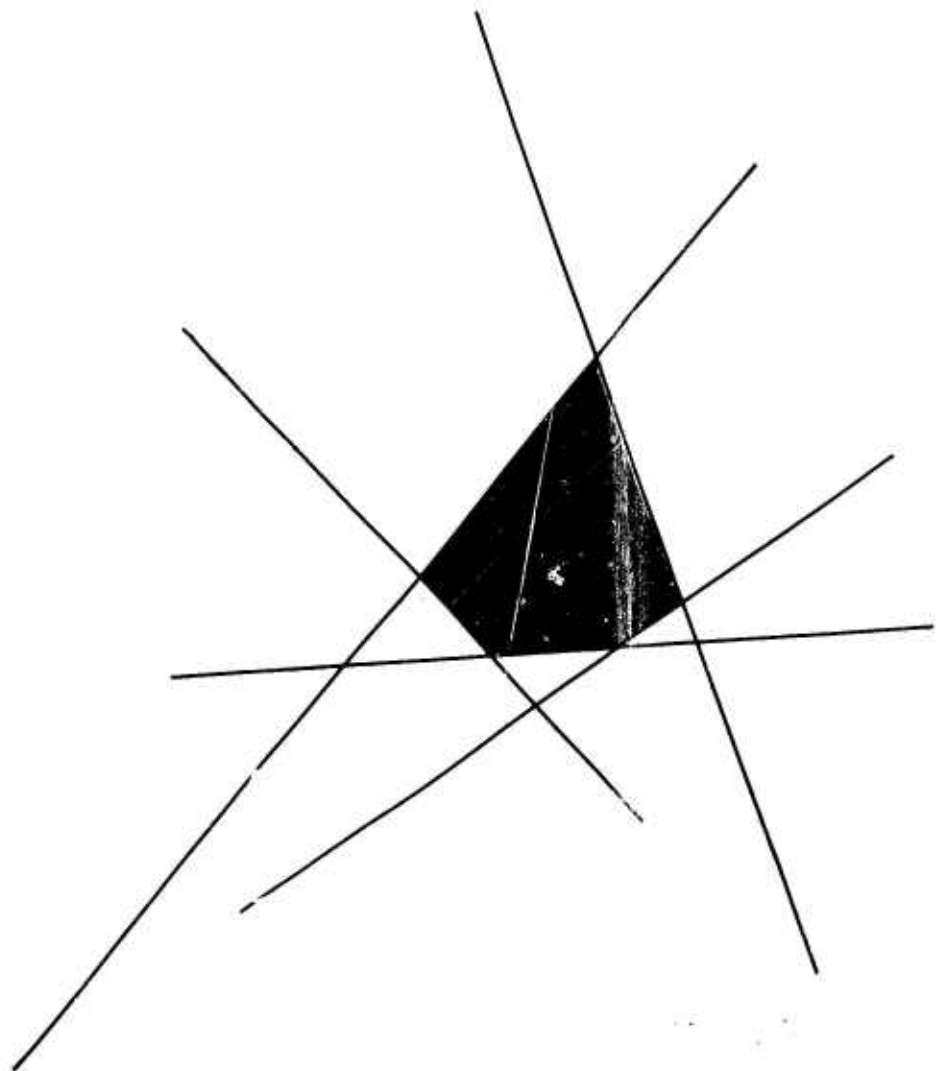


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MARK BROWN
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14

OPTIMAL ISSUING POLICIES

by

Mark Brown[†]
Department of Operations Research
Cornell University
Ithaca, New York

and

Sheldon M. Ross^{††}
Department of Industrial Engineering
and Operations Research
University of California, Berkeley

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ABSTRACT

Consider a stockpile consisting of n items where the i th item has a rating r_i , $i = 1, \dots, n$. An item with rating r , if kept in stockpile until time t and then released to the field, will have a field life of $L(r)d(t)$. Thus it is assumed that for any t , the field life for issuance at time t is proportional to the time 0 field life.

Items are to be issued one by one from the stockpile to the field until the stockpile is depleted. The i th issued item is placed in the field immediately upon the death in the field of the $(i-1)$ st issued item. Conditions for LIFO and FIFO optimality are obtained.

0. Introduction. Consider a stockpile consisting of n items where the i^{th} item has a rating r_i , $i = 1, \dots, n$. An item with rating r , if kept in stockpile until time t and then released to the field, will have a field life of $L(r)d(t)$. Thus it is assumed that for any t , the field life for issuance at time t is proportional to the time 0 field life.

Items are to be issued one by one from the stockpile to the field until the stockpile is depleted. The i^{th} issued item is placed in the field immediately upon the death in the field of the $(i-1)$ st issued item. The problem of interest is to find the order of item issue which maximizes the total field life obtained from the stockpile.

The policy which issues items in increasing order of the item ratings is called the LIFO issuing policy, while the policy which issue the items in decreasing order of the ratings is called the FIFO issuing policy. For example if there are three items which initially have ratings $0 < r_1 < r_2 < r_3$, then the total field life following LIFO is:

$$L(r_1) + L(r_2)d(L(r_1)) + L(r_3)d(L(r_1) + L(r_2)d(L(r_1)))$$

and the total field life following FIFO is:

$$L(r_3) + L(r_2)d(L(r_3)) + L(r_1)d(L(r_3) + L(r_2)d(L(r_3))) .$$

Derman and Klein [3] and Lieberman [9] have considered the problem of maximizing the total field life in a model under which the lifetime of an item at time t whose initial rating is r is given by $L(r+t)$.

In both papers conditions are derived for LIFO and FIFO optimality. Other papers on this model may be found in the references. The model which we discuss (which we call the multiplicative model) does not seem to have been previously considered in the literature.

In section 2 we obtain conditions for LIFO and FIFO optimality for the multiplicative model. In section 3 examples are presented.

1. Results

Given n ratings $r_1, \dots, r_n$ with $m \leq n$ distinct values $a_1, \dots, a_m$, with a_i appearing k_i times ($k_i \geq 1$, $\sum_{i=1}^m k_i = n$), an issuing policy is any one of the $n! / \prod_{i=1}^m k_i!$ distinguishable arrangements of the a_i 's. We therefore consider two policies to be the same if the n -tuples of ratings of items issued are the same, even though the indices of items may differ. For example if $r_1 = r_2 = 1$, $r_3 = 2$ the policy which issues item 1 first and item 2 second is considered to be the same as the one which issues item 2 first and item 1 second.

Define the following classes of functions on $[0, \infty)$:

$$C^+(C^-) = \{f : \alpha_f(a, b) = f(a) - \frac{a}{b-a} (f(b) - f(a)) < (>) 1 \text{ for all } 0 < a < b\}$$

$$L = \{f : f(0) = 0, f \text{ is monotone increasing}\}$$

$$M^+(M^-) = \{f : f \geq 0, f'' \text{ exists and is strictly positive (negative)}\}$$

Our main result is theorem 1 below:

Theorem 1. If $L \in L$, $d(0) = 1$, and $d \in M^+(M^-)$, then FIFO(LIFO) is the unique optimal policy.

We shall derive three lemmas before proving theorem 1.

Lemma 1. If $L \in L$, $d(0) = 1$, and $d \in C^+(C^-)$, then for $n = 2$ FIFO (LIFO) is the unique optimal policy.

Proof. FIFO is optimal iff for $0 < a < b$:

$$(1) \quad L(a) + L(b)d(L(a)) < L(b) + L(a)dL(b)$$

Note that (1) is equivalent to:

$$(L(b) - L(a))d(L(a)) - L(a)(d(L(b)) - d(L(a))) < L(b) - L(a)$$

which can be rewritten as $(L(b) - L(a))\alpha_d(L(a), L(b)) < L(b) - L(a)$. Since L is monotone increasing this is equivalent to $\alpha_d(L(a), L(b)) < 1$.

Lemma 2. Let $f_t(x) = f(t+xf(t))/f(t)$. Then if $f \in M^+(M^-)$ and $f(0) = 1$ then $f_t \in C^+(C^-)$ for all $t \geq 0$.

Proof. We have to show that if $f(0) = 1$ and $f \in M^+$ then $\alpha_{f_t}(a, b) < 1$ for all $t \geq 0$, $0 < a < b$. Note that:

$$(2) \quad f_t(b) - f_t(a) = (b-a)f'(a^*)$$

where $t+af(t) < a^* < t+bf(t)$, since f' exists and is continuous. Since f' is increasing it follows from (2) that:

$$(3) \quad \alpha_{f_t}(a,b) < (f(t+af(t))/f(t)) - af'(t+af(t)) = \beta(a)$$

Now $\beta(0) = 1$ and $\beta'(a) = -af(t)f''(t+af(t)) < 0$ since $f > 0$, $f'' > 0$

(Note that $f \geq 0$, $f(0) = 1$ and $f'' > 0$ imply $f > 0$). Therefore $\beta(a) < 1$ for all $a > 0$ so that $\alpha_{f_t}(a,b) < 1$ for all $0 < a < b$. A similar argument

holds for $f \in M^-$.

Lemma 3. Define $g_x(y) = y+xf(y)$. Then if $f(0) = 1$ and $f \in M^+$ then $g_x(z) > g_x(y)$ for all $z \geq x$, $y < x$, and g_x is monotone increasing on $[x, \infty)$. If $f(0) = 1$ and $f \in M^-$ then g_x is monotone increasing on $[0, \infty)$.

Proof. Let $f(0) = 1$ and $f \in M^+$. Then by lemma 2:

$$(4) \quad f(y) - \frac{y}{h} (f(y+h) - f(y)) < 1.$$

Therefore by letting $h \rightarrow 0$ we obtain:

$$(5) \quad f(y) - yf'(y) \leq 1$$

so that $g'_x(x) = 1 + xf'(x) \geq 1 + x(\frac{f(x)-1}{x}) = f(x) > 0$. Therefore $g'_x(x) > 0$

and $g''_x(y) = xf''(y) > 0$ for all y , so that $g_x(y)$ is increasing for $y \geq x$.

Now, there are two cases: $f'(0) > -\frac{1}{x}$ and $f'(0) \leq -\frac{1}{x}$. If $f'(0) > -\frac{1}{x}$ then $g'_x(y) > 0$ for all y and g_x is increasing on $[0, \infty)$. If $f'(0) \leq -\frac{1}{x}$ then since g'_x is increasing and continuous and $g'_x(x) > 0$, it follows that $g'_x(y) = 0$ for some $y \in (0, x)$. Therefore $\sup_{0 \leq y \leq x} g_x(y) = \max(g_x(0), g_x(x)) = \max(x, x+xf(x)) = x+xf(x) = g_x(x)$.

If $f \in M^-$ then $f'(x) > 0$ for all x , since if $f'(t) = -\delta < 0$, then $f(t+s) \leq f(t) - s\delta$ and $f(t) - s\delta$ is negative for $s > f(t)/\delta$, contradicting $f \geq 0$. Therefore $g'_x(y) = 1+xf'(y) > 0$ for all y and therefore g_x is increasing.

Proof of Theorem 1. Assume that if $d \in M^+$, $t \in L$, then FIFO is the unique optimal policy for the issuing of $n = k$ items. We shall prove that this implies that FIFO is an optimal policy for $n = k+1$. This, combined with lemmas 1 and 2, proves the FIFO part of theorem 1.

Given $k+1$ items, choose an optimal policy P (more than one may exist), and for this policy let t be the total field life of the first $k-1$ issued items, a the rating of the k th issued item, and x the rating of the last issued item. The total field life of this policy is $t+d(t)[L(a) + L(x)d_t(L(a))] = F(P)$.

Consider the policy P' , which issues the first $k-1$ items in the same order as P , but then issues x followed by a . The total field life, $F(P')$ of P' is equal to $t+d(t)[L(x) + L(a)d_t(L(x))]$. Since $F(P) \geq F(P')$ it follows from lemma 2 that $x \leq a$.

Let P^* be the policy which issues x last and issues the other k items according to FIFO. Then $F(P^*) = s^* + L(x)d(s^*)$ and $F(P) = s + L(x)d(s)$, where $s^*(s)$ is the total field life of the first k items issued under policies $P^*(P)$. By our induction assumption $s^* \geq s$. Moreover $s^* \geq L(a) \geq L(x)$, and therefore by lemma 3 $F(P^*) \geq F(P)$ with equality iff $s^* = s$, and thus by the induction assumption of uniqueness $P = P^*$. But if $P = P^*$ then a is the lowest rated item of the first k issued under P^* , and we showed that $x \leq a$. Therefore P^* is the FIFO policy and the result is proved. A similar argument works in the LIFO case.

2. Comments and Additions

(i) Note that our optimality conditions allow for d to be increasing in the FIFO case and require d to be increasing in the LIFO case (see the proof of lemma 3). The interpretation of increasing d is that items in stockpile may be constantly improved.

The more realistic case is where d is decreasing, since this indicates that items in stockpile are deteriorating. The intuition behind the FIFO optimality result is that if the rate of deterioration ($-d'$) decreases over time then we want to remove the higher rated items from the stockpile as soon as possible.

(ii) It follows from a slight modification of the proof of theorem 1 that if $L \in L$, $d(0) = 1$, $d \geq 0$ and $d'' \geq (<) 0$ then FIFO (LIFO) is an optimal (although perhaps not unique) policy.

(iii) If $L \in \mathcal{L}$ and $d(t) = 1+ct$, $c \geq 0$, then both LIFO and FIFO are optimal policies.

(iv) If $L \in \mathcal{L}$ and $d(t) = e^{\alpha t}$ then FIFO is an optimal policy. If $\alpha \neq 0$ then FIFO is the unique optimal policy. Thus for exponential deterioration ($\alpha < 0$) or exponential improvement ($\alpha > 0$) FIFO is the unique optimal policy.

(v) The condition $d(0) = 1$ can be replaced by $d(0) > 0$, since by defining $L^* = d(0)L$ and $d^* = d/d(0)$ the problem is transformed to the case $d^*(0) = 1$ and the essential structure of L and d is maintained.

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