

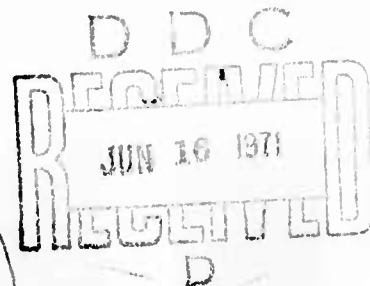
THE INVERSE OF A BOOLEAN MATRIX

by

Robert S. Ledley

National Biomedical Research Foundation
8600 Sixteenth Street, Silver Spring, Maryland

NBR-65071/3265
Contract Nonr-3265(00)



1. Partition of Matrices

Introduction. In this paper we shall develop the necessary and sufficient conditions for a Boolean matrix to have a left and a right inverse. Boolean matrices are arrays of Boolean functions just as ordinary matrices are arrays of numbers. The rules for Boolean matrix multiplication are the same as for ordinary matrix multiplication, except that summation is replaced by logical summation, and the product is replaced by a logical product.* Note, of course, that the elements of the matrices might just be 0 or 1, which would be treated as a "universally true" and a "universally false" Boolean function, i.e. the least upper bound and the greatest lower bound of all the Boolean functions. Use is made of the partition of the Boolean matrices, and hence we shall first briefly review the necessary elements of this subject.

Partitioning. A matrix can be partitioned into submatrices. For example the matrices

$$\underline{A} = \begin{pmatrix} A & \overline{A} \cdot \overline{B} & B \\ A \cdot \overline{B} & \overline{B} & A + B \\ \hline B & A \cdot B & \overline{A} + B \end{pmatrix} \quad \text{and} \quad \underline{B} = \begin{pmatrix} B & A + B & \overline{B} \\ A \cdot B & \overline{B} & \overline{A} \cdot \overline{B} \\ \hline \overline{A} & A \cdot \overline{B} & \overline{B} \end{pmatrix}$$

can be partitioned into submatrices as indicated by the dashed lines. The matrix $\underline{A}$ has been partitioned into the matrix and row vector

$$\underline{A}_1 = \begin{pmatrix} A & \overline{A} \cdot \overline{B} & B \\ A \cdot \overline{B} & \overline{B} & A + B \end{pmatrix} \quad \text{and} \quad \underline{A}_2 = (B, A \cdot B, \overline{A} + B)$$



* See page 474 in Programming and Utilizing Digital Computers by R. S. Ledley, McGraw-Hill Book Company, New York, 1962

Available for
Unlimited Distribution

Reproduced by
NATIONAL TECHNICAL
INFORMATION SERVICE
Springfield, Va. 22151

while the matrix $\underline{B}$ has been partitioned into

$$\underline{B}_1 = \begin{pmatrix} B & A + B \\ A \cdot B & \bar{B} \\ \bar{A} & A \cdot \bar{B} \end{pmatrix} \quad \text{and} \quad \underline{B}_2 = \begin{pmatrix} \bar{B} \\ \bar{A} \cdot \bar{B} \\ \bar{B} \end{pmatrix}$$

The importance of the notions of submatrix becomes clear with the following observations. From the definition, an element c_{ij} of a matrix product $\underline{C}$ is derived from the i^{th} row of $\underline{A}$ and the j^{th} column of $\underline{B}$. Thus it follows that the submatrix of $\underline{C}$ consisting of elements of rows $i_1, i_2, \dots, i_n$ and columns $j_1, j_2, \dots, j_n$ will be the product of the submatrix of $\underline{A}$ consisting of these rows and the submatrix of $\underline{B}$ consisting of these columns. For example for our matrices $\underline{A}$ and $\underline{B}$ above we have

$$\underline{A} \otimes \underline{B} = \begin{pmatrix} \underline{A}_1 \\ \underline{A}_2 \end{pmatrix} \otimes (\underline{B}_1, \underline{B}_2) = \begin{pmatrix} \underline{A}_1 \otimes \underline{B}_1 & \underline{A}_1 \otimes \underline{B}_2 \\ \underline{A}_2 \otimes \underline{B}_1 & \underline{A}_2 \otimes \underline{B}_2 \end{pmatrix} = \begin{pmatrix} B & A + \bar{B} & \bar{B} \\ \bar{A} \cdot B & \bar{B} & \bar{B} \\ \bar{A} + B & B & \bar{A} \cdot \bar{B} \end{pmatrix}$$

where, as the reader should independently verify, the dashed lines in this last result give the partitioned matrices corresponding to the submatrix products. Note that we can further partition the rows of $\underline{A}$ and the columns of $\underline{B}$ in a corresponding manner, and work with even smaller submatrices. For example, if

$$\begin{aligned} A_{11} &= \begin{pmatrix} A & \bar{A} \cdot \bar{B} \\ A \cdot \bar{B} & \bar{B} \end{pmatrix}, \quad A_{12} = \begin{pmatrix} B \\ A + B \end{pmatrix}, \quad A_{21} = (B, A \cdot B), \quad A_{22} = (\bar{A} + B) \\ B_{11} &= \begin{pmatrix} B & A + B \\ A \cdot B & \bar{B} \end{pmatrix}, \quad B_{12} = \begin{pmatrix} \bar{B} \\ \bar{A} \cdot \bar{B} \end{pmatrix}, \quad B_{21} = (\bar{A}, A \cdot \bar{B}), \quad B_{22} = (\bar{B}) \end{aligned}$$

then our above matrix product becomes

$$\begin{aligned} \underline{A} \otimes \underline{B} &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \otimes \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} (A_{11} \otimes B_{11} + A_{12} \otimes B_{21}) & (A_{11} \otimes B_{12} + A_{12} \otimes B_{22}) \\ (A_{21} \otimes B_{11} + A_{22} \otimes B_{21}) & (A_{21} \otimes B_{12} + A_{22} \otimes B_{22}) \end{pmatrix} \\ &= \begin{pmatrix} B & A + \bar{B} & \bar{B} \\ \bar{A} \cdot B & \bar{B} & \bar{B} \\ \bar{A} + B & B & \bar{A} \cdot \bar{B} \end{pmatrix} \end{aligned}$$

as the reader should independently verify. A partitioning which is consistent with the matrix product notation is called a conformable partitioning. There can be partitioning at several levels, the lowest level of which results in the actual matrix elements themselves.

Nonadjacent Partitioning. It is not necessary to use adjacent rows or columns when partitioning a matrix. However when adjacent rows or columns are not used, track must be kept of the original row or column positions. For example consider the matrices

$$\underline{\underline{A}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix} \quad \text{and} \quad \underline{\underline{B}} = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix}$$

where we have numbered the rows of $\underline{\underline{A}}$ and the columns of $\underline{\underline{B}}$. Now we will partition $\underline{\underline{A}}$ as follows

$$\underline{\underline{A}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} 1 \\ 3 \\ 5 \\ 6 \\ 2 \\ 4 \\ 7 \end{matrix} \quad \text{and} \quad \underline{\underline{B}} \text{ conformably as } \underline{\underline{B}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} 1 & 3 & 5 & 6 \\ 2 & 4 & 7 \end{matrix}$$

from which we have the matrix multiplication

$$\underline{\underline{A}} \otimes \underline{\underline{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

whence

$$\underline{\underline{A}} \otimes \underline{\underline{B}} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

where in this last step we have rearranged the rows and columns into their original order.

2. The Inverse of a Boolean Matrix

Definitions. We shall use our idea of partitioned matrices to develop the important concept of the inverse of a Boolean matrix. The left inverse $\underline{A}_L^{-1}$ of a Boolean matrix $\underline{A}$ is defined as being that matrix for which

$$\underline{A}_L^{-1} \otimes \underline{A} = \underline{H} \quad (\text{the identity matrix}) \quad [1]$$

Similarly the right inverse $\underline{A}_R^{-1}$ is a matrix such that

$$\underline{A} \otimes \underline{A}_R^{-1} = \underline{H}$$

An inverse matrix does not exist for every matrix $\underline{A}$. For instance, the simple matrix

$$\underline{A} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

has neither a left nor right inverse. When an inverse exists it is not necessarily unique. For example, the matrix

$$\underline{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

has four left inverses, namely

$$\underline{A}_L^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\text{i.e. } \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ etc.}$$

The properties of a permutation matrix play an important role in the development of an inverse for a matrix. For instance, we will show that a permutation matrix is the only type of matrix that can have both a left and right inverse. Also, a square matrix (i.e. of order $J \times J$) has no inverse unless it is a permutation. Let us begin with consideration of a left inverse, since the properties of a right inverse will then follow by analogy.

The Development of the Left Inverse. In the following series of theorems and corollaries we shall develop the necessary and sufficient conditions for the existence of a left inverse for a matrix, as well as a method for obtaining the left inverse if it exists.

Theorem 2-1 If a square Boolean matrix has a left inverse, then this left inverse must be a permutation matrix.

Proof. Let the square matrix $\underline{A}$ be of order $J \times J$; then in equation [1] $\underline{H}$ is of order $J \times J$ and the left inverse $\underline{A}^{-1}$ must be of order $J \times J$. Each row of $\underline{A}^{-1}$, say row i , contributes to a unit diagonal element of $\underline{H}$, namely $h_{ii} = 1$. This element arises from the i^{th} row of $\underline{A}^{-1}$ and the i^{th} column of $\underline{A}$, and in particular, say, from the unit in the k^{th} position (column) of the i^{th} row of $\underline{A}^{-1}$ and hence the k^{th} position (row) of the i^{th} column of $\underline{A}$. Consider any other of the rows of $\underline{A}^{-1}$, say the j^{th} row. This j^{th} row must have a zero in the k^{th} position, for if it had a unit in the k^{th} position, then $h_{ji} = 1$ contrary to the definition of $\underline{H}$. Thus the k^{th} column of $\underline{A}^{-1}$ can have only a single unit and this in the i^{th} row. Since this line of reasoning holds for each of the rows of $\underline{A}^{-1}$, then each of the J columns of $\underline{A}^{-1}$ must have a single unit, and each such unit must correspond to a different row of the J rows. Thus $\underline{A}^{-1}$ is a permutation, as desired.

Corollary 2-2. If a left inverse $\underline{A}^{-1}$ exists for a matrix $\underline{A}$ of order $I \times J$, then $J \leq I$.

Proof. Note that the order of $\underline{A}^{-1}$ must be $J \times I$ and the order of $\underline{H}$ must be $J \times J$. Now suppose that $J > I$, and partition $\underline{A}$ into

$$\underline{A} = (\underline{S}, \underline{E}) \text{ and } \underline{A}^{-1} \text{ conformably into } \underline{A}^{-1} = \begin{bmatrix} \underline{S}' \\ \underline{E}' \end{bmatrix}$$

where $\underline{S}$ and $\underline{S}'$ are square of order $I \times I$, and $\underline{E}$ is order $I \times (J - I)$ and $\underline{E}'$ is of order $(J - I) \times I$. Now

$$\underline{A}^{-1} \otimes \underline{A} = \begin{bmatrix} \underline{S}' \\ \underline{E}' \end{bmatrix} \otimes (\underline{S}, \underline{E}) = \begin{bmatrix} \underline{S}' \otimes \underline{S} & \underline{S}' \otimes \underline{E} \\ \underline{E}' \otimes \underline{S} & \underline{E}' \otimes \underline{E} \end{bmatrix} = \underline{H}$$

Since $\underline{S}' \otimes \underline{S}$ and $\underline{E}' \otimes \underline{E}$ are square, they must contain all the diagonal units of $\underline{H}$. Thus

$$\underline{S}' \otimes \underline{S} = \underline{H}'$$

where $\underline{H}'$ is a unit matrix of order $I \times I$. Hence $\underline{S}'$ is a left inverse and by Theorem 2-1 and Corollary 2-1 they are both permutations. Now

$$\underline{E}' \otimes \underline{S} = \underline{Z} \quad \text{and} \quad \underline{S}' \otimes \underline{E} = \underline{Z}'$$

where $\underline{Z}$ is a matrix of order $(J - I) \times I$ with all zero elements, and $\underline{Z}'$ is a matrix of order $I \times (J - I)$ with all zero elements. Since $\underline{S}$ and $\underline{S}'$ are permutations we can conclude that $\underline{E}'$ and $\underline{E}$ must have all zero elements. Hence

$$\underline{E}' \otimes \underline{E} = \underline{Z}''$$

where $\underline{Z}''$ is a matrix of order $(J - I) \times (J - I)$ with all zero elements, which is a contradiction to our hypothesis that $J > I$. Thus $J \leq I$ as desired.

Corollary 2-3. If a Boolean matrix $\underline{A}$ of order $I \times J$ has a left inverse $\underline{A}_L^{-1}$ of order $J \times I$, then $\underline{A}$ can be partitioned into a permutation matrix $\underline{S}$ of order $J \times J$ and a remaining matrix $\underline{E}$ of order $(I - J) \times J$.

Proof. By corollary 2-2, $J \leq I$ and hence $\underline{A}_L^{-1}$ can be partitioned as

$$\underline{A}_L^{-1} = (\underline{S}', \underline{E}') \text{ and } \underline{A} \text{ can be partitioned conformably as } \underline{A} = \begin{pmatrix} \underline{S} \\ \underline{E} \end{pmatrix}$$

where $\underline{S}$ and $\underline{S}'$ are square matrices of order $J \times J$, $\underline{E}$ is of order $(I - J) \times J$, and $\underline{E}'$ is of order $J \times (I - J)$. Then

$$\underline{A}_L^{-1} \otimes \underline{A} = (\underline{S}', \underline{E}') \otimes \begin{pmatrix} \underline{S} \\ \underline{E} \end{pmatrix} = (\underline{S}' \otimes \underline{S} + \underline{E}' \otimes \underline{E}) = \underline{H}$$

where $\underline{H}$ is of order $J \times J$. Let us consider in detail a method for choosing the columns of $\underline{S}'$. Since each row of $\underline{A}_L^{-1}$ contributes to its corresponding unit in $\underline{H}$, there must be a unit in the row which acts to form the unit in $\underline{H}$ under the matrix multiplication operation. Choose the column of this unit to be in $\underline{S}'$. Now no two rows can have their contributing unit in the same column for reasons analogous to those worked out in the proof of Theorem 2-1. Hence $\underline{S}'$ is a permutation, and when matrix multiplied by the conformable $\underline{S}$ gives $\underline{H}$, the unit matrix. Thus $\underline{S}'$ is a left inverse for $\underline{S}$, and hence $\underline{S}$ is a permutation matrix, as was to be proved. Note that $\underline{E}' \otimes \underline{E} \rightarrow \underline{H}$ and that $\underline{S}' = \underline{S}^t$ (where $\rightarrow$ indicates implication, or inclusion, and t represents transposition).

Theorem 2-2. A necessary and sufficient condition that a matrix $\underline{\underline{A}}$ of order $I \times J$ has a left inverse is that $\underline{\underline{A}}$ can be partitioned as follows:

$$\underline{\underline{A}} = \begin{pmatrix} \underline{\underline{S}} \\ \underline{\underline{E}} \end{pmatrix}$$

where $\underline{\underline{S}}$ is a permutation matrix of order $J \times J$ and $\underline{\underline{E}}$ is the remaining matrix. Then the left inverse is given by

$$\underline{\underline{A}}^{-1} = (\underline{\underline{S}}^t, \underline{\underline{E}}')$$

where $\underline{\underline{E}}'$ is any matrix such that $\underline{\underline{E}}' \otimes \underline{\underline{E}} \rightarrow \underline{\underline{H}}$.

Proof. The necessity of the condition follows from corollary 2-3. That the condition is sufficient can be shown by demonstrating that $\underline{\underline{A}}^{-1}$ is actually the left inverse:

$$\underline{\underline{A}}^{-1} \otimes \underline{\underline{A}} = (\underline{\underline{S}}^t, \underline{\underline{E}}') \otimes \begin{pmatrix} \underline{\underline{S}} \\ \underline{\underline{E}} \end{pmatrix} = (\underline{\underline{S}}^t \otimes \underline{\underline{S}} + \underline{\underline{E}}' \otimes \underline{\underline{E}})$$

But as $\underline{\underline{S}}$ is a permutation matrix, $\underline{\underline{S}}^t \otimes \underline{\underline{S}} = \underline{\underline{H}}$, here of order $J \times J$, whence

$$\underline{\underline{A}}^{-1} \otimes \underline{\underline{A}} = \underline{\underline{H}} + \underline{\underline{E}}' \otimes \underline{\underline{E}} = \underline{\underline{H}}$$

since by definition $\underline{\underline{E}}' \otimes \underline{\underline{E}} \rightarrow \underline{\underline{H}}$. This completes the proof.

As an example consider the matrix

$$\underline{\underline{A}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\text{Here } \underline{\underline{S}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$\underline{\underline{S}}^t = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\text{Now } \underline{\underline{E}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

and hence $\underline{\underline{E}}' =$

$$\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \\ e_{31} & e_{32} \\ e_{41} & e_{42} \end{pmatrix}$$

such that:

$$\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \\ e_{31} & e_{32} \\ e_{41} & e_{42} \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

By inspection we have that

$$\underline{E'} = \begin{pmatrix} \phi & 0 \\ \phi & 0 \\ \phi & 0 \\ \phi & 0 \end{pmatrix}$$

where ϕ stands for a zero or a unit. Thus there are $2^4 = 16$ possible inverses for A, namely

$$\underline{A}^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and so forth.

As a second example consider

$$\underline{A} = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

where we have numbered the rows. Here S becomes

$$\underline{S} = \begin{matrix} 1 \\ 3 \\ 6 \\ 7 \end{matrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and the conformable } \underline{S'} = \underline{S}^t = \begin{matrix} 1 & 3 & 6 & 7 \end{matrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

where we have numbered the rows and columns as required.

Now

$$\underline{E} = \begin{matrix} 2 \\ 4 \\ 5 \end{matrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \quad \text{whence } \underline{E'} = \begin{matrix} 2 & 4 & 5 \end{matrix} \begin{pmatrix} \phi & 0 & 0 \\ \phi & \phi & 0 \\ \phi & 0 & 0 \end{pmatrix}$$

Thus

$$A_2^{-1} = \begin{pmatrix} 0 & \phi & 0 & 0 & 0 & 1 & 0 \\ 1 & \phi & 0 & \phi & 0 & 0 & 0 \\ 0 & \phi & 0 & 0 & 0 & 0 & 1 \\ 0 & \phi & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which, due to the five occurrences of ϕ actually represents 2^5 different matrices. Note that we could have used row 4 instead of row 1 in $\underline{S}$. It makes no difference provided that whichever is chosen is used consistently. However, observe that if we had used row 4 in $\underline{S}$, then

$$A_4^{-1} = \begin{pmatrix} 0 & \phi & 0 & 0 & 0 & 1 & 0 \\ \phi & \phi & 0 & 1 & 0 & 0 & 0 \\ 0 & \phi & 0 & 0 & 0 & 0 & 1 \\ 0 & \phi & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which results in some different inverses than obtained with row 1 in $\underline{S}$. Altogether, then, there can be $2^4 \times 3 = 48$ different left inverses for $\underline{A}$.

The Right Inverse. Analogous statements to Theorem 2-1, Corollaries 2-1, 2-2 and 2-3 can be made for a right inverse of a matrix. The analogous statement to Theorem 2-2 is

Theorem 2-3. A necessary and sufficient condition that a matrix $\underline{A}$ of order $I \times J$ has a right inverse is that $\underline{A}$ can be partitioned as follows:

$$\underline{A} = (\underline{S}, \underline{E})$$

where $\underline{S}$ is a permutation matrix of order $I \times I$ and $\underline{E}$ is the remaining matrix. Then the right inverse is given by

$$\underline{A}^{-1} = \begin{pmatrix} \underline{S}^t \\ \underline{E}' \end{pmatrix}$$

where $\underline{E}'$ is any matrix such that

$$\underline{E} \otimes \underline{E}' \rightarrow \underline{H}$$

As an example consider

$$\underline{A} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Here $\underline{S} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $\underline{E} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}$ whence

$$\underline{S}' = \underline{S}^t = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } \underline{E}' = \begin{pmatrix} 0 & 0 & 0 \\ \phi & \phi & \phi \end{pmatrix}$$

or altogether $\underline{A}_{\underline{I}}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ \phi & \phi & \phi \end{pmatrix}$ which represents $2^3 = 8$ different matrices.