GEOMETRIC DIFFERENTIAL EQUATIONS

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My present topic refers to a scarcely recognized part of the vast topic of d.e. (= differential equations). The very title, however, presents an anomaly. While "d.e." is a clearly marked part of analysis, "geometry" offers considerable vagueness. However, while no question pertaining to d.e. can avoid a strong utilization of analysis, the "geometrical bent" will not always be as apparent.

Leaving aside the appropriateness of the title, my purpose is to present, however briefly, the profound contributions of a few outstanding authors, the first being Poincaré, the true founder of my topic.

Henri Poincaré. This work of foundation, dating almost a century, was accomplished by Poincaré in his great classic, Sur les courbes définies par une équation différentielle (Oervres, Vol. 2). For the first time the totality of the system of trajectories of a d.e. was studied. The particular first system selected was planar:

$$\dot{x} = f(x,y), \quad \dot{y} = g(x,y) \tag{1}$$

where f and g are real polynomials. Poincaré pointed out the

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importance of critical points where both f and g vanish and realized, correctly enough, that as a first step he should avoid all highly special cases. Therefore, he limited the study to the elementary type. At such a point A, taken as the origin, (1) assumes the form

$$(2) \qquad \dot{x} = ax + by + \cdots, \quad \dot{y} = cx + dy + \cdots$$

($\ldots$ terms of higher degree). Here a, b, c, d are real constants with $\delta = ad - bc \neq 0$ and characteristic roots defined by

$$\begin{vmatrix} a-r & b \\ c & d-r \end{vmatrix} = r^2 - (a+d)r + \delta = 0.$$

Everything revolves around the values of the roots λ, μ. The characterization is this:

λ, μ real: node; both negative, stable; both positive, unstable.

λ, μ real and of opposite signs: saddle point.

λ, μ complex, $\lambda = \lambda' + i\lambda''$, $\mu = \lambda' - i\lambda''$, $\lambda' \neq 0$:

focus $\lambda' < 0$, stable $\lambda' > 0$, unstable.

$\lambda' = 0$: center.

The center turns out to be the most troublesome. Basically, it begins with a succession of ovals and then turns into the preceding type, but no method has yet been found to say when this occurs.

Index. Let V be a vector system valid in the whole plane, for instance (f,g). Let A be an isolated critical point. In a small neighborhood of A the vector of V is only zero at A. Let γ be a small positive circle centered at A, and such that $V \neq 0$ on γ. Let P be a point of γ, $V(P)$ its vector, θ its positive angle with some fixed direction. As P describes γ once, $\operatorname{Var} \theta = m \cdot 2\pi$ and $m = \operatorname{Ind} A$. Its basic properties are;

(a) $\operatorname{Ind} A$ is independent of γ (if small enough).

(b) $\operatorname{Ind} A$ may also be defined topologically as follows. Take a large $\Gamma \supset A$. Let $V(P)$ cut Γ in P'. Then $P \to P'$ defines a map $\gamma \to \Gamma$ under which Γ is covered algebraically m times and $m = \operatorname{Ind} A$.

(Via a classic theorem of L. E. J. Brouwer, this offers the natural extension of the index to higher dimensions.)

(c) When the Jordan curve J surrounds c.ps. (= critical points) A_h, $1 \leq h \leq n$, then

$$\operatorname{Ind} J = \Sigma \operatorname{Ind} A_h.$$

(Note: $\operatorname{Ind} J$ defined like $\operatorname{Ind} A$.) Hence, if J surrounds no c.p. $\operatorname{Ind} A = 0$.

(d) Indices of elementary c.p. of $V(f,g)$ of (1) is $+1$, except for a saddle point when it is -1.

general conditions).

D. Treatment of solutions of d.e. on a projective plane. Application to the behavior of polynomial systems at ∞.

E. On a smooth orientable surface Φ one may define d.e. and related V with its indices. Then Σ Ind $A_h = \chi(\Phi)$ = Euler-Poincaré characteristic.

F. Extension to 3-dimensions (somewhat incomplete).

G. Method of sections. This method was later extensively exploited by Birkhoff. Let γ be a closed analytic trajectory of a d.e. system in 3-space. At some point P of γ let Φ be a portion of an analytic surface not tangent to γ at P. Then the trajectories δ of the system very near γ intersect Φ in a first point Q very near P. Let Q_1 be the second intersection of δ with Φ very near P. The nature of the transformation $T: Q \to Q_1$ may and often does serve to tell a great deal about the neighborhood of γ. In particular, the fixed points of T, likewise those of the recurrent $T^1, T^2, \ldots$, disclose the presence of new periodic trajectories of the system near γ. This is the famous method of sections.

H. Bifurcation theory for space analytic system which about a point split into several distinct systems (important in applications).

I. Poincaré's books on celestial mechanics (far ahead of their time) are replete with new theories and applications of d.e.

J. The Poincaré-Birkhoff theorem. Cosmology led Poincaré to surmise the following proposition: In the plane R, let Σ be the annular ring bounded by two circumferences C_1 and C_2, C_1 interior to C_2. Suppose that Σ has a topological area preserving transformation T (all important in dynamics) under which C_1 and C_2 rotate in opposite directions. Then T has generally two fixed points in Σ (the two may coincide).

A year or so before his death (aged 57), Poincaré published a long memoire describing his unfruitful endeavors to prove the full theorem, in the hope that a younger man might have more luck. The younger man G. D. Birkhoff published a remarkably short and ingenious proof of the theorem, about a year after Poincaré's death... .

General observation. It was characteristic of Poincaré that he constantly illustrated his results by concrete examples. In particular, he showed how his general 2-dimension theory could serve to provide a complete description of the full system of trajectories for comparatively simple systems, and in applications of various kinds his results turned out to be utterly useful.

G. D. Birkhoff (1884-1944). His work in our subject and related questions may be fairly described as a continuation of Poincaré's. However, in it the different directions pursued, analytical and geometric, are so mixed that the work of extricating the part belonging properly to my present topic is too difficult for me to separate. I shall, therefore, refer to the very competent article written by Marston Morse on Birkhoff's complete research (see beginning of Vol. I. of Birkhoff's complete works). I shall limit my discourse to the proof of the Poincaré-Birkhoff theorem, its extension, an application and a few striking contributions on details made by Birkhoff.

A. Proof of the Poincaré-Birkhoff theorem. Birkhoff utilized in a fundamental way the property of invariance of area. If T is the initial transformation he introduced a new transformation T_ϵ: a radial ϵ shrinking of the ring (ϵ small) towards the center. This enabled him to define an invariant arc λ of $T_\epsilon T$. Then considering the powers of $T_\epsilon T$ and variation along λ of a vector joining a point to its predecessor, followed by a return along $-\lambda$, he showed that no fixed point implied a contradiction, from which the theorem follows.

B. Application to the billiard ball problem. The ball is assumed to roll on a plane table bounded by a convex curve. As it hits the boundary, the ball is reflected through an angle of "reflection" equal to the angle of incidence. Birkhoff shows (complete works, II, p. 333) that upon looking to the closed polygon paths one succeeds in proving that the associated dynamical systems has an infinity of periodic motions, some stable, others unstable.

In the course of the discussion Birkhoff introduced the highly interesting notion of minimax (intermediary between a maximum and a minimum).

Regarding the collection of periodic motions (discovered through the Poincaré surface of sections scheme) see also the very interesting and extensive paper of Birkhoff: Complete works, II, p. 111.

C. Extension of the Poincaré-Birkhoff theorem. Let r, θ be plane polar coordinates and let C be the circle $r = a > 0$. Let R be a ring bounded by C and a curve $\Gamma \supset C$. Let R_1, Γ_1 be a second similar system and let T be a topological map $R \to R_1$.

Theorem. If Γ and Γ_1 are met only once by any line θ = const., and T carries points of Γ and Γ_1 in opposite directions then either (a) there are two distinct invariant points of R and R_1 under T or else (b) there is an annulus R_2 of R or R_1 (abutting on C) carried into part of itself by T (or T^{-1}).

D. Extension of the method of sections for dimension two. While Poincaré limited the method to a small open neighborhood Ω of a periodic solution Birkhoff extended the method to Ω's with boundary. This consisted of several disjoint periodic solutions of the basic system, or for that matter of the extension of the method to a surface of any genus.

E. This refers to a very interesting extension of the rotation character of a curve Γ in this direction. Let Γ be on an orientable surface Φ and thus let it have two distinct sides Γ_1 and Γ_2. Then Birkhoff showed that each of these two sides could have distinct rotation constants.

A. M. Liapunov. His name is indelibly attached to the concept of stability. His classical memoire, mainly known outside Russia by later French translation; Problème général de la stabilité du movement (Russian edition 1892, French translation, Annales de Toulouse, reproduced as Annals of Mathematics Studies) treats stability in a completely fundamental way. Much on the concept was known - and taken for granted before Liapunov but he not only organized the subject but also made many profound and new contributions to it. Evidently Lagrange, Poincaré and a number of others could not and did not dispense with a knowledge of stability. For instance while the motion of the planets is "apparently" stable - in some practical sense say 10^4 years - how long did it last? (Who knows?). Besides, for example, the 3-bodies problem evoked seemingly different stability... .

Well at any rate since Liapunov the answer is unique. We recall a couple of the basic statements.

Let Σ be a dynamical system in n-space $x = (x_1, \ldots x_n)$ and suppose that $x = 0$ is a solution. Then

0 is stable whenever given any open set $U \supset 0$ and time t_0, there exists another $V(U,t_0) \subset U$ such that if a motion starts in V at time t_0 it will remain in U for all $t > t_0$ (uniformly stable when V depends only on U) unstable whenever given U, t_0 as before, and whatever $V \subset U$, the motion at some $T > t_0$ reaches the boundary of U.

Asymptotically stable wherever with the same data the motion $\rightarrow 0$.

Conditional stability wherever stability holds only for some subset $W \subset U$.

There are obvious applications to elementary critical points. The center is the only stable one not a · s.

Note that Liapunov (a la 1892) limited his study to analytical systems, holomorphic at the origin - an unimportant restriction.

Hope: Would someone take his memoire and shorten it appreciably, via suitable vector notations!

Major result. It is in a sense a generalization of the Lagrange stability theorem by extremal values of the potential. However, Liapunov's result does not require knowledge of the solutions. I will not state the actual theorem but rather a geometric interpretation for dimension two. Let $W(x_1,x_2)$ be positive definite in a region $\Omega(0)$, that is within a cylinder $x_1^2 + x_2^2 = r^2$. Let y be a third coordinate. The surface F: $y = W$ is a cup over Ω. The curves $W = \text{const.}$ are the projections of the horizontal sections of F. The paths Γ in $\Omega(0)$ are imaged into paths Δ on F. The projection $F \to \Omega(0)$ is topological.

Stability: $W \leq 0$ implies that on F the path stays on the lower part of the cup, goes down sluggishly and need not reach 0.

A · st. $W \leq -a < 0$. The path tends rapidly to the origin.

Instability. Along some Δ: $W > \beta > 0$ - the path goes away from 0 and reaches the boundary of $\Omega(0)$.

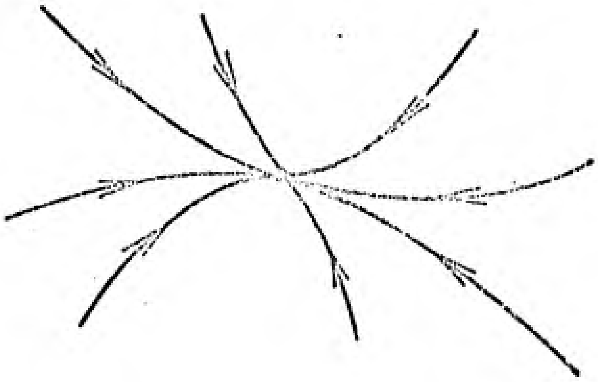
The extension is easy.

For rapid applications see my book: D.E. Geometric Theory, 2nd edition, p. 117.

Andronov and Pontryagin. These two authors have introduced in the late thirties a very novel and interesting concept: Structural Stability. Their announcing note in the Doklady gave no proofs. These were first provided by De Baggis and later considerably improved and expanded by Peixoto.

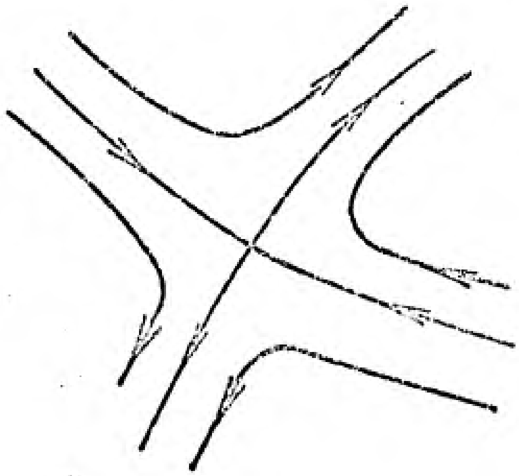
Briefly the idea is this. Given say an open bounded plane 2-cell region Ω with boundary B, which for a given d.e. has a finite set of elementary critical points and separatrices (lines emanating from saddle points). Suppose that (in some sense) the system undergoes an ϵ-deformation. Under what conditions does the phase - portrait of the system remain unchanged. This is known as structural stability. General conditions are: the system has only a finite number of elementary critical points, none a center; it has at most a finite number of closed paths in Ω, all limit - cycles, always stable or unstable on both sides; no separatrix joins two distinct saddle points. These are also sufficient conditions and (proved by Peixoto) they are also independent of ϵ.

It is understood throughout that B is analytical and crossed, not tangentially in the same direction by the solutions of the d.e.



stable node

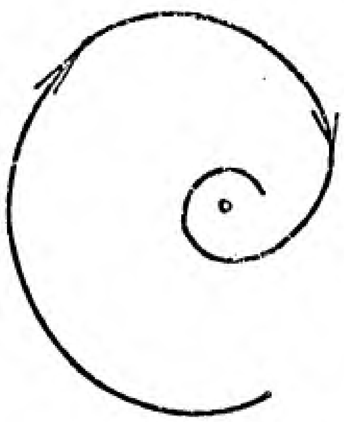
unstable; arrows reversed



saddle point

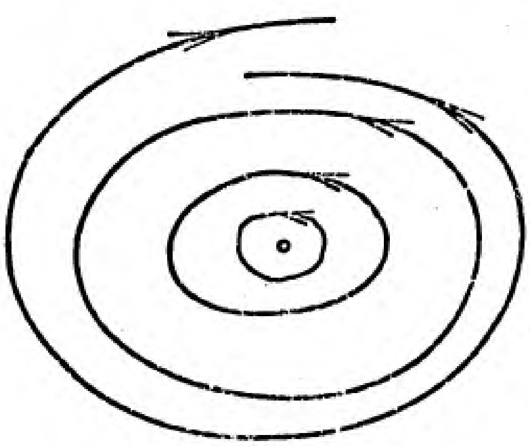
1 stable trajectory
1 unstable trajectory

all the rest unstable



stable focus

unstable; arrows reversed



center stable

unstable; arrows reversed

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