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A GENERALIZED UPPER BOUNDING ALGORITHM FOR MULTICOMMODITY NETWORK FLOW PROBLEMS

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SECTION I

INTRODUCTION

Multicommodity network flow problems require the selection of optimal flow patterns for each of a number of distinguishable commodities in a capacitated network. The objective can be either to minimize the cost of achieving given flows, or to maximize the sum of the flows. When a node-arc formulation is used, these problems may be written as block diagonal linear programs with coupling rows. In this paper a compact inverse version of the simplex method for solving multicommodity problems is described. By using the special structure of any basis matrix, the simplex method can be performed while maintaining the inverse of a working basis whose dimension is only the number of currently saturated arcs. Aside from multiplication by this inverse, all other simplex computations are performed using addition or graph theoretic operations. The algorithm is a specialization of the generalized upper bounding method for block angular problems [4], [5]. It is similar to Saigal's method [6] which was derived using an arc-circuit formulation.

The approach taken here has two important advantages. First it presents the algorithm as a direct specialization of a well known general procedure for linear programs. Second, in Saigal's work, at each iteration, several systems of linear equations must be solved and no procedures are given for updating the matrix inverses associated with these equations. Here we show that the only non graph theoretic or non additive operations required are multiplication by and updating of the working basis inverse. Hence all the nonunimodular aspects of the problem are condensed into a single matrix which appears to be of minimal size. Efficient relations for updating the working basis inverse are derived here as specializations of those in the generalized upper bounding method.

SECTION II

PROBLEM STATEMENT

Consider a network which has nodes 1, 2, ..., ..., N and directed arcs a_1, a_2, ..., ..., ..., a_M. The case with undirected arcs will be considered later. Arcs a_1, ..., a_ℓ ($\ell \leq M$) have capacities b_1, ..., b_ℓ. Let there be K commodities and define x_{km} as the flow of commodity k in arc a_m. Each commodity k has associated with it a source node s_k and a sink node t_k. The constraints are

1. flows are nonnegative

$$x_{km} \geq 0 \qquad \text{(all } k \text{ and } m) \tag{1}$$

2. capacity restrictions on arc a_m

$$\sum_{k=1}^{K} x_{km} \leq b_m \qquad (1 \leq m \leq \ell) \tag{2}$$

3. flow conservation for commodity k at node n.

$$\sum_{a_m \in B_n} x_{km} - \sum_{a_m \in A_n} x_{km} = \begin{cases} -f_k & \text{if } n = s_k \\ 0 & \text{if } n \neq s_k,\ n \neq t_k \\ +f_k & \text{if } n = t_k \end{cases} \tag{3}$$

where f_k is the amount of flow of commodity k in the network, B_n is the set of arcs terminating at node n, and A_n is the set of arcs originating at node n.

For the min-cost problem, the flows f_k are given and the objective is to minimize total cost

$$\min \quad Z = \sum_{k,m} c_{km}\, x_{km} \tag{4}$$

The max-flow problem views the f_k as variables and has objective

$$\max \quad \sum_{k} f_k \tag{5}$$

Since the max flow problem is a special case of the min cost problem, we will use (4) as the objective.

In matrix form (1) - (4) becomes

minimize Z

subject to

$$\begin{array}{c|c|cc|cc|c|cc|l}
Z & S_1 \dots S_\ell & x_{11} \cdots & x_{1M} & x_{21} \cdots & x_{2M} & & x_{K1} \cdots & x_{KM} & \\
\hline
1 & 0 & -C_{11} & -C_{1M} & -C_{21} & -C_{2M} & \cdots & -C_{K1} & -C_{KM} & = 0 \\
\hline
0 & I_{\ell\times\ell} & I_{\ell\times\ell} & 0 & I_{\ell\times\ell} & 0 & \cdots & I_{\ell\times\ell} & 0 & = b_1 \dots = b_\ell \\
\hline
 & & \multicolumn{2}{c|}{F} & & & & & & = d_1 \\
 & & & & \multicolumn{2}{c|}{F} & & & & = d_2 \\
 & & & & & & \ddots & & & \\
 & & & & & & & \multicolumn{2}{c|}{F} & = d_K
\end{array} \tag{6}$$

In the above linear program there are $\ell + 1$ coupling rows and K identical diagonal blocks. The matrix F is the node-arc incidence matrix of the network with the last row deleted. Hence F is N-1 x M and has rank N-1. The variables S_i are nonnegative slacks for the capacity constraints, and the vector d_k has $-f_k$ in position s_k, $+f_k$ in position t_k, and zeroes elsewhere.

SECTION III

THE GENERALIZED UPPER BOUNDING ALGORITHM FOR BLOCK ANGULAR PROBLEMS

Consider the general block diagonal problem with coupling rows.

$$
\begin{aligned}
&\text{minimize } Z \\
&\text{subject to } A_0 x_0 + A_1 x_1 + \cdots + A_K x_K = b \qquad (7) \\
&\qquad D_1 x_1 = b_1 \\
&\qquad \ddots \qquad \vdots \\
&\qquad D_K x_K = b_K \\
&x_i \geq 0
\end{aligned}
$$

where each A_i is an $m_0 \times n_i$ matrix, each D_i is $m_i \times n_i$, and Z is the first component of x_0. We assume throughout that the constraint matrix of (7) has full rank. Hence each D_i has rank m_i. The method is based on the following result proved in [5].

<u>Theorem 1</u> Any basis matrix <u>B</u> for (7) can partitioned to have the form:

$$\underline{B} = \left[\begin{array}{c|c} \overbrace{\hat{B}}^{m_0 \text{ columns}} & A_{11} A_{21} \cdots A_{K1} \\ \hline C & \begin{matrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_K \end{matrix} \end{array}\right] \begin{matrix} \} \; m_0 \text{ rows} \\ \\ \\ \\ \end{matrix} \qquad (8)$$

$$\underbrace{\qquad}_{\text{non-key columns}} \qquad \underbrace{\qquad\qquad}_{\text{key columns}}$$

where each B_i is an $m_i \times m_i$ nonsingular submatrix of D_i.

Using the fact that the B_i are nonsingular we develop a transformation matrix $\underline{T}$ such that $\underline{B}\ \underline{T}$ is block triangular. The simplest such $\underline{T}$ has the form

$$\underline{T} = \left[\begin{array}{c|c} I_1 & 0 \\ \hline V & I_2 \end{array}\right] \begin{matrix} \} \; m_0 \text{ rows} \\ \\ \end{matrix} \qquad (9)$$

$$\underbrace{\qquad}_{m_0 \text{ columns}}$$

where I_1 and I_2 are identity matrices and

$$V = - \begin{bmatrix} B_1^{-1} & & & 0 \\ & B_2^{-1} & & \\ & & \ddots & \\ 0 & & & B_K^{-1} \end{bmatrix} \begin{bmatrix} \\ C \\ \\ \end{bmatrix} \qquad (10)$$

Then

$$\underline{B}\,\underline{T} = \left[\begin{array}{c|cccc} \overbrace{B}^{m_0 \text{ columns}} & A_{11} & A_{21} & \cdots & A_{K1} \\ \hline 0 & B_1 & & & \\ & & B_2 & & \\ & & & \ddots & \\ & & & & B_K \end{array}\right] \begin{array}{l} \big\} \; m_0 \text{ rows} \\ \\ \\ \\ \\ \end{array} \tag{11}$$

is the block triangularized basis matrix. The submatrix B, given by

$$B = \hat{B} + [A_{11} \; A_{21} \; \ldots \; A_{K1}] \; V \tag{12}$$

is called the <u>working basis</u>. Since $\underline{B}\,\underline{T}$ is nonsingular, B is also nonsingular.

We now examine how the operations of the revised simplex method may be carried out using quantitites associated with the working basis. These operations require only that two sets of linear equations, with coefficient matrices $\underline{B}'$ and $\underline{B}$, be solved (one for the pricing vector, the other for the transform of the entering vector). Triangularizing $\underline{B}$ greatly simplifies their solution.

<u>Determining the Simplex Multipliers</u>. Here the vector of simplex multipliers $\pi = (\pi_0, \pi_1, \ldots, \pi_K)$ is to be computed. These satisfy

$$\pi \underline{B} = c_B$$

or, since only Z has a nonzero objective coefficient, and its column is the leftmost column of $\underline{B}$,

$$\pi \underline{B} = (1, 0, \ldots, 0) \tag{13}$$

Multiplying on the right by $\underline{T}$,

$$\pi(\underline{B}\ \underline{T}) = (1, 0, \ldots, 0)\ \underline{T} = (1, 0, \ldots, 0) \tag{14}$$

Since $\underline{B}\ \underline{T}$ is triangular, these are easily solved yielding

$$\pi_0 = \text{first row of } B^{-1} \tag{15}$$

$$\pi_i = -\pi_0 A_{i1} B_i^{-1} \qquad (i = 1, \ldots, K) \tag{16}$$

Thus if B^{-1} and B_i^{-1} are maintained, the vectors π_0 and π_i are easily computed.

<u>Determining the Column to Enter the Basis</u>. This is done as in the revised simplex method by computing

$$\bar{c}_j = -\pi \underline{P}_j \tag{17}$$

for each nonbasic column $\underline{P}_j$. Note that only 2 partitions of any column $\underline{P}_j$ are nonzero. If

$$\min \bar{c}_j = \bar{c}_s \geq 0$$

then the current solution is optimal. Otherwise $\underline{P}_s$ enters the basis. Suppose $\underline{P}_s$ is a column from the σ^{th} block so that $\underline{P}_s = [P_{s0}, 0 \ldots 0, P_{s\sigma}, 0 \ldots 0]'$.

Finding $\hat{\underline{P}}_s = \underline{B}^{-1}\,\underline{P}_s$

Here we must solve the linear system

$$\underline{B}\,\hat{\underline{P}}_s = \underline{P}_s \tag{18}$$

Let
$$\hat{\underline{P}}_s = \underline{T}\,\underline{Z} \tag{19}$$

Substituting (19) into (18) gives

$$(\underline{B}\,\underline{T})\,\underline{Z} = \underline{P}_s \tag{20}$$

which can be easily solved for $\underline{Z} = (Z_0, Z_1, \ldots, Z_K)'$ since $\underline{B}\,\underline{T}$ is block triangular;

$$Z_i = 0 \qquad i = 1, \ldots, K\ ;\ i \neq \sigma \tag{21}$$

$$Z_\sigma = B_\sigma^{-1}\,P_{s\sigma} \tag{22}$$

$$Z_0 = B^{-1}\left\{P_{s0} - A_{\sigma 1}\,Z_\sigma\right\} \tag{23}$$

Thus Z_σ and Z_0 can be computed if B^{-1} and B_σ^{-1} are known. Then $\hat{\underline{P}}_s = (\hat{P}_{s0}, \hat{P}_{s1}, \ldots, \hat{P}_{sK})'$ is computed from (19) giving

$$\hat{P}_{s0} = Z_0 \tag{24}$$

$$\hat{P}_{si} = V_i\,Z_0 \qquad i = 1, \ldots, K;\ i \neq \sigma \tag{25}$$

$$\hat{P}_{s\sigma} = V_\sigma Z_0 + Z_\sigma \tag{26}$$

where V_i is the i^{th} partition of V.

Choosing the Column to leave the Basis. This is done according to the standard simplex formulas. If the solution is not unbounded, then column r of $\underline{B}$, $\underline{P}_{j_r}$ leaves the basis. Assume that this column is from the ρ^{th} block of (7). Since computing the new values of the basic variables also proceeds as in the revised simplex method, we now consider updating the matrices B^{-1}, B_i^{-1} and any other quantities needed for the next iteration.

Updating Formulas. There are two cases which can occur. Only the results are stated here; derivations may be found in [5].

Case 1 The leaving column is non-key. Here the entering column can directly replace the one leaving without destroying the block diagonal structure of $\underline{B}$. Then none of the B_i^{-1} change, and B^{-1} is transformed to $*B^{-1}$ by a pivot operation.

$$*B^{-1} = E\,B^{-1}$$

where E is an $m_0 \times m_0$ elementary column matrix equal to the identity except in column r. Let $\bar{a}_{is}$ be the i^{th} component of $\hat{\underline{P}}_s$. Then column r of E has components

$$\eta_i = \begin{cases} -\bar{a}_{is}/\bar{a}_{rs} & i = 1, \dots, m_0;\ i \neq r \\ 1/\bar{a}_{rs} & i = r \end{cases} \qquad (17)$$

Case 2 The leaving column is a key column. Here when column $\underline{P}_{j_r}$ leaves the basis, the block B_ρ will have only $m_\rho - 1$ columns. Hence it is necessary to find another basic column from the ρ^{th} block to restore the basis structure. There are two subcases.

Case 2a There may be a basic non-key column from the ρ^{th} block which can be interchanged with $\underline{P}_{j_r}$ in the basis. Then the leaving column $\underline{P}_{j_r}$ will become non-key and Case 1 can be applied. Suppose $\underline{P}_{j_r}$ is the $i_2{}^{th}$ key column in the basis and that it will change places with the $i_1{}^{th}$ non-key column. Then the working basis is updated by

$$*B^{-1} = E\, B^{-1}$$

where E is an $m_0 \times m_0$ elementary row matrix equal to the identity except in the $i_1{}^{th}$ row. Row i_1 of E is just the $i_2{}^{th}$ row of the submatrix V in the transforming matrix $\underline{T}$ in (9). There is a non-key column which can be exchanged with $\underline{P}_{j_r}$ if and only if there is a nonzero element in this row. B_ρ^{-1} will change by a simple pivot, and all other B_i^{-1} will remain unchanged.

Case 2b If Case 2a cannot be performed, then by Theorem 1, the entering column $\underline{P}_s$ must be from the ρ^{th} block and a direct pivot is possible. In this case B_ρ^{-1} changes by a simple pivot, and the working basis will not change at all.

This completes the description of the algorithm for the general case. Note that at each iteration it is necessary to update at most an $m_0 \times m_0$ working basis inverse and an $m_i \times m_i$ diagonal block inverse. All updates can be performed using multiplication by an elementary row or column matrix.

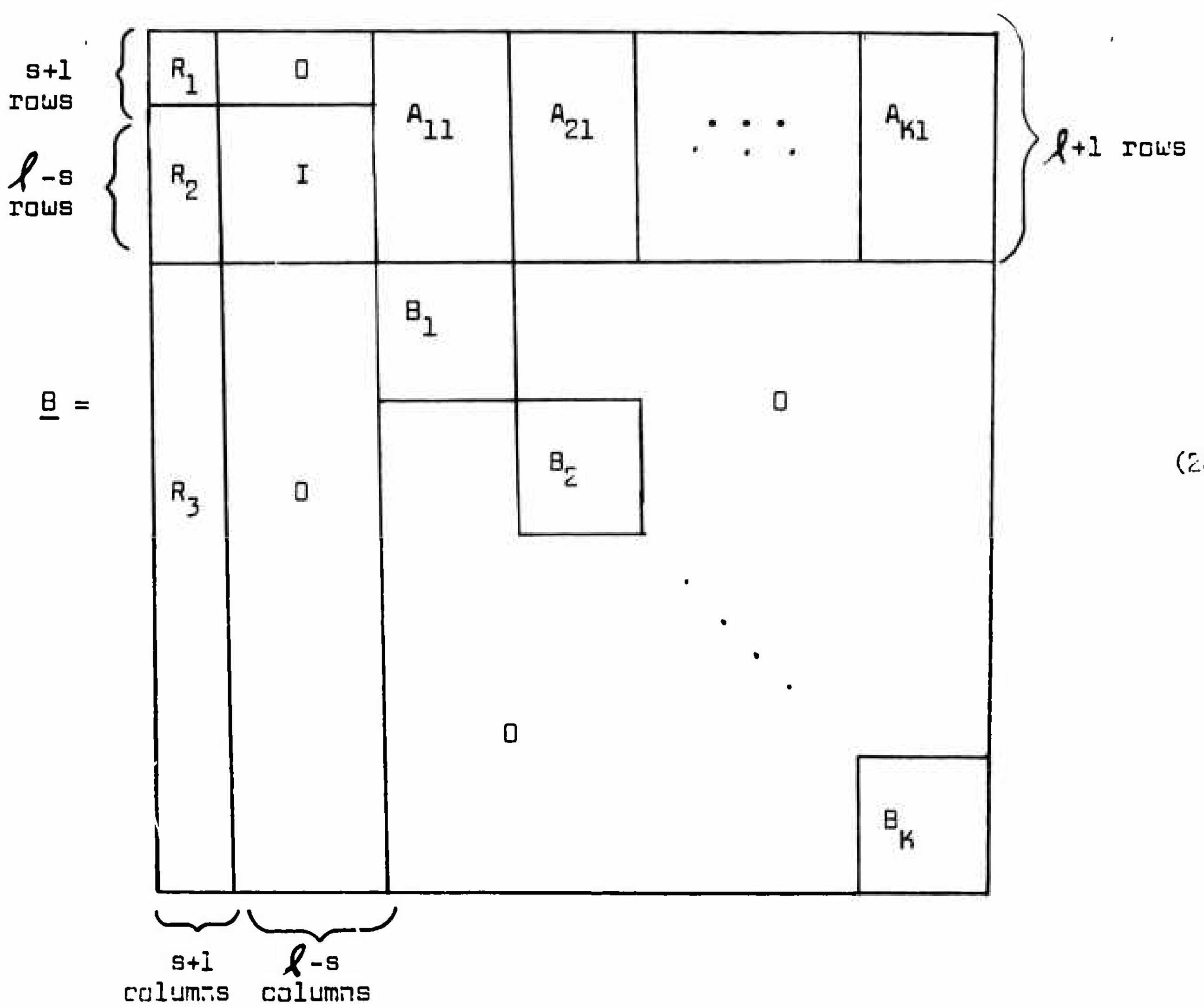
SECTION IV

WORKING BASIS STRUCTURE FOR THE MULTICOMMODITY PROBLEM

In the following sections the generalized upper bounding algorithm is applied to the multicommodity problem. Because of the special structure, significant simplifications occur.

Consider any basis matrix $\underline{B}$ for the multicommodity problem (6). By Theorem 1 the basis matrix can be partitioned as follows

$\underline{B}$ =



(28)

In this basis there are s saturated arcs, and hence $\ell - s$ slack variables in the basis. For each of the K commodities there is a diagonal block B_i which, by Theorem 1, is an $N-1 \times N-1$ nonsingular submatrix of the node arc incidence matrix F. The remaining $s+1$ columns in $[R_1\ R_2\ R_3]'$ consist of the cost variable (which is always the first basic variable) and s columns which are excess columns from some of the commodity blocks.

It is well known that the $N-1$ arcs corresponding to the columns of each matrix B_i form a spanning tree in the network [2]. Consequently we will be able to perform all the simplex operations which require B_i^{-1} by graph theoretic means, so it is not necessary to maintain these inverses (or the matrices B_i) explicitly.

The only portions of the algorithm which are not "graph theoretic" involve multiplication by the working basis inverse, so we now consider the structure of the working basis. It arises from the submatrix

R_1	0
R_2	I

of $\underline{B}$ in (28) when $\underline{B}$ is triangularized by driving R_3 to zero. Suppose $\underline{P}$ is one of the s excess columns in $[R_1\ R_2\ R_3]'$ of the basis, and that it is from the k^{th} commodity block, so

$$\underline{P} = (P_0, 0 \ldots 0, P_k, 0 \ldots 0)'$$

where P_0 has $\ell + 1$ components and P_k has N-1. The corresponding column in the working basis will then be given by

$$Q_0 = P_0 - A_{k1} B_k^{-1} F_k \qquad (29)$$

(see (12)). Here P_k is a column of F not contained in B_k, so it corresponds to an out-of-tree arc for the k^{th} commodity. Any such out-of-tree arc forms a unique circuit with the arcs of the spanning tree, and this circuit is described by the vector $- B_k^{-1} P_k$ whose j^{th} component is [1]

+1 if the tree arc corresponding to the j^{th} column of B_k is in the circuit and oriented the same as the out of tree arc.

-1 if the tree arc corresponding to the j^{th} column of B_k is in the circuit and oriented in the opposite direction as the out of tree arc.

0 if the tree arc corresponding to the j^{th} column of B_k is not in the circuit.

Hence the vector $- B_k^{-1} P_k$ can be calculated without knowing B_k^{-1} by a simple labeling process in the network:

(a) Label the destination node of the out-of-tree arc with the label +0. Go to Step b.

(b) Take some node n which has been labeled but not scanned and scan it. This means that every unlabeled node which is connected to node n by a tree arc (in the k^{th} spanning tree) is given a label. If the new node is reached by moving forward on arc a_m, then the new node is labeled $+m$. If the new node is reached by moving backward on arc a_m, then the new node is labeled $-m$. Go to Step c.

(c) If the origin node of the out-of-tree arc has been labeled, go to Step d. Otherwise go to Step b.

(d) Backtrack through the tree until the +0 label is found, recording the vector $-B_k^{-1} P_k$ as the backtracking is performed.

The submatrix A_{kl} in (29) has columns which contain a cost coefficient as the first component, and either zeroes or a unit vector as the remaining components. Essentially this matrix permates the arcs of the tree into the order in which they appear in the capacity constraints. Because $B_k^{-1} P_k$ is all 0 or ± 1, no multiplications are required to compute $A_{kl} B_k^{-1} P_k$ and hence Q_0 in (29) is readily computed. This column Q_0 of the working basis can be interpreted as follows. For $i = 1, \ldots, \ell$ let the i^{th} capacitated arc be the one corresponding to the $i+1^{th}$ row of $\underline{B}$. Then

a) The first component of Q_0 is the sum of the cost coefficients of arcs in the circuit for $\underline{P}$, with a plus sign for arcs oriented as $\underline{P}$'s arc and minus otherwise.

b) The remaining components are all zero or $\pm$ ones with the $i+1^{th}$ component being

+1 if the i^{th} capacitated arc is the arc associated with the column $\underline{P}$.

+1 if the i^{th} capacitated arc is in the unique circuit formed in the tree by the addition of $\underline{P}$ and oriented the same as $\underline{P}$.

-1 if the i^{th} capacitated arc is in the unique circuit formed in the tree by the addition of $\underline{P}$, but oriented opposite to $\underline{P}$.

0 otherwise.

As a result of this interpretation, Q_0 can be computed by a simple extension of the labeling algorithm for finding circuits.

The slack columns in the original basis are not affected by the triangularization. Hence the working basis B will have the form

$$B = \begin{array}{c} \\ \left[\begin{array}{c|c} S_1 & 0 \\ \hline S_2 & I \end{array}\right] \begin{array}{l} \}\, s+1 \\ \}\, \ell-s \end{array} \\ \underbrace{}_{s+1} \quad \underbrace{}_{\ell-s} \end{array} \tag{30}$$

where the columns of $\begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$ have the form of Q_0 in (29).

The algorithm presented in Section III involves the inverse of B in several places. In general, the elements of B^{-1} are not integers, and it is necessary to maintain B^{-1} explicitly. The presence of the slack columns lets us write

$$B^{-1} = \left[\begin{array}{c|c} S_1^{-1} & 0 \\ \hline -S_2 S_1^{-1} & I \end{array}\right] \tag{31}$$

and we will maintain only S_1^{-1} explicitly. Rows of $-S_2S_1^{-1}$ are just linear combinations of rows of S_1^{-1} with coefficients ± 1, so they are easily obtainable from S_1^{-1} wherever needed.

The result, then, of the special structure of the multicommodity problem is that it suffices to maintain and update a submatrix, S_1^{-1}, of the working basis inverse. The dimension of S_1^{-1} is s+1, where there are s saturated arcs in the current basis $\underline{B}$. Thus, considerable savings are obtained whenever the number of saturated arcs is small relative to the total number of capacitated arcs. All other computations are performed by graph theoretic means.

SECTION V

THE ALGORITHM FOR THE MULTICOMMODITY PROBLEM

Assume that at the beginning of some simplex iteration the following quantities are known:

1. The Submatrix S_1^{-1} of B^{-1} in (31)
2. The values and indices of the basic variables
3. The spanning tree for each commodity

In addition it may be desirable to maintain the submatrix V of $\underline{T}$ in (9) and the submatrix S_2 of B in (30) (see Section VI for further discussion). The simplex iteration proceeds as follows.

Determining the simplex multipliers. By (15) the multipliers π_0 for the capacity constraints are found in the first row of B^{-1}. Referring to (31) multipliers for saturated arcs are found in the first row of S_1^{-1} and multipliers for unsaturated arcs are zero. Uncapacitated arcs can be assigned a multiplier of zero. The vector π_k contains multipliers for the rows intersecting the k^{th} commodity block. By (16) these satisfy

$$\pi_k B_k = -\pi_0 A_{k1} \qquad (32)$$

The vector π_0 has a 1 as its first component, and the first row of A_{k1} contains the negatives of the coefficients for the arcs in the k^{th} tree. Hence for $1 \leq i \leq N - 1$ the i^{th} component of $-\pi_0 A_{k1}$ is the cost coefficient of the i^{th} arc of the k^{th} tree minus the component of π_0 corresponding to this arc*. We will call this the price, p_k^i, of the i^{th} tree arc. Since B_k is triangular, equations (32) can be solved by successive elimination. In graph theoretic terms the procedure is:

1. Assign node N a multiplier of 0 (the equation for this node has been dropped from F)

2. Suppose the multiplier $\pi_k^{n_1}$ for node n_1 has been evaluated and n_1 is connected to n_2 by an arc a_i in the tree with price p_k^i. Then

$$\pi_k^{n_2} = \pi_k^{n_1} + p_k^i \text{ if the arc is oriented } n_1 \longrightarrow n_2$$

$$\pi_k^{n_2} = \pi_k^{n_1} - p_k^i \text{ if the arc is oriented } n_2 \longrightarrow n_1$$

3. Continue branching along the k^{th} tree until all nodes have been assigned multipliers for the k^{th} commodity.

---

*Strictly speaking, uncapacitated arcs have no components in π_0. The multiplier for such an arc is taken to be zero.

Determining the column to enter the basis. Let $\bar{c}_{km}$ be the relative cost factor for x_{km}. Referring to (6), $\bar{c}_{km}$ has at most four non-zero terms:

$$\bar{c}_{km} = c_{km} - \pi_{0j_m} + \pi_k^{n_1} - \pi_k^{n_2}$$

where

n_1 is the origin node of arc a_m

n_2 is the destination node of arc a_m

and

π_{0j_m} is the component of π_0 corresponding to arc a_m if the arc is capacitated and zero otherwise.

The slack variable S_i has relative cost factor $-\pi_{0i}$.

Suppose column $\underline{P}_s = [P_{s0}, 0 \ldots 0, P_{sk}, 0 \ \ldots \ 0]'$ from the k^{th} commodity block is chosen to enter the basis. ($k = 0$ implies $\underline{P}_s$ is a slack column).

Finding $\hat{\underline{P}}_s = \underline{B}^{-1} \underline{P}_s$ The transformation of the entering column $\underline{P}_s$ in terms of the current basis is outlined in equations (21) - (26). In terms of the multicommodity problem these steps become

$$Z_i = 0 \qquad i = 1, \ldots, K \qquad i \neq k$$

$$Z_k = B_k^{-1} P_{sk}$$

$$Z_0 = B^{-1} (P_{s0} - A_{k1} Z_k) = B^{-1} Q_{s0}$$

Note that Z_k is just the negative of a circuit vector and Q_{s0} is a column like Q_0 in (29).

Hence both Z_k and Q_{s0} can be computed using the graph theoretic labeling process described in Section IV. To obtain Z_0 it is necessary to multiply by B^{-1}, a non-graph operation. The details of the computation are:

$$Z_0 = B^{-1} Q_{s0} = \left[\begin{array}{c|c} S_1^{-1} & 0 \\ \hline -S_2 S_1^{-1} & I \end{array}\right] \left[\begin{array}{c} Q_{s0}^1 \\ \hline Q_{s0}^2 \end{array}\right]$$

$$\left[\begin{array}{c} S_1^{-1} Q_{s0}^1 \\ \hdashline -S_2 S_1^{-1} Q_{s0}^1 + Q_{s0}^2 \end{array}\right]$$

so a matrix multiplication of order s+1 must be performed to get $S_1^{-1} Q_{s0}$. Then the rest of the column is generated by additive operations, since S_2 is a matrix of zeros and $\pm$ 1's.

Transforming back to $\hat{\underline{P}}_s$ is accomplished as in (24) - (26) by

$$\hat{P}_{s0} = Z_0$$

$$\hat{P}_{si} = V_i Z_0 \qquad i = 1, \ldots, K ; i \neq k$$

$$\hat{P}_{sk} = V_k Z_0 + Z_k$$

Here V_i is an $N-1 \times \ell+1$ matrix which is all zero except in columns corresponding to excess columns from commodity block i.

The nonzero columns contain the circuit vectors for those excess columns (see (10)). Thus this transformation from Z to $\hat{\underline{P}}_B$ is also accomplished using only additive operations. If the entering column is a slack column, then the computations are even simpler - all Z_i are zero ($i \neq 0$), and Z_0 is just a column of B^{-1}.

Choosing the Column to leave the Basis. This is done according to the standard simplex formulas. Assume that column r of $\underline{B}$ leaves the basis. Since computing the new values of the basic variables also proceeds as in the standard simplex method, we now consider updating the submatrix S_1^{-1}.

SECTION VI

UPDATING FORMULAS

In previous sections, we have maintained only a submatrix S_1^{-1} of B^{-1}. All other quantitites are calculated as needed by graph theoretic and additive methods. Hence, in the updating procedures for B^{-1}, it suffices to consider updating only S_1^{-1}. The cases are the same as in Section III.

Case 1 When the leaving column is non-key, B^{-1} is updated by

$$*B^{-1} = E\,B^{-1} \tag{33}$$

where E is an elementary column matrix. Since none of the diagonal blocks are affected, the spanning trees are unchanged. There are 4 subcases:

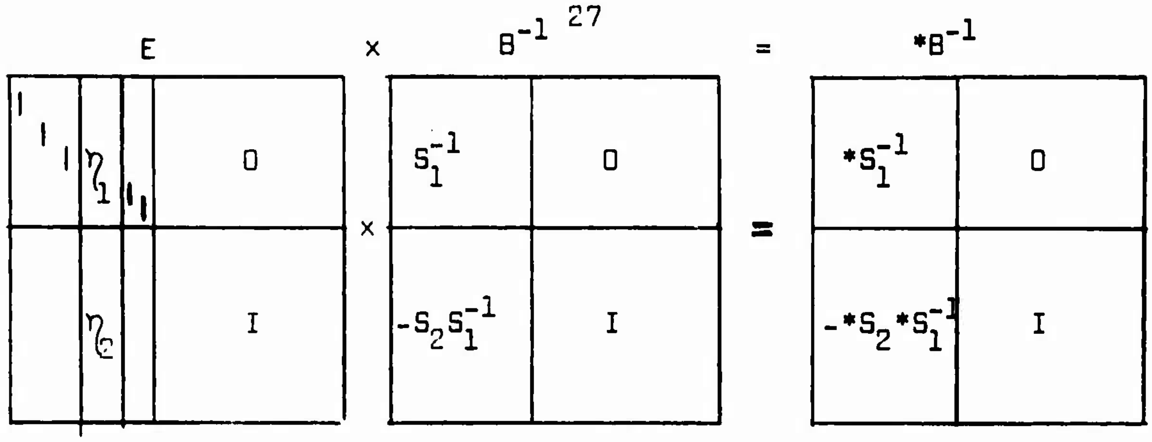
a) The leaving column is a flow column, and the entering column is a flow column.

b) The leaving column is a flow column, and the entering column is a slack column.

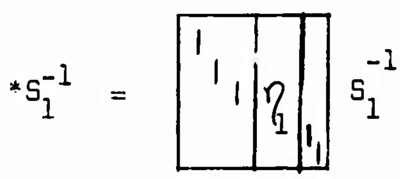
c) The leaving column is a slack column, and the entering column is a flow column.

d) The leaving column is a slack column, and the entering column is a slack column.

Consider first Cases 1a and 1b in which the leaving column is a flow column. Then, writing (33) in partitioned form gives



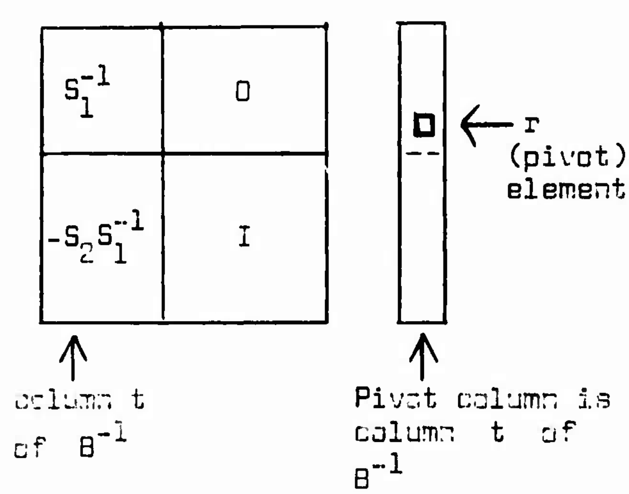
where



(34)

Hence S_1^{-1} is updated by an elementary column matrix.

If the entering column is a flow column, then the updating is complete. If the entering column is a slack column, then S_1^{-1} can be reduced in dimension by one, since $*S_1^{-1}$ in (34) will contain a unit vector column. To see this, suppose that the leaving column is in position r in the basis ($r \leq s+1$) and that the slack in row t ($t \leq s+1$) is entering. As shown in Section V the first $\ell+1$ components of the transformed entering column are the t^{th} column of B^{-1}. Updating the working basis is accomplished by pivoting on the r^{th} element of this column, as illustrated below:



Since the pivot reduces the pivot column to a unit vector, it will also reduce column t of B^{-1} to the r^{th} unit vector. Consequently we can reduce the dimension of $*S_1^{-1}$ by dropping its t^{th} column and r^{th} row.

In Cases 1c and 1d, the leaving column is a slack column in position r in the basis $(r > s + 1)$, so (33) becomes

$$\underset{E}{\left[\begin{array}{c|c|c|c} I & 0 & \eta_1 & 0 \\ \hline 0 & I & \eta_2 & I \end{array}\right]} \times \underset{B^{-1}}{\left[\begin{array}{c|c} S_1^{-1} & 0 \\ \hline -S_2 S_1^{-1} & I \end{array}\right]} = \underset{*B^{-1}}{\left[\begin{array}{c|c|c|c} *S_1^{-1} & 0 & \eta_1 & 0 \\ \hline -*S_2 *S_1^{-1} & I & \eta_2 & I \end{array}\right]} \qquad (35)$$

s+1 ... column r

Here

$$*S_1^{-1} = \left\{ I - \left[\begin{array}{c|c|c} 0 & \eta_1 & 0 \end{array}\right] S_2 \right\} S_1^{-1}$$

$$= \left\{ I - \eta_1 \left[\ v\ \right] \right\} S_1^{-1} \qquad (36)$$

where v is the $r-s-1^{th}$ row of S_2. As seen from (35), the block triangular structure of B^{-1} has been destroyed by the presence of the eta column. If the entering column is the slack in row t, (Case 1d) then just as in Case 1b, column t of $*B^{-1}$ will contain the r^{th} unit vector. The structure can then be restored by exchanging the r^{th} and t^{th} columns of $*B^{-1}$. This corresponds to replacing column t of $*S_1^{-1}$ (which is a zero column) with the column η_1.

Finally, if the entering column is a flow column (Case 1c), $*S_1^{-1}$ must increase in size by one since there is one less slack in the basis. To preserve the structure of $*B^{-1}$, the r^{th} column and row are moved to position $s+2$. Then $*S_1^{-1}$ is augmented by a border

$$\begin{bmatrix} *S_1^{-1} & \eta_1 \\ \omega & \eta \end{bmatrix}$$

where η is the $r\text{-}s\text{-}1^{th}$ element of η_2 and ω is the $r\text{-}s\text{-}1^{th}$ row of $-*S_2\,*S_1^{-1}$. To compute ω note from (35) that

$$-*S_2*S_1^{-1} = -\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & \eta_2 & \\ & & & & 1 \end{bmatrix} S_2\, S_1^{-1}$$

Hence its $r\text{-}s\text{-}1^{th}$ row is

$$-\begin{bmatrix} 0 & \eta & 0 \end{bmatrix} S_2\, S_1^{-1} = -\eta\, v\, S_1^{-1} \qquad (37)$$

where, as in (36), v is the $r\text{-}s\text{-}1^{th}$ row of S_2. If the calculations in (36) are carried out from right to left (as is clearly preferable), vS_1^{-1} will already have been computed.

<u>Case 2</u> When the leaving column is a key column, the corresponding arc is an arc in one of the spanning trees (say for commodity k). Removing it from the basis will destroy this tree, so the k^{th} spanning tree must be redefined. As in Case 2a of Section III, we first attempt to exchange the leaving column with a basic non-key column from block k. Consider the basic non-key columns from block k. The arc corresponding to each of these induces a unique circuit in the k^{th} tree. If one of these circuits contains the leaving column, then adding that arc to the tree and removing the leaving column will leave us with a new spanning tree. As in Section III, the working basis is then updated by an elementary row matrix,

$$*B^{-1} = \left[\begin{array}{c} I \\ \hline v \\ \hline I \end{array}\right] B^{-1} \qquad (38)$$

The vector v is a row of the submatrix V in (10). It contains zeroes except for ± 1 in columns corresponding to excess columns from block k whose circuits involve the leaving column. Hence, in particular, v is zero in the last ℓ-s columns, the slack columns. Hence, in partitioned form (38) is

$$\left[\begin{array}{c|c} *S_1^{-1} & 0 \\ \hline -*S_2 *S_1^{-1} & I \end{array}\right] = \left[\begin{array}{c|c} \begin{array}{c} I \\ \hline v_1 \\ \hline I \end{array} & 0 \\ \hline 0 & I \end{array}\right] \times \left[\begin{array}{c|c} S_1^{-1} & 0 \\ \hline -S_2 S_1^{-1} & I \end{array}\right]$$

where v_1 contains the first $s+1$ components of v. Then

$$*S_1^{-1} = \begin{bmatrix} I \\ \hline v_1 \\ \hline I \end{bmatrix} S_1^{-1}$$

gives the updating relation for $*S_1^{-1}$. Note that only one row of S_1^{-1} changes, and that row becomes a linear combination of rows of S_1^{-1} with coefficients $0, \pm 1$. Hence no multiplication is required for this update.

If no such exchange is possible, then, as in Case 2b of Section III, a direct pivot can be performed. A single spanning tree is redefined (one arc changes). There is no change in the working basis and hence no change in S_1^{-1}.

In each case, in addition to updating S_1^{-1} and one of the K spanning trees, we may wish to update the submatrices V and S_2. Each column of V contains at most one nonzero partition, and that partition is a circuit vector of the form $-B_k^{-1} P_k$.

When a non-key column leaves the basis, (Case 1), one column of V will change and a new circuit vector must be computed. When a key column leaves, a spanning tree (say the k^{th} tree) changes, so all circuit vectors in V_k must be recomputed. Since at most two partitions of V are changed at any iteration, it may be desirable to store the nonzero columns of V_k explicitly. Since these contain only zeroes and $\pm$ ones, they can be stored compactly. The alternative is to recompute them at each cycle. The best course of action depends on the amount of high speed storage available.

The matrix S_2 is a submatrix of B, and as shown in (29) B has columns of the form

$$Q_0 = P_0 - A_{k1} B_k^{-1} P_k$$

As shown in Section IV, Q_0 is essentially a permutation of the circuit vector $-B_k^{-1} P_k$. Hence S_2 probably should not be stored explicitly; it is easily generated as needed from the columns of V.

SECTION VII

MAX FLOW PROBLEMS AND UNDIRECTED ARCS

To solve the max flow problem, a column for the commodity flow variable f_k must be added to the k^{th} block. This column corresponds to a fictitious arc from the sink t_k to the source s_k for commodity k. The right hand side vectors d_k in (6) are all zero, and the cost coefficients are unity for the f_k and zeroes otherwise. Aside from the change from minimization to maximization, the algorithm proceeds as before.

As shown in [3] problems with undirected arcs can be formulated by defining new variables y^+_{km} and y^-_{km} satisfying

$$x_{km} = y^+_{km} - y^-_{km}$$

$$y^+_{km} \geq 0, \qquad y^-_{km} \geq 0$$

Then the capacity constraint

$$\sum_k |x_{km}| \leq b_m$$

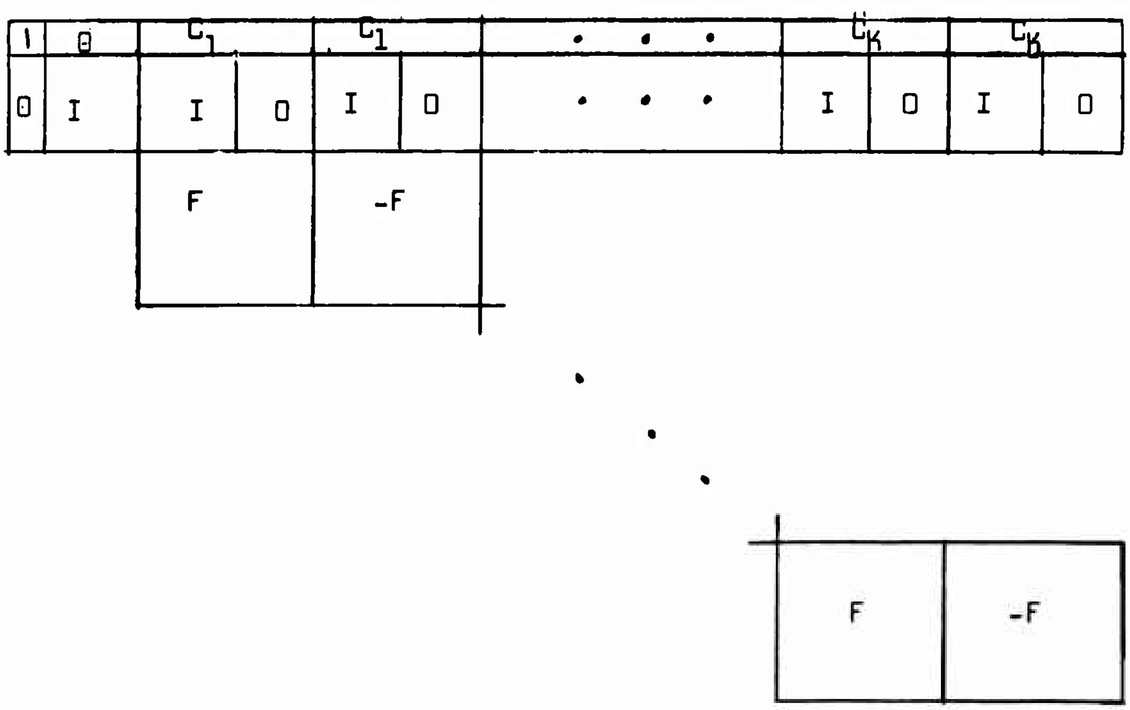
becomes

$$\sum_k (y^+_{km} + y^-_{km}) \leq b_m$$

provided that

$$y_{km}^{+} \; y_{km}^{-} = 0 \qquad (39)$$

If the problem has an optimal solution, then it has a solution in which (39) is satisfied. The constraint matrix then takes the form



The algorithm described above can be applied directly to this case. The structure of the working basis is exactly the same. The only change is that the extra columns must be considered in the pricing operation.

BIBLIOGRAPHY

1. Berge, C., and A. Ghouila-Houri, Programming, Games, and Transportation Networks, John Wiley and Sons, New York, 1965.

2. Dantzig, G. B., Linear Programming and Extensions, Princeton University Press, 1963.

3. Jewell, W.S., "Multi-commodity Network Solutions", ORC 66-23, Operations Research Center, University of California, Berkeley (1966).

4. Kaul, R. N., "An Extension of Generalized Upper Bounded Techniques for Linear Programs", ORC-65-27, Operations Research Center, University of California, Berkeley, (1965).

5. Lasdon, L. S., Optimization Theory for Large Systems, Macmillan Company, New York, 1970.

6. Saigal, R., "Multicommodity Flows in Directed Networks", ORC 67-38, Operations Research Center, University of California, Berkeley, 1967.