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A NEW APPROACH TO EVALUATION OF INFINITE PROCESSES

By
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NAVAL ORDNANCE LABORATORY, WHITE OAK, SILVER SPRING, MARYLAND

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ABSTRACT: The simplest forms of discrete infinite processes, such as infinite series, products, continued fractions and their generalizations are considered. It is shown that by associating such processes with "equivalent" linear difference equations with boundary conditions at infinity a means of classifying them in a unified way is provided, as well as a means of evaluating asymptotic approximations to remainder sequences. If the approximate remainder sequences are introduced at the definitional level, so that the "value" of the infinite process is defined as a limit of successive stages of the finite process with an approximate remainder term included at each stage, two benefits result. First, where the process converges by the Cauchy definition (zero remainder terms), convergence is speeded, so that numerical computations of "value" are aided. Secondly, where the process is Cauchy-divergent, it may nevertheless be "summed" to a useful value. A broad class of processes, termed "asymptotically tractable," is identified for which these benefits are obtained. This class appears to include most cases of interest in classical analysis. When applied to infinite series, the method appears to exceed in convergence-forcing power all other known approaches to "summability." When applied to continued fractions and their generalizations, it reveals a possibility of multiple-valuedness in such processes that apparently has not hitherto been recognized. Examples are given to illustrate the implications and advantages of the new definitional approach. These should be of interest to physicists and engineers concerned with the convergence of infinite processes or with the solution of linear recurrence relations arising in physical problems.

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GEORGE G. BALL
Captain, USN
~~Commander~~

C. M. SCHOMAN
By direction

CONTENTS

ILLUSTRATIONS

TABLES

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Chapter 1

CLASSIFICATION OF DISCRETE INFINITE PROCESSES

1-1 INTRODUCTION

The purpose of this report is to describe an unusual definitional approach to discrete infinite processes, based on a formal "equivalence" of the infinite process to a linear difference equation with boundary conditions at infinity. This is a preliminary account, aimed at stimulating comment that would permit the ideas to be more fully evaluated. No formal publication has been attempted, although the basic concept has been presented verbally (reference (a)) to physicists -- who have possibly the most pressing need for more effective ways of "summing" nominally divergent infinite processes.

Applications of the mathematical tools we shall develop are of particular importance in the treatment of three- or more-term recurrence relations, of the type that often arise in mathematical physics in the solution of linear, homogeneous, ordinary differential equations. This report, however, is concerned with the development of ideas rather than applications.

1-2 BACKGROUND

The present investigation has its origin, like all modern concern with the "summability" of infinite series, in dissatisfaction with the numerous infinite process divergences encountered in classical mathematics. These mar the simplicity and generality of theorems and hamper formal manipulative freedom. Before accepting such consequences, one would wish to explore alternatives, with the object of determining to what extent the divergences might prove to be artifacts of definition.

For example, consider the binominal series,

$$(1+\alpha)^{z} = \sum_{s=0}^{\infty} \alpha^{s} \binom{z}{s} . \qquad (1)$$

According to Cauchy's definition of the infinite process symbolized on the right, this has a "value" equal to the limit as $n \to \infty$ of a sequence of one-sided partial sums, each with a remainder term set equal to zero. That is,

$$\sum_{i=0}^{\infty} a_i \overset{(\text{Cauchy})}{\equiv} \lim_{n\to\infty} \left(S_n + R_n \right) \qquad (2)$$

where

$$S_n \equiv \sum_{i=0}^{n} a_i \quad \text{and} \quad R_n \equiv 0 , \quad n = 1, 2, \cdots .$$

By this definition the right-hand side of (1) diverges for $|\alpha| > 1$ everywhere in the z-plane except at the points z = 0, 1, 2, ..., where the series terminates. But the mathematical object on the left side of (1) exhibits no such pathology for $|\alpha| > 1$. Observations of this sort have long suggested that the Cauchy definition — despite a widespread belief that Cauchy "robbed infinity of its mystery" -- may not be the best one available in all circumstances. Indeed, we know that any competent mathematical definition strips all mystery from the thing defined. This observation puts Cauchy's definition in perspective as one of a number of available contenders.

A broader conception of the range of useful meanings of infinite process might reduce the total inventory of mathematical anomalies, e.g., by enabling the two sides of (1) to become equal for all α and z. Clearly, without prejudice to the Cauchy approach, there is every aesthetic and practical motivation for perennial re-examination of the definitional foundations of infinite processes.

The history of attempts to exploit alternatives to the Cauchy definition is a long and honorable one which goes back at least to a correspondence between Leibniz and Christian Wolf (reference (b)). On the basis of Equation (1) in the specialized form[1]

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots,$$

Leibniz, in 1713, proposed the summation

$$\frac{1}{2} = 1 - 1 + 1 - 1 + \cdots \qquad (\text{for } x = 1).$$

Since the sequence of finite summands on the right oscillates, the Cauchy limit specified in Equation (2) does not exist. Hence, Leibniz's result is an example of "summability" by non-Cauchy methods. Fired with enthusiasm for Leibniz's suggestion, Wolf proposed to carry the matter to logical conclusions, such as

$$\frac{1}{3} = 1 - 2 + 4 - 8 + 16 - \cdots \qquad (\text{for } x = 2),$$

$$\frac{1}{4} = 1 - 3 + 9 - 27 + 81 - \cdots \qquad (\text{for } x = 3),$$

etc.

However, Leibniz threw cold water on this. He was willing to go to the limit circle of convergence ($x = 1$), but declined to venture beyond.

There the matter has rested, since scholars were soon diverted by work of Euler and Daniel Bernoulli toward mean-value and convergence-factor methods of summing "divergent" series. These methods were not powerful enough to support Wolf's proposal. The reason is that they all employ the (here so-called) "Cauchy" conception of a discrete infinite process, according to which a zero remainder term is used at each stage of the limiting process. The traditional summability methods just mentioned involve sequence-to-sequence transformation; but the sequences transformed are

[1] Footnotes are in Appendix A

always the finite summand sequences S_n, n = 1, 2, ..., never the remainder sequences R_n. Tampering with finite summands is allowed, but tampering with remainder sequences is forbidden. Apart from the historical evolution of the subject, one seeks in vain for a rational explanation of this exclusion.

The present report establishes that a more fruitful summability theme involves leaving the finite summand sequences S_n untransformed, and concentrating instead on a search for minimal yet adequate restrictions on remainder sequences representing the terminus of the infinite process, or "part at infinity." We shall refer to the resulting infinite-process evaluation procedures as "terminal summation."

An agreeable outcome of our investigation will be the confirmation of Christian Wolf's speculations: By terminal summation of infinite series we can indeed reproduce his results. However, this is only the beginning of the benefits resulting from a new definitional approach. The full advantage emerges when we look beyond infinite series to infinite processes in general.

1-3 CLASSIFICATION OF DISCRETE INFINITE PROCESSES

For purposes of systematic classification (cf. Table 1-1) it is convenient to introduce the "difference equation viewpoint," according to which any discrete infinite process is assigned the order of, and is in a sense identified with, an "equivalent" difference equation. That is, the process is viewed as a notationally different way of writing the difference equation.

DEFINITION 1. A discrete infinite process (i.e., a specified unending sequence of algebraic operations) will be termed *linear* if it can be represented as the formal development of a linear difference equation, referred to as the *equivalent difference equation*.

In the present paper, all references to discrete infinite processes will be confined to linear processes. As examples, consider processes of the first order, infinite products and series. The infinite product $\prod_{i=1} p_i$ is a formal development of the first

Table 1-1

A CLASSIFICATION OF LINEAR DISCRETE INFINITE PROCESSES, BASED ON ONE-ONE CORRESPONDENCE OF SUCH PROCESSES WITH "EQUIVALENT" LINEAR DIFFERENCE EQUATIONS HAVING BOUNDARY CONDITIONS APPLICABLE AS $n \to \infty$ $(n = 1,2,\ldots)$

Nature of Equivalent Linear D.E. / D.E. or Process Order m	Homogeneous	Inhomogeneous, Expressing an mth Difference	General Inhomogeneous
m = 1	Infinite Product $R_{n+1} - p_n^{-1} R_n = 0$ $R_1 = \left(\prod_{i=1}^{n} p_i\right) R_{n+1}$	Infinite Series $R_{n+1} - R_n \equiv \Delta R_n = -q_n$ $R_1 = \sum_{i=1}^{n} q_i + R_{n+1}$	Infinite Serduct (or Ascending Continued Fraction) $R_{n+1} - p_n^{-1} R_n = -q_n$ $R_1 = \sum_{l=1}^{n} q_l \left(\prod_{i=1}^{l} p_i\right) + \left(\prod_{i=1}^{n} p_i\right) R_{n+1}$
m = 2	(Decending) Continued Fraction $C_{n+2} + b_n C_{n+1} - a_n C_n = 0$ $R_n \equiv \frac{C_{n+1}}{C_n}$ $\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \propto \left(\prod_{i=1}^{n} P_i\right) \begin{pmatrix} 1 \\ R_{n+1} \end{pmatrix}$, $P_i \equiv \begin{pmatrix} b_i & 1 \\ a_i & 0 \end{pmatrix}$	Second-order Infinite Series $R_{n+2} - 2R_{n+1} + R_n \equiv \Delta^2 R_n = -q_n$ $R_1 = \sum_{i=0}^{n-1} (n-i) q_i + R_{n+1}$ (q_0 arbitrary)	Second-order Serduct $C_{n+2} + b_n C_{n+1} - a_n C_n = -q_n$ (Theory not treated in this report)
m > 2	Generalized Continued Fraction $C_{n+m} + j_n C_{n+m-1} + \cdots + b_n C_{n+1} - a_n C_n = 0$ (See text, equation (8))	mth-order Infinite Series $\Delta^m R_n = -q_n$ $R_1 = \sum_{i=0}^{n-m+1} \binom{n-i}{m-1} q_i + R_{n+1}$ (Note: $\sum_{i=0}^{\alpha} \equiv 0$ for $\alpha < 0$)	mth-order Serduct $C_{n+m} + j_n C_{n+m-1} + \cdots + b_n C_{n+1} - a_n C_n = -q_n$ (Theory little developed)

NOTE: The above classification is limited to single-subscript processes. Multiple processes are not considered in this report.

order, linear, homogeneous difference equation,

$$R_{n+1} - p_n^{-1} R_n = 0, \quad n = 1, 2, \cdots, \qquad (3a)$$

inasmuch as $\quad R_1 = p_1 p_2 \cdots p_n R_{n+1}.$

The infinite series $\sum_{i=1}^{\infty} q_i$ can be developed from the first order, linear, inhomogeneous equation,

$$R_{n+1} - R_n = -q_n, \quad n = 1, 2, \cdots, \qquad (3b)$$

since $\quad R_1 = \sum_{i=1}^{n} q_i + R_{n+1}.$

We postpone consideration of boundary conditions on these difference equations, apart from the remark that such conditions clearly must apply at infinity (i.e., in the limit $n \to \infty$), if the value problem of the formally equivalent infinite process is to be nontrivial.

From the most general first-order linear difference equation,

$$R_{n+1} - p_n^{-1} R_n = -q_n, \qquad (3c)$$
$$n = 1, 2, \cdots,$$

may be developed an infinite process representable either as an ascending continued fraction[2] or as combination of product and series formation (see Table 1-1), for which one might coin the name "serduct." Although theorems on infinite serducts or ascending continued fractions would provide a compact codification of the features of first-order infinite processes, there are practical advantages in retaining the classical distinctions embodied in (3a) and (3b).

Turning to second-order infinite processes, we observe the natural generalization of an infinite product to be a descending continued fraction. The second-order counterpart of (3a) is a linear, homogeneous difference equation,

$$C_{n+2} + b_n C_{n+1} - a_n C_n = 0, \qquad n = 1, 2, \cdots. \tag{4}$$

If division by C_n is permissible, we may bring (4) into the form

$$R_{n+1} R_n + b_n R_n - a_n = 0, \qquad n = 1, 2, \cdots, \tag{5a}$$

where $R_n \equiv C_{n+1}/C_n$. This yields

$$R_n = \frac{a_n}{b_n + R_{n+1}}, \tag{5b}$$

with the formal development embodying downwardly-continuing quotient formation,

$$R_1 = \frac{C_2}{C_1} = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cdots + \cfrac{a_n}{b_n + R_{n+1}}}}. \tag{6}$$

It is useful to note an alternative way of writing (6) patterned on the method of Milne-Thomson (ref (c)); <u>viz</u>.,

$$R_1 = \frac{C_2}{C_1} = \frac{A_n + A_{n-1} R_{n+1}}{B_n + B_{n-1} R_{n+1}}, \tag{7a}$$

where the numerator and denominator on the right are expressible by a continuing product of 2 x 2 matrices,

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \propto \begin{pmatrix} B_n + B_{n-1} R_{n+1} \\ A_n + A_{n-1} R_{n+1} \end{pmatrix} = P_1 P_2 \cdots P_n \begin{pmatrix} 1 \\ R_{n+1} \end{pmatrix}, \tag{7b}$$

$$P_k \equiv \begin{pmatrix} b_k & 1 \\ a_k & 0 \end{pmatrix}, \qquad k, n = 1, 2, \cdots.$$

Here A_n, B_n are the solutions of well-known[3] linear difference equations and (7a) is a standard identity (ref (d), Equation (1.3)). The value of the infinite process is $R_1 = C_2/C_1$. The symbol $\propto$ in Equation (7b) denotes proportionality, which arises because the homogeneity of (4) leaves the C_n determined only within an arbitrary multiplier.

Equation (7b), rather than (6), is the form that generalizes readily to higher-order processes. Just as an infinite product, equivalent to a first-order linear homogeneous difference equation can be viewed as a product of 1 x 1 matrices, an m^{th}-order linear homogeneous difference equation can be considered equivalent to a product of m x m matrices. That is, the difference equation

$$C_{n+m} + j_n C_{n+m-1} + \cdots + b_n C_{n+1} - a_n C_n = 0, \qquad (8a)$$

$$n = 1, 2, \cdots,$$

can be developed into

$$\begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_m \end{pmatrix} \propto P_1 P_2 \cdots P_n \begin{pmatrix} 1 \\ R_{n+1} \\ R_{n+1} R_{n+2} \\ \vdots \\ \prod_{i=1}^{m-1} R_{n+i} \end{pmatrix}, \qquad (8b)$$

where

$$P_i = \begin{pmatrix} b_i & c_i & \cdots & i_i & j_i & 1 \\ a_i & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_i & \cdots & 0 & 0 & 0 \\ 0 & 0 & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_i & 0 & 0 \\ 0 & 0 & \cdots & 0 & a_i & 0 \end{pmatrix}, \qquad R_n \equiv \frac{C_{n+1}}{C_n}. \qquad (8c)$$

The "value" of the infinite process thus symbolized can be considered to be $R_1 = C_2/C_1$, as before, or it might be considered to be the set of initial coefficient ratios, C_{i+1}/C_i, $i = 1, 2, \ldots, m$. This is immaterial, since such higher-order forms are not customarily studied as infinite processes in their own right, but only as intermediaries in the solution of many-term recurrence relations.

Milne-Thomson, (ref (c)), refers to such matrix products as "generalized continued fractions," although Table 1-1 suggests that they could just as aptly be termed generalized infinite products. In all cases, it is the homogeneity of the equivalent linear difference equation that links such processes and distinguishes them from infinite series or serducts and their higher-order generalizations.

To summarize the information in Table 1-1, it is possible through the "difference equation viewpoint" to systematize the classification of linear discrete infinite processes by assigning them the orders of equivalent linear difference equations. Homogeneous equations give rise to infinite products and their generalizations. Inhomogeneous equations expressing m^{th} differences give rise to infinite series and their (trivial) generalizations. General inhomogeneous equations give rise to serducts and their generalizations, with which the present investigation is not concerned. In general, the simpler processes tend to be more interesting, especially for illustrating the concept of terminal summation on which we now wish to focus attention.

The particular forms in which the infinite process value $R_1 = C_2/C_1$, is expressed in Table 1-1 have been chosen to exhibit explicitly n^{th}-stage remainder terms, *e.g.*, R_{n+1} in Equation (6). The infinite sequence R_{n+1}, $n = 1, 2, \ldots,$ of such remainder terms will play a central role in our new approach to summability. We formalize this by the following:

<u>DEFINITION 2</u>. <u>A linear, discrete, finite or infinite process will be referred to as terminated if it exhibits a finite remainder</u>

term[4] or set of remainder terms associated with its last (n^{th}) stage, prior to and throughout the application of any limiting procedure to n.

The field of study of terminated infinite processes will be referred to as terminal summability. Terminated infinite processes are illustrated throughout Table 1-1 wherein a limiting procedure $n \to \infty$ is to be understood in each case. Note that the definition does not require the remainder term to be in any sense "exact." Suitably terminated processes are of greater importance than has been widely recognized. Examples will be given in Section 1 of Chapter 2 to illustrate the types of penalty that can result from failure to consider process terminations.

We shall begin by considering continued fractions, since these have the right degree of complexity to suggest generalizations without obscuring the basic simplicity of the conceptual issues. In a subsequent chapter infinite series will be treated.

Chapter 2

TERMINAL SUMMABILITY OF CONTINUED FRACTIONS

2-1 EXAMPLES: CONTINUED FRACTIONS WITH REMAINDERS

We begin with two examples which illustrate in particularly transparent ways what can happen if the remainder term of a continued fraction (c.f.) is ignored. The significance of these examples lies in the fact that the conventional (or, as we here term it, the Cauchy) definition of the process in question, viz.,

$$\text{"Value" (Cauchy)} = \lim_{n\to\infty} \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cdots + \cfrac{a_n}{b_n + R_{n+1}}}}, \tag{9}$$

where $R_{n+1} \equiv 0$ or ∞ for all n,
involves the ignoring of remainder terms at the definitional level.

EXAMPLE 1. QUADRATIC EQUATION WITH REAL COEFFICIENTS

We noted that a continued fraction is a second-order process. The simplest second-order algebraic process is a quadratic equation,

$$z^2 + bz - a = 0, \quad \text{a,b real.} \tag{10a}$$

We should therefore anticipate the existence of a c.f. development of the roots. In fact

$$z(z+b) = a, \qquad z = \frac{a}{b+z},$$

$$z = \cfrac{a}{b + \cfrac{a}{b + \cdots + \cfrac{a}{b+z}}}. \tag{10b}$$

As long as the remainder term $R_{n+1} = z$, n = 1,2,..., is retained in this formal development, (10b) can represent both roots of the quadratic, wherever they may lie in the complex plane and regardless of the number of process stages n. But, if we convert the right side of (10b) into a conventional c.f. by imposing the definition (9), the possibility of c.f. representation of non-real roots is lost, since the right side of (10b) becomes a pure-real quantity. Only the presence of a non-real remainder term can prevent this loss of representation power. Such an example, trivial though it may be, properly suggests that the study of c.f. remainder sequences other than 0,0,0,... or $\infty,\infty,\infty,\ldots$ is of utmost importance for enhancing the domain of c.f. convergence.

EXAMPLE 2. EIGENVALUE REPRESENTATION.[5]

The Schroedinger radial wave equation for the nonrelativistic hydrogenlike atom,

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \left(\frac{\lambda}{x} - \frac{1}{4}\right)y = 0, \tag{11a}$$

$\lambda > 0$, y bounded in $0 \leq x < \infty$,

is satisfied by the Laguerre polynomials for positive eigenvalues $\lambda = 1,2,3,\ldots$. This is commonly proved by a change of variable, $y = v \exp(-x/2)$, which leads to a two-term recurrence relation. If instead we choose to make a direct series substitution,

$$y = \sum_{n=1}^{\infty} C_n x^{n-1}, \tag{11b}$$

in (11a), we derive a three-term recurrence relation identical with equation (4), wherein

$$a_n \equiv \frac{1}{4(n+1)(n+2)}, \quad b_n \equiv \frac{\lambda}{(n+1)(n+2)}.$$

After formal development, as before, we obtain the terminated c.f. of equation (6). Specifically, we derive the so-called indicial equation in the form

$$-\frac{\lambda}{2} = \frac{C_2}{C_1} = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cdots + \cfrac{a_n}{b_n + R_{n+1}}}}, \tag{11c}$$

where $R_{n+1} = C_{n+2}/C_{n+1}$.

General techniques for evaluating limits of terminated c.f.'s as $n \to \infty$ will be developed later in this chapter and applied to the present problem as an illustration. For immediate purposes no limiting process need be introduced, since we can without difficulty discover exact remainder sequences associated with individual eigenvalues, e.g.,

$$R_{n+1} = \frac{C_{n+2}}{C_{n+1}} = -\frac{1}{2(n+1)} \quad \text{for } \lambda = 1,$$
$$R_{n+1} = -\frac{(n+2)}{2(n+1)^2} \quad \text{for } \lambda = 2, \text{ etc.} \tag{12}$$

These are valid for n = 1,2,... . Thus, there is no question about the capacity of the terminated c.f. in (11c) to represent the eigenvalues of equation (11a). However, if by following dubious practices long common among physicists (e.g., in the treatment of the Mathieu equation -- cf. reference (g)), we replace the terminated c.f. on the right side of (11c) by a conventional c.f., as defined in equation (9), the power to represent eigenvalues is lost. We see this from the fact that for the physically interesting case $\lambda > 0$ the left side of (11c) is negative, whereas the right side, converted to a conventional c.f., is positive, viz.,

$$\lim_{n\to\infty} \cfrac{\frac{1}{4\cdot2\cdot3}}{\frac{\lambda}{2\cdot3} + \cfrac{\frac{1}{4\cdot3\cdot4}}{\frac{\lambda}{3\cdot4} + \cdots + \cfrac{\frac{1}{4(n+1)(n+2)}}{\frac{\lambda}{(n+1)(n+2)} + 0}}} > 0,$$

so the indicial equation has no solution $\lambda > 0$. Thus we have a second example in which "the tail wags the dog" · · i.e., the information relevant to the solution of the problem is contained in a non-disposable remainder term or sequence.

Perhaps because of the dominance of the Cauchy conception of the meaning of an infinite process, which led to equation (9) as the universally accepted definition of "continued fraction" (cf. reference (d)), there exists today no established body of theory concerning the mathematical objects (11c) or their limiting properties as $n \to \infty$. We must therefore develop the theory of terminated c.f.'s and other terminated infinite processes more or less ab initio. In so doing we shall develop a way of "summing" c.f. processes which, as might be expected from these examples, will in general prove more powerful than the Cauchy method.

2-2 EXCEPTIONAL REMAINDER SEQUENCES

To determine under what conditions remainder dominance of the type illustrated by our examples can occur, we introduce the notation

$$F_n(w_n) \equiv \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_n}{b_n + w_n}}} \tag{13a}$$

and observe that according to equation (7a)

$$F_n(w_n) = \frac{A_n + A_{n-1} w_n}{B_n + B_{n-1} w_n}, \qquad n = 1, 2, \cdots, \tag{13b}$$

where the A_n, B_n are as before.[3] In accordance with (9), the conventional or Cauchy value of the associated continued fraction is defined as

$$\lim_{n \to \infty} F_n(0) = \lim_{n \to \infty} F_n(\infty) = L. \tag{14}$$

LEMMA 1. Given that $\lim\limits_{n \to \infty} \frac{A_n}{B_n} = L$, and given any sequence w_n, $n = 1, 2, \ldots,$ such that

$$\lim_{n \to \infty} \left(\frac{\frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}}}{1 + w_n \frac{B_{n-1}}{B_n}} \right) = V;$$

then

$$\lim_{n\to\infty} F_n(w_n) = L + V.$$

The proof is immediate from the formal identity[6]

$$F_n(w_n) = \frac{A_n + A_{n-1} w_n}{B_n + B_{n-1} w_n}$$
$$= \frac{A_{n-1}}{B_{n-1}} + \left(\frac{\frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}}}{1 + w_n \frac{B_{n-1}}{B_n}} \right). \tag{15}$$

Although the lemma is trivial, its consequences are noteworthy. Any sequence of remainders w_1, w_2, ... for which V vanishes produces convergence of the terminated c.f., equation (13a), in the limit $n \to \infty$ to the conventional value L of the (zero-terminated) c.f., equation (9), if the latter exists. The vanishing of V is a very lenient condition. The hypothesis that L exists implies that

$$\frac{A_n}{B_n} \to \frac{A_{n-1}}{B_{n-1}} \quad \text{as } n \to \infty,$$

so the numerator in the expression for V vanishes in the limit. Hence, it is clear that V = 0 and $\lim_{n\to\infty} F_n(w_n) = \lim_{n\to\infty} F_n(0) = L$. We shall designate as exceptional sequences those for which the denominator of the V-expression goes to zero at least as rapidly as the numerator.

DEFINITION 3. A sequence of c.f. remainders w_n, n = 1,2,..., will be termed exceptional if[7]

$$\left(1 + w_n \frac{B_{n-1}}{B_n}\right) = O\left(\frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}}\right) \quad \text{as } n \to \infty. \tag{16}$$

All other remainder sequences are termed nonexceptional.

With the exclusion of exceptional sequences, our lemma indicates that an arbitrary remainder sequence will produce convergence to the conventional (Cauchy) value L of a c.f. whenever that value exists. Thus there is a great natural tendency of the dog to wag the tail. By the same token there is nothing unique or especially advantageous about the sequences 0,0,..., or $\infty,\infty,\ldots$ of the customary definition. As we shall see, in parameter domains where L exists the latter sequences are generally not best from the standpoint of speed of convergence; and where L does not exist they become altogether useless. Whether one's interest derives from a practical desire to improve the computability of infinite process values, or from pure mathematical concern with enhancing domains of process convergence, in either case one's attention is naturally drawn to the possibilities of terminal summability through suitable choices of non-Cauchy remainder sequences.

2-3 ASYMPTOTIC TRACTABILITY

In both the examples of Section 2-1 it was possible to determine exact remainders for finite n, and thus to avoid consideration of limits. In general such exact remainder evaluation will not be feasible, because of the difficulty of obtaining closed-form solutions of the equivalent difference equation, (4) or (5a). However, asymptotically approximate solutions, valid with increasing accuracy as $n\rightarrow\infty$, can be obtained in a very broad class of cases. To identify for working purposes a useful portion of the class, we adduce a concept of asymptotic tractability, applicable not only to continued fractions but to other infinite processes as well. This will be introduced via a somewhat specialized definition of asymptotic expansion.

DEFINITION 4. An infinite sequence of functions $\varphi_i(n)$, i = 1,2,..., will be termed a standard asymptotic sequence if it has the five properties listed below. A function f(n), n = 1,2,..., will be said to possess a standard asymptotic expansion, denoted $f(n) \sim \mathcal{S}(\varphi(n))$ as $n\rightarrow\infty$, if it possesses an asymptotic expansion[8] to an arbitrary number of terms in a set of basis functions forming a subsequence of a standard asymptotic sequence.

(1) Asymptotic sequence property:

$$\varphi_{i+1}(n) = o\big(\varphi_i(n)\big) \quad \text{as } n \to \infty, \quad i = 1, 2, \cdots .$$

(2) Multiplicative group property:

$$\varphi_i(n) \cdot \varphi_j(n) = \varphi_k(n) \quad \text{for all } i, j \text{ and for some } k.$$

(3) Group unit:

$$\varphi_i(n) = 1 \quad \text{for some } i .$$

(4) Reflexive property:

$$\varphi_i(n+\nu) \sim \mathcal{S}(\varphi(n)) \quad \text{as } n \to \infty, \quad i = 1, 2, \cdots,$$

where "$\sim \mathcal{S}(\varphi(n))$" is a generic notation[9] for any standard asymptotic expansion in the $\varphi_j(n)$, $j = 1,2,\ldots,$ and $\nu = \pm 1, \pm 2, \ldots$.

(5) Stability property:

$$\varphi_i(n+\nu) \simeq k_\nu \varphi_i(n) \quad \text{as } n \to \infty, \quad i = 1, 2, \cdots,$$

where "$\simeq$" denotes equality merely of asymptotically dominant terms[10] as $n \to \infty$, $\nu = \pm 1, \pm 2, \ldots,$ and k_ν is a constant independent of n and i.

In this chapter all discussion of asymptotic properties will refer to standard asymptotic expansions, series, or sequences, and the symbol "$\sim$" will refer exclusively to these. We note for later use that by the properties of the basis φ's specified in Definition 4 any linear combination of products of functions $\sim \mathcal{S}(\varphi(n))$ is also $\sim \mathcal{S}(\varphi(n))$ as $n \to \infty$.

DEFINITION 5. A linear discrete infinite process will be termed asymptotically tractable if for any sufficiently large n at least one remainder term R_{n+1} or set of remainder terms associated with its last (n^{th}) stage (prior to or concurrently with taking a limit on n) [R_{n+1} being formally expressible as the ratio of successive coefficients of the equivalent difference equation (4), viz., $R_{n+1} = C_{n+1}/C_n$, n = 1,2,...], exists and possesses a standard asymptotic expansion, $r_{n+1} \sim C_{n+1}/C_n$ as $n \to \infty$. A function f(n), n = 1,2,..., will be termed asymptotically tractable, abbreviated [A], if it possesses a standard asymptotic expansion as $n \to \infty$.

For convenience in treating the case of continued fractions we shall also require for asymptotic tractability that the c.f. be such as to allow an equivalence transformation to the form

$$\frac{\lambda_1}{1 + \frac{\lambda_2}{1 + \cdots}} \quad . \tag{17}$$

This is accomplished by the formal substitution

$$\mathcal{R}_{n+1} \equiv \frac{\mathcal{C}_{n+2}}{\mathcal{C}_{n+1}} = \frac{C_{n+1}}{C_n b_n}, \quad n = 1, 2, \cdots, \tag{18}$$

which converts the equivalent difference equation (4) into

$$\mathcal{C}_{n+2} + \mathcal{C}_{n+1} - \lambda_n \mathcal{C}_n = 0, \quad n = 1, 2, \cdots, \tag{19a}$$

where

$$\lambda_n \equiv \frac{a_n}{b_{n-1} b_n}, \quad b_0 \equiv 1, \quad n = 1, 2, \cdots. \tag{19b}$$

The transformed c.f. with remainder takes the form

$$\mathcal{F}_n(\mathcal{R}_{n+1}) = \cfrac{\lambda_1}{1 + \cfrac{\lambda_2}{1 + \cdots + \cfrac{\lambda_n}{1 + \mathcal{R}_{n+1}}}} \tag{20}$$

$$= \frac{\mathcal{A}_n + \mathcal{A}_{n-1} \mathcal{R}_{n+1}}{\mathcal{B}_n + \mathcal{B}_{n-1} \mathcal{R}_{n+1}}, \quad n = 1, 2, \cdots,$$

where

$$\mathcal{A}_n = \mathcal{A}_{n-1} + \lambda_n \mathcal{A}_{n-2},$$
$$\mathcal{B}_n = \mathcal{B}_{n-1} + \lambda_n \mathcal{B}_{n-2}, \tag{21}$$
$$\mathcal{A}_{-1} = 1,\ \mathcal{B}_{-1} = 0,\ \mathcal{A}_0 = 0,\ \mathcal{B}_0 = 1.$$

Evidently $\mathcal{F}, \mathcal{A}, \mathcal{B}$ are interchangeable with F, A, B in the statements of Lemma 1, Definition 1, etc.

The definition of asymptotic tractability by this restriction acquires the following more specialized form for c.f.'s:

DEFINITION 5a. A continued fraction will be termed asymptotically tractable if it is expressible in the form of equation (17) and if for all sufficiently large n at least one remainder term $\mathcal{R}_{n+1}$ (equation (20)) [equal to the coefficient ratio $\mathcal{C}_{n+2}/\mathcal{C}_{n+1}$, n = 1,2,..., of the equivalent difference (equation (19a)], exists and possesses a standard asymptotic expansion, $\rho_{n+1} \sim \mathcal{C}_{n+2}/\mathcal{C}_{n+1}$ as $n \to \infty$.

Although asymptotic tractability demands that for any n at least one standard asymptotic expansion of $\mathcal{R}_n$ exist, it is not implied by the definition that the same expansion necessarily applies for all n. On the contrary, provision must be made to treat the periodic properties often encountered in c.f.'s.

DEFINITION 6. A function sequence ξ_n, n = 1,2,..., each member of which possesses an asymptotic expansion as $n \to \infty$ will be termed[11] μ- periodic if $\mu > 1$ is the smallest integer such that, for all sufficiently large n, $\xi_{n+\mu} \simeq \xi_n$ as $n \to \infty$. For the case $\mu = 1$ the expansion will be termed nonperiodic. A similar terminology will be applied to continued fractions; i.e., the c.f. of equation (17) will be termed asymptotically tractable of period μ, abbreviated $[A_\mu]$ for $\mu > 1$, or asymptotically tractable nonperiodic, abbreviated $[A_1]$ for $\mu = 1$, if it is asymptotically tractable and if in addition any asymptotic remainder sequence ρ_n, n = 1,2,..., is

μ-periodic or nonperiodic, respectively. That is, $\rho_{n+\mu} \simeq \rho_n$ as $n \to \infty$.[12]

Attention will be confined in this report to c.f.'s $[A_\mu]$ for $\mu \geqslant 1$. This is not a severe restriction, since it appears to include most of the c.f.'s treated in current texts or encountered in classical analysis. We now give criteria by which c.f. asymptotic tractability can be established on the basis of knowledge only of the "given" of the problem, the λ_n, n = 1,2,..., without reference to remainder sequences.

THEOREM 1. Sufficient conditions[13] for the μ- periodic asymptotic tractability $[A_\mu]$, μ= 1,2,..., of the c.f. (17), are that for all sufficiently large n

(a) the quantity λ_n, defined by equation (19b), shall possess a μ-periodic standard asymptotic expansion as $n \to \infty$, i.e.,

$$\lambda_n \sim \mathcal{S}(\varphi(n)) \quad \text{as } n \to \infty,$$
$$\lambda_{n+\mu} \simeq \lambda_n \quad \text{as } n \to \infty, \tag{22}$$

where in general different standard asymptotic expansions in the common set of basis functions $\varphi_i(n)$, i = 1,2,..., are implied for different n-values within a period of length μ; and

(b) the formal expression

$$t_n = \cfrac{\lambda_n}{1 + \cfrac{\lambda_{n+1}}{1 + \dots + \cfrac{\lambda_{n+\mu-1}}{1 + t_n}}}, \tag{23}$$

when converted to a quadratic in t_n by clearing of fractions, shall yield roots, $t_n^{(1)}$ and $t_n^{(2)}$, having dominant terms that satisfy

$$t_n^{(i)} \simeq K^{(i)} \varphi_{k^{(i)}}(n) \quad \text{as } n \to \infty, \; i = 1, 2, \tag{24}$$

where $K^{(i)}$ is some (n-independent) constant and $k^{(i)}$ is an integer designating one of the functions of the same basis set as that in which the λ_n are expansible.

Proof. Consider the formal identity

$$\mathcal{R}_n = \cfrac{\lambda_n}{1 + \cfrac{\lambda_{n+1}}{1 + \cdots \cfrac{\lambda_{n+\mu-1}}{1 + \mathcal{R}_{n+\mu}}}}, \tag{25}$$

(cf. equation (20)) valid for n = 1,2,... . Clearing of fractions, we convert this to an equation of the form

$$\mathcal{R}_{n+\mu}\mathcal{R}_n G_n + \mathcal{R}_{n+\mu} H_n + \mathcal{R}_n J_n + K_n = 0, \quad n = 1, 2, \cdots, \tag{26}$$

where G_n, H_n, J_n, K_n are linear combinations of products of the λ's. We note for later reference that by the group properties of the basis $\varphi(n)$'s, each of these quantities possesses a standard asymptotic expansion; i.e., $G_n \sim \mathcal{S}(\varphi(n))$, $H_n \sim \mathcal{S}(\varphi(n))$, etc., as $n \to \infty$; and since the λ's are μ-periodic, the G_n, H_n, J_n, K_n share this property. The theorem will be proved if we can establish the existence of one or more standard asymptotic expansions of $\mathcal{R}_n$, designated $\rho_n \sim \mathcal{S}(\varphi(n))$ as $n \to \infty$, such that the sequence ρ_n, n = 1,2,..., is μ-periodic; i.e. $\rho_{n+\mu} \simeq \rho_n$ as $n \to \infty$. This in turn will be established if for any n in a period of length μ we can evaluate the undetermined coefficients a_1, a_2,..., in a trial standard asymptotic expansion of the form

$$\rho_n = \sum_{j=1}^{M} a_j \varphi_j(n) + o(\varphi_M(n)) \sim \mathcal{S}(\varphi(n)) \quad \text{as } n \to \infty \tag{27}$$

for arbitrary integral M, such that the sequence ρ_n, n = 1,2,..., is μ-periodic.

Evaluation of a_1: The assumed μ-periodicity of ρ_n is permitted by the stability property of the φ's (property (5) of Definition 2). We have $\rho_{n+\mu} \simeq \rho_n \simeq a_1 \varphi_1(n)$ as $n \to \infty$, since (27) implies the numbering of the $\varphi_i(n)$ to be chosen so that $i = 1$ for the dominant term of ρ_n as $n \to \infty$. On comparing (23) and (25) and noting that $\mathcal{R}_n \simeq \rho_n \simeq \rho_{n+\mu} \simeq \mathcal{R}_{n+\mu}$ as $n \to \infty$, we observe that $\mathcal{R}_n \simeq t_n$ as $n \to \infty$; i.e., t_n is another notation for the dominant term of $\mathcal{R}_n$ as $n \to \infty$. The same comparison shows that t_n obeys a "characteristic equation,"

$$t_n^2 G_n + t_n (H_n + J_n) + K_n \simeq 0 \qquad \text{as } n \to \infty, \tag{28}$$

where G_n, H_n, J_n, K_n are the same functions appearing in (26), but are here conveniently replaceable by their dominant terms or by at most a few leading terms of their asymptotic expansions as $n \to \infty$. There being two roots of (28) that by hypothesis obey $t_n^{(i)} \simeq K^{(i)} \varphi_{k^{(i)}}(n)$ as $n \to \infty$, $i = 1,2$, we derive, in the case of root distinctness, not one but two μ-periodic standard asymptotic expansions of $\mathcal{R}_n$, the dominant terms of which are

$$\rho_n^{(i)} \simeq a_1^{(i)} \varphi_1^{(i)}(n) \simeq t_n^{(i)} \simeq K^{(i)} \varphi_{k^{(i)}}(n) \quad \text{as } n \to \infty, \; i = 1,2. \tag{29}$$

For each of these we evaluate a_1 as $a_1^{(i)} = K^{(i)}$, $i = 1,2$, and determine the dominant basis function in the asymptotic expansion of $\mathcal{R}_n$ to be $\varphi_1^{(i)}(n) = \varphi_{k^{(i)}}(n)$, $i = 1,2$.

Evaluation of a_j, $j>1$: We pick a root ($i = 1$ or 2) and represent all quantities in (26) by their μ-periodic standard asymptotic expansions. For instance, through the use of (27) $\mathcal{R}_n$ and $\mathcal{R}_{n+\mu}$ in (26) are represented by parallel expansions (having the same dominant terms), $\rho_n \sim \mathcal{R}_n$ and $\rho_{n+\mu} \sim \mathcal{R}_{n+\mu}$ as $n \to \infty$. In the special case of nonperiodicity ($\mu=1$), the result is

$$\rho_{n+1} \cdot \rho_n + \rho_n - \lambda_n \sim 0 \qquad \text{as } n \to \infty, \tag{30a}$$

and for $\mu > 1$

$$\rho_{n+\mu} \cdot \rho_n \cdot g_n + \rho_{n+\mu} h_n + \rho_n j_n + k_n \sim 0 \quad \text{as } n \to \infty, \tag{30b}$$

where g_n, h_n, etc., are standard asymptotic expansions of G_n, H_n, etc. By the group and reflexive properties of the expansion basis functions, (30) has the form

$$\mathcal{S}(\varphi(n)) \sim 0 \quad \text{as } n \to \infty , \tag{31}$$

where $\mathcal{S}$ contains one of the undetermined a_2, a_3,..., in association with each successively decreasing order of the φ's following the dominant term as $n \to \infty$. Since a_1 is known, relation (30) or (31) provides recursion relations by which the remaining a's may be evaluated. This establishes for any sufficiently large n the existence of a standard asymptotic expansion (actually in general two such expansions) $\rho_n^{(i)}$ of R_n as $n \to \infty$, i = 1,2, such that each $\rho_n^{(i)}$ -sequence is μ-periodic, q.e.d.

Hypotheses (a) and (b) of Theorem 1, together with the five properties specified in Definition 4, may be regarded as seven working requirements (sufficient but not necessary) to be met by a suitable expansion basis set $\varphi_i(n)$, i = 1,2,... . A single example of the application of these requirements will be given: Suppose $\lambda_n = a_n/(b_{n-1}b_n)$ is nonperiodic, $\lambda_{n+1} \simeq \lambda_n$ as $n \to \infty$, and, let us say, has the form $\lambda_n = A^2 n^\alpha$, $\alpha > 0$, A^2 constant. By nonperiodicity ($\mu = 1$), equation (23) yields

$$t_n = \frac{\lambda_n}{1 + t_n} .$$

The equivalent quadratic, or "characteristic equation" (reference (e)), is

$$t_n^2 + t_n - \lambda_n = 0 , \tag{32a}$$

The roots have dominant terms

$$t_n^{(i)} \simeq \frac{1}{2}\left[-1 \pm \sqrt{1+4\lambda_n}\right] \qquad \text{as } n \to \infty, \qquad (32b)$$

where i = 1 or 2 corresponds to choice of the + or - sign. In our example we suppose $\lambda_n = A^2 n^{\alpha}$; hence $t_n^{(i)} = \pm A\, n^{\alpha/2}$. This and the requirement of group properties suggests that a standard asymptotic sequence, $\varphi_1(n), \varphi_2(n), \ldots$, suited to the needs of the problem consists of powers of n, the exponent being $[m + (k\alpha/2)]$, where $m,k = 0, \pm 1, \pm 2, \ldots$. For such a set each of the seven requirements just mentioned is satisfied. For instance, the reflexive property (4) of Definition 4 is established by binomial expansion; i.e., if $\varphi_i(n) = n^{\beta}$, where $\beta = m + \frac{k\alpha}{2}$, we have $\varphi_i(n+\nu) = (n+\nu)^{\beta} = n^{\beta}(1+\frac{\nu}{n})^{\beta} = n^{\beta}(1+\frac{\beta\nu}{n}+..) = n^{m+\frac{k\alpha}{2}} + \nu\beta n^{(m-1+\frac{k\alpha}{2})} + \ldots \sim \mathcal{S}(\varphi(n)) \simeq \varphi_i(n)$ as $n \to \infty$, which also confirms the stability property (5). The particular $\varphi_{k^{(i)}}(n)$ appearing in (24) in this case has the value $n^{\alpha/2}$ for both roots, and $K^{(i)} = \pm A$ (i=1,2 specifies the sign choice). The function $\varphi_{k^{(i)}}(n)$ is in all cases the dominant member of the relevant φ-sequence; i.e., all other φ's to be considered are $o(\varphi_{k^{(i)}}(n))$ as $n \to \infty$. The coefficients of higher-order terms in the asymptotic expansion of $\rho_n^{(i)}$ are evaluated from (30a).

EXERCISE: How is the above discussion modified when $\alpha \leq 0$?

The seven-requirement method of establishing asymptotic tractability just exemplified is not so restrictive as it might appear. For example, since a_n, b_n appear only in the combination $\lambda_n \equiv a_n/(b_{n-1}b_n)$, it is possible to have b_n proportional to an essentially arbitrary function of n, $b_n \simeq b_{n-1} \simeq f(n)$, provided $a_n \simeq f^2(n)\, g(n)$, and provided the resulting λ_n can be expanded in a series of basis functions related to g(n) that satisfy the stated requirements. We conclude that asymptotic tractability is a workable concept, readily applied in practice to c.f.'s and covering a broad class of embodiments. It doubtless has "growth potential" in respect to liberalization and generalization.

2-4 THE C.F. VALUE PROBLEM

We now give a series of results bearing on the "value" problem of c.f. terminal summation.

LEMMA 2. Given sequences w_n, z_n, $n = 1,2,\ldots,$ such that

$$z_n = w_n + o\left[\Psi(z_n, w_n)\right] \quad \text{as } n\to\infty, \text{ where}$$

$$\Psi(z_n, w_n) \equiv \frac{\mathcal{B}_n}{\mathcal{B}_{n-1}}\left(\frac{\mathcal{A}_n}{\mathcal{B}_n} - \frac{\mathcal{A}_{n-1}}{\mathcal{B}_{n-1}}\right)^{-1}\left(1 + z_n\frac{\mathcal{B}_{n-1}}{\mathcal{B}_n}\right)\left(1 + w_n\frac{\mathcal{B}_{n-1}}{\mathcal{B}_n}\right);$$

then $\mathcal{F}_n(z_n) = \mathcal{F}_n(w_n) + o(1)$ as $n\to\infty$, $\mathcal{F}_n$ being defined by equation (20).

This follows from expressing the difference $[\mathcal{F}_n(z_n) - \mathcal{F}_n(w_n)]$ by means of the identity (15) (transformed to $\mathcal{F}$ instead of F) and examining the limit $n\to\infty$. By this lemma we prove at once

THEOREM 2. Given that $\lim_{n\to\infty} \mathcal{F}_n(\mathcal{C}_{n+2}/\mathcal{C}_{n+1}) = V$, where $\mathcal{F}_n$ is defined by equation (20), and $\mathcal{C}_n$, $n = 1,2,\ldots,$ is an exact solution of the equivalent difference equation (equation (19b)); and given that $\mathcal{C}_{n+2}/\mathcal{C}_{n+1}$ possesses an asymptotic expansion, designated ρ_{n+1}, extended sufficiently far that the remaining (omitted) terms are each

$$o\left[\frac{\mathcal{B}_n}{\mathcal{B}_{n-1}}\left(\frac{\mathcal{A}_n}{\mathcal{B}_n} - \frac{\mathcal{A}_{n-1}}{\mathcal{B}_{n-1}}\right)^{-1}\left(1 + \rho_{n+1}\frac{\mathcal{B}_{n-1}}{\mathcal{B}_n}\right)\left(1 + \frac{\mathcal{C}_{n+2}}{\mathcal{C}_{n+1}}\frac{\mathcal{B}_{n-1}}{\mathcal{B}_n}\right)\right] \quad \text{as } n\to\infty;$$

then $\lim_{n\to\infty} \mathcal{F}_n(\rho_{n+1}) = V$. Conversely, if for such ρ_{n+1} $\lim_{n\to\infty} \mathcal{F}_n(\rho_{n+1}) = V$, then $\lim_{n\to\infty} \mathcal{F}_n(\mathcal{C}_{n+2}/\mathcal{C}_{n+1}) = V$.

The import of this theorem is that an exact solution of the equivalent difference equation need not be available; it suffices to know an "accurate enough" asymptotic approximation ρ_{n+1} to the coefficient ratio $\mathcal{C}_{n+2}/\mathcal{C}_{n+1}$ (which for $n = 1,2,\ldots$ constitutes an exact remainder sequence). In short, asymptotically approximate remainder sequences, if they exist, are as effective for evaluating the terminal sum of a c.f. as are exact remainder sequences.

Asymptotically tractable c.f.'s $[A_\mu]$, $\mu = 1,2,\ldots$, in particular those that satisfy the hypotheses of Theorem 1, meet the requirements of existence and stability ($\rho_{n+\mu} \simeq \rho_n$ as $n\to\infty$) of ρ_n and thus render this class of infinite processes terminally summable, regardless of the exact solubility or insolubility of the equivalent difference equation. We formalize this by the following:

DEFINITION 7. The terminal sum or value V of a μ-periodic (or nonperiodic) asymptotically tractable c.f. $[A_\mu]$, $\mu = 1,2,\ldots$, written

$$V \overset{(T)}{=} \cfrac{\lambda_1}{1 + \cfrac{\lambda_2}{1 + \cdots}}\,, \tag{33}$$

is

$$\begin{aligned} V &\equiv \lim_{n\to\infty} \cfrac{\lambda_1}{1 + \cfrac{\lambda_2}{1 + \cdots + \cfrac{\lambda_n}{1+\rho_{n+1}}}} \\ &= \lim_{n\to\infty} \frac{\mathcal{A}_n + \mathcal{A}_{n-1}\rho_{n+1}}{\mathcal{B}_n + \mathcal{B}_{n-1}\rho_{n+1}} = \lim_{n\to\infty} \mathcal{F}_n(\rho_{n+1}), \end{aligned} \tag{34}$$

where $\mathcal{F}_n, \mathcal{A}_n, \mathcal{B}_n$ are defined by equations (20), (21) and where ρ_{n+1} is any μ-periodic (or nonperiodic) standard asymptotic expansion of the ratio $\mathcal{C}_{n+2}/\mathcal{C}_{n+1}$ of successive solutions of the equivalent difference equation (19a), extended sufficiently far in the sense of Theorem 2.

A formal finite development of (19a),

$$\frac{\mathcal{C}_2}{\mathcal{C}_1} = \cfrac{\lambda_1}{1 + \cfrac{\lambda_2}{1 + \cdots + \cfrac{\lambda_{n-1}}{1 + (\mathcal{C}_{n+1}/\mathcal{C}_n)}}}\,, \quad n = 2,3,\cdots, \tag{35}$$

shows that if the roots of the "characteristic equation" (28), $t_n^{(1)} \simeq \rho_n^{(1)}$ and $t_n^{(2)} \simeq \rho_n^{(2)}$ as $n \to \infty$, are distinct then for n sufficiently large (but finite) the remainder term $(\mathcal{C}_{n+1}/\mathcal{C}_n) \sim \rho_n$ must have two distinct values. Evaluating the right-hand side of (35) by a finite number of algebraic operations, we must therefore in the case of root distinctness find two distinct values of $\mathcal{C}_2/\mathcal{C}_1$, hence of the terminal sum or value V of the process. Where the roots coalesce the terminal values coalesce. Consequently we have the important result:[14]

<u>THEOREM 3</u>. <u>The terminal sum of a c.f. $[A_\mu]$, $\mu = 1,2,\ldots,$ always exists and corresponds to at most two distinct value points in the complex plane</u>.

An example illustrating root coalescence is the c.f.

$$\cfrac{(-\frac{1}{4})}{1 + \cfrac{(-\frac{1}{4})}{1 + \cdots}}$$

which terminally sums to a single value, -1/2, in the complex plane, equal to either of the roots of the characteristic equation, $t^2 + t - (-1/4) = (t + 1/2)^2 = 0$.

Although invariable existence of a "value" for such a broad class of c.f.'s as $[A_\mu]$ may be a worthwhile discovery, still, it might appear that too high a price has been paid in departing from the traditional conception of a c.f. as a univalent process by accepting bi-valuedness. However, this objection is untenable. The recognition of bi-valuedness is in fact the most significant qualitative accomplishment of the theory; for, if we recall the elementary examples given in Section 2-1, both of these clearly illustrated the advantage of bi-valuedness of c.f.'s. Thus, in order to represent both roots of a quadratic equation by a c.f., bi-valuedness cannot be avoided. And in order to accomplish c.f. representation of eigenvalues in the Legendre polynomial problem the existence of a <u>second value</u> of the c.f., different from that yielded by the Cauchy definitional approach, is essential.

Examples could be multiplied indefinitely. It is therefore a confirmation of the validity of the definitional theme embodied in terminal summability that bi-valuedness is allowed by Theorem 2.

2-5 TERMINAL SUM VS. CONVENTIONAL VALUE

There remains the question of the ability of terminal summation to reproduce the conventional value, $L = \lim_{n\to\infty} \mathcal{J}_n(0)$, of a c.f. when the latter exists. The following theorem establishes that if the terminal sum and the conventional value of a c.f. both exist then the two must agree, in the sense that at least one value of the terminal summation process must coincide with the conventional value.

THEOREM 4. Given that $\lim_{n\to\infty} \mathcal{J}_n(0) = \lim_{n\to\infty} \mathcal{J}_n(\infty) = L$ for a c.f. $[A_\mu]$, and given that ρ_{n+1}, $n = 1,2,\ldots$, is a μ-periodic standard remainder sequence associated with a root of the characteristic equation (28); them $\lim_{n\to\infty} \mathcal{J}_n(\rho_{n+1}) = L$ (i.e., the process terminally sums to L) for one root or the other of the characteristic equation, or for both in the case of root coalescence.

Proof. First consider the roots of (28) to be distinct. Under this hypothesis the remainder sequences $\rho^{(i)}_{n+1}$, $i = 1,2$, $n = 1,2,\ldots$, associated with the two roots exhibit dominantly dissimilar behaviors as $n\to\infty$. Consequently, it cannot be true for both $i = 1$ and 2 that $\lim_{n\to\infty} \rho^{(i)}_{n+1}(\mathcal{B}_{n-1}/\mathcal{B}_n) = -1$. That is, by Definition 3, not both $\rho^{(i)}_{n+1}$ and $\rho^{(2)}_{n+1}$ can be exceptional sequences. From Lemma 1 it follows that at least one root must produce convergence to the conventional value L, hypothesized to exist. By an argument already given, root distinctness implies value distinctness of the c.f. Hence one and only one of the two roots produces terminal summation to the value L. Root coalescence can be treated by a limiting procedure. Consider $\rho^{(1)}_{n+1} = \rho^{(2)}_{n+1} + \epsilon(n)$, where $|\epsilon(n)|$ is small for all n. For $\epsilon \neq 0$ the roots remain distinct, so by the above argument the remainder sequence associated with one root or the other must terminally sum the c.f. to L. As $\epsilon \to 0$ the roots coalesce and the process values must finally coincide at L.

Since $0,0,\ldots$, and $\infty,\infty,\ldots$ by Definition 3 exemplify nonexceptional remainder sequences, it is clear that the class C of Cauchy-convergent c.f.'s (those for which $\lim_{n\to\infty} \mathcal{A}_n/\mathcal{B}_n$ or $\lim_{n\to\infty} A_n/B_n$ exists) is contained within the class of all c.f.'s with nonexceptional remainder sequences, which in turn is a sub-class of the class of all c.f.'s with remainder sequences. It cannot be asserted that all of C is contained within the class $[A_\mu]$, $\mu = 1, 2, \ldots$. However, most of it certainly is, and whatever is left out is probably of a somewhat pathological or "concocted" nature. Therefore for practical purposes it can be loosely asserted that whatever is accomplished by the Cauchy convergence of c.f.'s can also be accomplished by terminal summability.

Theorem 3 indicates that terminal summability will in fact accomplish a great deal more, since within the class of continued fractions $[A_\mu]$, $\mu = 1,2,\ldots$, the terminal sum always exists, whereas the Cauchy sum or value often does not. It remains to establish by the following results what will doubtless be intuitively obvious to the reader, that the convergence <u>rate</u> in terminal summability to the Cauchy value exceeds that of Cauchy convergence. It is entirely reasonable that even the crudest asymptotic approximation to a sequence of exact remainders should provide a convergence rate improvement over such still cruder remainder-sequence approximations as $0,0,\ldots$, or $\infty,\infty,\ldots$. Indeed in view of the fact that both of the remainder sequences $\mathcal{R}_n \equiv 0$ and $\mathcal{R}_n \equiv \infty$, $n = 1,2,\ldots$, violate the characteristic formal relation (25) of any μ-periodic c.f. process, which follows from equation (19a) and is ultimately based on no deeper assumption than that a definite c.f. process "value" of some kind exists, the remarkable thing is that the conventional c.f. definition so often yields a quite satisfactory convergence rate. (For this we have the situation revealed by Lemma 1 to thank.)

LEMMA 3. Given sequences w_n and z_n, n= 1,2,..., such that $z_n = w_n + o[\Phi(z_n,w_n)]$ as $n\to\infty$, where $\Phi(z_n,w_n) = w_n(1 + z_n(\mathcal{B}_{n-1}/\mathcal{B}_n))$; then $[\mathcal{F}_n(w_n) - \mathcal{F}_n(z_n)] = o[\mathcal{F}_n(w_n) - \mathcal{F}_n(0)]$ as $n\to\infty$, $\mathcal{F}_n$ being defined by equation (20).

Proof. By the identity of equation (15), with $\mathcal{F}$ for F, we establish that

$$\frac{\mathcal{F}_n(w_n) - \mathcal{F}_n(z_n)}{\mathcal{F}_n(w_n) - \mathcal{F}_n(0)} = -\frac{(z_n - w_n)}{\Phi(z_n, w_n)} .$$

On taking the limit as $n\to\infty$, the right-hand side goes to zero and the stated result follows.

THEOREM 5. Given that $\lim_{n\to\infty} \mathcal{F}_n(0) = L$, where $\mathcal{F}_n$ is defined by equation (20); given that $\lim_{n\to\infty} \mathcal{F}_n(\mathcal{C}_{n+2}/\mathcal{C}_{n+1}) = L$, where $\mathcal{C}_n$, n = 1,2,..., is a solution of the equivalent difference equation, (19b); and given that ρ_{n+1} is an asymptotic expansion of $\mathcal{C}_{n+2}/\mathcal{C}_{n+1}$ (for this same solution) extended sufficiently far that the remaining (omitted) terms are each $o[\frac{\mathcal{C}_{n+2}}{\mathcal{C}_{n+1}}(1 + \rho_{n+1}\frac{\mathcal{B}_{n-1}}{\mathcal{B}_n})]$ as $n\to\infty$; then $[\mathcal{F}_n(\rho_{n+1}) - L] = o[\mathcal{F}_n(0) - L]$ as $n\to\infty$.

Proof. By Lemma 3, with $w_n = \mathcal{C}_{n+2}/\mathcal{C}_{n+1}$, $z_n \equiv \rho_{n+1}$, we have $[\mathcal{F}_n(\mathcal{C}_{n+2}/\mathcal{C}_{n+1}) - \mathcal{F}_n(\rho_{n+1})] = o[\mathcal{F}_n(\mathcal{C}_{n+2}/\mathcal{C}_{n+1}) - \mathcal{F}_n(0)]$ as $n\to\infty$. If $\lim_{n\to\infty} \mathcal{F}_n(\mathcal{C}_{n+2}/\mathcal{C}_{n+1}) = L$, as hypothesized, the fact that $\mathcal{C}_n$ is an exact solution implies that compatible initial conditions on the equivalent difference equation can always be found such that $\mathcal{C}_2/\mathcal{C}_1 = L = \mathcal{F}_n(\mathcal{C}_{n+2}/\mathcal{C}_{n+1})$ identically for all n. The stated result follows.

Theorem 5 establishes that when L exists and when the c.f. is $[A_\mu]$, $\mu = 1,2,\ldots$, the rate of terminal convergence $\mathcal{F}_n(\rho_{n+1}) \to L$

for that solution $t_n \simeq \rho_n$ of equation (28) which converges to L (i.e., for the appropriate root of the characteristic equation) exceeds the rate at which $\mathcal{F}_n(0) \rightarrow L$. However, it should be noted that this superiority is in respect to ultimate rate of convergence only. To the practical calculator this means that, if his requirements in evaluating a Cauchy-convergent c.f. demand a sufficient number of decimal places of accuracy, it will always pay him to use an asymptotic approximation to a remainder term. But, if his accuracy requirements are not great, the extra effort involved in determining such an approximation may not be repaid. On the other hand, if he deals with a Cauchy-divergent c.f. that happens to be $[A_\mu]$, $\mu = 1,2,\ldots$, or if (as in Example 2 of Section 2-1) the Cauchy convergence is to the "wrong" value, for physical or other reasons, the analyst will always find the use of terminal summability methods advantageous. For ready reference we now summarize and recapitulate the systematic computational steps to be employed in the terminal summation of a c.f. $[A_\mu]$.

2-6 C.F. TERMINAL SUMMATION PROCEDURE (SUMMARY)

The following refers to a nonperiodic c.f. $[A_1]$. Modifications for the periodic case will be noted subsequently.

Step 1. (Optional). Conversion to standard form.

It is convenient, but seldom necessary, to begin by casting the c.f. into the standard form

$$\cfrac{\lambda_1}{1 + \cfrac{\lambda_2}{1 + \cdots}}$$

This is done by the substitution $\lambda_n \equiv a_n/(b_{n-1}b_n)$, $n = 1,2,\ldots$, $b_o \equiv 1$.

Step 2. Evaluation and selection of roots.

With λ_n thus determined as an explicit function of n, verify the nonperiodicity of the c.f., which holds if $\lambda_n \simeq \lambda_{n+1}$ as $n \rightarrow \infty$. For the nonperiodic case calculate the characteristic equation roots,

$$t_n \simeq \frac{1}{2}\left[-1 \pm \sqrt{1 + 4\lambda_n}\right] \qquad \text{as } n\to\infty,$$

retaining only the dominant term on the right for each sign choice. Select the root(s) of interest for the problem by adducing any "boundary condition" or "remainder condition" information available to assist in root selection.

<u>Step 3</u>. Preliminary verification of asymptotic tractability.

This requires the use of ingenuity to discover a set of basis functions $\varphi_i(n)$ in which λ_n can be asymptotically expanded. If this step fails, the method fails. In many cases powers of n will serve. The requirements of Definition 4 and Theorem 1 provide guidance. At this stage the verification of asymptotic tractability is only tentative, since full verification awaits completion of step 5, below.

<u>Step 4</u>. Introduction of trial series.

With the expansion basis determined, introduce a trial series for ρ_{n+1}, the n^{th}-stage c.f. remainder term, in the form of a linear combination of the $\varphi_i(n)$ with undetermined constant (n-independent) coefficients. Use the dominant term determined in step 2, with all other terms o(this term) as $n\to\infty$.

<u>Step 5</u>. Evaluation of coefficients.

Insert the trial series for ρ_{n+1} into

$$\rho_{n+1}\cdot\rho_n + \rho_n \sim \lambda_n \qquad \text{as } n\to\infty,$$

where now all terms are represented by their complete asymptotic expansions. Equate coefficients of successive members of the basis set $\varphi_i(n)$ for increasing i, thereby determining as many of the coefficients of the ρ_{n+1} series as computational needs of the problem demand. Success in this step completes the verification of asymptotic tractability. If the series is semi-convergent, always cut it off where the terms cease to decrease in magnitude.

<u>Step 6</u>. Evaluation of c.f.

Compute for increasing values of n the value of V_n, where either

$$V_n = \cfrac{\lambda_1}{1 + \cfrac{\lambda_2}{1 + \cdots + \cfrac{\lambda_n}{1 + \rho_{n+1}}}}$$

or else

$$V_n = \frac{C_2}{C_1}, \quad \text{with}$$

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \propto \begin{pmatrix} 1 & 1 \\ \lambda_1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \lambda_2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \lambda_3 & 0 \end{pmatrix} \cdots \begin{pmatrix} 1 & 1 \\ \lambda_n & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \rho_{n+1} \end{pmatrix}.$$

The choice between these computational techniques is strictly one of convenience. If $V_n \to$ some fixed value for increasing n, the job is done. If not, and n becomes so large that computational errors dominate, return to step 5, evaluate more of the ρ_{n+1} asymptotic series, and try step 6 again. Note that if the ρ_{n+1} series is semi-convergent steps 5 and 6 are in principle interdependent, since the optimum number of terms m to be retained in the series depends on n. However, in practice it seldom incurs a serious penalty to take m smaller than its "optimum" value. In fact, any remainder term at all, even the dominant term alone, often speeds convergence markedly.

<u>Modifications for μ-Periodicity, $\mu > 1$</u>:

For periodic c.f.'s the foregoing steps are modified as follows:

<u>Step 1</u>. Determine the value of μ by observing the minimum number of c.f. stages at which the law of n-dependence of the λ's repeats itself cyclically for all large n; <u>i</u>.<u>e</u>., $\lambda_{n+\mu} \simeq \lambda_n$ as $n \to \infty$.

<u>Step 2</u>. Determine the characteristic equation by clearing of fractions the expression

$$t_n = \cfrac{\lambda_n}{1 + \cfrac{\lambda_{n+1}}{1 + \cdots + \cfrac{\lambda_{n+\mu-1}}{1 + t_n}}} .$$

The result, in which only dominant terms as $n\to\infty$ are significant, is a quadratic,

$$t_n^2 G_n + t_n (H_n + J_n) + K_n \simeq 0 \quad \text{as } n \to \infty,$$

wherein G_n, H_n, J_n, K_n are functions of the λ's, and all quantities can be replaced by their dominant terms or by a few leading terms in their asymptotic expansions as $n\to\infty$ (in case cancellation of dominant terms occurs). Solve for and select roots as before.

Steps 3 and 4. As before.

Step 5. Clear the formal expression[15]

$$\rho_n = \cfrac{\lambda_n}{1 + \cfrac{\lambda_{n+1}}{1 + \cdots + \cfrac{\lambda_{n+\mu-1}}{1 + \rho_{n+\mu}}}}$$

of fractions and introduce the trial series of Step 4 for ρ_n, $\rho_{n+\mu}$, as well as the standard asymptotic expansions of the λ's. The result is of the form

$$\rho_{n+\mu} \cdot \rho_n \cdot g_n + \rho_{n+\mu} h_n + \rho_n j_n + k_n \sim 0 \quad \text{as } n\to\infty,$$

where g_n is a standard asymptotic expansion of the previous G_n, etc. Proceed as before to evaluate undetermined constants.

Step 6. As before. If desired, consistency can be checked by repeating the c.f. terminal value calculation for different n-values within the c.f. period of length μ. That is, with n successively replaced by n + 1, n + 2, ..., n + μ -1, the terminal value should remain invariant, provided the same root of the characteristic equation is chosen in each case. This provides a numerical check on the calculation.

2-7 GENERALIZED CONTINUED FRACTIONS

An m^{th}-order c.f., $m > 2$, as described by equation (8), can be terminally summed by similar methods. A condition for asymptotic tractability is the asymptotic expansibility of all the coefficients a_n, b_n, ..., j_n in a common set of basis functions $\varphi_i(n)$, $i = 1,2,\ldots$. (We could reduce the number of coefficients by one by suitable transformation, but it hardly seems worthwhile in the general case.) The characteristic equation, instead of being quadratic, is of m^{th} degree. From equation (8a) we see that for a nonperiodic generalized c.f. $[A_1]$ it takes the form

$$t_n^m + j_n t_n^{m-1} + \cdots + b_n t_n - a_n \simeq 0, \qquad \text{as } n \to \infty \quad (36)$$

$$t_n \simeq \rho_n \simeq \rho_{n+1} \simeq \cdots \simeq \rho_{n+m-1} \simeq R_{n+m-1} \equiv C_{n+m}/C_{n+m-1};$$

i.e., an m^{th}-degree algebraic equation, having a maximum of m roots. Hence, the infinite process can have associated with its "value" (defined, let us say, as C_2/C_1) as many as m different value points. The choice among these would again have to be made in conformity with any other data of the problem. The computational steps in refining the asymptotic approximation ρ_n parallel those outlined for the case m = 2, and periodicity of the coefficients can be treated in a similar way. The calculations involve repeated m x m matrix products, in conformity with equations (8b), (8c).

2-8 EXAMPLES OF C.F. TERMINAL SUMMABILITY

In Section 2-6 we outlined a 6-step procedure by which any asymptotically tractable c.f. could be evaluated or "terminally summed." We now give several examples of the reduction of this method to practice.

EXAMPLE 1. C. F. REPRESENTATION OF THE ERROR INTEGRAL

A well-known formal representation of the Gaussian error integral (reference (d)), is

$$\int_x^\infty e^{-u^2/2}\,du = \frac{e^{-x^2/2}}{x + G(x)}, \quad x > 0, \tag{37a}$$

where

$$G(x) = \cfrac{1}{x + \cfrac{2}{x + \cfrac{3}{x + \cdots}}} \tag{37b}$$

Applying our method to the terminal summation of this c.f., we have:

Step 1. (Optional). Conversion to standard form.

In the c.f. as given, $a_n = n$, $b_n = x$, $n = 1,2,\ldots$. Hence $\lambda_n \equiv a_n/(b_{n-1}b_n) = n/x^2$. Therefore

$$G(x) = \cfrac{(1/x^2)}{1 + \cfrac{(2/x^2)}{1 + \cfrac{(3/x^2)}{1 + \cdots}}} \tag{38}$$

Step 2. Evaluation and selection of characteristic equation roots.

Since λ_n is nonperiodic ($\lambda_n \simeq \lambda_{n+1}$ as $n\to\infty$), we have for the roots of the characteristic equation

$$t_n \simeq \frac{1}{2}\left[-1 \pm \sqrt{1 + 4\lambda_n}\right] = \frac{1}{2}\left[-1 \pm \sqrt{1 + (4n/x^2)}\right]$$

as $n\to\infty$,

or, retaining only the dominant term,

$$t_n \simeq \pm \frac{n^{1/2}}{x} \tag{39}$$

as $n \to \infty$.

Root selection is simplified in this case by the observation that the integrand is everywhere positive. Hence for any (real) x we require $G(x) > 0$. This selects the root with the + sign in (39), because the use of a negative n^{th}-stage remainder, $\rho_{n+1} \simeq -\frac{n^{1/2}}{x}$, in the terminal sum,

$$G(x) \stackrel{(T)}{=} \lim_{n\to\infty} \cfrac{(1/x^2)}{1 + \cfrac{(2/x^2)}{1 + \cdots + \cfrac{(n/x^2)}{1 + \rho_{n+1}}}} \tag{40}$$

would make $G(x) < 0$. Hence the proper root is $t_n \simeq n^{1/2}/x$ for the case $x > 0$, to which we confine attention here.

Step 3. Preliminary verification of asymptotic tractability.

By the sufficient condition (b) of Theorem 1 the set of expansion basis functions must (in accordance with equation (39)) have as its dominant member $\varphi_1(n) = n^{1/2}$. The remaining φ's are then $\varphi_2(n) = 1$, $\varphi_3 = n^{-1/2}$, $\varphi_4 = n^{-1}$, $\varphi_5 = n^{-3/2}$, etc. Asymptotic tractability $[A_1]$ follows, if we adjoin to the basis set a member $\varphi_0(n) = n$, in order to meet the requirement (condition (a) of Theorem 1) that $\lambda_n \sim \mathcal{S}(\varphi(n))$ as $n \to \infty$.

Step 4. Introduction of trial series.

Let

$$\rho_{n+1} \sim \frac{n^{1/2}}{x}\left[1 + \frac{a_1}{n^{1/2}} + \frac{a_2}{n} + \frac{a_3}{n^{3/2}} + \cdots\right] \tag{41a}$$

as $n \to \infty$.

Expressing ρ_n in the same basis, we have

$$\begin{aligned}\rho_n &\sim \frac{(n-1)^{1/2}}{x}\left[1 + \frac{a_1}{(n-1)^{1/2}} + \frac{a_2}{(n-1)} + \frac{a_3}{(n-1)^{3/2}} + \cdots\right] \\ &\sim \frac{n^{1/2}}{x}\left[1 + \frac{a_1}{n^{1/2}} + \frac{(a_2 - \frac{1}{2})}{n} + \frac{a_3}{n^{3/2}} + \cdots\right]\end{aligned} \tag{41b}$$

as $n \to \infty$.

Step 5. Evaluation of coefficients.

The equation to be satisfied is

$$\rho_{n+1} \cdot \rho_n + \rho_n \sim \lambda_n \quad \text{as} \quad n \to \infty .$$

Introducing the expansions (41), we find

$$\frac{n}{x^2} + \frac{1}{x}\left(1 + \frac{2a_1}{x}\right) n^{1/2} + \frac{1}{x}\left(a_1 + \frac{(2a_2 + a_1^2 - \frac{1}{2})}{x}\right)$$

$$+ \frac{1}{x}\left[a_2 - \frac{1}{2} + \frac{(2a_3 + 2a_1 a_2 - \frac{a_1}{2})}{x}\right] n^{-\frac{1}{2}} + \cdots \sim \lambda_n = \frac{n}{x^2}$$

as $n \to \infty$.

On equating coefficients of successive powers of n, we obtain

Coefficient of n: Identically satisfied.

Coefficient of $n^{1/2}$: $\frac{1}{x}\left(1 + \frac{2a_1}{x}\right) = 0$,

whence

$$a_1 = -\frac{x}{2} .$$

Coefficient of n^0: $\frac{1}{x}\left(a_1 + \frac{(2a_2 + a_1^2 - \frac{1}{2})}{x}\right) = 0$,

whence

$$a_2 = \frac{1}{4}\left(1 + \frac{x^2}{2}\right) .$$

Coefficient of $n^{-1/2}$: $\frac{1}{x}\left[a_2 - \frac{1}{2} + \frac{(2a_3 + 2a_1 a_2 - \frac{a_1}{2})}{x}\right] = 0$,

whence

$$a_3 = \frac{x}{8} .$$

Therefore

$$\rho_{n+1} = \frac{n^{\frac{1}{2}}}{x}\left[1 - \frac{(\frac{x}{2})}{n^{1/2}} + \frac{\frac{1}{4}(1 + \frac{x^2}{2})}{n} + \frac{(\frac{x}{8})}{n^{3/2}}\right]$$

$$+ o(n^{-1}) \quad \text{as } n \to \infty . \tag{42}$$

The expansion could be extended indefinitely.

Step 6. Evaluation of c.f.

We have $G(x) = \lim_{n\to\infty} F_n(\rho_{n+1})$ where

$$F_n(\rho_{n+1}) = \cfrac{(1/x^2)}{1 + \cfrac{(2/x^2)}{1 + \cdots + \cfrac{(n/x^2)}{1 + \rho_{n+1}}}} \quad ,$$

with ρ_{n+1} given by (42). Better results near x = 0 are obtained in this case by undoing our original equivalence transformation (step 1) and going back to the form

$$G(x) \overset{(T)}{=} \lim_{n\to\infty} \cfrac{1}{x + \cfrac{2}{x + \cdots + \cfrac{n}{x + r_{n+1}}}} \quad , \tag{43}$$

where

$$r_{n+1} = x\,\rho_{n+1} = n^{\frac{1}{2}}\left[1 + \frac{\left(\frac{x}{2}\right)}{n^{1/2}} + \frac{\frac{1}{4}\left(1 + \frac{x^2}{2}\right)}{n} + \frac{\left(\frac{x}{8}\right)}{n^{3/2}}\right] + o(n^{-1})$$

as $n \to \infty$,

since in this case the process is well-behaved at x = 0. In fact we have from equation (43) at x = 0

$$G(0) \overset{(T)}{=} \lim_{n\to\infty} \cfrac{1}{0 + \cfrac{2}{0 + \cdots + \cfrac{n}{0 + r_{n+1}}}}$$

$$= \begin{cases} \lim\limits_{\substack{n\to\infty \\ (\text{even})}} \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{(n-1)}{n}\, r_{n+1} & \text{for } n \text{ even} \\ \lim\limits_{\substack{n\to\infty \\ (\text{odd})}} \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{n}{r_{n+1}} & \text{for } n \text{ odd}\,. \end{cases}$$

Using Stirling's formula and retaining only the dominant term of the remainder, $r_{n+1} \simeq n^{1/2}$ as $n \to \infty$, we find for n even

$$G(0) \overset{(T)}{=} \lim_{\substack{n\to\infty \\ (\text{even})}} \frac{\sqrt{2\pi n}\; n^{n}\, e^{-n} \sqrt{n}}{\left[2^{n/2}\sqrt{\pi n}\left(\frac{n}{2}\right)^{n/2} e^{-n/2}\right]^2} = \sqrt{\frac{2}{\pi}}\,, \tag{44}$$

with a similar result for n odd. Thus G(x) is terminally summed to the proper value to agree with the integral in equation (37a) for $x \gg 0$, and the rate of convergence near $x = 0$ is excellent. By contrast the Cauchy definition yields an expression for G(x),

$$G(x) \overset{(\text{Cauchy})}{=} \lim_{n\to\infty} \cfrac{1}{x + \cfrac{2}{x + \cdots + \cfrac{n}{x}}}\,,$$

that does not exist at $x = 0$ and is slowly convergent for x small.

EXAMPLE 2. A PERIODIC C.F.

Another integral representation from reference (d) is related to the exponential integral:

$$I(z) \equiv \int_0^{\infty} \frac{e^{-u}\,du}{z+u} = \cfrac{1}{z + \cfrac{1}{1 + \cfrac{1}{z + \cfrac{2}{1 + \cfrac{2}{z + \cfrac{3}{1 + \cdots}}}}}} \tag{45a}$$

(z not on the negative real axis).

Step 1. (Optional). Conversion to standard form.

$$I(z) = \cfrac{(1/z)}{1 + \cfrac{(1/z)}{1 + \cfrac{(1/z)}{1 + \cfrac{(2/z)}{1 + \cfrac{(2/z)}{1 + \cfrac{(3/z)}{1 + \cdots}}}}}} \tag{45b}$$

In the c.f. on the right side $\lambda_1 = 1/z$; $\lambda_n = n/2z$ for $n = 2,4,6,\ldots$; $\lambda_n = (n-1)/2z$ for $n = 3,5,7,\ldots$. Technically, this transformed c.f.is $[A_1]$, but we shall treat it as $[A_2]$ to show the versatility of the method.[11]

Step 2. Evaluation and selection of roots.

Suppose n is even. Then $\lambda_n = \frac{n}{2z}$, $\lambda_{n+1} = \frac{n}{2z}$. Hence, from equation (23), for $\mu = 2$, the roots of the characteristic equation are

$$t_n = \frac{1}{2}\left[-1 \pm \sqrt{(\lambda_{n+1} - \lambda_n + 1)^2 + 4\lambda_n}\right]$$

$$= \frac{1}{2}\left[-1 \pm \sqrt{1 + \frac{2n}{z}}\right] \simeq \pm\sqrt{\frac{n}{2z}}$$

as $n \to \infty$.

The same result holds if n is odd; however, this is somewhat coincidental. In more general cases (e.g., equation (45a) untransformed) the dominant terms for even- and odd-n remainder sequences can differ. For $z > 0$, $I(z) > 0$ and the positive root is evidently the one to select; i.e.,

$$\rho_n \simeq \sqrt{\frac{n}{2z}} \quad \text{for} \quad z > 0. \tag{46}$$

For $z < 0$ the roots become conjugate imaginaries and other considerations must enter[16] if the divergent integral is to be evaluated by terminal summation. We limit attention here to the case of real $z > 0$. (Cf. the discussion in Chapter 3 of an "equivalent" infinite series.)

Step 3. Preliminary verification of asymptotic tractability.

As before, a set of integral and half-integral powers of n will serve as the trial expansion basis.

Step 4. Introduction of trial series.

Consider n even and let

$$\rho_n \sim \frac{n^{\frac{1}{2}}}{\sqrt{2z}}\left[1 + \frac{a_1}{n^{1/2}} + \frac{a_2}{n} + \frac{a_3}{n^{3/2}} + \cdots\right] \tag{47a}$$

as $n \to \infty$.

From this we find

$$\rho_{n+2} \sim \frac{n^{1/2}}{\sqrt{2z}}\left[1 + \frac{a_1}{n^{1/2}} + \frac{(a_2+1)}{n} + \frac{a_3}{n^{3/2}} + \cdots\right] \tag{47b}$$

as $n \to \infty$.

Step 5. Evaluation of coefficients.

Equation (25), with $\rho_n \sim \mathcal{R}_n$, $\rho_{n+1} \sim \mathcal{R}_{n+1}$, yields formally

$$\rho_n \sim \cfrac{\lambda_n}{1 + \cfrac{\lambda_{n+1}}{1 + \rho_{n+2}}}$$

as $n \to \infty$,

where ρ_n, ρ_{n+2} are expansions of the same function.[11] This is best re-written in a form cleared of fractions before the asymptotic expansions are introduced. That is,

$$\rho_{n+2} \cdot \rho_n + \rho_n(1 + \lambda_{n+1}) \sim \lambda_n + \lambda_n \rho_{n+2} \quad \text{as } n \to \infty. \tag{48}$$

For n even, $\lambda_n = \frac{n}{2z}$, $\lambda_{n+1} = \frac{n}{2z}$. Hence we find on equating coefficients of successive powers of n

$$\rho_n \sim \left(\frac{n}{2z}\right)^{\frac{1}{2}}\left[1 + \frac{\left(\frac{1-2z}{2\sqrt{2z}}\right)}{n^{1/2}} + \frac{\left(\frac{z}{4} - \frac{1}{2} - \frac{1}{16z}\right)}{n} + \cdots\right] \tag{49a}$$

as $n \to \infty$, n even.

Similarly, for n odd, $\lambda_n = \frac{(n-1)}{2z}$, $\lambda_{n+1} = \frac{(n+1)}{2z}$, and (48) yields

$$\rho_n \sim \left(\frac{n}{2z}\right)^{\frac{1}{2}}\left[1 - \frac{\left(\frac{1+2z}{2\sqrt{2z}}\right)}{n^{1/2}} + \frac{\left(\frac{z}{4} - \frac{1}{2} + \frac{3}{16z}\right)}{n} + \cdots\right] \tag{49b}$$

as $n \to \infty$, n odd.

Step 6. Evaluation of c.f.

We have

$$I(z) \overset{(T)}{=} \lim_{n\to\infty} \cfrac{\lambda_1}{1+\cfrac{\lambda_2}{1+\cdots+\cfrac{\lambda_{n-1}}{1+\rho_n}}},$$

where ρ_n is given by (49a) or (49b), depending on whether n is even or odd. Calculation confirms that, whether the limit is taken through even or odd values of n, the c.f. terminally sums to the value of the integral,

$$I(z) = -e^{z} Ei(-z)$$

(cf. reference (j)), for z real and positive. The "value" $+\infty$ is obtained for z = 0, so no improvement in range of convergence is offered by terminal summation over Cauchy convergence in this example. More interesting results might be obtained for $z < 0$, but this requires a different treatment[16] of the remainder and has not been investigated.

EXAMPLE 3. REVIEW OF INTRODUCTORY EXAMPLES

Example 1 of Section 2-1, the quadratic equation, is too trivial to need attention, but skeptics may wish to try the 6-step method on it. Since the characteristic equation of any (descending) c.f. is quadratic, one is using a quadratic to solve a quadratic -- but at least consistency can be established.

Example 2 of Section 2-1, the eigenvalue representation problem, has been treated in reference (f). Consequently we merely summarize results here.

Step 1. (Optional). Conversion to standard form.

This step will be omitted, to illustrate the procedure in dealing with the unmodified c.f.,

$$F_n(r_{n+1}) = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_n}{b_n + r_{n+1}}}}, \tag{50}$$

where

$$a_n = \frac{1}{4(n+1)(n+2)}, \qquad b_n = \frac{\lambda}{(n+1)(n+2)}, \qquad \lambda > 0.$$

There being no periodicity of the coefficients, the c.f. is $[A_1]$.

<u>Step 2</u>. Evaluation and selection of roots.

From

$$R_n = \frac{a_n}{b_n + R_{n+1}}, \qquad R_n \sim r_n$$

as $n \to \infty$,

we derive

$$r_{n+1} \cdot r_n + b_n r_n - a_n \sim 0 \quad \text{as } n \to \infty, \tag{51}$$

which for nonperiodic remainder sequences,

$$r_n \simeq r_{n+1} \simeq t_n, \tag{52}$$

yields the characteristic equation

$$t_n^2 + b_n t_n - a_n \simeq 0 \quad \text{as } n \to \infty. \tag{53}$$

Using dominant terms, $a_n \simeq \frac{1}{4n^2}$, $b_n \simeq \frac{\lambda}{n^2}$, we derive

$$t_n \simeq \frac{1}{2}\left[-b_n \pm \sqrt{b_n^2 + 4a_n}\right]$$

$$\simeq \frac{1}{2}\left[-\frac{\lambda}{n^2} \pm \sqrt{\frac{\lambda^2}{n^4} + \frac{1}{n^2}}\right] \simeq \pm\frac{1}{2n}$$

as $n \to \infty$.

Since we have to satisfy the indicial equation (12), viz., $F_n(r_{n+1}) = -\frac{\lambda}{2}$, $\lambda > 0$, we must choose r_{n+1} to be negative. Hence

$$r_{n+1} \simeq r_n \simeq t_n \simeq -\frac{1}{2n} \qquad \text{as } n \to \infty, \quad (54)$$

is the proper root to yield eigenvalues. (The Cauchy value of the c.f. exists and corresponds to the "wrong root," $+\frac{1}{2n}$.)

Steps 3-5.

We omit discussion, since the calculations, based on putting a trial series into (51), are straightforward. The result is

$$r_{n+1} = -\frac{1}{2n}\left[1 + \frac{(\lambda-2)}{n} + \frac{\left(4 - \frac{7\lambda}{2} + \frac{\lambda^2}{2}\right)}{n^2} + \frac{(-8 + 9\lambda - 2\lambda^2)}{n^3}\right] + o(n^{-4}) \qquad (55)$$

as $n \to \infty$.

Step 6. Calculation of c.f.

By using (55) in the indicial equation and solving the resulting transcendental equation for its λ-roots, as many eigenvalues can be numerically approximated as need be. The simplest check on this is to expand the exact remainders given in equation (11c) in inverse powers of n and to verify agreement with (55) when λ is assigned its appropriate eigenvalue.

The method of terminal summation in this case does not yield exact eigenvalues, nor does it yield the whole infinite eigenvalue spectrum by finite amounts of numerical calculation. But it does yield the important lowest eigenvalues to any required degree of numerical accuracy. We do not offer it as competition for the normal method of treating Laguerre polynomials, but have shown that at least the c.f. capacity to represent eigenvalues is provided by terminal summability, whereas it definitely is not provided by the conventional definition of a c.f. in this instance.

Chapter 3

TERMINAL SUMMATION OF INFINITE SERIES

3-1 HEURISTIC DISCUSSION

The terminal summability methods developed for continued fractions and their generalizations in Chapter 2 can be applied with minor alterations to the terminal summation of infinite series. We shall denote the value so obtained by

$$V \overset{(T)}{=} a_1 + a_2 + a_3 + \cdots$$

or

$$V \overset{(T)}{=} \sum_{n=1}^{\infty} a_n \, . \tag{56}$$

The main difference from the previous case is that V is now univalent, rather than bivalent, since the equivalent difference equation for the process, which we shall write as

$$S_n - S_{n-1} = a_n \, , \quad n = 1, 2, \cdots, \tag{57a}$$

where

$$S_n \equiv \sum_{i=1}^{n} a_i \, , \tag{57b}$$

or alternatively as

$$(-R_{n+1}) - (-R_n) = a_n \, , \quad n = 1, 2, \cdots, \tag{57c}$$

where

$$R_{n+1} \equiv \sum_{i=n+1}^{\infty} a_i \, , \tag{57d}$$

is of the first order.

By adding μ equations of the form (57a) with successive replacements of n by n + 1, we obtain the form appropriate to the case of μ-periodicity $(a_{n+\mu} \simeq a_n \text{ as } n \to \infty)$ of the infinite series; viz.,

$$S_{n+\mu-1} - S_{n-1} = \sum_{i=n}^{n+\mu-1} a_i , \qquad (57e)$$

$$\mu = 1, 2, \cdots , \qquad n = 1, 2, \cdots .$$

The quantity a_n now plays the role previously played by λ_n. The already-developed concepts of standard asymptotic sequence and μ-periodicity can be adopted without alteration. However, considerable liberalization of the concept of asymptotic tractability is possible. In particular, we can allow the addition to any standard asymptotic expansion of R_n or S_n of a function $\psi(n)$, not in general a member of the $\varphi_i(n)$ group, having the property

$$\psi(n+\nu) \sim \psi(n) + \mathcal{L}(\varphi(n)) \quad \text{as } n \to \infty, \qquad (58a)$$

$$\nu = \pm 1, \pm 2, \cdots ,$$

since such a function subtracts out from equation (57a) or (57c). An example of a function having this property is $\psi(n) = \log_e n$, where the $\varphi_i(n)$ are inverse powers of n. Further, we can allow the asymptotic expansions of a_n and of S_n or R_n to be of the form of a standard asymptotic expansion multiplied by a function $\theta(n)$, not in general a member of the $\varphi_i(n)$ group, having the property

$$\theta(n+\nu) \sim \theta(n) \cdot \mathcal{L}(\varphi(n)) \quad \text{as } n \to \infty \qquad (58b)$$

$$\nu = \pm 1, \pm 2, \cdots ,$$

since such a function cancels from equation (57a) or (57c) by division. An example of a function having this property is $\theta(n) = n!$, where the $\varphi_i(n)$ are powers of n. Modification (58b) can apply also to infinite serducts, whereas (58a) is unique to infinite series.

In the discussion that follows we shall sketch the theme of development, insofar as it differs from that of c.f.'s, and in subsequent sections will give a few formal results, followed by examples. Attention will be confined in this section to the nonperiodic case.

Comparing Equations (57a) and (57c), we observe that $(-R_{n+1})$ and S_n obey the same difference equation. Since any solution of this equation is determinate only within an additive constant, it follows that $(-R_{n+1})$ and S_n differ at most by such a constant. If we assume asymptotic tractability and introduce asymptotic expansions,

$$R_{n+1} \sim \rho_{n+1}, \qquad S_n \sim \sigma_n \quad \text{as } n \to \infty, \tag{59a}$$

both in the same $\varphi_i(n)$ expansion basis, we can express these quantities alternatively as

$$R_{n+1} \sim \bar{\rho}_{n+1} + B, \qquad S_n \sim \bar{\sigma}_n + A \quad \text{as } n \to \infty, \tag{59b}$$

where A, B are for the moment arbitrary constants and the bars denote elimination of constant terms from the corresponding asymptotic expansions. That is, $\bar{\rho}_{n+1}$ is the same as ρ_{n+1} except for the deletion of any term in the expansion that does not depend explicitly on n. Since $(-R_{n+1})$ and (S_n) differ at most by a constant, the same must be true of their asymptotic expansions. Consequently, because a common expansion basis is used,

$$-\bar{\rho}_{n+1} = \bar{\sigma}_n, \qquad n = 1, 2, \cdots . \tag{59c}$$

The "value" of the infinite process (56), if we assume it (however defined) to <u>exist</u>, must formally be equal to

$$\text{Value} = V = S_n + R_{n+1}, \tag{60}$$

for finite n, or to an equivalent limiting expression if asymptotic expansions are introduced. From (59b) and (60) we obtain

$$V \sim \bar{\sigma}_n + \bar{\rho}_{n+1} + A + B \quad \text{as } n \to \infty,$$

or from (59c),

$$V \sim A + B, \tag{61}$$

where "$\sim$" can be replaced by an equality sign.

It remains to evaluate the constants A, B. Here it is necessary to introduce an arbitrary definition. We <u>define</u> the value of the process (56) to be that given by the Cauchy definition when the latter value exists. The Cauchy definition is

$$V \overset{(\text{Cauchy})}{=} \lim_{n \to \infty} S_n . \tag{62}$$

This value exists only when $R_{n+1} \to 0$ as $n \to \infty$ (as follows from (60)). By equation (59b), this means $\bar{\rho}_{n+1} \to 0$ as $n \to \infty$ and also

$$B = 0. \tag{63}$$

We shall take it as a matter of definition that (63) holds even when the Cauchy value V of the process, given by (62), does not exist. Hence (61) yields

$$V = A. \tag{64}$$

From (59b) we evaluate A numerically by means of

$$V = A \sim S_n - \bar{\sigma}_n \quad \text{as } n \to \infty, \tag{65a}$$

or

$$V \overset{(T)}{=} \lim_{n \to \infty} \left[S_n - \bar{\sigma}_n \right]. \tag{65b}$$

We can now cast aside the heuristic scaffolding by which we arrived at (65b) and take that equation as our definition of the terminal sum of (56). To use (65b) for numerical evaluation of V (whether or not a value exists by the Cauchy definition), we pick a suitably large value of n, evaluate the exact sum S_n from Equation (57b), and subtract off the asymptotic expansion of S_n, modified by deletion of any constant term. That is, we evaluate

$$V_n \equiv \sum_{i=1}^{n} a_i - \bar{\sigma}_n , \qquad (65c)$$

repeat this for n replaced by n + 1, and continue until stabilization occurs, $V_n \rightarrow V$.

Such a "value" will in general exist for all asymptotically tractable infinite series, because any divergence depending on n is subtracted out, by the definition (65b). It can happen that the magnitude of the value so obtained will exceed any preassigned number, but this in general occurs only near singular points in the range of some parameter. The property of asymptotic tractability ensures the existence almost everywhere of a finite terminal sum of the process. The present method of eliminating divergences by subtracting them out repeatedly in the course of a limiting process is therefore a much more powerful summability technique than any based on the modification of finite summands. However, we shall see that the technique has its peculiarities, in that there can exist certain "points of ambiguity," not necessarily singularities of any parameter, at which the terminal sum is not unique. We shall illustrate this phenomenon and attempt a qualitative elucidation in Section 3-5.

The necessary adaptations to treat the case of μ-periodicity, $a_{n+\mu} \simeq a_n$ as $n \rightarrow \infty, \mu = 1, 2, \cdots$, are readily accomplished with the help of Equation (57e). We then perceive that the terminal sum of an infinite series, wherein periodicities are synthetically introduced by including groups of zeros, is invariant under any such modification. This is in strong distinction to most other types of summability. In this particular respect terminal summation most nearly resembles Cauchy summation.

If (65b) is rewritten by means of (59c) in the form of a terminated infinite process (Definition 2),

$$V \overset{(T)}{=} \lim_{n \rightarrow \infty} \left[S_n + \bar{\rho}_{n+1} \right] , \qquad (65d)$$

we see that our present results are consistent with those of Chapter 2. That is, an asymptotic approximation to a remainder term, representing an imaged point, is included at each step in a double limiting process on transformation index and imaged point sequence. This is to be compared, for instance, with the Cauchy definition, Equation (2) of Chapter 1. Equation (65d) could have been written down by inspection, except that the role of the arbitrary additive constant, undetermined by the equivalent difference equation (57a), had to be clarified. The fixing of this constant, to repeat, was accomplished by requiring that whenever an asymptotically tractable infinite series possesses a Cauchy value V, the terminal sum of the process shall be V. Since the definition of terminal value has thus been contrived to agree with the Cauchy value when the latter exists, we shall need no separate theorem establishing the "regularity" (reference (b)) of our summability definition.

3-2 FORMAL RESULTS

In this section we present more formally some of the matters outlined in the preceding section. The definitions of standard asymptotic sequence and expansion (Definition 4) remain unaltered. However, we find it convenient in dealing with infinite series to introduce a more general type of asymptotic expansion.

DEFINITION 8. A function f(n), n = 1, 2, ..., will be said to possess a broad asymptotic expansion if it possesses an asymptotic expansion expressible either (1) as the product of a standard asymptotic expansion and a function $\theta(n)$ obedient to Equation (58b), or (2) as the sum of a standard asymptotic expansion and the product of any constant c and a function $\psi(n)$ obedient to Equation (58a). That is, either

$$(1)\quad f(n) \sim \theta(n)\cdot \mathcal{S}(\varphi(n)) \quad \text{as } n\to\infty, \qquad (66a)$$

or

$$(2)\quad f(n) \sim c\psi(n) + \mathcal{S}(\varphi(n)) \quad \text{as } n\to\infty, \qquad (66b)$$

where

$$\psi(n+\nu) \sim \psi(n) + \mathcal{S}(\varphi(n))$$
$$\theta(n+\nu) \sim \theta(n)\cdot \mathcal{S}(\varphi(n))$$
as $n\to\infty$, $\nu = \pm 1, \pm 2, \cdots$.

Here $\mathcal{S}(\varphi(n))$ denotes any standard asymptotic expansion[17] in basis functions $\varphi_i(n)$, $i = 1, 2, \ldots$, having the properties enumerated in Definition 4. Equations (66a) and (66b) will be referred to as expansion method 1 and method 2, respectively.

In this chapter all discussion of asymptotic properties will refer to broad asymptotic expansions, by one or the other or both expansion methods, and the symbol "$\sim$" will refer to these or to standard asymptotic expansions, which are a special case ($c = 0$, $\theta(n) = 1$). We now extend the concept of asymptotic tractability in a similar way.

DEFINITION 9. An infinite series will be termed broadly asymptotically tractable if for all sufficiently large n a remainder term R_{n+1} (Equations (57c), (60)) exists and possesses a broad asymptotic expansion, $\rho_{n+1} \sim R_{n+1}$ as $n \to \infty$. A function f(n), n = 1, 2, ..., will be termed broadly asymptotically tractable, abbreviated $\{A\}$, if it possesses a broad asymptotic expansion.

The definition (Definition 6) of a μ-periodic function sequence requires no modification.[11] On the basis of our introductory discussion we note that the existence of $\rho_{n+1} \sim R_{n+1}$ as $n \to \infty$ implies the existence of $\sigma_n \sim S_n$ as $n \to \infty$, and indeed $\sigma_n = -\rho_{n+1} + \text{constant}$, if the same expansion basis functions are used for both. Hence $-\bar{\sigma}_n$ and $\bar{\rho}_{n+1}$ are freely interchangeable in any of the following discussion, where a bar denotes deletion of any constant (n-dependent) term in an expansion.

DEFINITION 10. An infinite series will be termed broadly asymptotically tractable of period μ, abbreviated $\{A_\mu\}$ for $\mu > 1$, or broadly asymptotically tractable nonperiodic, abbreviated $\{A_1\}$ for $\mu = 1$, if it is broadly asymptotically tractable and if the asymptotic remainder sequence ρ_{n+1}, n = 1, 2, ..., is μ-periodic or nonperiodic, respectively; i.e., $\rho_{n+\mu} \simeq \rho_n$ as $n \to \infty$.

We now state criteria by which broad asymptotic tractability of an infinite series can be established on the basis of knowledge only of the summands a_n, n = 1, 2, ..., without reference to remainder sequences.

<u>THEOREM 6</u>. <u>Sufficient conditions for the μ-periodic broad asymptotic tractability $\{A_\mu\}$, μ = 1, 2, ..., of the infinite series</u>

$$\sum_{n=1}^{\infty} a_n ,$$

<u>except possibly at isolated singularities in the ranges of parameters contained in a_n, are that for all sufficiently large n (a) the sequence a_n, a_{n+1}, a_{n+2}, ..., shall be μ-periodic; i.e., $a_{n+\mu} \simeq a_n$ as $n \to \infty$; and (b) each a_n shall possess a broad asymptotic expansion by method 1 of Definition 8 [Equation (66a)] in a common set of basis functions $\varphi_i(n)$, i = 1, 2, ..., with a common multiplier $\theta(n)$ conforming to Equation (58b).</u>

<u>Proof</u>. The stated conditions imply that for all sufficiently large n the function

$$f(n) \equiv \sum_{i=n}^{n+\mu-1} a_i$$

possesses a μ-periodic broad asymptotic expansion as $n \to \infty$, in which no additive ψ-term appears; <u>i.e.</u>,

$$f(n) \sim \theta(n) \cdot \mathcal{S}(\varphi(n)) \quad \text{as } n \to \infty, \tag{67}$$

(where θ may be unity), and

$$f(n+\mu) \simeq f(n) \quad \text{as } n \to \infty.$$

Introduce a trial asymptotic expansion of S_n with undetermined coefficients. As determined by the specific form of a_n, this may conform to either expansion method 1 or method 2 of Definition 8. In the case $\theta(n) \neq$ constant, it is necessary to use method 1:

$$S_n \sim \sigma_n \sim \theta(n) \cdot \mathcal{S}(\varphi(n)) \quad \text{as } n \to \infty,$$

where θ is the same function appearing in Equation (67).

Replacing all quantities in Equation (57e) formally by their asymptotic expansions, we obtain a relationship of the form

$$\theta(n+\mu)\cdot \mathcal{S}(\varphi(n+\mu)) - \theta(n)\cdot \mathcal{S}(\varphi(n)) \sim f(n) \sim \theta(n)\cdot \mathcal{S}(\varphi(n)) \quad \text{as } n \to \infty . \tag{68}$$

By (58b) and the group and reflexive properties of the φ's, this reduces (for $\theta \neq 0$) to an equivalence of two standard asymptotic expansions in the same set of basis functions,

$$\mathcal{S}(\varphi(n)) \sim \mathcal{S}(\varphi(n)) \quad \text{as } n \to \infty , \tag{69}$$

which can in general be satisfied if the undetermined coefficients on the left obey recurrence relations. Thus a broad asymptotic expansion σ_n usually exists, although it can fail to exist at singular points of parameters that may enter the a_n. In the special case $\theta(n) = 1$, if expansion method 1 fails, there is an option to use method 2, which introduces an extra adjustable constant. Thus in method 2 we use a trial series of the form

$$\sigma_n \sim c\psi(n) + \mathcal{S}(\varphi(n)) \quad \text{as } n \to \infty ,$$

where c and the coefficients in $\mathcal{S}$ are initially undetermined . Introducing this into (57e), as before, we obtain

$$c\cdot\psi(n+\mu) + \mathcal{S}(\varphi(n+\mu)) - c\psi(n) - \mathcal{S}(\varphi(n)) \sim \mathcal{S}(\varphi(n)) \quad \text{as } n \to \infty ,$$

which with the help of (58a) reduces again to the form (69). By either of the methods just outlined, except possibly at particular points of a parameter range, σ_n and $\bar{\sigma}_n$, hence ρ_n and $\bar{\rho}_n$, possess broad asymptotic expansions as $n \to \infty$. These, by virtue of the μ-periodicity of f(n), are μ-periodic, q.e.d. We note in passing that μ-periodicity, $\sigma_{n+\mu} \simeq \sigma_n$ as $n \to \infty$, implies that in the asymptotic form of (57e), viz., $\sigma_{n+\mu} - \sigma_n \sim f(n)$, the dominant terms on the left cancel by subtraction, so that $f(n) = o(\sigma_n)$ as $n \to \infty$, as would be expected for a process of summation or integration.

DEFINITION 11. The terminal sum or value V of a μ-periodic (or nonperiodic) broadly asymptotically tractable infinite series $\{A_\mu\}$, $\mu = 1, 2, \ldots$, written

$$V \overset{(T)}{=} a_1 + a_2 + a_3 + \cdots = \sum_{i=1}^{\infty} a_i \, ,$$

is

$$V \equiv \lim_{m \to \infty} \left[\sum_{i=1}^{m} a_i - \bar{\sigma}_m \right] ,$$

where $\bar{\sigma}_m$ is a μ- periodic (or nonperiodic) broad asymptotic expansion of the function $S_n \equiv \sum_1^m a_i$, from which any constant (n-independent) term has been deleted. The expansion must be carried sufficiently far for each n-value, n = 1,2,..., that the sequence of first omitted terms (i.e., of dominant error terms) approaches zero as $n \to \infty$.

The proviso about carrying the $\bar{\sigma}_n$ expansion "sufficiently far" means that, if $\lim_{n\to\infty} |\theta(n)\, \varphi_i(n)| = \infty$ for all i, the minimum number of terms m to be retained must increase with n. In fact m = m(n) must in this case approach infinity more rapidly than n does, in such a manner that the product of $\theta(n)$ and the first omitted expansion term $\to 0$ as $n \to \infty$. The latter requirement can always be met, by virtue of the hypothesized Cauchy convergence[17] of the series $\mathcal{S}(\varphi(n))$ multiplying $\theta(n)$. For less rapidly growing $|\theta(n)|$ a fixed value of m, sufficiently large to make the error (first omitted) term o(1) as $n \to \infty$, can be chosen independently of n; so in this case $\mathcal{S}(\varphi(n))$ can be merely semiconvergent.

The proviso just mentioned implies that $S_n = \bar{\sigma}_n + o(1)$ as $n \to \infty$, or $S_n - \bar{\sigma}_n = o(1)$. Hence $\lim_{n\to\infty} [S_n - \bar{\sigma}_n]$ can differ from zero by at most a constant. This establishes the existence of a terminal sum in general.

THEOREM 7. The terminal sum of an infinite series $\{A_\mu\}$, $\mu = 1,2,\ldots,$ always exists and corresponds to a single value point in the complex plane.

Where asymptotic tractability fails, the method of terminal summability fails. In general, as already noted, the failure of asymptotic tractability (of an otherwise tractable process) occurs at isolated points of parametric singularity. At such points alternative asymptotic expansions (possibly based on different expansion methods, as described in Definition 8) can sometimes be found. Thus a local "failure" of asymptotic tractability is primarily a challenge to ingenuity.

It should be stated as a reservation about all the results in this section, in particular Theorem 7, that "infinite series" refers to a process that is presumed to be well-defined, in the sense that if more than one limiting process is involved the orders in which the limits are to be taken must be specified. Otherwise, we shall see in Section 3-5 that uniqueness of a terminal sum can be lost at certain "points of ambiguity."

We now enumerate the steps of a systematic computational procedure for evaluating terminal sums.

3-3 SERIES TERMINAL SUMMATION

The following refers to a μ-periodic infinite series $\{A_\mu\}$, $\mu = 1, 2, \ldots$.

Step 1. Preliminary verification of asymptotic tractability.

Consider the quantity $f(n) \equiv \sum_{i=n}^{n+\mu-1} a_i$, where μ is the period, equal to the least integer for which $a_{n+\mu} \simeq a_n$ as $n \to \infty$ for all sufficiently large n. By ingenuity discover a function $\theta(n)$ that is a factor of the dominant term as $n \to \infty$ of each of the a_i entering $f(n)$, and discover a set of expansion basis functions $\varphi_i(n)$, $i = 1, 2, \ldots$, such that

$$\theta(n+\mu) \sim \theta(n) \cdot \mathcal{S}(\varphi(n)) \quad \text{as} \quad n \to \infty$$

and

$$a_k \sim \theta(n) \cdot \mathcal{S}(\varphi(n)) \quad \text{as} \quad n \to \infty$$

$$k = n, n+1, \cdots, n+\mu-1,$$

where $\mathcal{S}$ denotes any standard asymptotic expansion. Let τ_n denote the broad asymptotic expansion of f(n). Express τ_n as

$$\tau_n \sim \theta(n) \cdot \mathcal{S}(\varphi(n)) \quad \text{as} \quad n \to \infty.$$

If this step fails the method fails.

<u>Step 2</u>. Introduction of trial series.

Two cases are to be considered.

<u>Case A</u>. The function $\theta(n)$ found in step 1 is not a constant. In this case the trial series for σ_n takes the form

$$\sigma_n \sim \theta(n) \cdot \mathcal{S}(\varphi(n)) \quad \text{as} \quad n \to \infty,$$

where S involves an undetermined constant multiplying each of the $\varphi_i(n)$, i = 1, 2, ...

<u>Case B</u>. $\theta(n)$ is a constant.

In this case a trial series of the form

$$\sigma_n \sim c\,\psi(n) + \mathcal{S}(\varphi(n)) \quad \text{as} \quad n \to \infty$$

may be introduced, where ψ is a function such that

$$\psi(n+\mu) \sim \psi(n) + \mathcal{S}(\varphi(n)) \quad \text{as} \quad n \to \infty,$$

and where c and the constants present in the trial series $\mathcal{S}$ are arbitrary. Alternatively, except in special cases, it should be possible to take c = 0 and omit the ψ-term. An indication as to whether a ψ-term will be needed is obtained by examining $\int$(dominant term of τ_n as $n \to \infty$)dn. If this is a member of the $\varphi_i(n)$ group, it is safe to take c = 0.

<u>Step 3</u>. Evaluation of coefficients.

Insert the trial series for σ_n into the asymptotic form of Equation (57d); <u>viz</u>.,

$$\sigma_{n+\mu-1} - \sigma_{n-1} \sim \tau_n \quad \text{as} \quad n \to \infty. \tag{70}$$

Cancel the multiplier $\theta(n)$, if it is present, and equate coefficients of successive members of the basis set $\varphi_i(n)$ for increasing i, thereby determining as many of the coefficients of the σ_n series as computational needs of the problem require. Success in this step completes the verification of asymptotic tractability, unless $\lim_{n\to\infty} |\theta(n)\,\varphi_i(n)| = \infty$ for all i, in which case it is also necessary to establish the Cauchy convergence[17] of $\sigma_n / \theta(n)$.

<u>Step 4</u>. Evaluation of infinite series.

Compute for increasing n the values of V_n, where

$$V_n = \sum_{i=1}^{n} a_i - \bar{\sigma}_n .$$

Here $\bar{\sigma}_n$ is the series σ_n determined in Step 3, from which the constant (n-independent) term, if any, has been deleted. If $V_n \to V$ for increasing n, V is the terminal sum $\sum_1^\infty a_i$. If not, recycle through steps 3 and 4 with more terms in the expansion and larger n-values. If $|\theta(n)|$, the multiplier in $|\bar{\sigma}_n|$, is a strongly increasing function of n, a very delicate balance must be maintained between the number of expansion terms m and the value of n for minimum absolute error. In this connection the use of the first omitted or last included term in the asymptotic expansion as an error estimate is a useful device.

3-4 <u>EXAMPLES OF SERIES TERMINAL SUMMABILITY</u>

We now give some examples of the reduction to practice of the method just described.

<u>EXAMPLE 1. Christian Wolf's Conjecture</u>.

This example is too trivial to exhibit much of the power of the method, but we give it in full detail because of its historical interest. The problem is to evaluate the terminal sum of

$$1 - z + z^2 - z^3 + \cdots$$
$$= \sum_{n=0}^{\infty} (-z)^n$$

for z any point in the complex plane. We follow the method outlined in the preceding section.

<u>Step 1</u>. Because of the algebraic sign alternation we could take $\mu = 2$ and treat this as a 2-periodic process. However, in such a simple case it is easier and more instructive to treat it as nonperiodic, $\mu = 1$. Then $f(n) = a_n = (-z)^n$.

Take $\theta(n) = (-z)^n$. We have $\theta(n+\nu) = (-z)^{n+\nu} = \theta(n)\cdot(-z)^\nu$, which is of the required form, Equation (58b). Consequently

$$f(n) \sim \tau_n \sim (-z)^n ,$$

where we could use equality, since there is no expansion.

Step 2. Case A applies, since θ(n) is not a constant. We try a series of the form

$$\sigma_n \sim (-z)^n\left[a_0 n + a_1 + \frac{a_2}{n} + \frac{a_3}{n^2} + \cdots\right]$$

and calculate

$$\sigma_{n-1} \sim (-z)^n\left[\frac{a_0}{(-z)} n + \frac{(a_1 - a_0)}{(-z)} + \frac{a_2/(-z)}{n} + \frac{(a_2+a_3)/(-z)}{n^2} + \cdots\right].$$

(We include the a_0-term merely to confirm its disappearance.)

Step 3. For $\mu = 1$, Equation (70) becomes

$$\sigma_n - \sigma_{n-1} \sim \tau_n \quad \text{as } n \to \infty . \tag{71}$$

This yields

$$(-z)^n\left[\left(a_0 - \frac{a_0}{(-z)}\right)n + \left(a_1 - \frac{(a_1-a_0)}{(-z)}\right) + \frac{\left(a_2 - \frac{a_2}{(-z)}\right)}{n} + \frac{\left(a_3 - \frac{(a_2+a_3)}{(-z)}\right)}{n^2} + \cdots\right] \sim (-z)^n \quad \text{as } n\to\infty .$$

Equating coefficients of equal powers of n, we have

Coefficient of n:

$$a_0\left(\frac{1+z}{z}\right) = 0 .$$

If we impose the requirement

$$z \neq -1 ,$$

then

$$a_0 = 0 .$$

Coeff. of n^0:

$$a_1 - \frac{(a_1 - a_0)}{(-z)} = 1 ,$$

or

$$a_1 = \frac{z}{1+z} .$$

<u>Coeff. of n^{-1}:</u>

$$a_2 - \frac{a_2}{(-z)} = 0,$$

or

$$a_2 = 0.$$

<u>Coeff. of n^{-2}:</u>

$$a_3 - \frac{(a_2+a_3)}{(-z)} = 0,$$

or

$a_3 = 0$, etc.

All the remaining a's vanish. Consequently, we have equality:

$$\sigma_n = (-z)^n\left(\frac{z}{1+z}\right) = -\frac{(-z)^{n+1}}{1+z},$$

predicated on $z \neq -1$.

<u>Step 4.</u> We have $\bar{\sigma}_n = \sigma_n$, since there is no n-independent term in the "expansion." The quantity $S_n = \sum_{i=0}^{n}(-z)^i$ is readily evaluated in closed form as

$$S_n = \frac{1-(-z)^{n+1}}{1+z}$$

for all finite n and $z \neq -1$. Thus the terminal sum is

$$V \overset{(T)}{=} \lim_{n\to\infty}\left[S - \bar{\sigma}_n\right]$$

$$= \lim_{n\to\infty}\left[\frac{1-(-z)^{n+1}}{1+z} + \frac{(-z)^{n+1}}{1+z}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{1+z}\right] = \frac{1}{1+z}, \qquad z \neq -1.$$

Putting $z = 2$, we have

$$1 - 2 + 4 - 8 + \cdots \overset{(T)}{=} \frac{1}{3},$$

as Wolf conjectured.

The case $z = -1$, leading to

$V = 1 + 1 + 1 + 1 + \ldots,$

is trivially solved by a trial "expansion" $\sigma_n \sim b_0 n + b_1$, b_0, b_1 constants. Since $\sigma_{n-1} \sim b_0 n + (b_1 - b_0)$, we have $\sigma_n - \sigma_{n-1} \sim b_0 \sim \tau_n \sim 1$ as $n \to \infty$, or $b_0 = 1$. Hence $\sigma_n = n$ and $V \overset{(T)}{=} \sum_1^n 1 - n = 0$. (This is a special case of the series $a + a + a + \ldots$, which, consistently with the above, terminally sums to zero.) Thus a terminal sum of $1 - z + z^2 - z^3 + \cdots$ exists at every point in the complex z-plane and comforms to Wolf's conjecture outside the Cauchy convergence circle $|z| = 1$. One is as free to manipulate with the series as with its terminal sum-function, $1/(1+z)$.

EXAMPLE 2. Riemann Zeta Function.

Amusingly enough, we can spare ourselves the trouble of carrying through our computational steps in this case by borrowing from Jahnke-Emde (reference (k), p. 269) the following interesting formula, which they give for $|s| \ll 1$ as "very well suited for numerical computation:"

$$\zeta(s) = \sum_{n=1}^{m} \frac{1}{n^s} + \frac{1}{m^s}\left\{ \frac{m}{s-1} - \frac{1}{2} + \frac{s/12}{m} - \frac{s(s+1)(s+2)}{720\, m^3} + \frac{s(s+1)(s+2)(s+3)(s+4)}{3024\, m^5} - \frac{s(s+1)\cdots(s+6)}{1{,}209{,}600\, m^7} + \frac{s(s+1)\cdots(s+8)}{47{,}900{,}160\, m^9} - \cdots \right\}. \quad (72)$$

This will be recognized as having precisely the form of our definition of the terminal sum of

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

except that the limit on n is not indicated. (There is no computational distinction, of course, since no computer can "take a limit.") If the steps of our method are followed, exactly the same formula for the terminal value of $\zeta(s)$ is obtained, for s any point in the complex plane, except that we must exclude s = 1 because of an obvious divergence. It happens also that the expansion coefficients are not uniquely determined at the points s = 0, -1, -2, -3, ... However, the limiting value given by (72) for $\zeta(s)$ as s approaches any of these points is well-defined, unique, and numerically correct. The ambiguity regarding the terminal sum at such points will be discussed in Section 3-5.

Because of the singularity in (72) at s = 1, a different type of expansion must be used there. The series is

$$\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} .$$

(Note that s = 1 is not a "point of ambiguity," so considerations of the type discussed in Section 3-5 do not arise.) We apply our method to this series.

Step 1. The series is nonperiodic, $\mu = 1$. $f(n) = a_n = \frac{1}{n}$. If we take the φ-group to be the integral powers of n, we have $\theta(n) = 1$.

Step 2. Case B applies; hence a ψ-function is to be considered. The dominant term of τ_n or f(n) is $1/n$. The integral of this, $\int dn/n = \log_e n \equiv \ln n$, gives an indication of the proper ψ-function to try. We take $\psi(n) = \ln n$, verify that

$$\begin{aligned}\psi(n+\nu) &= \ln(n+\nu) = \ln\left(n\left[1+\frac{\nu}{n}\right]\right) \\ &= \ln n + \ln\left(1+\frac{\nu}{n}\right) \\ &\sim \ln n + \frac{\nu}{n} - \frac{\nu^2}{2n^2} + \frac{\nu^3}{3n^3} - \cdots \\ &\sim \psi(n) + \mathcal{L}(\varphi(n)) \quad \text{as } n\to\infty,\end{aligned}$$

and try a series

$$\begin{aligned}\sigma_n &\sim c\psi(n) + \mathcal{L}(\varphi(n)) \\ &\sim c\ln n + b_0 + \frac{b_1}{n} + \frac{b_2}{n^2} + \frac{b_3}{n^3} + \cdots .\end{aligned}$$

Step 3. Putting this in (71), we obtain in the usual way

$$\sigma_n \sim \ln n + b_0 + \frac{1}{2n} - \frac{1}{12n^2} + \cdots \quad \text{as } n\to\infty$$

or

$$\bar{\sigma}_n \sim \ln n + \frac{1}{2n} - \frac{1}{12n^2} + \cdots \quad \text{as } n\to\infty .$$

Step 4. The terminal sum of $\zeta(s)$ at $s = 1$ is thus found to be

$$\zeta(1) \overset{(T)}{=} \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{k} - \ln n - \frac{1}{2n} + \frac{1}{12n^2} - \cdots\right] \tag{73}$$

= Euler's constant = 0.5772...

(See reference (ℓ), Equation (70), p. 14. We could have borrowed this formula, as well.) As a check on this evaluation of the infinite process, we note that it happens to agree with the "value" conceived as the limit of an arithmetic average of the values on either side of the singularity,

$$\text{Value of } \zeta(1) \equiv \lim_{n\to\infty}\lim_{\epsilon\to 0}\frac{1}{2}\left[\zeta_n(1+\epsilon) + \zeta_n(1-\epsilon)\right], \tag{74}$$

where $\zeta_n(s)$ is the quantity on the right side of (72) and ϵ is real. Thus from (72) we calculate

$$\lim_{\epsilon\to 0}\frac{1}{2}\left[\zeta_n(1+\epsilon) + \zeta_n(1-\epsilon)\right] = \sum_{k=1}^{n}\frac{1}{k} + \frac{1}{n}\left\{-\frac{1}{2} + \frac{(1/12)}{n} + \cdots\right\}$$
$$+ \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\left(n^{-\epsilon} - n^{\epsilon}\right).$$

The limit of the last term is $-\ln n$, so (74) becomes formally identical with (73). Hence (74) is another way of defining the terminal sum at $s = 1$. Note that the ϵ-limit here precedes the n-limit. It is suggested by the graph of Figure 3-1 that the orders of these limiting processes could be interchanged, but this seems difficult to verify either analytically or computationally.

We see that the $\lim_{n\to\infty}$ of the right-hand side of (72) is in fact not at all restricted to $|s| \ll 1$ in its ability to represent the ζ-function. Jahnke-Emde were clearly unaware of the broader significance of their formula. It is interesting to note that for $|s/n| < 1$ the series in (72) appears convergent in the Cauchy sense. Whether it is or not, the question arises, could this series itself be terminally summed? That is, in general can the theme of terminal summability be repeatedly applied? The answer is yes, as long as asymptotic tractability can be established. This means in practice, however, that a closed-form expression for the n^{th} summand, a_n, must be available. The trial-series method yields values of coefficients through recurrence relations, but in general no closed-form expression for the coefficients as functions of n. From surveying the numerical coefficient sequence 720, 3024, 1209600, 47900160, ... in (72) it seems difficult to guess the law of formation. Hence there is no evident way to reapply the terminal summation machinery to (72).

The results of numerical computations of $\zeta(s)$ for real s, using Equations (72) and (73), are shown in Figure 1. It must be emphasized that these evaluations, which agree with the more extensive results in standard tables (references (j), (k)), are obtained by real-variable methods exclusively. Hitherto, methods of contour integration, etc., have been used to obtain such startling results as

$$\zeta(-2) = 1 + 4 + 9 + 16 + \cdots = 0. \quad (75)$$

The writer is not aware of any strictly real-variable methods of "summability" that have yielded an evaluation of the ζ-function for real arguments $s < 1$. Indeed, (75) is much too strongly Cauchy-divergent to be amenable to any of the traditional methods of summability.[18] Equation (75) follows from setting s = -2 in (72) and expressing $\sum_{n=1}^{\infty} n^2$ in closed form. The result is a terminal sum identically equal to zero, without need for taking a limit on n.

EXAMPLE 3. Integral Representation.

In Chapter 2, Example 2, we considered a continued-fraction representation of the integral

$$I(z) \equiv \int_0^\infty \frac{e^{-u}\,du}{z+u} \quad . \qquad (76)$$

For $|\arg z| < \pi$ this equals $-e^{z}\, Ei(-z)$ (see reference (j), p. 311), a tabulated function. The power series expansion of the continued fraction (reference (d), p. 367), or the result of developing the integrand into power series and integrating term by term, is the everywhere (Cauchy-) divergent power series

$$I(z) = \sum_{m=0}^{\infty} \frac{(-1)^m m!}{z^{m+1}} . \tag{77}$$

This is a famous semi-convergent series considered by Euler in 1754. (Reference (i), p. 1.) We shall terminally sum it by our method.

Step 1. We treat the series as nonperiodic, $\mu = 1$. Trying $\theta(m) = (-1)^m m!/z^{m+1}$, we find

$$\theta(m+\nu) = (-1)^{\nu} z^{-\nu} (\text{polynomial in } n \text{ of degree } |\nu|)^{k}\, \theta(m),$$

where k = 1 if $\nu > 0$ and $k = -1$ if $\nu < 0$. This meets criterion (58b), if we take the $\varphi_i(n)$ to be integral powers of n.

Step 2. Case A applies, since $\theta(n) \neq$ constant. We try

$$\sigma_m \sim \frac{(-1)^m m!}{z^{m+1}} \left[a_0 + \frac{a_1}{m} + \frac{a_2}{m^2} + \frac{a_3}{m^3} + \cdots \right] \text{ as } m \to \infty.$$

Step 3. Putting this in (71), we obtain the following set of recurrence relations involving the well-known "triangle numbers" of Pascal (reference (m), p. 135):

$$a_0 = 1, \qquad a_1 = -z a_0, \qquad a_2 = -z a_1,$$

$$a_3 = -z(a_1 + a_2), \qquad a_4 = -z(a_1 + 2a_2 + a_3),$$

$$a_5 = -z(a_1 + 3a_2 + 3a_3 + a_4),$$

$$a_6 = -z(a_1 + 4a_2 + 6a_3 + 4a_4 + a_5),$$

$$a_7 = -z(a_1 + 5a_2 + 10a_3 + 10a_4 + 5a_5 + a_6),$$

$$a_8 = -z(a_1 + 6a_2 + 15a_3 + 20a_4 + 15a_5 + 6a_6 + a_7),$$

etc.

From these we derive

$$\sigma_m = \bar{\sigma}_m \sim \frac{(-1)^n n!}{z^{n+1}}\Big[1 - \frac{z}{n} + \frac{z^2}{n^2} - \frac{(z^3-z^2)}{n^3} + \frac{(z^4-3z^3+z^2)}{n^4} - \frac{(z^5-6z^4+7z^3-z^2)}{n^5}$$
$$+ \frac{(z^6-10z^5+25z^4-15z^3+z^2)}{n^6}$$
$$- \frac{(z^7-15z^6+65z^5-90z^4+31z^3-z^2)}{n^7}$$
$$+ \cdots\Big] \quad \text{as } n \to \infty. \tag{78}$$

Step 4. Let m denote the exponent on n in the last term retained in the asymptotic expansion σ_m of S_n ($m = 7$, above). Then because of the strong ultimate (Cauchy) divergence of the original series there is a close relationship between m and n for best computational results. That is, a double limiting process on m and n must occur, the nature of which depends on $|z|$. Formally, we have

$$I(z) \overset{(T)}{=} \lim_{\substack{n,m\to\infty \\ \text{(for minimum error)}}} \left\{ \sum_{k=0}^{n} \frac{(-1)^k k!}{z^{k+1}} - \bar{\sigma}_m \right\}, \tag{79}$$

where $\bar{\sigma}_m$ is given by Equation (78), and where the indicated limit is of such a nature as to cause the magnitude of the error term (first omitted or last retained term in the expansion) to go to zero. For $|z| >> 1$ this error term is of the order of

$$f(n,m) = \frac{n!}{|z|^{n+1}} \frac{|z|^m}{n^m} \approx \sqrt{2\pi n}\, e^{-n}\, n^{n-m} |z|^{m-n-1}, \quad n \text{ large}.$$

In this case if we take n of the order of m we obtain

$$f(m,m) \approx \frac{\sqrt{2\pi m}\, e^{-m}}{|z|},$$

so that a reasonably small error can be obtained for $m \approx n \approx 10$ or thereabouts. This is eminently satisfactory for computational purposes.

In case $|z| \ll 1$, the situation is computationally much less satisfactory. We have then an error term of the order of

$$g(n,m) = \frac{n!}{|z|^{n+1}} \frac{|z|^2}{n^m} \approx \frac{\sqrt{2\pi n}\; e^{-n}}{|z|^{n-1}\; n^{m-n}} , \qquad n \text{ large}.$$

In this case we can no longer maintain $m \approx n$, but must instead require that $m \to \infty$ at a faster rate than $n \to \infty$. If we require $m \approx \alpha n$, where $\alpha > 1$, it is not difficult to show that satisfactory convergence is obtained provided m is large enough to satisfy

$$\left(\frac{m}{\alpha}\right)^{\alpha - 1} \gg \frac{1}{|z|} .$$

A value of α in the vicinity of 2 or 3 seems satisfactory for most purposes, though one can evaluate an "optimum" $\alpha(m)$. The fact that the triangle numbers appear in a regular way in the evaluation of expansion coefficients means that there is no practical obstacle to making m as large as we please. Hence — in principle — we can evaluate $I(z)$ to any required accuracy for $|z|$ as small as desired. For $|z|$ of the order of 1 the computational difficulties are intermediate between those for very large and very small $|z|$.

To illustrate the latitude available in formulating asymptotic expansions (without altering the value of the terminal sum), we remark that instead of (78) it is equally feasible to use an expansion multiplier $\theta(n)$ in which n! is replaced by its Stirling's approximation. The result is that $I(z)$ can be evaluated from the alternative formula

$$I(z) \overset{(T)}{=} \lim_{\substack{n,m\to\infty \\ (\text{for min. error})}} \left\{ \sum_{k=0}^{m} \frac{(-1)^k k!}{z^{k+1}} - \left(\frac{-n}{ze}\right)^n \frac{\sqrt{2\pi n}}{z} \left[1 - \frac{\left(z - \frac{1}{12}\right)}{n} + \frac{\left(z^2 - \frac{z}{12} + \frac{1}{288}\right)}{n^2} - \frac{\left(z^3 - \frac{13}{12} z^2 + \frac{z}{288} + \frac{139}{51840}\right)}{n^3} + \cdots \right] \right\} . \tag{80}$$

Because of the simpler l·w of formation of the coefficients,[19] (78) is a more useful expansion. However, (80) gives satisfactory results, and has in fact been used in calculating the results shown in Figure 3-2. The last term ($O(n^{-3})$) in (80), with its $\Theta(n)$ multiplier included, was used as an error estimate. This represented the first omitted term, so that all calculations were made using a fixed m-value, m = 2.

The results for $z > 5$ show quite satisfactory convergence of the terminal sum of the infinite series to the tabulated value of the integral. The latter is shown as a solid curve for $z > 0$. The shaded region about this curve indicates the order of magnitude of uncertainty in evaluation of the terminal sum. Specifically it represents the spread between V_{n1} and V_{n2}, where V_{n1} is the value of $S_n - \sigma_n$ for $n = n_1$ = n-value yielding minimum magnitude of the error term, and $n_2 = n_1 \pm 1$ = n-value yielding next smallest error-term magnitude. From the foregoing discussion it is clear that a fixed m-value as small as 2 should yield very poor results for small $|z|$, and this is confirmed by the figure.

We now call attention to the remarkable results for real $z < 0$ displayed in Figure 3-2. It is apparent that for $z < 0$ the process $\sum_0^\infty \frac{(-1)^n n!}{z^{n+1}}$ possesses a well-defined terminal sum. In fact for $z < -5$ the "value" (which turns out to be negative) is very easily computed, even for m = 2. For z real and negative, the integral $I(z)$ has a singularity in the path of integration. The integral is therefore divergent and is not tabulated. Customarily, the exponential integral to which $I(z)$ is related is defined in terms of a cut in the z-plane along the negative real axis. With such a cut the function becomes single-valued. Here, however, we see that without introduction of complex variable methods, or even admission of the existence of a "z-plane", $I(z)$ can be assigned well-defined, unique functional "values" for real $z < 0$ by terminal summation of a formally equivalent infinite series. (The value $I(-|z|)$ thus obtained is not simply the negative of the corresponding $I(|z|)$.)

Only at the isolated essential singularity z = 0 does terminal summability fail.

It would appear from this example that the method of terminal summability can in the right circumstances be used to sum divergent integrals. This involves strictly real-variable methods, where the path of integration and the integrand are real. One is encouraged to surmise that the method of expanding an integrand into series, formally integrating term by term into (in general) Cauchy-divergent series, and terminally summing the latter, may be of quite general applicability. We conclude that terminal summability appears to make a valuable contribution to analysis through helping to free it from the limitations imposed by Cauchy divergence and through offering a wide range of formal manipulative freedoms (perhaps more so than other methods of "summability.") It remains for future investigation to determine, within the realm of complex-variable function theory, what the implications may be of a capability uniquely to "sum" a series along a cut or at a singularity. It also remains to explore the exact extent of the new manipulative freedoms just mentioned.

3-5 POINTS OF AMBIGUITY

The examples of the preceding section illustrated that at certain isolated points in the complex plane an ambiguity can arise, such that the terminal sum of an infinite series appears to be nonunique at these points. We shall show that this is not a failure of terminal summability but a failure of problem definition. It arises out of ambiguity as to the order in which limits are to be taken. Uniqueness is restored when the problem is sufficiently well-posed to make clear the appropriate sequential ordering of all limiting process.

Specifically, what is under discussion is the following phenomenon: In Example 1 of the preceding section it was established that

$$V = 1 + 1 + 1 + \cdots \overset{(T)}{=} 0,$$

whereas in Example 2 we note that equation (72) yields

$$\zeta(0) \overset{(T)}{=} -\frac{1}{2} .$$

But of course

$$\zeta(0) = \sum_{n=1}^{\infty} \frac{1}{n^0} = 1+1+1+\cdots ,$$

so that the formal expression

$$1+1+1+\cdots$$

terminally sums to either $-\frac{1}{2}$ or 0. We remarked that for $\zeta(s)$ such nonuniqueness could arise at s = 0, -1, -2, ..., and we termed these "points of ambiguity." The clue to the existence of such points is to be found in the recurrence relations that determine the coefficients of the asymptotic expansion σ_n. In deriving equation (72) by our method we find on equating the coefficient of n^{-s-1} to zero that $a_1 s = \frac{1}{2}s$, where a_1 = coeff. of n^{-s} in trial expansion of σ_n. This implies that either $a_1 = \frac{1}{2}$ (the value used in (72)) for $s \neq 0$, or else s = 0 and we cannot say what a_1 is. Similarly, on equating the coefficient of n^{-s-2} to zero, we obtain $a_2(1+s) = -\frac{1}{12}s(1+s)$, which implies either $a_2 = -\frac{s}{12}$ for $s \neq -1$, or else s = -1 and we cannot say what a_2 is. And so on. For each of the points s = 0, -1, -2, ..., such an ambiguity arises.

One way to deal with this problem is to say that at such isolated points the terminal sum is nonunique. A better approach seems to be to use this, in the present example, as an occasion for defining more sharply what is meant by the ζ-function. We may take it as a definition that ζ is an analytic function of its argument such that for s_o any point of regularity in the complex plane

$$\zeta(s_0) \equiv \lim_{s\to s_0} \lim_{M\to\infty} \sum_{n=1}^{M} \frac{1}{n^s} , \tag{81}$$

the value so obtained being (by regularity) independent of the manner of approach of s to s_o. Since the points s = 0, -1, -2, ...

are points of regularity of $\zeta(s)$, it follows that the coefficients a_1, a_2, ... in the asymptotic expansion (72) cannot jump discontinuously to different values at any of these points. Hence (72) remains a valid representation of the (analytic) ζ-function at these points, even though we have seen that the coefficients become formally indeterminate there -- and even though our prescription for replacing σ_n by $\bar{\sigma}_n$ whenever a constant (n-independent) term appears would seem to call for such a jump. In short, we let the requirement to represent an analytic function, where the problem so indicates, be the determining factor in restoring uniqueness at points of ambiguity.

By contrast with the ζ-function, the quantity represented solely by

$$V = 1 + 1 + 1 + \cdots$$

is probably best interpreted -- if we choose to relate it to the ζ-function at all -- in terms of a reversed order of taking the limits. That is,

$$V_{s_0} \equiv \lim_{M\to\infty} \lim_{s\to s_0} \sum_{n=1}^{M} \frac{1}{n^s} , \tag{82}$$

where in this case $s_o = 0$. The quantity V_o so obtained is not related to $\zeta(0)$, as defined by (81), hence answers to no requirement of regularity as a function of s. (Indeed the first limit eliminates all s-dependence.) Consequently, it is no surprise that the terminal sum,

$$V = 1 + 1 + 1 + \cdots \overset{(T)}{=} 0 ,$$

bears no relation to the terminal sum of $\zeta(0) \overset{(T)}{=} -\frac{1}{2}$. Similarly, $1 + 2 + 3 + \ldots$, interpreted according to (82) with $s_o = -1$, terminally sums to zero, whereas $\zeta(-1) \overset{(T)}{=} -\frac{1}{12}$. Only for $\zeta(-2)$, $\zeta(-4)$, ..., do we get coincidentally a terminal summation by both (81) and (82) to the value zero.

From the above we see that there is indeed at certain isolated "points of ambiguity" an indeterminacy of the terminal sum of an infinite series. But this ambiguity simply mirrors the multiplicity of orders of parametric limiting processes by which the given series is expressible. We may say of a problem in series evaluation that it is "well-posed" if the problem statement eliminates such ambiguities through the provision of extra information (or, in default of such extra information, if it can be assumed that there are no extra considerations, e.g., of regularity in a parameter). For well-posed problems in infinite series, the terminal sum not only always exists (under the proviso of asymptotic tractability) but is unique. Thus terminal summation fulfills several of the historic goals that have been sought under the name of "summability."

Our considerations have been limited here primarily to real variables, though we have referred to the complex plane on occasion. It will be interesting to see, since function theory is so firmly built on power series, what may be the contribution to analytic continuation, etc., of a method of summing such series throughout regions of Cauchy divergence. The ζ-function has a single pole with residue 1 at s = 1. What, if anything, does it mean for function theory that the process terminally sums to Euler's constant at this point? The writer has no idea of the answer. Perhaps it makes a theory more interesting to other investigators that it raises questions beyond the originator's grasp.

Euler played fast and loose with infinite processes, using divergent series in formal operations rather freely in accordance with the (since much ridiculed) "spirit of the times." He did this deliberately, not in ignorance of what he was doing. We close with a question as to whether Euler may have had a premonition of something resembling terminal summability when he wrote (Opera Omnia, p 587, translated in reference (b), p 3), "Whenever an infinite series is obtained as the development of some closed expression, it may be used in mathematical operations as the equivalent of that expression, even for values of the variable for which the series diverges."

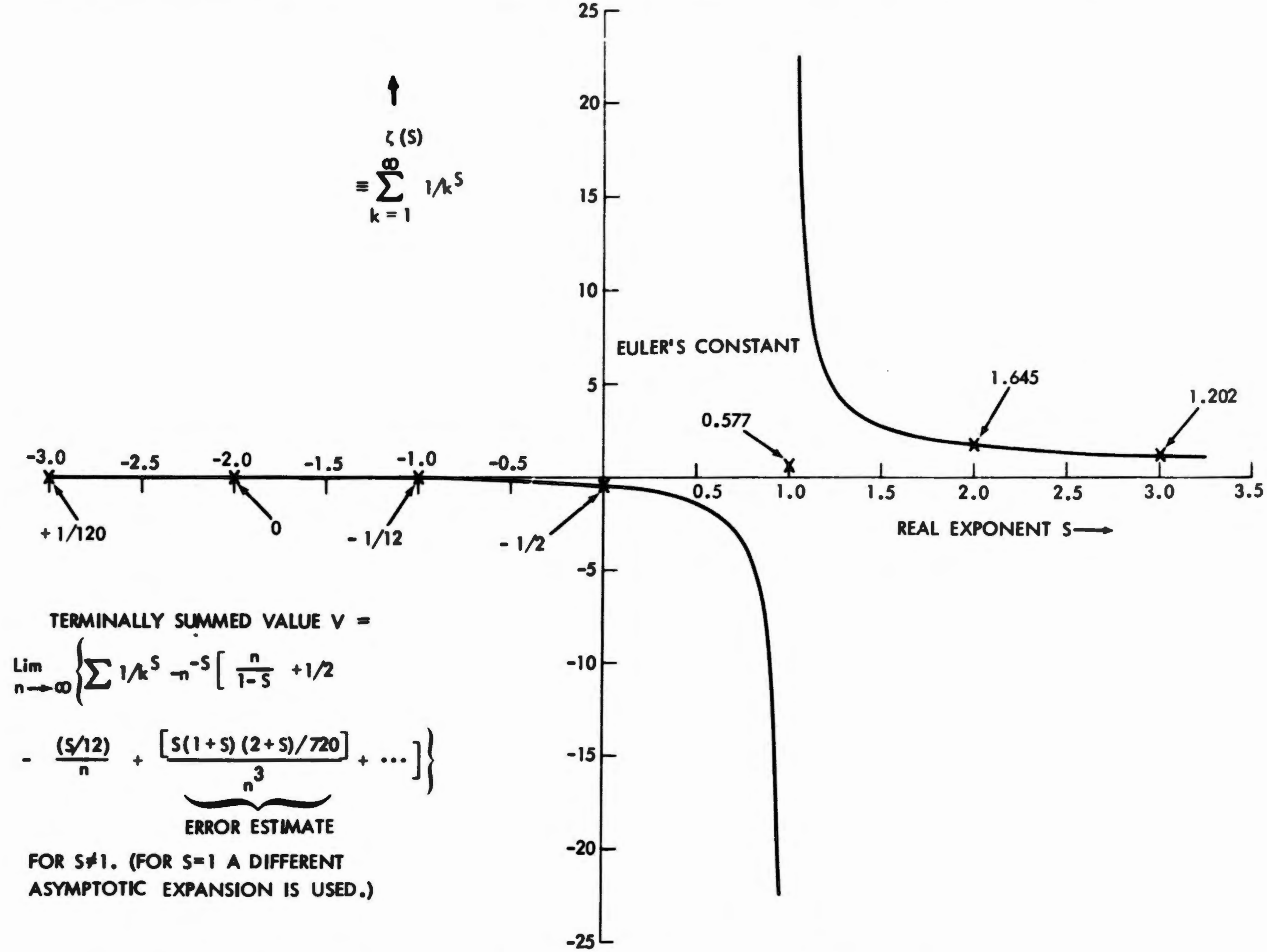
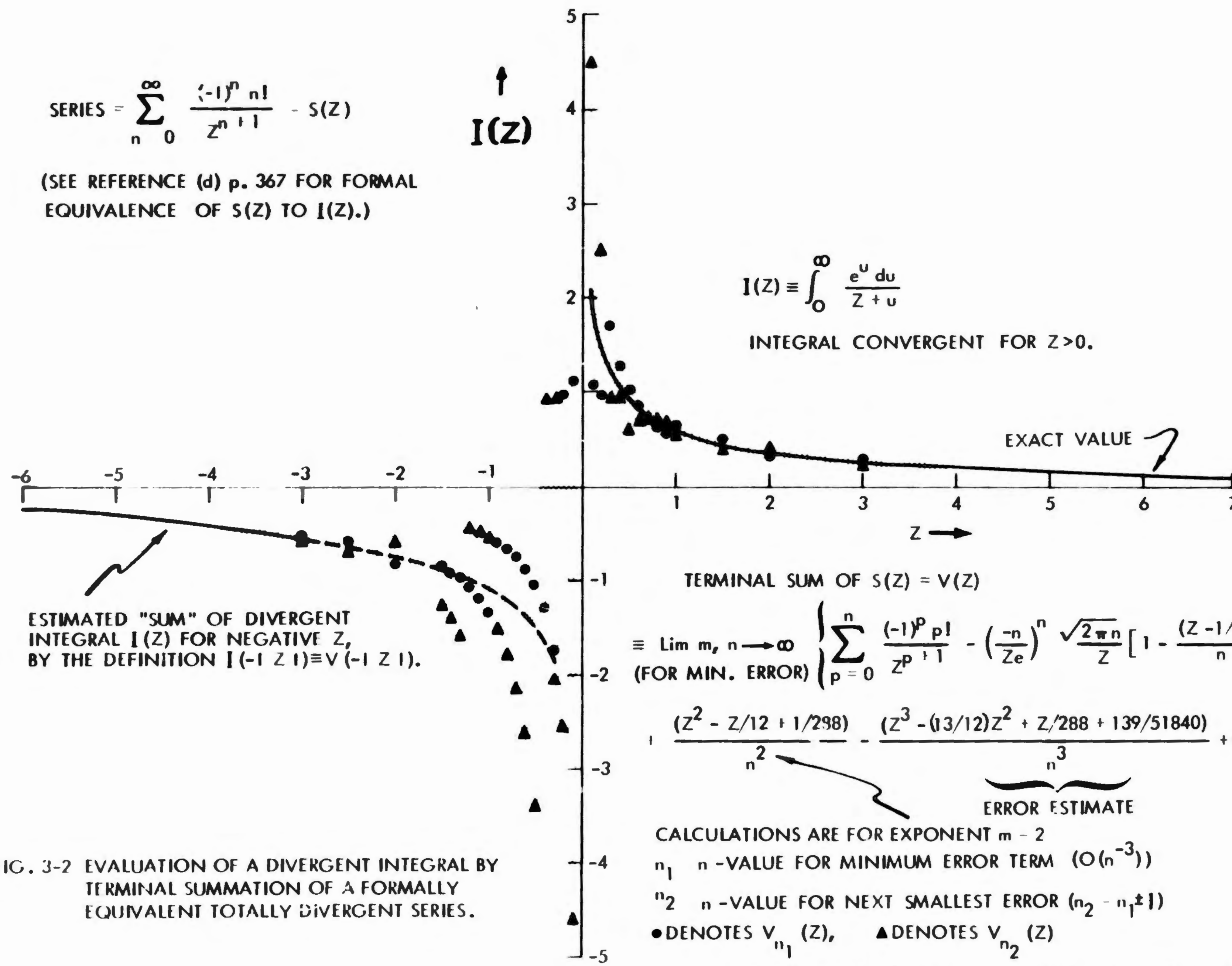


FIG. 3-1 RIEMANN ZETA FUNCTION FOR REAL ARGUMENT, EVALUATED BY TERMINAL SU: MATION



$$\text{SERIES} = \sum_{n=0}^{\infty} \frac{(-1)^n\, n!}{Z^{n+1}} = S(Z)$$

(SEE REFERENCE (d) p. 367 FOR FORMAL EQUIVALENCE OF S(Z) TO I(Z).)

$$I(Z) \equiv \int_0^{\infty} \frac{e^{-u}\,du}{Z+u}$$

INTEGRAL CONVERGENT FOR Z>0.

TERMINAL SUM OF S(Z) = V(Z)

$$\equiv \lim_{\substack{m,\, n \to \infty \\ \text{(FOR MIN. ERROR)}}} \left\{ \sum_{p=0}^{n} \frac{(-1)^p\, p!}{Z^{p+1}} - \left(\frac{-n}{Ze}\right)^n \sqrt{\frac{2\pi n}{Z}} \left[1 - \frac{(Z-1/12)}{n} + \frac{(Z^2 - Z/12 + 1/288)}{n^2} - \underbrace{\frac{(Z^3-(13/12)Z^2 + Z/288 + 139/51840)}{n^3}}_{\text{ERROR ESTIMATE}} + \cdots \right] \right\}$$

CALCULATIONS ARE FOR EXPONENT $m = 2$

n_1 n-VALUE FOR MINIMUM ERROR TERM $(O(n^{-3}))$

n_2 n-VALUE FOR NEXT SMALLEST ERROR $(n_2 = n_1 \pm 1)$

● DENOTES $V_{n_1}(Z)$, ▲ DENOTES $V_{n_2}(Z)$

DENOTES ERROR RANGE IN EVALUATION OF TERMINAL SUM V(Z)

FIG. 3-2 EVALUATION OF A DIVERGENT INTEGRAL BY TERMINAL SUMMATION OF A FORMALLY EQUIVALENT TOTALLY DIVERGENT SERIES.

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APPENDIX A

FOOTNOTES

1 To derive the geometrical progression from Equation (1) requires an interpretation of the binomial coefficient for negative argument. If

$$\binom{z}{s} \equiv \frac{\Gamma(z+1)}{\Gamma(s+1)\cdot\Gamma(z-s+1)},$$

we have for s a non-negative integer

$$\binom{-1}{s} = \frac{\Gamma(0)}{\Gamma(s+1)\cdot\Gamma(-s)} = \frac{1}{s!}\,\frac{\Gamma(0)}{\Gamma(-s)}.$$

The ratio of the divergent quantities $\Gamma(0) : \Gamma(-s)$ may be identified with the ratio of the corresponding simple pole residues, 1: $(-1)^s/s!$ Thus

$$\binom{-1}{s} = (-1)^s$$

and (1) yields formally $(1+\alpha)^{-1} = 1 - \alpha + \alpha^2 - \alpha^3 + \cdots.$

2 Equation (3c) can be developed (reference (c), p. 330) into the ascending continued fraction

$$R_1 = \cfrac{q_1 + \cfrac{q_2 + \cdots + \cfrac{q_m + R_{m+1}}{p_m^{-1}}}{p_2^{-1}}}{p_1^{-1}}.$$

This is seen to be equivalent to the serduct development given in Table 1-1. Such a first-order linear process should not be confused with a "true" (descending) continued fraction, equivalent to the second-order linear Equation (4) or the first-order non-linear Equation (5a). It is well-known (reference (d)) that descending continued fractions can in general be associated by one-one correspondence with infinite series; but this does not mean that these are processes of equal order. In somewhat the same way, linear and quadratic algebraic equations can share a root, but this coincidence does not imply an equivalence of process order or complexity.

3 The equations in question, which appear in numerous standard works (e.g. references (c), (d)), viz.,

$$A_n = b_n A_{n-1} + a_n A_{n-2},$$
$$B_n = b_n B_{n-1} + a_n B_{n-2}, \qquad n = 1, 2, \cdots$$
$$A_{-1} = 1, \quad B_{-1} = 0, \quad A_0 = 0, \quad B_0 = 1.$$

are distinguished from Equation (4) by an algebraic sign difference and by the presence of specified boundary conditions for finite n. We note that the difference equation treated, e.g., by Milne-Thomson (reference (c), p. 378) is in no sense "equivalent" to the continued fraction he considers, but rather to the numerators or denominators of approximants (reference (d), p. 15) of that continued fraction. On the other hand Nörlund (reference (e), pp. 438-441) does formally develop a difference equation into a continued fraction, but uses an identity (attributed by Milne-Thomson [reference (c), pp. 532-534] to Thiele) without a remainder term and thus appropriate to the treatment of finite boundary conditions, rather than those "at infinity." Terminal summability methods require consideration of remainder sequences and consequently must be based on identities, such as those of the text and Table 1-1 that display remainder terms explicitly.

4 In general what is referred to here is a remainder term that is not identically 0 or ∞ , or, in the case of the multiplicative remainders occurring in infinite products, not identically unity. If the i^{th} application of the repeated algebraic operation involved in the linear process is denoted t_i and the point imaged in the n^{th} stage of this process is denoted w_n, then a terminated discrete infinite process is expressed as

$$\lim_{n\to\infty} t_n\left[\cdots t_2\left[t_1\left[t_0(w_n)\right]\right]\cdots\right],$$

whereas the corresponding statically terminated or "Cauchy" process is expressed as

$$\lim_{n\to\infty} t_n\left[\cdots t_2\left[t_1\left[t_0(\alpha)\right]\right]\cdots\right],$$

where α is a specified fixed point in the complex plane. w_n, $n = 1,2,...$ is a sequence of exact or approximate remainders subject to further definition.

5 The solution of this problem is more extensively discussed in reference (f).

6 In connection with this identity we call attention to an omission in the work of Perron, which is significant because it seems to represent an oversight by all workers in the field of continued fractions. In reference (h), p. 98, Perron gives the equation

$$\frac{x_0}{x_1} = \frac{A_{\nu-1}x_\nu + a_\nu A_{\nu-2}x_{\nu+1}}{B_{\nu-1}x_\nu + a_\nu B_{\nu-2}x_{\nu+1}}$$

and states that it implies "dass x_0/x_1 zwischen $A_{\nu-2}/B_{\nu-2}$ und $A_{\nu-1}/B_{\nu-1}$ liegt." However, by a formal rearrangement exactly like that of Equation (15), we bring Perron's expression into the form

$$\frac{x_0}{x_1} = \frac{A_{\nu-2}}{B_{\nu-2}} + \left(\frac{\dfrac{A_{\nu-1}}{B_{\nu-1}} - \dfrac{A_{\nu-2}}{B_{\nu-2}}}{1 + \left(\dfrac{x_{\nu+1}}{x_\nu}\right) a_\nu \dfrac{B_{\nu-2}}{B_{\nu-1}}} \right).$$

Clearly, exceptional sequences $(x_{\nu+1}/x_\nu)$, $\nu = 1,2,\ldots$, can be found that give x_0/x_1 a value entirely different from $\lim_{n\to\infty} A_\nu/B_\nu$. (Example: The sequences of Equation (12) are of this type.) Perron's conclusion, "Der Grenzwert $\lim_{n\to\infty} A_\nu/B_\nu$ kann also, wenn er existiert, nicht von x_0/x_1 verschieden sein," is therefore invalidated. (This is fortunate; otherwise the hydrogen atom in our second example would lack eigenvalues, a serious consequence for the consistency of classical mathematics. For in that example $\lim_{n\to\infty} A_\nu/B_\nu$ existiert and is positive for $\lambda > 0$, a circumstance that disqualifies it from representing the value of a c.f. satisfying the indicial equation, wherein x_0/x_1 is negative.) Perron overlooks exceptional sequences through requiring $x_\nu > 0$.

7 For the O- and o-notations used throughout this report see, for example, reference (i).

8 For a definition of asymptotic expansion see reference (i).

9 By this we mean that "$\mathcal{L}$" denotes in general a different linear combination of basis functions wherever it appears in the text.

10 Another more common notation is $\varphi_i(n+\nu) = \varphi_i(n) + o(\varphi_i(n))$ as $n\to\infty$.

11 A more thorough exposition would introduce in addition to the dominant term μ-periodicity defined here a concept of complete asymptotic μ'-periodicity. Specifically, the function sequence ξ_n, $n = 1,2,\ldots$, each member of which possesses an asymptotic expansion as $n\to\infty$, displays complete asymptotic μ'-periodicity if for all sufficiently large n there exists a μ' (equal to a

positive integral multiple of μ) which is the smallest integer such that both $\xi_n \sim \mathcal{S}(\varphi(n))$ and $\xi_{n+\mu'} \sim \mathcal{S}(\varphi(n+\mu'))$ as $n \to \infty$, where $\mathcal{S}$ denotes the same asymptotic expansion (i.e., an identical expansion in the same basis functions φ_i, $i = 1,2,\ldots$, with the same numerical coefficients). Complete asymptotic μ'-periodicity will obtain if for n large enough there is a cyclic repetition of functional forms ($\xi_n = u(n)$, $\xi_{n+\mu'} = u(n+\mu')$) as well as dominant terms ($\xi_n \simeq \xi_{n+\mu}$ as $n \to \infty$), the period μ of the latter cycle being in general a sub-multiple of the period μ' of the former. To accommodate this generalization, Theorem 1 should include among its sufficient conditions for μ-μ' periodic asymptotic tractability the condition that the λ_n-sequence, $n = 1,2,\ldots$, exhibit complete asymptotic μ'-periodicity; and μ' should replace μ in equation (30b), etc. Example 2 in Section 2-8 treats a case in which μ' differs from μ ($\mu=1, \mu'=2$). The reader can regard the text as specialized to the case $\mu' = \mu$, and can readily supply the modifications needed to treat the more general case.

[12]The requirement of μ- periodicity of the remainder sequence of the c.f. in equation (17) is equivalent to the imposition of μ-periodic "boundary conditions at infinity" on the (ratios of) solutions of the equivalent difference equation (19a).

[13]Any analytic function of functions [A] can also be considered asymptotically tractable in an extended sense. For example, if $\lambda_n = e^{f(n)}\, g(n)$, $n = 1,2,\ldots$, where f(n) and g(n) both possess standard asymptotic expansions (not necessarily in the same basis), one can find a trial c.f. remainder sequence ρ_n, $n = 1,2,\ldots$, of the form $\rho_n \sim e^{F(n)}\, G(n) + \text{constant}$, where F, G are standard asymptotic expansions, with undetermined coefficients, expressed in expansion bases simply related to those of f, g, respectively. In this case the definition of μ-periodicity has to be extended, so that $\rho_{n+\mu} \simeq \rho_n$ as $n \to \infty$, $\mu = 1,2,\ldots$, is interpreted to mean equality merely of dominant terms of the dominant expansion; i.e., in this case, if

$\lim_{n\to\infty}|F(n)| = \infty$, then $F(n+\mu) \simeq F(n)$ as $n \to \infty$. (Similarly, $\lambda_{n+\mu} \simeq \lambda_m$ means, if $\lim_{n\to\infty}|f(n)| = \infty$, that $f(n+\mu) \simeq f(n)$ as $n \to \infty$.) The special case F = polynomial is the one of principal interest. We omit such generalizations from a first exposition because they tend to obscure the simplicity of the subject, and because they amount to little more than variations on the basic theme.

[14]By similar reasoning one can establish that the terminal sum of a c.f. $[A_\mu]$, $\mu = 1,2,\ldots$, is invariant under any equivalence transformation not involving division by zero.

[15]As discussed in footnote 11, μ' should replace μ in this step, if the two differ.

[16]It should be noted that our summation method, based on the characteristic equation, may have to be modified in certain cases, e.g., when the roots of the characteristic equation are conjugate pure imaginaries. In its most fundamental aspect, the remainder term $\mathcal{R}_{n+1}$ is a ratio of general solutions of the equivalent difference equation,

$$\frac{A\,C^{(1)}_{n+1} + B\,C^{(2)}_{n+1}}{A\,C^{(1)}_{n} + B\,C^{(2)}_{n}},$$

where $C^{(1)}_n$, $C^{(2)}_n$ comprise a fundamental system of solutions of the equivalent difference equation and A,B are arbitrary constants. Usually, "standard remainder conditions," $A = 0$, $B \neq 0$, or $A \neq 0$, $B = 0$, serve to convert the problem to its simpler form, treated in the text, in which only ratios of fundamental solutions enter -- these corresponding (in their dominant terms as $n \to \infty$) to the roots of the characteristic equation. It is speculated that, even when such a simplification does not occur, it may prove possible to accomplish c.f. terminal summation by direct use of asymptotic expansions of solutions of the equivalent difference equation (4), with nonstandard remainder conditions dictated by the problem. In this direction the theory is presently undeveloped.

[17] In the special case that $\lim_{n} \left|\theta(n)\,\varphi_i(n)\right| = \infty$ for all i (i.e., $|\theta(n)|$ very rapidly increasing as $n \to \infty$), it is expedient, for reasons that will emerge in connection with Definition 11, to add the requirement that the series symbolized by $\mathcal{L}(\varphi(n))$ in equation (66a) be convergent in the Cauchy sense. We shall treat this extra requirement, at least in application to remainder sequences of infinite series, as part of the definition of broad asymptotic tractability. The condition might be weakened from true Cauchy convergence to a requirement that the series be sufficiently rapidly convergently beginning, since this would permit accurate numerical evaluations to be made and would relieve us of the necessity to verify Cauchy convergence (which can be very difficult in practice, where the coefficients in the series are determined by complicated recurrence relations). However, we do not pursue this line of development here.

[18] Powerful methods of summability by nonlinear transformations of the finite summand sequence have been advanced by S. Lubkin (reference (n)) and by D. Shanks (reference (o)). By his W-transformation Lubkin finds 1 + 2 + 3 + 4 + ... = 0, which agrees with one of the results obtained by our method at the "point of ambiguity" (cf. Section 3.5), but not with $\zeta(-1) = -1/12$, as yielded by $s \to -1$ in our equation (78).

[19] Note that equations (78) and (80) both exemplify the case $\lim_{n\to\infty} \left|\theta(n)\,\varphi_i(n)\right| = \infty$ for all i. Hence we should verify Cauchy convergence of the series in square brackets. For (78) this is easily done, but for (80) the case $|z| \ll 1$ presents a problem, since the formation law of the coefficients in the limit $|z| \to 0$ is not obvious. This illustrates the desirability of being able to relax the Cauchy convergence requirement in favor of a criterion involving the properties only of an initial portion of the series, as mentioned in footnote 17.

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13. ABSTRACT The simplest forms of discrete infinite processes, such as infinite series, products, continued fractions and their generalizations are considered. It is shown that by associating such processes with "equivalent" linear difference equations with boundary conditions at infinity a means of classifying them in a unified way is provided, as well as a means of evaluating asymptotic approximations to remainder sequences. If the approximate remainder sequences are introduced at the definitional level, so that the "value" of the infinite process is defined as a limit of successive stages of the finite process <u>with an approximate remainder term included</u> at each stage, two benefits result. First, where the process converges by the Cauchy definition (zero remainder terms), convergence is speeded, so that numerical computations of "value" are aided. Secondly, where the process is Cauchy-divergent, it may nevertheless be "summed" to a useful value. A broad class of processes, termed "asymptotically tractable," is identified for which these benefits are obtained. This class appears to include most cases of interest in classical analysis. When applied to infinite series, the method appears to exceed in convergence-forcing power all other known approaches to "summability." When applied to continued fractions and their generalizations, it reveals a possibility of multiple-valuedness in such processes that apparently has not hitherto been recognized. Examples are given to illustrate the implications and advantages of the new definitional approach. These should be of interest to physicists and engineers concerned with convergence of infinite processes or with the solution of linear recurrence relations arising in physical problems.

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