NAVAL SHIP RESEARCH AND DEVELOPMENT CENTER
Washington, D.C. 20034

INTEGRAL EQUATION METHODS IN SOUND RADIATION AND SCATTERING FROM ARBITRARY SURFACES

by
George Chertock

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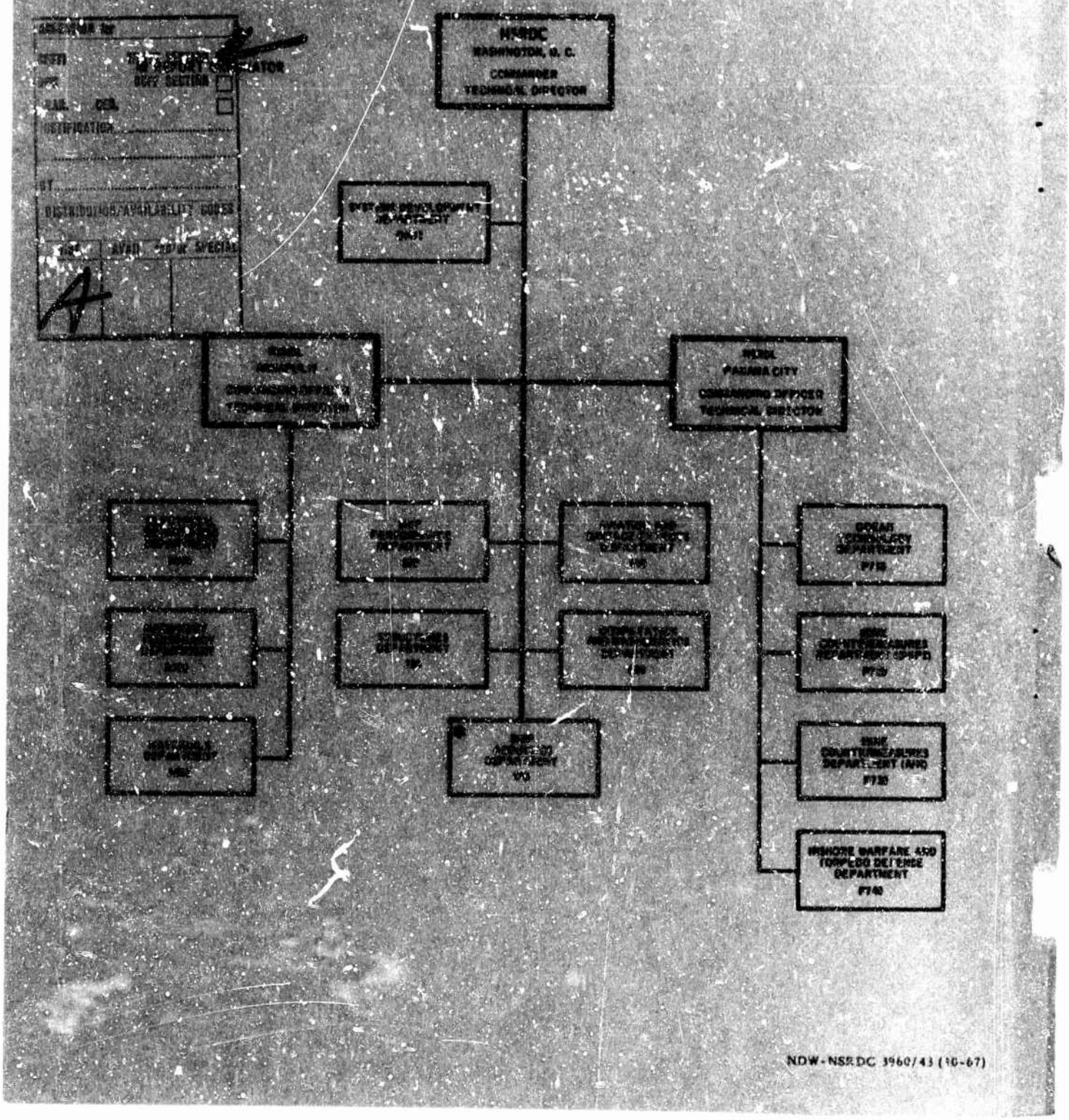

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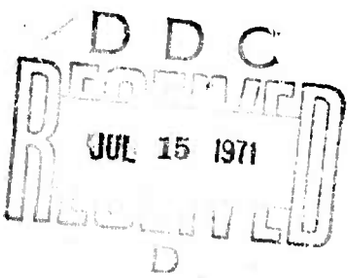

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TABLE OF CONTENTS

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INTRODUCTION

Some integral equation methods that are currently being used in the numerical solution of radiation and scattering problems are described and analyzed. The intention is not to provide a comprehensive review of the very extensive literature but rather to give a unified derivation of the methods which stresses their common features and compares their relative merits as they appear to the author.

These methods have been used by the author in a variety of specific engineering problems, e.g., the distribution of sound radiation from a vibrating ship hull, the directivity pattern of an array of sound transducers, the scattering of explosive sound by the resonant vibrations of a ship hull, and the thrust-deduction factor of a marine propeller. All of these problems required a solution to the wave equation for the region outside an arbitrary closed surface, given some specified relation between the sound pressure and its normal gradient at the surface.

However, we will consider here only a restricted version of this problem, namely, where the pressure is harmonic in time and where either the pressure or its normal gradient is specified at the surface. We take the sound pressure to be the real part of $p \exp(-ikct)$ and the gradient to be the real part of $q \exp(-ikct)$, where p and q are complex. We denote an arbitrary point inside, on, and outside the closed surface by the vectors $\underline{x}$, $\underline{y}$, and $\underline{z}$, respectively. Hereafter we will omit the time factor from the equations and also we will tacitly assume that the real parts are taken. Then the differential equation is the Helmholtz equation

$$(\nabla^2 + k^2)\, p(\underline{z}) = 0 \tag{1}$$

If the normal gradient of the pressure is specified at the surface (Neumann boundary conditions), the boundary condition is

$$\left.\frac{\partial p(\underline{z})}{\partial n}\right|_{\underline{z}=\underline{y}} = q(\underline{y}) \tag{2}$$

where $\underline{n}$ is a unit normal, pointing outward from the surface point $\underline{y}$. Or alternatively, if the surface values of the sound pressure are specified (Dirichlet boundary conditions), the boundary condition is

$$p(\underline{z}) = p(\underline{y}) \; ; \; \underline{z} \to \underline{y} \; . \tag{3}$$

We also assume some radiation condition at infinity, most simply in the form

$$p(\underline{z}) \approx \frac{e^{ikz}}{z} \; ; \; z \to \infty \tag{4}$$

where $z = |\underline{z}|$.

Of course, these simple boundary-value problems occur in many fields of applied physics but we shall continue to interpret the equations solely in terms of sound pressure waves. Even in these terms, the equations have many useful alternative physical interpretations in sound radiation and scattering. For example, there are two direct interpretations as radiation problems (i) The surface $\underline{y}$ is the outer surface of some solid body which vibrates in an arbitrary pattern at steady circular frequency $\omega = kc$, $v(\underline{y}) = q(\underline{y})/(i\rho kc)$ is the distribution of vibration velocity normal to the surface, and $p(\underline{z})$ is the sound pressure distribution in the external space; (ii) The surface $\underline{y}$ is any geometric surface which encloses all the sources of sound in the fluid, the sound sources all have circular frequency $\omega = kc$, $p(\underline{z})$ is the external sound pressure at $\underline{z}$, and $q(\underline{y})/(i\rho kc)$ is the normal component of the particle velocity of the fluid at $\underline{y}$.

There are also two straightforward interpretations as simple scattering problems (i) The surface $\underline{y}$ is a rigid impenetrable surface that scatters an incoming sound wave of frequency $\omega = kc$, $p(\underline{z})$ is the sound pressure of only the scattered or reflected wave (omitting the incident wave), and $-q(\underline{y})/(i\rho kc)$ is the normal component of particle velocity in the incoming wave at $\underline{y}$, provided the rigid surface and its interior are replaced by fluid; (ii) The surface $\underline{y}$ is a free (pressure-release) surface that scatters an incoming pressure wave, $p(\underline{z})$ is the sound pressure of only the scattered

or reflected wave, and $-p(\underline{y})$ is the sound pressure at $\underline{y}$ in the incoming wave, provided the free surface and its interior are replaced by fluid.

Finally there are two straightforward interpretations in terms of special source distributions in a bounded region (i) The surface $\underline{y}$ is a rigid impenetrable surface with a layer of simple sources, $q(\underline{y})$ per unit area, placed an infinitesimal distance outside the surface, and $p(\underline{z})$ is the external sound pressure; (ii) The surface $\underline{y}$ is a free (pressure-release) surface with a layer of dipoles, $\underline{n}\, p(\underline{y})$ per unit area, placed an infinitesimal distance outside the surface, and $p(\underline{z})$ is the external sound pressure. The last two interpretations are based on the fact (which we prove subsequently) that the normal gradient of the field of a layer of simple sources and the field of a layer of dipole sources are each discontinuous as the field point crosses the layer.

The mathematical methods to be described for solving the boundary value problems use integral equations to determine "equivalent" monopole and/or dipole layers whose external field is the same as in the original boundary value problem. Hence, we first review the properties of the fields of such monopole and dipole layers and derive an indefinite number of appropriate integral equations. We then show that each integral equation has an infinite sequence of characteristic wave numbers, at which the solution becomes infinite or indeterminate, and discuss how the original boundary value problem can be solved in such cases. We then describe specific techniques for approximating each of these integral equations by finite order matrix equations and for obtaining numerical solutions to each of the matrix equations. These techniques are specifically designed for problems having the arbitrary geometry and boundary conditions that commonly occur in real engineering situations. We also compare some of the relative merits of the various methods.

EQUIVALENT SURFACE LAYERS

The methods of solution are based on the possibility of finding some surface distribution of simple sources (monopoles), and/or some surface distribution of double sources (dipoles), which, if placed at

the position of the vibrating surface in an unbounded fluid, would have exactly the same pressure field outside the surface as does the vibrating surface. It turns out that there are an infinite number of such equivalent surface distributions of monopoles and dipoles;[1] we shall discuss only three special distributions.

The first equivalent surface distribution is one having monopoles only. We will take the field at $\underline{z}$ due to monopole of unit strength at $\underline{y}$ to be $e^{ikR}/4\pi R$, where $\underline{R} = \underline{z} - \underline{y}$. Then the field due to the entire surface distribution is

$$p(\underline{z}) = \iint \frac{\sigma(\underline{y}) e^{ik|\underline{z} - \underline{y}|}}{4\pi|\underline{z} - \underline{y}|} \, dS \tag{5}$$

where $\sigma(\underline{y})$ is the source strength per unit area; $\sigma(\underline{y})$ must be specified so as to satisfy the boundary condition of either Equation (2) or (3). It can be shown[1] that $\sigma(\underline{y})$ must equal $Q(\underline{y}) - q(\underline{y})$, where $q(\underline{y})$ is the normal gradient at $\underline{y}$ of the external field $p(\underline{z})$, and $Q(\underline{y})$ is the normal gradient at $\underline{y}$ of an internal pressure wave $P(\underline{x})$ whose surface values match those of the external field $p(\underline{z})$, i.e., $P(\underline{y}) = p(\underline{y})$. However, this interpretation does not define $\sigma(\underline{y})$ explicitly because the boundary condition does not define both $q(\underline{y})$ and $Q(\underline{y})$ explicitly.

The second equivalent surface distribution is made up only of dipoles which are oriented in a direction normal to the surface. Hence, since we take the field at $\underline{z}$ due to unit dipole in the $\underline{n}$ direction at $\underline{y}$ to be $(\partial/\partial n)(e^{ikR}/4\pi R)$; where $R = |\underline{z} - \underline{y}|$, the field due to the entire surface distribution is

$$p(\underline{z}) = \iint \mu(\underline{y}) \frac{\partial}{\partial n} \left[\frac{e^{ik|\underline{z} - \underline{y}|}}{4\pi|\underline{z} - \underline{y}|} \right] dS \tag{6}$$

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[1] References are listed on pages 55-57

where $\mu(\underline{y})$ is the local dipole density per unit area and must be specified so as to satisfy the boundary condition of either Equation (2) or (3). It can be shown[1] that $\mu(\underline{y})$ must equal $p(\underline{y}) - P'(\underline{y})$. where $p(\underline{y})$ is the surface value of the external pressure wave $p(\underline{z})$, and $P'(\underline{y})$ is the surface value of an internal pressure wave $P'(\underline{x})$ whose normal gradient at $\underline{y}$, $Q'(\underline{y})$, matches that of $p(\underline{z})$, i.e., $Q'(\underline{y}) = q(\underline{y})$. But again this interpretation does not define $\mu(\underline{y})$ explicitly.

The third equivalent surface distribution is made up of a particular monopole layer, whose surface density is equal to the negative gradient $-q(\underline{y})$, and a particular dipole layer, whose surface density is equal to the surface pressure $p(\underline{y})$. The statement that the combined external field of these two particular layers is equivalent to $p(\underline{z})$ is called the Helmholtz equation

$$p(\underline{z}) = -\iint q(\underline{y}) \frac{e^{ik|\underline{z} - \underline{y}|}}{4\pi|\underline{z} - \underline{y}|} dS + \iint p(\underline{y}) \frac{\partial}{\partial n}\left[\frac{e^{ik|\underline{z} - \underline{y}|}}{4\pi|\underline{z} - \underline{y}|}\right] dS \qquad (7)$$

and is commonly derived from Green's theorem.[1,2] Ordinarily, either $p(\underline{y})$ or $q(\underline{y})$ is specified explicitly by the boundary condition and the other function remains to be determined.

We shall call $\sigma(\underline{y})$ and $\mu(\underline{y})$, Green's equivalent monopole and dipole layer, respectively, since they were apparently first used by George Green in electrostatic problems about 1830. We may call $q(\underline{y})$ and $p(\underline{y})$, the Helmholtz monopole and dipole layers, respectively. If either $\sigma(\underline{y})$ or $\mu(\underline{y})$ were known, or if <u>both</u> $p(\underline{y})$ and $q(\underline{y})$ were known, then $p(\underline{z})$ could be calculated by simple quadratures. But in all the present problems, the unknown surface layer, either $\sigma(\underline{y})$, $\mu(\underline{y})$, $p(\underline{y})$, or $q(\underline{y})$, as the case may be, must be determined from the boundary condition in some way. In every case this may be done by solution of an appropriate integral equation. We will discuss at least five different integral equations which can be used for this purpose when Dirichlet boundary conditions are prescribed and at least five integral equations which can be used when Neumann boundary conditions are prescribed.

INTEGRAL EQUATIONS FOR MONOPOLE AND DIPOLE LAYERS

In mathematical terms, the integral equations can all be interpreted as limiting forms of Equations (5) and (6), or their derivatives, as the field point $\underline{z}$ approaches a surface point $\underline{y}'$. However, the mathematical processes must be carefully defined, and restrictions must be placed on the continuity of the surface layers because the integrands, the integrals, and the derivatives which occur are sometimes unbounded and/or discontinuous. A classical, rigorous and detailed analysis of these processes was originally given by Kellogg.[3] His analysis was confined to a case equivalent to $k \equiv 0$ in Equations (5) through (7). But the additional factor $\exp(ik|\underline{z} - \underline{y}|)$ is only a minor complication because this factor is continuous for all $\underline{z}$. Appendix A adapts and summarizes some of this analysis. The arguments are not meant to be rigorous but mainly to give operational and detailed meaning to the abbreviated notation which follows and which would otherwise be ambiguous. A star before the integral sign identifies terms that are written in this special abbreviated notation. The star may imply that the integral is "improper," that a "principal value" is intended, that some other special limiting process is intended, or that some other special definition applies which can be determined by reference to Appendix A.

The integral equations generally state that when you cross a monopole surface layer the pressure $p(\underline{r})$ is continuous, whereas the normal gradient $q(\underline{y})$ is discontinuous by an amount equal to the local monopole surface density. And, when you cross a dipole surface layer, the normal gradient is continuous, whereas the pressure is discontinuous by an amount equal to the local dipole surface density.

There are two integral equations for Green's monopole layer $\sigma(\underline{y})$, depending on the boundary condition. When $q(\underline{y})$ is specified apriori, an appropriate integral equation follows from the discontinuity in the normal gradient of a monopole field and can be given in the form

$$*\iint \sigma(\underline{y}) \frac{\partial}{\partial n'} \left[\frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|} \right] dS - \frac{\sigma(\underline{y}')}{2} = q(\underline{y}') \qquad (8)$$

where $\underline{n}'$ is a unit normal at $\underline{y}'$ into the region $\underline{z}$. The integral is an abbreviated notation for the principal value integral which is defined more precisely by (A10). The second term is the discontinuous component to $q(\underline{y}')$ and changes sign when $\underline{y}'$ is approached from region $\underline{x}$.

When $p(\underline{y})$ is specified apriori, an appropriate integral equation for $\sigma(\underline{y})$ follows from the continuity of a monopole field as the field point passes through the surface. Hence

$$-\iint \sigma(\underline{y}) \frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|} dS = p(\underline{y}') \tag{9}$$

where the integral is an improper integral which must be evaluated as in (A6).

There are likewise two integral equations for Green's dipole layer $\mu(\underline{y})$, depending on the boundary conditions. When $q(\underline{y})$ is specified apriori, an appropriate integral equation for $\mu(\underline{y})$ follows from the continuity of the normal gradient of a dipole field at the surface.[3] Thus, in an abbreviated notation,

$$*\iint \mu(\underline{y}) \frac{\partial^2}{\partial n' \partial n}\left[\frac{e^{ik|\underline{y}' - y|}}{4\pi|\underline{y}' - \underline{y}|}\right] dS = q(\underline{y}') \tag{10}$$

More explicit and detailed forms for this integral are given by the right hand side of either (A17) or (A18).

When $p(\underline{y})$ is specified apriori, the appropriate integral equation for the dipole density $\mu(\underline{y})$ follows from the discontinuity in the field of a dipole layer and can be given in the form

$$*\iint \mu(\underline{y}) \frac{\partial}{\partial n}\left[\frac{e^{ik|\underline{y}' - y|}}{4\pi|\underline{y}' - \underline{y}|}\right] dS + \frac{\mu(\underline{y}')}{2} = p(\underline{y}') \tag{11}$$

where $\underline{n}$ is the unit normal at $\underline{y}$. The integral term is an abbreviated notation for the principal value integral which is defined more precisely by (A14). The second term is the discontinuous component to $p(\underline{y}')$ and changes sign when the point $\underline{y}'$ is approached from within S.

There are at least three independent integral equations for the Helmholtz layers $p(\underline{y})$ and $q(\underline{y})$, and each can be used for either Dirichlet or Neumann boundary conditions. The most commonly used integral equation is the Helmholtz surface pressure equation, which is derived in Appendix A in Equation (A22). In the abbreviated notation this equation is

$$\frac{p(\underline{y}')}{2} - *\iint p(\underline{y}) \frac{\partial}{\partial n}\left[\frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\right] dS = -*\iint q(\underline{y}) \frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|} dS \qquad (12)$$

An alternative integral equation, which may be called the surface velocity equation is derived in Appendix A, Equation (A25), and can be stated in the abbreviated notation by

$$\frac{q(\underline{y}')}{2} + *\iint q(\underline{y}) \frac{\partial}{\partial n'}\left[\frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\right] dS$$

$$= *\iint p(\underline{y}) \frac{\partial^2}{\partial n' \partial n}\left[\frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\right] dS \qquad (13)$$

A third relation which can be used as an integral equation for the Helmholtz layers is the Helmholtz "interior" equation[1,2]

$$\iint p(\underline{y}) \frac{\partial}{\partial n}\left[\frac{e^{ik|\underline{x} - \underline{y}|}}{|\underline{x} - \underline{y}|}\right] dS = \iint q(\underline{y}) \frac{e^{ik|\underline{x} - \underline{y}|}}{|\underline{x} - \underline{y}|} dS \qquad (14)$$

where $\underline{x}$ is a field point <u>inside</u> the closed surface. The last relation differs from all the other integral equations because the two variable points $\underline{x}$ and $\underline{y}$ do not have a common domain.

Finally, we note the possibility of obtaining an indefinite number of additional integral equations for either $p(\underline{y})$ or $q(\underline{y})$ by a combination of (12) and (13). For example, if we multiply (13) by $\lambda(\underline{y}')$ and add to (12) we get

$$p(\underline{y}') - {}^{*}\iint p\,dS\left\{\frac{\partial}{\partial n}\left[\frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\right] - \lambda(\underline{y}')\,\frac{\partial^2}{\partial n \partial n'}\left[\frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\right]\right\}$$

$$= \frac{\lambda q(\underline{y}')}{2} - {}^{*}\iint q\,dS\left\{\frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|} - \lambda(\underline{y}')\,\frac{\partial}{\partial n'}\left[\frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\right]\right\} \qquad (15)$$

$\lambda(\underline{y}')$ could be any continuous function of $\underline{y}'$, e.g., the constant k^{-1}. However, it should have the physical dimensions of p/q, i.e., of length, in order to maintain dimensional homogeneity.

Equations (8) through (15) are all integral equations because the unknown function, either $\sigma(\underline{y})$, $\mu(\underline{y})$, $p(\underline{y})$, or $q(\underline{y})$, appears under an integral sign. When the unknown function also appears outside the integral, as in (8) and (11), and possibly (12), (13), and (15), the equation is then an integral equation of the second kind, for which the theory is complete and well documented.[4,5]

The properties of an integral equation for a given domain of integration are determined essentially by the properties of the kernel function. There are four kernel functions that appear in all of these integral equations. They all have simple physical interpretations, and they are all simply related to each other. These kernel functions are defined as follows in the abbreviated notation as well as in a detailed notation, with $\underline{R} = \underline{y}' - \underline{y}$; $R = |R|$; $\hat{\underline{R}} = \underline{R}/R$.

$$M(\underline{y}', \underline{y}) = \frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|} = \frac{e^{ikR}}{4\pi R} \qquad (16)$$

is the field at $\underline{y}'$ due to a unit monopole at $\underline{y}$ in a free unbounded field and is a function of the two vector field points.

$$D(\underline{y}',\underline{y}/\underline{n}) = \frac{\partial}{\partial n}\left[\frac{e^{ik|\underline{y}'-\underline{y}|}}{4\pi|\underline{y}'-\underline{y}|}\right] = \underline{n}\cdot\nabla_2 M(\underline{y}',\underline{z})\Big|_{\underline{z}=\underline{y}}$$

$$= \frac{e^{ikR}}{4\pi R}\left(ik - \frac{1}{R}\right)\left(-\underline{n}\cdot\hat{\underline{R}}\right) \tag{17}$$

is the field at $\underline{y}'$ due to a unit dipole $\underline{n}$ at $\underline{y}$ in an unbounded field and is a function of the two positions and the direction $\underline{n}$. The subscript 2 indicates that a derivative is taken with respect to the second variable.

$$N(\underline{y}'/\underline{n}',\underline{y}) = \frac{\partial}{\partial n'}\left[\frac{e^{ik|\underline{y}'-\underline{y}|}}{4\pi|\underline{y}'-\underline{y}|}\right] = \underline{n}'\cdot\nabla_1 M(\underline{z},\underline{y})\Big|_{\underline{z}=\underline{y}'}$$

$$= \frac{e^{ikR}}{4\pi R}\left(ik - \frac{1}{R}\right)\left(\underline{n}'\cdot\hat{\underline{R}}\right) \tag{18}$$

is the field gradient in the $\underline{n}'$ direction at $\underline{y}'$ due to a unit monopole at $\underline{y}$ in an unbounded field and is a function of the two points and the direction $\underline{n}'$.

$$E(\underline{y}'/\underline{n}',\underline{y}/\underline{n}) = \frac{\partial^2}{\partial n'\partial n}\left[\frac{e^{ik|\underline{y}'-\underline{y}|}}{4\pi|\underline{y}'-\underline{y}|}\right] = \underline{n}'\cdot\nabla_1 D(\underline{z},\underline{y}/\underline{n})\Big|_{\underline{z}=\underline{y}'}$$

$$= \frac{e^{ikR}}{4\pi R^2}\left[\left(3ik - \frac{3}{R} + k^2R\right)(\underline{n}\cdot\hat{\underline{R}})(\underline{n}'\cdot\hat{\underline{R}}) - \left(ik - \frac{1}{R}\right)(\underline{n}\cdot\underline{n}')\right] \tag{19}$$

is the gradient in the $\underline{n}'$ direction of the field at $\underline{y}'$ due to a unit dipole $\underline{n}$ at $\underline{y}$ in an unbounded fluid and depends on the two points and the two directions.

These kernels all have weak singularities at $\underline{y}' = \underline{y}$ but a principal value integral exists in all cases. Note that the two functions M and E are each self-symmetric (but not complex-conjugate-symmetric)

to an interchange in the two field points and the two directions, i.e., $M(\underline{y}',\underline{y}) = M(\underline{y},\underline{y}')$ and $E(\underline{y}'/\underline{n}',\underline{y}/\underline{n}) = E(\underline{y}/\underline{n},\underline{y}'/\underline{n})$. However, the two functions D and N are mutually symmetric (but not complex-conjugate-symmetric) to an interchange in the same variables, i.e., $D(\underline{y},\underline{y}'/\underline{n}') = N(\underline{y}'/\underline{n}',\underline{y})$.

The integral equations for these equivalent monopole and dipole layers may have no finite solution, or possibly an indeterminate solution, under certain commonly occurring practical conditions. This may seem surprising because the boundary value problem, for the space external to an arbitrary surface, posed by Equations (1) through (4) always has a solution and the solution is unique.[2] However, the same integral equations can also be intermediate steps in solving a boundary value problem for the region interior to the surface, and it should be obvious on physical grounds that the solution to the interior problem becomes indeterminate at the resonance frequencies of interior standing waves. Accordingly, we will discuss in the next section the possible standing-wave solutions in the space interior to any arbitrary closed surface.

INTERNAL STANDING WAVES AND ASSOCIATED EXTERNAL TRAVELING WAVES

We shall assume without proof the assertion[6] that for every closed surface, an unbounded sequence of wave numbers exists at which standing waves can exist inside the surface with any prescribed combination of field and normal gradient, $\alpha P(\underline{y}) + \beta Q(\underline{y}) = 0$, at the surface where α and β are real constants. We consider, in particular, the two independent series of characteristic wave numbers, those for which $\beta \equiv 0$ and $P(\underline{y}) = 0$ and those for which $\alpha \equiv 0$ and $Q(\underline{y}) = 0$. We denote the characteristic wave numbers of the first kind by k_i; $i = 1,2,\ldots$ the interior standing wave by $P_i^{(1)}(\underline{x})$ (normalized in any convenient way), and its normal gradient at the surface by $Q_i^{(1)}(\underline{y})$. Also we define an associated traveling wave $p_i^{(1)}(\underline{z})$ for the external space by the condition that the normal gradients match at $\underline{y}$, i.e.,

$$q_i^{(1)}(\underline{y}) = \left[\frac{\partial p_i^{(1)}(\underline{z})}{\partial n}\right]_{\underline{y}} = \left[\frac{\partial p_i^{(1)}(\underline{x})}{\partial n}\right]_{\underline{y}} = Q_i^{(1)}(\underline{y}) \tag{20}$$

Likewise, we denote the characteristic wave numbers of the second kind by m_i and the internal normalized standing wave by $P_i^{(2)}(\underline{x})$, and we define an associated traveling wave $p_i^{(2)}(\underline{z})$ for the external space by the condition that the fields match at $\underline{y}$, i.e.,

$$P_i^{(2)}(\underline{y}) = p_i^{(2)}(\underline{y}) \tag{21}$$

Note that for the interior of any closed surface, the fundamental characteristic standing wave of the second class is $P_1^{(2)}(\underline{x})$ = constant with $k = m_1 = 0$. For this wave function clearly satisfies the Helmholtz equation and the boundary condition $Q_i^{(2)} = 0$.

More generally, for any surface shape, the characteristic wave number may be degenerate, i.e., there may exist a finite number of independent wave functions $P_{ij}(\underline{x})$ and $p_{ij}(\underline{z})$; $j = 1,2,\ldots j_{max}$, where j_{max} depends on the surface shape and wave number. However, we shall omit any further reference to the degeneracy of the wave functions in order to simplify the notation.

We show in Appendix B that the equivalent monopole and dipole layers for these four series of characteristic wave functions have some simple and remarkable properties. Thus for characteristic functions of the first kind, i.e., if $k = k_i$, at which there can exist internal standing waves with zero pressure at the surface, then

$$*\iint q_i^{(1)}(\underline{y}) \frac{e^{ik_i|\underline{r} - \underline{y}|}}{4\pi|\underline{r} - \underline{y}|}\, dS = P_i^{(1)}(\underline{x}) \qquad \text{if } \underline{r} = \underline{x} \tag{22a}$$

$$= 0 \qquad \text{if } \underline{r} = \underline{y}' \tag{22b}$$

$$= 0 \qquad \text{if } \underline{r} = \underline{z} \tag{22c}$$

$$*\iint p_i^{(1)}(\underline{y}) \frac{\partial}{\partial n}\left[\frac{e^{ik_i|\underline{r}-\underline{y}|}}{4\pi|\underline{r}-\underline{y}|}\right] dS = p_i^{(1)}(\underline{x}) \quad \text{if } \underline{r} = \underline{x} \tag{23a}$$

$$= \frac{p_i^{(1)}(\underline{y}')}{2} \quad \text{if } \underline{r} = \underline{y}' \tag{23b}$$

$$= p_i^{(1)}(\underline{z}) \quad \text{if } \underline{r} = \underline{z} \tag{23c}$$

$$*\iint q_i^{(1)}(\underline{y}) \frac{\partial}{\partial n'}\left[\frac{e^{ik_i|\underline{y}'-\underline{y}|}}{4\pi|\underline{y}'-\underline{y}|}\right] dS = \frac{q_i^{(1)}(\underline{y}')}{2} \tag{24}$$

$$*\iint p_i^{(1)}(\underline{y}) \frac{\partial^2}{\partial n' \partial n}\left[\frac{e^{ik_i|\underline{y}'-\underline{y}|}}{4\pi|\underline{y}'-\underline{y}|}\right] dS = q_i^{(1)}(\underline{y}') \tag{25}$$

For characteristic functions of the second kind, i.e., if $k = m_i$, (at which there can exist internal standing waves with zero normal gradient at the surface)

$$*\iint q_i^{(2)}(\underline{y}) \frac{e^{im_i|\underline{r}-\underline{y}|}}{4\pi|\underline{r}-\underline{y}|} dS = -p_i^{(2)}(\underline{x}) \quad \text{if } \underline{r} = \underline{x} \tag{26a}$$

$$= -p_i^{(2)}(\underline{y}') \quad \text{if } \underline{r} = \underline{y}' \tag{26b}$$

$$= -p_i^{(2)}(\underline{z}) \quad \text{if } \underline{r} = \underline{z} \tag{26c}$$

$$*\iint p_i^{(a)}(\underline{y}) \frac{\partial}{\partial n}\left[\frac{e^{im_i|\underline{r}-\underline{y}|}}{4\pi|\underline{r}-\underline{y}|}\right] dS = -p_i^{(a)}(\underline{x}) \quad \text{if } \underline{r} = \underline{x} \tag{27a}$$

$$= -\frac{p_i^{(a)}(\underline{y})}{2} \quad \text{if } \underline{r} = \underline{y}' \tag{27b}$$

$$= 0 \quad \text{if } \underline{r} = \underline{z} \tag{27c}$$

$$*\iint q_i^{(a)}(\underline{y}) \frac{\partial}{\partial n'}\left[\frac{e^{im_i|\underline{y}'-\underline{y}|}}{4\pi|\underline{y}'-\underline{y}|}\right] dS = -\frac{q_i^{(a)}(\underline{y}')}{2} \tag{28}$$

$$*\iint p_i^{(a)}(\underline{y}) \frac{\partial^2}{\partial n \partial n'}\left[\frac{e^{im_i|\underline{y}'-\underline{y}|}}{4\pi|\underline{y}'-\underline{y}|}\right] dS = 0 \tag{29}$$

Note that the integrals in (22a), (22c), (23a), (23c), (26a), (26c), (27a), and (27c) need not be starred.

Equation (22c) shows that for an arbitrary closed surface there exists one sequence of wave numbers (the characteristic wave numbers of the first kind) at which an associated surface distribution of monopoles, $q_i^{(1)}(\underline{y})$ produces no external field, and (27c) shows that there is a second sequence of wave numbers at which an associated surface distribution of normal dipoles $p_i^{(a)}(\underline{y})$ produces no external field. These results generalize a remark of Lamb[7] who showed that when sin ka = 0, "a uniform distribution of simple sources over a sphere of radius a produces no effect at external points." Thus the surface distributions that produce a null external field exist for every surface shape, for dipole distributions as well as simple source distributions, and they exist in an infinite number of characteristic patterns. In fact at k = 0, a uniform distribution of normal dipoles over <u>any</u> closed surface produces no external field.

However, these results are principally of consequence in the solution of the integral equations because (22b) and (29) demonstrate a family of solutions to the homogeneous parts of the integral equations

of the first kind, whereas (23b), (24), (27b), and (28) demonstrate a family of solutions to the homogeneous parts of the integral equations of the second kind. And it is well-known that the solution to any linear integral equation becomes indeterminate under conditions in which the homogeneous part of the equation has a solution.

EXISTENCE AND UNIQUENESS

The mathematical condition for the existence of a unique solution to any linear integral equation of the second kind is known as the Fredholm Alternative.[4,5] As applied to (8), for example, this states that either the integral equation with a specified k and a specified surface shape has a unique and regular solution for all choices of the known function $q(\underline{y})$ or the homogeneous part of the integral equation has a nontrivial solution and the adjoint homogeneous integral equation has a nontrivial solution. In the alternative case the integral equation has a regular (but not unique) solution only when the known function is orthogonal to every solution of the adjoint homogeneous equation.

In the present context, two functions $f_1(\underline{y})$ and $f_2(\underline{y})$ are orthogonal if

$$\iint f_1 \bar{f}_2 \, dS = \iint \bar{f}_1 f_2 \, dS = 0 \tag{30}$$

where the "overbar" denotes the complex conjugate. And by the adjoint homogeneous equation we mean a homogeneous integral equation of the second kind whose kernel function is obtained from the original kernel by interchanging the two variables and taking the complex conjugate. Thus adjoint homogeneous equations to: (i) Equation (8) for $\sigma(\underline{y})$, (ii) (11) for $\mu(\underline{y})$, (iii) (12) for $p(\underline{y})$, and (iv) (13) for $q(\underline{y})$ are the complex conjugates of (i) (23b), (ii) (27b), (iii) (24), and (iv) (28), respectively.

The special relations of (22) through (29) demonstrate that solutions to the homogeneous integral equations for sound radiation do occur and that they occur at the characteristic wave numbers of the first and second kind ℓ_i and m_i. Also, they identify the solutions of the homogeneous equations to be the surface values of the characteristic

traveling waves of the first and second kind--$p_i^{(1)}(y)$, $q_i^{(1)}(\underline{y})$, $p_i^{(2)}(\underline{y})$, and $q_i^{(2)}(\underline{y})$. Furthermore, we show in Appendix B that solutions to these homogeneous integral equations, excluding (14) and (15), can occur only at the characteristic wave numbers k_i or m_i.

Accordingly, we list below the conditions for the existence and uniqueness of solutions to the integral equations for $\sigma(\underline{y})$, $\mu(\underline{y})$, $p(\underline{y})$, and $q(\underline{y})$. These are stated as conditions on the prescribed wave number k and on the prescribed boundary condition $p(\underline{y})$ or $q(\underline{y})$. The conditions may be verified by systematic use of the Fredholm Alternative and by appeal to the special relations of (22) to (29).

1. A unique equivalent monopole layer $\sigma(\underline{y})$ exists as a solution to either integral equation (8) or (9) unless k is equal to a characteristic wave number of the first kind, k_i. If $k = k_i$ and if either $\iint q\overline{p}_i^{(1)}dS = 0$ or $\iint p\overline{q}_i^{(1)}dS = 0$, see Equation (B21); then a particular solution to either (8) or (9) for $\sigma(\underline{y})$ exists. Any finite multiple of $q_i^{(1)}(\underline{y})$, however, can be added to $\sigma(\underline{y})$ with no effect on the external field as calculated from (5). If $k = k_i$ and $\iint q\overline{p}_i^{(1)}dS = \iint p\overline{q}_i^{(1)}dS \neq 0$, no finite $\sigma(\underline{y})$ exists.

2. A unique equivalent dipole layer $\mu(\underline{y})$ exists as a solution to either integral equation (10) or (11) unless k is equal to a characteristic wave number of the second kind, m_i. If $k = m_i$ and if either $\iint p\overline{q}_i^{(2)}dS = 0$, or, equivalently, if $\iint q\overline{p}_i^{(2)}dS = 0$, then a particular solution $\mu(\underline{y})$ exists. However, any multiple of $p_i^{(2)}(\underline{y})$ can be added to $\mu(\underline{y})$ with no effect on the external field as calculated from (6). If $k = m_i$ and $\iint p\overline{q}_i^{(2)}dS = \iint q\overline{p}_i^{(2)}dS \neq 0$, then no finite $\mu(\underline{y})$ exists.

3. A particular solution to the Helmholtz surface-pressure equation, (12), for either $p(\underline{y})$ or $q(\underline{y})$, exists for all k and is unique unless $k = k_i$. When $k = k_i$, then any finite multiple of $q_i^{(1)}$ can be added to a particular solution for $q(\underline{y})$ with no effect on the external field as calculated from (6). But also when $k = k_i$ the particular solution to (12) for $p(\underline{y})$ is indeterminate by an arbitrary multiple of $p_i^{(1)}(y)$, and the corresponding formal solution of (6) for $p(\underline{z})$ is indeterminate by the same multiple of $p_i^{(1)}(\underline{z})$. The correct multiple of $p_i^{(1)}(\underline{y})$ which fits the specified boundary condition must be determined by some supplementary condition.

4. A particular solution to the surface-gradient equation (13), for either $p(\underline{y})$ or $q(\underline{y})$, exists for all k and is unique unless $k = m_i$. When $k = m_i$, any finite multiple of $p_i^{(2)}(\underline{y})$ can be added to the particular solution of (13) for $p(\underline{y})$ with no effect on the external field $p(\underline{z})$ as calculated from (6). But also when $k = m_i$ the particular solution to Equation (13) for $q(\underline{y})$ is indeterminate by an arbitrary multiple of $q_i^{(2)}(\underline{y})$, and the consequent solution of (6) for $p(\underline{z})$ is indeterminate by the same multiple of $p_i^{(2)}(\underline{z})$. Again the correct multiple of $p_i^{(2)}(\underline{z})$ which fits the specified boundary condition may be determined by some supplementary condition.

5. The existence and uniqueness of solutions to the Helmholtz interior equation (14) depend upon the position of the interior points $\underline{x}$ as well as on the wave number k. We can list only some of the conditions under which the solution of (14) is indeterminate. These occur when $k = \ell_i$ or m_i, and all $\underline{x}$ are on a nodal surface of the internal standing waves $P_i^{(1)}(\underline{x})$ or $P_i^{(2)}(\underline{x})$, respectively. If $k = \ell_i$, and $P_i^{(1)}(\underline{x}) = 0$, then any finite multiple of $q_i^{(1)}(\underline{y})$ can be added to the particular solution of $q(\underline{y})$ with no effect on the external field as calculated from (6). Under the same conditions the particular solution to (14) for $p(\underline{y})$ is indeterminate by an arbitrary multiple of $p_i^{(1)}(\underline{y})$, and the consequent solution of (6) for $p(\underline{z})$ is indeterminate by the same multiple of $p_i^{(1)}(\underline{z})$. If $k = m_i$ and $P_i^{(2)}(\underline{x}) = 0$, the particular solution to Equation (14) for $q_i(\underline{y})$ is indeterminate by an arbitrary multiple of $q_i^{(2)}(\underline{y})$, and the consequent solution to (6) for $p(\underline{z})$ is indeterminate by the same multiple of $p_i^{(2)}(\underline{z})$. Under the same conditions ($k = m_i$ and $P_i^{(2)}(\underline{x}) = 0$) the particular solution to (14) for $p(\underline{y})$ is indeterminate by an arbitrary multiple of $p_i^{(2)}(\underline{y})$; ideally this indeterminate component has no external field.

6. The existence and uniqueness of solutions to the combination equation (15) depend upon the choice of the function $\lambda(\underline{y})$ as well as on the wave number k. However, if $\lambda(\underline{y})$ is chosen as a positive constant independent of $\underline{y}$, it seems plausible that the integral equation would have an infinite sequence of characteristic wave numbers with values that interlace the ℓ_i and interlace the m_i.

7. The conditions for the existence and uniqueness of any equivalent monopole layer to be used in Equation (5) are the same as the conditions for the solutions of integral equations (8) and (9). Likewise the conditions for an equivalent dipole layer to be used in (6) are the same as the conditions for the solutions of integral equations (10) and (11). However, there always exist correct and unique Helmholtz layers $p(\underline{y})$ and $q(\underline{y})$ which can be used in (7), even though the solutions to one of the integral equations (12) through (15) may be indeterminate.

In principle, the characteristic wave numbers for any one of these integral equations are discrete line values; i.e., the singularities in the solutions would occur only if k were exactly equal to m_i or ℓ_i, and, in the case of (14), only if $\underline{x}$ were exactly on a nodal surface. However, in practice, the integral equations are solved by numerical methods which necessarily generate an indeterminate error, depending on the precision of the computing machine and the nature of the computation algorithms. Thus, there will be a band of values for k surrounding each ℓ_i and m_i, within which the solution to the integral equation has an indeterminate component or a singularity; for (14) there will be a layer of positions about each nodal surface. Furthermore, for every surface shape, the average spacing in k between successive ℓ_i or m_i decreases as k^{-2}, and, eventually, the bands (or layers) must overlap. Thus, for every surface shape and every integral equation, there is a value of k beyond which k is arbitrarily close to a characteristic value and beyond which a numerical solution of an integral equation will generate an indeterminate error.

In summary, the numerical solutions for $p(\underline{y})$ or $q(\underline{y})$ will be indeterminate by an arbitrary multiple of some particular surface distribution when the wave number k is too large or when k is too close to some characteristic value. Under similar conditions on k, a finite solution for $\sigma(\underline{y})$ or $\mu(\underline{y})$ will exist only for special boundary conditions; even then, the indeterminacy will occur.

However, these difficulties do not mean that the integral equations cannot be used in the neighborhood of these characteristic wave numbers. In some cases, e.g., the solution of (12) for $q(\underline{y})$ or the solution of (13)

for $p(\underline{y})$, the indeterminate component in the solution, has no external field and, hence, will not cause an error in the subsequent calculation of the external field from either (5), (6), or (7). In other cases where the indeterminate component does have an external field, it has been possible[8] to supplement the original integral equation with enough additional conditions to fix the correct amplitude of this component. Even in those cases where no finite solution exists, the experience described in the next section suggests that it may be possible to factor out the infinite component from the external field.

UNIFORMLY PULSATING SPHERE

To illustrate all of these methods, and to demonstrate the complications which occur at the resonance wave numbers, we will consider the very simple problem of a sphere of radius a, pulsating with uniform radial velocity in a uniform unbounded medium. Assume that either the surface pressure $p_o \exp(-i\omega t)$ or the normal gradient at the surface $q_o \exp(-i\omega t)$ is specified apriori. Then explicit analytical solutions to any one of the integral equations (8) through (14) can be obtained because all the unknown functions are constants which can be factored outside the integral sign and because all of the integrals can be reduced to functions of the single variable $R = |\underline{r} - \underline{y}|$ by taking the surface element in the shape of a spherical zone between the two circles at distances R and R + dR from the field point $\underline{r}$. The method of solution is demonstrated in Appendix C.

By these methods the explicit solutions to (8) and (9) for the equivalent monopole density are readily found to be

$$\sigma(\underline{y}') = \left(\frac{ka}{\sin ka}\right) \frac{e^{-ika}}{ika - 1} q_o \tag{31}$$

$$\sigma(\underline{y}') = \left(\frac{ka}{\sin ka}\right) \frac{e^{-ika}}{a} p_o \tag{32}$$

Likewise, the solutions to (10) and (11) for the equivalent dipole densities are

$$\mu(\underline{y}') = \frac{ka^2}{ika - 1} \left(\frac{1 - i \tan ka}{ka - \tan ka}\right) q_o \tag{33}$$

$$\mu(\underline{y}') = ka \left(\frac{1 - i \tan ka}{ka - \tan ka}\right) p_o \tag{34}$$

A formal solution to the Helmholtz surface equation (12) for either of the Helmholtz densities $p(\underline{y})$ or $q(\underline{y})$ is

$$(\sin ka)\, q(\underline{y}') = (\sin ka)\, (ik - \tfrac{1}{a})\, p(\underline{y}') \tag{35}$$

A formal solution to the surface gradient equation (13), for either of the two Helmholtz densities is

$$(ka - \tan ka)\, q(\underline{y}') = (ka - \tan ka)\, (ik - \tfrac{1}{a})\, p(\underline{y}') \tag{36}$$

and, finally, a formal solution to the Helmholtz interior equation (14) for either of the two Helmholtz layers is

$$(\sin kx)\ \ q(\underline{y}') = (\sin kx)\, (ik - \tfrac{1}{a})\, p(\underline{y}') \tag{37}$$

Hence in this problem, every one of the integral equations has a formal solution for all values of k. However, each solution has some singularity at those special values of k where the homogeneous part of the integral equation would have a solution. Thus, the formal solutions for $\sigma(\underline{y})$ become infinite where sin ka = 0, and this condition identifes the characteristic wave numbers of the first kind ℓ_i, where there can exist internal standing waves with zero surface pressure. The formal solutions for $\mu(\underline{y})$ become infinite where tan ka = ka, and these identify the characteristic wave numbers of the second kind where there can exist internal standing waves with zero surface gradient. The solutions to the Helmholtz surface pressure equation (12) become formally indeterminate when sin ka = 0, again at the characteristic wave numbers of the first kind. However, the solutions to the surface gradient equation (13) for the

same Helmholtz layers become indeterminate at the characteristic wave numbers of the second kind. And the solutions to the Helmholtz interior equation are formally indeterminate when $\sin kx = 0$. It is interesting that this can occur not only at the characteristic wave numbers ℓ_i and m_i but also at any wave number $k \geq \pi/a$.

If we disregard the apparent singularities and simply substitute either (31) or (32) into (5), or substitute (33) or (34) into (6), or substitute either (35), (36), or (37) into (7), we will get expressions for the external field $p(\underline{z})$ which are formally indeterminate at the critical wave numbers. But in each case we can simply divide out the indeterminate factor, or, more generally, we can require that the solution for $p(\underline{z})$ be a continuous function of k. In this way we obtain the correct and unique external field pressure.

$$p(\underline{z}) = \frac{q_o a}{ika - 1} \frac{a}{z} e^{ik(z - a)} \tag{38}$$

$$p(\underline{z}) = p_o \frac{a}{z} e^{ik(z - a)} \tag{39}$$

which is valid for all k, even at those critical wave numbers where either a finite equivalent monopole or dipole layer does not exist.

COMPARISON OF INTEGRAL EQUATION METHODS

We have derived at least five different integral equation methods for solving the Dirichlet problem and at least five different integral equation methods for solving the Neumann problem. Equation (10) for the Neumann problem and (13) and (15) for either the Dirichlet or Neumann problems appear to be new in sound radiation and scattering problems; however, all of these methods have a long and continuous history of application, development, and rediscovery in problems in fluid flow, elasticity, acoustic waves, and electromagnetic waves. Mikhlin[5] cites a long bibliography of older publications by Soviet authors, and there is no doubt that an equally long bibliography can be prepared

with non-Soviet authors. However, the great power of these methods could not be realized until the development of high speed computers, and we particularly wish to stress the more recent methods and applications using these computers. Practical experience in such applications to acoustic problems seems to be confined to solutions of the Neumann problem, most often by calculating $\sigma(\underline{y})$ from (8)[9-13] or by calculating $p(\underline{y})$ from (12);[14-19] however, there is some limited experience with solutions of the Helmholtz interior equation (14) for $p(\underline{y})$.[20]

It seems generally preferable to base a solution of either the Dirichlet or Neumann problems on a calculation of the unknown $q(\underline{y})$ or $p(\underline{y})$ rather than on a calculation of the equivalent monopole layer $\sigma(\underline{y})$ or the equivalent dipole layer $\mu(\underline{y})$. This is because: (i) the functions $p(\underline{y})$ or $q(\underline{y})$ have immediate physical significance and may even be determined experimentally; (ii) in many practical applications, e.g., low-frequency radiation by longitudinal vibrations of a slender body, the Helmholtz integral term in $p(\underline{y})$ contributes negligibly to the far field, and there is no need to solve an integral equation to determine $p(\underline{y})$; (iii) in cases of high-frequency radiation by an arbitrary body to the far field there is a simple approximation to the unknown function $p(\underline{y})$ or $q(\underline{y})$ which again makes it unnecessary to solve an integral equation; and (iv) when the wave number k is too large or too close to a characteristic wave number, a finite $\sigma(\underline{y})$ or $\mu(\underline{y})$ may not exist but a unique and finite $p(\underline{y})$ or $q(\underline{y})$ always exists.

However, the relative merits of integral equations of the first kind and of the second kind are not so easy to assess. Equations (8) and (12) are both equations of the second kind for the Neumann problem, having similar kernel functions and with the same spectrum of critical wave numbers. Equations (11) and (13) are both integral equations of the second kind for the Dirichlet problem, again having similar kernels and with the same spectrum of critical wave numbers. Ordinarily, we would expect that the equations of the second kind are preferable and easier to solve than the first kind because the theory of the former is complete and well documented, whereas it is still possible that unexpected difficulties can arise in solutions of the latter equations. However, the equations of the first kind may have some special advantages at the critical wave numbers. For

example, if we solve the Dirichlet problem at $k = \ell_i$ by using (12), an equation of the first kind, then the solution is indeterminate by some multiple of $q_i^{(1)}$. However, this indeterminate component has no effect on the external field. Likewise a solution to the Neumann problem at $k = m_i$, obtained by using an equation of the first kind (13), has an indeterminate component $p_i^{(2)}$ without an external field.

It may sometimes be useful to seek a simultaneous solution of two of these integral equations rather than only one. For example, one method[16] for solving the Neumann problem when $k \simeq \ell_i$ is by a simultaneous solution of the Helmholtz surface equation (12) and the Helmholtz interior equation (14). The latter must be satisfied at a sufficient number of interior points $\underline{x}$ to fix the amplitude of any indeterminate component to the solution of (12). The scheme has value only when the wave number k is close enough to a characteristic wave number of the first kind ℓ_i to generate an indeterminate component in the solution to (12) but when the interior points $\underline{x}$ can be picked not too close to the nodal surface of an internal standing wave. When k is sufficiently large, the layers of critical positions about each nodal surface will overlap, and this method cannot be used.

An alternative, and preferable, scheme is to solve either the Neumann or Dirichlet problems by a simultaneous solution of both surface integral equations (12) and (13). The critical wave numbers for these equations are the characteristic wave numbers of the first and second kind, respectively. They do not coincide except for special surface shapes, e.g., a rectangular box, and then the characteristic wave functions are orthogonal on the surface and need cause no difficulty. Nevertheless, it is likely that any algorithm, and Equation (15) can be viewed as one such algorithm, which "selects" the "accurate" solution from an overdetermined combination of two integral equations, will tend to introduce its own set of characteristic wave numbers. In any case the numerical calculation will undoubtedly become ill-conditioned at sufficiently high wave numbers where the critical bandwidths about ℓ_i and m_i overlap.

REDUCTION OF INTEGRAL EQUATIONS TO MATRIX EQUATIONS

The first step in obtaining a numerical solution to any of these integral equations for either the Neumann or Dirichlet problem is to convert the equation into a matrix equation, i.e., into a set of simultaneous algebraic equations. The procedures that we will describe are designed for the most general case, where the surface has some arbitrary, nonanalytical shape, and where the known boundary condition in $q(\underline{y})$ or $p(\underline{y})$ is likewise arbitrary and nonanalytical. Even in those cases where the surface is a spheroid or an ellipsoid, and where in principle the kernel function can be expanded in terms of separable wave functions, it may be more accurate to solve the integral equation by these general, nonanalytical, techniques because the higher order wave functions are not tabulated and are too difficult to compute.[17]

The basic procedure is to approximate the integral equation that is specified over a continuous range of positions on the surface by a matrix equation that is specified at only a finite number of stations on the surface. To do this we first pick a finite number N of representative stations $\underline{y}_j$; $j = 1,2,\ldots N$ on the surface and then associate each station with an element of area S_j, a normal direction $\underline{n}_j$, and some representative value c_j for the local radius of curvature of the surface. The stations need not be spaced with uniform density, and the elements of surface area need not have any special or uniform shape.[18] The density of stations on the surface should be high in regions of high surface curvature and in regions where $p(\underline{y})$ or $q(\underline{y})$ changes most rapidly, except that no station should be placed at a discontinuity of $p(\underline{y})$ or $q(\underline{y})$. The shape of each surface element need not be specified; the magnitude of each area S_j should be determined by distributing the total surface area ΣS_j among the N elements in approximately inverse proportion to the local station density and in accordance with any convenient bookkeeping scheme. In practice, it may be convenient to assume that the elements have some particular shape, e.g., quadrilateral or triangular; however, the shape has no effect on the calculations. The general prescription is that $p(\underline{y})$, $q(\underline{y})$, the direction of the normal $\underline{n}(\underline{y})$, and the curvature of the surface should each be reasonably uniform over a compact area S_j about the point $\underline{y}_j$.

We now approximate each of the integrals that occurs in (8) through (14) by a sum of terms evaluated only at the N stations $\underline{y}_j$. For example, if the unknown function is $f(\underline{y}_j)$, and the kernel function is $K(\underline{y}_j,\underline{y})$, we set

$$\iint f(\underline{y})\ K(\underline{y}_j,\underline{y})\ dS = \sum_{k=1}^{N} W_{jk}\ f(\underline{y}_k)\ K_{jk} \tag{40}$$

for $j = 1,2,\ldots N$. The weighting factor W_{jk} depends upon the quadrature formula being used. In the simplest and most common case $W_{jk} = S_k$; however, for special surface shapes or for special ways of ordering the points $\underline{y}_j$, it is possible to make a more sophisticated choice for the weighting factor. The value for the kernel depends on whether or not $j = k$, i.e., whether the point $\underline{y}_j$ is identical with the point $\underline{y}_k$.

When $j \neq k$, then K_{jk} is taken to be the value of the kernel function $K(\underline{y}',\underline{y})$ at $\underline{y}' = \underline{y}_j$ and $\underline{y} = \underline{y}_k$. In particular the four kernel functions of interest have the following forms, with $\underline{R} = \underline{y}_j - \underline{y}_k$.

$$M_{jk} = \frac{e^{ikR}}{4\pi R} \tag{41}$$

$$D_{jk} = \frac{e^{ikR}}{4\pi R}\left(ik - \frac{1}{R}\right)\left(-\underline{n}_k\cdot\hat{\underline{R}}\right) \tag{42}$$

$$N_{jk} = \frac{e^{ikR}}{4\pi R}\left(ik - \frac{1}{R}\right)\left(\underline{n}_j\cdot\hat{\underline{R}}\right) \tag{43}$$

$$E_{jk} = \frac{e^{ikR}}{4\pi R^2}\left[\left(3ik - \frac{3}{R} + k^2R\right)\left(\underline{n}_j\cdot\hat{\underline{R}}\right)\left(\underline{n}_k\cdot\hat{\underline{R}}\right) - \left(ik - \frac{1}{R}\right)\left(\underline{n}_j\cdot\underline{n}_k\right)\right] \tag{44}$$

When $j = k$ these forms have singularities and are inapplicable. Instead, in conformity with the original definitions of these functions

given in Appendix A, we assume that K_{jj} is a mean value of the particular kernel function over the area S_j surrounding the point $\underline{y}_j$. Furthermore, we assume that if S_j is sufficiently small, the mean value of K_{jj} will be the same as though the surface element were a spherical cap with $\underline{y}_j$ at its pole, with surface area $\pi b_j^2 = S_j$, and with the same mean radius of curvature c_j as the actual surface at $\underline{y}_j$. Then by special calculations for such a spherical cap (the procedures are similar to the derivations in Appendix C) we obtain the following results which are correct to the first order in kb_j:

$$M_{jj} \simeq \frac{1}{2\pi b_j} \tag{45}$$

$$D_{jj} = N_{jj} \simeq \frac{ikb_j - 1}{4\pi b_j c_j} \tag{46}$$

$$E_{jj} \simeq \frac{-1}{2\pi b_j^3} \tag{47}$$

Note that the values of D_{jj} and N_{jj} are ordinarily negligible, regardless of the shape assumed for the element of area, because the curvature of the element is usually small. In fact, it is common to approximate the neighborhood of each station by a plane surface element for which $c_j = \infty$ and $D_{jj} = N_{jj} = 0$.

In summary then, the recommended procedure is: (i) pick N stations on the surface; (ii) associate an area S_j, a normal $\underline{n}_j$, and a curvature S^{-1} with each station; and (iii) replace each integral by a finite sum of the form of (40), where $W_{jk} = S_k$; and where K_{jk} is given by (41) to (47).

This is probably the simplest and most universally practical of possible procedures, but most calculators have used more complex schemes. Usually they prescribe the shape of each area element to be a plane quadrilateral or a triangle and then prescribe the location of the station within the element of area. This is equivalent to picking the weighting factor W_{jk} so that it depends on the shape of S_k as well as the size of S_k. The

weighting factor may also depend on some assumed form of variation of the integrand in the neighborhood of y_k. These more complex schemes can be more accurate in special geometries, or they may be more accurate when calculating the main value of each separate term K_{jj}. However, for a body of arbitrary shape, it is doubtful that the accuracy of the total calculation will be improved at all. In fact, the procedure recommended here is so much more economical in computing time, etc., that it is often practical to increase the accuracy by increasing the number of stations.

One special geometry occurs so commonly in real problems, however, as to justify the development of a special calculation scheme. These are problems where the vibrating surface can be approximated by a surface of revolution, and where the specified boundary condition can be written in the form of $\cos(n\varphi)$ times an arbitrary function of the axial coordinate, where φ is the angle about the axis of revolution. A common and simple example of this problem would be scattering an incoming plane wave by a surface of revolution. For such a problem the integral equations can all be reduced to equations having only one dimension. Each integral term can be replaced by a sum over a finite number of stations in which each station is not a point location on the surface but a circular ring on the surface about the axis of revolution. The elements of the kernel matrices then have much more complicated forms and are more laborious to compute, although their accuracy is greatly increased. Also it is feasible to use more sophisticated quadrature formulas. A detailed explanation of the solution of the Helmholtz surface equation for this type of problem is given in Reference 19.

Reverting to the general procedures, we find that each of the integral equations for sound radiation may be replaced by a matrix equation. We will denote a matrix by an underbar. Take $\underline{p}$, $\underline{q}$, $\underline{\sigma}$, and $\underline{\mu}$ to be column matrices, each with N complex elements $p(\underline{y}_j)$, $q(\underline{y}_j)$, etc; $\underline{S}$ is a diagonal matrix with N diagonal elements S_j; and $\underline{M}$, $\underline{D}$, $\underline{N}$, and $\underline{E}$ are square matrices with N^2 complex elements defined in Equations (41) through (47), and $\underline{I}$ is the identity matrix.

Thus Equations (8) and (9) for the equivalent monopole density $\sigma(\underline{y})$ are replaced by

$$(2\underline{NS} - \underline{I})\underline{\sigma} = 2\underline{q} \tag{48}$$

and

$$\underline{MS\sigma} = \underline{p} \tag{49}$$

Equations (10) and (11) for the equivalent dipole density $\mu(\underline{y})$ are replaced by

$$\underline{ES\mu} = \underline{q} \tag{50}$$

and

$$(2\underline{DS} + \underline{I})\underline{\mu} = 2\underline{p} \tag{51}$$

The two independent integral equations (12) and (13) for the Helmholtz surface layers may be approximated by

$$(2\underline{DS} - \underline{I})\underline{p} = 2\underline{MSq} \tag{52}$$

and

$$(2\underline{NS} + \underline{I})\underline{q} = 2\underline{ESp} \tag{53}$$

Furthermore, we can form an arbitrary linear combination of (52) and (53) to obtain an indefinite number of matrix equations for $\underline{p}$ or $\underline{q}$.

Finally the Helmholtz interior equation (14) may be replaced by

$$\underline{gSq} = \underline{hSp} \tag{54}$$

where $\underline{g}$ and $\underline{h}$ are row matrices, not column matrices, with elements

$$g_x = \frac{e^{ikR}}{4\pi R} \tag{55}$$

$$h_x = \frac{e^{ikR}}{4\pi R}\left(ik - \frac{1}{R}\right)(-\underline{n}_k \cdot \hat{\underline{R}}) \tag{56}$$

with $\underline{R} = \underline{x} - \underline{y}_k$, and $\underline{x}$ is an arbitrary fixed point within the surface. If $\underline{p}$ or $\underline{q}$ is to be calculated completely from (54), there must be at least N interior points $\underline{x}$ and N equations such as (54).

A formal solution can be written for each of these equations in terms of the inverse matrix. Thus the formal solutions to (48) and (49) are

$$\underline{\sigma} = (2\underline{NS} - \underline{I})^{-1}\, 2\underline{q} \tag{57}$$

$$\underline{\sigma} = (\underline{MS})^{-1}\, \underline{p} \tag{58}$$

where the superscript -1 denotes an inverse matrix. However, when k is equal to a characteristic wave number of the kernel matrix, the matrix is singular, and the inverse does not exist. The characteristic wave numbers of the matrix are the same (except for some shifting due to the finite approximations) as the characteristic wave numbers of the kernel function in the integral equation. The characteristic wave numbers ℓ_i and m_i of the first and second kind and the characteristic wave functions $p_i^{(1)}$, $q_i^{(1)}$, $p_i^{(2)}$, and $q_i^{(2)}$ which we have previously defined in Equations (22) through (28) are related to the eigenvalues and eigenvectors of the matrices of (48) through (53). We may consider that an element $A_{mn}(k)$ of a typical matrix $\underline{A}$ is a function of the wave number k as well as of the two surface points $\underline{y}_m$ and $\underline{y}_n$, and we define the eigenvalue $\gamma_j(k)$ and the eigenvector $\underline{\varphi}_j(k)$ by the equation

$$\underline{A}(k)\ \underline{\varphi}_j(k) = \gamma_j(k)\ \underline{\varphi}_j(k) \tag{59}$$

Then, according to (22) through (25), when $k = \ell_i$

$2\underline{MS}$ has eigenvalue 0 and eigenvector $\underline{q}_i^{(1)}$

$2\underline{DS}$ has eigenvalue +1 and eigenvector $\underline{p}_i^{(1)}$

$2\underline{NS}$ has eigenvalue +1 and eigenvector $\underline{q}_i^{(1)}$.

Also, according to (26) through (28), when $k = m_i$

$2\underline{DS}$ has eigenvalue -1 and eigenvector $\underline{p}_i^{(a)}$

$2\underline{NS}$ has eigenvalue -1 and eigenvector $\underline{q}_i^{(a)}$

$2\underline{ES}$ has eigenvalue 0 and eigenvector $\underline{p}_i^{(a)}$

A more complete analysis of the eigenvalues and eigenvectors of these matrices for arbitrary values of the wave number k will be given in a subsequent report.

NUMERICAL SOLUTIONS TO MATRIX EQUATIONS

The matrix equations discussed here are all, except for (54), conventional linear matrix equations of the form $\underline{Af} = \underline{g}$. There is a vast and growing literature, e.g., References (21) and (22), about numerical methods of solution, and most of their analyses are applicable here. It is well-known, for example, that it is not efficient to obtain a solution by calculating the inverse matrix $\underline{A}^{-1}$. The methods of solutions are commonly classified into direct methods, where the solution is obtained after a definite number of prescribed operations, and iterative methods, where the solution is obtained by an indefinite number of successively improved approximations. In general the choice of an optimum method depends upon such factors as the size, i.e., the order, of the matrix A; whether the matrix is sparse, i.e., has many zero elements; whether the matrix is nearly singular or whether the problem is otherwise "ill conditioned"; and whether an approximate solution is available.

In the particular problems of sound radiation and scattering, all the matrix elements are finite, complex, and generally nonzero, although the elements of the principal diagonal of A are generally dominant. The numbers of stations $\underline{y}_j$ on the surface often exceeds 100, which means that the order of the matrix exceeds 200. Also, in these problems, the wave number k is often greater than ℓ_1, the lowest characteristic wave number of the first kind; in most cases the characteristic wave numbers ℓ_i and m_i in the neighborhood of k are not known apriori, and it is very laborious to compute them.

When $k \ll \ell_1$, then it is a simple matter to start with an integral equation such as (12), (8), or (9) that does not have a singularity at $k = 0$. All conventional direct and iterative methods of solution are then applicable, and the optimum choice is mostly a matter of convenience. Sometimes in this low-frequency case, a solution to the Laplace equation, $k = 0$ in Equation (1), may be available as an initial approximation in an iterative solution to the matrix equation; however, when k is not too close to a characteristic value, the problem is well conditioned--even without an approximate solution.

As already discussed, when the wave number k is sufficiently large, k will be arbitrarily close to one or more characteristic wave numbers, the matrix A will be almost singular, and the problem will be ill-conditioned. Even when the solution must simultaneously satisfy two matrix equations having no characteristic wave numbers in common, it appears likely that the problem will be ill-conditioned for sufficiently high k.

However, in the high-frequency case, if only the far-field is ultimately required, then there is an approximation for the surface pressure that can be used directly in (7) without solving any integral equation.

$$p(\underline{y}) \left(ik - \frac{1}{c}\right) = q(\underline{y}) \tag{60}$$

where $c(\underline{y})$ is a mean radius of curvature of the local surface at $\underline{y}$. Presumably a more accurate solution can be obtained by using this equation as the initial approximation in an iterative solution of matrix equations (52) or (53) and by terminating the iteration before the process diverges.

When the wave number has some intermediate value greater than ℓ_1 but less than the wave numbers at which there is a high density of characteristic values, then either direct or iterative methods of solution may be used. The principal advantage of the direct methods is that many library programs are available which can be used automatically. These direct methods generally require more storage space and have larger computing time (proportional to N^3) than the iterative methods (proportional to N^2). The

more common library routines are, however, restricted to approximately 100 elements. A more serious disadvantage of a direct method is that the apparent solution could be grossly inaccurate when the wave number is too close to a characteristic value without an indication that the solution is incorrect. For example, if an indeterminate solution exists at the characteristic value, then the direct method will produce some unknown multiple of the indeterminate component; if no finite solution exists at the characteristic value, then the direct method will tend to produce a finite apparent solution, having an incorrect component.

A iterative method can easily be programmed for any one of these matrix equations. The solution is ordinarily faster than a direct method and requires a smaller storage space, so that the number of stations could be increased with a consequent gain in accuracy. When the matrix equation results from an integral equation of the second kind, e.g. (48), it is already in a form to fit the simplest iteration scheme, one which is the direct analog of the classical Neumann series for solving the integral equation.

$$\underline{f}^{(n+1)} = \underline{B}\underline{f}^{(n)} + \underline{g} \tag{61}$$

where $\underline{f}^{(n)}$ is the n^{th} approximation to $\underline{f}$, and $\underline{f}^{(1)} = \underline{g}$. For example, in (48), $\underline{F} = \underline{\sigma}$, $\underline{g} = -2\underline{q}$, and $\underline{B} = 2\underline{NS}$. When the matrix equation results from an integral equation of the first kind, e.g. (49), it is in the form $\underline{AF} = \underline{q}$, which can be transformed into the form of (61) by the scheme

$$\underline{f}^{(n+1)} = \underline{B}\underline{F}^{(n)} + \underline{L}^{-1}\,\underline{g} \tag{62}$$

where $\underline{B} = \underline{I} - \underline{L}^{-1}\,\underline{A}$, and $\underline{L}$ is a diagonal matrix whose diagonal elements are equal to the diagonal elements of $\underline{A}$. Note that (45) and (47) show that the diagonal elements of $\underline{M}$ or $\underline{E}$ are real, finite, and of one sign, so that the inverse matrix $\underline{L}^{-1}$ always exists for matrix equations like (49), (50), and (52).

Equations (61) and (62) are called Jacobi iteration schemes, implying that the (n+1) approximation to every element of $\underline{f}$ is based on the $(n)^{th}$ approximation to every element of $\underline{f}$. In practice, some amount of Gauss-Seidel iteration is used, implying that the (n+1) approximation to the j^{th} element of $\underline{f}$ is based on the (n+1) approximation to the first j-1 elements of $\underline{f}$ (which have already been calculated) and to the n^{th} approximation to the remaining elements of $\underline{f}$. The Gauss-Seidel iteration usually converges faster than does the Jacobi iteration. The Gauss-Seidel scheme can also be written in the form of (61) but with a modified iteration matrix instead of $\underline{B}$.

The principal disadvantage of an iterative method is that the iteration may not converge, even in those cases where a finite solution exists. Conversely, an important advantage of an iterative method is that when a finite, and possibly indeterminate, solution exists, and when the iteration does converge, it normally converges on the correct and unique solution that fits the boundary condition. An iterative process such as (61) converges if and only if the largest eigenvalue of the matrix B, i.e., the largest root γ_i of the equation $|\underline{B} - \gamma\underline{I}| = 0$, has a modulus less than unity. This condition is difficult to apply because the eigenvalues are difficult to compute, unless the surface has some special shape, e.g., a quadric surface. A sufficient, but not necessary, condition that the iteration converge is that some norm of $\underline{B}$ be less than unity, $||B|| < 1$. In our limited experience with solutions of the matrix equations for sound radiation, a sufficient condition for the convergence of either the Jacobi or Gauss-Seidel processes is that the wave number k be sufficiently removed from a characteristic wave number at which the determinant of $\underline{A} = 0$. In particular, iterative solutions to (48) and (52) always seem to converge as $k \to 0$.

There are also more sophisticated schemes, which expand or otherwise modify the range of k, for which the iteration will converge on a correct and unique solution. In particular, there is the iteration scheme

$$f^{(n+1)} = h\underline{f}^{(n)} + (1 - h)\ (\underline{B}\underline{f}^{(n)} + \underline{g}) \qquad (63)$$

where h is a complex number. In Reference 17 we prove that this process converges to the correct solution provided all the eigenvalues γ_i of the matrix $\underline{B}$ are within a circle in the complex plane passing through the point (1,0), having a center at h/(h-1), and having a radius equal to the modulus of 1/(1-h). When h = 0, this reduces to the simple scheme of (61). When h is a real and positive number less than unity, the domain of γ_i within which the process converges is a large circle that includes the unit circle of the simple iteration scheme. In fact, (63) with h = 1/2, or perhaps a Gauss-Seidel variation of (63) with h = 1/2, is a good wide-range iteration scheme that is worth trying when the eigenvalues γ_i are not known apriori.

SPECIAL METHODS NEAR CHARACTERISTIC WAVE NUMBERS

There probably is no general method for obtaining the correct unique solution when k is close to a k_i for which the solution is indeterminate. But there are always special methods in special situations. We have already mentioned several such methods:

(i) Shift the nearby k_i by changing to an integral equation with a different spectrum of characteristic wave numbers.

(ii) Use the iteration scheme of Equation (63) and pick h so that the process converges.

(iii) Seek a simultaneous solution of two integral equations.

(iv) Use the high-frequency approximation of (60) and avoid the solution of any integral equation.

There are several other methods:

(v) The correct and unique solution is clearly continuous in k. Hence, if the characteristic k_i are not too close together, we may solve the matrix equation and obtain unique solutions with two different values of the wave number on either side of the nearest k_i, and then we can interpolate to get the correct solution at k_i.

(vi) Another method, which apparently has been tried in electromagnetic wave theory,[23] is to eliminate the possibility of internal standing waves by adding an energy dissipation term to the integral equation. Presumably, if the damping or dissipation is small enough, the solution to the equation

with damping will not differ appreciably from the correct solution without damping. A simple way to introduce damping is to treat k as a complex number with a small imaginary part. However, when the real part of k comes too close to a characteristic k_i, then we can expect that the matrix problem will be ill-conditioned.

(vii) There are many situations where a low-frequency approximation to the unknown function $p(\underline{y})$ can be used with negligible error in Equation (7) for $p(\underline{z})$. The approximation might be obtained by solution of the integral equation with a value of $k \ll k_i$. Or the approximation might be estimated from some incompressible potential flow ($k = 0$) result which does not entail the solution of any integral equation.

(viii) As a final example of a special method we solve the matrix equation (52) for $\underline{p}$ by the iteration scheme of (63), and use matrix equation (53) to determine a non-stationary value for the iteration parameter h. If $\underline{p}^{(n)}$ is the n^{th} approximation to $\underline{p}$, and $\underline{p}^{(+)}$ is a tentative value for $\underline{p}^{(n+1)}$,

$$\underline{p}^{(+)} = 2\underline{DS}\underline{p}^{(n)} - 2\underline{MS}\underline{q} \tag{64}$$

$$\underline{p}^{(n+1)} = \underline{p}^{(+)} + h(\underline{p}^{(n)} - \underline{p}^{(+)}) \tag{65}$$

The parameter h is picked so that $\underline{p}^{(n+1)}$ approximately satisfies (53) at any m, $1 \le m \le N$, stations. We truncate the $\underline{N}$, $\underline{I}$, and $\underline{E}$ matrices to m x N rectangular matrices and denote them by $\underline{N}_m$, $\underline{I}_m$, and $\underline{E}_m$, respectively. Then

$$(2\underline{N}_m \underline{S} + \underline{I}_m)\underline{q} = 2\underline{E}_m \underline{S}\underline{p}^{(n+1)}$$

$$= 2\underline{E}_m \underline{S}\underline{p}^{(+)} + h\,2\underline{E}_m \underline{S}(\underline{p}^{(n)} - \underline{p}^{(+)}) \tag{66}$$

This equation gives m different values for the complex number h. To define a "best" value, we multiply on the left by $\underline{f}\underline{S}_m$ where $\underline{S}_m$ is a diagonal matrix with m diagonal elements S_j, and $\underline{f}$ is a row vector

whose elements are the complex conjugate-transpose (hermitian transpose) of the column vector in the last term of (66). That is, if $\underline{f}^H$ denotes a hermitian transpose

$$\underline{f}^H = 2\underline{E}_m \, \underline{S}(\underline{p}^{(n)} - \underline{p}^{(+)}) \tag{67}$$

and

$$\underline{f} \, \underline{S}_m \, (2\underline{N}_m \, \underline{S} + \underline{I}_m)\underline{q} = \underline{f} \, \underline{S}_m \, 2\underline{E}_m \, \underline{S}\underline{p}^{(+)} + h \, \underline{f} \, \underline{S} \, \underline{f}^H \tag{68}$$

and

$$h = \frac{\underline{f} \, \underline{S}_m [(2\underline{N}_m \, \underline{S} + \underline{I}_m)\underline{q} - 2\underline{E}_m \, \underline{S}\underline{p}^{(+)}]}{\underline{f} \, \underline{S}_m \, \underline{f}^H} \tag{69}$$

The iteration factor h could be computed in this way for each iteration, but it seems safer to do the first few iterations with h = 1/2. The m stations for the calculation should be selected at positions where the component of $\underline{f}$, as defined in (67), has a maximum modulus. If the iteration process has already converged except for the contribution of a single characteristic mode of indeterminate amplitude, then in principle it would be sufficient to select only a single station (m = 1) for the calculation of h, but in practice it would be safer to use more stations.

In almost all of these special methods, it would be helpful, but not necessary, to know the values for the nearby characteristic wave numbers k_i, and to have some knowledge of their associated wave functions $p_i(\underline{y})$ and $q_i(\underline{y})$. For example, in this last method, it would be preferable to pick the stations for the calculation of the iteration factor h at these points at which $p_i(\underline{y})$ is a maximum.

SUMMARY AND CONCLUSIONS

We have derived and discussed many different methods for solving sound-radiation and scattering problems associated with arbitrary surfaces.

The common features of all of these methods are: (1) they replace the radiating or scattering surface by an "equivalent" surface distribution of monopoles and/or dipoles, (2) they specify the density of this equivalent layer by an integral equation, (3) they approximate the integral equation with a matrix equation of finite order, (4) they solve the matrix equation by the general numerical techniques which have been developed for matrix operations on high-speed computers. We have also derived and discussed the conditions for the nonexistence or indeterminacy of solutions to these equations, and we have suggested several techniques for obtaining a correct and unique solution when such conditions obtain.

Some general recommendations have been made for these methods which are likely to be faster, more accurate, or more elegant or to have a wider range of application. However, it is recognized that every one of these methods is complicated and laborious to implement and that each method may have special advantages for special classes of problems. Furthermore, it happens that in many real engineering problems, the physical parameters which affect the solution--e.g., the details of the boundary conditions or the detailed specification of the surface shape--can be known only approximately or only in a statistical sense. In such cases there is little point in obtaining a solution whose mathematical precision exceeds the possible error due to uncertainties in the parameters. Hence an overall recommendation is to use the simplest successful method that is easily available and to give only minor consideration to such factors as speed, accuracy, or wide range.

This has not been a comprehensive or an historical review. For example, we have not covered what seems to have been a parallel historical development of the same methods--but applied to problems in electromagnetic radiation. A small bibliography on this application is given in Reference 24. Also other possible "equivalent" distribtuions of multipoles have not been covered, besides the surface distributions. For example, it is possible to find a finite <u>linear</u> distribution of monopoles[25] and/or dipoles[26] <u>inside</u> a body which solves arbitrary Neumann or Dirichlet

problems on the body. And it is possible to express the far field of any vibrating body by a suitable combination of point multipoles.[27] In each of these equivalent distributions, the multipole strength can probably be specified by an integral equation similar to those used here.

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APPENDIX A

DERIVATION OF INTEGRAL EQUATIONS

A common expression that recurs in all the equations to be derived has the form $\underline{m}\cdot\nabla_{z'}f(R)$, where $f(R)$ is an arbitrary function of the distance R, $\underline{R} = \underline{z}' - \underline{z}$, $\underline{m}$ is an arbitrary unit vector, and $\nabla_{z'}$ means that the gradient is to be taken with point $\underline{z}'$ varied and point $\underline{z}$ fixed. Then

$$\underline{m}\cdot\nabla_{z'}f(R) = (\underline{m}\cdot\hat{\underline{R}})\frac{df}{dR} = -\underline{m}\cdot\nabla_{z}f(R) \tag{A1}$$

where $\hat{\underline{R}}$ is a unit vector in direction $\underline{R}$. When the two unit vectors $\underline{n}'$ and $\underline{n}$ are associated with points $\underline{z}'$ and $\underline{z}$, respectively, it is sometimes convenient to use the abbreviated notation

$$\frac{\partial f}{\partial n} = \underline{n}\cdot\nabla_{z}f = -(\underline{n}\cdot\hat{\underline{R}})\frac{df}{dR} \tag{A2}$$

$$\frac{\partial f}{\partial n'} = \underline{n}'\cdot\nabla_{z'}f = (\underline{n}'\cdot\hat{\underline{R}})\frac{df}{dR} \tag{A3}$$

To derive the integral equations for the monopole layer $\sigma(\underline{y})$, we consider first the field at a point $\underline{z}'$, which is a small distance h off surface point $\underline{y}'$ in the direction of the normal $\underline{n}'$, i.e., $\underline{z}' = \underline{y}' + h\underline{n}'$. We write the field in two parts, an integral over the surface A within a sphere of infinitesimal radius ϵ centered at $\underline{y}'$, and an integral over the remaining surface area S-A.

$$p(\underline{z}') = \iint_{A}\frac{\sigma e^{ik|\underline{z}' - \underline{y}|}}{4\pi|\underline{z}' - \underline{y}|}\,dS + \iint_{S-A}\frac{\sigma e^{ik|\underline{z}' - \underline{y}|}}{4\pi|\underline{z}' - \underline{y}|}\,dS \tag{A4}$$

Take $\underline{R} = \underline{z}' - \underline{y}$, $dS = R^2 d\Omega/(\underline{n}\cdot\hat{\underline{R}})$, where $d\Omega$ is an element of a solid angle and evaluate the first integral as $\epsilon \to 0$, and subsequently, as $h \to 0$.

$$\iint_A \frac{\sigma e^{ik|\underline{z}' - \underline{y}|}}{4\pi|\underline{z}' - \underline{y}|}\, dS = \iint \frac{\sigma e^{ikR}}{4\pi R} \frac{R^2 d\Omega}{(\underline{n}\cdot\hat{\underline{R}})} \xrightarrow[\epsilon\to 0]{} \iint \frac{\sigma e^{ikh}}{4\pi}\, h d\Omega \xrightarrow[h\to 0]{} 0 \tag{A5}$$

The limit of the second integral as ϵ and h approach zero in either order will be written in an abbreviated form as

$$*\iint \frac{\sigma e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\, dS = \lim_{\epsilon\to 0} \iint_{S-A} \frac{\sigma e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\, dS \tag{A6}$$

Hence, an integral equation for $\sigma(\underline{y})$, when $p(\underline{y})$ is specified apriori, is given by

$$p(\underline{y}') = *\iint \sigma(\underline{y}) \frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\, dS \tag{A7}$$

which is the same as (9).

When $q(\underline{y})$ is specified apriori, we proceed from (A4) by taking the gradient in the $\underline{n}'$ direction. Then

$$q(\underline{y}') = \lim_{z'\to y'} \iint_A \sigma \frac{\partial}{\partial n'} \left[\frac{e^{ik|\underline{z}' - \underline{y}|}}{4\pi|\underline{z}' - \underline{y}|}\right] dS$$

$$+ \lim_{z'\to y'} \iint_{S-A} \sigma \frac{\partial}{\partial n'} \left[\frac{e^{ik|\underline{z}' - \underline{y}|}}{4\pi|\underline{z}' - \underline{y}|}\right] dS \tag{A8}$$

The first term becomes

$$\iint_A \sigma \frac{d}{dR}\left(\frac{e^{ikR}}{4\pi R}\right)(\underline{n}'\cdot\hat{\underline{R}}) \frac{R^2 d\Omega}{(\underline{n}\cdot\hat{\underline{R}})} \xrightarrow{\epsilon\to 0} \iint_A \frac{\sigma e^{ikh}}{4\pi}(ikh - 1)\, d\Omega$$

$$\xrightarrow{h\to 0} -\frac{\sigma(\underline{y}')}{2} \tag{A9}$$

This term changes sign when $\underline{y}'$ is approached from the region $\underline{x}$. The second integral in (A8) has the same unique limiting value, regardless of the order in which ϵ and h approach zero and regardless of whether $\underline{y}'$ is approached from the $\underline{z}$ or $\underline{x}$ side. The limiting value of this second integral will be written in an abbreviated notation as (with $R = |\underline{y}' - \underline{y}|$)

$$*\iint \sigma \frac{\partial}{\partial n'}\left[\frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\right] dS = \lim_{\epsilon \to 0} \iint_{S-A} \frac{\sigma e^{ikR}}{4\pi R} \left(ik - \frac{1}{R}\right)(\underline{n}' \cdot \hat{\underline{R}})\, dS \tag{A10}$$

Hence the integral equation becomes

$$q(\underline{y}') = -\frac{\sigma(\underline{y}')}{2} + *\iint \sigma \frac{\partial}{\partial n'}\left[\frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\right] dS \tag{A11}$$

which is the same as (8).

To derive the integral equations for the dipole density $\mu(\underline{y})$, we start with the field at the point $\underline{z}' = \underline{y}' + h\underline{n}'$ and again separate the form into two parts, an integral over the infinitesimal area A about $\underline{y}'$ and an integral over the remaining surfaces S-A.

$$p(\underline{z}') = \iint_A \mu \frac{\partial}{\partial n}\left[\frac{e^{ik|\underline{z}' - \underline{y}|}}{4\pi|\underline{z}' - \underline{y}|}\right] dS + \iint_{S-A} \mu \frac{\partial}{\partial n}\left[\frac{e^{ik|\underline{z}' - \underline{y}|}}{4\pi|\underline{z}' - \underline{y}|}\right] dS \tag{A12}$$

We now follow the same procedures as in (A9) and the first integral becomes

$$\iint_A = \iint_A \frac{\mu e^{ikR}}{4\pi R}\left(ik - \frac{1}{R}\right)\left(\frac{-\underline{n}\cdot\hat{\underline{R}}}{\underline{n}\cdot\hat{\underline{R}}}\right) R^2 d\Omega$$

$$\xrightarrow[\epsilon \to 0]{} \iint_A \frac{\mu e^{ikh}}{4\pi}(1 - ikh)\, d\Omega \xrightarrow[h \to 0]{} + \frac{\mu(\underline{y}')}{2} \tag{A13}$$

The limiting form of the second integral in (A12) will be written in the abbreviated form as

$$*\iint \mu \frac{\partial}{\partial n}\left[\frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\right]dS = \lim_{\epsilon \to 0} \iint_{S-A} \frac{\mu e^{ikR}}{4\pi R}\left(ik - \frac{1}{R}\right)(-\underline{n}\cdot\hat{\underline{R}})\, dS \tag{A14}$$

Hence, the integral equation becomes

$$p(\underline{y}') = \frac{\mu(\underline{y}')}{2} + *\iint \mu \frac{\partial}{\partial n}\left[\frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\right]dS \tag{A15}$$

which is the same as (11).

We return now to the field of a dipole layer at the point $\underline{z}' = \underline{y}' + h\underline{n}'$

$$p(\underline{z}') = \iint \mu(\underline{y}) \frac{\partial}{\partial n}\left[\frac{e^{ikR}}{4\pi R}\right]dS \tag{A16}$$

where $\underline{R} = \underline{z}' - \underline{y}$. We take the gradient in the $\underline{n}'$ direction at $\underline{z}'$ and then take the limit as $z' \to y'$, assuming that the limit exists,

$$q(\underline{y}') = \lim_{\underline{z}' \to \underline{y}'}\left[\frac{\partial p(\underline{z}')}{\partial n'}\right] = \lim_{\underline{z}' \to \underline{y}'}\left[\frac{\partial}{\partial n'}\iint \mu \frac{\partial}{\partial n}\left(\frac{e^{ikR}}{4\pi R}\right)dS\right]$$

$$= \lim_{\underline{z}' \to \underline{y}'} \iint \mu(\underline{y}) \frac{\partial^2}{\partial n' \partial n}\left(\frac{e^{ikR}}{4\pi R}\right)dS \tag{A17}$$

And by repeated use of (A2) and (A3),

$$q(\underline{y}') = \lim_{\underline{z}' \to \underline{y}'} \iint \frac{\mu e^{ikR}}{4\pi R^2}$$

$$\left\{\left(3ik - \frac{3}{R} + k^2R\right)(\underline{n}\cdot\hat{\underline{R}})(\underline{n}'\cdot\hat{\underline{R}}) - \left(ik - \frac{1}{R}\right)(\underline{n}\cdot\underline{n}')\right\}dS \tag{A18}$$

In fact, the gradient of the field of a dipole layer is continuous as the field point crosses the dipole layer. Thus, the limit of (A17) exists and is unique. A rigorous proof of this result[3] would be long and tedious to develop; thus, it will be omitted. However, the result should be plausible because the dipole layer may be considered as the superposition of two monopole layers, $\pm\,\sigma(\underline{y}) = \pm\,\mu(\underline{y})/d$, of equal amplitude but having an opposite sign and being displaced an infinitesimal distance d apart. The gradient of the field due to each monopole layer has a discontinuity at the surface, as given in (A9); however, in the combined field the two discontinuities cancel; therefore, the gradient of the field of a dipole layer has no discontinuity at the surface.

Hence, an appropriate integral equation for $\mu(\underline{y})$ when $q(\underline{y})$ is specified can be stated in an abbreviated notation by

$$q(\underline{y}') = {}^{*}\iint \mu(\underline{y})\,\frac{\partial^2}{\partial n'\partial n}\left[\frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\right]dS \tag{A19}$$

which is the same as (10). More explicit and detailed forms for the term on the right are given by the right-hand sides of (A17) or (A18), where it must be emphasized that the integral term in square brackets must be evaluated with $R = |\underline{z}' - \underline{y}|$ <u>before</u> passing to the limit $\underline{z}' = \underline{y}'$. However, if the surface of integration is divided into two parts, a small <u>finite</u> area A about $\underline{y}'$ and the remaining surface S-A, the second part may be evaluated with $R = |\underline{y}' - \underline{y}|$, and no limit is necessary in the second part.

One derivation of the Helmholtz surface equation obtained by using Green's transformation theorem is given in Reference 15. However, once we have established (A7) and (A15) for the fields of arbitrary monopole and dipole layers, we can derive the Helmholtz surface equation quite simply. We consider the combined field $p(\underline{y}) = p_m(\underline{y}) + p_d(\underline{y})$, where $p_m(\underline{y})$ is the field at the surface of a monopole layer with surface density minus $q(\underline{y})$

$$p_m(\underline{y}') = -\,{}^{*}\iint \frac{q(\underline{y})e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\,dS \tag{A20}$$

and $p_d(\underline{y})$ is the field at the surface of a dipole layer with surface density $p(\underline{y})$ on the positive side.

$$p_d(\underline{y}') = \frac{p(\underline{y}')}{2} + *\iint p \frac{\partial}{\partial n}\left[\frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\right] dS \tag{A21}$$

Hence, the combined field is

$$\frac{p(\underline{y}')}{2} = - *\iint \frac{q e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|} dS + *\iint p \frac{\partial}{\partial n}\left[\frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\right] dS \tag{A22}$$

which is the same as (12).

The new integral relation between the Helmholtz surface layers $p(\underline{y})$ and $q(\underline{y})$ as given by (13) can be derived by a slight variation of this last method. We consider the combined normal gradient at the surface, $q(\underline{y}') = q_m(\underline{y}') + q_d(\underline{y}')$, where $q_m(\underline{y}')$ is the normal gradient at $\underline{y}'$ of the field of a layer of monopoles with surface density $-q(\underline{y})$, and $q_d(\underline{y}')$ is the normal gradient at $\underline{y}'$ of the field of a layer of dipoles with surface density $p(\underline{y})$. Hence by (A11) and (A17)

$$q_m(\underline{y}') = \frac{q(\underline{y}')}{2} - *\iint q(\underline{y}) \frac{\partial}{\partial n'}\left[\frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\right] dS \tag{A23}$$

$$q_d(\underline{y}) = *\iint p(\underline{y}) \frac{\partial^2}{\partial n' \partial n}\left[\frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\right] dS \tag{A24}$$

$$\frac{q(\underline{y}')}{2} = - *\iint q(\underline{y}) \frac{\partial}{\partial n'}\left[\frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\right] dS$$

$$+ *\iint p(\underline{y}) \frac{\partial^2}{\partial n' \partial n}\left[\frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\right] dS \tag{A25}$$

which is the same as (13).

APPENDIX B

CHARACTERISTIC WAVE FUNCTIONS

The simple homogeneous relations, (22) through (29), that apply to the characteristic wave functions can all be derived from the Green transformation theorems as applied to an arbitrary external wave function $p(\underline{z})$ with gradient $q(\underline{y})$ at surface point $\underline{y}$ in the direction of the outward normal $\underline{n}$.

$$- {}^{*}\iint \frac{q(\underline{y})e^{ik|\underline{r} - \underline{y}|}}{4\pi|\underline{r} - \underline{y}|}\, dS + {}^{*}\iint p(\underline{y})\, \frac{\partial}{\partial n}\left[\frac{e^{ik|\underline{r} - \underline{y}|}}{4\pi|\underline{r} - \underline{y}|}\right] dS$$

$$= 0 \qquad \text{if } \underline{r} = \underline{x} \tag{B1}$$

$$= \frac{p(\underline{y}')}{2} \qquad \text{if } \underline{r} = \underline{y}' \tag{B2}$$

$$= p(\underline{z}) \qquad \text{if } \underline{r} = \underline{z} \tag{B3}$$

$$- {}^{*}\iint q(\underline{y})\, \frac{\partial}{\partial n'}\left[\frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\right] dS + {}^{*}\iint p(\underline{y})\, \frac{\partial^2}{\partial n' \partial n}\left[\frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\right] dS$$

$$= \frac{q(\underline{y}')}{2} \tag{B4}$$

The first three equations are those we have previously called the Helmholtz interior, surface, and exterior equations (14), (12), and (7), respectively. Equation (B4) is the new surface gradient equation derived in (A25). Note that the integrals in (B1) and (B3) need not be starred because no abbreviated notation is implied. However, the possibility of writing the set of equations in this unified notation was the principal reason for selecting the particular form of abbreviated notation.

We also write similar equations for any internal wave function $P(\underline{x})$, with gradient $Q(\underline{y})$ in the direction of the outward normal $\underline{n}$.

$$*\iint \frac{Q(\underline{y})e^{ik|\underline{r}-\underline{y}|}}{4\pi|\underline{r}-\underline{y}|}\,dS - *\iint P(\underline{y})\,\frac{\partial}{\partial n}\left[\frac{e^{ik|\underline{r}-\underline{y}|}}{4\pi|\underline{r}-\underline{y}|}\right]dS$$

$$= P(\underline{x}) \qquad \text{if } \underline{r} = \underline{x} \tag{B5}$$

$$= \frac{P(\underline{y}')}{2} \qquad \text{if } \underline{r} = \underline{y}' \tag{B6}$$

$$= 0 \qquad \text{if } \underline{r} = \underline{z} \tag{B7}$$

$$*\iint Q(\underline{y})\,\frac{\partial}{\partial n'}\left[\frac{e^{ik|\underline{y}'-\underline{y}|}}{4\pi|\underline{y}'-\underline{y}|}\right]dS - *\iint P(\underline{y})\,\frac{\partial^2}{\partial n'\partial n}\left[\frac{e^{ik|\underline{y}'-\underline{y}|}}{4\pi|\underline{y}'-\underline{y}|}\right]dS \tag{B8}$$

$$= \frac{Q(\underline{y}')}{2} \tag{B8}$$

Note that the forms of (B1) through (B4) differ in sign from (B5) through (B8) only because in the former set, the normal points into the region where the wave function $p(\underline{r})$ obeys the differential equation without singularities; whereas, in the latter set, the normal points into the region where there must be some singularities in the wave function $P(\underline{r})$. If there were no singularities anywhere, $P(\underline{r})$ would be identically zero.

The special relations for the characteristic wave functions of the first kind would then result from supplementing (B1) through (B8) with the defining characteristics for functions of the first kind.

$$Q_i^{(1)}(\underline{y}) = q_i^{(1)}(\underline{y}) \tag{B9}$$

$$P_i^{(1)}(\underline{y}) = 0 \tag{B12}$$

For example, (22a) through (22c), respectively, follow from substituting (B9) and (B10) into (B5) through (B7), respectively.

The special relations for the characteristic wave functions of the second kind follow in like manner from (B1) through (B8), plus the defining characteristics for functions of the second kind

$$P_i^{(2)}(\underline{y}) = p_i^{(2)}(\underline{y}) \tag{B11}$$

$$Q_i^{(2)}(\underline{y}) = 0 \tag{B12}$$

We now show that the homogeneous equation

$$*\iint p(\underline{y}) \frac{\partial}{\partial n}\left[\frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\right] dS = \frac{p(\underline{y}')}{2} \tag{B13}$$

has a solution only when $k = \ell_i$, a characteristic wave number of the first kind, i.e., only when an internal wave $P(\underline{x})$ exists with $P(\underline{y}) = 0$. For $p(\underline{y}')$ in (B13) can be interpreted, according to (A11), as the field at $\underline{z} \to \underline{y}'$ of a special dipole layer $\mu(\underline{y})$

$$p(\underline{y}') = *\iint \mu(\underline{y}) \frac{\partial}{\partial n}\left[\frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\right] dS + \frac{\mu(\underline{y}')}{2} \tag{B14}$$

provided $\mu(\underline{y})$ is taken equal to $p(\underline{y})$. Also, the field of the same dipole layer as the point $\underline{y}'$ is approached from the interior region $\underline{x}$ is given by

$$P(y') = *\iint \mu(\underline{y})\left[\frac{\partial}{\partial n} \frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\right] dS - \frac{\mu(\underline{y}')}{2} = \frac{p(\underline{y})}{2} - \frac{p(\underline{y}')}{2} = 0 \tag{B15}$$

In like manner, we show that the homogeneous equation

$$*\iint p \frac{\partial}{\partial n}\left[\frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\right]dS = -\frac{p(\underline{y}')}{2} \tag{B16}$$

has a solution only when $k = m_i$, i.e., only when an interior standing wave $P(\underline{x})$ exists with $Q(\underline{y}) = (\partial P/\partial n)_y = 0$. First, note that if a solution to (B16) exists for a specific k, then according to the Fredholm Alternative a solution must also exist to the adjoint homogeneous equation

$$*\iint f(\underline{y}) \frac{\partial}{\partial n'}\left[\frac{e^{-ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\right] dS = -\frac{f(\underline{y}')}{2} \tag{B17}$$

and by taking complex conjugate of (B17), a solution must also exist to the homogeneous equation

$$*\iint \bar{f}(\underline{y}) \frac{\partial}{\partial n'}\left[\frac{e^{+ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\right] dS = -\frac{\bar{f}(\underline{y}')}{2} \tag{B18}$$

Now, according to (8), $\bar{f}(\underline{y}')$ can be interpreted as the field gradient of a special monopole layer $\sigma(\underline{y}')$ as $\underline{y}'$ is approached from $\underline{z}$.

$$\bar{f}(\underline{y}') = *\iint \sigma(\underline{y}) \frac{\partial}{\partial n'}\left[\frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\right]dS - \frac{\sigma(\underline{y}')}{2} \tag{B19}$$

provided $\sigma(\underline{y})$ is taken equal to $-\bar{f}(\underline{y})$. However, the gradient at $\underline{y}'$ of the internal field $P(\underline{x})$ due to this same monopole layer is given by

$$Q(\underline{y}') = *\iint \sigma(\underline{y}) \frac{\partial}{\partial n'}\left[\frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\right] dS + \frac{\sigma(\underline{y}')}{2} = \frac{\bar{f}}{2} - \frac{\bar{f}}{2} = 0 \tag{B20}$$

Finally, we show that if $p_1(\underline{z})$ and $p_2(\underline{z})$ are two independent solutions of the Helmholtz equation with the same wave number k, then

$$\iint p_1(\underline{y})\, q_2(\underline{y})\, dS = \iint p_2(\underline{y})\, q_1(\underline{y})\, dS$$

Consider the Green formula as being applied to the two functions $p_1(\underline{z})$ and $p_2(\underline{z})$ in the infinite region outside the surface $\underline{y}$ and inside a sphere at infinity.

$$\iint (p_1 q_2 - p_2 q_1)\, dS = \iiint (p_1 \nabla^2 p_2 - p_2 \nabla^2 p_1)\, dV$$

$$= k^2 \iiint (p_1 p_2 - p_2 p_1)\, dV$$

$$\therefore \iint p_1 q_2\, dS = \iint p_2 q_1\, dS$$

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APPENDIX C

SOLUTIONS TO THE INTEGRAL EQUATIONS FOR THE PULSATING SPHERE

To solve the various integral equations for the pulsating sphere it is convenient to first calculate values for the integrals of the four kernel functions over the surface of the sphere. We take point $\underline{z}'$ to be outside the sphere at a distance r from the origin at the center. We take R and R_o as indicated in Figure 1,

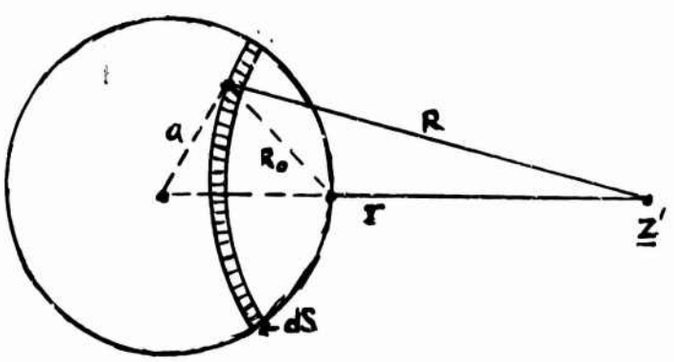

Figure 1 - Geometry of Pulsating Sphere

and take the element of area to be in the shape of an annular zone with

$$dS = 2\pi a^2 \sin\theta \, d\theta = 2\pi a \, RdR/r = 2\pi R_o dR_o \tag{C1}$$

We can now calculate the four integrals M_o, D_o, N_o, and E_o.

$$M_o = {}^{*}\!\iint \frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|y' - y|} dS = \int_o^{2a} \frac{e^{ikR_o}}{4\pi R_o} 2\pi R_o dR_o = \frac{e^{ika}}{k} \sin ka \tag{C2}$$

$$D_o = *\iint \frac{\partial}{\partial n}\left[\frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\right]dS = \int_o^{2a} \frac{d}{dR_o}\left(\frac{e^{ikR_o}}{4\pi R_o}\right)\frac{R_o}{2a}\,2\pi R_o\,dR_o$$

$$= e^{ika}\cos ka - \frac{e^{ika}\sin ka}{ka} - \frac{1}{2} \qquad (C3)$$

$$N_o = *\iint \frac{\partial}{\partial n'}\left[\frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\right]dS = *\int_o^{2a} \frac{d}{dR_o}\left(\frac{e^{ikR_o}}{4\pi R_o}\right)\frac{R_o}{2a}\,2\pi R_o\,dR_o$$

$$= e^{ika}\cos ka - \frac{e^{ika}\sin ka}{ka} - \frac{1}{2} \qquad (C4)$$

$$E_o = *\iint \frac{\partial^2}{\partial n'\partial n}\left[\frac{e^{ik|\underline{y}' - \underline{y}|}}{4\pi|\underline{y}' - \underline{y}|}\right]dS = \lim_{r\to a}\left[\frac{\partial}{\partial r}\int_{r-a}^{r+a} \frac{d}{dR}\left(\frac{e^{ikR}}{4\pi R}\right)\frac{\partial R}{\partial a}\,\frac{2\pi aR\,dR}{r}\right]$$

$$= \left(ik - \frac{1}{a}\right)e^{ika}\left(\cos ka - \frac{\sin ka}{ka}\right) \qquad (C5)$$

The integral equations (8) through (14) for this problem all reduce to simple algebraic forms in terms of these four values. For example, (13) becomes

$$q_o(ka - \tan ka) = p_o\left(ik - \frac{1}{a}\right)(ka - \tan ka) \qquad (C6)$$

or

$$q_o + 2q_oI_o = 2p_oJ_o \qquad (C7)$$

which is the same as (35). The remaining equations of the series, (30) through (36), can be obtained in a similar way.

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