ON MODELS FOR 1/f NOISE PROCESSES

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I. INTRODUCTION

A noise phenomenon which is often dominant at low frequencies is comprised of those processes which are called 1/f noise, flicker effect, and sporadic processes. These processes are characterized by a power spectral density which varies as $1/|f|^{\alpha}$ where $0 \le \alpha \le 2$. They have been observed over a wide range of frequencies in such diverse situations as vacuum tube and semiconductor noise, the spectrum of sea waves, and ultra-precise time standards. Some of these processes are not admissable in the usual Wiener-Khinchin theory of random processes. On the other hand, their extensive occurrence suggests that some accommodation must be made for them in the organized theory.

It is usually assumed that noise processes are sample functions of Wiener-Khinchin (i.e., stationary second-order) random processes. Since such processes are well categorized (in a mean square sense) by their power spectral density, or, equivalently, by their autocorrelation function, it is of interest to see what this assumption implies about 1/f processes.

For the process $x(t)$ the autocorrelation function will be written as

$$R_x(t, t+\tau) = E\{x(t)x(t+\tau)\} \tag{1}$$

In the case where $x(t)$ is at least wide-sense stationary, the autocorrelation function depends only on the time difference and will be written as $R_x(\tau)$. In this case the Fourier transform of $R_x(\tau)$ will be called the power spectral density of the process and will be denoted by $\varphi_x(\omega)$.

The 1/f processes to be investigated will be assumed to have a power spectral density

$$\varphi_x(\omega) = k|\omega|^{-\alpha} \tag{2}$$

at least for small $|\omega|$ and may be divided into three classes of interest by the parameter α. Since both high and low frequency problems will be encountered if it is assumed that Eq. (2) holds for all ω and since low frequency problems are of primary interest here, it will be assumed that $\varphi_x(\omega)$ decreases sufficiently rapidly

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as a function of $|\omega|$ that the high frequency behavior of x(t) is not pathological (e.g., it will typically be assumed that

$$\int_{|\omega|>b} \varphi_x(\omega)\, d\omega < \infty \tag{3}$$

for all b>0). The three regions of interest are $\alpha=0$, $0<\alpha<1$, and $\alpha\geq 1$.

If $\alpha=0$, we have

$$\lim_{\omega\to 0} \varphi_x(\omega) = k$$

Now assuming that $\varphi_x(\omega)$ has no singularities, $R_x(\tau)$ is an integrable function of τ. Under the assumption that $\varphi_x(\omega)$ is integrable, $R_x(\tau)$ is finite and therefore $E\{x^2(t)\}<\infty$ Of course, in the case where $\varphi_x(\omega)=k\omega^0=k$ for all ω, the autocorrelation function is $R_x(\tau)=k\delta(\tau)$ and one encounters the high frequency difficulties associated with the white noise idealization of random processes even though there are no low frequency difficulties.

In the case where $0<\alpha<1$, the power spectral density $\varphi_x(\omega)$ given by Eq. (2) has a Fourier transform

$$R_x(\tau) = \frac{k|\tau|^{\alpha-1}}{2\Gamma(\alpha)\cos\frac{\pi\alpha}{2}} \tag{4}$$

This function is unbounded at the origin, implying that $E\{x^2(t)\}=\infty$. This difficulty, however, is generated by the high frequency behavior of $\varphi_x(\omega)$ and may be eliminated by an assumption such as Eq. (3). Processes of interest with $0<\alpha<1$ have power spectral density given by

$$\varphi_x(\omega) = k|\omega|^{-\alpha}\, g(\omega) \tag{5}$$

where $g(\omega)$ is slowly varying for small $|\omega|$ and decreases at least as fast as $|\omega|^{\alpha-2}$ for large $|\omega|$. It can be seen that Eq. (4) approaches $k\delta(\tau)$ as $\alpha\to 0$ but is degenerate as $\alpha\to 1$. The case $0<\alpha<1$ is thus consistent with Wiener-Khinchin theory although some special results of the theory (i.e., those which depend on $\varphi_x(0)$ being finite) cannot necessarily be applied..

The last region of interest is where $\alpha\geq 1$. In this case, since

$$\int_0^b k|\omega|^{-\alpha} d\omega = \infty, \text{ all } b>0 \tag{6}$$

the function

$$F(\omega) = \int_0^{\omega} \varphi_x(\lambda)\, d\lambda \tag{7}$$

is unbounded. The Fourier transform of $\varphi_x(\omega)$ is not positive definite [1] and thus cannot be the autocorrelation function of a Wiener-Khinchin random process. Power spectral densities, then, which are proportional to $|\omega|^{-\alpha}$, where $\alpha \geq 1$ for all $|\omega| < b$, are not admissible. In order to accommodate such processes in the Wiener-Khinchin theory, it must be assumed that there is a non-zero frequency W below which

$$\varphi_x(\omega) \leq k|\omega|^{-\alpha} \quad , \qquad \alpha < 1 \tag{8}$$

It is this third class which presents the greatest difficulty. The unbounded amount of low frequency energy is analogous to the unbounded high frequency energy of white noise processes.

2. OBSERVATIONS OF 1/f PROCESSES

The sample power spectral density of a wide class of processes has demonstrated a 1/f type behavior over a wide range of frequencies. Some areas in which these processes have been observed are noise in solid state devices [2], fluctuations of precise frequency standards [3], and the spectrum of ocean waves [4]. In some cases the behavior has been observed over many decades of frequency.

Since the minimum frequency at which the power spectral density of a physical process can be estimated is limited by the necessarily finite observation time, a common approach toward accommodating $1/f^{\alpha}$ type processes into the second order theory is to impose a non-zero low frequency limit on the 1/f behavior of the process and to assume that the power spectral density approaches a limit as f approaches zero. As was seen in the previous section, this assumption is unnecessary where $\alpha < 1$ and even in cases where $\alpha \geq 1$ the assumption is not completely satisfactory. The assumed low frequency limit has never been observed and measurement of extremely precise crystal oscillators have indicated that there are processes for which any low-frequency limit would have to be less than 3×10^{-8} Hz. An additional difficulty associated with the low-frequency limit W is that the process variance

$$\sigma_x^2 = E\{x^2(t)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_x(\omega)\, d\omega \tag{9}$$

is strongly dependent on the value of W.

Estimation of sample power spectral densities of continuous records at very low frequencies requires very narrow-band frequency estimates and therefore very long lag windows or averaging times [5]. In practical cases the assumption of stationarity of either the measurement or the measured process is often questionable for the time interval required. For intervals of more acceptable length the variance of the estimate of the power spectral density is so high as to make its value worthless.

Digital processing of sampled data relieves many of the constraints on the measurement process but relies on exact knowledge of the mean of the process or on techniques to remove the unknown mean. Unlike the continuous time case in which the inherent high pass nature of the measurement apparatus removes the d-c value of the process $x(t)$, digitally processed samples of $x(t)$ will assign all of the energy due to the average value of $x(t)$ to the lowest frequency values, thus artificially biasing the low frequency estimates of the power spectral density. Techniques have been developed [5] to reduce the effect of $E\{x(t)\}$ on the spectral estimates. However determination of such parameters as α from low frequency sample spectra has been only moderately successful.

One approach to the estimation of 1/f type power spectral densities relies on the fact that the expected value of the variance of the process with a power spectral density of the form of Eq. (2) for low frequencies is a function of N, the number of samples, and of α. The following relationships can be shown to hold [6]:

$$\langle x(t)\rangle = \frac{1}{N}\sum_{n=0}^{N-1} x(t_0+nT) \tag{10}$$

and

$$\langle \sigma^2(N,T)\rangle = \frac{a(\mu)N|T|^{\mu}}{N-1}\left[1-N^{\mu}\right] \tag{11}$$

where $\mu=\alpha-1$, $\langle\cdot\rangle$ indicates time average, and $a(\mu)$ is a constant depending on the value of μ. If the quantity

$$S(N,\mu) = \frac{\langle\sigma^2(N,T)\rangle}{\langle\sigma^2(2,T)\rangle} \tag{12}$$

is known for several values of N, the values of μ (and therefore α) can be approximated for the process. Using this procedure, Allan [6] found that, for "flicker" noise in cesium beam frequency standards, $\mu=-2/3$. In other words, for low frequencies,

$$\varphi_x(\omega) \approx k|\omega|^{-1/3} \tag{13}$$

The function $S(N,\mu)$ is an increasing function of N and, as $\mu\to 0$ ($\alpha\to 1$), we have

$$S(N,0) = \frac{N\ell nN}{2(N-1)\ell n2}$$

and as $N\to\infty$, the quantity $S(N,0)$ is unbounded .

Thus, with proper consideration of the variance of finite length samples, processes have been found which correspond to the second case $0<\alpha<1$.

3. PHYSICALLY MOTIVATED MODELS FOR SHOT PROCESSES

Shot process models have been proposed as physically motivated models for many noise processes, in particular, for noise generated in vacuum tubes and in solid-state devices [7]. The shot process $F(t)$ is given by

$$F(t) = \sum_i f_i(t - t_i) \qquad (14)$$

where $f_i(t)$ are the basic events of the process and the t_i are the occurrence times. The t_i will be considered to constitute a Poisson process. The $f_i(t)$ may be deterministic time functions or may themselves depend on a random parameter. In the particular case to be considered $f_i(t)$ will be given by $f(t,\tau_i)$ where the τ_i are independent identically distributed random variables which will be related to the "duration" of the event $f_i(t)$. The shot process to be considered is thus given by

$$F(t) = \sum_i f(t - t_i, \tau_i) \qquad (15)$$

It is shown in [7] that the power spectral density $\varphi_F(\omega)$ of $F(t)$ is given by

$$\varphi_F(\omega) = \nu E\{|\varphi_f(\omega,\tau)|^2\} \qquad (16)$$

where ν is the average rate of the Poisson process and $\varphi_f(\omega,\tau)$ is the Fourier transform of $f(t,\tau)$.

Let us assume that

$$m_1^2(\tau)\ p(\tau) = c \qquad , \ a_1<\tau<a_2 \qquad (17)$$

where $p(\tau)$ is the common probability density function of the τ_i and $m_1(\tau)$ is given by

$$m_1(\tau) = \int_{-\infty}^{\infty} f(t,\tau)dt \qquad (18)$$

Then it is shown in [7] that, for $(1/a_2)<<\omega<<(1/a_1)$,

$$\varphi_F(\omega) \approx K/\omega \qquad (19)$$

Thus there is a shot process model which approximates a 1/f process over an arbitrary large frequency interval (depending on the constants a_1 and a_2). It is instructive to consider an example of such a process. Let

$$f(t,\tau) = \begin{cases} e^{-t/\tau} & , t\geq 0 \\ 0 & , t<0 \end{cases} \tag{20}$$

and assume that the random variable τ has probability density

$$p(\tau) = \begin{cases} \frac{a_1 a_2}{a_2 - a_1} \frac{1}{\tau^2} & , a_1 \leq \tau \leq a_2 \\ 0 & , \text{elsewhere} \end{cases} \tag{21}$$

Then the power spectral density $\varphi_f(\omega,\tau)$ becomes

$$\varphi_f(\omega,\tau) = \frac{\tau}{1+j\omega\tau} \tag{22}$$

and $\varphi_F(\omega)$ becomes

$$\varphi_F(\omega) = \frac{\nu a_1 a_2}{a_2 - a_1} \int_{a_1}^{a_2} \frac{1}{1+\omega^2\tau^2} d\tau$$

$$= \frac{\nu a_1 a_2}{a_2 - a_1} \frac{1}{\omega} [\tan^{-1}(\omega a_2) - \tan^{-1}(\omega a_1)] \tag{23}$$

For $(1/a_2) << \omega << (1/a_1)$, this last expression becomes

$$\varphi_F(\omega) \approx \frac{\pi \nu a_1 a_2}{2(a_2 - a_1)} \frac{1}{\omega} \tag{24}$$

It should be noted that, as $\omega \to 0$, $\varphi_F(\omega) < K/\omega$ as reaquired in Section 1.

The exact form of $\varphi_F(\omega)$, given by Eq. (23), is also of interest. In a consideration of oxide traps as a source of flicker noise in solid state devices, van der Ziel has shown that oxide traps give rise to a noise process with a power spectral density given by

$$\varphi_N(\omega) = \frac{K}{\ln(\tau_1/\tau_2)} \frac{1}{\omega} [\tan^{-1}(\omega\tau_1) - \tan^{-1}(\omega\tau_2)] \tag{25}$$

where K, τ_1 and τ_2 are physical parameters depending upon the material and surface involved. This power spectral density is precisely that given for the Poisson shot process just discussed. An indication of the behavior of $f(t,\tau)$ is shown in Fig. 2 and a typical density $p(\tau)$ is given in Fig. 1. A comparison of $S_F(\omega)$ for the process with this $p(\tau)$ and a K/ω spectrum is shown in Fig. 3.

The existence of processes consisting of events having widely distributed duration time has been described by several authors. In a report on 1/f noise in thin evaporated films, for example, Osman [8] has determined the pulse duration due to election tunneling to be given approximately by

$$\tau = 10^{-15+s} \text{ seconds} \tag{26}$$

where s is the barrier thickness in Angstrom units (1 angstrom unit = 10^{-10} meter). A variation in barrier thickness of 5-20 angstroms is sufficient to account for pulse durations ranging over fifteen orders of magnitude (10^{-10} to 10^{5} seconds). Thus reasonable physical mechanisms have been described which can lead to 1/f type behavior of a shot process over a very large frequency range.

Shot models, then, can provide reasonable physically motivated models for some 1/f type processes. These models are particularly attractive for modeling noise in solid state devices where appropriate physical mechanisms for generation of these processes have been described. The particular model discussed is appropriate for 1/f processes in the third class ($\alpha \geq 1$) and has the low frrquency limit W inherent in the model.

4. SPORADIC RANDOM PROCESSES

The term "flicker processes" was ascribed to processes with $1/f^{\alpha}$ type power spectral densities because of the apparent nonstationarity of these processes when observed over finite time intervals. This behavior is due to the dominance of low frequency energy in these processes but is not typical of the behavior expected for stationary second order random processes. The previous discussions have been concerned primarily with accomodating these processes to the familiar framework. From a finite sample of a process $x(t)$, however, it is impossible to determine if the process is a stationary, second order process and even the practical value of specifying the behavior of the process over all time may be questioned. Thus a process which is stationary over an interval which is long compared with the sampling interval may for all practical purposes be considered to be a stationary process. The concept of a sporadic random process was introduced by Mandelbrot [9] to characterize processes whose behavior can be divided into periods of activity and periods of inactivity. These processes cannot always be accomodated in the Wiener-Khinchin theory. For example, a random process of the type described below is not a stationary second order random process:

1) $x(t)$ is almost surely constant in every finite interval (t_1,t_2)
2) $x(t)$ almost surely varies for some t, $-\infty<t<\infty$.

Sporadic random processes do not necessarily possess a Wiener-Khinchin power spectral density but do possess a conditional spectral density (conditioned on the periods of activity of the process) which may vary as $\omega^{-\alpha}L(\omega)$ for small ω where $0\leq\alpha\leq 2$ and $L(\omega)$ is a function which varies slowly at the origin*. Some sporadic random processes have been shown [10] to possess the erratic sampling behavior associated with 1/f processes. A basic sporadic random process is the process which is constant with a single discontinuity at the point t=T. This point is uniformly distributed over all t (the random variable T is a generalized random variable in the sense of Mandelbrot [9]). This process can be represented by

$$x(t) = \begin{cases} X_1 & , t\leq T \\ X_2 & , t>T \end{cases}$$

where X_1 and X_2 are independent identically distributed random variables. This process does not possess a Wiener-Khinchin spectral density but has a conditional spectral density which is proportional to $|\omega|^{-2}$ for small ω. Sporadic random processes therefore offer a generalization of Wiener-Khinchin random processes and the conditional spectral density of these processes may provide an interpretation of measurements intended to measure the Wiener-Khinchin spectral density.

5. CONCLUSION

Several different approaches have been followed in an investigation of 1/f processes. While each of these appear particularly suited to one type of process (i.e. one particular range of α) the approaches are not mutually exclusive and provide among them an interpretation for the majority of the low frequency noise problems observed to date.

These models are physical in the sense that the primary rationale behind the modelling is to explain observed phenomena. However, the universality of the $1/f^{\alpha}$ phenomena would argue against the applicability of a single model closely associated with the physics of a particular situation.

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*$L(\omega)$ varies slowly at the origin in the sense of Karamata if, for every $h>0$

$$\lim_{\omega\to 0} L(h\omega)/L(\omega) = 1$$

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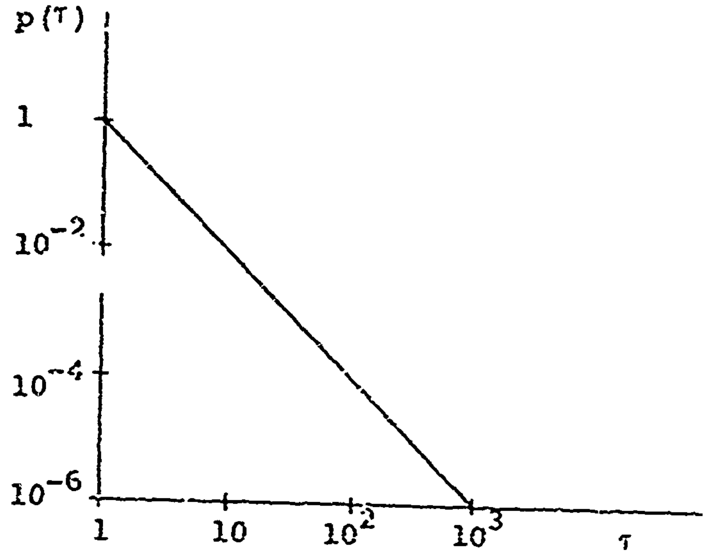

Fig. 1 - The Density Function p(τ)

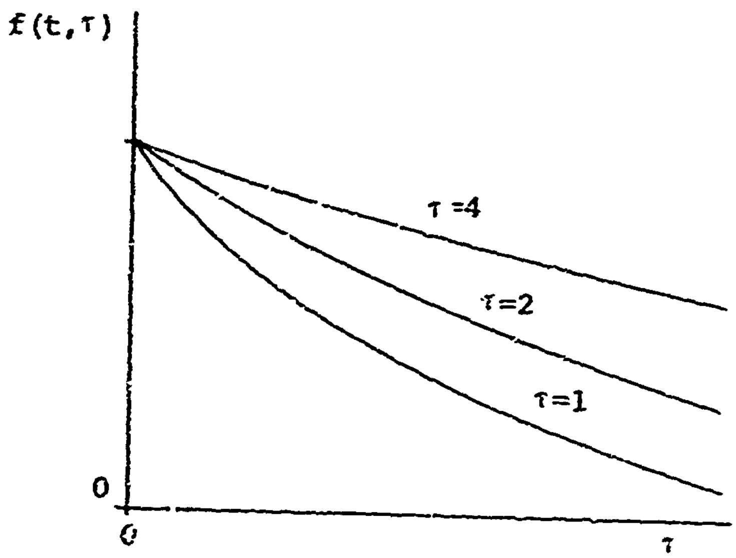

Fig. 2 - The Basic Pulse f(t,τ)

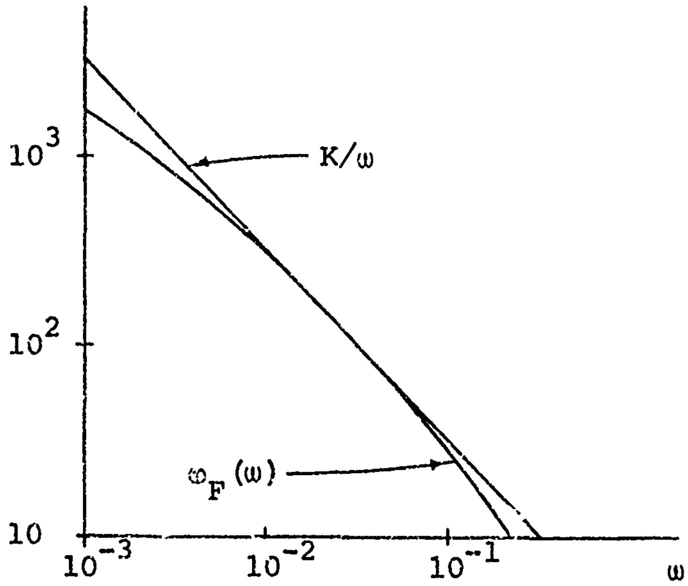

Fig. 3 - The Power Spectral Density $\Phi_F(\omega)$

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13. ABSTRACT

A noise phenomenon which is often dominant at low frequencies is comprised of those processes which are called 1/f noise, flicker effect, and sporadic processes. These processes are characterized by a power spectral density which varies as $1/|f|^{\alpha}$ where $0<\alpha\leq 2$. They have been observed over a wide range of frequencies in such diverse situations as vacuum tube and semiconductor noise, the spectrum of sea waves, and ultra-precise time standards. Some of these processes are not admissable in the usual Wiener-Khinchin theory of random processes. On the other hand, their extensive occurrence suggests that some accomodation must be made for them in the organized theory.

It is the purpose of this paper to consider several approaches to the modeling of such 1/f processes. It will be shown that models do exist which satisfactorily explain observed phenomena while retaining adequate connection with the underlying physics of the model.