U. S. ARMY TEST AND EVALUATION COMMAND
COMMON ENGINEERING TEST PROCEDURE

SERVOMECHANISMS

1. OBJECTIVE

The object of this procedure is to measure the ability of a servomechanism to perform within the criteria set forth in applicable MDP, Mil Specs, and TC's.

2. BACKGROUND

In recent years, automatic control systems (servomechanisms) have assumed an important role in the development of modern technology. In modern weapons systems, the applications of control systems have become overwhelmingly important. The basic control system may be described by the simple block diagram shown in Figure 1. The output variable C's controlled by the input variable r through the elements of the control system.

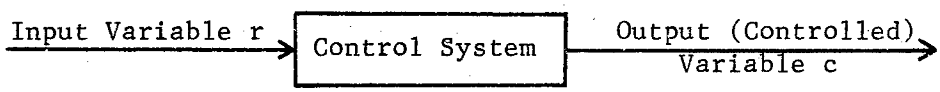


Figure 1. Basic Control System Block Diagram

The name "servomechanism", or "servo", for short, has been used quite freely in the literature of feedback control systems. Strictly speaking, servomechanisms merely represent a particular group of feedback control systems whose controlled outputs are mechanical positions. From the basic concept of closed-loop control systems, feedback control systems may be defined.

Systems comprising one or more feedback loops which compare the controlled signal c, with the command signal r; the difference ($e = r - c$) is used to drive c into correspondence with r. A servomechanism, on the other hand, is defined as a feedback control system in which one or more system signals represent mechanical motion. However, since "servomechanism" and "servo", have been frequently used in a very broad sense, in this test "servo system" and "feedback control system", are used interchangeably. The elements of a basic servo system can be schematically represented by the closed-loop block diagram shown in Figure 2. The block diagram consists of a forward path, a feedback path, and an error sensing device.

In general, the forward path of a servo system may consist of the following elements: (1) error-sensing device (error detector), (2) amplifier, (3) servomotor, (4) compensating networks, etc. The feedback path usually consists of transducers and compensating networks. The error detector compares the reference input with the actual output or some function of the output signal, and sends out a signal proportional to the difference. Compensating networks are often needed in the forward path, the feedback path, or both, to improve the performance of the system. A servo system with only its

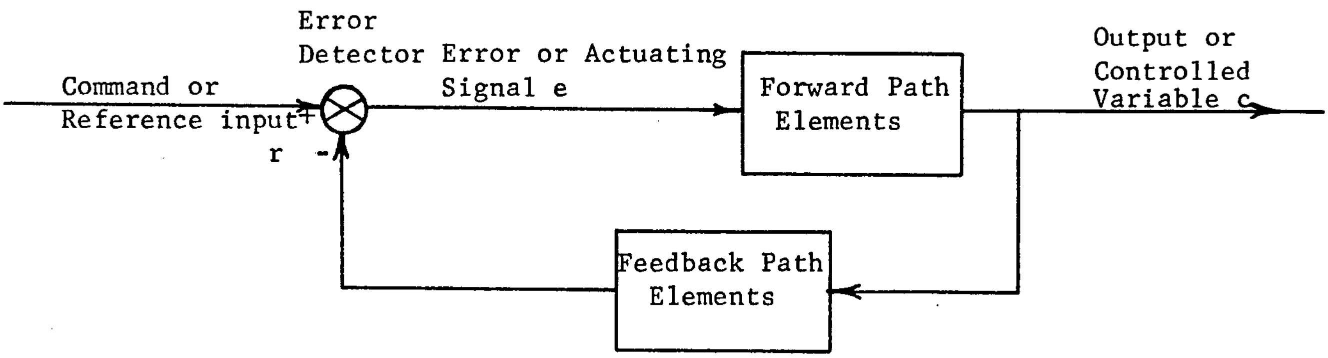


Figure 2. Basic block diagram of a servo system

A more detailed diagram of a servo system is illustrated in Figure 3.

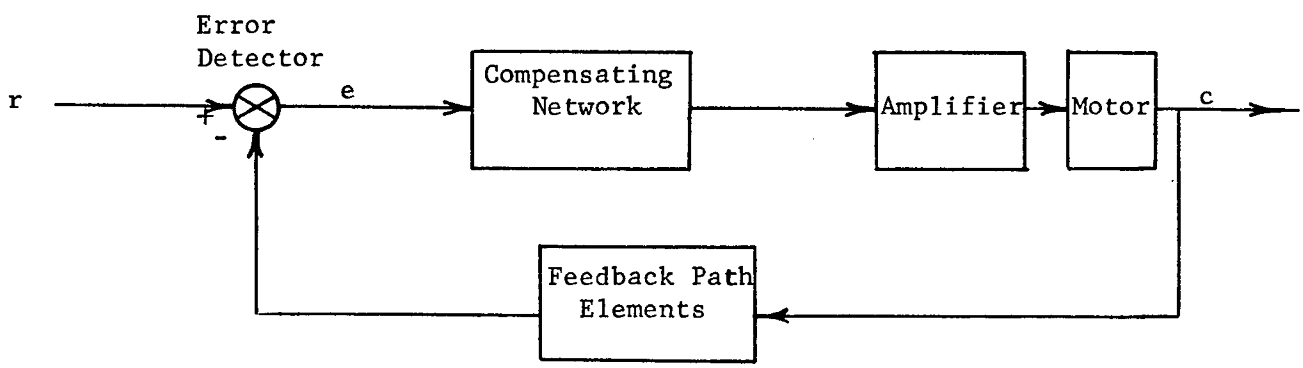


Figure 3. Block diagram illustrating the basic components of a Servo System.

minimum required components seldom give satisfactory performance.

In general, the testing of servo mechanisms involves taking measurements that will determine whether or not the control system's output will follow its input within the required limits.

3. REQUIRED EQUIPMENT

a. 2 channel X-Y plotter
b. Dual Base - Dual Trace oscilloscope
c. Function generators of that are capable of generating the functions described in test conduct in a form compatable to the servo mechanism under test.
d. Transducers that are capable of converting the outputs of the signal generator and the servo mechanism into a form that can be recorded on the plotter or oscilloscope
e. Oscilloscope recording camera and mount
f. Gerber Scale
g. Desk calculator
h. Drawing instruments
i. Assorted graph paper

4. REFERENCES

A. Chestnut, Harold and Mayer, Robert W., Servomechanisms and Regulating System Design, John Wiley and Sons, Inc., New York, 1951
B. Benjamin C. Kuo, Automatic Control Systems; Prentice-Hall, Inc., Englewood Cliffs, N.J., 1962
C. Thaler, George J. and Brown, Robert G., Servomechanism Analysis, McGraw-Hill Book Company, Inc., New York, 1953

5. SCOPE

5.1 SUMMARY

This Materiel Test Procedure describes the following response tests conducted on servomechanism.

a. Time-domain tests: The evaluation of the transient and steady state responses when typical test signals, such as step, ramp, and parabolic functions, are applied to the system. The transient response refers to the manner in which the system arrives at the steady-state conditions. In particular, for a step function input, the servo system should approach the steady-state in the most rapid but least oscillatory way. A response with a slow time or excess over shoot and oscillation about the steady-state value is undesirable. The steady-state response indicates the accuracy of the servo system. An ideal steady-state response of a servo system should correspond with the reference input.

b. Frequency-Domain tests: The steady-state response of the system is a sinusoidal signal. Because any periodic input function can be represented by a Fourier series and an aperiodic input function can be expressed by a Fourier integral, the steady-state response of a given linear control system to any arbitrary function of time can be determined by the principle of superposition once the frequency response of the system is known. The essential characteristic of a linear system is that when the input is a sinusoid, the output response also varies sinusoidally, and at the same frequency. For a non-linear system, the output will, in addition, contain higher harmonics or sub-harmonics.

5.2 LIMITATIONS

The tests outlined in this Materiel Test Procedure are limited in scope to those necessary to determine the stability and accuracy of a servomechanism. They do not include those necessary to evaluate the performance of the system under conditions of environment extremes. These may be evaluated by referring to the appropiate MTP's.

6. <u>PROCEDURES</u>

6.1 PREPARATION FOR TEST

a. The response of the mathmetical model of the servo to the inputs to be used must be known or specified in the MPD's, TC's, and Military Specifications of the servomechanism.

b. Personnel conducting this test must be qualified technical or engineering personnel familiar with the use and operation of control systems.

c. Safety precautions as applicable to the equipment will be observed during the conduct of the test.

d. Care should be taken to insure that the control system is loaded as it would be during actual operation.

6.2 TEST CONDUCT

6.2.1 <u>Time Domain Tests</u>

In practice, the input excitation to a feedback control systemis not known ahead of time. In most cases, the actual inputs vary in random fashions with respect to time. For instance, in a radar tracking system, the position and speed of the target to be tracked may vary in any unpredictable manner, so they cannot be expressed mathematically by any simple equation. However, for the purpose of test and evaluation, it is necessary to assume some basic types of input functions so that the performance of a system can be analyzed with at least these test signals. In a design problem, performance criteria are derived with respect to these test signals, and linear systems are designed to meet the criteria. In general, the following three types of input are used:

a. Step displacement input

This is the instantaneous change in the reference input variable; e.g., a sudden rotation of an input shaft. The mathematical representation of a step function is

$$r(t) = R \qquad t > 0$$
$$r(t) = 0 \qquad t < 0$$

and at t = 0, the function is not defined. The step function is shown in Figure 4.

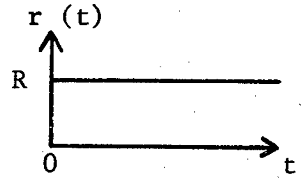


Figure 4. Step Displacement Input

b. Step Velocity Input (Ramp Function)

In this case, the reference input variable is considered to have a constant change in position with respect to time. Mathematically a ramp function is represented by

$$r(t) = Rt \qquad t > 0$$
$$r(t) = 0 \qquad t < 0$$

The ramp function is shown in Figure 5.

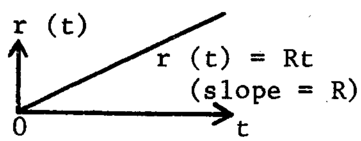


Figure 5. Step Velocity Input

c. Acceleration input (parabolic function): The mathematical representation of an acceleration input is

$$r(t) = Rt^2 \qquad t > 0$$
$$r(t) = 0 \qquad t < 0$$

The graphical representation of an acceleration function is shown in Figure 6.

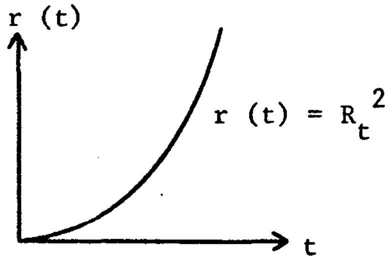


Figure 6. Acceleration Input (Parabolic Function)

1) The servomechanism is connected so that its input is excited by the appropriate input function generator. Its output is connected to a recording device that is compatible with the expected response. A 2 channel X-Y plotter is used for systems with expected response of greater than .5 seconds and in dual base, dual trace oscilloscope is used for systems with expected response less than .5 sec. The test setup is shown in Figure 7.

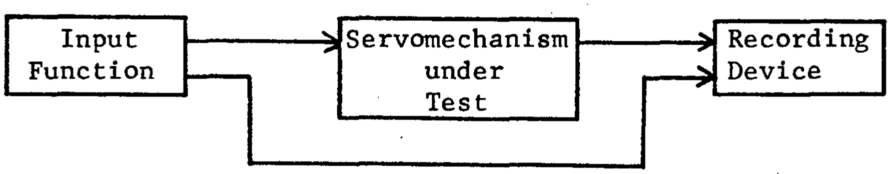


Figure 7. Test Setup for Time Response Measurements

2) Adjust the X-Y plotter to the scale setting, so that the response of the servo may be recorded until it reaches a condition of final value. When using an oscilloscope, adjust the time base to a similar condition.
3) Adjust the amplitude of the X-Y plotter and the vertical input oscilloscope to a setting 10% greater than the maximum expected overshoot.
4) Set the servo input to zero initial conditions, and ensure that the output is correct for zero input conditions.
5) Ensure that the output of the servo is connected tighter to the load under which it will operate or to a corresponding simulated load.
6) Apply the test inputs to the servomechanism in turn, by means of appropriate function generators.
7) Record the waveform of the input and the waveform of the output, on the X-Y plotter or by means of an oscilloscope camera. The recording equipment must be set to zero conditions prior to the start of the test.
8) Repeat the test at least ten times to ensure consistency and to account for random responses.

6.2.2 Frequency Domain Tests

The essential feature of the frequency response method is that the description of the system is given in terms of its response to a sinusoidally varying input signal. If the system is linear, the output will be a sine wave of the same frequency as the input, if the system is nonlinear, the output will, in addition, contain higher harmonics, and some times sub-harmonics. The frequency response method is basically a graphical method. The graphs which are obtained experimentally are output level vs applied frequency and phase output signals vs applied frequency. The plots required in the evaluation of the servo are obtained from these two.

a. The input of the servo to be tested is connected to a sinusoidal input generator whose frequency can be swept automatically across the range of frequencies required to evaluate the system.

b. Connect the output of the servo to a device such as a rectifier that will allow the amplitude of the output signal to be applied to one "Y" channel of a 2 channel X-Y plotter.

c. Connect the output of the signal generator to one input of a phase measuring device.

d. Connect the output of the servomechanism to the other phase measuring device input.

e. Connect the output of the phase comparator to the other "Y" channel of the X-Y plotter.

f. Connect the sweep of the signal generator to the "X" channel of the plotter.

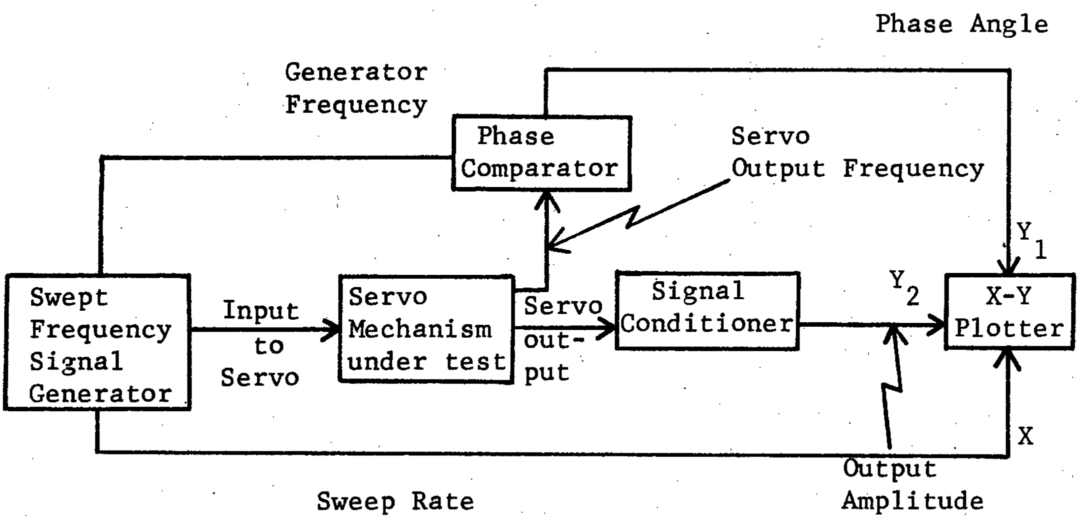


Figure 8. Connection of Equipment for Frequency Domain Tests

g. Set the input to zero initial conditions for the system.

h. Ensure that the output of the servo is that required for zero input.

i. Set the X-Y plotter to zero conditions.

j. Adjust the signal generator amplitude to a level that will produce a discernable output from the servo.

k. Determine the scaling factors for each direction of sweep such that the output of the servo is normalized to the input from the signal generator, i.e., for an input of ten from the signal generator, the output of the servo is also ten at the starting frequency and that this level is zero. The X-plotter for amplitude is one and zero phase shift the range of frequency to be swept should be normalized to the starting frequency.

NOTE: Some plotters can also be scaled to read out logarithmically if desired.

l. Adjust the generator starting output frequency to that as specified for the servo, and allow the generator to sweep through the range of specified frequency response.

m. Remove the trace from the X-Y plotter and repeat the test ten times.

n. Repeat the test for values of input amplitude varying in steps of one up to a value ten times greater than the initial value or until the servo exhibits and obvious instability, whichever occurs first.

6.2.2.1 Determination of Non-linearities

Although nonlinearities in feedback control systems may generally be due to shortcomings of the physical components, such as saturation, hysteresis, dead-zone, backlash, or variation of parameters as a function of operating conditions. There is a class of systems in which nonlinear elements are intentionally introduced in order to improve system performance. The on-off or relay-type servo systems, such as automobile voltage regulators and on-off furnace controls, are common examples of this class of nonlinear systems. The methods of evaluation for nonlinear systems are not as well defined as for linear systems, by the methods in this procedure will provide some measure of the nonlinearities which may be present in a given servomechanism.

a. Connect the equipment as shown in Figure 9., taking care to observe the precautions for control system measurements as described in the preceding test.

b. The signal generator is such that it will produce signals of sufficient amplitude and frequency to exceed the designated limits of the servo. It should also be capable of providing triangularly amplitude and frequency modulated signals as shown in Figure 10.

c. Start the equipment, set the carrier level. Set the modulating frequency to a rate that it will not exceed the response limitations of the X-Y plotter, apply the signal to the servo. Repeat the testing using AM then FM.

d. Record the trace from the X-Y plotter, the curve obtained will be that of input versus output.

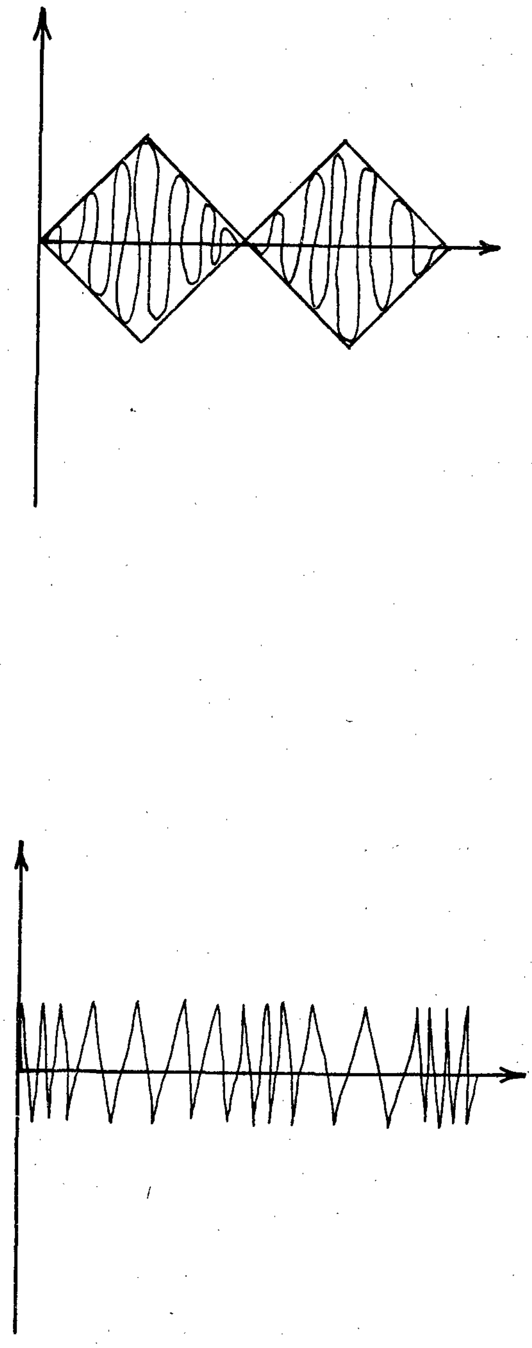


Figure 10. Input Signals for Test

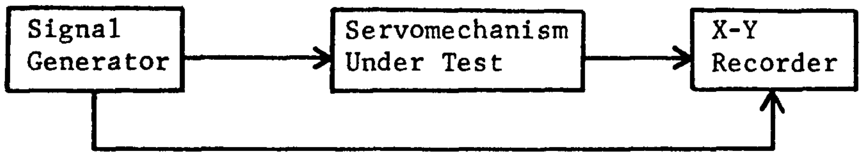


Figure 9. Configuration for measurement of Non-linearities

e. Vary the carrier frequency over the range of operation as specific for the servomechanism dividing the range into ten parts and measuring at each point.

f. Vary the modulating amplitude over the specific range of operation as frequency varied in e.

g. Repeat each test ten times to determine the random deviations.

6.3 TEST DATA

6.3.1 Time Domain Tests

a. Record the type and function of the servomechanism under test.

b. Record the type of loading used actual and simulated.

c. Record the type of function generator or method of function generation used.

d. Record the make, model, type and serial number of recording device used.

e. Record the scaling factors used in obtaining the plots in amplitude/cm on the vertical or Y channel and in seconds/cm, on the horizontial channel or X channel.

f. Record the number of times the test is run, the date, and any external conditions that prevailed during the test that may have affected the performance of the system.

6.3.2 Frequency Domain Tests

a. Record data as in time domain test except X channel scaling factor will now be in Hz/cm and Y channel scaling factors will be in amplitude/cm and degrees/cm respectively.

b. Record the range of amplitudes over which the test is run and note the point if any, where the system exhibited instability.

6.3.2.1 Determination of Non-linearities

a. Record the scaling factors of the X-Y plotter as in preceding tests in amplitude/cm on both axis.

b. Record modulating frequency in Hz.

c. Record the amplitude of the modulation.

d. Record the amplitude of the carrier.

e. Record the frequency of the carrier in Hz.

f. Record the test run number and type.

g. Record the type of equipment used to conduct the test, nomenclature, model number, and serial number.

6.4 DATA REDUCTION AND PRESENTATION

6.4.1 Time Domain Tests

a. Transient performance takes the recorded response of the servo system to the step displacement or unit step input, and utilizing the appropriate scaling factors. Label the time and response axis as shown in Figure 11. The response axis step input showing time-domain specifications of feedback control systems.

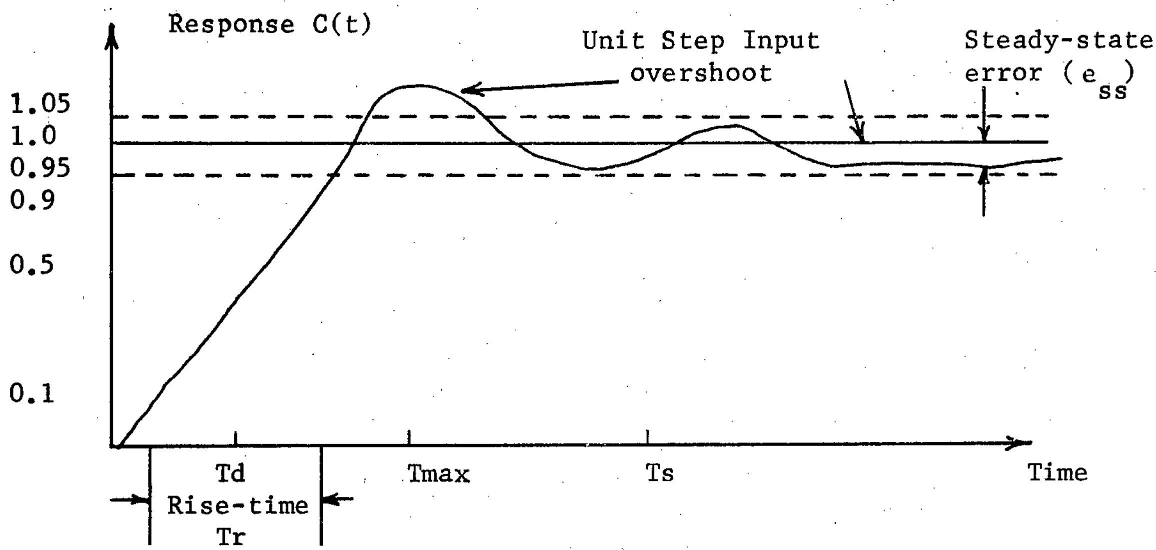


Figure 11. Response to Unit

These should be normalized to the input level by considering the input as having a relative value of one for all cases. The results of the calculation should be compared to the values specified for the system under test.

Using the appropriately scaled graph as shown in Figure 11, determine the following:

1) Overshoot - the overshoot is an indication of the largest error between input and output during the transient state. It is also recognized as a measure of the relative stability of a system. The amplitude is measured at the point indicated at Tmax in Figure 11. The overshoot is calculated as a % of the desired final value, i.e., magnitude of the input level. The calculation is as shown below:

$$\% \text{ Overshoot} = \frac{\text{Maximum Overshoot}}{\text{Final desired value}} \times 100$$

2) Time Delay - The time delay Td is normally defined as the time required for the response to reach 50% of the final value.

as shown in Figure 11.

3) Rise Time - The rise time Tr is defined as the time required for the response to rise from 10% to 90% of its final value as shown in Figure 11. This may also be represented as the reciprocal of the slope of the response at the instant the response is equal to 50 % of its final value.

4) Settling Time - The settling time Ts is defined as the time required for the response to decrease to and stay within specified percentage of its final value. A frequently used figure is 5 per cent.

5) Steady-State Performance - The steady-state error is a measure of the system accuracy when a specific type of input is applied to a feedback control system. In a physical system, because of friction and other related factors, the steady-state output response of a system seldom agrees exactly with the reference input. The steady-state performance of a feedback control system is generally judged by evaluating the steady-state error due to the three test signals used. A detailed description of the three types of error and the calculations associated with them is given below.

a) Steady-state error of systems with step displacement input. The steady-state error (e_{ss}) is measured from the response as shown in Figure 11. The positional error constant is then calculated as shown.

$$Kp = \frac{R - e_{ss}}{e_{ss}}$$

Where:

Kp = Positional error constant
R = Desired output
e_{ss} = Measured error from the graph

The value arrived at should be compared with the specified Kp.

b) Steady-State error of systems with step velocity Input. The error is measured from the graph as shown in Figure 12.

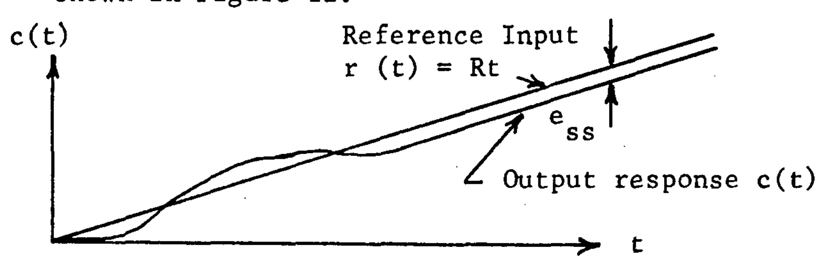


Figure 12. Output Response of Feedback Control System With Ramp Input

The velocity error constant Kv is calculated as follows, and is compared to the value specified for the system.

Where: $Kv = \frac{R}{e_{ss}}$

Kv = Velocity error constant
e_{ss} = Velocity error as measured from the graph
R = Desired output velocity

c) Steady-state error of system with acceleration input. The acceleration steady-state error is measured as shown as in figure 13.

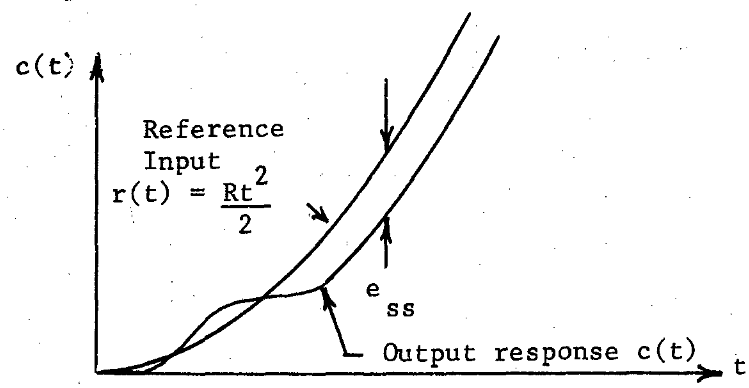


Figure 13. Output with Acceleration Input

The acceleration error constant Ka is computed as shown and compared to the value specified for the system.

$$Ka = \frac{R}{e_{ss}} = \frac{\text{Desired output acceleration}}{\text{Steady-state error}}$$

The errors described in b and c, are to be expected and are due to errors in displacement due to the type of input and are not to be considered errors in velocity or accelerations. It is only when the output does not differ from the input, by a constant value or when this value exceeds the specific criteria, that the system is not meeting its designed specifications.

6.4.2 Frequency Domain Tests

a. Determination of specification directly from the frequency response plots. There are several specifications that are usually given for a servomechanism that may be compared directly with the measurements taken directly from the frequency vs amplitude plots. These criteria may also be determine if the amplitude is expressed logarthimically. The specifications are as described and as shown in Figure 14.

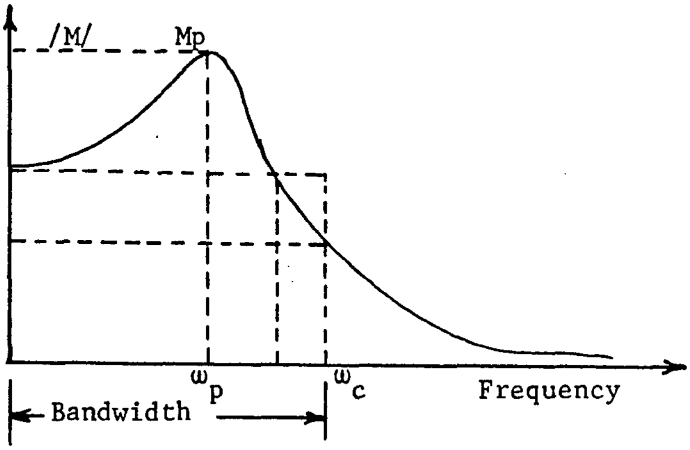


Figure 14. The Frequency Response Curve of a Servo System

1) Bandwidth: The bandwidth is defined with the conventional definitions, it is shown in Figure 14. The bandwidth roughly indicates the filtering characteristics of the system.
2) Resonance Peak: This is the maximum value of M, which indicates the relative stability of the system. A large resonance peak Mp corresponds to a large overshoot in the transient response. An optimum value of Mp is usually some value between 1.1 and 1.5.
3) Resonant frequency : Wp: This is the frequency at which the resonance peak Mp occurs
4) Cut-off rate: In some cases, the rate of cut-off of the frequency response at high frequencies is important, since it indicates the characteristics of the system in distinguishing the signal from noise. However, in general, sharp. cut-off characteristics are accompanied by large Mp, which means a less stable system.

b. Conversion of plotted data to determine stability of the servo system.

The following plots are reduced from the raw data off of the X-Y plotter:

1) Polar Plot. A plot of magnitude versus phase shift on polar coordinates. It is used to evaluate the system by means of Nyquist criterion.
2) Bode Plot (Corner plot). A plot of magnitude in decibel versus log w (or W)and phase angle versus log W (or W) in rectangular coordinates.
3) Magnitude versus phase shift plot. A plot of magnitude in decibels versus phase shift on rectangular coordinates with frequency as a varying parameter on the curve. Since the polar plots are determined from the Bode plots and since the analysis is usually

easiest from the standpoint of the Bode plot. The quantities to be determined from the Bode plot are the gain margin, which is the amount of the gain in decibels of the system can be allowed to increase before the system reaches instability, and the phase margin, which is defined as the angle the 180° point and the gain cross-over point as determined from the magnitude versus frequency Bode plot. To construct the Bode plots take the response plots from the X-Y plotter and scale the magnitude vs frequency plot logarithmically. If the plot was not taken on a logarithmically recording plotter then the bode plot may be constructed by substituting the value of the curve at a sufficient number of points into the formula:

$$db = 20 \log_{10} /M/$$

Where:

M = The magnitude of the curve at a given point.

NOTE: This procedure assumes that the plots were taken with the output normalized and reconstructing the curve at the points taken. The phase angle versus frequency plot is then constructed at the same points. The plots are as shown in Figure 15.

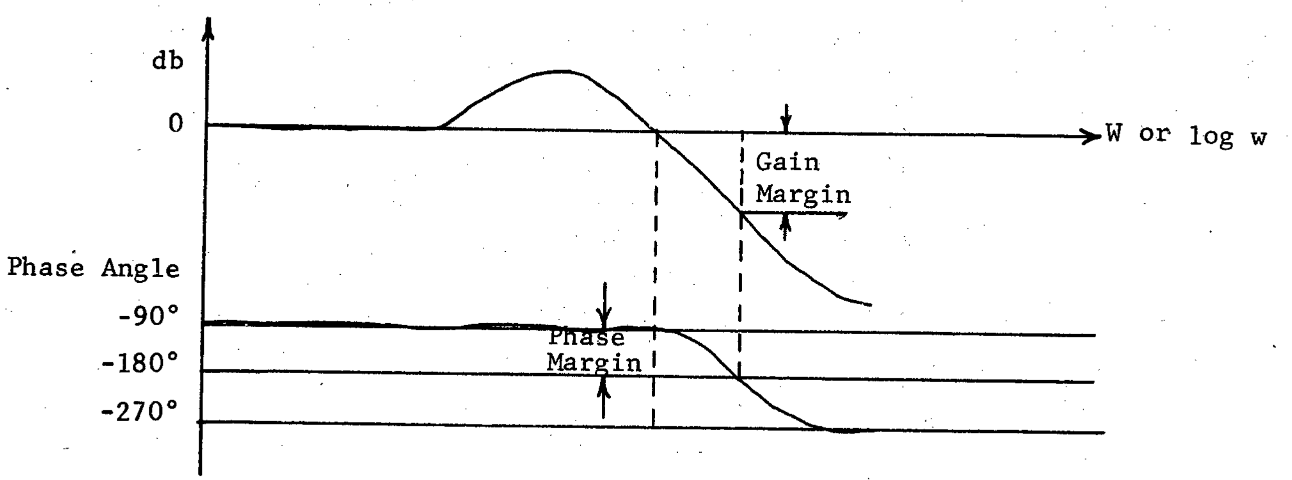


Figure 15. Bode Plots of Gain in db vs frequency and phase angle in degrees vs frequency.

To obtain the gain margin, first locate the point at which the phase shift curve crosses the 180 degree axis. This point is usually called the "phase cross-over", and the corresponding frequency is called the "phase cross-over frequency". The magnitude of the gain curve in decibels at the phase cross-over frequency is the gain margin for, or system gain equal to one. For any other values of K, the gain margin is simply equal to the difference between the gain at the phase cross-over and the value of K in decibels.

For example, if K is set at 10 db, and the gain at phase cross-over is 30 db, then the gain margin is 30 db- 10 db = 20 db. Sometimes, the gain margin corresponding to K = 1 is called the "gain limit". Then the gain margin in decibels for any gain K is simply:

Gain Margin (G.M.) in dg = Gain Limit G_1 db - Gain K db

In terms of absolute values, the last equation is written as
$G.M. = G_1/K$

Furthermore, if the phase shift curve never crosses the -180 degree axis from above, the system is always stable. To obtain the phase margin first locate the point where the magnitude curve crosses the zero decibels axis. This point is usually called the "gain cross-over", and the corresponding frequency is called the "gain cross-over frequency". The phase curve and the -180 degree axis, is the phase margin for K = 1. If the phase shift curve is above the -180 degree axis, the phase margin is positive; otherwise, it is negative. For any other value K, the phase margin is obtained by shifting the zero decibels axis to -K, in decibels and following the same procedure. For instance, in the last example, K = 10 db, the magnitude curve is shifted up by 10 db. This is as if the zero decibels axis were shifted down by 10 db.
As previously stated, the polar plot is used chiefly in conjunction with the Nyquist Criteria. The plot is obtained from points taken from both gain and phase plots at corresponding frequencies, and plotted as gain vs shift.
The gain for this plot is non-logarthmic normalized. The plot is constructed as shown in Figure 16.

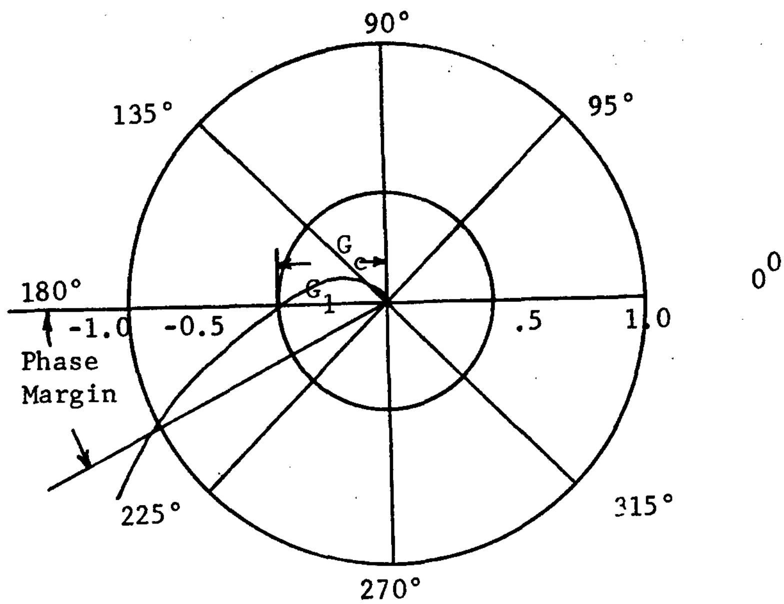


Figure 16. Polar Plot Showing Utilization of Nyquist Criteria

To evaluate the stability of the system in accordance with the Nyquist Criteria, the point (-1.0, (180°) first defined as the critical point for stability. i.e., If the plot crosses the 180° line beyond the -1.0 point, the system is unstable; if the plot crosses at the -1.0 point, the system is said to be stable oscillatory, and if it is less than -1.0, then it is said to be "relative stable", for this condition. The term "relative stability" is used to indicate the degree of stability of a system, or, in simple words, how close the Nyquist plot is to the critical point. To be more specific, the gain margin and the phase margin have been generally used to define the amount of relative stability of a closed-loop system. The gain margin of the system is derined as the magnitude of the gain at the point that the plot crosses the 180° line. It is computed as below:

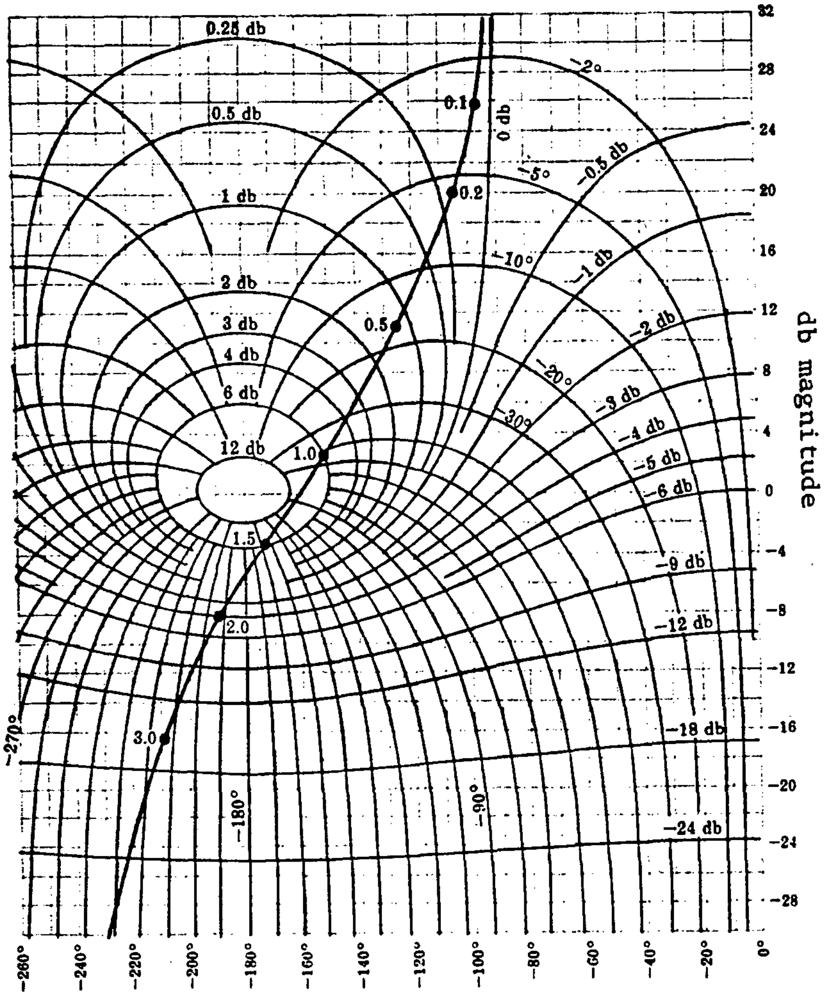
$$\text{Gain Margin (G.M.)} = 20 \log_{10} \frac{1}{G_c} \text{ db}$$

The phase margin is defined as the angle through which the Nyquist locus must be rotated in order that the unity magnitude point on the locus passes through the critical point. In other words, the phase margin is the angle between the 180° line and the point of intersection with the plot and the unity gain circle.

The magnitude versus phase shift plot or gain-phase plot is constructed by plotting gain db versus phase shift in degrees, in rectangular coordinates. This plot is usually constructed on a chart called a Nichols chart and the gain and phase margins read directly from it. The intersection of the G-plot and the zero decibels axis in the gain-phase plane represents the gain crossover, the phase margin is read directly as the phase angle between this intersection and the -180° axis. Similarly, the intersection of the plot and the -180° axis in the gain-phase diagram is the phase crossover point, and gives margin in decibels.

The constant M loci in the Nichols Chart, as shown in Figure 17, superposed on the gain-phase diagram, enables the determination of the closed-loop frequency response of the system. The intersections of the constant M loci give the values of M at the corresponding frequencies W, read on the G(s) curve. The constant M locus that is tangent only to the G-locus gives the response peak Mp; the corresponding frequency is Wp.

The bandwidth of the closed-loop system can also be determined from the gain-phase diagram and the Nichols chart. It is easy to see that the frequency at the M = 0.707 (-3 db) locus and the G(s) plot is the bandwidth in rad/sec. The values determined should be compared with those specified for the system.



Phase Angle
Figure 17. Nichols Chart Analysis

Table I

ω	20 $\log_{10}\lvert M \rvert$	∅
0	0db	0°
0.2	0.2	-6°
0.5	1.2	-15°
1.0	6.0	-42°
1.25	10.0	-90°
1.5	6.0	-155°
2.0	-4.0	-194°
3.0	-15.0	-212°

Table II

Parameter	Value
Gain Margin (G.M.)	-5db
Phase Margin (P.M.)	-20°
Bandwidth	1.20 rad/sec
Mp at (W = 1.25 Rad/Sec)	db
Gain Crossover Frequency	1.20 rad/sec
Phase Crossover Frequency	1.70 rad/sec

The G-plot shown in Figure 17 gives the data shown in Table I. From this data or by reading directly off the graph the values in Table II were obtained.

6.4.2.1 Determination of Non-linearities

The data reduction consists largely of measuring the variations of the plots obtained from the plot of the ideal linear servo as shown in Figure 18. The form that these variations take can be used to some extent to identify the type of non-linearity encountered. The amount of the deviation will be used to give a measure of the magnitude of the non-linearity encountered.

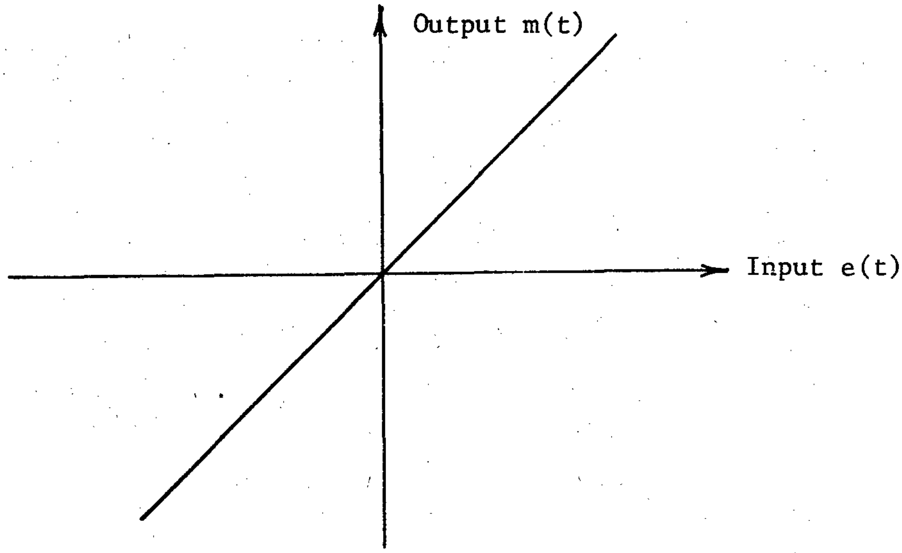


Figure 18. Ideal Plot of a Linear Servomechanism.

A brief table of the types of non-linearities that may be encountered and their characteristics and describing functions is given in Table I. This will serve to illustrate the measurement that may be taken from the plots obtained.

6.4.3 Summary of Reduction

In general, as indicated throughout the test, the data taken are to be compared with the servomechanism parameters as stated in the applicable MPD's, TC's, Mil Specs, SDR's, and QMR's. If these are not given or available the parameters must be developed from the techniques as given in the references. The response functions of a servomechanism may be simulated by use of digital or analog computers, if the transfer and describing functions can be found. The transfer function of a servo can usually be found from the response plots. These methods are however, not within the scope of this MTP, and may be found in MTP 5-2-596.

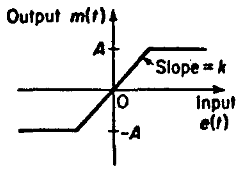
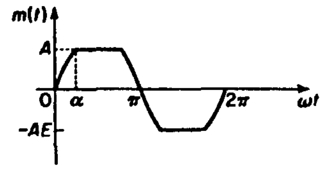
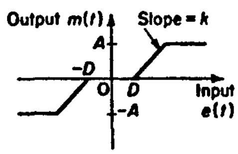
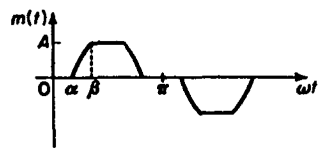
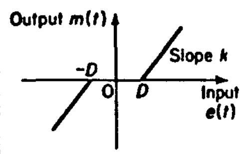
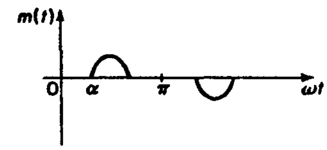
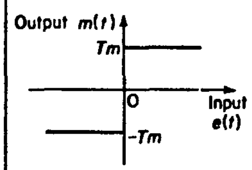
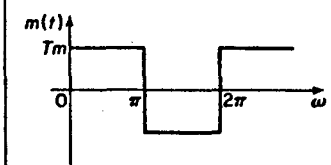
Type of nonlinearity	*Input-output transfer characteristic*	*Output waveform corresponding to sinusoidal input*	*Describing function N*
(a) *Saturation*			$N = k \quad E < A$ $N = \frac{M_1}{E} = \frac{2k}{\pi}\left(\alpha + \frac{\sin 2\alpha}{2}\right) \quad E > A$ where $\alpha = \sin^{-1}\frac{A}{E}$
(b) *Saturation with dead zone*			$N = 0 \quad E < D$ $N = \frac{M_1}{E} = \frac{2k}{\pi}\left(\beta - \alpha - \frac{\sin 2\alpha - \sin 2\beta}{2}\right) \quad E > \frac{A}{k}$ $N = \frac{2k}{\pi}\left(\frac{\pi}{2} - \alpha - \frac{\sin 2\alpha}{2}\right) \quad A > E \geq D$ where $\alpha = \sin^{-1}\frac{D}{E} \quad \beta = \sin^{-1}\frac{A}{E}$
(c) *Dead zone but no saturation*			$N = 0 \quad E < D$ $N = \frac{2k}{\pi}\left(\frac{\pi}{2} - \alpha - \frac{\sin 2\alpha}{2}\right) \quad E > D$
(d) *Ideal relay*			$N = \frac{4T_m}{\pi E}$

Table III. Characteristics and Describing Functions of Non-linear Elements.

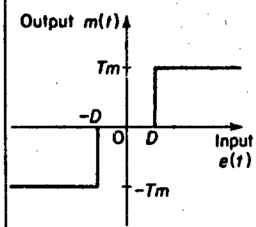
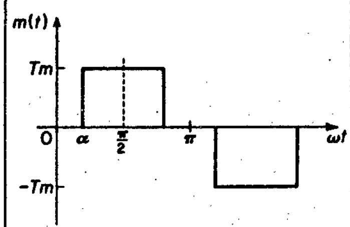
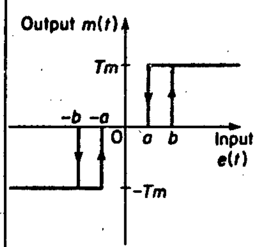
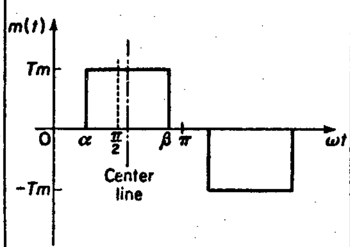
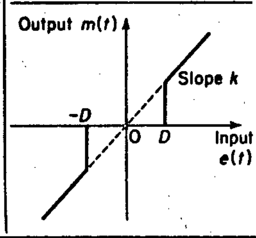
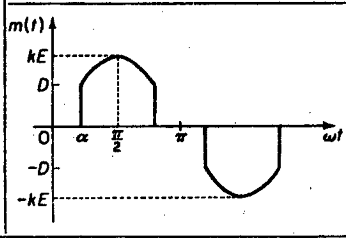
Type of nonlinearity	*Input-output transfer characteristic*	*Output waveform corresponding to sinusoidal input*	*Describing function N*
(e) ***Relay with dead zone***			$N = 0 \quad E < D$ $N = \frac{4T_m}{\pi E}\sqrt{1-\left(\frac{D}{E}\right)^2}$
(f) ***Relay with dead zone and hysteresis***			$\bar{N} = \frac{4T_m}{\pi E}\sin\left(\frac{\beta-\alpha}{2}\right)\exp\left[j\left(\frac{\pi}{2}+\frac{\alpha+\beta}{2}\right)\right]$ $\alpha = \sin^{-1}\frac{b}{E} \quad \beta = \pi - \sin^{-1}\frac{a}{E}$
(g) ***Dead zone with linear transfer characteristic***			$N = 0 \quad E < D$ $N = \frac{2k}{\pi}\left(\frac{\pi}{2} - \alpha + \frac{\sin 2\alpha}{2}\right)$ $\alpha = \sin^{-1}\frac{D}{E}$

Table III (continued)

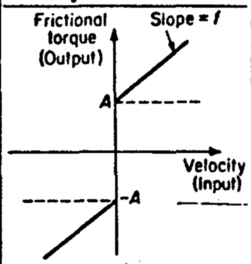
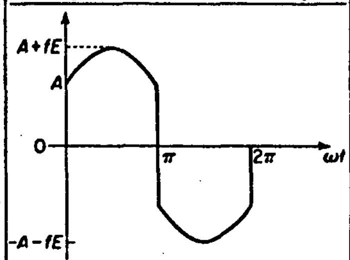
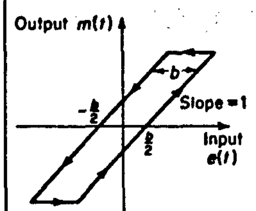
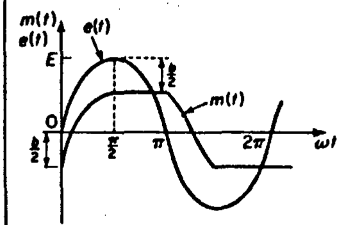
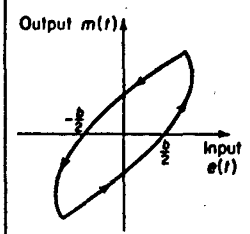
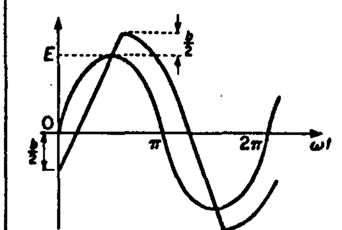
<table>
<tr><th>Type of nonlinearity</th><th>Input-output transfer characteristic</th><th>Output waveform corresponding to sinusoidal input</th><th>Describing function N</th></tr>
<tr><td>(h) Coulomb friction plus viscous friction</td><td>
</td><td>
</td><td>$N = \frac{4A}{\pi E} + f$</td></tr>
<tr><td>(i) Backlash element with viscous friction</td><td>
</td><td>
</td><td>$\overline{N} = \frac{\overline{M}_1}{E} = |N|e^{j\phi}$
$|N| = \sqrt{A_1^2 + B_1^2} \quad \phi = \tan^{-1}\frac{A_1}{B_1}$
$A_1 = \frac{E}{\pi}\left[\left(\frac{\pi}{2} + \beta\right) + \frac{\sin 2\beta}{2}\right]$
$B_1 = \frac{2}{\pi}\left[\left(\frac{b}{E}\right)^2 - \left(\frac{b}{E}\right)\right]$
$\beta = \sin^{-1}\left(1 - \frac{b}{E}\right)$</td></tr>
<tr><td>(j) Backlash element with inertia</td><td>
</td><td>
</td><td>$\overline{N} = \frac{\overline{M}_1}{E} = |N|e^{j\phi}$
$|N| = \sqrt{A_1^2 + B_1^2} \quad \phi = \tan^{-1}\frac{A_1}{B_1}$
$A_1 = \frac{2E}{\pi}\left(-\frac{3}{4} + \cos\alpha - \frac{\cos 2\alpha}{4}\right)$
$B_1 = \frac{2E}{E}\left(\sin\alpha + \frac{\pi}{2} - \frac{\sin 2\alpha}{4} - \frac{\alpha}{2}\right)$
$\alpha = \sin\alpha + b/E$</td></tr>
</table>

Table III (continued)