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Technical Note

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A New Geometrical Theorem Discovered with the Aid of a Computer

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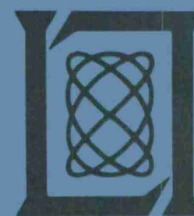
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A NEW GEOMETRICAL THEOREM DISCOVERED
WITH THE AID OF A COMPUTER

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ABSTRACT

In the course of a computer study of a new form of ball bearing, a curious invariance was noted. This led to a new theorem in the geometry of circles. A proof for this theorem, together with a useful lemma, is the subject of this Technical Note.

Accepted for the Air Force
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A NEW GEOMETRICAL THEOREM DISCOVERED WITH THE AID OF A COMPUTER

Let Γ and $\tilde{\Gamma}$ be two circles in euclidean 3-space, R^3 . Suppose there is a number x such that: (1) every point on either circle is distance x from exactly two points on the other circle. We can then select a point Z_1 on Γ and draw a zig-zag line between the two circles as follows:

Select $\tilde{Z}_1$ on $\tilde{\Gamma}$ with $|\tilde{Z}_1 - Z_1| = x$

Select $Z_2 \neq Z_1$ on Γ with $|Z_2 - \tilde{Z}_1| = x$

Select $\tilde{Z}_2 \neq \tilde{Z}_1$ on $\tilde{\Gamma}$ with $|\tilde{Z}_2 - Z_2| = x$

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Select $Z_{n+1} \neq Z_n$ on Γ with $|Z_{n+1} - \tilde{Z}_n| = x$

Select $\tilde{Z}_{n+1} \neq \tilde{Z}_n$ on $\tilde{\Gamma}$ with $|\tilde{Z}_{n+1} - Z_{n+1}| = x$.

It may happen (illustrated for the case $n = 3$ in Fig. 1) that $Z_{n+1} = Z_1$. We show that if this occurs, the zig-zag line can be started at any point on Γ and it will still close.

This remarkable fact was observed while performing certain calculations about ball bearings on a computer¹. The theorem bears a superficial resemblance to Steiner's Porism² but cannot be proved the same way.

Condition (1) is not as formidable as it may appear. If both circles lie in the same plane, with radii r and $\bar{r}$ and with centers separated by δ , elementary calculus shows that (1) is equivalent to the two inequalities:

$$\begin{aligned} |r - \bar{r}| &< x - \delta \\ x + \delta &< r + \bar{r} \end{aligned}$$

Thus suitable x 's will exist provided the smaller circle encloses the center of the larger one.

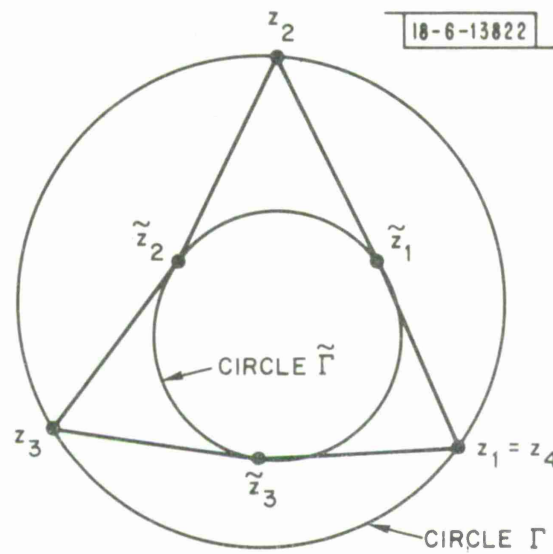


Fig. 1. Each straight line has length x , $n = 3$.

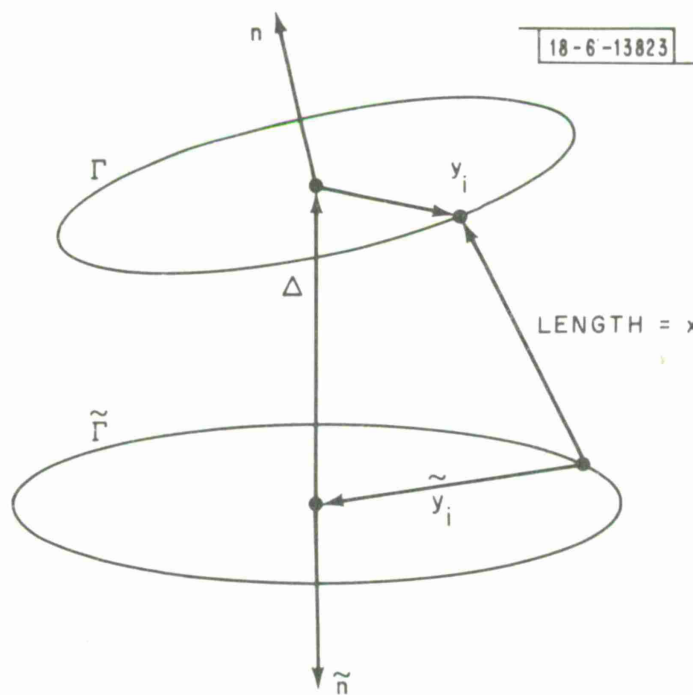


Fig. 2. Illustrating the notation used in the computation of f .

The proof proceeds as follows: Suppose for some choice of Z_1 , say $Z_1 = p$, $Z_{n+1} = Z_1$. We parametrize Γ by s , the directed arc length measured from p . Thus we can view Z_1 , and thus Z_{n+1} , and ultimately t , the arc length from p to Z_{n+1} as functions of s . Below we show that there is a smooth function f of s and t with properties:

$$(i.) f(s,s) = 1 \text{ for all } s$$

$$(ii.) f(s,t) = \frac{dt}{ds}$$

Application of a well known uniqueness theorem³ assures us that this ordinary differential equation:

$$\frac{dt}{ds} = f(s,t)$$

$$t(0) = 0$$

has only one solution. By (i), $t(s) = s$ is the solution. But this implies that $Z_{n+1} = Z_1$ for all Z_1 .

In constructing f we will need the following lemma:

Lemma: Let B, n, Z_1 and Z_2 be vectors in R^3 , satisfying:

$$(a.) |y_1| = |y_2|$$

$$(b.) |B + y_1| = |B + y_2|$$

$$(c.) n \cdot y_1 = n \cdot y_2$$

$$(d.) y_1 \neq y_2$$

Then:

$$B \times n \cdot (y_1 + y_2) = 0$$

Proof: If B and n are dependent the result is obvious. If not, by squaring (b) and (a) and taking the difference we obtain:

$$(e.) B \cdot y_1 = B \cdot y_2$$

In conjunction with (c), (e) shows that y_1 and y_2 have the same orthogonal projection on the plane spanned by B and n . By (a) and (d) the components of y_1 and y_2 normal to this plane must be equal and opposite, so $y_1 + y_2$ lies in the B, n plane. From this the conclusion is evident.

We can now compute f . We will use the following notation:

n is the unit normal to circle Γ

$\tilde{n}$ " " " " $\tilde{\Gamma}$

y_i is the vector from the center of Γ to Z_i

$\tilde{y}_i$ is the negative of the vector from the center of $\tilde{\Gamma}$ to $\tilde{Z}_i$.

Δ is the vector from the center of $\tilde{\Gamma}$ to the center of Γ .

Figure two illustrates this notation. Imagine a slight motion of y_1 along the circle. Since dy_1 is perpendicular to both n and y_1 we can write:

$$dy_1 = n \times \frac{y_1}{|y_1|} |dy_1| \quad (2)$$

As y_1 moves, $\tilde{y}_1$ also must move, keeping

$$|\Delta + y_1 + \tilde{y}_1| = x \quad (3)$$

Squaring (3) and differentiating gives:

$$(\Delta + y_1 + \tilde{y}_1) \cdot (dy_1 + d\tilde{y}_1) = 0 \quad (4)$$

Substituting into (4), (2) and (5) where:

$$d\tilde{y}_1 = \tilde{n} \times \frac{\tilde{y}_1}{|\tilde{y}_1|} |d\tilde{y}_1| \quad (5)$$

gives:

$$(\Delta + \tilde{y}_1) \cdot n \times \frac{y_1}{|y_1|} |dy_1| + (\Delta + y_1) \cdot \tilde{n} \times \frac{\tilde{y}_1}{|\tilde{y}_1|} |d\tilde{y}_1| = 0 \quad (6)$$

Thus:

$$\frac{|d\tilde{y}_1|}{|dy_1|} = - \frac{|\tilde{y}_1| (\Delta + \tilde{y}_1) \cdot n \times y_1}{|y_1| (\Delta + y_1) \cdot \tilde{n} \times \tilde{y}_1} \quad (7)$$

Similarly,

$$\frac{|dy_2|}{|d\tilde{y}_1|} = - \frac{|y_2| (\Delta + y_2) \cdot \tilde{n} \times \tilde{y}_1}{|\tilde{y}_1| (\Delta + \tilde{y}_1) \cdot n \times y_2} \quad (8)$$

Multiplying (7) by (8) and using the lemma in the forms:

$$(\Delta + \tilde{y}_1) \cdot n \times y_1 = - (\Delta + \tilde{y}_1) \cdot n \times y_2$$

$$(\Delta + y_2) \cdot \tilde{n} \times \tilde{y}_1 = - (\Delta + y_2) \cdot \tilde{n} \times \tilde{y}_2$$

gives:

$$\frac{|dy_2|}{|dy_1|} = \frac{(\Delta + y_2) \cdot \tilde{n} \times \tilde{y}_2}{(\Delta + y_1) \cdot \tilde{n} \times \tilde{y}_1} \quad (9)$$

Multiplying n similar expressions gives:

$$\frac{dt}{ds} = \frac{|dy_{n+1}|}{|dy_1|} = \frac{(\Delta + y_{n+1}) \cdot \tilde{n} \times \tilde{y}_{n+1}}{(\Delta + y_1) \cdot \tilde{n} \times \tilde{y}_1} = f(s, t)$$

Clearly if $z_{n+1} = z_1$, then $y_{n+1} = y_1$ and $\tilde{y}_{n+1} = \tilde{y}_1$, whence:

$$f(s, s) = 1 \quad .$$

This completes the proof.

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2. H. S. M. Coxeter and M. C. Greitzer, Geometry Revisited, Random House, pp. 124, 125 (1967).
3. E. A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York, p. 10 (1955).

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