partial x\,\partial t^2}\right]dx + \int_a^b x\frac{\partial^2 v}{\partial t^2}\mu\, dx =$$

$$= \int_a^b xq_v\, dx - w_{0v}\int_a^b x\mu\, dx - \frac{d\omega_z}{dt}\int_a^b x^2\mu\, dx -$$

$$- \int_a^b x\frac{\partial}{\partial x}\left(m_z - j_z\frac{d\omega_z}{dt}\right)dx.$$

Having integrated this relationship by parts, it is possible to convert it in the following manner:

$$-\int_a^b\left[\frac{\partial}{\partial x}\left(B_v\frac{\partial^2 v}{\partial x^2}\right) - j_z\frac{\partial^3 v}{\partial x\,\partial t^2}\right]dx + \left\{x\left[\frac{\partial}{\partial x}\left(B_v\frac{\partial^2 v}{\partial x^2}\right) -\right.\right.$$

$$\left.\left.- j_z\frac{\partial^3 v}{\partial x\,\partial t^2}\right]\right\}_{x=a}^{x=b} + \int_a^b x\frac{\partial^2 v}{\partial t^2}\mu\, dx = \int_a^b xq_v\, dx - w_{0v}\int_a^b x\mu\, dx -$$

$$-\frac{d\omega_z}{dt}\int_a^b x^2\mu\, dx + \int_a^b\left(m_z - j_z\frac{d\omega_z}{dt}\right)dx + \left[x\left(-m_z + j_z\frac{d\omega_z}{dt}\right)\right]_{x=a}^{x=b},$$

or according to (4.2.21)

$$-\int_a^b\left[\frac{\partial}{\partial x}\left(B_v\frac{\partial^2 v}{\partial x^2}\right) - j_z\frac{\partial^3 v}{\partial x\,dt^2}\right]dx + \int_a^b x\frac{\partial^2 v}{\partial t^2}\mu\, dx =$$

$$= \int_a^b xq_v\, dx - w_{0v}\int_a^b x\mu\, dx - \frac{d\omega_z}{dt}\int_a^b x^2\mu\, dx + \int_a^b\left(m_z - j_z\frac{d\omega_z}{dt}\right)dx. \quad (4.3.6)$$

In accordance with the first of boundary conditions (4.2.21) there should exist equality

$$\int_a^b \frac{\partial}{\partial x}\left(B_y \frac{\partial^2 v}{\partial x^2}\right) dx = \left(B_y \frac{\partial^2 v}{\partial x^2}\right)_{x=a}^{x=b} = 0 \tag{4.3.7}$$

and thus, relationship (4.3.6) can be given the form

$$\begin{gathered}\int_a^b \left(x \frac{\partial^2 v}{\partial t^2} \mu + j_z \frac{\partial^3 v}{\partial x \partial t^2}\right) dx = \int_a^b (x q_y + m_z) dx - \\ - w_{0y} \int_a^b x\mu\, dx - \frac{d\omega_z}{dt} \int_a^b (x^2\mu + j_z) dx.\end{gathered} \tag{4.3.8}$$

By using equations (4.3.3), it is possible to convert equalities (4.3.5) and (4.3.8):

$$\begin{gathered}\int_a^b \frac{\partial^2 v}{\partial t^2} \mu\, dx = w_{0y}\left(m - \int_a^b \mu\, dx\right) - \frac{d\omega_z}{dt} \int_a^b x\mu\, dx, \\ \int_a^b \left(x \frac{\partial^2 v}{\partial t^2} \mu + j_z \frac{\partial^3 v}{\partial x \partial t^2}\right) dx = -w_{0y} \int_a^b x\mu\, dx + \\ + \frac{d\omega_z}{dt} \left[J_z - \int_a^b (x^2\mu + j_z) dx \right].\end{gathered} \tag{4.3.9}$$

Total mass of rocket m, coordinate of its center of mass x_C and moment of inertia J_z can be expressed by integrals:

$$m = \int_V \varrho\, d\tau, \quad x_C = \frac{1}{m} \int_V x\varrho\, d\tau, \quad J_z = \int_V (x^2 + y^2) \varrho\, d\tau, \tag{4.3.10}$$

where V - region occupied by the rocket with nondeformed state of its body (deformation of the body do not affect the coordinate of center of mass x_C because of our selection of mobile coordinate system x, y, z; we will everywhere disregard the effect of deformations of the body at moment of inertia J_z, since this moment of inertia enters equation (4.3.3) with small factor $\frac{d\omega_z}{dt}$). Integration of some function with respect to cross section of region S and having then completed integration of the obtained result with respect to variable x from a to b. Thus, formulas (4.3.10) can be given the form

$$m=\int_a^b\int_S \varrho\, ds\, dx,$$

$$x_C=\frac{1}{m}\int_a^b x\int_S \varrho\, ds\, dx,\ J_z=\int_a^b\left(x^2\int_S \varrho\, ds+\int_S y^2\varrho\, ds\right)dx,$$

or according to (4.2.16)

$$m=\int_a^b \mu\, dx,\quad x_C=\frac{1}{m}\int_a^b x\mu\, dx,\quad J_z=\int_a^b (x^2\mu+j_z)\, dx. \tag{4.3.11}$$

In accordance with equalities (4.3.11) relationships (4.3.9) can be reduced to form

$$\int_a^b \frac{\partial^2 v}{\partial t^2}\mu\, dx=0,\quad \int_a^b\left(x\frac{\partial^2 v}{\partial t^2}\mu+j_z\frac{\partial^3 v}{\partial x\,\partial t^2}\right)dx=0 \tag{4.3.12}$$

($x_C \equiv 0$, since the origin of coordinate system x, y, z is combined by us with the center of mass of the rocket). According to (4.3.12) there should exist equalities:

$$\frac{d^2}{dt^2}\left(\int_a^b v\mu\, dx\right)=0,\quad \frac{d^2}{dt^2}\left[\int_a^b\left(xv\mu+j_z\frac{\partial v}{\partial x}\right)dx\right]=0 \tag{4.3.13}$$

[when performing time differentiation in the left sides of relationships (4.3.13) the slow change of form of functions μ and j_z, appearing in the process of rocket flight, can be disregarded, since these functions figure in (4.3.13) with small factors v and $\frac{\partial v}{\partial x}$].

Let us assume that before the initial moment of time of rocket flight $t = t_0$ there are no deflections of its body $v(x, t)$. In this instance there will take place relationships:

$$\left.\begin{aligned} &\int_a^b v\mu\,dx=0, \quad \int_a^b\left(xv\mu+j_z\frac{\partial v}{\partial x}\right)dx=0,\\ &\frac{d}{dt}\left(\int_a^b v\mu\,dx\right)=0,\\ &\frac{d}{dt}\left[\int_a^b\left(xv\mu+j_z\frac{\partial v}{\partial x}\right)dx\right]=0.\end{aligned}\right|\ \text{when } t=t_0 \qquad (4.3.14)$$

By twice integrating the left and right sides of equalities (4.3.13) with respect to time t from t_0 to t, according to (4.3.14) we obtain integral relationships (4.3.4).

§ 4. Differential Equations of Motion of the Rocket in Pitching Plane Considering the Elasticity of the Rocket Body

Let us now construct expanded expressions for lateral linear force q_y and linear moment m_z, which enter equations of forces and moments and equations of flexural vibrations (4.2.19). Successively let us examine linear forces and linear moments, being generated by the force of gravity, aerodynamic forces, reactive forces and controlling forces and moments.

1. *Force of gravity*. In accordance with the first of formulas (4.2.16) function $\mu(x)$ represents the mass of rocket, taken per unit of its length in cross section S, having abscissa x. In connection with this, it is accepted to call function $\mu(x)$ the linear mass of

the rocket. Vector of linear force $\vec{q}$, being generated by force of gravity, will be equal to $\mu\vec{g}$, where $\vec{g}$ - vector of acceleration of force of gravity. By projecting vector $\vec{q}$ to axis y, we obtain relationship

$$q_y = \mu g_y. \quad (4.4.1)$$

The center of mass of any cross section of the rocket will lie on its longitudinal axis, since we assumed the rocket symmetric relative to planes $y = 0$ and $z = 0$. Thus, the force of gravity cannot generate linear moments and in this case there will take place equality

$$m_z = 0. \quad (4.4.2)$$

2. *Aerodynamic forces*. In this case we will consider lateral linear force proportional to the local angle of attack in accordance with the original prerequisites, formulated in § 1. By designating the proportionality factor through $c(x, t)$, we obtain equality

$$q_y = c\alpha^*, \quad (4.4.3)$$

where α^* - local angle of attack. As can be seen from Fig. 4.1 angle ϕ, to which the cross section of the rocket is turned in the process of bending, during small flexural vibrations can be considered equal to derivative $\partial v/\partial x$. In this instance the local angle of attack α^* will be determined by equality

$$\alpha^* = \alpha + \frac{\partial v}{\partial x}, \quad (4.4.4)$$

where α - angle of attack, corresponding to nondeformed state of rocket body, i.e., angle between direction of velocity vector of center of mass of the rocket and positive direction of axis x. By substituting (4.4.4) in (4.4.3) we obtain relationship

$$q_y = c\left(\alpha + \frac{\partial v}{\partial x}\right). \qquad (4.4.5)$$

In the considered case it is possible to assume

$$m_z = 0, \qquad (4.4.6)$$

since during investigation of the stability of motion the linear aerodynamic moments for all practical purposes can be disregarded.

3. *Reactive forces.* Lateral linear force q_y, corresponding to the action of concentrated force Q_y on the elastic rod applied in the section of rod with abscissa $x = \xi$, in strength of materials is considered as the limit, to which when $\varepsilon \to 0$ there approaches linear lateral force $q_y^*(x)$, determined by equalities:

$$q_y^*(x) = \frac{Q_y}{\varepsilon} \text{ when } \xi - \frac{\varepsilon}{2} < x < \xi + \frac{\varepsilon}{2},$$
$$q_y^*(x) = 0 \text{ when } a < x < \xi - \frac{\varepsilon}{2} \text{ and when } \xi + \frac{\varepsilon}{2} < x < b. \qquad (4.4.7)$$

Graph of function $q_y^*(x)$ is shown in Fig. 4.3. Formulas (4.4.7) can be given the form

$$q_y^*(x) = Q_y f(x, \xi), \qquad (4.4.8)$$

where $f(x, \xi)$ - function determined in interval $a < x < b$ by relationships:

$$f(x, \xi) = \frac{1}{\varepsilon} \text{ when } \xi - \frac{\varepsilon}{2} < x < \xi + \frac{\varepsilon}{2},$$
$$f(x, \xi) = 0 \text{ when } a < x < \xi - \frac{\varepsilon}{2} \text{ and when } \xi + \frac{\varepsilon}{2} < x < b. \qquad (4.4.9)$$

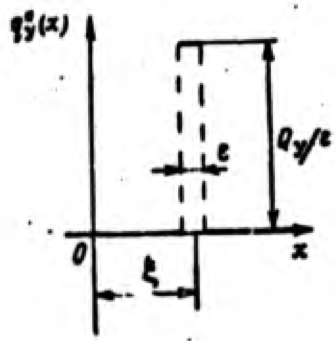

Fig. 4.3.

The limit which function $f(x, \xi)$ approaches when $\varepsilon \to 0$ is called delta function and is designated by $\delta(x, \xi)$. Thus, according to (4.4.8), in the considered case lateral linear force $q_y(x)$ can be expressed by relationship

$$q_y(x) = \lim_{\varepsilon \to 0} q_y^\varepsilon(x) = Q_y \delta(x, \xi). \tag{4.4.10}$$

In accordance with formulas (4.4.9) at any small ε there is observed equality

$$\int_a^b f(x, \xi)\, dx = \frac{1}{\varepsilon} \int_{\xi - \frac{\varepsilon}{2}}^{\xi + \frac{\varepsilon}{2}} dx = 1. \tag{4.4.11}$$

According to (4.4.9) and (4.4.11) delta function $\delta(x, \xi)$ possesses properties

$$\delta(x, \xi) = 0 \text{ when } x \neq \xi, \quad \int_a^b \delta(x, \xi)\, dx = 1. \tag{4.4.12}$$

In case of the action of concentrated moment M_z on the elastic rod, applied in the section of the rod with abscissa $x = \xi$, the corresponding linear moment $m_z(x)$ can be expressed by equality

$$m_z(x) = M_z \delta(x, \xi), \tag{4.4.13}$$

Analogous to equality (4.4.10).

Let us now designate by x_T the abscissa of the section of the rocket, in which to the supporting body there is fastened the power plant. Total thrust force $\vec{P}$ will have the line of action shown in Fig. 4.4. Projections of vector $\vec{P}$ to coordinate axes x and y will be determined by equalities:

$$P_x = P\cos\varphi,\ P_y = P\sin\varphi, \quad (4.4.14)$$

where φ angle between the direction of vector $\vec{P}$ and positive direction of axis x. In accordance with approximate equalities (4.2.7), established for angle φ in § 2, formulas (4.4.14) can be given the form

$$P_x = P,\quad P_y = P\left(\frac{\partial v}{\partial x}\right)_{x=x_T}. \quad (4.4.15)$$

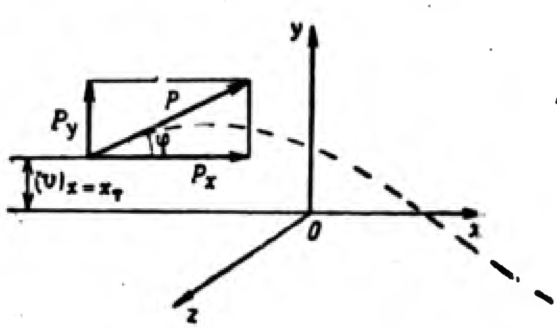

Fig. 4.4.

As can be seen from Fig. 4.4, reactive forces, acting on the rocket body in its section having abscissa $x = x_T$ will be reduced to lateral concentrated force Q_y and to concentrated moment M_z, which according to (4.4.15) can be found by formulas:

$$Q_y = P_y = P\left(\frac{\partial v}{\partial x}\right)_{x=x_T},\quad M_z = -P_x(v)_{x=x_T} = -P(v)_{x=x_T}. \quad (4.4.16)$$

In accordance with equalities (4.4.10) and (4.4.13) for linear lateral force q_y and linear moment m_z in the considered case there will exist relationships:

$$\left.\begin{aligned} q_y &= P\left(\frac{\partial v}{\partial x}\right)_{x=x_T}\delta(x, x_T), \\ m_z &= -P(v)_{x=x_T}\delta(x, x_T). \end{aligned}\right\} \qquad (4.4.17)$$

In this case it is necessary to make the following observation. In the process of derivation of formulas (4.4.17) as the point of reduction of reactive forces we used point $x = x_T$ of nondeformed axis of the rocket in accordance with the meaning, which is usually placed in linear force q_y and linear moment m_z in original equations of equilibrium (4.2.1). The account of the influence, which is shown on flexural vibrations of the rocket body by change of line of action of total thrust force $\vec{P}$ by means of formulas (4.4.17), is very approximate. However, in the problem about stability of flexural vibrations the given effect plays a comparatively unessential role in connection with this, its approximate account by means of formulas (4.4.17) is entirely sufficient for all practical purposes.

4. *Controlling forces and moments.* By x_y let us designate the abscissa of cross section of the rocket, in which its body is affected by controlling force, or controlling moment. If for motion of the rocket in the pitching plane the system of control creates controlling force $F_{y\ ynp}$, the appropriate linear lateral force q_y according to (4.4.10) can be determined by expression

$$q_y = F_{y\ ynp}\delta(x, x_y). \qquad (4.4.18)$$

If the control system forms concentrated controlling moment $M_{z\ ynp}$ (case of control of thrust misalignment), the corresponding linear moment m_z according to (4.4.13) can be represented by relationship

$$m_z = M_{z\,упр}\delta(x,\ x_y). \qquad (4.4.19)$$

In order not to subsequently separate at any time the above-indicated two cases of control of rocket motion in the pitching plane, we will assume below that the control system simultaneously generates linear lateral force q_y and linear moment m_z, expressed by equalities:

$$q_y = F_{y\,упр}\delta(x, x_y),\ m_z = M^{(0)}_{z\,упр}\delta(x, x_y), \qquad (4.4.20)$$

having agreed to consider that $M^{(0)}_{z\,упр} = 0$ when the control system forms controlling force, and assuming $F_{y\,упр} = 0$, $M^{(0)}_{z\,упр} = M_{z\,упр}$ for the case when actuating elements of the control system create concentrated controlling moment.

By uniting linear lateral forces, determined by formulas (4.4.1), (4.4.5), (4.4.17) and (4.4.20), for total lateral linear force q_y we obtain expression

$$q_y = \mu g_y + c\left(\alpha + \frac{\partial v}{\partial x}\right) + P\left(\frac{\partial v}{\partial x}\right)_{x=x_T}\delta(x, x_T) + F_{y\,упр}\delta(x, x_y). \qquad (4.4.21)$$

According to (4.4.2), (4.4.6), (4.4.17) and (4.4.20) the total linear moment m_z can generally be expressed by relationship

$$m_z = -P(v)_{x=x_T}\delta(x, x_T) + M^{(0)}_{z\,упр}\delta(x, x_y). \qquad (4.4.22)$$

By substituting (4.4.21) and (4.4.22) in equations of forces and moments (4.3.3), we obtain equations:

$$m w_{0y} = g_y \int_a^b \mu dx + \int_a^b \left(\alpha + \frac{\partial v}{\partial x}\right) c dx + P\left(\frac{\partial v}{\partial x}\right)_{x=x_T} \int_a^b \delta(x, x_T) dx +$$
$$+ F_{y\,y\text{пр}} \int_a^b \delta(x, x_y) dx,$$

$$J_z \frac{d\omega_z}{dt} = g_y \int_a^b \mu x dx + \int_a^b \left(\alpha + \frac{\partial v}{\partial x}\right) c x dx + P\left(\frac{\partial v}{\partial x}\right)_{x=x_T} \times$$
$$\times \int_a^b \delta(x, x_T) x dx - P(v)_{x=x_T} \int_a^b \delta(x, x_T) dx + F_{y\,y\text{пр}} \int_a^b \delta(x, x_y) x dx +$$
$$+ M^{(0)}_{z\,y\text{пр}} \int_a^b \delta(x, x_y) dx,$$

or according to (4.3.11) and (4.4.12)

$$m(w_{0y} - g_y) = \int_a^b \left(\alpha + \frac{\partial v}{\partial x}\right) c dx + P\left(\frac{\partial v}{\partial x}\right)_{x=x_T} + F_{y\,y\text{пр}},$$
$$J_z \frac{d\omega_z}{dt} = \int_a^b \left(\alpha + \frac{\partial v}{\partial x}\right) c x dx + P\left(\frac{\partial v}{\partial x}\right)_{x=x_T} \int_a^b \delta(x, x_T) \times \qquad (4.4.23)$$
$$\times x dx - P(v)_{x=x_T} + F_{y\,y\text{пр}} \int_a^b \delta(x, x_y) x dx + M^{(0)}_{z\,y\text{пр}}$$

(let us recall that the coordinate of center of gravity of the rocket x_C always remains equal to zero).

In accordance with formulas (4.4.12) for any function $f(x)$, assigned in interval $a \leq x \leq b$, at any small positive ε there should be observed equality

$$\int_a^b \delta(x, \xi) f(x) dx = f(\xi) \int_a^b \delta(x, \xi) dx + \int_a^b \delta(x, \xi)[f(x) - f(\xi)] dx =$$
$$= f(\xi) + \int_{\xi-\varepsilon}^{\xi+\varepsilon} \delta(x, \xi)[f(x) - f(\xi)] dx. \qquad (4.4.24)$$

If function $f(x)$ is continuous at point $x = \xi$, at any small η it is possible to find such a sufficiently small ε, so that there would exist inequality

$$|f(x)-f(\xi)|<\eta \text{ when } \xi-\varepsilon<x<\xi+\varepsilon. \qquad (4.4.25)$$

In this instance we obtain estimation

$$\left|\int_{\xi-\varepsilon}^{\xi+\varepsilon} \delta(x,\xi)[f(x)-f(\xi)]dx\right| < \eta\int_{\xi-\varepsilon}^{\xi+\varepsilon} \delta(x,\xi)dx=\eta. \qquad (4.4.26)$$

Inequality (4.4.26) can exist at any small η only when

$$\int_{\xi-\varepsilon}^{\xi+\varepsilon} \delta(x,\xi)[f(x)-f(\xi)]dx=0. \qquad (4.4.27)$$

From relationships (4.4.24) and (4.4.27) ensues equality

$$\int_a^b \delta(x,\xi)f(x)dx=f(\xi), \qquad (4.4.28)$$

valid for any function $f(x)$, continuous at point $x = \xi$.

According to (4.4.28) equations (4.4.23) can be given the form

$$\begin{aligned} m(w_{0y}-g_y) &= \int_a^b \left(a+\frac{\partial v}{\partial x}\right) c\,dx + P\left(\frac{\partial v}{\partial x}\right)_{x=x_t} + F_{y\,\text{упр}}, \\ J_z\frac{d\omega_z}{dt} &= \int_a^b \left(a+\frac{\partial v}{\partial x}\right) cx\,dx + P\left(x\frac{\partial v}{\partial x}-v\right)_{x=x_t} + \\ &\quad + x_y F_{y\,\text{упр}} + M^{(0)}_{z\,\text{упр}}. \end{aligned} \qquad (4.4.29)$$

By introducing meanings

$$F_y = \alpha \int_a^b c\,dx + F_{y\,\text{упр}},\quad M_z = \alpha \int_a^b c x\,dx + x_T F_{y\,\text{упр}} + M^{(0)}_{z\,\text{упр}}, \qquad (4.4.30)$$

it is possible to reduce equations (4.4.29) to form

$$\begin{aligned} m(w_{0y} - g_y) &= F_y + \int_a^b \frac{\partial v}{\partial x} c\,dx + P\left(\frac{\partial v}{\partial x}\right)_{x=x_T}, \\ J_z \frac{d\omega_z}{dt} &= M_z + \int_a^b \frac{\partial v}{\partial x} c x\,dx + P\left(x\frac{\partial v}{\partial x} - v\right)_{x=x_T}. \end{aligned} \qquad (4.4.31)$$

Equations (4.4.31) represent equations of forces and moments, coinsidering the elasticity of the rocket body. In these equations according to (4.4.30) force F_y is the projection of principal vector of aerodynamic and controlling forces, acting on the rocket in the absence of flexural vibrations of its body to axis y, moment M_z is the projection of principal moment of these forces to axis z.

Direct substitution of linear force q_y and linear moment m_z, determined by formulas (4.4.21) and (4.4.22), in the equation of flexural vibrations (4.2.19) leads to the necessity of differentiation of delta function. In order to avoid this operation, let us replace equation (4.2.19) by a system of two differential equations in partial derivatives for deflection $v(x, t)$ and shearing force $Q(x, t)$, which is determined by formula (4.2.18). We obtain a system of two equations

$$\left.\begin{aligned} &\mu \frac{\partial^2 v}{\partial t^2} + \frac{\partial Q}{\partial x} = q_y - \mu w_{0y} - \mu x \frac{d\omega_z}{dt}, \\ &\frac{\partial}{\partial x}\left(B_v \frac{\partial^2 v}{\partial x^2}\right) - j_z \frac{\partial^3 v}{\partial x \partial t^2} - Q = -m_z + j_z \frac{d\omega_z}{dt}. \end{aligned}\right| \qquad (4.4.32)$$

In accordance with formula (4.2.18) let us reduce boundary conditions (4.2.21) to the form

$$B_y \frac{\partial^2 v}{\partial x^2} = 0, \; Q = 0 \text{ when } x = a \text{ and when } x = b. \tag{4.4.33}$$

By substituting q_y and m_z from (4.4.21) and (4.4.22) in (4.4.32), we obtain system of equations

$$\begin{aligned} &\mu \frac{\partial^2 v}{\partial t^2} + \frac{\partial Q}{\partial x} = -\mu(w_{0y} - g_y) - \mu x \frac{d\omega_z}{dt} + c\left(\alpha + \frac{\partial v}{\partial x}\right) + \\ &+ P\left(\frac{\partial v}{\partial x}\right)_{x=x_T} \delta(x, x_T) + F_{y\,ynp} \delta(x, x_y), \\ &\frac{\partial}{\partial x}\left(B_y \frac{\partial^2 v}{\partial x^2}\right) - J_z \frac{\partial^3 v}{\partial x \partial t^2} - Q = J_z \frac{d\omega_z}{dt} + P(v)_{x=x_T} \times \\ &\times \delta(x, x_T) - M^{(0)}_{z\,ynp} \delta(x, x_y). \end{aligned} \tag{4.4.34}$$

By uniting equations of forces and moments (4.4.31), differential equations in partial derivatives (4.4.34) and boundary conditions (4.4.33), we obtain equations:

$$\left.\begin{aligned} &m(w_{0y} - g_y) = F_y + \int_a^b \frac{\partial v}{\partial x} c\,dx + P\left(\frac{\partial v}{\partial x}\right)_{x=x_T}, \\ &J_z \frac{d\omega_z}{dt} = M_z + \int_a^b \frac{\partial v}{\partial x} cx\,dx + P\left(x \frac{dv}{dx} - v\right)_{x=x_T}, \\ &\mu \frac{\partial^2 v}{\partial t^2} + \frac{\partial Q}{\partial x} = -\mu(w_{0y} - g_y) - \mu x \frac{d\omega_z}{dt} + \\ &+ c\left(\alpha + \frac{\partial v}{\partial x}\right) + P\left(\frac{\partial v}{\partial x}\right)_{x=x_T} \delta(x, x_T) + F_{y\,ynp} \delta(x, x_y), \\ &\frac{\partial}{\partial x}\left(B_y \frac{\partial^2 v}{\partial x^2}\right) - J_z \frac{\partial^3 v}{\partial x \partial t^2} - Q = J_z \frac{d\omega_z}{dt} + \\ &+ P(v)_{x=x_T} \delta(x, x_T) - M^{(0)}_{z\,ynp} \delta(x, x_y), \\ &B_y \frac{\partial^2 v}{\partial x^2} = 0, \; Q = 0 \text{ when } x = a \text{ and when } x = b. \end{aligned}\right| \tag{4.4.35}$$

Equations (4.4.35) represent motion equations of the rocket in plane pitching, considering elasticity of the rocket body. These equations determine the motion, being accomplished in the pitching plane by mobile system of coordinates x, y, z, and functions $v(x, t)$, $Q(x, t)$, characterizing flexural vibrations of the rocket body in pitching plane.

System of equations (4.4.35) includes differential equations in partial derivatives. As we will show below, equations of motion of the rocket in pitching plane, listed in this paragraph, can be converted into infinite system of ordinary differential equations. This conversion is based on the investigation of so-called natural flexural vibrations of the rocket body, which we will examine in the following paragraph.

§ 5. Flexural Natural Vibrations of the Rocket Body in the Pitching Plane

Flexural vibrations of the rocket body in the pitching plane, proceeding in the absence of linear lateral force q_y and linear moments m_z, are called *natural oscillations of the body* in the pitching plane. According to (4.3.3) in the considered case accelerations w_{0y} and $\frac{d\omega_z}{dt}$ are absent,and thus, differential equation of flexural vibrations (4.2.19) in this case assumes the form

$$\frac{\partial^2}{\partial x^2}\left(B_y\frac{\partial^2 v}{\partial x^2}\right)-\frac{\partial}{\partial x}\left(J_z\frac{\partial^3 v}{\partial x\partial t^2}\right)+\mu\frac{\partial^2 v}{\partial t^2}=0. \qquad (4.5.1)$$

Boundary conditions (4.2.21) in this case will change to boundary conditions:

$$\left.\begin{aligned}&B_y\frac{\partial^2 v}{\partial x^2}=0\\&\frac{\partial}{\partial x}\left(B_y\frac{\partial^2 v}{\partial x^2}\right)-J_z\frac{\partial^3 v}{\partial x\partial t^2}=0\end{aligned}\right\}\ \text{when } x=a \text{ and when } x=b. \qquad (4.5.2)$$

In examining differential equation (4.5.1) with boundary conditions (4.5.2) we will assume below that functions μ and j_z depend on only one variable x, disregarding the slow change of the form of these functions, appearing in the considered time interval because of emptying of fuel tanks.

Homogeneous differential equation (4.5.1) with homogeneous boundary conditions (4.5.2) possesses obvious solution $v(x, t) \equiv 0$ which it is accepted to call trivial solution of this boundary value problem. Nontrivial solutions of differential equation (4.5.1), satisfying boundary conditions (4.5.2), will be sought in the form of the product of two functions

$$v(x, t)=V(x)g(t), \tag{4.5.3}$$

applying in the considered case the method called Fourier method, or method of separation of variables. By substituting (4.5.3) in (4.5.1), we obtain equality

$$\frac{d^2}{dx^2}\left(B_y \frac{d^2V}{dx^2}\right)g+\left[-\frac{d}{dx}\left(j_z \frac{dV}{dx}\right)+\mu V\right]\frac{d^2g}{dt^2}=0,$$

which can be reduced to form

$$\frac{\frac{d^2}{dx^2}\left(B_y \frac{d^2V}{dx^2}\right)}{-\frac{d}{dx}\left(j_z \frac{dV}{dx}\right)+\mu V}=-\frac{\frac{d^2g}{dt^2}}{g}. \tag{4.5.4}$$

The left side of equality (4.5.4) does not depend on variable t, the right side of this equality does not depend on variable x.

Thus, the ratios, figuring in the left and right sides of equality (4.5.4), should represent a certain constant, which we designate through λ. According to, function $V(x)$ must satisfy differential equation

$$\frac{d^2}{dx^2}\left(B_y\frac{d^2V}{dx^2}\right)+\lambda\left[\frac{d}{dx}\left(j_z\frac{dV}{dx}\right)-\mu V\right]=0, \qquad (4.5.5)$$

and function $g(t)$ - differential equation

$$\frac{d^2g}{dt^2}+\lambda g=0. \qquad (4.5.6)$$

According to (4.5.2) and (4.5.3) there must be fulfilled boundary conditions:

$$\left.\begin{aligned}&B_y\frac{d^2V}{dx^2}g=0\\&\frac{d}{dx}\left(B_y\frac{d^2V}{dx^2}\right)g-j_z\frac{dV}{dx}\frac{d^2g}{dt^2}=0\end{aligned}\right\}\ \text{when } x=a \text{ and when } x=b,$$

which can be given the form

$$\left.\begin{aligned}&B_y\frac{d^2V}{dx^2}g=0\\&\left[\frac{d}{dx}\left(B_y\frac{d^2V}{dx^2}\right)+\lambda j_z\frac{dV}{dx}\right]g=0\end{aligned}\right\}\ \text{when } x=a \text{ and when } x=b \qquad (4.5.7)$$

in accordance with differential equation (4.5.6). In order that condition (4.5.7) would be fulfilled at any moment of time t, function $V(x)$ must satisfy boundary conditions:

$$\left.\begin{aligned} & B_y\frac{d^2V}{dx^2}=0 \\ & \frac{d}{dx}\left(B_y\frac{d^2V}{dx^2}\right)+\lambda j_z\frac{dV}{dx}=0 \end{aligned}\right\} \text{when } x=a \text{ and when } x=b. \quad (4.5.8)$$

Differential equation (4.5.5) with boundary conditions (4.5.8) with any value of numerical parameter λ has trivial solution $V(x) \equiv 0$. The values of parameter λ, at which the boundary value problem formed by differential equation (4.5.5) as boundary conditions (4.5.8), has nontrivial solutions, are called *eigenvalues* of this boundary value problem. Nontrivial solutions of differential equation (4.5.5) with boundary (4.5.8), corresponding to eigenvalues of the given boundary value problem, are called its *eigenfunctions*. Let us show that eigenvalues of the boundary value problem, formed by differential equation (4.5.5) and boundary conditions (4.5.8), are real and nonnegative.

Let us examine two eigenfunctions of the considered boundary value problem $V(x)$ and $V^*(x)$, corresponding to eigenvalues λ and λ^*. Function $V(x)$ will satisfy differential equation (4.5.5) and boundary conditions (4.5.8), function $V^*(x)$ - differential equation

$$\frac{d^2}{dx^2}\left(B_y\frac{d^2V^*}{dx^2}\right)+\lambda^*\left[\frac{d}{dx}\left(j_z\frac{dV^*}{dx}\right)-\mu V^*\right]=0 \quad (4.5.9)$$

and boundary conditions:

$$\left.\begin{aligned} & B_y\frac{d^2V^*}{dx^2}=0 \\ & \frac{d}{dx}\left(B_y\frac{d^2V^*}{dx^2}\right)+\lambda^* j_z\frac{dV^*}{dx}=0 \end{aligned}\right\} \text{when } x=a \text{ and when } x=b. \quad (4.5.10)$$

According to (4.5.5) there should exist equality

$$\int_a^b V^*\frac{d}{dx}\left[\frac{d}{dx}\left(B_y\frac{d^2V}{dx^2}\right)+\lambda j_z\frac{dV}{dx}\right]dx-\lambda\int_a^b \mu VV^*dx=0. \quad (4.5.11)$$

By carrying out integration by parts in (4.5.11), we obtain relationship

$$-\int_a^b \frac{dV^*}{dx}\left[\frac{d}{dx}\left(B_y\frac{d^2V}{dx^2}\right)+\lambda J_z\frac{dV}{dx}\right]dx+\left\{V^*\left[\frac{d}{dx}\left(B_y\frac{d^2V}{dx^2}\right)+\right.\right.$$
$$\left.\left.+\lambda J_z\frac{dV}{dx}\right]\right\}_{x=a}^{x=b}-\lambda\int_a^b \mu VV^*dx=0,$$

or according to (4.5.8)

$$-\int_a^b \frac{dV^*}{dx}\,\frac{d}{dx}\left(B_y\frac{d^2V}{dx^2}\right)dx-\lambda\int_a^b\left(J_z\frac{dV}{dx}\,\frac{dV^*}{dx}+\mu VV^*\right)dx=0. \qquad (4.5.12)$$

By means of integration by parts equality (4.5.12) can be reduced to form

$$\int_a^b B_y\frac{d^2V}{dx^2}\,\frac{d^2V^*}{dx^2}\,dx-\left(B_y\frac{dV^*}{dx}\,\frac{d^2V}{dx^2}\right)_{x=a}^{x=b}-$$
$$-\lambda\int_a^b\left(J_z\frac{dV}{dx}\,\frac{dV^*}{dx}+\mu VV^*\right)dx=0,$$

or in accordance with boundary conditions (4.5.8)

$$\int_a^b B_y\frac{d^2V}{dx^2}\,\frac{d^2V^*}{dx^2}dx=\lambda\int_a^b\left(J_z\frac{dV}{dx}\,\frac{dV^*}{dx}+\mu VV^*\right)dx. \qquad (4.5.13)$$

Analogously, having multiplied both sides of equation (4.5.9) by V and having integrated them with respect to variable x from a to b, we obtain the equality, which by means of integration by parts can be given the form

$$\int_a^b B_y \frac{d^2V}{dx^2}\frac{d^2V^*}{dx^2}dx=\lambda^*\int_a^b\left(j_z\frac{dV}{dx}\frac{dV^*}{dx}+\mu VV^*\right)dx. \qquad (4.5.14)$$

If for a certain complex value of parameter λ differential equation (4.5.5) has nontrival solution $V(x)$, which satisfies boundary conditions (4.5.8), then when $\lambda^* = \bar{\lambda}$ the boundary value problem, formed by differential equation (4.5.9) and boundary conditions (4.5.10), will have nontrivial solution $V^*(x) = \bar{V}(x)$. In the considered case equality (4.5.13) takes the form

$$\int_a^b B_y \frac{d^2V}{dx^2}\frac{d^2\bar{V}}{dx^2}dx=\lambda\int_a^b\left(j_z\frac{dV}{dx}\frac{d\bar{V}}{dx}+\mu V\bar{V}\right)dx,$$

or

$$\int_a^b B_y\left|\frac{d^2V}{dx^2}\right|^2dx=\lambda\int_a^b\left(j_z\left|\frac{dV}{dx}\right|^2+\mu|V|^2\right)dx. \qquad (4.5.15)$$

In accordance with formulas (4.2.16) there should always be observed inequalities:

$$\mu>0,\ j_z>0. \qquad (4.5.16)$$

According to (4.5.16) equality

$$\int_a^b\left(j_z\left|\frac{dV}{dx}\right|^2+\mu\left|V\right|^2\right)dx=0$$

can exist only for function $V(x)$, identically equal to zero in interval $a \leq x \leq b$. By assumptions, function $V(x)$ represents nontrivial solution of differential equation (4.5.5), and, thus, integral, figuring in the right side of relationship (4.5.15), cannot be equal to zero. From (4.5.15) ensues equality

$$\lambda = \frac{\int_a^b B_y \left| \frac{d^2V}{dx^2} \right|^2 dx}{\int_a^b \left(J_z \left| \frac{dV}{dx} \right|^2 + \mu |V|^2 \right) dx}, \qquad (4.5.17)$$

in accordance with which eigenvalue λ must be real. Having assumed that the boundary value problem, formed by differential equation (4.5.5) and boundary conditions (4.5.8), has complex eigenvalue, we arrived at a contradiction. Thus, we proved the substantiality of eigenvalues of the considered boundary value problem; from equality (4.5.17) there also ensues nonnegative character of these eigenvalues, since the denominator of the fraction, figuring in the right side of relationship (4.5.17), is positive, the numerator of this fraction is nonnegative.

From formulas (4.5.13) and (4.5.14) ensues equality

$$(\lambda - \lambda^*) \int_a^b \left(J_z \frac{dV}{dx} \frac{dV^*}{dx} + \mu V V^* \right) dx = 0, \qquad (4.5.18)$$

in accordance with which the eigenfunctions of the considered boundary value problem $V(x)$ and $V^*(x)$, corresponding to its various eigenvalue λ and λ^*, must always satisfy condition

$$\int_a^b \left(J_z \frac{dV}{dx} \frac{dV^*}{dx} + \mu V V^* \right) dx = 0. \qquad (4.5.19)$$

The property of eigenfunctions, expressed by integral relationship (4.5.19), is accepted to call the property of their orthogonality.

When $\lambda = 0$ differential equation (4.5.5) degenerates into equation

$$\frac{d^2}{dx^2}\left(B_y \frac{d^2V}{dx^2}\right) = 0, \tag{4.5.20}$$

boundary conditions (4.5.8) when $\lambda = 0$ take the form

$$B_y \frac{d^2V}{dx^2} = 0, \ \frac{d}{dx}\left(B_y \frac{d^2V}{dx^2}\right) = 0 \text{ when } x = a \text{ and when } x = b. \tag{4.5.21}$$

According to (4.5.20) there should exist equality

$$B_y \frac{d^2V}{dx^2} = C_1 + C_2 x, \tag{4.5.22}$$

where C_1 and C_2 - arbitrary constants. In accordance with boundary conditions (4.5.21) both constants C_1 and C_2 be equal to zero. By assuming in (4.5.22) $C_1 = C_2 = 0$, we obtain a differential equation having two linearly independent solutions:

$$V = 1 \text{ and } V = x. \tag{4.5.23}$$

Both these solutions satisfy boundary conditions (4.5.21), and, thus, the boundary value problem, formed by differential equation (4.5.5) and boundary conditions (4.5.8), has eigenvalue λ equal to zero. To this eigenvalue correspond two eigenfunctions of the boundary value problem determined by formulas (4.5.23). All the remaining eigenvalues of the considered boundary value problem must be positive. These positive eigenvalues will be designated by λ_j, $j = 1, 2, \ldots$, having agreed to number them in ascending order, eigenfunctions corresponding to them - by $V_j(x)$, $j = 1, 2, \ldots$

According to (4.5.3) to the first of eigenfunctions (4.5.23) correspond identical movements of cross sections of the body, to the second of these eigenfunctions - movements proportional to abscissa of cross section x. In the first case function $v(x, t)$ determines the forward movements of the rocket body in the pitching plane, to the second case corresponds rotation of the body around axis z. Both these motions are not accompanied by deformations of the body, and, thus, the eigenvalue of the considered boundary value problem, equal to zero, of flexural natural vibrations of the rocket body is not determined.

To the positive eigenvalue of boundary value problem λ_j according to (4.5.3) and (4.5.6) there will correspond function $v(x, t)$, determined by formula

$$v(x,t)=V_j(x)g_j(t), \tag{4.5.24}$$

where $g_j(t)$ - solution of differential equation

$$\frac{d^2g_j}{dt^2}+\lambda_j g_j=0. \tag{4.5.25}$$

Assuming

$$\omega_j=\sqrt{\lambda_j}, \tag{4.5.26}$$

it is possible to reduce equation (4.5.25) to the form

$$\frac{d^2g_j}{dt^2}+\omega_j^2 g_j=0. \tag{4.5.27}$$

General solution of differential equation (4.5.27) can be presented in the form

$$g_j(t)=C_j\cos(\omega_j t+\varphi_j), \tag{4.5.28}$$

where C_j and ϕ_j - arbitrary constants.

By substituting (4.5.28) in (4.5.24) for deflection of rocket body $v(x, t)$ we obtain expression

$$v(x, t) = C_j V_j(x) \cos(\omega_j t + \varphi_j). \tag{4.5.29}$$

According to (4.5.29) in the considered case at any moment of t the movements of cross sections of the body is proportional to corresponding values of function $V_j(x)$. Thus, functions $V_j(x)$, $j = 1, 2, \ldots$ determine the forms of natural flexural vibrations of the rocket body, numbers ω_j, $j = 1, 2, \ldots$ in accordance with formula (4.5.29) determine the frequencies of these natural oscillations. It is accepted to call differential equation (4.5.27) the differential equation of natural oscillations.

We showed above that differential equation (4.5.5) with boundary conditions (4.5.8) has two eigenfunctions $V = 1$ and $V = x$, corresponding to eigenvalue $\lambda = 0$, and the sequence of eigenfunctions $V_j(x)$, $j = 1, 2, \ldots$, corresponding to positive eigenvalues $\lambda_j = \omega_j^2$, $j = 1, 2, \ldots$ Assuming in (4.5.19) $V = V_j$, $V^* = 1$, for functions $V_j(x)$, $j = 1, 2, \ldots$ we obtain integral relationships:

$$\int_a^b V_j \mu \, dx = 0, \quad j = 1, 2, \ldots. \tag{4.5.30}$$

When $V = V_j$, $V^* = x$ from equality (4.5.19) there will ensue integral relationships:

$$\int_a^b \left(x V_j \mu + J_z \frac{dV_j}{dx} \right) dx = 0, \quad j = 1, 2, \ldots. \tag{4.5.31}$$

When $\omega_j \neq \omega_k$, assuming in (4.5.19) $V = V_j$, $V^* = V_k$, for the forms of natural oscillations of *condition of orthogonality* we obtain

$$\int_0^l \left(J_x \frac{dV_j}{dx} \frac{dV_k}{dx} + \mu V_j V_k\right) dx = 0 \text{ when } \omega_j \neq \omega_k. \qquad (4.5.32)$$

With derivation of differential equation of flexural vibrations of the rocket body we did not consider energy dissipation inevitably accompanying any real oscillatory process. In accordance with this, the obtained differential equation of natural oscillations (4.5.27) predetermines the harmonic character of the considered natural oscillations of the rocket body. In order to take into account damping of natural oscillations, it is necessary to replace equation (4.5.27) by differential equation

$$\frac{d^2 g_j}{dt} + \varepsilon_j \frac{dg_j}{dt} + \omega_j^2 g_j = 0. \qquad (4.5.33)$$

Having repeatedly conducted natural dynamic tests of various rockets, and also dynamic tests of structurally similar rocket models (see [14]), attest to the fact that for elastic oscillations of the rocket body the logarithmic decrement of natural oscillations for all practical purposes does not depend on the number of their tone, designated in equation (4.5.33) through j. The logarithmic decrement of natural oscillations is proportional to the ratio of damping factor ε_j to the frequency of oscillations ω_j. Thus, for equation (4.5.33) it is possible to approximately assume

$$\frac{\varepsilon_j}{\omega_j} = \eta, \qquad (4.5.34)$$

where η - dimensionless coefficient, independent of number of tone j. Coefficient η usually lies within 0.01 to 0.02. According to (4.5.34) differential equation (4.5.33) can be given the form

$$\frac{d^2 g_j}{dt^2} + \eta_{(\omega)j} \frac{dg_j}{dt} + \omega_j^2 g_j = 0. \qquad (4.5.35)$$

In conclusion of this paragraph let us note that differential equation (4.5.5) with boundary conditions (4.5.8) determines functions $V_j(x)$, $j = 1, 2, \ldots$ to within arbitrary constant factors. For elimination of this indeterminancy functions $V_j(x)$, $j = 1, 2, \ldots$ are always subordinate to some additional condition. As such an additional condition we usually use initial condition

$$V_j(x) = 1 \text{ when } x = a. \qquad (4.5.36)$$

§ 6. Calculation of Forms and Frequencies of Natural Flexural Vibrations by Method of Successive Approximations

For calculation of the forms and frequencies of natural flexural vibrations it is possible to use the method of successive approximations discussed in this paragraph.

Let us assume approximation $V_j^{(k-1)}(x) \dot{-} (k-1)$ for j-th form of natural oscillations $V_j(x)$. Relative to function $V_j^{(k-1)}(x)$ we will assume that it satisfies conditions:

$$\left.\begin{aligned} &\int_a^b V_j^{(k-1)} \mu \, dx = 0, \\ &\int_a^b \left(x V_j^{(k-1)} \mu + j_z \frac{dV_j^{(k-1)}}{dx} \right) dx = 0, \\ &V_j^{(k-1)} = 1 \text{ when } x = a, \end{aligned}\right\} \qquad (4.6.1)$$

in accordance with integral relationships (4.5.30) and (4.5.31), and to initial condition (4.5.36), which we will adhere to everywhere in the subsequent discussion. The following approximation $V_j^{(k)}(x)$ for function $V_j(x)$ will be conditionally determined from differential equation

$$\frac{d^2}{dx^2}\left(B_y \frac{d^2V_j^{(k)}}{dx^2}\right)+\lambda_j^{(k)}\left[\frac{d}{dx}\left(j_z \frac{dV_j^{(k-1)}}{dx}\right)-\mu V_j^{(k-1)}\right]=0 \qquad (4.6.2)$$

with boundary conditions:

$$\left.\begin{aligned} &B_y \frac{d^2V_j^{(k)}}{dx^2}=0 \\ &\frac{d}{dx}\left(B_y \frac{d^2V_j^{(k)}}{dx^2}\right)+\lambda_j^{(k)} j_z \frac{dV_j^{(k-1)}}{dx}=0 \end{aligned}\right\} \begin{aligned} &\text{when } x=a \text{ and} \\ &\text{when } x=b \end{aligned} \qquad (4.6.3)$$

[see (4.5.5) and (4.5.8)], having required in this case that the sought function $V_j^{(k)}(x)$ satisfy conditions:

$$\begin{aligned} &\int_a^b V_j^{(k)}\mu\, dx=0, \\ &\int_a^b \left(xV_j^{(k)}\mu + j_z \frac{dv_j^{(k)}}{dx}\right)dx=0, \\ &V_j^{(k)}=1 \text{ when } x=a \end{aligned} \qquad (4.6.4)$$

[see (4.6.1)]. As we will see subsequently, differential equation (4.6.2) with boundary conditions (4.6.3) and additional conditions (4.6.4) uniquely determines function $V_j^{(k)}(x)$ and numerical parameter $\lambda_j^{(k)}$, which figures in differential equation (4.6.2) and in boundary conditions (4.6.3).

According to (4.6.2) and (4.6.3) there should exist equality

$$\frac{d}{dx}\left(B_y \frac{d^2V_j^{(k)}}{dx^2}\right)+\lambda_j^{(k)} j_z \frac{dV_j^{(k-1)}}{dx}=\lambda_j^{(k)}\int_a^x V_j^{(k-1)}\mu\, dx. \qquad (4.6.5)$$

In accordance with the first of relationships (4.6.1) boundary condition

$$\frac{d}{dx}\left(B_y \frac{d^2V_j^{(k)}}{dx^2}\right)+\lambda_j^{(k)} j_z \frac{dV_j^{(k-1)}}{dx}=0 \text{ when } x=a \text{ и when } x=b$$

will be fulfilled.

Differential equation (4.6.5) can be reduced to form

$$\frac{d}{dx}\left(B_y \frac{d^2V_j^{(k)}}{dx^2}\right)-\lambda_j^{(k)} \int_a^x V_j^{(k-1)}\mu dx-\lambda_j^{(k)} x V_j^{(k-1)}\mu =$$
$$= -\lambda_j^{(k)} x V_j^{(k-1)}\mu-\lambda_j^{(k)} j_z \frac{dV_j^{(k-1)}}{dx},$$

or

$$\frac{d}{dx}\left(B_y \frac{d^2V_j^{(k)}}{dx^2}-\lambda_j^{(k)} x \int_a^x V_j^{(k-1)}\mu dx\right)=$$
$$= -\lambda_j^{(k)}\left(x V_j^{(k-1)}\mu + j_z \frac{dV_j^{(k-1)}}{dx}\right). \qquad (4.6.6)$$

According to (4.6.6) and (4.6.3) there should exist equality

$$B_y \frac{d^2V_j^{(k)}}{dx^2}=\lambda_j^{(k)} x \int_a^x V_j^{(k-1)}\mu dx-\lambda_j^{(k)} \int_a^x \left(x V_j^{(k-1)}\mu + j_z \frac{dV_j^{(k-1)}}{dx}\right)dx. \qquad (4.6.7)$$

Boundary conditions

$$B_y \frac{d^2V_j^{(k)}}{dx^2}=0 \text{ when } x=a \text{ и when } x=b$$

will be fulfilled in accordance with relationships (4.6.1).

Equation (4.6.7) can be reduced to form

$$\frac{d^2V_j^{(k)}}{dx^2}=\lambda_j^{(k)}\frac{M_j^{(k)}(x)}{B_y(x)}, \qquad (4.6.8)$$

where

$$M_j^{(k)}(x)=x\int_a^x V_j^{(k-1)}\mu dx-\int_a^x\left(xV_j^{(k-1)}\mu+J_z\frac{dV_j^{(k-1)}}{dx}\right)dx. \qquad (4.6.9)$$

Assuming in (4.6.8)

$$V_j^{(k)}(x)=\lambda_j^{(k)}F_j^{(k)}(x), \qquad (4.6.10)$$

for function $F_j^{(k)}(x)$ we obtain differential equation

$$\frac{d^2F_j^{(k)}}{dx^2}=\frac{M_j^{(k)}(x)}{B_y(x)}. \qquad (4.6.11)$$

According to (4.6.4) and (4.6.10) function $F_j^{(k)}(x)$ must satisfy conditions:

$$\int_a^b F_j^{(k)}\mu dx=0,\quad \int_a^b\left(xF_j^{(k)}\mu+J_z\frac{dF_j^{(k)}}{dx}\right)dx=0. \qquad (4.6.12)$$

Differential equation (4.6.11) has particular solution $F_j^{(k)}=f_j^{(k)}(x)$, where

$$f_j^{(k)}(x)=x\int_a^x\frac{M_j^{(k)}}{B_y}dx-\int_a^x\frac{M_j^{(k)}}{B_y}xdx, \qquad (4.6.13)$$

which can be checked by twice differentiating both sides of equality (4.6.13) with respect to x. Thus, general solution of differential equation (4.6.11) can be presented in the form

$$F_j^{(k)}(x)=f_j^{(k)}(x)+C_1+C_2x, \tag{4.6.14}$$

where C_1 and C_2 - arbitrary constants.

By substituting (4.6.14) in (4.6.12), for constants C_1 and C_2 we obtain system of equations

$$C_1\int_a^b \mu dx+C_2\int_a^b \mu x dx=-\int_a^b f_j^{(k)}\mu dx,$$
$$C_1\int_a^b \mu x dx+C_2\int_a^b (\mu x^2+j_z)dx=-\int_a^b\left(xf_j^{(k)}\mu+j_z\frac{df_j^{(k)}}{dx}\right)dx,$$

or according to (4.3.11)

$$mC_1=-\int_a^b f_j^{(k)}\mu dx,$$
$$J_zC_2=-\int_a^b\left(xf_j^{(k)}\mu+j_z\frac{df_j^{(k)}}{dx}\right)dx \tag{4.6.15}$$

(let us recall that the coordinate of center of mass of the rocket x_C is always equal to zero).

By substituting C_1 and C_2 from (4.6.15) into (4.6.14), for function $F_j^{(k)}(x)$ we obtain expression

$$F_j^{(k)}(x)=f_j^{(k)}(x)-\frac{1}{mJ_z}\Big[J_z\int_a^b f_j^{(k)}\mu dx+$$
$$+mx\int_a^b\left(xf_j^{(k)}\mu+j_z\frac{df_j^{(k)}}{dx}\right)dx\Big]. \tag{4.6.16}$$

According to (4.6.4) and (4.6.10) there should exist equality

$$\lambda_j^{(k)} F_j^{(k)}(a) = 1. \qquad (4.6.17)$$

From formulas (4.6.13) and (4.6.16) we find

$$F_j^{(k)}(a) = -\frac{1}{mJ_z}\int_a^b \left[f_j^{(k)}\mu(J_z + max) + maJ_z\frac{df_j^{(k)}}{dx}\right]dx. \qquad (4.6.18)$$

In accordance with equalities (4.6.17) and (4.6.18) numerical factor $\lambda_j^{(k)}$ should have a value determined by formula

$$\lambda_j^{(k)} = -\frac{mJ_z}{\int_a^b \left[f_j^{(k)}\mu(J_z + max) + maJ_z\frac{df_j^{(k)}}{dx}\right]dx}. \qquad (4.6.19)$$

By substituting (4.6.16) and (4.6.19) in (4.6.10), for the sought function $V_j^{(k)}(x)$ we obtain expression

$$V_j^{(k)}(x) = \frac{J_z\int_a^b f_j^{(k)}\mu dx + mx\int_a^b \left(xf_j^{(k)}\mu + J_z\frac{df_j^{(k)}}{dx}\right)dx - mJ_z f_j^{(k)}(x)}{\int_a^b \left[f_j^{(k)}\mu(J_z + max) + maJ_z\frac{df_j^{(k)}}{dx}\right]dx}. \qquad (4.6.20)$$

Thus, by knowing $(k - 1)$ approximation of $V_j^{(k-1)}(x)$ for function $V_j(x)$, it is possible to calculate the following approximation $V_j^{(k)}(x)$ by formulas (4.6.9), (4.6.13) and (4.6.20). For determining the first form of natural oscillations as initial approximation $V_1^{(0)}(x)$ there can be taken the arbitrary function, according to (4.6.1) satisfying conditions:

$$\int_a^b V_1^{(0)}\mu dx=0,$$
$$\int_a^b \left(xV_1^{(0)}\mu+j_z\frac{dV_1^{(0)}}{dx}\right)dx=0, \qquad (4.6.21)$$
$$V_1^{(0)}=1 \text{ when } x=a.$$

It is possible to assume, for example,

$$V_1^{(0)}(x)=C_0+C_1x+C_2x^2, \qquad (4.6.22)$$

having determined coefficients C_0, C_1 and C_2 from conditions (4.6.21). The sequence of functions $V_1^{(k)}(x)$, $k=1, 2, \ldots$ will always converge at function $V_1(x)$, which determines the first form of natural oscillations, and numerical sequence $\lambda_1^{(k)}$, $k=1, 2, \ldots,$ constructed in accordance with formula (4.6.19), will always have limit λ_1, according to (4.5.26) determining the square of the first frequency of natural oscillations of the rocket body. For finding the second form of natural oscillations as initial approximation $V_2^{(0)}(x)$ there should be taken the function, according to (4.6.1) satisfying conditions:

$$\int_a^b V_2^{(0)}\mu dx=0,$$
$$\int_a^b \left(xV_2^{(0)}\mu+j_z\frac{dV_2^{(0)}}{dx}\right)dx=0, \qquad (4.6.23)$$
$$V_2^{(0)}=1 \text{ when } x=a$$

and furthermore, condition

$$\int_a^b \left(j_z\frac{dV_1}{dx}\frac{dV_2^{(0)}}{dx}+\mu V_1V_2^{(0)}\right)dx=0 \qquad (4.6.24)$$

in accordance with properties of orthogonality (4.5.32). By knowing function $V_1(x)$, it is possible to construct initial approximation for function $V_2(x)$, having assumed

$$V_2^{(0)}(x)=C_1V_1^{(0)}(x)+C_2V_1(x). \tag{4.6.25}$$

According to (4.5.30), (4.5.31), (4.5.36) and (4.6.21) the first two conditions from (4.6.23) will be fulfilled at any values of coefficients C_1 and C_2, the third condition will be fulfilled, if constants C_1 and C_2 satisfy equation

$$C_1+C_2=1. \tag{4.6.26}$$

Having substituted (4.6.25) in (4.6.24), for coefficients C_1 and C_2 we obtain equation

$$C_1\int_a^b\left(J_x\frac{dV_1}{dx}\frac{dV_1^{(0)}}{dx}+\mu V_1V_1^{(0)}\right)dx+ \\ +C_2\int_a^b\left[J_x\left(\frac{dV_1}{dx}\right)^2+\mu V_1^2\right]dx=0. \tag{4.6.27}$$

Having determined constants C_1 and C_2 from equations (4.6.26) and (4.6.27), by formula (4.6.25) we find function $V_2^{(0)}(x)$, which satisfies all the necessary conditions. The sequence of functions $\overset{k}{V_2}(x)$, k = 1, 2, ... will converge at function $V_2(x)$, which determines the second form of natural oscillations, and numerical sequence $\lambda_2^{(k)}$, $k=1, 2, \ldots$ will have limit λ_2, determining the second frequency of natural oscillations. As initial approximation $V_3^{(0)}(x)$ for function $V_3(x)$ there can be taken function

$$V_3^{(0)}(x)=C_1V_2^{(0)}(x)+C_2V_2(x), \tag{4.6.28}$$

where C_1 and C_2 - constants, determined by equations:

$$\begin{aligned}
&C_1+C_2=1,\\
&C_1\int_a^b\left(J_z\frac{dV_2}{dx}\frac{dV_2^{(0)}}{dx}+\mu V_2V_2^{(0)}\right)dx+\\
&+C_2\int_a^b\left[J_z\left(\frac{dV_2}{dx}\right)^2+\mu V_2^2\right]dx=0
\end{aligned}\qquad(4.6.29)$$

and so forth.

Let us note two specific cases, which can occur in the process of performing the above-indicated calculations.

If point $x = a$ is nodal for the j-th form of natural oscillations, condition (4.5.36) for this value of index j becomes impracticable. Then the determined integral, which plays the role of denominator in formulas (4.6.19) and (4.6.20), when $k\to\infty$ will approach zero and the process of successive approximations will become divergent. In the considered case condition (4.5.36) should be replaced by some other additional condition for function $V_j(x)$.

If function $V_1^{(0)}(x)$ is randomly orthogonal to function $V_j(x)$, i.e., if at certain j there appears equality

$$\int_a^b\left(J_z\frac{dV_j}{dx}\frac{dV_1^{(0)}}{dx}+\mu V_jV_1^{(0)}\right)dx=0,\qquad(4.6.30)$$

the j-th form of natural oscillations in the process of the above-indicated calculations will be omitted. In connection with this, during calculation of the forms and frequencies of natural oscillations by the considered method of successive approximations, it is useful to perform computations twice, proceeding from two different initial approximations

§ 7. Conversion of Equations of Motion of the Rocket in Pitching Plane to Infinite System of Ordinary Differential Equations

By knowing the forms and frequencies of natural flexural vibrations of the rocket body, it is possible to convert equations (4.4.35), containing differential equations in partial derivatives, into infinite system of ordinary differential equations.

We showed above that the boundary value problem, formed by differential equation (4.5.5) and boundary conditions (4.5.8), possesses two eigenfunctions 1 and x, corresponding to eigenvalue $\lambda = 0$, and sequence of eigenfunctions $V_j(x)$, $j = 1, 2 \ldots,$ corresponding to positive eigenvalues of the given boundary value problem. Sequence of eigenfunctions

$$1, x, V_1(x), V_2(x), \ldots \quad (4.7.1)$$

possesses so-called property of completeness, which is expressed in the fact that any function $f(x)$, continuous in interval $a \leq x \leq b$, can be expanded into series according to eigenfunctions (4.7.1)

$$f(x) = C_1 + C_2 x + \sum_{k=1}^{\infty} g_k V_k(x), \quad (4.7.2)$$

converging at any point of interval $a < x < b$ to the corresponding particular value of function $f(x)$.

As a result of emptying of fuel tanks, functions μ and j_z, which figure in differential equation (4.5.5) and boundary conditions (4.5.8), change their form in the process of rocket flight. In accordance with this, different forms of natural flexural vibrations of the rocket body will correspond to different moments of time t. In other words, eigenfunctions V_j of boundary value problem, formed by differential equation (4.5.5) and boundary conditions (4.5.8), will depend on time t. At any moment of time t the sequence of functions

$$1,\ x,\ V_1(x,\ t),\ V_2(x,\ t),\ \ldots \qquad (4.7.3)$$

will possess the property of completeness, and, thus, at each moment of time t the deflection of the rocket body $v(x,\ t)$ can be represented by series

$$v(x,t)=C_1(t)+C_2(t)x+\sum_{k=1}^{\infty} g_k(t)V_k(x,t), \qquad (4.7.4)$$

converging at any point of interval $a < x < b$ to the corresponding value of deflection v.

Let us first determine coefficients of expansion (4.7.4) $C_1(t)$, $C_2(t)$. In § 3 we showed that function $v(x,\ t)$, found from differential equation (4.2.19) with boundary conditions (4.2.21), will always satisfy integral conditions (4.3.4). By substituting (4.7.4) in (4.3.4), we obtain equalities:

$$C_1\int_a^b \mu dx+C_2\int_a^b \mu x dx+\sum_{k=1}^{\infty} g_k\int_a^b V_k\mu dx=0,$$

$$C_1\int_a^b \mu x dx+C_2\int_a^b (\mu x^2+J_z)\,dx+\sum_{k=1}^{\infty} g_k\int_a^b \left(x\mu V_k+\frac{\partial V_k}{\partial x}\,j_z\right)dx=0,$$

or according to (4.3.11), (4.5.30) and (4.5.31)

$$mC_1=0,\ J_zC_2=0 \qquad (4.7.5)$$

($x_C \equiv 0$). Thus, in expansion (4.7.4) coefficients $C_1(t)$ and $C_2(t)$ will always be identically equal to zero and in accordance with this, deflection of rocket body $v(x,\ t)$ will be sought in the form of series

$$v(x,t)=\sum_{k=1}^{\infty} g_k(t)V_k(x,\ t). \qquad (4.7.6)$$

By substituting series (4.7.6) in the first two equations of (4.4.35), we obtain equations:

$$m(w_{0y}-g_y)=F_y+\sum_{k=1}^{\infty} g_k\left[\int_a^b \frac{\partial V_k}{\partial x}cdx+P\left(\frac{\partial V_k}{\partial x}\right)_{x=x_T}\right],$$

$$J_z\frac{d\omega_z}{dt}=M_z+\sum_{k=1}^{\infty} g_k\left[\int_a^b \frac{\partial v_k}{\partial x}cxdx+P\left(x\frac{\partial v_k}{\partial x}-V_k\right)_{x=x_T}\right],$$

or

$$m(w_{0y}-g_y)+\sum_{k=1}^{\infty} c_{yk}g_k=F_y,$$
$$J_z\frac{d\omega_z}{dt}+\sum_{k=1}^{\infty} c_{\varphi k}g_k=M_z, \qquad (4.7.7)$$

where

$$c_{yk}=-\int_a^b \frac{\partial V_k}{\partial x}cdx-P\left(\frac{\partial V_k}{\partial x}\right)_{x=x_T},$$
$$c_{\varphi k}=-\int_a^b \frac{\partial V_k}{\partial x}cxdx+P\left(V_k-x\frac{\partial V_k}{\partial x}\right)_{x=x_T}. \qquad (4.7.8)$$

Let us now substitute series (4.7.6) in differential equations in partial derivatives, contained in equations of motion (4.4.35). In the process of motion of the rocket the forms of flexural natural vibrations of its body change slowly and in connection with this during calculation of partial derivatives in time t of deflection $v(x, t)$ for all practical purposes it is possible to disregard the relationship of function $V_k(x, t)$, $k=1, 2, \ldots$ to time t. In this instance, by substituting series (4.7.6) in differential equations in partial derivatives, shown in (4.4.35), we obtain relationships:

$$\mu\sum_{k=1}^{\infty}V_k\frac{d^2g_k}{dt^2}+\frac{\partial Q}{\partial x}=-\mu(w_{0y}-g_y)-\mu x\frac{d\omega_z}{dt}+c(\alpha+$$
$$+\sum_{k=1}^{\infty}\frac{\partial V_k}{\partial x}g_k\Big)+P\sum_{k=1}^{\infty}\left(\frac{\partial V_k}{\partial x}\right)_{x=x_T}g_k\delta(x,x_T)+F_{y\,ynp}\delta(x,x_y),$$
$$\sum_{k=1}^{\infty}\frac{\partial}{\partial x}\left(B_y\frac{\partial^2 v_k}{\partial x^2}\right)g_k-j_z\sum_{k=1}^{\infty}\frac{\partial v_k}{\partial x}\frac{d^2g_k}{dt^2}-Q=j_z\frac{d\omega_z}{dt}+ \qquad (4.7.9)$$
$$+P\sum_{k=1}^{\infty}(V_k)_{x=x_T}g_k\delta(x,x_T)-M^{(0)}_{z\,ynp}\delta(x,x_y).$$

In both sides of the first equation from (4.7.9) by multiplying by $V_j(x,\ t)$ and integrating with respect to variable x from a to b, we obtain equality

$$\sum_{k=1}^{\infty}\int_a^b V_jV_k\mu dx\frac{d^2g_k}{dt^2}+\int_a^b V_j\frac{\partial Q}{\partial x}dx=-\int_a^b V_j\mu dx(w_{0y}-g_y)-$$
$$-\int_a^b xV_j\mu dx\frac{d\omega_z}{dt}+\alpha\int_a^b V_jcdx+\sum_{k=1}^{\infty}\int_a^b V_j\frac{\partial V_k}{\partial x}cdxg_k+ \qquad (4.7.10)$$
$$+P\sum_{k=1}^{\infty}\left(\frac{\partial V_k}{\partial x}\right)_{x=x_T}g_k\int_a^b V_j\delta(x,x_T)dx+F_{y\,ynp}\int_a^b V_j\delta(x,x_y)dx.$$

Having multiplied both sides of the second equation from (4.7.9) by $-\partial V_j/\partial x$ and integrated them with respect to variable x from a to b, we obtain relationship

$$-\sum_{k=1}^{\infty}\int_a^b\frac{\partial V_j}{\partial x}\frac{\partial}{\partial x}\left(B_y\frac{\partial^2V_k}{\partial x^2}\right)dxg_k+\sum_{k=1}^{\infty}\int_a^b\frac{\partial V_j}{\partial x}\frac{\partial V_k}{\partial x}j_zdx\frac{d^2g_k}{dt^2}+ \qquad (4.7.11)$$
$$+\int_a^b Q\frac{\partial V_j}{\partial x}dx=-\int_a^b\frac{\partial V_j}{\partial x}j_zdx\frac{d\omega_z}{dt}-P\sum_{k=1}^{\infty}(V_k)_{x=x_T}\times$$
$$\times\int_a^b\frac{\partial V_j}{\partial x}\delta(x,x_T)dxg_k+M^{(0)}_{z\,ynp}\int_a^b\frac{\partial V_j}{\partial x}\delta(x,x_y)dx.$$

According to (4.7.10) and (4.7.11) there should exist equality

$$\sum_{k=1}^{\infty}\int_a^b\left(V_jV_k\mu+\frac{\partial V_j}{\partial x}\frac{\partial V_k}{\partial x}j_z\right)dx\frac{d^2g_k}{dt^2}-\sum_{k=1}^{\infty}\int_a^b\frac{\partial V_j}{\partial x}\frac{\partial}{\partial x}\left(B_y\times\right.$$

$$\left.\times\frac{\partial^2V_k}{\partial x^2}\right)dxg_k+\int_a^b\left(V_j\frac{\partial Q}{\partial x}+Q\frac{\partial V_j}{\partial x}\right)dx=-\int_a^b V_j\mu dx(w_{0y}-$$

$$-g_y)-\int_a^b\left(xV_j\mu+\frac{\partial V_j}{\partial x}j_z\right)dx\frac{d\omega_z}{dt}+a\int_a^b V_jc dx+$$

$$+\sum_{k=1}^{\infty}\left\{\int_a^b V_j\frac{\partial V_k}{\partial x}cdx+P\left[\left(\frac{\partial V_k}{\partial x}\right)_{x=x_\tau}\int_a^b V_j\delta(x,x_\tau)dx-\right.\right. \tag{4.7.12}$$

$$\left.\left.-(V_k)_{x=x_\tau}\int_a^b\frac{\partial V_j}{\partial x}\delta(x,x_\tau)dx\right]\right\}g_k+F_{y\,ynp}\int_a^b V_j\delta(x,x_y)dx+$$

$$+M^{(0)}_{z\,ynp}\int_a^b\frac{\partial V_j}{\partial x}\delta(x,x_y)dx.$$

According to (4.5.5) and (4.5.8) function $V_k(x, t)$ must satisfy differential equation

$$\frac{\partial^2}{\partial x^2}\left(B_y\frac{\partial^2V_k}{\partial x^2}\right)+\lambda_k\left[\frac{\partial}{\partial x}\left(j_z\frac{\partial V_k}{\partial x}\right)-\mu V_k\right]=0 \tag{4.7.13}$$

and boundary conditions:

$$\left.\begin{aligned}&B_y\frac{\partial^2V_k}{\partial x^2}=0,\\&\frac{\partial}{\partial x}\left(B_y\frac{\partial^2V_k}{\partial x^2}\right)+\lambda_kj_z\frac{\partial V_k}{\partial x}=0\end{aligned}\right\}\ \text{when } x=a \text{ and when } x=b. \tag{4.7.14}$$

By integrating in (4.7.13) with respect to variable x from a to x and in this case taking into account boundary conditions (4.7.14), we obtain equality

$$\frac{\partial}{\partial x}\left(B_y \frac{\partial^2 V_k}{\partial x^2}\right) + \lambda_k J_x \frac{\partial V_k}{\partial x} - \lambda_k \int_a^x V_k \mu dx = 0. \tag{4.7.15}$$

In accordance with equality (4.7.15) there should exist relationship

$$\int_a^b \frac{\partial V_j}{\partial x} \frac{\partial}{\partial x}\left(B_y \frac{\partial^2 V_k}{\partial x^2}\right) dx = -\lambda_k \int_a^b \frac{\partial V_j}{\partial x} \frac{\partial V_k}{\partial x} J_x dx + \lambda_k \int_a^b \frac{\partial V_j}{\partial x} \int_a^x V_k \mu dx dx. \tag{4.7.16}$$

By integrating both sides of identity

$$\frac{\partial}{\partial x}\left(V_j \int_a^x V_k \mu dx\right) = \frac{\partial V_j}{\partial x} \int_a^x V_k \mu dx + V_j V_k \mu$$

with respect to variable x from a to b, we obtain equality

$$(V_j)_{x=b} \int_a^b V_k \mu dx = \int_a^b \frac{\partial V_j}{\partial x} \int_a^x V_k \mu dx dx + \int_a^b V_j V_k \mu dx. \tag{4.7.17}$$

According to (4.5.30) and (4.7.17) there should exist relationship

$$\int_a^b \frac{\partial V_j}{\partial x} \int_a^x V_k \mu dx dx = -\int_a^b V_j V_k \mu dx, \tag{4.7.18}$$

in accordance with which formula (4.7.16) can be reduced to form

$$\int_a^b \frac{\partial V_j}{\partial x}\frac{\partial}{\partial x}\left(B_y\frac{\partial^2 V_k}{\partial x^2}\right)dx=-\lambda_k\int_a^b\left(V_jV_k\mu+\frac{\partial V_j}{\partial x}\frac{\partial V_k}{\partial x}j_z\right)dx \qquad (4.7.19)$$

By using the boundary condition, shown for function Q in equations of motion (4.4.35), we find

$$\int_a^b\left(V_j\frac{\partial Q}{\partial x}+Q\frac{\partial V_j}{\partial x}\right)dx=\int_a^b\frac{\partial(V_jQ)}{\partial x}dx=V_jQ\Big|_{x=a}^{x=b}=0. \qquad (4.7.20)$$

According to (4.7.19) and (4.7.20) equation (4.7.12) can be given the form

$$\sum_{k=1}^{\infty}\int_a^b\left(V_jV_k\mu+\frac{\partial V_j}{\partial x}\frac{\partial V_k}{\partial x}j_z\right)dx\left(\frac{d^2g_k}{dt^2}+\lambda_kg_k\right)=-\int_a^b V_j\mu dx\times$$
$$\times(w_{0y}-g_y)-\int_a^b\left(xV_j\mu+\frac{\partial V_j}{\partial x}j_z\right)dx\frac{d\omega_z}{dt}+a\int_a^b V_jcdx+$$
$$+\sum_{k=1}^{\infty}\left\{\int_a^b V_j\frac{\partial V_k}{\partial x}cdx+P\left[\left(\frac{\partial V_k}{\partial x}\right)_{x=x_T}\int_a^b V_j\delta(x,x_T)dx-\right.\right.$$
$$\left.\left.-(V_k)_{x=x_T}\int_a^b\frac{\partial V_j}{\partial x}\delta(x,x_T)dx\right]\right\}g_k+F_{y\,\text{ynp}}\int_a^b V_j\delta(x,x_y)dx+$$
$$+M^{(0)}_{z\,\text{ynp}}\int_a^b\frac{\partial V_j}{\partial x}\delta(x,x_y)dx,$$

or in accordance with equalities (4.5.30), (4.5.31) and (4.5.32)

$$\int_a^b\left[V_j^2\mu+\left(\frac{\partial V_j}{\partial x}\right)^2j_z\right]dx\left(\frac{d^2g_j}{dt^2}+\lambda_jg_j\right)=a\int_a^b V_jcdx+$$
$$+\sum_{k=1}^{\infty}\left\{\int_a^b V_j\frac{\partial V_k}{\partial x}cdx+P\left[\left(\frac{\partial V_k}{\partial x}\right)_{x=x_T}\int_a^b V_j\delta(x,x_T)dx-\right.\right.$$
$$\left.\left.-(V_k)_{x=x_T}\int_a^b\frac{\partial V_j}{\partial x}\delta(x,x_T)dt\right]\right\}g_k+F_{y\,\text{ynp}}\int_a^b V_j\delta(x,x_y)dx+ \qquad (4.5.21)$$
$$+M^{(0)}_{z\,\text{ynp}}\int_a^b\frac{\partial V_j}{\partial x}\delta(x,x_y)dx.$$

By using formula (4.4.28) and equality (4.5.26), it is possible to reduce equation (4.7.21) to the form

$$\int_a^b \left[V_j^2\mu + \left(\frac{\partial V_j}{\partial x}\right)^2 j_z\right] dx \left(\frac{d^2 g_j}{dt^2} + \omega_j^2 g_j\right) = \alpha \int_a^b V_j c\, dx +$$

$$+ \sum_{k=1}^{\infty} \left[\int_a^b V_j \frac{\partial V_k}{\partial x} c\, dx + P\left(V_j \frac{\partial V_k}{\partial x} - V_k \frac{\partial V_j}{\partial x}\right)_{x=x_T}\right] g_k +$$

$$+ (V_j)_{x=x_y} F_{y\,ynp} + \left(\frac{\partial V_j}{\partial x}\right)_{x=x_y} M^{(0)}_{z\,ynp},$$

or

$$m_j \left(\frac{d^2 g_j}{dt^2} + \omega_j^2 g_j\right) = \alpha \int_a^b V_j c\, dx - \sum_{k=1}^{\infty} c_{jk} g_k + (V_j)_{x=x_y} F_{y\,ynp} + \left(\frac{\partial V_j}{\partial x}\right)_{x=x_y} M^{(0)}_{z\,ynp}, \qquad (4.7.22)$$

where

$$m_j = \int_a^b \left[V_j^2\mu + \left(\frac{\partial V_j}{\partial x}\right)^2 j_z\right] dx,$$

$$c_{jk} = P\left(V_k \frac{\partial V_j}{\partial x} - V_j \frac{\partial V_k}{\partial x}\right)_{x=x_T} - \int_a^b V_j \frac{\partial V_k}{\partial x} c\, dx. \qquad (4.7.23)$$

Equation (4.7.22) can be treated as a differential equation of forced oscillations of the rocket body, caused by aerodynamic forces, controlling forces and loads, appearing as a result of the effect of bending of the body on aerodynamic and reactive forces, acting on the rocket. If we do not consider the effect of bending of the rocket body on the line of action of total thrust force of the engines, differential equation of forced oscillations (4.7.22) in the absence of aerodynamic and controlling forces will degenerate into differential equation of natural oscillations (4.5.27).

Equation of forced oscillations (4.7.22) is obtained by us without allowing for energy dissipation, appearing in the process of oscillations. In § 5 we showed that energy dissipation, which accompanies natural oscillations of the rocket body, can be taken into account by substituting the equation of natural oscillations (4.5.27) by differential equation (4.5.35). In accordance with this, energy dissipation can be taken into account in equation of forced oscillations (4.7.22), by substituting this equation by differential equation

$$m_j\left(\frac{d^2g_j}{dt^2}+\eta\omega_j\frac{dg_j}{dt}+\omega_j^2g_j\right)=a\int_a^b V_jc\,dx-\sum_{k=1}^{\infty}c_{jk}g_k+ \\ +(V_j)_{x=x_y}F_{y\,ynp}+\left(\frac{\partial V_j}{\partial x}\right)_{x=x_y}M^{(0)}_{z\,ynp}. \tag{4.7.24}$$

By uniting the equations of forced oscillations of the rocket body corresponding to various tones of these oscillations, and adding equations of forces and moments (4.7.7) to this infinite system of differential equations, we obtain equations

$$\begin{gathered}m(w_{0y}-g_y)+\sum_{k=1}^{\infty}c_{1k}g_k=F_y,\\ J_z\frac{d\omega_z}{dt}+\sum_{k=1}^{\infty}c_{0k}g_k=M_z,\\ m_j\left(\frac{d^2g_j}{dt^2}+\eta\omega_j\frac{dg_j}{dt}+\omega_j^2g_j\right)+\sum_{k=1}^{\infty}c_{jk}g_k=a\int_a^b V_jc\,dx+ \\ +(V_j)_{x=x_y}F_{y\,ynp}+\left(\frac{\partial V_j}{\partial x}\right)_{x=x_y}M^{(0)}_{z\,ynp},\ j=1,2,\ldots\end{gathered} \tag{4.7.25}$$

Infinite system of ordinary differential equations (4.7.25) determines the motion, being accomplished by mobile coordinate system x, y, z, and infinite sequence of functions $g_1(t)$, $g_2(t)$, ...

Deflection of rocket body $v(x, t)$, appearing in the process of flexural vibrations in the pitching plane, can be determined by functions $g_j(t)$, $j = 1, 2, \ldots$, by summarizing series (4.7.6).

§ 8. Differential Equations of Disturbed Motion in the Pitching Plane, Considering the Elasticity of the Rocket Body

By changing to construction of differential equations of disturbed motion, through $g_1, g_2, \ldots$ let us designate time functions, determining flexural vibrations of the rocket body, corresponding to undisturbed motion, and through $g_1', g_2', \ldots$ let us designate time functions determining flexural vibrations of the body, appearing in the process of disturbed motion of the rocket. In Chapter I, proceeding from equations of forces and moments:

$$\begin{aligned} & m(w_{0y} - g_y) = F_y, \\ & J_z \frac{d\omega_z}{dt} = M_z, \end{aligned} \tag{4.8.1}$$

we obtained equations of disturbed motion of the rocket in pitching plane:

$$\begin{aligned} & m\frac{d\Delta V_y}{dt} + v_y \frac{d\Delta\theta}{dt} + c_{yy}\Delta V_y + c_{y\theta}\Delta\dot{\theta} = c_{y\delta}\Delta\delta_0 + \Delta F_y, \\ & J_z \frac{d^2\Delta\vartheta}{dt^2} + \mu_z \frac{d\Delta\theta}{dt} + c_{\theta y}\Delta V_y + c_{\theta\theta}\Delta\theta = c_{\theta\delta}\Delta\delta_0 + \Delta M_z. \end{aligned} \tag{4.8.2}$$

Thus, to the first two equations of system (4.7.25) there will correspond equations of disturbed motion:

$$\begin{aligned} & m\frac{d\Delta V_y}{dt} + v_y \frac{d\Delta\theta}{dt} + c_{yy}\Delta V_y + c_{y\theta}\Delta\theta + \sum_{k=1}^{\infty} c_{yk}\Delta g_k = \\ & = c_{y\delta}\Delta\delta_0 + \Delta F_y, \\ & J_z \frac{d^2\Delta\theta}{dt^2} + \mu_z \frac{d\Delta\theta}{dt} + c_{\theta y}\Delta V_y + c_{\theta\theta}\Delta\theta + \\ & + \sum_{k=1}^{\infty} c_{\theta k}\Delta g_k = c_{\theta\delta}\Delta\delta_0 + M_z. \end{aligned} \tag{4.8.3}$$

$$\Delta g_k = g'_k - g_k \tag{4.8.4}$$

(small changes, which coefficients $c_{\nu k}$ and $c_{\nu k}$ during transition from undisturbed motion to disturbed, can be disregarded, since these coefficients in equations (4.7.25) are multiplied by functions g_k, being small in accordance with our assumed smallness of flexural vibrations of the rocket body).

According to (4.7.25) functions g'_j, $j = 1, 2, \ldots$ will be determined by equations:

$$m_j\left(\frac{d^2g'_j}{dt^2} + \eta_{\omega j}\frac{dg'_j}{dt} + \omega_j^2 g'_j\right) + \sum_{k=1}^{\infty} c_{jk} g'_k = \alpha' \int_a^b V'_j c' dx +$$
$$+ (V'_j)_{x=x_y} F'_{y'\,\text{ynp}} + \left(\frac{\partial V'_j}{\partial x}\right)_{x=x_y} M^{(0)'}_{z\,\text{ynp}},\ j=1,2,\ldots, \tag{4.8.5}$$

where α', c', V'_j, $F'_{y'\text{ynp}}$ and $M^{(0)'}_{z\,\text{ynp}}$ - functions determining the angle of attack α, coefficient c, form of flexural vibrations V_j, controlling force $F_{y\,\text{ynp}}$ and controlling moment $M^{(0)}_{z'\text{ynp}}$ for the case of disturbed motion of the rocket (in view of the assumed smallness of functions $g_j, \frac{dg_j}{dt}$ and $\frac{d^2g_j}{dt^2}$, $j=1,2,\ldots$ the small changes, which coefficients with these functions can undergo in the case of transition from undisturbed motion to disturbed, can be disregarded). Assuming in (4.8.5)

$$g'_j = g_j + \Delta g_j,\ j=1,2,\ldots, \alpha' = \alpha + \Delta\alpha,\ c' = c + \Delta c,$$
$$V'_j = V_j + \Delta V_j,\ j=1,2,\ldots \tag{4.8.6}$$

and disregarding the products of small quantities $\Delta\alpha$, ΔV_j, $\frac{\partial \Delta V_j}{\partial x}$, $j=1,2,\ldots$, Δc, $F'_{y'\,\text{ynp}} - F_{y\,\text{ynp}}$ and $M^{(0)'}_{z\,\text{ynp}} - M^{(0)}_{z\,\text{ynp}}$, we obtain equations

$$m_j\left(\frac{d^2g_j}{dt^2}+\eta_{\omega j}\frac{dg_j}{dt}+\omega_j^2g_j\right)+\sum_{k=1}^{\infty}c_{jk}g_k+m_j\left(\frac{d^2\Delta g_j}{dt^2}+\eta_{\omega j}\frac{d\Delta g_j}{dt}+\right.$$

$$\left.+\omega_j^2\Delta g_j\right)+\sum_{k=1}^{\infty}c_{jk}\Delta g_k=(\alpha+\Delta\alpha)\int_a^b V_jc\,dx+\alpha\int_a^b \Delta V_jc\,dx+\alpha\int_a^b V_j\Delta c\,dx+ \qquad (4.8.7)$$

$$+(V_j)_{x=x_y}F'_{y'\,ynp}+(\Delta V_j)_{x=x_y}F_{y\,ynp}+\left(\frac{\partial V_j}{\partial x}\right)_{x=x_y}M^{(0)'}_{z\,ynp}+$$

$$+\left(\frac{\partial\Delta V_j}{\partial x}\right)_{x=x_y}M^{(0)}_{z\,ynp},\ j=1,2,\ldots$$

According to (4.7.25) equations (4.8.7) can be given the form

$$m_j\left(\frac{d^2\Delta g_j}{dt^2}+\eta_{\omega j}\frac{d\Delta g_j}{dt}+\omega_j^2\Delta g_j\right)+\sum_{k=1}^{\infty}c_{jk}\Delta g_k=\Delta\alpha\int_a^b V_jc\,dx+$$

$$+(V_j)_{x=x_y}F'_{y'\,ynp}+\left(\frac{\partial V_j}{\partial x}\right)_{x=x_y}M^{(0)'}_{z\,ynp}+\alpha\int_a^b\Delta V_jc\,dx+$$

$$+\alpha\int_a^b V_j\Delta c\,dx+[(\Delta V_j)_{x=x_y}-(V_j)_{x=x_y}]F_{y\,ynp}+\left[\left(\frac{\partial\Delta V_j}{\partial x}\right)_{x=x_y}-\right. \qquad (4.8.8)$$

$$\left.-\left(\frac{\partial V_j}{\partial x}\right)_{x=x_y}\right]M^{(0)}_{z\,ynp},\ j=1,2,\ldots$$

By using transition matrix (1.12.5) and approximate equalities $\alpha=-\frac{v_{0y}}{v_{0x}},\ \alpha'=-\frac{v_{0y'}}{v_{0x}},\ v_{0y}'-v_{0y}\approx\Delta V_y$, we find:

$$\Delta\alpha=\alpha'-\alpha=-\frac{v_{0y'}-v_{0y}}{v_{0x}}=-\frac{v_{0y}-v_{0x}\Delta\vartheta-v_{0y}}{v_{0x}}=\Delta\vartheta-\frac{\Delta V_y}{v_{0x}}. \qquad (4.8.9)$$

In accordance with equality (4.8.9) equations (4.8.8) can be reduced to the form

$$m_j\left(\frac{d^2\Delta g_j}{dt^2}+\eta_{\omega j}\frac{d\Delta g_j}{dt}+\omega_j^2\Delta g_j\right)+\sum_{k=1}^{\infty}c_{jk}\Delta g_k+c_{jv}\Delta V_y+ \qquad (4.8.10)$$

$$+c_{j\vartheta}\Delta\vartheta=(V_j)_{x=x_y}F'_{y'\,ynp}+\left(\frac{\partial V_j}{\partial x}\right)_{x=x_y}M^{(0)'}_{z\,ynp}+\Delta F_j,\quad j=1,2,\ldots$$

where

$$c_{jv} = \frac{1}{v_{0x}} \int_a^b V_j c\,dx, \quad c_{j\vartheta} = -\int_a^b V_j c\,dx, \tag{4.8.11}$$

$$\Delta F_j = a \int_a^b \Delta V_j c\,dx + a \int_a^b V_j \Delta c\,dx + [(\Delta V_j)_{x=x_y} - \\ -(V_j)_{x=x_y}] F_{y\,\text{упр}} + \left[\left(\frac{\partial \Delta V_j}{\partial x}\right)_{x=x_y} - \left(\frac{\partial V_j}{\partial x}\right)_{x=x_y}\right] M^{(0)}_{z\,\text{упр}}. \tag{4.8.12}$$

When the actuating elements of the control system form controlling force

$$F'_{y'\,\text{упр}} = c_{y\delta} \Delta\delta_{\text{в}}, \quad M^{(0)'}_{z\,\text{упр}} = 0. \tag{4.8.13}$$

If the actuating elements create concentrated controlling moment, then

$$F'_{y\,\text{упр}} = 0, \quad M^{(0)'}_{z\,\text{упр}} = M'_{z\,\text{упр}} = c_{\vartheta\delta} \Delta\delta_{\text{в}}. \tag{4.8.14}$$

Thus, equations (4.8.10) can be given the form

$$m_j\left(\frac{d^2 \Delta g_j}{dt^2} + \eta \omega_j \frac{d \Delta g_j}{dt} + \omega_j^2 \Delta g_j\right) + \sum_{k=1}^{\infty} c_{jk} \Delta g_k + c_{jv} \Delta V_y + \\ + c_{j\vartheta} \Delta\vartheta = c_{j\delta} \Delta\delta_{\text{в}} + \Delta F_j, \quad j = 1, 2, \ldots, \tag{4.8.15}$$

where

$$c_{j\delta} = c_{y\delta} (V_j)_{x=x_y} \tag{4.8.16}$$

when the actuating elements of the control system create controlling force, and

$$c_{j\delta} = c_{\vartheta\delta} \left(\frac{\partial V_j}{\partial x}\right)_{x=x_y} \tag{4.8.17}$$

when actuating elements of the control system create concentrated controlling moment, i.e., in case of control by thrust misalignment.

By uniting equations (4.8.3) and (4.8.15) we obtain infinite system of differential equations

$$\begin{gathered}
m\frac{d\Delta V_y}{dt}+v_y\frac{d\Delta\vartheta}{dt}+c_{yy}\Delta V_y+c_{y\vartheta}\Delta\vartheta+\sum_{k=1}^{\infty}c_{yk}\Delta g_k=\\
=c_{y\delta}\Delta\delta_\vartheta+\Delta F_y,\\
J_z\frac{d^2\Delta\vartheta}{dt^2}+\mu_z\frac{d\Delta\vartheta}{dt}+c_{\vartheta y}\Delta V_y+c_{\vartheta\vartheta}\Delta\vartheta+\\
+\sum_{k=1}^{\infty}c_{\vartheta k}\Delta g_k=c_{\vartheta\delta}\Delta\delta_\vartheta+\Delta M_z,\\
m_j\left(\frac{d^2\Delta g_j}{dt^2}+\varepsilon_{\omega j}\frac{d\Delta g_j}{dt}+\omega_j^2\Delta g_j\right)+c_{jy}\Delta V_y+c_{j\vartheta}\Delta\vartheta+\\
+\sum_{k=1}^{\infty}c_{jk}\Delta g_k=c_{j\delta}\Delta\delta_\vartheta+\Delta F_j,\quad j=1,\,2,\ldots
\end{gathered}\tag{4.8.18}$$

Differential equations of disturbed motion (4.8.18) determine disturbances $\Delta V_y, \Delta\vartheta$ and $\Delta g_j, j=1, 2, \ldots$ according to assigned function $\Delta\delta_\vartheta(t)$, which describes the operation of actuating elements of the control system, and according to assigned functions $\Delta F_y(t)$, $\Delta M_z(t)$ and $\Delta F_j(t), j=1, 2, \ldots$, which determine the disturbing force, disturbing moment and sequence of so-called reduced disturbing forces.

By knowing the forms of natural flexural vibrations of the rocket body, we can determine the so-called reduced masses m_j, $j = 1, 2, \ldots$ and coefficients of differential equations of disturbed motion

$$c_{yj},\ c_{\vartheta j},\ c_{jy},\ c_{j\vartheta},\ j=1,\,2,\ldots \text{ and } c_{jk},\ j=1,\,2,\ldots,\ k=1,\,2,\ldots$$

by formulas (4.7.8), (4.7.23) and (4.8.11). When the actuating elements of the control system form controlling force, for calculation of coefficients $c_{j\delta}, j=1, 2, \ldots$ formula (4.8.16) should be used, in case of control by thrust misalignment coefficients $c_{j\delta}, j=1, 2, \ldots$ should be calculated by formula (4.8.17).

§ 9. Calculation of Frequency Characteristics of the Rocket as the Object of Automatic Control by the Method of Summation of Series

During investigation of the stability of motion of the rocket with allowance for elasticity of its body we usually apply the same frequency methods as in the question about stabilization of oscillations of liquid propellant components.

By changing to consideration of the frequency characteristics of the rocket, considering the elasticity of its body, let us first determine the effect that flexural vibrations of the rocket have on the input signal of the automatic stabilization control. As a result of flexural vibrations of the body the gyroscopic instrument, designed for measurement of disturbance of angle of pitch $\Delta\vartheta$, in actuality records angle $\Delta\vartheta^*$ made up of disturbance $\Delta\vartheta$, corresponding to nondeformed state of the rocket body, and additional angle of rotation of the gyro instrument, appearing with bending of the body.

As can be seen from Fig. 4.1, at small flexural vibrations this additional angle of rotation can be considered equal to partial derivative $\partial v/\partial x$, calculated for the cross section of the rocket, in which the gyro instrument is located. By designating the abscissa of this cross section through x_Γ, for input signal of automatic stabilization control $\Delta\vartheta^*$ we obtain expression

$$\Delta\vartheta^* = \Delta\vartheta + \left(\frac{\partial v}{\partial x}\right)_{x=x_\Gamma}. \qquad (4.9.1)$$

In examining the disturbed motion of the rocket in expansion (4.7.6) it is necessary to replace functions $g_1(t)$, $g_2(t)$, ... by functions $g_1'(t)_1$, $g_2'(t)_1$, ..., corresponding to the case of disturbed motion. Assuming in (4.9.1)

$$v(x, t) = \sum_{k=1}^{\infty} g_k'(t)_1 V_k(x, t), \qquad (4.9.2)$$

for input signal of automatic stabilization control $\Delta\vartheta^*$ we obtain expansion

$$\Delta\vartheta^* = \Delta\vartheta + \sum_{k=1}^{\infty} \left(\frac{\partial V_k}{\partial x}\right)_{x=x_\tau} g_k'$$

or according to (4.8.4)

$$\Delta\vartheta^* = \Delta\vartheta + \sum_{k=1}^{\infty} \left(\frac{\partial V_k}{\partial x}\right)_{x=x_\tau} \Delta g_k + \sum_{k=1}^{\infty} \left(\frac{\partial V_k}{\partial x}\right)_{x=x_\tau} g_k \tag{4.9.3}$$

(in view of the assumed smallness of functions g_k', $k = 1, 2, \ldots$ in expansion (4.9.2) it is possible to disregard small changes of functions V_k, $k = 1, 2, \ldots$, which can appear during transition from undisturbed motion of the rocket to disturbed).

In accordance with formula (4.9.3) the input signal of automatic stabilization control, generated by disturbances $\Delta\vartheta, \Delta g_1, \Delta g_2, \ldots$, can be represented in the form of series

$$\Delta\vartheta^* = \Delta\vartheta + \sum_{k=1}^{\infty} \left(\frac{\partial V_k}{\partial x}\right)_{x=x_\tau} \Delta g_k \tag{4.9.4}$$

(in formula (4.9.3) series

$$\sum_{k=1}^{\infty} \left(\frac{\partial V_k}{\partial x}\right)_{x=x_\tau} g_k$$

determines the input signal of automatic stabilization control, generated by bending of the construction of the rocket during its undisturbed motion).

During investigation of the stability of motion of the rocket in the vicinity of moment of time $t = \tau$ as before we will use the method of quenched coefficients, substituting variable coefficients of differential equations of disturbed motion (4.8.18) by their particular values, corresponding to moment of time $t = \tau$. In formula (4.9.4) let us replace variable coefficients $\left(\frac{\partial V_k}{\partial x}\right)_{x=x_r}$, $k=1, 2, \ldots$ by their values, calculated for moment of time $t = \tau$. For determination of frequency characteristics of the rocket as an object of automatic control in equations of disturbed motion (4.8.18) it is necessary to assume

$$\Delta F_y=0,\ \Delta M_z=0,\ \Delta F_j=0,\ j=1,\ 2,\ldots,\ \Delta\delta_\theta=e^{i\omega t}, \tag{4.9.5}$$

find disturbances ΔV_y, $\Delta\vartheta$, Δg_1, $\Delta g_2, \ldots$ from the obtained equations and then by formula (4.9.4) determine the input signal of automatic stabilization control $\Delta\vartheta^*$, corresponding to case (4.9.5). By substituting (4.9.5) in (4.8.18) and finding the solution of obtained equations in the form

$$\Delta V_y=V_y e^{i\omega t},\ \Delta\vartheta=\theta e^{i\omega t},\ \Delta g_j=G_j e^{i\omega t},\ j=1,\ 2,\ldots, \tag{4.9.6}$$

for coefficients V_y, θ and G_j, $j = 1, 2, \ldots$ we obtain infinite system of algebraic equations

$$\begin{gathered}
(c_{yy}+i\omega m)V_y+(i\omega v_y+c_{y\theta})\,\theta+\sum_{k=1}^{\infty}c_{yk}G_k=c_{y\delta},\\
c_{\theta y}V_y+(c_{\theta\theta}+i\omega\mu_z-\omega^2 J_z)\theta+\sum_{k=1}^{\infty}c_{\theta k}G_k=c_{\theta\delta},\\
m_j(\omega_j^2+i\eta\omega\omega_j-\omega^2)G_j+c_{jy}V_y+c_{j\theta}\theta+\sum_{k=1}^{\infty}c_{jk}G_k=\\
=c_{j\delta},\ j=1,\ 2,\ldots
\end{gathered} \tag{4.9.7}$$

According to (4.9.4) and (4.9.6) in the considered case there will exist equality

$$\Delta\vartheta^*=\theta^*(\omega)e^{i\omega t}, \tag{4.9.8}$$

where

$$\theta^*(\omega)=\theta(\omega)+\sum_{k=1}^{\infty}\left(\frac{\partial V_k}{\partial x}\right)_{x=x_r} G_k(\omega). \tag{4.9.9}$$

By designating through $\theta_1^*(\omega)$ and $\theta_2^*(\omega)$ the real and imaginary parts of complex-valued function $\theta^*(\omega)$ and assuming

$$\theta_1^*(\omega)=k_0(\omega)\cos\varphi_0(\omega),\quad \theta_2^*(\omega)=k_0(\omega)\sin\varphi_0(\omega), \tag{4.9.10}$$

it is possible to reduce equality (4.9.8) to form

$$\Delta\theta^*=k_0(\omega)e^{i[\omega t+\varphi_0(\omega)]}. \tag{4.9.11}$$

Functions $k_0(\omega)$ and $\varphi_0(\omega)$ determine the amplitude-frequency and phase-frequency characteristics of the rocket respectively as the object of automatic control. According to (4.9.10) functions $k_0(\omega)$ and $\varphi_0(\omega)$ can be found by formulas:

$$k_0(\omega)=\sqrt{[\theta_1^*(\omega)]^2+[\theta_2^*(\omega)]^2},\quad \varphi_0(\omega)=\operatorname{arctg}\frac{\theta_2^*(\omega)}{\theta_1^*(\omega)}. \tag{4.9.12}$$

Thus, calculation of frequency characteristics of the rocket as the controllable object is reduced to determination of functions $\theta(\omega)$ and $G_j(\omega)$, $j = 1, 2, \ldots$ from infinite system of algebraic equations (4.9.7) and to summation of series (4.9.9). Solution of infinite system of equations (4.9.7) can be found by the method of successive approximations stated below, possessing rapid convergence.

By introducing meanings:

$$c_{jk}^*=c_{jk}-\sum_{k=1}^{\infty} c_{jk}G_k, \tag{4.9.13}$$

$$c_{\theta k}^{*}=c_{\theta k}-\sum_{k=1}^{\infty} c_{\theta k} G_{k},$$

$$c_{j k}^{*}=c_{j k}-c_{j y} V_{y}-c_{j \theta} \theta-\sum_{k=1}^{\infty} c_{j k} G_{k}, \quad j=1,2, \ldots, \qquad \text{(4.9.13) (Cont'd)}$$

it is possible to reduce equations (4.9.7) to the form

$$\begin{gathered}
\left(c_{y y}+i \omega m\right) V_{y}+\left(c_{y \theta}+i \omega \nu_{y}\right) \theta=c_{y k}^{*}, \\
c_{\theta y} V_{y}+\left(c_{\theta \theta}+i \omega \mu_{z}-\omega^{2} J_{z}\right) \theta=c_{\theta k}^{*}, \\
m_{j}\left(\omega_{j}^{2}+i \eta \omega \omega_{j}-\omega^{2}\right) G_{j}=c_{j k}^{*}, \quad j=1,2, \ldots
\end{gathered} \qquad (4.9.14)$$

According to (4.9.14) there should take place equalities

$$\begin{aligned}
V_{y} &=\frac{-\left(c_{y \theta}+i \omega \nu_{y}\right) c_{\theta k}^{*}+\left(c_{\theta \theta}+i \omega \mu_{z}-\omega^{2} J_{z}\right) c_{y k}^{*}}{\left(c_{y y}+i \omega m\right)\left(c_{\theta \theta}+i \omega \mu_{z}-\omega^{2} J_{z}\right)-c_{\theta y}\left(c_{y \theta}+i \omega \nu_{y}\right)}, \\
\theta &=\frac{\left(c_{y y}+i \omega m\right) c_{\theta k}^{*}-c_{\theta y} c_{y k}^{*}}{\left(c_{y y}+i \omega m\right)\left(c_{\theta \theta}+i \omega \mu_{z}-\omega^{2} J_{z}\right)-c_{\theta y}\left(c_{y \theta}+i \omega \nu_{y}\right)}, \\
G_{j} &=\frac{c_{j k}^{*}}{m_{j}\left(\omega_{j}^{2}+i \eta \omega \omega_{j}-\omega^{2}\right)}, \quad j=1,2, \ldots
\end{aligned} \qquad (4.9.15)$$

By substituting (4.9.15) in (4.9.13), we obtain equations:

$$\begin{aligned}
c_{y k}^{*}&=c_{y k}-\sum_{k=1}^{\infty} \frac{c_{y k} c_{k k}^{*}}{m_{k}\left(\omega_{k}^{2}+i \eta \omega \omega_{k}-\omega^{2}\right)}, \\
c_{\theta k}^{*}&=c_{\theta k}-\sum_{k=1}^{\infty} \frac{c_{\theta k} c_{k k}^{*}}{m_{k}\left(\omega_{k}^{2}+i \eta \omega \omega_{k}-\omega^{2}\right)}, \\
c_{j k}^{*}&=c_{j k}-\frac{\left[c_{j \theta}\left(c_{y y}+i \omega m\right)-c_{j y}\left(c_{y \theta}+i \omega \nu_{y}\right)\right] c_{\theta k}^{*}}{\left(c_{y y}+i \omega m\right)\left(c_{\theta \theta}+i \omega \mu_{z}-\omega^{2} J_{z}\right)-c_{\theta y}\left(c_{y \theta}+i \omega \nu_{y}\right)}- \\
&-\frac{\left[c_{j y}\left(c_{\theta \theta}+i \omega \mu_{z}-\omega^{2} J_{z}\right)-c_{j \theta} c_{\theta y}\right] c_{y k}^{*}}{\left(c_{y y}+i \omega m\right)\left(c_{\theta \theta}+i \omega \mu_{z}-\omega^{2} J_{z}\right)-c_{\theta y}\left(c_{y \theta}+i \omega \nu_{y}\right)}- \\
&-\sum_{k=1}^{\infty} \frac{c_{j k} c_{k k}^{*}}{m_{k}\left(\omega_{k}^{2}+i \eta \omega \omega_{k}-\omega^{2}\right)}, \quad j=1,2, \ldots
\end{aligned} \qquad (4.9.16)$$

Formula (4.9.9) according to (4.9.15) can be converted in the following manner:

$$\theta^{*}(\omega)=\frac{(c_{yy}+i\omega m)\,c^{*}_{\vartheta\delta}-c_{\vartheta y}\,c^{*}_{y\delta}}{(c_{yy}+i\omega m)\,(c_{\vartheta\vartheta}+i\omega\mu_{z}-\omega^{2}J_{z})-c_{\vartheta y}(c_{y\vartheta}+i\omega\nu_{y})}+$$

$$+\sum_{k=1}^{\infty}\frac{c^{*}_{k\delta}\left(\frac{\partial V_{k}}{\partial x}\right)_{x=x_{r}}}{m_{k}\left(\omega_{k}^{2}+i\gamma_{k}\omega\omega_{k}-\omega^{2}\right)}. \qquad (4.9.17)$$

Coefficients $c^{*}_{y\delta}$, $c^{*}_{\vartheta\delta}$, $c^{*}_{j\delta}$, $j=1, 2, \ldots$ determined by infinite system of algebraic equations (4.9.16), usually little differ from coefficients $c_{y\delta}$, $c_{\vartheta\delta}$, $c_{j\delta}$, $j=1, 2, \ldots$ Assuming in the first approximation

$$c^{*}_{y\delta}=c_{y\delta},\quad c^{*}_{\vartheta\delta}=c_{\vartheta\delta},\quad c^{*}_{j\delta}=c_{j\delta},\quad j=1, 2, \ldots \qquad (4.9.18)$$

and substituting (4.9.18) in the right sides of equations (4.9.16), for coefficients $c^{*}_{y\delta}$, $c^{*}_{\vartheta\delta}$, $c^{*}_{j\delta}$, $j=1, 2, \ldots$ we obtain series, which determine the refined values of the shown coefficients. We will consider these refined values of coefficients as their second approximations. Having calculated the second approximations for coefficients $c^{*}_{y\delta}$, $c^{*}_{\vartheta\delta}$, $c^{*}_{j\delta}$, $j=1, 2, \ldots$ for the assigned value of frequency ω and having substituted them in the right sides of equations (4.9.16), for the sought coefficients we obtain new series, summation of which for original value of frequency ω will give the third approximation for these coefficients, and so forth. Having determined by the shown method of successive approximations the coefficients $c^{*}_{y\delta}$, $c^{*}_{\vartheta\delta}$, $c^{*}_{j\delta}$, $j=1, 2, \ldots$, it is possible to construct according to (4.9.17) the expansion, which determines the corresponding particular value of function $\theta^{*}(\omega)$ for the accepted value of frequency ω. For this particular value of function θ^{*} having separated the real part θ^{*}_{1} and imaginary part θ^{*}_{2} further by formulas (4.9.12) we can determine the amplification factor k_{0} and phase shift ϕ_{0} for original value of frequency ω. By carrying out the shown calculation for series of successive values of frequency, it is possible to construct, thus, the sought frequency characteristics of the rocket as the object of automatic control.

Method of calculation of frequency characteristics described in this paragraph requires summation of numerical series, figuring in equations (4.9.16) and in formula (4.9.17). In the following paragraph we will indicate another method of calculation of the frequency characteristics of the rocket, not requiring determination of the forms and frequencies of natural flexural vibrations of the rocket and summation of numerical series.

§ 10. Determination of Frequency Characteristics of the Rocket as the Controllable Object from Ordinary Differential Equation

By changing to discussion of the second method of calculation of frequency characteristics of the rocket as the controllable object, let us preliminarily note that in expressions $\omega_j^2+i\eta\omega\omega_j-\omega^2$, $j=1, 2, \ldots$, figuring in equations (4.9.7), each of components in view of the smallness of coefficient η plays a role only with values of frequency ω_j. In connection with this, for all practical purposes without loss of accuracy of determination of sought frequency characteristics in equation (4.9.7) it is possible to substitute expressions

$$\omega_j^2+i\eta\omega\omega_j-\omega^2,\ j=1, 2,\ldots \quad (4.10.1)$$

by expressions

$$\omega_j^2+i\eta\omega_j^2-\omega^2,\ j=1, 2,\ldots$$

or

$$\omega_j^{*2}-\omega^2,\ j=1, 2,\ldots, \quad (4.10.2)$$

where

$$\omega_j^{*2}=(1+i\eta)\,\omega_j^2,\ j=1, 2,\ldots \quad (4.10.3)$$

In this case equations (4.9.7) will change into equations

$$
\begin{aligned}
&(c_{yy}+i\omega m)V_y+(c_{y\theta}+i\omega\nu_y)\theta+\sum_{k=1}^{\infty}c_{yk}G_k=c_{y\delta},\\
&c_{\theta y}V_y+(c_{\theta\theta}+i\omega\mu_z-\omega^2J_z)\theta+\sum_{k=1}^{\infty}c_{\theta k}G_k=c_{\theta\delta},\\
&m_j(\omega_j^{*2}-\omega^2)G_j+c_{jy}V_y+c_{j\theta}\theta+\sum_{k=1}^{\infty}c_{jk}G_k=c_{j\delta},\ j=1,2,\ldots
\end{aligned}
\tag{4.10.4}
$$

As we will show below, the frequency characteristics, determined by equations (4.10.4) and formulas (4.9.9) and (4.9.12), can be constructed, while not resorting to solution of infinite system of algebraic equations (4.9.16) and to summation of series, figuring in formula (4.9.9).

In the absence of damping, i.e., when $n = 0$, fictitious complex frequencies ω_j^*, entering equations (4.10.4), according to (4.10.3) will change into actual real frequencies of natural flexural vibrations of the rocket body ω_j. The method of calculation of frequency characteristics, discussed in this paragraph, will be analyzed first for particular case $n = 0$, after this we will indicate the method permitting taking into account the damping, characterized by assigned value of coefficient n during determination of the sought frequency characteristics.

We obtained differential equations of disturbed motion (4.8.18) having converted motion equations of the rocket in the pitching plane (4.4.35) into infinite system of ordinary differential equations, formed by equations (4.7.7) and (4.7.22), and then taking into account damping by means of transition from equations (4.7.22) to equations (4.7.24). Without allowing for damping the differential equations of disturbed motion of the rocket in the pitching plane can be obtained by direct transition from undisturbed motion to disturbed in the original equations of motion (4.4.35).

By $v'(x, t)$ let us designate deflection of the rocket body, corresponding to its disturbed motion, and by $\Delta v(x, t)$ - the difference between deflection $v'(x, t)$ and deflection $v(x, t)$, corresponding to undisturbed motion,

$$\Delta v = v' - v. \tag{4.10.5}$$

According to (4.4.35) when taking into account the elasticity of the rocket body the equations of forces and moments (4.8.2), constructed in Chapter I on the basis of motion equations (4.8.1), will change into equations:

$$\begin{gathered}
m\frac{d\Delta V_y}{dt}+c_{yy}\Delta V_y+v_y\frac{d\Delta\vartheta}{dt}+c_{y\vartheta}\Delta\vartheta-\\
-\int_a^b\frac{\partial\Delta v}{\partial x}c\,dx-P\left(\frac{\partial\Delta v}{\partial x}\right)_{x=x_\tau}=c_{y\delta}\Delta\delta_\vartheta+\Delta F_y,\\
J_z\frac{d^2\Delta\vartheta}{dt^2}+\mu_z\frac{d\Delta\vartheta}{dt}+c_{\vartheta y}\Delta V_y+c_{\vartheta\vartheta}\Delta\vartheta-\\
-\int_a^b\frac{\partial\Delta v}{\partial x}cx\,dx-P\left(x\frac{\partial\Delta v}{\partial x}-\Delta v\right)_{x=x_\tau}=c_{\vartheta\delta}\Delta\delta_\vartheta+\Delta M_z.
\end{gathered} \tag{4.10.6}$$

During the transition from undisturbed motion of the rocket to disturbed, the boundary value problem for functions v and Q, shown in (4.4.35), will change into boundary value problem, formed by differential equations:

$$\begin{gathered}
\mu\frac{\partial^2 v'}{\partial t^2}+\frac{\partial Q'}{\partial x}=-\mu'\left(w'_{0y'}-g_{y'}\right)-\mu' x\frac{d\omega'_z}{dt}+c'\alpha'+\\
+c\frac{\partial v'}{\partial x}+P\left(\frac{\partial v'}{\partial x}\right)_{x=x_\tau}\delta(x, x_\tau)+F'_{y'\,\text{упр}}\delta(x, x_y),\\
\frac{\partial}{\partial x}\left(B_y\frac{\partial^2 v'}{\partial x^2}\right)-j_z\frac{\partial^3 v'}{\partial x\,\partial t^2}-Q'=j'_z\frac{d\omega'_z}{dt}+P(v')_{x=x_\tau}\times\\
\times\delta(x, x_\tau)-M^{(0)'}_{z\,\text{упр}}\delta(x, x_y)
\end{gathered} \tag{4.10.7}$$

and boundary conditions:

$$B_y \frac{\partial^2 v'}{\partial x^2} = 0, \; Q' = 0 \text{ when } x = a \text{ and when } x = b. \tag{4.10.8}$$

By primes at the sign of functions we designated the membership of these functions to disturbed motion of the rocket. For coefficients with function v' and with its partial derivatives we retained their values everywhere, corresponding to undisturbed motion, in accordance with the assumption about smallness of flexural vibrations of the rocket.

Assuming in (4.10.7) and (4.10.8)

$$v' = v + \Delta v, \; Q' = Q + \Delta Q, \; \mu' = \mu + \Delta\mu, \; c' = c + \Delta c, \; \alpha' = \alpha + \Delta\alpha,$$

by using formulas (3.24.5) and (4.8.9) and disregarding small quantities of higher orders of smallness, we obtain relationships:

$$\mu \frac{\partial^2 v}{\partial t^2} + \frac{\partial Q}{\partial x} + \mu \frac{\partial^2 \Delta v}{\partial t^2} + \frac{\partial \Delta Q}{\partial x} = -\mu (w_{0y} - g_y) - \mu x \frac{d\omega_z}{dt} +$$
$$+ c\left(\alpha + \frac{\partial v}{\partial x}\right) + P\left(\frac{\partial v}{\partial x}\right)_{x=x_\text{т}} \delta(x, x_\text{т}) - \mu \left[\frac{d\Delta V_y}{dt} - (w_{0x} - g_x)\Delta\vartheta\right] -$$
$$- \mu x \frac{d^2\Delta\vartheta}{dt^2} + c\left(\Delta\vartheta - \frac{\Delta V_y}{v_{0x}}\right) + c\frac{\partial \Delta v}{\partial x} + P\left(\frac{\partial \Delta v}{\partial x}\right)_{x=x_\text{т}} \times$$
$$\times \delta(x, x_\text{т}) - \Delta\mu (w_{0y} - g_y) - \Delta\mu x \frac{d\omega_z}{dt} + \Delta c\alpha + F'_{y'\,\text{упр}} \delta(x, x_y),$$

$$\frac{\partial}{\partial x}\left(B_y \frac{\partial^2 v}{\partial x^2}\right) - J_z \frac{\partial^3 v}{\partial x \, \partial t^2} - Q + \frac{\partial}{\partial x}\left(B_y \frac{\partial^2 \Delta v}{\partial x^2}\right) - J_z \frac{\partial^3 \Delta v}{\partial x \, \partial t^2} -$$
$$- \Delta Q = J_z \frac{d\omega_z}{dt} + P(v)_{x=x_\text{т}} \delta(x, x_\text{т}) + J_z \frac{d^2 \Delta\vartheta}{dt^2} + P(\Delta v)_{x=x_\text{т}} \times$$
$$\times \delta(x, x_\text{т}) + \Delta J_z \frac{d\omega_z}{dt} - M^{(0)'}_{z\,\text{упр}} \delta(x, x_y),$$

$$B_y \frac{\partial^2 v}{\partial x^2} + B_y \frac{\partial^2 \Delta v}{\partial x^2} = 0, \; Q + \Delta Q = 0 \text{ when } x = a \text{ and when } x = b,$$

or according to (4.4.35)

$$\begin{aligned}
&\mu \frac{\partial^2 \Delta v}{\partial t^2}+\frac{\partial \Delta Q}{\partial x}=-\mu\left[\frac{d \Delta V_y}{dt}+x\frac{d^2\Delta\vartheta}{dt^2}-(w_{0x}-g_x)\Delta\vartheta\right]+\\
&+c\left(\frac{\partial \Delta v}{\partial x}+\Delta\vartheta-\frac{\Delta V_y}{v_{0x}}\right)+P\left(\frac{\partial \Delta v}{\partial x}\right)_{x=x_\tau}\delta(x, x_\tau)+\\
&+F'_{y'\,ynp}\delta(x, x_y)+\Delta q_y,\\
&\frac{\partial}{\partial x}\left(B_y\frac{\partial^2\Delta v}{\partial x^2}\right)-j_z\frac{\partial^3\Delta v}{\partial x\,\partial t^2}-\Delta Q=j_z\frac{d^2\Delta\vartheta}{dt^2}+\\
&+P(\Delta v)_{x=x_\tau}\delta(x, x_\tau)-M^{(0)'}_{z\,ynp}\delta(x, x_y)-\Delta m_z,\\
&B_y\frac{\partial^2\Delta v}{\partial x^2}=0,\ \Delta Q=0 \text{ when } x=a \text{ and when } x=b,
\end{aligned} \tag{4.10.9}$$

where

$$\begin{aligned}
&\Delta q_y=-\Delta\mu\left(w_{0y}-g_y+x\frac{d\omega_z}{dt}\right)+\Delta ca-\\
&-F_{y\,ynp}\delta(x, x_y)\\
&\Delta m_z=-\Delta j_z\frac{d\omega_z}{dt}-M^{(0)}_{z\,ynp}\delta(x, x_y).
\end{aligned} \tag{4.10.10}$$

By uniting equations (4.10.6) and (4.10.9), we obtain equations of disturbed motion:

$$\begin{aligned}
&m\frac{d\Delta V_y}{dt}+c_{yy}\Delta V_y+r_y\frac{d\Delta\vartheta}{dt}+c_{y\vartheta}\Delta\vartheta-\int_a^b\frac{\partial\Delta v}{\partial x}c\,dx-\\
&-P\left(\frac{\partial\Delta v}{\partial x}\right)_{x=x_\tau}=c_{y\delta}\Delta\delta_\varphi+\Delta F_y,\\
&J_z\frac{d^2\Delta\vartheta}{dt^2}+\mu_z\frac{d\Delta\vartheta}{dt}+c_{\vartheta y}\Delta V_y+c_{\vartheta\vartheta}\Delta\vartheta-\int_a^b\frac{\partial\Delta v}{\partial x}cx\,dx-\\
&-P\left(x\frac{\partial\Delta v}{\partial x}-\Delta v\right)_{x=x_\tau}=c_{\vartheta\delta}\Delta\delta_\varphi+\Delta M_z,\\
&\mu\frac{\partial^2\Delta v}{\partial t^2}+\frac{\partial\Delta Q}{\partial x}+\mu\left[\frac{d\Delta V_y}{dt}+x\frac{d^2\Delta\vartheta}{dt^2}-(w_{0x}-g_x)\Delta\vartheta\right]-\\
&-c\left(\frac{\partial\Delta v}{\partial x}+\Delta\vartheta-\frac{\Delta V_y}{v_{0x}}\right)-P\left(\frac{\partial\Delta v}{\partial x}\right)_{x=x_\tau}\delta(x, x_\tau)=\\
&=F'_{y'\,ynp}\delta(x, x_y)+\Delta q_y,\\
&\frac{\partial}{\partial x}\left(B_y\frac{\partial^2\Delta v}{\partial x^2}\right)-j_z\frac{\partial^3\Delta v}{\partial x\,\partial t^2}-\Delta Q-j_z\frac{d^2\Delta\vartheta}{dt^2}-P(\Delta v)_{x=x_\tau}\times\\
&\times\delta(x, x_\tau)=-M^{(0)'}_{z\,ynp}\delta(x, x_y)-\Delta m_z,\\
&B_y\frac{\partial^2\Delta v}{\partial x^2}=0,\ \Delta Q=0 \text{ when } x=a \text{ and when } x=b,
\end{aligned} \tag{4.10.11}$$

determining disturbances $\Delta\vartheta(t), \Delta V_y(t)$ and $\Delta v(x, t)$ according to predetermined controlling actions and disturbing forces and moments.

According to (4.9.1) disturbances $\Delta V_y(t), \Delta v(x, t)$ will generate input signal of automatic stabilization control, determined by formula

$$\Delta\vartheta^* = \Delta\vartheta + \left(\frac{\partial \Delta v}{\partial x}\right)_{x=x_r}. \tag{4.10.12}$$

Thus, having substituted time dependent coefficients of equations (4.10.11) by their values, corresponding to moment of time $t = \tau$, for this moment of time it is possible to calculate the sought frequency characteristics of the rocket, having assumed in (4.10.11)

$$\Delta F_y = 0,\ \Delta M_z = 0,\ \Delta q_y = 0,\ \Delta m_z = 0,\ \Delta\delta_\vartheta = e^{i\omega t}, \tag{4.10.13}$$

having determined the disturbances $\Delta\vartheta$ and Δv from the obtained equations and substituted them in (4.10.12).

In accordance with formulas (4.8.13) and (4.8.14) to functions $\Delta\delta_\vartheta$, shown in relationships (4.10.13), there will correspond equalities:

$$F'_{y' \text{упр}} = c_{y\delta} e^{i\omega t},\ M^{(0)'}_{z\,\text{упр}} = 0 \tag{4.10.14}$$

in the case when the actuating element of the control system form controlling force, and equalities:

$$F'_{y' \text{упр}} = 0,\ M^{(0)'}_{z\,\text{упр}} = c_{\delta\delta} e^{i\omega t} \tag{4.10.15}$$

in the case when the actuating elements create concentrated controlling moment. By uniting these two cases, in equations (4.10.11) we assume

$$F'_{y'\,\text{упр}}=c_{y\delta}e^{i\omega t},\quad M^{(0)'}_{z\,\text{упр}}=c^{(0)}_{\theta\delta}e^{i\omega t}, \tag{4.10.16}$$

having agreed that in the first case

$$c^{(0)}_{\theta\delta}=0, \tag{4.10.17}$$

and in the second case

$$c_{y\delta}=0,\quad c^{(0)}_{\theta\delta}=c_{\theta\delta}. \tag{4.10.18}$$

We will seek the solution of equations (4.10.11) in the form

$$\begin{gathered}\Delta v(x,t)=V(x)e^{i\omega t},\quad \Delta Q(x,t)=Q(x)e^{i\omega t},\\ \Delta\theta(t)=\theta e^{i\omega t},\quad \Delta V_y=V_y e^{i\omega t}.\end{gathered} \tag{4.10.19}$$

By substituting (4.10.13), (4.10.16) and (4.10.19) in (4.10.11), for functions $V(x)$, $Q(x)$ and for constants θ and V_y we obtain equations:

$$\begin{gathered}
(c_{yy}+i\omega m)V_y+(c_{y\theta}+i\omega\nu_y)\theta-\int_a^b\frac{dV}{dx}c\,dx-\\
-P\left(\frac{dV}{dx}\right)_{x=x_\text{т}}=c_{y\delta},\\
c_{\theta y}V_y+(c_{\theta\theta}+i\omega\mu_z-\omega^2J_z)\theta-\int_a^b\frac{dV}{dx}cx\,dx-\\
-P\left(x\frac{dV}{dx}-V\right)_{x=x_\text{т}}=c_{\theta\delta},\\
-\omega^2\mu V+\frac{dQ}{dx}+\mu\left[i\omega V_y-(w_{0x}-g_x+\omega^2x)\theta\right]-\\
-c\left(\frac{dV}{dx}+\theta-\frac{V_y}{v_{0x}}\right)-P\left(\frac{dV}{dx}\right)_{x=x_\text{т}}\delta(x,x_\text{т})=c_{y\delta}\delta(x,x_y),\\
\frac{d}{dx}\left(B_y\frac{d^2V}{dx^2}\right)+\omega^2j_z\frac{dV}{dx}-Q+\omega^2j_z\theta-\\
-P(V)_{x=x_\text{т}}\delta(x,x_\text{т})=-c^{(0)}_{\theta\delta}\delta(x,x_y),\\
B_y\frac{d^2V}{dx^2}=0,\ Q=0 \text{ when } x=a \text{ and when } x=b.
\end{gathered} \tag{4.10.20}$$

According to (4.10.12) and (4.10.19) in the considered case the input signal of automatic stabilization control will be determined by equality

$$\Delta\theta^* = \theta^* e^{i\omega t}, \tag{4.10.21}$$

where

$$\theta^* = \theta + \left(\frac{dV}{dx}\right)_{x=x_r}. \tag{4.10.22}$$

Thus, without allowing for damping, peculiar to flexural vibrations of the rocket body, i.e., when $\eta = 0$, function $\theta^*(\omega)$, determining the sought frequency characteristics of the rocket, can be constructed from formula (4.10.22), having found function $V(x)$ and constant θ from equations (4.10.20).

Let us now show that the presence of damping, determined by predetermined value of coefficient η, can be taken into account having substituted the actual flexural rigidity of the rocket body $B_y(x)$ in equations (4.10.20) by fictitious complex rigidity $B_y^*(x)$, determined by relationship

$$B_y^*(x) = (1 + i\eta) B_y(x). \tag{4.10.23}$$

For this it is sufficient for us to prove that function $\theta^*(\omega)$, found from relationships (4.10.20) and (4.10.22) with replacement of rigidity B_y by fictitious rigidity B_y^*, becomes identical function $\theta^*(\omega)$, found from infinite system of algebraic equations (4.10.4) by summation of series (4.9.9).

We showed above that function $v(x, t)$, found from original equations of motion (4.4.35), can always be represented by series (4.7.6). In accordance with this, function $V(x)$, found from equations (4.10.20), can be represented in the form of series

$$V(x)=\sum_{k=1}^{\infty} G_k V_k(x, \tau) \tag{4.10.24}$$

[equations (4.10.20) are obtained from equations of disturbed motion (4.10.11) after replacement of time dependent coefficients of system (4.10.11) by their values, corresponding to moment of time $t = \tau$].

By substituting series (4.10.24) in the first two equations from (4.10.20), according to (4.7.8) we obtain equations:

$$\begin{aligned} &(c_{yy}+i\omega m)V_y+(c_{y\theta}+i\omega v_y)\theta+\sum_{k=1}^{\infty} c_{yk}G_k=c_{yt},\\ &c_{\theta y}V_y+(c_{\theta\theta}+i\omega\varphi_z-\omega^2 J_z)\theta+\sum_{k=1}^{\infty} c_{\theta k}G_k=c_{\theta t},\end{aligned} \tag{4.10.25}$$

where c_{yk} and $c_{\theta k}$, $k=1, 2, \ldots$ we should mean the values of these coefficients, corresponding to moment of time $t = \tau$.

By substituting series (4.10.24) in the differential equations, shown in (4.10.20), and replacing rigidity B_y by rigidity B_y^*, we obtain relationships:

$$\begin{aligned} &-\omega^2\mu\sum_{k=1}^{\infty} G_k V_k+\frac{dQ}{dx}+\mu\left[i\omega V_y-(\dot{w}_{0x}-g_x+\omega^2 x)\theta\right]-\\ &\quad-c\left(\sum_{k=1}^{\infty} C_k\frac{\partial V_k}{\partial x}+\theta-\frac{V_y}{v_{0x}}\right)-\\ &\quad-\sum_{k=1}^{\infty} G_k\left(\frac{\partial V_k}{\partial x}\right)_{x=x_\tau}\delta(x, x_\tau)=c_{yt}\delta(x, x_y),\\ &\sum_{k=1}^{\infty} G_k\left[\frac{\partial}{\partial x}\left(B_y^*\frac{\partial^2 V_k}{\partial x^2}\right)+\omega^2 J_z\frac{\partial V_k}{\partial x}\right]-Q+\omega^2 J_z\theta-\\ &\quad-P\sum_{k=1}^{\infty} G_k(V_k)_{x=x_\tau}\delta(x, x_\tau)=-c_{\theta t}^{(0)}\delta(x, x_y),\end{aligned} \tag{4.10.26}$$

in which for all functions, depending on time t, we bear in mind their values, corresponding to moment of time $t = \tau$.

Having multiplied both sides of the first equation from (4.10.26) by V_j and integrated them with respect to x from a to b, we obtain relationship

$$
\begin{gathered}
-\omega^2 \sum_{k=1}^{\infty} G_k \int_a^b V_j V_k \mu dx + \int_a^b V_j \frac{dQ}{dx} dx + [i\omega V_y - \\
-(w_{0x} - g_x)\theta] \int_a^b V_j \mu dx - \omega^2 \theta \int_a^b V_j \mu x dx - \sum_{k=1}^{\infty} G_k \int_a^b V_j \frac{\partial V_k}{\partial x} c dx - \\
-\left(\theta - \frac{V_y}{w_{0x}}\right) \int_a^b V_j c dx - P \sum_{k=1}^{\infty} G_k \left(\frac{\partial V_k}{\partial x}\right)_{x=x_\tau} \int_a^b V_j \delta(x, x_\tau) dx = \\
= c_{y\delta} \int_a^b V_j \delta(x, x_\eta) dx.
\end{gathered} \tag{4.10.27}
$$

Having multiplied both sides of the second equation from (4.10.26) by $-\frac{\partial V_j}{\partial x}$ and integrated them with respect x from a to b, we obtain relationship

$$
\begin{gathered}
-\sum_{k=1}^{\infty} G_k \int_a^b \frac{\partial V_j}{\partial x} \frac{\partial}{\partial x} \left(B_y^* \frac{\partial^2 V_k}{\partial x^2}\right) dx - \omega^2 \sum_{k=1}^{\infty} G_k \int_a^b \frac{\partial V_j}{\partial x} \frac{\partial V_k}{\partial x} J_z dx + \\
+\int_a^b Q \frac{\partial V_j}{\partial x} dx - \omega^2 \theta \int_a^b \frac{\partial V_j}{\partial x} J_z dx + P \sum_{k=1}^{\infty} G_k (V_k)_{x=x_\tau} \int_a^b \frac{\partial V_j}{\partial x} \delta(x, x_\tau) dx = \\
= c_{y\delta}^{(\delta)} \int_a^b \frac{\partial V_j}{\partial x} \delta(x, x_y) dx.
\end{gathered} \tag{4.10.28}
$$

In accordance with formulas (4.10.23), (4.10.27) and (4.10.28) there should exist equality

$$
-\omega^2 \sum_{k=1}^{\infty} G_k \int_a^b \left(V_j V_k \mu + \frac{\partial V_j}{\partial x} \frac{\partial V_k}{\partial x} j_z \right) dx -
$$

$$
-(1+i\eta) \sum_{k=1}^{\infty} G_k \int_a^b \frac{\partial V_j}{\partial x} \frac{\partial}{\partial x} \left(B_y \frac{\partial^2 V_k}{\partial x^2} \right) dx + \int_a^b \left(V_j \frac{dQ}{dx} + Q \frac{\partial V_j}{\partial x} \right) dx +
$$

$$
+ [i\omega V_y - (w_{0x} - g_x)\theta] \int_a^b V_j \mu dx - \omega^2 \theta \int_a^b \left(V_j \mu x + \frac{\partial V_j}{\partial x} j_z \right) dx -
$$

$$
-\left(\theta - \frac{V_y}{v_{0x}} \right) \int_a^b V_j c dx - \sum_{k=1}^{\infty} G_k \left\{ \int_a^b V_j \frac{\partial V_k}{\partial x} c dx + \right.
$$

$$
+ P \left[\left(\frac{\partial V_k}{\partial x} \right)_{x=x_\tau} \int_a^b V_j \delta(x, x_\tau) dx - \right.
$$

$$
\left. \left. -(V_k)_{x=x_\tau} \int_a^b \frac{\partial V_j}{\partial x} \delta(x, x_\tau) dx \right] \right\} = c_{y\delta} \int_a^b V_j \delta(x, x_y) dx +
$$

$$
+ c_{\theta\delta}^{(0)} \int_a^b \frac{\partial V_j}{\partial x} \delta(x, x_y) dx,
$$

or according to (4.4.28), (4.5.30), (4.5.31) and (4.7.19)

$$
\begin{aligned}
&\sum_{k=1}^{\infty} G_k [(1+i\eta)\lambda_k - \omega^2)] \int_0^b \left(V_j V_k \mu + \frac{\partial V_j}{\partial x} \frac{\partial V_k}{\partial x} j_z \right) dx + \\
&+ \int_a^b \left(V_j \frac{dQ}{dx} + Q \frac{dV_j}{dx} \right) dx - \left(\theta - \frac{V_y}{v_{0x}} \right) \int_a^b V_j c dx - \\
&- \sum_{k=1}^{\infty} G_k \left[\int_a^b V_j \frac{\partial V_k}{\partial x} c dx + P \left(V_j \frac{\partial V_k}{\partial x} - V_k \frac{\partial V_j}{\partial x} \right)_{x=x_\tau} \right] = \\
&= c_{y\delta} (V_j)_{x=x_y} + c_{\theta\delta}^{(0)} \left(\frac{\partial V_j}{\partial x} \right)_{x=x_y}.
\end{aligned}
\tag{4.10.29}
$$

By using relationships (4.5.26) and (4.5.32) and meanings (4.7.23) and (4.8.11), we can reduce equality (4.10.29) to the form

$$[(1+i\eta)\omega_j^2-\omega^2]m_jG_j+\int_a^b\left(V_j\frac{dQ}{dx}+Q\frac{\partial V_j}{\partial x}\right)dx+c_{jv}V_v+ \\ +c_{j\theta}\theta+\sum_{k=1}^{\infty}c_{jk}G_k=c_{y\delta}(V_j)_{x=x_y}+c_{\delta\delta}^{(0)}\left(\frac{\partial V_j}{\partial x}\right)_{x=x_y}. \quad (4.10.30)$$

In accordance with boundary conditions shown for function Q in formulas (4.10.20), there should take place equality

$$\int_a^b\left(V_j\frac{dQ}{dx}+Q\frac{\partial V_j}{\partial x}\right)dx=\int_a^b\frac{\partial}{\partial x}(QV_j)dx=QV_j\Big|_{x=a}^{x=b}=0. \quad (4.10.31)$$

According to (4.8.16) and (4.10.17) when the actuating elements of the control system create controlling force

$$c_{y\delta}(V_j)_{x=x_y}+c_{\delta\delta}^{(0)}\left(\frac{\partial V_j}{\partial x}\right)_{x=x_y}=c_{j\delta}. \quad (4.10.32)$$

Equality (4.10.32) will exist when the actuating elements of the control system create concentrated controlling moment, since in this instance there will take place relationships (4.8.17) and (4.10.18). According to (4.10.3), (4.10.30), (4.10.31) and (4.10.32) the coefficients of series (4.10.24) must satisfy equations

$$m_j(\omega_j^{*2}-\omega^2)G_j+c_{jv}V_v+c_{j\theta}\theta+\sum_{k=1}^{\infty}c_{jk}G_k=c_{j\delta},\ j=1,2,\ldots \quad (4.10.33)$$

Infinite system of algebraic equations, formed by equations (4.10.25) and (4.10.33), coincides with the infinite system of equations (4.10.4). By substituting series (4.10.24) in formula (4.10.22), for function $\theta^*(\omega)$ we obtain expansion (4.9.9). Thus, by replacing rigidity B_y in (4.10.20) by rigidity B_y^*, by solving equations (4.10.20) and further determining function $\theta^*(\omega)$ by formula (4.10.22), we obtain the same result as in the case of construction of this function by means of solution of infinite system of equations (4.10.4) and summation of series (4.9.9), which it was required to prove.

For determination of function $V(x)$ and constant θ from equations (4.10.20) it is possible to use the following calculation scheme. Differential equations, shown in (4.10.20), after replacement of rigidity B_y by rigidity B_y^*, can be given the form

$$\begin{gathered} c\frac{dV}{dx}+\omega^2\mu V-\frac{dQ}{dx}=f_0(x)+V_y f_1(x)+\theta f_2(x)+ \\ +(V)_{x=x_\tau} f_3(x)+\left(\frac{dV}{dx}\right)_{x=x_\tau} f_4(x), \\ \frac{d}{dx}\left(B_y^*\frac{d^2V}{dx^2}\right)+\omega^2 j_z\frac{dV}{dx}-Q=g_0(x)+V_y g_1(x)+ \\ +\theta g_2(x)+(V)_{x=x_\tau} g_3(x)+\left(\frac{dV}{dx}\right)_{x=x_\tau} g_4(x), \end{gathered} \tag{4.10.34}$$

where

$$\begin{gathered} f_0=-c_{yt}\delta(x,x_y),\ f_1=i\omega\mu+\frac{c}{v_{0x}}, \\ f_2=-(w_{0x}-g_x+\omega^2x)\mu-c, \\ f_3=0,\ f_4=-P\delta(x,x_\tau), \\ g_0=-c_{\delta\delta}^{(0)}\delta(x,x_y),\ g_1=0,\ g_2=-\omega^2 j_z, \\ g_3=P\delta(x,x_\tau),\ g_4=0. \end{gathered} \tag{4.10.35}$$

Boundary conditions for functions V and Q, shown in (4.10.20), with replacement of rigidity B_y by rigidity B_y^* take the form

$$B_y^*\frac{d^2V}{dx^2}=0,\ Q=0 \text{ when } x=a \text{ and when } x=b. \tag{4.10.36}$$

Let us introduce into consideration functions $V_j(x)$ and $Q_j(x)$, $j = 0, 1, \ldots, 4$, satisfying differential equations:

$$\begin{aligned} & c\frac{dV_j}{dx}+\omega^2\mu V_j-\frac{dQ_j}{dx}=f_j(x) \\ & \frac{d}{dx}\left(B_y^\circ \frac{d^2V_j}{dx^2}\right)+\omega^2 j_z\frac{dV_j}{dx}-Q_j=g_j(x) \end{aligned} \qquad j=0, 1, \ldots, 4 \tag{4.10.37}$$

and boundary conditions:

$$B_y^\circ \frac{d^2V_j}{dx^2}=0, \quad Q_j=0, \; j=0, 1, \ldots, 4, \text{ when } x=a \text{ and when } x=b. \tag{4.10.38}$$

According to (4.10.37) and (4.10.38) functions:

$$\begin{aligned} V(x) &= V_0(x)+V_y V_1(x)+\theta V_2(x)+ \\ &\quad +(V)_{x=x_T} V_3(x)+\left(\frac{dV}{dx}\right)_{x=x_T} V_4(x), \\ Q(x) &= Q_0(x)+V_y Q_1(x)+\theta Q_2(x)+ \\ &\quad +(V)_{x=x_T} Q_3(x)+\left(\frac{dV}{dx}\right)_{x=x_T} Q_4(x) \end{aligned} \tag{4.10.39}$$

will form the solution of system of differential equations (4.10.34), satisfying boundary conditions (4.10.36). By substituting function $V(x)$ from (4.10.39) in the first two equations of system (4.10.20), we obtain equations:

$$\left.\begin{aligned}
&\left[c_{yy}+i\omega m-\int_a^b \frac{dV_1}{dx}c\,dx-P\left(\frac{dV_1}{dx}\right)_{x=x_T}\right]V_y+\\
&\quad+\left[c_{y\theta}+i\omega v_y-\int_a^b \frac{dV_2}{dx}c\,dx-P\left(\frac{\partial V_2}{\partial x}\right)_{x=x_T}\right]\theta-\\
&\quad-\left[\int_a^b \frac{dV_3}{dx}c\,dx+P\left(\frac{dV_3}{dx}\right)_{x=x_T}\right](V)_{x=x_T}-\\
&\quad-\left[\int_a^b \frac{dV_4}{dx}c\,dx+P\left(\frac{dV_4}{dx}\right)_{x=x_T}\right]\left(\frac{dV}{dx}\right)_{x=x_T}=c_{y0}+\\
&\quad+\int_a^b \frac{dV_0}{dx}c\,dx+P\left(\frac{dV_0}{dx}\right)_{x=x_T},\\
&\left[c_{\theta y}-\int_a^b \frac{dV_1}{dx}cx\,dx-P\left(x\frac{dV_1}{dx}-V_1\right)_{x=x_T}\right]V_y+\\
&\quad+\Bigg[c_{\theta\theta}+i\omega p_z-\omega^2 J_z-\int_a^b \frac{dV_2}{dx}cx\,dx-\\
&\quad-P\left(x\frac{dV_2}{dx}-V_2\right)_{x=x_T}\Bigg]\theta-\Bigg[\int_a^b \frac{dV_3}{dx}cx\,dx+\\
&\quad+P\left(x\frac{dV_3}{dx}-V_3\right)_{x=x_T}\Bigg](V)_{x=x_T}-\Bigg[\int_a^b \frac{dV_4}{dx}cx\,dx+\\
&\quad+P\left(x\frac{dV_4}{dx}-V_4\right)_{x=x_T}\Bigg]\left(\frac{dV}{dx}\right)_{x=x_T}=\\
&\quad=c_{\theta 0}+\int_a^b \frac{dV_0}{dx}cx\,dx+P\left(x\frac{dV_0}{dx}-V_0\right)_{x=x_T}.
\end{aligned}\right\} \qquad (4.10.40)$$

According to (4.10.39) there should exist equalities:

$$\left.\begin{aligned}
&(V_1)_{x=x_T}V_y+(V_2)_{x=x_T}\theta+[(V_3)_{x=x_T}-1](V)_{x=x_T}+\\
&\qquad+(V_4)_{x=x_T}\left(\frac{dV}{dx}\right)_{x=x_T}=-(V_0)_{x=x_T},\\
&\left(\frac{dV_1}{dx}\right)_{x=x_T}V_y+\left(\frac{dV_2}{dx}\right)_{x=x_T}\theta+\left(\frac{dV_3}{dx}\right)_{x=x_T}(V)_{x=x_T}+\\
&\qquad+\left[\left(\frac{dV_4}{dx}\right)_{x=x_T}-1\right]\left(\frac{dV}{dx}\right)_{x=x_T}=-\left(\frac{dV_0}{dx}\right)_{x=x_T}.
\end{aligned}\right\} \qquad (4.10.41)$$

Equations (4.10.40) and (4.10.41) form a system of four linear algebraic equations for four unknown constants V_y, θ, $(V)_{x=x_T}$ and $\left(\frac{dV}{dx}\right)_{x=x_T}$.

Thus, calculation of sought frequency characteristics is reduced to integration of the system of ordinary differential equations

$$c\frac{dV}{dx}+\omega^2\mu V-\frac{dQ}{dx}=f(x),$$
$$\frac{d}{dx}\left(B_y^*\frac{d^2V}{dx^2}\right)+\omega^2 J_z\frac{dV}{dx}-Q=g(x) \qquad (4.10.42)$$

with boundary conditions:

$$B_y^*\frac{d^2V}{dx^2}=0,\quad Q=0 \text{ when } x=a \text{ and when } x=b. \qquad (4.10.43)$$

With prescribed value of frequency ω, by successively assuming in (4.10.42) $f(x) = f_j(x)$, $g(x) = g_j(x)$, $j = 0, 1, \ldots, 4$, we find functions $V_j(x)$ and $Q_j(x)$, $j = 0, 1, \ldots, 4$, satisfying equations (4.10.37) and boundary conditions (4.10.38). Having determined further constants V_y, θ, $(V)_{x=x_T}$ and $\left(\frac{dV}{dx}\right)_{x=x_T}$ from equations (4.10.40) and (4.10.41) and having substituted their values in the first of formulas (4.10.39), we find function $V(x)$. Having then found the real part θ_1^* and imaginary part θ_2^* of complex number θ^*, determined by relationship (4.10.22), by formulas (4.9.12) it is possible to further calculate amplification factor k_0 and phase shift ϕ_0 for original calculated value of frequency ω. By carrying out the shown calculation for series of successive values of frequency ω, it is possible to construct, thus, the amplitude-frequency and phase-frequency characteristics of the rocket as the object of automatic control.

Solution of nonhomogeneous system of differential equations (4.10.42), satisfying boundary conditions (4.10.43), can be found, having constructed four linearly independent particular solutions of homogeneous system of differential equations

$$\begin{aligned} & c\frac{dV}{dx}+\omega^2\mu V-\frac{dQ}{dx}=0, \\ & \frac{d}{dx}\left(B^{*}_{v}\frac{d^2V}{dx^2}\right)+\omega^2 I_s\frac{dV}{dx}-Q=0. \end{aligned} \quad (4.10.44)$$

By using the method of variation of arbitrary constants, it is possible by these particular solutions of system (4.10.44) to construct general solution of system of differential equations (4.10.42); arbitrary constants will be determined by boundary conditions (4.10.43).

The method of calculation of frequency characteristics of the rocket as the controllable object, discussed in this paragraph, is less laborious than the method shown in § 9, and it provides higher accuracy of calculations. It is necessary to resort to the method described in § 9 during solution of the problem about some "shortening" of infinite system of differential equations of disturbed motion (4.8.18) during investigation of stability of motion by means of electromodeling with actual equipment of the control system. Shortening of equations of disturbed motion (4.8.18) can be considered permissible if the frequency characteristics, found from appropriately shortened equations (4.9.7) and from appropriate particular sum of series (4.9.9), for the investigated range of frequencies gives errors, which can be considered permissible in the question of stability of motion.

§ 11. Stabilization of Motion of the Rocket with Account of Elasticity of its Construction

During examination of the question about stabilization of flexural vibrations of the rocket body we will proceed from approximate construction of frequency characteristics of the rocket, as a controllable object, for frequencies ω close to frequencies of natural flexural vibrations of the body ω_k, $k = 1, 2, \ldots$ With frequencies ω close to frequency ω_j, in expansion (4.9.9) the basic role is played by component $\left(\frac{\partial V_j}{\partial x}\right)_{x=x_r} G_j(\omega)$, the remaining terms of this series in the considered case for all practical purposes can be disregarded. In connection with this, for frequencies ω close to frequency ω_j, it is approximately possible to assume

$$\theta^*(\omega)=\left(\frac{\partial V_j}{\partial x}\right)_{x=x_r} G_j(\omega). \qquad (4.11.1)$$

As was already indicated in § 9, coefficients $c^*_{j\delta}$, $j=1, 2, \ldots$, figuring in formulas (4.9.15), are close to corresponding coefficients $c_{j\delta}$, $j=1, 2, \ldots$ and it is possible to approximately assume $c^*_{j\delta}=c_{j\delta}$. In this instance according to (4.9.15) there will exist equality

$$G_j(\omega)=\frac{c_{j\delta}}{m_j\left(\omega_j^2+i\eta\omega\omega_j-\omega^2\right)}. \qquad (4.11.2)$$

By substituting (4.11.2) in (4.11.1), for function $\theta^*(\omega)$ we obtain approximate formula

$$\theta^*(\omega)=\frac{c_{j\delta}\left(\frac{\partial V_j}{\partial x}\right)_{x=x_r}}{m_j\left(\omega_j^2+i\eta\omega\omega_j-\omega^2\right)}. \qquad (4.11.3)$$

By separating the real and imaginary parts of complex expressions in (4.11.3), we obtain relationships:

$$\left.\begin{aligned}\theta_1^*(\omega)&=\frac{c_{j\delta}\left(\frac{\partial V_j}{\partial x}\right)_{x=x_r}\left(\omega_j^2-\omega^2\right)}{m_j\left[\left(\omega_j^2-\omega^2\right)^2+\eta^2\omega^2\omega_j^2\right]},\\ \theta_2^*(\omega)&=-\frac{c_{j\delta}\left(\frac{\partial V_j}{\partial x}\right)_{x=x_r}\eta\omega\omega_j}{m_j\left[\left(\omega_j^2-\omega^2\right)^2+\eta^2\omega^2\omega_j^2\right]}.\end{aligned}\right\} \qquad (4.11.4)$$

where in accordance with earlier accepted meanings $\theta_{1}^{*}(\omega)$ - real and $\theta_{r}^{*}(\omega)$ - imaginary parts of complex-valued function $\theta^{*}(\omega)$. According to (4.9.10) and (4.11.4) in the considered case there will exist equalities:

$$k_{o}(\omega)\cos\varphi_{o}(\omega)=\frac{c_{jt}\left(\frac{\partial V_{j}}{\partial x}\right)_{x=x_{r}}(\omega_{j}^{2}-\omega^{2})}{m_{j}\left[(\omega_{j}^{2}-\omega^{2})+\eta^{2}\omega^{2}\omega_{j}^{2}\right]},$$
$$k_{o}(\omega)\sin\varphi_{o}(\omega)=-\frac{c_{jt}\left(\frac{\partial V_{j}}{\partial x}\right)_{x=x_{r}}\eta\omega\omega_{j}}{m_{j}\left[(\omega_{j}^{2}-\omega^{2})^{2}+\eta^{2}\omega^{2}\omega_{j}^{2}\right]}. \quad (4.11.5)$$

In accordance with formulas (3.26.25) and (4.11.5) the equation of hodograph of open system of automatic control in this case will have the form

$$X=\frac{c_{jt}\left(\frac{\partial V_{j}}{\partial x}\right)_{x=x_{r}}k_{a}(\omega)\left[(\omega_{j}^{2}-\omega^{2})\cos\varphi_{a}(\omega)+\eta\omega\omega_{j}\sin\varphi_{a}(\omega)\right]}{m_{j}\left[(\omega_{j}^{2}-\omega^{2})^{2}+\eta^{2}\omega^{2}\omega_{j}^{2}\right]},$$
$$Y=\frac{c_{jt}\left(\frac{\partial V_{j}}{\partial x}\right)_{x=x_{r}}k_{a}(\omega)\left[(\omega_{j}^{2}-\omega^{2})\sin\varphi_{a}(\omega)-\eta\omega\omega_{j}\cos\varphi_{a}(\omega)\right]}{m_{j}\left[(\omega_{j}^{2}-\omega^{2})^{2}+\eta^{2}\omega^{2}\omega_{j}^{2}\right]}, \quad (4.11.6)$$

where $k_{a}(\omega)$ and $\varphi_{a}(\omega)$ - functions determining the amplitude-frequency and phase-frequency characteristics cf automatic stabilization control.

Assuming in (4.11.6)

$$\omega=\omega_{j}+\delta \quad (4.11.7)$$

and disregarding quantities of the second order of smallness in numerators of the obtained fractions and quantities of the third order of smallness in their denominators in comparison with smallness of quantities δ and η, for frequencies ω close to frequency ω_{j}, we obtain approximate equation of hodograph

$$X = -\frac{c_{js}\left(\frac{\partial V_j}{\partial x}\right)_{x=x_r} k_a(\omega_j)\left[2\delta\cos\varphi_a(\omega_j) - \eta\omega_j \sin\varphi_a(\omega_j)\right]}{m_j\omega_j\left(4\delta^2 + \eta^2\omega_j^2\right)}.$$

$$Y = -\frac{c_{js}\left(\frac{\partial V_j}{\partial x}\right)_{x=x_r} k_a(\omega_j)\left[2\delta\sin\varphi_a(\omega_j) + \eta\omega_j \cos\varphi_a(\omega_j)\right]}{m_j\omega_j\left(4\delta^2 + \eta^2\omega_j^2\right)}. \qquad (4.11.8)$$

According to (4.11.8) there should exist equalities:

$$X^2 + Y^2 = \frac{c_{js}^2\left(\frac{\partial V_j}{\partial x}\right)_{x=x_r}^2 k_a^2(\omega_j)}{m_j^2\omega_j^2\left(4\delta^2 + \eta^2\omega_j^2\right)},$$

$$X\sin\varphi_a(\omega_j) - Y\cos\varphi_a(\omega_j) = \frac{\eta c_{js}\left(\frac{\partial V_j}{\partial x}\right)_{x=x_r} k_a(\omega_j)}{m_j\left(4\delta^2 + \eta^2\omega_j^2\right)}, \qquad (4.11.9)$$

in accordance with which the equation of hodograph can be presented in the form

$$X^2 + Y^2 = \frac{c_{js}\left(\frac{\partial V_j}{\partial x}\right)_{x=x_r} k_a(\omega_j)\left[X\sin\varphi_a(\omega_j) - Y\cos\varphi_a(\omega_j)\right]}{\eta m_j\omega_j^2}. \qquad (4.11.10)$$

By introducing meanings

$$X_0 = \frac{c_{js}\left(\frac{\partial V_j}{\partial x}\right)_{x=x_r} k_a(\omega_j)\sin\varphi_a(\omega_j)}{2\eta m_j\omega_j^2},$$

$$Y_0 = -\frac{c_{js}\left(\frac{\partial V_j}{\partial x}\right)_{x=x_r} k_a(\omega_j)\cos\varphi_a(\omega_j)}{2\eta m_j\omega_j^2}, \qquad (4.11.11)$$

it is possible to reduce equation (4.11.10) to the form

$$X^2 + Y^2 = 2(XX_0 + YY_0),$$

or

$$(X - X_0)^2 + (Y - Y_0)^2 = R^2, \qquad (4.11.12)$$

where

$$R=\sqrt{X_0^2+Y_0^2}=\frac{k_a(\omega_j)\left|c_{j\beta}\left(\frac{\partial V_j}{\partial x}\right)_{x=x_r}\right|}{2\eta_j m_j \rho_j^2}. \tag{4.11.13}$$

Equation (4.11.12) represents the equation of circumference of radius R with center at the point with coordinates $X = X_0$, $Y = Y_0$. This circumference passes through the origin of coordinates, since $X_0^2 + Y_0^2 = R^2$. According to (4.11.11) when

$$c_{j\beta}\left(\frac{\partial V_j}{\partial x}\right)_{x=x_r} \sin \varphi_a(\omega_j)<0 \tag{4.11.14}$$

The center of this circumference does not exceed the bounds of the left half-plane and in this case at frequencies close to frequency ω_j, the hodograph of open system of automatic control has the form shown in Fig. 4.5 by solid line for case $Y_0 > 0$ and broken line for case $Y_0 < 0$. In the considered case the circumference, determined by equation (4.11.12), cannot encompass point $X = 1$, $Y = 0$, no matter how large its radius R. Thus, conditions (4.11.4) are sufficient conditions of stability of flexural vibrations of the rocket body.[1] Inequality (4.11.14) does not maintain the amplification factor of automatic stabilization control, and in connection with this, it has been accepted to call it the condition of phase stabilization of j-th tone of flexural vibrations.

According to (4.11.11) when

$$c_{j\beta}\left(\frac{\partial V_j}{\partial x}\right)_{x=x_r} \sin \varphi_a(\omega_j)>0 \tag{4.11.15}$$

[1]With the simplifications made by us the transfer function does not have poles in the right half-plane.

the center of the circumference, determined by equation (4.11.12), lies in the right half-plane and in this case at frequencies close to frequency ω_j the hodograph of open system of automatic control has the form shown in Fig. 4.6 by solid line for case $Y_0 > 0$ and broken line for case $Y_0 < 0$. According to (4.11.12) the point of intersection of positive semiaxis x by the hodograph has coordinate x, equal to $2X_0$. In order that at frequencies close to frequency ω_j the hodograph would not encompass point $X = 1$, $Y = 0$, inequality $2X_0 \leq 1$ must be fulfilled. According to (4.11.11) this inequality can be given the form

$$k_a(\omega_j) \leqslant \frac{\eta m_j \omega_j^2}{c_{jS}\left(\frac{\partial V_j}{\partial x}\right)_{x=x_r} \sin\varphi_a(\omega_j)}. \quad (4.11.16)$$

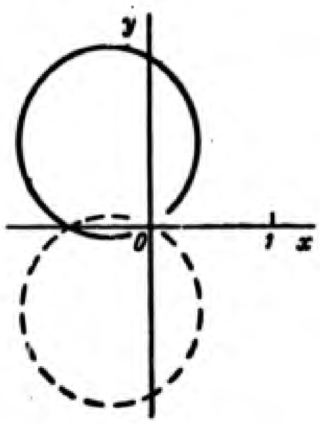

Fig. 4.5.

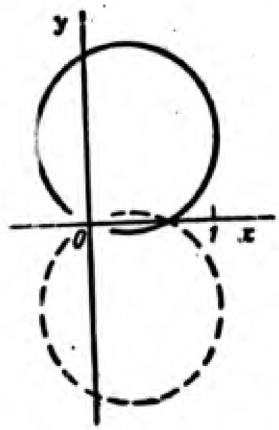

Fig. 4.6.

Condition (4.11.16) in the considered case is a necessary condition of stability for the j-th tone of flexural vibrations of the rocket body; this condition is usually called the condition of amplitude stabilization.

Thus, for j-th tone of flexural vibrations of the rocket body the stability can be provided for either by fulfilling the condition of phase stabilization (4.11.14), or by fulfilling the condition of amplitude stabilization (4.11.16). Coefficients m_j and $c_{j\delta}$, derivatives $\left(\frac{\partial V_j}{\partial x}\right)_{x=x_T}$ and frequencies of natural flexural vibrations of the body ω_j, figuring in conditions of stability (4.11.14) and (4.11.16), in the process of rocket flight undergo substantial changes. In connection with this, for all powered-flight phases of the rocket the phase stabilization is usually realized only for the first tone of flexural vibrations. Providing the amplitude stabilization of flexural vibrations also frequently turns out to be difficultly feasible. In such cases on the rocket it is necessary to install additional sensing devices of the control system, as which there are usually applied angular velocity transmitters. By successfully combining the signals of these transmitters with the basic input signal of automatic stabilization control, it is possible to significantly facilitate stabilization of flexural vibrations of the rocket body.

After preliminary selection of frequency characteristics of automatic stabilization control the stability of flexural vibrations of the rocket body is checked by calculation by means of accurate construction of a hodograph of the open system of automatic control and experimentally by means of electromodeling with a mockup of the actual equipment of the control system.

APPENDIX

CANONICAL TRANSFORMATIONS OF EQUATIONS OF DISTURBED MOTION

In practice usually the stability and controllability of the rocket is finally checked on simulating devices — electronic analog computers combined with actual equipment of the control system. On such devices there is reproduced the behavior of the rocket under prescribed initial conditions. For complete assurance there are required the same types of investigations at various combinations of initial conditions, the minimum necessary number of which sharply increases with increase of the order of system of equations of disturbed motion of the rocket and are usually so great that examination of all variants is sometimes not possible. Furthermore, with increase of the order of system of equations with utilization of analog computers the errors in results and complexity of debugging noticeably increase.

These difficulties to a considerable degree can be eliminated, if, using high-speed discrete computers, we preliminary perform "diagonalization" of the original system of equations of disturbed motion of the rocket as the object of control. This appendix is devoted to an account of algorithms of similarity transformation of equations. In § 2 there is listed the algorithm of transformation of equations, and in subsequent paragraphs there is indicated the method of such transformation with allowance for variability of coefficients of equations; there are also given corresponding transformations of the expression of control signal of automatic stabilization control.

The account is given in convenient and economical language of the theory of matrices. The necessary reductions from theory of matrices are listed in § 1.

Generally there are examined linearized equations of disturbed motion of the rocket

$$\sum_{j=1}^{n}\left[m_{ij}(t)\frac{d^2q_j}{dt^2}+r_{ij}(t)\frac{dq_j}{dt}+n_{ij}(t)q_j\right]= =a_i(t)\delta+h_i(t) \qquad (i=1, 2, \ldots, n). \tag{A}$$

In matrix writing these equations have the form

$$m(t)\frac{d^2q}{dt^2}+r(t)\frac{dq}{dt}+n(t)q=a(t)\delta+h(t), \tag{A'}$$

where

$$m=\begin{bmatrix} m_{11} & m_{12} & \ldots & m_{1n} \\ m_{21} & m_{22} & \ldots & m_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ m_{n1} & m_{n2} & \ldots & m_{nn} \end{bmatrix}, \quad r=\begin{bmatrix} r_{11} & r_{12} & \ldots & r_{1n} \\ r_{21} & r_{22} & \ldots & r_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ r_{n1} & r_{n2} & \ldots & r_{nn} \end{bmatrix};$$

$$n=\begin{bmatrix} n_{11} & n_{12} & \ldots & n_{1n} \\ n_{21} & n_{22} & \ldots & n_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ n_{n1} & n_{n2} & \ldots & n_{nn} \end{bmatrix},$$

$$q=\begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}, \quad a=\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad h=\begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix}.$$

Control signal of automatic stabilization control is examined in the form

$$v=b_1\frac{dq}{dt}+b_2q, \tag{B}$$

where b_1 and b_2 - row matrices.

§ 1. Elements of Matrix Calculation

1.1. Basic Determinations and Meanings

Let us assume there is given a certain numerical field K, i.e., aggregate of numbers, within which four operations are always feasible and simple: summation, subtraction, multiplication and division by a number, nonzero, for example, field of complex numbers.

Matrix is a rectangular table of numbers of field K, consisting of m lines and n columns:

$$\begin{bmatrix} a_{11} & a_{12} \dots a_{1n} \\ a_{21} & a_{22} \dots a_{2n} \\ \dots & \dots \\ a_{m1} & a_{m2} \dots a_{mn} \end{bmatrix}.$$

If $m = n$, then the matrix is called *square*, and number m, equal to n, – is its *order*. Generally the matrix is called *rectangular* of type $m \times n$. The numbers, from which the matrix is compiled, are called its *elements*. Concisely the matrix is designated so:

$$(a_{ik}) \quad (i=1, \dots, m;\ k=1, \dots, n),$$

or by one letter (capital or small); for example, matrix A, bearing in mind that $A = (a_{ik})$.

Rectangular matrix, consisting of one column

$$\begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_m \end{bmatrix},$$

is called *column* matrix. Rectangular matrix, consisting of one line $(x_1 \ldots x_n)$, is called *row* matrix. Column and row matrices are sometimes called *vectors*.

The matrix is called *zero*, if all its elements are equal to zero.

Matrices $A = (a_{ij})$ and $B = (b_{ij})$ are equal to each other, if they are of the same type $m \times n$ and, furthermore,

$$a_{ij} = b_{ij} \ (i = 1, \ldots, m; \ j = 1, \ldots, n).$$

1.2. Summation and Multiplication of Rectangular Matrices

1. The operation of *summation of matrices* is applicable only to matrices of the same type. The sum of two rectangular matrices $A = (a_{ij})$ and $B = (b_{ij})$ of type $m \times n$ is called matrix $C = (c_{ij})$ of type $m \times n$, elements of which are equal to the sums of corresponding matrix elements A and B, i.e.,

$$C = A + B,$$

if

$$c_{ij} = a_{ij} + b_{ij} \ (i = 1, \ldots, m; \ j = 1, \ldots, n).$$

The operation of summation of matrices possesses communative and associative properties :

1) $A + B = B + A$,

2) $(A + B) + C = A + (B + C)$.

2. *The product of matrix* $A = (a_{ij})$ *by number* α from K is called matrix $C = (c_{ij})$, elements of which are obtained from corresponding matrix elements A by multiplication by number α, i.e.,

$$C = \alpha A,$$

if

$$c_{ij} = \alpha a_{ij} \ (i = 1, \ldots, m; \ j = 1, \ldots, n).$$

The operation of multiplication of the matrix by a number possesses the following properties :

1) $\alpha(A + B) = \alpha A + \alpha\beta$,

2) $(\alpha + \beta)A = \alpha A + \beta A$,

3) $(\alpha\beta)A = \alpha(\beta A)$.

Here A and B – rectangular matrices of the same type, α, β – numbers from field K.

3. *The difference* of $A - B$ of two rectangular matrices of the same type is determined by equality

$$A - B = A + (-1)B.$$

4. *The product of two matrices* A and B of type $m \times n$ and $m' \times n'$ respectively and such that $n = m'$, is called matrix C, whose element c_{ij}, standing on the intersection of i-th line and j-th column, is equal to "product" of i-th line of the first matrix A by j-th column of the second matrix B, i.e.,

$$C = AB,$$

where

$$A = (a_{ik}) \ (i = 1, \ldots m; \ k = 1, \ldots, n),$$
$$B = (b_{ik}) \ (i = 1, \ldots, m'; \ k = 1, \ldots, n'),$$

moreover $n = m'$, if

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \quad (i = 1, \ldots, m; \quad j = 1, \ldots, n').$$

Matrix C has as many lines as there are lines in matrix A, and as many columns as columns in matrix B, i.e., C is the matrix of type $m \times n'$.

The operation of multiplication of two matrices, introduced above, is inapplicable if the number of columns of the first matrix is not equal to the number of lines of the second matrix. For square matrices of the same order multiplication is always feasible.

Multiplication of matrices possesses associative and distributive properties :

1) $(AB)C = A(BC)$;

2) $(A + B)C = AC + BC$,

3) $A(B + C) = AB + AC$.

Multiplication of matrices does not possess associative property. If $AB = BA$, then matrices A and B are called exchanging or commutating.

1.3. Partitioned Matrices

Rectangular matrix $A = (a_{ik})$ of type $m \times n$ with the aid of horizontal and vertical lines can be cut to rectangular blocks.

$$A = \begin{bmatrix} A_{11} & \ldots & A_{1t} \\ \ldots & \ldots & \ldots \\ A_{s1} & \ldots & A_{st} \end{bmatrix} \quad (s \leqslant m; \quad t \leqslant n).$$

Here A_{ij} - rectangular matrices or numbers. Specifically, the matrix can be cut by only horizontal lines or only vertical lines. In this case we will have respectively

$$A=\begin{bmatrix} A_1 \\ \cdot \\ \cdot \\ \cdot \\ A_s \end{bmatrix} \text{ or } A=(A_1 \ldots A_t).$$

The *partitioned* matrix can concisely be designated so:

$$A=(A_{\alpha\beta}) \quad (\alpha=1, \ldots, s; \ \beta=1, \ldots, t).$$

The operations of *summation* and *multiplication* over partitioned matrices are formally performed by the same rules as when instead of blocks there are numerical elements.

Two rectangular matrices of identical sizes and with identical division into blocks are made up so:

$$A+B=(A_{\alpha\beta}+B_{\alpha\beta}) \quad (\alpha=1, \ldots, s; \ \beta=1, \ldots, t).$$

For the possibility of multiplication of two partitioned matrices by each other it is necessary that with division into blocks all the horizontal dimensions of the blocks in the first factor would coincide with corresponding vertical dimensions in the second. Let us assume

$$A=\begin{bmatrix} A_{11} & \ldots & A_{1t} \\ \cdot & \cdot & \cdot \\ A_{s1} & \ldots & A_{st} \end{bmatrix}, \quad B=\begin{bmatrix} B_{11} & \ldots & B_{1u} \\ \cdot & \cdot & \cdot \\ B_{t1} & \ldots & B_{tu} \end{bmatrix}$$

and the number of columns of block $A_{\alpha\delta}$ is equal to the number of lines of block (α = 1, ..., s; β = 1, ..., u; δ = 1, ..., t). Then

$$AB=C=(C_{\alpha\beta}),$$

$$C_{\alpha\beta}=\sum_{\delta=1} A_{\alpha\delta}B_{\delta\beta} \quad (\alpha=1, \ldots, s; \quad \beta=1, \ldots, u).$$

1.4. Square Matrices

1. Square matrix, for which all elements except those located on the main diagonal, are equal to zero, is called *diagonal* matrix. Diagonal matrix, thus, has the form

$$\begin{bmatrix} a_{11} & 0 & \ldots & 0 \\ 0 & a_{22} & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & a_{nn} \end{bmatrix}.$$

Diagonal matrix of order n, for which all diagonal elements are equal to one, is called *unit* matrix and is designated through E_n, or simply E.

For any rectangular matrix A of type $m \times n$ there exist equalities

$$E_m A = A E_n = A.$$

Square matrix $A = (a_{ik})$ is called *symmetric*, if $a_{ik} = a_{ki}$.

2. Determinant of square matrix $A = (a_{ij})$ is called the determinant, composed of elements of matrix A. This determinant is designated so: $|a_{ij}|$ or $|A|$.

Square matrix A is called *degenerate* (or *particular*), if $|A| = 0$. Otherwise square matrix A is called *nondegenerate* (or *ordinary*).

If A – nondegenerate matrix, then there exists unique nondegenerate matrix B, which satisfies equalities

$$AB = BA = E.$$

Such a matrix is called *inverse* matrix for matrix A and is designated through A^{-1}.

1.5. Eigenvectors and Eigenvalues of Square Matrix

1. Column matrix $x \neq 0$ and number λ are respectively called *eigenvector* and *eigenvalue* of square matrix A, if

$$Ax = \lambda x. \tag{1.1}$$

To matrix equality (1.1) there corresponds system of algebraic equations

$$\left.\begin{array}{l} (a_{11}-\lambda)x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0, \\ a_{21}x_1 + (a_{22}-\lambda)x_2 + \ldots + a_{2n}x_n = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + (a_{nn}-\lambda)x_n = 0. \end{array}\right\} \tag{1.2}$$

In order that system of linear homogeneous equations (1.2) would have nonzero solution, it is necessary and sufficient that the determinant of this system be equal to zero:

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-\lambda & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn}-\lambda \end{vmatrix} = 0. \tag{1.3}$$

Equation (1.3) represents algebraic equation of n-th of degree relative to λ and is called *characteristic equation* of matrix A.

Let us assume in equations (1.2) λ is the root of characteristic equation (1.3). Then system (1.2) has nonzero solution $x_1, \ldots, x_n$,

i.e., number λ corresponds eigenvector $\begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$.

Eigenvector is determined to within arbitrary factor, not equal to zero.

Eigenvectors of matrix A, corresponding in pairs to various eigenvalues, are always linearly independent.

2. Matrix A of order n is called matrix of *simple structure*, if it has exactly n linearly independent eigenvectors.

Matrix A of order n has simple structure, if its characteristic equation has n different roots. This condition is sufficient, but not necessary. There exist matrices of simple structure, characteristic equations of which have multiple roots. It is possible to show that the symmetric matrix has simple structure. Matrix $u = m^{-1}n$ also has simple structure, where m and n – symmetric matrices.

1.6. Expansion of Matrix of Simple Structure into Components and Its Reduction to Diagonal Form

1. Let us assume A – matrix of simple structure of order n, and $K_1, \ldots, K_n$ – eigenvectors, so that

$$AK_j = \lambda_j K_j \ (j = 1, \ldots, n). \qquad (1.4)$$

Square matrix $K = (K_1 \ldots K_n)$, composed of n linearly independent eigenvectors of matrix A, is a nondegenerate matrix and therefore has inverse matrix K^{-1}, which for convenience we will designate through M ($K^{-1} = M$).

Thus, we have

$$MK = KM = E_n. \tag{1.5}$$

Let us designate rows of matrixes M by $M_1, \ldots, M_n$:

$$M = \begin{bmatrix} M_1 \\ \vdots \\ M_n \end{bmatrix}.$$

Between rows of matrix M and columns of matrix K there exist the following relationships:

$$M_s K_\sigma = \begin{cases} 1 & (s = \sigma) \\ 0 & (s \neq \sigma). \end{cases} \tag{1.6}$$

On the basis of (1.4) and (1.5)

$$A = AE_n = AKM = A(K_1 \ldots K_n)M = (\lambda_1 K_1 \ldots \lambda_n K_n)\begin{bmatrix} M_1 \\ \vdots \\ M_n \end{bmatrix} =$$

$$= \lambda_1 K_1 M_1 + \ldots + \lambda_n K_n M_n.$$

Matrices

$$A_j = \lambda_j K_j M_j \ (j = 1, \ldots, n)$$

are called *components* of matrix A. These components are mutually orthogonal, i.e.,

$$A_i \cdot A_j = 0 \ (i \neq j),$$

which follows from property (1.6).

2. Through Λ let us designate diagonal matrix

$$\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

for which diagonal elements are eigenvalues of matrix A. By using (1.4), we obtain

$$AK=K\Lambda, \tag{1.7}$$

and after multiplication of equality (1.7) on the right by K^{-1} we will have

$$A=K\Lambda K^{-1}. \tag{1.8}$$

Two matrices B and C are called *similar,* if there exists nondegenerate matrix T, so that

$$B=TCT^{-1}.$$

Thus, matrix A, having simple structure, is similar to diagonal matrix.

1.7. Differentiation of Matrices

1. Matrix, elements of which are functions of some parameter, is called *functional:*

$$A(t)=(a_{ij}(t)) \ (i=1,\ldots,m;\ j=1,\ldots,n).$$

2. Derivative of matrix $A(t)$ with respect to parameter t is called a matrix, elements of which are derivatives of corresponding matrix elements $A(t)$, i.e.,

$$\frac{dA}{dt}=\left(\frac{da_{ij}}{dt}\right).$$

The derivative of the matrix possesses the following properties :

1) if C - constant of matrix, then

$$\frac{d[CA(t)]}{dt}=C\frac{dA(t)}{dt}, \quad \frac{d[A(t)C]}{dt}=\frac{dA(t)}{dt}C.$$

and generally

$$\frac{d[CA(t)]}{dt}\neq\frac{d[A(t)C]}{dt};$$

2) $$\frac{d[A(t)+B(t)]}{dt}=\frac{dA(t)}{dt}+\frac{dB(t)}{dt};$$

3) $$\frac{d[A(t)B(t)]}{dt}=\frac{dA(t)}{dt}B(t)+A(t)\frac{dB(t)}{dt}.$$

§ 2. Reduction of Equations of Disturbed Motion of the Rocket to a Form Convenient for Modeling, with Quenched Coefficients of Equations

Let us examine equations of disturbed motion of the rocket as the object of control with quenched coefficients

$$m\frac{d^2q}{dt^2}+r\frac{dq}{dt}+nq=a\delta+h(t). \qquad (2.1)$$

Here m, r, n, a - matrices with constant elements.

For simplicity let us omit the terms proportional to generalized velocities, which determine damping in the system and are usually small. In this instance we will have

$$m\frac{d^2q}{dt^2}+nq=a\delta+h. \qquad (2.2)$$

Further, we will assume that matrix $u = m^{-1}n$ has simple structure.

Let us assume $\nu_1, \ldots, \nu_n$ – eigenvalues of matrix u. To these eigenvalues correspond respectively n linearly independent eigenvectors $\kappa_1, \kappa_2, \ldots, \kappa_n$. Therefore square matrix

$$\kappa = (\kappa_1 \ldots \kappa_n)$$

is nondegenerate matrix and, thus, has inverse matrix

$$\kappa^{-1} = \mu = \begin{bmatrix} \mu_1 \\ \cdot \\ \cdot \\ \cdot \\ \mu_n \end{bmatrix}.$$

Rows of matrix μ and columns of matrix κ are connected by relationships

$$\mu_s \kappa_\sigma = \begin{cases} 1 & (s = \sigma) \\ 0 & (s \neq \sigma). \end{cases} \tag{2.3}$$

Let us recall that elements of column matrix q are generalized coordinates $q_1, \ldots, q_n$. Let us introduce into examination new coordinates $z_1, \ldots, z_n$, connected to generalized coordinates by relationships

$$q = \sum_{\sigma=1}^{n} \kappa_\sigma z_\sigma = \kappa z \left(z = \begin{bmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_n \end{bmatrix} \right). \tag{2.4}$$

Let us substitute (2.4) in (2.2), having preliminarily multiplied both sides of equality (2.2) by m^{-1}. We obtain

$$\kappa \frac{d^2 z}{dt^2} + u\kappa z = m^{-1} a\delta + m^{-1} h. \tag{2.5}$$

Here

$$\mu x = \mu(x_1 \ldots x_n) = (\mu x_1 \ldots \mu x_n) =$$

$$= (v_1 x_1 \ldots v_n x_n) = (x_1 \ldots x_n) \begin{bmatrix} v_1 & & 0 \\ & \ddots & \\ 0 & & v_n \end{bmatrix},$$

or

$$\mu x = x v, \tag{2.6}$$

where

$$v = \begin{bmatrix} v_1 & & 0 \\ & \ddots & \\ 0 & & v_n \end{bmatrix}.$$

Considering (2.6), from (2.5), after multiplication of both sides of this equality from the left by μ, we obtain

$$\frac{d^2 z}{dt^2} + vz = \mu m^{-1}(a\delta + h). \tag{2.7}$$

The last relationship in expanded form is written so:

$$\begin{bmatrix} \frac{d^2 z_1}{dt^2} \\ \vdots \\ \frac{d^2 z_n}{dt^2} \end{bmatrix} + \begin{bmatrix} v_1 & & 0 \\ & \ddots & \\ 0 & & v_n \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix} m^{-1}(a\delta + h),$$

or

$$\begin{bmatrix} \frac{d^2 z_1}{dt^2} + v_1 z_1 \\ \cdots\cdots \\ \frac{d^2 z_n}{dt^2} + v_n z_n \end{bmatrix} = \begin{bmatrix} \mu_1 m^{-1}(a\delta + h) \\ \cdots\cdots \\ \mu_n m^{-1}(a\delta + h) \end{bmatrix}.$$

Hence it is clear that matrix equation (2.7) is equivalent to n equations

$$\frac{d^2z_\sigma}{dt^2}+\nu_\sigma z_\sigma=\mu_\sigma m^{-1}(a\delta+h) \quad (\sigma=1,\ldots,n). \tag{2.8}$$

The left sides of these equations contain only one coordinate each, and the connection between equations remains only through their right sides, containing common function $\delta(t)$. In such form it is considerably simpler to realize electromodeling of equations.

When $h \equiv \delta \equiv 0$ system of equations (A) describes free motion of the rocket. In this instance the converted system (2.8) is expanded into n independent equations

$$\frac{d^2z_\sigma}{dt^2}+\nu_\sigma z_\sigma=0 \quad (\sigma=1,\ldots,n), \tag{2.9}$$

each of which determines the change of corresponding coordinate z_σ. In the case when m and n -- symmetric matrices, eigenvalues ν_σ of matrix $u = m^{-1}n$, it is possible to show, – real numbers. In this case, if $\nu_\sigma > 0$, then coordinate z_σ accomplishes harmonic oscillations with frequency $\sqrt{\nu_\sigma}$; when $\nu_\sigma = 0$ coordinate z_σ is changed according to linear law. Finally, when $\nu_\sigma < 0$, z_σ is presented in the form of linear combination of function $\exp\sqrt{\nu_\sigma}t$ and $\exp(-\sqrt{\nu_\sigma}t)$. Each of the generalized coordinates q_σ is linear combination of coordinates z_σ. Coordinates z_σ are called normal coordinates of the system, and $\sqrt{\nu_\sigma}$ (when $\nu_\sigma > 0$) – eigen or normal frequency. In connection with this, frequently the operation of reduction of equations (2.2) to form (2.8) is called reduction of equations of disturbed motion to normal coordinates.

The control signal of regulator also must be expressed through coordinates z_σ ($\sigma = 1, \ldots, n$). For this purpose let us substitute expression (2.4) in equality (B). Then

$$v(t)=b_1\sum_{\sigma=1}^{n}x_\sigma\frac{dz_\sigma}{dt}+b_2\sum_{\sigma=1}^{n}x_\sigma z_\sigma, \tag{2.10}$$

or in another form

$$v(t)=b_{1}^{*}\frac{dz}{dt}+b_{2}^{*}z. \qquad (2.11)$$

§ 3. Reduction of Equations of Free Oscillations to Canonical Form

System of equations

$$m(t)\frac{d^2q}{dt^2}+r(t)\frac{dq}{dt}+n(t)q=0, \qquad (3.1)$$

describing free oscillations of the rocket, is converted so as to obtain a system of independent linear differential second order equations. Our target – to change from generalized coordinates $q_1, \ldots, q_n$ to new coordinates $z_1, \ldots, z_n$, relative to which the equations of free oscillations have the form

$$\frac{d^2z_\sigma}{dt^2}+\tilde{a}_{1\sigma}\frac{dz_\sigma}{dt}+\tilde{a}_{2\sigma}z_\sigma=0 \quad (\sigma=1, \ldots, n). \qquad (3.2)$$

The connection between old and new coordinates will be sought in the form

$$q=\sum_{\sigma=1}^{n}\left(\tilde{\varkappa}_{1\sigma}\frac{dz_\sigma}{dt}+\tilde{\varkappa}_{2\sigma}z_\sigma\right), \qquad (3.3)$$

where $\tilde{\varkappa}_{1\sigma}$, $\tilde{\varkappa}_{2\sigma}$ – some column matrices.

Having introduced so-called "slow time" $\tau = \varepsilon t$, let us examine system

$$m(\tau)\frac{d^2q}{dt^2}+\varepsilon r(\tau)\frac{dq}{dt}+n(\tau)q=0 \quad (\tau=\varepsilon t), \qquad (3.4)$$

which when $\varepsilon = 1$ will change to system (3.1).

Let us first reduce system (3.4) to form (3.2), considering $\tilde{x}_{i\sigma}$, $\tilde{\alpha}_{i\sigma}$ ($i = 1, 2$) as functions of slow time τ and parameter ε.

By differentiating (3.3) with respect to t and excluding $\frac{d^2 z_\sigma}{dt^2}$ each time with the aid of equalities (3.2), we will have

$$\frac{dq}{dt}=\sum_{e=1}^{n}\left[\left(\varepsilon\frac{d\tilde{x}_{1e}}{d\tau}-\tilde{x}_{1e}\tilde{\alpha}_{1e}+\tilde{x}_{2e}\right)\frac{dz_e}{dt}+\left(\varepsilon\frac{d\tilde{x}_{2e}}{d\tau}-\tilde{x}_{1e}\tilde{\alpha}_{2e}\right)z_e\right],$$

$$\frac{d^2q}{dt^2}=\sum_{e=1}^{n}\left\{\left[\varepsilon^2\frac{d^2\tilde{x}_{1e}}{d\tau^2}+\varepsilon\left(2\frac{d\tilde{x}_{2e}}{d\tau}-2\frac{d\tilde{x}_{1e}}{d\tau}\tilde{\alpha}_{1e}-\tilde{x}_{1e}\frac{d\tilde{\alpha}_{1e}}{d\tau}\right)+\right.\right.$$
$$\left.+\tilde{x}_{1e}\tilde{\alpha}_{1e}^2-\tilde{x}_{2e}\tilde{\alpha}_{1e}-\tilde{x}_{1e}\tilde{\alpha}_{2e}\right]\frac{dz_e}{dt}+\left[\varepsilon^2\frac{d^2\tilde{x}_{2e}}{d\tau^2}+\right.$$
$$\left.\left.+\varepsilon\left(-2\frac{d\tilde{x}_{1e}}{d\tau}\tilde{\alpha}_{2e}-\tilde{x}_{1e}\frac{d\tilde{\alpha}_{2e}}{d\tau}\right)+\tilde{x}_{1e}\tilde{\alpha}_{1e}\tilde{\alpha}_{2e}-\tilde{x}_{2e}\tilde{\alpha}_{2e}\right]z_e\right\}.$$

Let us substitute values q, $\frac{dq}{dt}$ and $\frac{d^2q}{dt^2}$ in (3.4) :

$$m\sum_{e=1}^{n}\left\{\left[\varepsilon^2\frac{d^2\tilde{x}_{1e}}{d\tau^2}+\varepsilon\left(2\frac{d\tilde{x}_{2e}}{d\tau}-2\frac{d\tilde{x}_{1e}}{d\tau}\tilde{\alpha}_{1e}-\tilde{x}_{1e}\frac{d\tilde{\alpha}_{1e}}{d\tau}\right)+\right.\right.$$
$$\left.+\tilde{x}_{1e}\tilde{\alpha}_{1e}^2-\tilde{x}_{2e}\tilde{\alpha}_{1e}-\tilde{x}_{1e}\tilde{\alpha}_{2e}\right]\frac{dz_e}{dt}+$$
$$+\left[\varepsilon^2\frac{d^2\tilde{x}_{2e}}{d\tau^2}+\varepsilon\left(-2\frac{d\tilde{x}_{1e}}{d\tau}\tilde{\alpha}_{2e}-\tilde{x}_{1e}\frac{d\tilde{\alpha}_{2e}}{d\tau}\right)+\right.$$
$$\left.\left.+\tilde{x}_{1e}\tilde{\alpha}_{1e}\tilde{\alpha}_{2e}-\tilde{x}_{2e}\tilde{\alpha}_{2e}\right]z_e\right\}+\varepsilon r\sum_{e=1}^{n}\left[\left(\varepsilon\frac{d\tilde{x}_{1e}}{d\tau}-\tilde{x}_{1e}\tilde{\alpha}_{1e}+\tilde{x}_{2e}\right)\frac{dz_e}{dt}+\right.$$
$$\left.+\left(\varepsilon\frac{d\tilde{x}_{2e}}{d\tau}-\tilde{x}_{1e}\tilde{\alpha}_{2e}\right)z_e\right]+n\sum_{e=1}^{n}\left(\tilde{x}_{1e}\frac{dz_e}{dt}+\tilde{x}_{2e}z_e\right)=0. \qquad (3.5)$$

In the obtained equality let us equate coefficients at z_σ and $\frac{dz_\sigma}{dt}$.

We will have

$$m\left[\tilde{\varkappa}_{1\sigma}\tilde{\alpha}_{1\sigma}^2-\tilde{\varkappa}_{2\sigma}\tilde{\alpha}_{1\sigma}-\tilde{\varkappa}_{1\sigma}\tilde{\alpha}_{2\sigma}+\varepsilon\left(2\frac{d\tilde{\varkappa}_{2\sigma}}{d\tau}-2\frac{d\tilde{\varkappa}_{1\sigma}}{d\tau}\tilde{\alpha}_{1\sigma}-\tilde{\varkappa}_{1\sigma}\frac{d\tilde{\alpha}_{1\sigma}}{d\tau}\right)+\right.$$
$$\left.+\varepsilon^2\frac{d^2\tilde{\varkappa}_{1\sigma}}{d\tau^2}\right]+\varepsilon r\left(\tilde{\varkappa}_{2\sigma}-\tilde{\varkappa}_{1\sigma}\tilde{\alpha}_{1\sigma}+\varepsilon\frac{d\tilde{\varkappa}_{1\sigma}}{d\tau}\right)+n\tilde{\varkappa}_{1\sigma}=0\quad(\sigma=1,\ldots,n), \tag{3.6}$$

$$m\left[\tilde{\varkappa}_{1\sigma}\tilde{\alpha}_{1\sigma}\tilde{\alpha}_{2\sigma}-\tilde{\varkappa}_{2\sigma}\tilde{\alpha}_{2\sigma}+\varepsilon\left(-2\frac{d\tilde{\varkappa}_{1\sigma}}{d\tau}\tilde{\alpha}_{2\sigma}-\tilde{\varkappa}_{1\sigma}\frac{d\tilde{\alpha}_{2\sigma}}{dt}\right)+\varepsilon^2\frac{d^2\tilde{\varkappa}_{2\sigma}}{d\tau^2}\right]+$$
$$+\varepsilon r\left(-\tilde{\varkappa}_{1\sigma}\tilde{\alpha}_{2\sigma}+\varepsilon\frac{d\tilde{\varkappa}_{2\sigma}}{d\tau}\right)+n\tilde{\varkappa}_{2\sigma}=0\quad(\sigma=1,\ldots,n). \tag{3.7}$$

Column matrices $\tilde{\kappa}_{i\sigma}$ and scalar functions $\tilde{\alpha}_{i\sigma}$, which should become identities, let us construct equalities (3.6) and (3.7) in the form of series :

$$\tilde{\varkappa}_{i\sigma}(\tau,\varepsilon)=\sum_{k=0}^{\infty}\varepsilon^k\varkappa_{i\sigma}^{[k]}(\tau),\quad \tilde{\alpha}_{i\sigma}(\tau,\varepsilon)=\sum_{k=0}^{\infty}\varepsilon^k\alpha_{i\sigma}^{[k]}(\tau). \tag{3.8}$$

Let us substitute (3.8) in equalities (3.6), (3.7) and let us equate coefficients at ε°. We obtain

$$\left.\begin{aligned}u\varkappa_{1\sigma}^{[0]}&=\varkappa_{1\sigma}^{[0]}\alpha_{2\sigma}^{[0]}+\varkappa_{2\sigma}^{[0]}\alpha_{1\sigma}^{[0]}-\varkappa_{1\sigma}^{[0]}\alpha_{1\sigma}^{[0]2},\\ u\varkappa_{2\sigma}^{[0]}&=\varkappa_{2\sigma}^{[0]}\alpha_{2\sigma}^{[0]}-\varkappa_{1\sigma}^{[0]}\alpha_{1\sigma}^{[0]}\alpha_{2\sigma}^{[0]}.\end{aligned}\right\} \tag{3.9}$$

Let us assume

$$\alpha_{1\sigma}^{[0]}=0,\quad \varkappa_{1\sigma}^{[0]}=0. \tag{3.10}$$

Then the left and right sides of the first equality (3.9) become zero, and the second equality assumes the form

$$u\varkappa_{2\sigma}^{[0]}=\varkappa_{2\sigma}^{[0]}\alpha_{2\sigma}^{[0]}.$$

The last relationship will be fulfilled if $\alpha_{2\sigma}^{[0]}$ is same eigenvalue of matrix u, for example ν_σ, and $\kappa_{2\sigma}^{[0]}$ – corresponding eigenvector of this matrix. Thus,

$$\alpha_{2\sigma}^{[0]}=\nu_\sigma, \quad \kappa_{2\sigma}^{[0]}=\kappa_\sigma. \tag{3.11}$$

Further, in equalities (3.6) and (3.7) let us equate coefficients at ε^k ($k = 1, 2, \ldots$). Taking into account (3.10) and (3.11) we obtain

$$u\kappa_{1\sigma}^{[1]}=\kappa_{1\sigma}^{[1]}\nu_\sigma+\kappa_\sigma\alpha_{1\sigma}^{[1]}-m_0^{-1}r\kappa_\sigma-2\frac{d\kappa_\sigma}{d\tau},$$

$$u\kappa_{2\sigma}^{[1]}=\kappa_{2\sigma}^{[1]}\nu_\sigma+\kappa_\sigma\alpha_{2\sigma}^{[1]},$$

$$u\kappa_{1\sigma}^{[2]}=\kappa_{1\sigma}^{[2]}\nu_\sigma+\kappa_\sigma\alpha_{1\sigma}^{[2]}-m_0^{-1}r\kappa_{2\sigma}^{[1]}+\kappa_{2\sigma}^{[1]}\alpha_{1\sigma}^{[1]}+\kappa_{1\sigma}^{[1]}\alpha_{2\sigma}^{[1]}-$$
$$-2\frac{d\kappa_{2\sigma}^{[1]}}{d\tau},$$

$$u\kappa_{2\sigma}^{[2]}=\kappa_{2\sigma}^{[2]}\nu_\sigma+\kappa_\sigma\alpha_{2\sigma}^{[2]}+m_0^{-1}r\left(\kappa_{1\sigma}^{[1]}\nu_\sigma-\frac{d\kappa_\sigma}{d\tau}\right)-\kappa_{1\sigma}^{[1]}\alpha_{1\sigma}^{[1]}\nu_\sigma+$$
$$+\kappa_{2\sigma}^{[1]}\alpha_{2\sigma}^{[1]}+2\frac{d\kappa_{1\sigma}^{[1]}}{d\tau}\nu_\sigma+\kappa_{1\sigma}^{[1]}\frac{d\nu_\sigma}{d\tau}-\frac{d^2\kappa_\sigma}{d\tau^2},$$

and generally

$$u\kappa_{1\sigma}^{[k]}=\kappa_{1\sigma}^{[k]}\nu_\sigma+\kappa_\sigma\alpha_{1\sigma}^{[k]}+d_{1\sigma}^{[k-1]} \quad (k=1, 2, \ldots), \tag{3.12}$$

$$u\kappa_{2\sigma}^{[k]}=\kappa_{2\sigma}^{[k]}\nu_\sigma+\kappa_\sigma\alpha_{2\sigma}^{[k]}+d_{2\sigma}^{[k-1]} \quad (k=1, 2, \ldots). \tag{3.13}$$

Here

$$d_{1\sigma}^{[0]}=-m_0^{-1}r\kappa_\sigma-2\frac{d\kappa_\sigma}{d\tau},$$
$$d_{2\sigma}^{[0]}=0,$$
$$d_{1\sigma}^{[1]}=-m_0^{-1}r\kappa_{2\sigma}^{[1]}+\kappa_{2\sigma}^{[1]}\alpha_{1\sigma}^{[1]}+\kappa_{1\sigma}^{[1]}\alpha_{2\sigma}^{[1]}-2\frac{d\kappa_{2\sigma}^{[1]}}{d\tau},$$
$$d_{2\sigma}^{[1]}=m_0^{-1}r\left(\kappa_{1\sigma}^{[1]}\nu_\sigma-\frac{d\kappa_\sigma}{d\tau}\right)-\kappa_{1\sigma}^{[1]}\alpha_{1\sigma}^{[1]}\nu_\sigma+\kappa_{2\sigma}^{[1]}\alpha_{2\sigma}^{[1]}+$$
$$+2\frac{d\kappa_{1\sigma}^{[1]}}{d\tau}\nu_\sigma+\kappa_{1\sigma}^{[1]}\frac{d\nu_\sigma}{d\tau}-\frac{d^2\kappa_\sigma}{d\tau^2}, \tag{3.14}$$

etc.

Let us assume that $x_{i\circ}^{[r]}, a_{i\circ}^{[r]}$ $(i=1, 2;\ r=0, 1, 2, \ldots, k-1)$ are already found. Then equalities

$$u x_{i\circ}^{[k]} = x_{i\circ}^{[k]} v_\circ + x_\circ a_{i\circ}^{[k]} + d_{i\circ}^{[k-1]} \quad (i=1, 2), \tag{3.15}$$

where $d_{i\circ}^{[k-1]}$ already known column matrices, completely determine the values of the following approximation. Let us show this.

Let us multiply equality (3.15) from the left by matrix μ. Taking into account (2.6), we obtain

$$\nu\mu x_{i\circ}^{[k]} = \mu x_{i\circ}^{[k]} v_\circ + \mu x_\circ a_{i\circ}^{[k]} + \mu d_{i\circ}^{[k-1]} \quad (i=1, 2). \tag{3.16}$$

Let us introduce into examination column matrix

$$q_{i\circ}^{[k]} = \begin{bmatrix} q_{1i\circ}^{[k]} \\ \vdots \\ q_{ni\circ}^{[k]} \end{bmatrix},$$

determined by equality

$$q_{i\circ}^{[k]} = \mu x_{i\circ}^{[k]}. \tag{3.17}$$

Then

$$\nu\mu x_{i\circ}^{[k]} = \begin{bmatrix} \nu_1 & & & 0 \\ & \nu_2 & & \\ & & \ddots & \\ 0 & & & \nu_n \end{bmatrix} \begin{bmatrix} q_{1i\circ}^{[k]} \\ q_{2i\circ}^{[k]} \\ \vdots \\ q_{ni\circ}^{[k]} \end{bmatrix} = \begin{bmatrix} \nu_1 q_{1i\circ}^{[k]} \\ \nu_2 q_{2i\circ}^{[k]} \\ \vdots \\ \nu_n q_{ni\circ}^{[k]} \end{bmatrix},$$

and

$$\mu x|_{\sigma}^{[k]}\nu_{\sigma}+\mu x_{\sigma}\alpha|_{\sigma}^{[k]}+\mu d|_{\sigma}^{[k-1]}=\begin{bmatrix}\nu_{\sigma}q|_{1\sigma}^{[k]}\\ \nu_{\sigma}q|_{2\sigma}^{[k]}\\ \vdots\\ \nu_{\sigma}q|_{n\sigma}^{[k]}\end{bmatrix}+\begin{bmatrix}\mu_1 x_{\sigma}\\ \mu_2 x_{\sigma}\\ \vdots\\ \mu_n x_{\sigma}\end{bmatrix}\alpha|_{\sigma}^{[k]}+$$

$$+\begin{bmatrix}\mu_1\\ \mu_2\\ \vdots\\ \mu_n\end{bmatrix}d|_{\sigma}^{[k-1]}=\begin{bmatrix}\nu_{\sigma}q|_{1\sigma}^{[k]}+\mu_1 x_{\sigma}\alpha|_{\sigma}^{[k]}+\mu_1 d|_{\sigma}^{[k-1]}\\ \nu_{\sigma}q|_{2\sigma}^{[k]}+\mu_2 x_{\sigma}\alpha|_{\sigma}^{[k]}+\mu_2 d|_{\sigma}^{[k-1]}\\ \cdots\cdots\cdots\cdots\\ \nu_{\sigma}q|_{n\sigma}^{[k]}+\mu_n x_{\sigma}\alpha|_{\sigma}^{[k]}+\mu_n d|_{\sigma}^{[k-1]}\end{bmatrix}.$$

From the last relationship it is evident that matrix equality (3.16) is equivalent to n algebraic (not matrix) equalities

$$\nu_s q|_{s\sigma}^{[k]}=\nu_{\sigma}q|_{s\sigma}^{[k]}+\mu_s x_{\sigma}\alpha|_{\sigma}^{[k]}+\mu_s d|_{\sigma}^{[k-1]}\quad(s=1,2,\ldots,n). \tag{3.18}$$

In view of (2.3) when $s = \sigma$ from (3.18) we have

$$(\nu_{\sigma}-\nu_{\sigma})q|_{\sigma\sigma}^{[k]}=\alpha|_{\sigma}^{[k]}+\mu_{\sigma}d|_{\sigma}^{[k-1]}=0. \tag{3.19}$$

Hence we find

$$\alpha|_{\sigma}^{[k]}=-\mu_{\sigma}d|_{\sigma}^{[k-1]}. \tag{3.20}$$

When $s\neq\sigma$ $\mu_s x_{\sigma}=0$, a $\nu_s\neq\nu_{\sigma}$. Therefore, s equality (3.18) is solvable relative to $q|_{s\sigma}^{[k]}$:

$$q|_{s\sigma}^{[k]}=\frac{\mu_s d|_{\sigma}^{[k-1]}}{\lambda_s-\lambda_{\sigma}}\quad(s\neq\sigma). \tag{3.21}$$

Only scalar function $q|_{\sigma\sigma}^{[k]}$ remained indeterminate. As can be seen from (3.19), as $q|_{\sigma\sigma}^{[1]}$ it is possible to take arbitrary, sufficient number of times differentiable function.

Now it is easy to determine $x_{\sigma}^{[k]}$. For this both sides of equality (3.17) must be multiplied from the left by $\varkappa$. Then,

$$x_{\sigma}^{[k]}=\varkappa q_{\sigma}^{[k]}=(\varkappa_1\ldots\varkappa_n)\begin{bmatrix} q_{1\sigma}^{[k]}\\ \cdot\\ \cdot\\ \cdot\\ q_{n\sigma}^{[k]}\end{bmatrix}=\sum_{s=1}^{n}\varkappa_s q_{s\sigma}^{[k]},$$

or, considering (3.21),

$$x_{\sigma}^{[k]}=P_{\sigma}d_{\sigma}^{[k-1]}+\varkappa_{\sigma}q_{\sigma\sigma}^{[k]}, \tag{3.22}$$

where

$$P_{\sigma}=\sum_{s\neq\sigma}\frac{\varkappa_s\mu_s}{\nu_s-\nu_{\sigma}}; \tag{3.23}$$

$q_{\sigma\sigma}^{[k]}$ – arbitrary, sufficient number of times differentiable function of τ.

When $k = 1$

$$x_{\sigma}^{[1]}=-P_{\sigma}\left(m_0^{-1}r\varkappa_{\sigma}+2\frac{d\varkappa_{\sigma}}{d\tau}\right)+\varkappa_{\sigma}q_{\sigma\sigma}^{[1]},$$
$$\alpha_{\sigma}^{[1]}=\mu_{\sigma}\left(m_0^{-1}r\varkappa_{\sigma}+2\frac{d\varkappa_{\sigma}}{d\tau}\right). \tag{3.24}$$

Since $d_{2\sigma}^{[0]} = 0$, then

$$\alpha_{2\sigma}^{[1]}=0. \tag{3.25}$$

Assuming $q_{2\sigma\sigma}^{[1]}=0$, we will also have

$$x_{2\sigma}^{[1]}=0. \tag{3.26}$$

By considering the last relationships, we obtain [see (3.14)]

$$
\begin{aligned}
&d_{1\sigma}^{[1]}=0,\\
&d_{2\sigma}^{[1]}=m_0^{-1}r\left(x_{1\sigma}^{[1]}\nu_\sigma-\frac{dz_\sigma}{d\tau}\right)-x_{1\sigma}^{[1]}\alpha_{1\sigma}^{[1]}\nu_\sigma+2\frac{dx_{1\sigma}^{[1]}}{d\tau}\nu_\sigma+\\
&\qquad+x_{1\sigma}^{[1]}\frac{d\nu_\sigma}{d\tau}-\frac{d^2z_\sigma}{d\tau^2}.
\end{aligned}
\tag{3.27}
$$

In accordance with this, quantities of the next approximation are determined so:

$$x_{1\sigma}^{[2]}=0,\quad \alpha_{1\sigma}^{[2]}=0, \tag{3.28}$$

$$x_{2\sigma}^{[2]}=P_\sigma d_{2\sigma}^{[1]}+z_\sigma q_{2\sigma\sigma}^{[2]},\quad \alpha_{2\sigma}^{[2]}=-\mu_\sigma d_{2\sigma}^{[1]}. \tag{3.29}$$

Here arbitrary function $q_{\sigma\sigma}^{[2]}$ is taken equal to zero.

By this method there can successively be determined all terms of series (3.8). By retaining in these series a finite number of first terms, we obtain approximately converted system of equations. Thus, if we are confined only to the first terms of series, then we will have

$$\frac{d^2z_\sigma}{dt^2}+\nu_\sigma z_\sigma=0\quad (\sigma=1,\ldots,n). \tag{3.30}$$

By retaining the two first terms, we obtain

$$\frac{d^2z_\sigma}{dt^2}+\varepsilon\alpha_{1\sigma}^{[1]}\frac{dz_\sigma}{dt}+\nu_\sigma z_\sigma=0\quad (\sigma=1,\ldots,n). \tag{3.31}$$

The next approximation gives

$$\frac{d^2z_\sigma}{dt^2}+\varepsilon\alpha_{1\sigma}^{[1]}\frac{dz_\sigma}{dt}+(\nu_\sigma+\varepsilon^2\alpha_{2\sigma}^{[2]})z_\sigma=0\quad (\sigma=1,\ldots,n) \tag{3.32}$$

and so forth.

For application of these results to original system (3.1') it is necessary in all relationships to assume $\varepsilon = 1$.

§ 4. Reduction of Equations of Disturbed Motion of the Rocket to a Form Convenient for Simulation

For reduction of equations (A') to the form convenient for simulation, let us introduce into examination system

$$m(\tau)\frac{d^2q}{dt^2}+\varepsilon r(\tau)\frac{dq}{dt}+n(\tau)q=\varepsilon a(\tau)\delta+h(t) \quad (\tau=\varepsilon t), \tag{4.1}$$

coinciding with (A') when $\varepsilon = 1$.

For the purpose of convenience of computations let us write system (4.1) in the form

$$m(\tau)\frac{d^2q}{dt^2}+\varepsilon r(\tau)\frac{dq}{dt}+n(\tau)q=h(t)+\varepsilon h_1(t, \tau) \quad (\tau=\varepsilon t), \tag{4.2}$$

where

$$h_1(t, \tau)=a(\tau)\delta(t). \tag{4.3}$$

In system (4.2) let us perform substitution

$$q=\sum_{\sigma=1}^{n}\left(\tilde{\varkappa}_\sigma(\tau, \varepsilon)\frac{dz_\sigma}{dt}+\tilde{\varkappa}_{2\sigma}(\tau, \varepsilon)z_\sigma\right), \tag{4.4}$$

assuming that scalar functions z_σ satisfy equations

$$\frac{d^2z_\sigma}{dt^2}+\tilde{\alpha}_{1\sigma}(\tau, \varepsilon)\frac{dz_\sigma}{dt}+\tilde{\alpha}_{2\sigma}(\tau, \varepsilon)z_\sigma=\psi_{0\sigma}(t, \tau, \varepsilon)+\varepsilon\psi_{1\sigma}(t, \tau, \varepsilon)$$
$$(\sigma=1, \ldots,). \tag{4.5}$$

We have

$$\frac{dq}{dt}=\sum_{\sigma=1}^{n}\left[\left(\varepsilon\frac{d\tilde{x}_{1\sigma}}{d\tau}-\tilde{x}_{1\sigma}\tilde{a}_{1\sigma}+\tilde{x}_{2\sigma}\right)\frac{dz_\sigma}{dt}+\left(-\tilde{x}_{1\sigma}\tilde{a}_{2\sigma}+\varepsilon\frac{d\tilde{x}_{2\sigma}}{d\tau}\right)z_\sigma\right]+$$

$$+\sum_{\sigma=1}^{n}\tilde{x}_{1\sigma}(\psi_{0\sigma}+\varepsilon\psi_{1\sigma}),$$

$$\frac{d^2q}{dt^2}=\sum_{\sigma=1}^{n}\left\{\left[\varepsilon^2\frac{d^2\tilde{x}_{1\sigma}}{d\tau^2}+\varepsilon\left(2\frac{d\tilde{x}_{2\sigma}}{d\tau}-2\frac{d\tilde{x}_{1\sigma}}{d\tau}\tilde{a}_{1\sigma}-\tilde{x}_{1\sigma}\frac{d\tilde{a}_{1\sigma}}{d\tau}\right)+\right.\right.$$

$$\left.+\tilde{x}_{1\sigma}\tilde{a}_{1\sigma}^2-\tilde{x}_{2\sigma}\tilde{a}_{1\sigma}-\tilde{x}_{1\sigma}\tilde{a}_{2\sigma}\right]\frac{dz_\sigma}{dt}+$$

$$+\left[\varepsilon^2\frac{d^2\tilde{x}_{2\sigma}}{d\tau^2}+\varepsilon\left(-2\frac{d\tilde{x}_{1\sigma}}{d\tau}\tilde{a}_{2\sigma}-\tilde{x}_{1\sigma}\frac{d\tilde{a}_{2\sigma}}{d\tau}\right)+\right.$$

$$\left.\left.+\tilde{x}_{1\sigma}\tilde{a}_{1\sigma}\tilde{a}_{2\sigma}-\tilde{x}_{2\sigma}\tilde{a}_{2\sigma}\right]z_\sigma\right\}+$$

$$+\sum_{\sigma=1}^{n}\left[\left(2\varepsilon\frac{d\tilde{x}_{1\sigma}}{d\tau}-\tilde{x}_{1\sigma}\tilde{a}_{1\sigma}+\tilde{x}_{2\sigma}\right)(\psi_{0\sigma}+\varepsilon\psi_{1\sigma})+\right.$$

$$\left.+\tilde{x}_{1\sigma}\left(\frac{\partial\psi_{0\sigma}}{\partial t}+\varepsilon\frac{\partial\psi_{0\sigma}}{\partial\tau}+\varepsilon\frac{\partial\psi_{1\sigma}}{\partial t}+\varepsilon^2\frac{\partial\psi_{1\sigma}}{\partial\tau}\right)\right].$$

Having substituted the obtained expressions in (4.2) and having equated coefficients at z_σ, dz_σ/dt and free terms, we obtain again equalities (3.6), (3.7), which, as we saw, become identities, if terms of series (3.8) are determined by formulas (3.20) and (3.22), and equalities

$$m\sum_{\sigma=1}^{n}\left[\left(2\varepsilon\frac{d\tilde{x}_{1\sigma}}{d\tau}-\tilde{x}_{1\sigma}\tilde{a}_{1\sigma}+\tilde{x}_{2\sigma}\right)\psi_{0\sigma}+\tilde{x}_{1\sigma}\left(\frac{\partial\psi_{0\sigma}}{\partial t}+\varepsilon\frac{\partial\psi_{0\sigma}}{\partial\tau}\right)\right]+$$

$$+\varepsilon r\sum_{\sigma=1}^{n}\tilde{x}_{1\sigma}\psi_{0\sigma}=h. \qquad (4.6)$$

$$m\sum_{\sigma=1}^{n}\left[\left(2\varepsilon\frac{d\tilde{x}_{1\sigma}}{d\tau}-\tilde{x}_{1\sigma}\tilde{a}_{1\sigma}+\tilde{x}_{2\sigma}\right)\psi_{1\sigma}+\tilde{x}_{1\sigma}\left(\frac{\partial\psi_{1\sigma}}{\partial t}+\varepsilon\frac{\partial\psi_{1\sigma}}{\partial\tau}\right)\right]+$$

$$+\varepsilon r\sum_{\sigma=1}^{n}\tilde{x}_{1\sigma}\psi_{1\sigma}=h_1. \qquad (4.7)$$

Thus, the problem is reduced to construction of functions $\psi_{0\sigma}$ and $\psi_{1\sigma}$ ($\sigma = 1, \ldots, n$), which satisfy relationships (4.6) and (4.7).

We will construct these functions in the form of series

$$\psi_{0\sigma}(t, \tau, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \psi_{0\sigma}^{[k]}(t, \tau), \quad \psi_{1\sigma}(t, \tau, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \psi_{1\sigma}^{[k]}(t, \tau). \tag{4.8}$$

Relationships (4.6) and (4.7) coincide with each other to within meanings, therefore, it is sufficient to show, for example, how from (4.6) to determine $\psi_{0\sigma}$ depending on h.

Let us introduce square matrices

$$\tilde{x}_l = \left(\tilde{x}_{l1} \ldots \tilde{x}_{ln}\right) (l = 1, 2), \quad \tilde{\alpha}_1 = \begin{bmatrix} \tilde{\alpha}_{11} & & 0 \\ & \ddots & \\ 0 & & \tilde{\alpha}_{1n} \end{bmatrix}$$

and column matrices

$$\psi_j = \begin{bmatrix} \psi_{j1} \\ \vdots \\ \psi_{jn} \end{bmatrix} \quad (j = 0, 1).$$

By means of these matrices equality (4.6) can be written so:

$$m\left[\left(2\varepsilon \frac{d\tilde{x}_1}{d\tau} - \tilde{x}_1\tilde{\alpha}_1 + \tilde{x}_2\right)\psi_0 + \tilde{x}_1\left(\frac{\partial \psi_0}{\partial t} + \varepsilon \frac{\partial \psi_0}{\partial \tau}\right)\right] + \\ + \varepsilon r \tilde{x}_1 \psi_0 = h. \tag{4.9}$$

In accordance with (3.8) and (4.8) we have

$$\tilde{x}_i(\tau, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k x_i^{[k]}(\tau),$$

$$\tilde{\alpha}_1(\tau, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \alpha_1^{[k]}(\tau), \quad (4.10)$$

$$\psi_j(t, \tau, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \psi_j^{[k]}(t, \tau) \quad (j=0, 1),$$

where

$$x_i^{[k]} = (x_{i1}^{[k]} \ldots x_{in}^{[k]}) \quad (i=1, 2),$$

$$\alpha_1^{[k]} = \begin{bmatrix} \alpha_{11}^{[k]} & & 0 \\ & \ddots & \\ 0 & & \alpha_{1n}^{[k]} \end{bmatrix}, \quad \psi_j^{[k]} = \begin{bmatrix} \psi_{j1}^{[k]} \\ \vdots \\ \psi_{jn}^{[k]} \end{bmatrix} \quad (j=0, 1).$$

Let us substitute series (4.10) in matrix equation (4.9). Taking into account that $x_1^{[0]} = 0, \alpha_1^{[0]} = 0$, we obtain

$$m\left[\left(2\varepsilon \sum_{k=1}^{\infty} \varepsilon^k \frac{dx_1^{[k]}}{d\tau} - \sum_{k=1}^{\infty} \varepsilon^k x_1^{[k]} \sum_{k=1}^{\infty} \varepsilon^k \alpha_1^{[k]} + \sum_{k=0}^{\infty} \varepsilon^k x_2^{[k]}\right) \sum_{k=0}^{\infty} \varepsilon^k \psi_0^{[k]} + \right.$$
$$\left. + \sum_{k=1}^{\infty} \varepsilon^k x_1^{[k]} \left(\sum_{k=0}^{\infty} \varepsilon^k \frac{\partial \psi_0^{[k]}}{\partial t} + \varepsilon \sum_{k=0}^{\infty} \varepsilon^k \frac{\partial \psi_0^{[k]}}{\partial \tau}\right)\right] +$$
$$+ \varepsilon r \sum_{k=1}^{\infty} \varepsilon^k x_1^{[k]} \sum_{k=0}^{\infty} \varepsilon^k \psi_0^{[k]} = h. \quad (4.11)$$

In equality (4.11) let us equate the terms, not containing ε in the form of factor. Bearing in mind that $x_2^{[0]} = x$ [see (3.11)], we obtain

$$mx\psi_0^{[0]} = h. \quad (4.12)$$

Hence

$$\psi_0^{[0]} = \varkappa^{-1} m^{-1} k = \mu m^{-1} k. \tag{4.13}$$

In accordance with this

$$\psi_{0*}^{[0]} = \mu_* m^{-1} k. \tag{4.14}$$

Now in equality (4.11) let us equate coefficients at ε in the first power. We will have

$$m\varkappa\psi_0^{[1]} = -m\left(\varkappa_2^{[1]}\psi_0^{[0]} + \varkappa_1^{[1]}\frac{\partial\psi_0^{[0]}}{\partial t}\right). \tag{4.15}$$

Hence

$$\psi_0^{[1]} = -\mu\left(\varkappa_2^{[1]}\psi_0^{[0]} + \varkappa_1^{[1]}\frac{\partial\psi_0^{[0]}}{\partial t}\right). \tag{4.16}$$

By substituting here the value of $\psi_0^{[0]}$ and taking into account that

$$\frac{\partial\psi_0^{[0]}}{\partial t} = \frac{\partial}{\partial t}[\mu(\tau) m^{-1}(\tau) k(t)] = \mu m^{-1}\frac{dk}{dt},$$

and $\varkappa_2^{[1]} = 0$ [see (3.26)], we obtain

$$\psi_0^{[1]} = -\mu\varkappa_1^{[1]}\mu m^{-1}\frac{dk}{dt} \tag{4.17}$$

and accordingly

$$\psi_{0*}^{[1]} = -\mu_*\varkappa_1^{[1]}\mu m^{-1}\frac{dk}{dt}. \tag{4.18}$$

By this method there can be successively determined all terms of expansion $\psi_{0*}(t, \tau, \varepsilon)$.

Terms of expansion of function $\psi_{1\sigma}(t, \tau, \varepsilon)$ are expressed by the same formulas as terms of expansion of function $\psi_{0\sigma}(t, \tau, \varepsilon)$, only in the latter instead of h it is necessary to substitute function (4.3). Thus,

$$\psi_{1\sigma}^{[0]} = \mu_\sigma m^{-1} a\delta, \tag{4.19}$$

$$\psi_{1\sigma}^{[1]} = -\mu_\sigma \varkappa^{[1]} \mu m^{-1} a \frac{d\delta}{dt}, \tag{4.20}$$

etc.

Thus, system of equations (4.1) can be reduced to form (4.5). Coefficients of equations of converted system are represented in the form of series with respect to powers ε. By retaining in coefficients the terms containing ε not higher than in the first power, we will have

$$\frac{d^2 z_\sigma}{dt^2} + \varepsilon r_\sigma^{[1]} \frac{dz_\sigma}{dt} + \nu_\sigma z_\sigma = \varepsilon \mu_\sigma m^{-1} a\delta + + \mu_\sigma \left(m^{-1} h - \varepsilon \varkappa^{[1]} \mu m^{-1} \frac{dh}{dt}\right) \quad (\sigma = 1, \ldots, n). \tag{4.21}$$

It is possible to refine the converted system by retaining in coefficients of equations the terms containing ε^2. However, calculations show that in this case the values of coefficients are changed insignificantly, and therefore it is entirely possible to be limited to examination of system (4.21).

In order to apply the obtained results to original system (A'), it is necessary in all relationships to assume $\varepsilon = 1$. By assuming $\varepsilon = 1$ in (4.21), we will have

$$\frac{d^2 z_\sigma}{dt^2} + \alpha_\sigma^{[1]} \frac{dz_\sigma}{dt} + \nu_\sigma z_\sigma = \mu_\sigma m^{-1} a\delta + + \mu_\sigma \left(m^{-1} h - \varkappa^{[1]} \mu m^{-1} \frac{dh}{dt}\right) \quad (\sigma = 1, \ldots, n). \tag{4.22}$$

§ 5. Conversion of Control Signal of Automatic Stabilization Control

In the expression of control signal of automatic stabilization control (*Б*) let us turn to new variables $z_1, \ldots, z_n$ by means of substitution (4.4):

$$v = b_1 \frac{dq}{dt} + b_2 q = b_1 \left\{ \sum_{\nu=1}^{n} \left[\left(\varepsilon \frac{d\tilde{x}_{1\nu}}{d\tau} - \tilde{x}_{1\nu}\tilde{a}_{1\nu} + \tilde{x}_{2\nu} \right) \frac{dz_\nu}{dt} + \right.\right.$$

$$\left.\left. + \left(\varepsilon \frac{d\tilde{x}_{2\nu}}{d\tau} - \tilde{x}_{1\nu}\tilde{a}_{2\nu} \right) z_\nu \right] + \varepsilon \sum_{\nu=1}^{n} \tilde{x}_{1\nu}\tilde{\varphi}_\nu \right\} + b_2 \sum_{\nu=1}^{n} \left(\tilde{x}_{1\nu} \frac{dz_\nu}{dt} + \tilde{x}_{2\nu} z_\nu \right).$$

Let us designate

$$\tilde{v}_{1\nu} = b_1 \left(\varepsilon \frac{d\tilde{x}_{1\nu}}{d\tau} - \tilde{x}_{1\nu}\tilde{a}_{1\nu} + \tilde{x}_{2\nu} \right) + b_2 \tilde{x}_{1\nu},$$
$$\tilde{v}_{2\nu} = b_1 \left(\varepsilon \frac{d\tilde{x}_{2\nu}}{d\tau} - \tilde{x}_{1\nu}\tilde{a}_{2\nu} \right) + b_2 \tilde{x}_{2\nu}. \qquad (5.1)$$

Then

$$v = \sum_{\nu=1}^{n} \left(\tilde{v}_{1\nu} \frac{dz_\nu}{dt} + \tilde{v}_{2\nu} z_\nu + \varepsilon \tilde{x}_{1\nu}\tilde{\varphi}_\nu \right). \qquad (5.2)$$

Let us substitute series (3.8) in (5.1):

$$\tilde{v}_{1\nu} = b_1 \left(\varepsilon \sum_{k=1}^{\infty} \varepsilon^k \frac{dx_{1\nu}^{[k]}}{d\tau} - \sum_{k=1}^{\infty} \varepsilon^k x_{1\nu}^{[k]} \sum_{k=1}^{\infty} \varepsilon^k a_{1\nu}^{[k]} + \right.$$
$$\left. + \sum_{k=0}^{\infty} \varepsilon^k x_{2\nu}^{[k]} \right) + b_2 \sum_{k=1}^{\infty} \varepsilon^k x_{1\nu}^{[k]}.$$

$$\tilde{v}_{2\nu} = b_1 \left(\varepsilon \sum_{k=0}^{\infty} \varepsilon^k \frac{dx_{2\nu}^{[k]}}{d\tau} - \sum_{k=1}^{\infty} \varepsilon^k x_{1\nu}^{[k]} \sum_{k=0}^{\infty} \varepsilon^k a_{2\nu}^{[k]} \right) + b_2 \sum_{k=0}^{\infty} \varepsilon^k x_{2\nu}^{[k]}$$

By uniting in the right sides of the last equalities the terms containing ε in equal powers, let us represent functions $\tilde{v}_{1\sigma}$ and and $\tilde{v}_{2\sigma}$ in the form

$$\tilde{v}_{l\sigma}=\sum_{k=0}^{\infty}\varepsilon^{k}v_{l\sigma}^{[k]}\quad (l=1,2). \tag{5.3}$$

Here

$$\begin{aligned} &v_{1\sigma}^{[0]}=b_1\varkappa_{\sigma}, \quad v_{2\sigma}^{[0]}=b_2\varkappa_{\sigma},\\ &v_{1\sigma}^{[1]}=b_2\varkappa_{\sigma}^{[1]}, \quad v_{2\sigma}^{[1]}=b_1\left(\frac{d\varkappa_{\sigma}}{d\tau}-\varkappa_{\sigma}^{[1]}\nu_{\sigma}\right), \end{aligned} \tag{5.4}$$

etc. In formulas (5.4) there are taken into account equalities (3.11) and (3.26).

On the basis of obtained relationships we have

$$\begin{aligned} v=\sum_{\sigma=1}^{n}\Big\{&[b_1\varkappa_{\sigma}+\varepsilon b_2\varkappa_{\sigma}^{[1]}+\varepsilon^2\ldots]\frac{dz_{\sigma}}{dt}+\\ &+\left[b_2\varkappa_{\sigma}+\varepsilon b_1\left(\frac{d\varkappa_{\sigma}}{d\tau}-\varkappa_{\sigma}^{[1]}\nu_{\sigma}\right)+\varepsilon^2\ldots\right]z_{\sigma}+\varepsilon^2\ldots\Big\}. \end{aligned} \tag{5.5}$$

Hence, by retaining only the terms containing ε in not higher than the first power we obtain approximately

$$\begin{aligned} v=\sum_{\sigma=1}^{n}\Big\{&[b_1\varkappa_{\sigma}+\varepsilon b_2\varkappa_{\sigma}^{[1]}]\frac{dz_{\sigma}}{dt}+\\ &+\left[b_2\varkappa_{\sigma}+\varepsilon b_1\left(\frac{d\varkappa_{\sigma}}{dt}-\varkappa_{\sigma}^{[1]}\nu_{\sigma}\right)\right]z_{\sigma}\Big\}. \end{aligned} \tag{5.6}$$

Relationship (5.6) when $\varepsilon = 1$ represents approximate expression of control signal of automatic stabilization control through new coordinates $z_1, \ldots, z_n$, which must be considered together with converted equations of disturbed motion of the rocket as the object of control.

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13. ABSTRACT

In the book are discussed fundamentals of dynamics of rocket operating on solid and liquid propellants. There are listed equations of undisturbed and disturbed motion of rockets as bodies of variable composition, methods of linearization of equations and decomposition of linearized equations into separate groups. There are examined stability and controllability of the rocket, its transfer functions and dynamic characteristics as the object of control. There are comprehensively examined questions of stabilization of the rocket taking into account the elasticity of its construction and mobility of liquid propellant in the tanks. The book is intended for students of engineering colleges and can be useful for engineers of the aviation and rocket industry. Orig. art. has: 2 tables, 56 illustrations.

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