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AN INTEGRATED APPROACH TO THE DETERMINATION OF PROPELLER-GENERATED VIBRATORY FORCES ACTING ON A SHIP HULL

William S. Vorus

This research was carried out in part under the Naval Ship Systems Command General Hydromechanics Research Program Subproject SR 009 01 01, administered by the Naval Ship Research and Development Center. Contract No. N00014-67-A-0181-0033.

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The formula by which the generalized forces can be calculated is derived using potential flow theory. The basic ingredient in the derivation is a theorem which permits the problem to be solved without need for the solution of a hull-propeller interaction boundary value problem.

To use the method one must have available: a) data pertaining to the hull and propeller design, b) the hull normal mode shapes, c) an estimate of the propeller blade loading distribution, and d) the fluid velocity-potential function corresponding to free vibration of the bare hull in each normal mode. Very simple, yet accurate, strip formulas for estimating d), the bare hull velocity potential, and associated velocity components, are derived for the case of hull vertical vibration.

Fluid velocity-potential function.

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NOTATION

a	dimensional constant in paraboloidal coordinate transformation
B	ship beam
$B(\vec{R}_1)$	propeller blade (helix) surfaces
$f(x, t)$	hull forcing function
$f_n(x)$	hull forcing function harmonic amplitude
(f_n, ψ_i)	amplitude of n-th harmonic component of generalized force exciting i-th hull normal mode
FS	free surface of water
F_{nj}	bearing force harmonic amplitude
$G(\vec{R}; \vec{R}_1)$	Green's function
H	hull surface
$H_i(\vec{R})$	amplitude of bare hull vibratory velocity potential
i	$\sqrt{-1}$ and subscript defining hull normal mode
$\vec{i}, \vec{j}, \vec{k}$	unit vectors parallel to three cartesian axes
$K_m(arg)$	modified Bessel function of the second kind
ℓ	ship length and focal length of spheroid
L	linear operator and ship half-length
M	linear operator
n	excitation harmonic number
$\vec{n}_n(x, y, z_0)$	hull surface unit normal vector
$\vec{n}_p(\vec{R}_1)$	propeller blade (helix) surface unit normal vector
N	number of propeller blades
$p(\vec{R}, t)$	fluid pressure
$p_n(\vec{R})$	fluid pressure harmonic amplitude

$P(\vec{R}_1)$	propeller blade surfaces
$P(\vec{R}, t)$	total fluid pressure
r	radius coordinate in cylindrical coordinate system
$r_0(x)$	surface definition of slender body of revolution - cylindrical coordinates
$\vec{R}$	position vector, $\vec{R} = x\vec{i} + y\vec{j} + z\vec{k}$
$\vec{R}_1(t)$	position vector to a point on a propeller blade, $\vec{R}_1(t) = x_1\vec{i} + y_1(t)\vec{j} + z_1(t)\vec{k}$
S_1, S_2, S_3	surfaces bounding fluid volume
S	integral operator
t	time
$\bar{t}(\vec{R}_1)$	propeller blade thickness distribution
u_r, u_x, u_θ	fluid velocity components
U	ship speed (or stream speed)
$V(x)$	vertical vibratory velocity of hull surface
$\vec{V}_s(\vec{R})$	steady component of fluid velocity field
$\vec{V}(\vec{R}_1)$	velocity of a point on propeller blade surface
x, y, z	cartesian coordinates
$z_0(x, y)$	hull surface definition
$\vec{\alpha}_i(x, y, z_0)$	hull surface displacement vector in mode i
$\delta(arg)$	Dirac - delta function
η	paraboloidal coordinates and dummy variable
μ	spheroidal coordinate
$\mu(\vec{R}_1, t)$	propeller blade net pressure distribution
$\vec{\nu}(j)$	vectors defining direction of hull excitation, $j = 1 \rightarrow 6$
ω	propeller angular velocity
ω_i	hull i-th mode natural frequency

$\psi_i(x)$	hull i-th mode shape
$\phi(\vec{R}, t)$	perturbation velocity potential
$\phi_u(\vec{R})$	hull steady perturbation velocity potential
$\tilde{\phi}(\vec{R})$	steady component of propeller perturbation velocity potential
$\bar{\phi}(\vec{R}, t)$	unsteady component of propeller perturbation velocity potential
$\Phi(\vec{R}, t)$	fluid total velocity potential
ρ	water density
$\sigma(\vec{R}_1, t)$	source-sink intensity representing propeller blade thickness effect
θ	angle coordinate in cylindrical coordinate system
$\xi(x, t)$	hull vibratory displacement
$\xi_n(x)$	hull vibratory displacement harmonic amplitude
ξ	paraboloidal coordinate and dummy variable
$\xi_o(\eta)$	surface definition of slender body of revolution - paraboloidal coordinates
ζ	spheroidal coordinates and dummy variable
$\zeta_o(\mu)$	surface definition of slender body of revolution - spheroidal coordinates

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I INTRODUCTION

A vibrating ship traveling on or beneath the water surface involves many complicated physical processes interacting together in complicated ways. The ship may be vibrating in a steady or non-steady manner with local plating, machinery component, or deck and bulkhead vibrations superimposed on global beam-like flexural motions of the hull. The vibration may be excited internally by periodic or transient forces associated with machinery systems. It may be excited externally by propeller related fluid pressure fluctuations, hydroaeroelastic action in way of hull appendages, or by the ambient sea state. Damping of the vibration may result from combinations of fluid and structural dissipation complicated by such completely unpredictable effects as the shifting of cargo in the ship's holds [1][1]. The overall problem of ship vibration, like most real processes, defies precise analysis.

Nevertheless, much work has been done and much has been accomplished in developing analytical schemes for predicting ship vibration. The most notable achievements begin perhaps with F. M. Lewis' famous 1929 paper on hull added mass [2]. Successes have been achieved in areas where physically justifiable assumptions and simplifications have rendered the problem amenable to mathematical analysis. As with all physical problems, in situations where the simplifications needed to permit a mathematical solution have strong physical justification, the mathematics to follow seems to be nearly infallible in producing valid results. Two areas of this problem which are found to fall into this rather exclusive category are free undamped vibration of the hull in its beam-like modes and the periodic fluid pressure fluctuations associated with the lightly loaded free-running propeller.

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[1]Numbers in brackets denote references at end of text.

Tractable analysis in either of these areas requires a linear theory: a) linear ideal-fluid theory to describe the propeller agitation and, b) elementary beam theory as well as linear ideal-fluid theory to describe the hull free vibration. This work will attempt to take advantage of the ability of mathematics to perform effectively in these two areas to construct a method for predicting the propeller generated forces exciting steady-state vibration of the ship's hull. The highly developed theory of ship motions in waves will be repeatedly relied upon. Many aspects of ship motion theory carry-over directly to the flexural degrees of freedom of the vibrating ship hull.

The beam-like vibrations of a ship's hull may be combinations of vertical, athwartship, axial, or torsional modes. These modes may be excited in a steady manner by the ship's propeller in essentially two ways: a) indirectly by forces originating on the propeller blades and transmitted to the hull via the propeller shaft bearings and b) directly by forces on the surface of the ship stern arising from the periodic pressure field associated with the rotating propeller.

Assuming that the propeller is balanced, the indirect effect (bearing force excitation) is due entirely to the non-uniform hull wake in which the propeller must operate. The rotating propeller blade sees a non-steady inflow which produces non-steady lift and hence thrust and torque fluctuations on each blade. Bearing force excitation has received substantial attention and a number of methods are available for its analysis. Many workers in the past have relied on Burrill's method [3] which is based on a quasi-steady approach using blade element theory and measured wake data. Recently Tsakonas has developed a procedure [4] for estimating the distribution of unsteady blade loading based on finite-aspect-ratio foil theory, also using measured wake data as input.

The more difficult problem, which is no less important, is the excitation arising from the periodic variation of pressure on the hull surface. This may be thought of as a "fanning" or "washing" action as a propeller blade sweeps past adjacent surfaces of the hull or hull appendages. As with bearing force excitation, considerable theoretical work has been done in way of surface force excitation, but less has been accomplished from the standpoint of development of analytical methods of analysis. The approach to this problem has been to formulate an ideal-fluid boundary-value problem involving the hull and propeller, to solve the problem and obtain a fluid pressure, and to integrate the pressure over the hull to produce a time-dependent exciting force. The solution of this type of boundary-value problem is very difficult to obtain for a configuration which is at all general. Breslin has been successful with this approach for very idealized ship-like forms [5,6,7]. These solutions are important in that they establish trends to be expected but are less valuable in studying refinements to the stern configuration of a real ship.

A numerical solution of the integral equation formulation of the hull-propeller interaction boundary-value problem has also been attempted [8]. This procedure should be capable of giving good results but it is unattractive from a practical point of view due to the high costs associated with long computing times and large computer storage requirements.

These attempts at analyzing surface force excitation have aimed at finding a total integrated time dependent force on the hull. But such forces alone are not necessarily indicative of the degree of ship vibration which might occur. Certainly, if total forces are made very small, which may not be possible due to other considerations, then the resulting vibration will likewise be small. However, situations can exist, and always do to some extent, where large exciting forces do not produce significant hull vibration, even when the exciting frequencies lie near critical frequencies of the hull. Consider a situation where some hull mode produces a node at a longitudinal position x on the hull. The propeller generated surface force has some longitudinal distribution. Suppose that the center of force of this force distribution is also located at position x . Then this mode of hull vibration will not be excited by the resultant surface force, even though it may be large. The problem is not quite so simple, however, for the same mode will be excited by the moment of force at x . The situation will be reversed for an anti-node at position x . Obviously, all situations lie between these extremes.

It appears that, in order to evaluate the vibratory characteristics of a particular hull-propeller configuration in an accurate way, inter-relationships between the hydrodynamic and structural dynamic characteristics of the vibrating system must be properly considered. A proper measure of propeller excitation is expressed by the generalized forces relating the hydrodynamic disturbance of the propeller to the normal modes of hull free vibration.

This work develops a method for predicting the generalized exciting forces arising from the propeller generated periodic variation of pressure on the surface of a ship stern. The major ingredient in the mathematical derivation is the very elegant Chertock reciprocity theorem [9], which eliminates the major obstacle to success in this area: a fluids boundary value problem for the propeller operating in the presence of the hull and water surfaces. This same theorem is also related to Haskind's formula [10], well known in ship motions work; wave forces exciting the rigid body motions of a ship are determined without solving a diffraction problem. Chertock in [11] has used the same approach for analyzing the response of floating beams to under-water explosions. In that problem the excitation is provided by an oscillating gas bubble. Chertock's comparisons with experiment are remarkable [12]. Chertock has also studied propeller-induced axial vibration of submarine hulls in this same way, [13].

II ANALYSIS OF PROPELLER-INDUCED VIBRATION

1. Fluid Pressure Variation

Consider a ship traveling over the water surface and vibrating in response to propeller induced pressure fluctuations on the under-stern surfaces. For an N bladed propeller with rotational speed ω the predominant exciting frequency, for the "fanning" action of the propeller against adjacent surfaces, is $N\omega$. This excitation is not precisely sinusoidal so that harmonics, $nN\omega$, $n = 2, 3, \ldots$, although weaker, will also exist. For the general case of non-uniform hull wake, wake harmonics at all multiples of ω are present. Tsakonas points out in reference 4 that, even in the case of a non-uniform wake, hull surface excitation is constituted entirely of blade-rate harmonic components. The wake non-uniformity contributes by the coincidence of sums and differences of the individual wake harmonics with the blade-rate components. With the excitation frequency spectrum so established, the periodic variation of pressure exciting the hull vibration can be written,

$$p(\vec{R}, t) = \text{Re} \sum_{n=1}^{\infty} p_n(\vec{R}) e^{inN\omega t} \qquad (1)$$

where $\vec{R}$ is the position vector to a point in the fluid (in subsequent work the real part will be assumed in operations involving the complex exponential).

2. Hull Girder Dynamics

Let the ship hull be replaced by a one-dimensional beam with axially (x) varying properties which duplicates the characteristics of the hull vibrating in any combination of the beam modes: vertical, athwartship, axial, or torsional. The distributed forcing function exciting the beam is also one-dimensional and is obtained by integrating the hull surface pressure sectionally. Denote this distributed forcing function by $f(x, t)$. This may represent a true force or it may represent a moment, as in the case of torsional vibration. Let the vibratory displacement of the beam in any mode be denoted as $\xi(x, t)$; again, this displacement may be either a translation or a rotation. The forcing function $f(x, t)$, of course, retains the spectral character of the excitation pressure,

$$f(x, t) = \mathrm{Re} \sum_{n=1}^{\infty} f_n(x) e^{inN\omega t} \tag{2}$$

Then $\xi(x, t)$ can be represented as a sum of responses to each harmonic of the forcing function,

$$\xi(x, t) = \mathrm{Re} \sum_{n=1}^{\infty} \xi_n(x) e^{inN\omega t} \tag{3}$$

where ξ_n may be complex in the case of a damped system.

With this notation the mathematical model for the structural system vibrating in response to n-th order propeller excitation is as depicted in Figure 1.

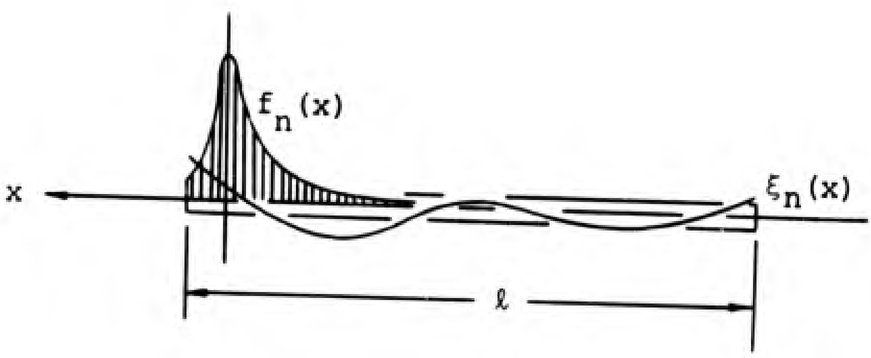

FIGURE 1
Hull Structural Model

For the linear theory assumed, the equation governing the vibratory amplitude of the beam, ξ_n , is an ordinary linear differential equation,

$$L\ \xi_n(x) = M\ f_n(x) \tag{4}$$

where L and M denote linear operators in x . The equation is subject to boundary conditions corresponding to zero restraining forces on the ends of the beam. The forms of the operators L and M are well documented in a number of references [1,14,15].

In general, the coefficients of L are non-constant and complex and an analytical solution of (4) cannot be written down. However, many methods for accurately approximating small amplitude forced vibration response of non-uniform beams are available, once the operator coefficients and forcing function have been established. If the differential equation and boundary conditions are reduced in such a way that the eigenfunctions of the homogeneous undamped differential equation are orthogonal, then a solution can be expanded in terms of the eigenfunctions,

$$\xi_n(x) = \sum_{i=1}^{\infty} \frac{\left\{\int_0^{\ell} f_n(x)\ \psi_i(x)\ dx\right\}\psi_i(x)}{n^2N^2\omega^2 - \omega_i^{\ 2}} \tag{5}$$

where ψ_i and ω_i are the mode shape (eigenfunction) and natural frequency corresponding to the i-th normal mode. The integral term in the numerator is a generalized force, that is, the integral over the length, ℓ , of the hull of force distribution times mode shape.[2]

---

[2]Note that the relationship of nodal position and centers of resultant force and moment previously discussed is properly considered by the generalized force.

A modal expansion generally requires that damping be excluded from the formulation, although Rayleigh-type viscous damping, proportional to mass, is permitted. If included, Rayleigh damping would appear as an additional term in the denominator of the solution, (5). This restriction on damping is rather severe, since vibratory response near resonance is usually of primary interest. But ship damping cannot be estimated with nearly the degree of confidence needed to allow a meaningful prediction of resonance amplitudes, regardless of the method used. It might appear, due to this inability to estimate ship damping, that steady-state response is not an area of the ship vibration problem where mathematical analysis can be applied to advantage. This may be true in cases where the magnitude of the response is considered to be the only useful result. However, response magnitude should not be considered to be the only useful result.

3. Generalized Forces

For given hull normal modes, ψ_i and ω_i , and a given exciting frequency, $nN\omega$, the generalized forces are a measure of the vibration response. To minimize a set of generalized forces would be to minimize the vibration response of a given hull at some operating condition. The generalized forces, being a good measure of propeller excitation, could be used to advantage in minimization processes or trade-off studies during design stage efforts to control ship vibration.

Obviously, in order to estimate the generalized force exciting some normal mode of hull vibration, the mode shape must be known. This is an aspect of ship vibrations which lends itself well to rational analysis, at least for the important lower natural modes. The homogeneous differential equation (4) is solved for the orthogonal mode shapes by any of many methods [1,14]. Of course, the coefficients in the differential equation must first be determined. Accepted methods exist for estimating these coefficients. An up-to-date method for estimating the hydrodynamic components of the coefficients, allowing for such effects as ship forward speed, could be adapted from the highly developed theory of ship motions [16]. Much of the latter, developed for rigid body motions, carries over directly to flexural vibratory motion of the ship.

In the following section a formula for predicting the generalized force, as defined in equation (5), will be developed from hydrodynamic considerations. It will be assumed that the hull normal mode shapes are available from separate analysis.

III THE GENERALIZED FORCE

As discussed, the propeller generated forces exciting hull vibration are a distributed surface force, with harmonic amplitude $f_n(x)$, and concentrated bearing forces. For m shaft bearings carrying vibratory load, define the complex amplitude of the n-th order concentrated force on the j-th bearing as F_{nj}. Then the complete n-th order generalized force exciting the i-th normal mode can be written,

$$\text{Re}\left\{ e^{inN\omega t} \int_0^{\ell} \left[f_n(x) + \sum_{j=1}^{m} F_{nj}\, \delta(x - x_j) \right] \psi_i(x)\, dx \right\} \quad (6)$$

where δ is the Dirac-delta function and x_j is the axial coordinate of the j-th bearing. The F_{nj} have been made available by reference 4. It would be a simple matter, with the F_{nj} in hand, to compute the generalized force component arising from the bearing forces. It would be added to the generalized force due to surface excitation in accordance with (6), once the surface force component had been found. The remainder of this work is concerned with finding the surface force component of the complete generalized force. A scalar-product notation is adopted to denote the amplitude of this component which will hereafter be called just "the generalized force." It is,

$$(f_n, \psi_i) = \int_0^{\ell} f_n(x)\ \psi_i(x)\ dx \tag{7}$$

which is the same factor as that in the numerator of (5). Note that the i-th generalized force, (7), is effectively the total n-th order surface force weighted according to the ability of its distribution to excite the i-th hull normal mode.

1. <u>Derivation of the Generalized Force Formula</u>

With the hull surface defined by $z - z_o(x, y) = 0$, Figure 2, the obvious way to attack (f_n, ψ_i) is to determine $f_n(x)$ from hull sectional integration of the pressure harmonic, $p_n(x, y, z_o(x, y))$, multiply it with the mode shape, which is assumed to be known, and integrate the product over the length of the hull.

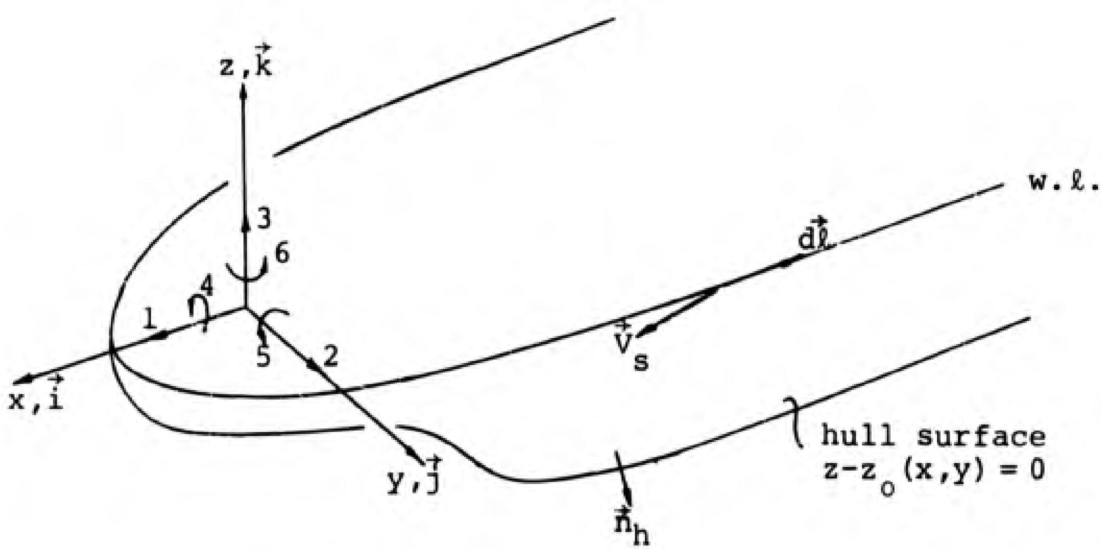

FIGURE 2
Hull Coordinate System

The force harmonic $f_n(x)$ can act to produce displacement in any of the six directions, numbered 1 to 6, on Figure 2. One corresponds to axial vibration, 2 and 6 to athwartship vibration, 3 and 5 to vertical vibration, and 4 to torsional vibration. Of course, coupled vibrations also exist; athwartship vibration is usually coupled with torsional vibration in the case of surface ships. Using a superscript to define the direction of excitation, 1 to 6,

$$f_n^{(j)} = \int_{\text{breadth}} p_n(x, y, z_o(x, y))(\vec{v}^{(j)} \cdot \vec{n}_h)\, dy \tag{8}$$

$\vec{n}_h$ is the unit normal vector to the hull surface and the $\vec{v}^{(j)}$ are:

$$\begin{aligned} \vec{v}^{(1)} &= \vec{i} & \vec{v}^{(4)} &= y\vec{k} - z\vec{j} \\ \vec{v}^{(2)} &= \vec{j} & \vec{v}^{(5)} &= z\vec{i} \\ \vec{v}^{(3)} &= \vec{k} & \vec{v}^{(6)} &= -y\vec{i} \end{aligned} \tag{9}$$

Let $\psi_i^{(j)}$ denote the 6 components in directions 1 to 6 of the i-th normal mode shape. Then, by (8), the generalized force component in that direction can be written

$$\begin{aligned} (f_n^{(j)}, \psi_i^{(j)}) &= \int_0^{\ell} \psi_i^{(j)}(x) \int_{\text{breadth}} p_n(x, y, z_o(x, y)) \\ &\qquad (\vec{v}^{(j)} \cdot \vec{n}_h)\, dy\, dx \\ &= \iint_H p_n(\psi_i^{(j)}\vec{v}^{(j)} \cdot \vec{n}_h)\, dS \\ &\equiv (f_n, \psi_i)^{(j)} \end{aligned} \tag{10}$$

where H denotes the hull surface. The generalized force exciting the mode is the sum over the 6 components,

$$(f_n, \psi_i) = \sum_{j=1}^{6} (f_n, \psi_i)^{(j)}$$

but from (10),

$$(f_n, \psi_i) = \iint_H p_n \left(\sum_{j=1}^{6} \psi_i^{(j)} \vec{v}^{(j)} \right) \cdot \vec{n}_h \, dS$$

Now define,

$$\vec{\alpha}_i = \sum_{j=1}^{6} \psi_i^{(j)} \vec{v}^{(j)} \tag{11}$$

so that,

$$(f_n, \psi_i) = \iint_H p_n (\vec{\alpha}_i \cdot \vec{n}_h) \, dS \tag{12}$$

This is an integral over the hull surface of the surface pressure amplitude times a function defining a normal mode of hull girder vibration.

So far, nothing significant has really been accomplished, for p_n comes from the unknown solution of a difficult boundary value problem. This problem describes a ship traveling along the water surface with forward speed U, as depicted in Figure 3. The fluid region is the volume of water bounded by the hull surface, H, the propeller surface, P, the free surface of the water, FS, and the bounding surfaces at infinity, S_1, S_2, and S_3. With the assumption of an ideal incompressible fluid the problem can be formulated mathematically using potential theory. For this situation the flow is representable as a potential flow everywhere except in the rotational slip-stream of the propeller. With the coordinate system fixed in the ship the velocity potential for the translating hull-propeller system can be expressed as,

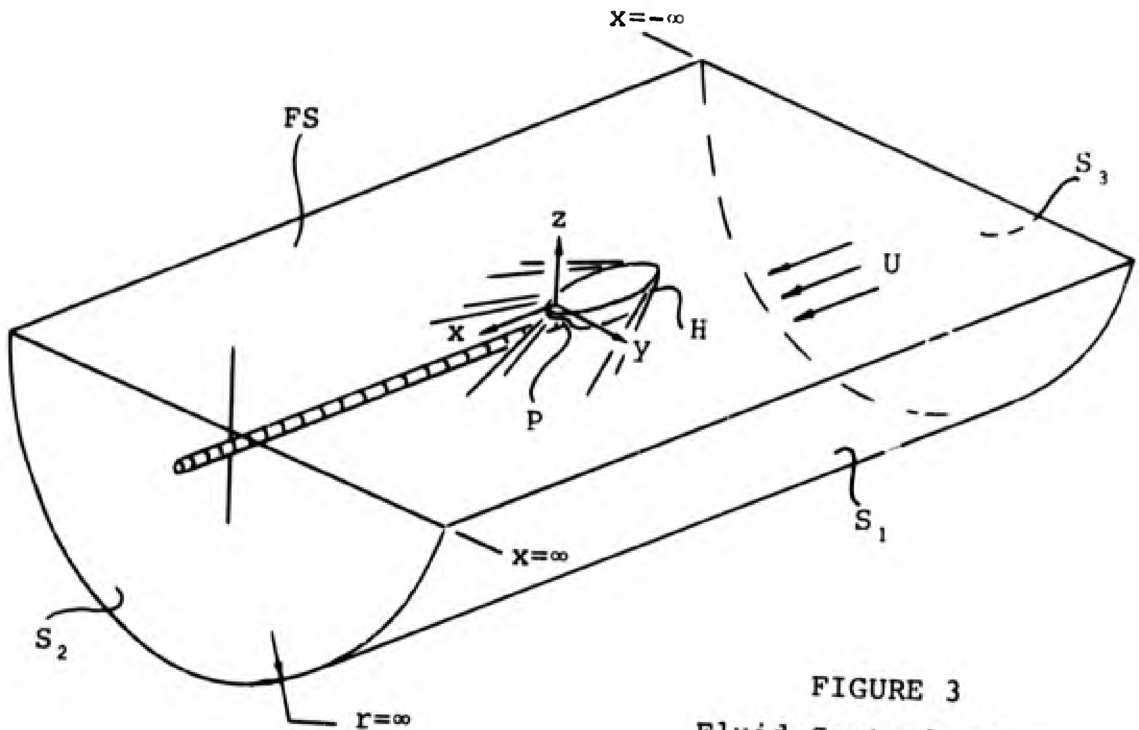

FIGURE 3
Fluid Control Volume

$$\Phi = Ux + \phi_u + \phi \qquad (13)$$

where Ux is just the uniform stream potential,

$\phi_u(\vec{R})$ is the perturbation potential due to the bare hull (no propeller), and

$\phi(\vec{R}, t)$ is the perturbation potential due to the propeller operating in the presence of the hull.

The fluid pressure $P(\vec{R}, t)$ is related to $\Phi(\vec{R}, t)$ by the Bernoulli equation,

$$-\frac{P}{\rho} + \frac{1}{2}U^2 = \frac{\partial\Phi}{\partial t} + \frac{1}{2}[\vec{\nabla}\Phi \cdot \vec{\nabla}\Phi] \qquad (14)$$

where ρ is the water density. Separate $\phi(\vec{R}, t)$ into its steady and unsteady parts,

$$\phi(\vec{R}, t) = \tilde{\phi}(\vec{R}) + \overline{\phi}(\vec{R}, t) \qquad (15)$$

The linearized unsteady component of fluid pressure is,

$$p(\vec{R}, t) = -\rho\left[\frac{\partial\Phi}{\partial t} + (U\vec{i} + \vec{\nabla}\phi_u + \vec{\nabla}\tilde{\phi}) \cdot \vec{\nabla}\overline{\Phi}\right]$$

$p(\vec{R}, t)$ is the pressure inducing the hull to vibrate and is also expressed by equation (1) as a Fourier series,

$$p(\vec{R}, t) = \text{Re} \sum_{n=1}^{\infty} p_n(\vec{R}) e^{inN\omega t} \tag{1}$$

Express $\overline{\Phi}$ accordingly,

$$\overline{\Phi}(\vec{R}, t) = \text{Re} \sum_{n=1}^{\infty} \overline{\Phi}_n(\vec{R}) e^{inN\omega t}$$

so that

$$p_n(\vec{R}) = -\rho\left\{inN\omega\overline{\Phi}_n + \vec{V}_s \cdot \vec{\nabla}\overline{\Phi}_n\right\} \tag{16}$$

where $\vec{V}_s = U\vec{i} + \vec{\nabla}\phi_u + \vec{\nabla}\tilde{\phi}$, the total steady fluid velocity vector.
Equation (16) re-expresses the pressure component needed in (12) in terms of an unknown velocity potential, $\overline{\Phi}_n$.

It should be pointed out that for the case where the hull is vibrating as well as translating, Φ contains an additional term, the potential corresponding to vibration of the hull. Then the total unsteady pressure is $p(\vec{R}, t)$, as given by (1), plus an additional pressure term due to the vibratory motion of the hull. For the linear situation assumed, this additional pressure does not influence the excitation force; it is the pressure which relates to the hydrodynamic coefficients in the L operator of the structural differential equation (4). $p(\vec{R}, t)$, as given in (1), is the pressure which excites the vibration.

Now combining (12) and (16)

$$(f_n, \psi_i) = -\rho \iint_H (inN\omega\overline{\Phi}_n + \vec{V}_s \cdot \vec{\nabla}\overline{\Phi}_n)(\vec{\alpha}_i \cdot \vec{n}_h)\, dS \tag{17}$$

Following Newman's treatment [17] of the exciting forces on a ship moving through waves, a function $H_i(\vec{R})$ is introduced which satisfies the following condition on the surface of the hull,

$$\frac{\partial H_i}{\partial n_h} = \left[inN\omega\vec{\alpha}_i - \vec{\nabla} \times (\vec{\alpha}_i \times \vec{V}_s) \right] \cdot \vec{n}_h \qquad \text{on } H \tag{18}$$

It is not necessary at this time that H_i have physical significance. However, it may be noted that if the sign on the second term in the condition is changed to plus, the condition is the hull boundary condition on the velocity potential amplitude corresponding to i-th mode vibration of the bare hull. (18) can be rearranged as,

$$(\vec{\alpha}_i \cdot \vec{n}_h) = \frac{1}{inN\omega} \left[\frac{\partial H_i}{\partial n_h} + \vec{\nabla} \times (\vec{\alpha}_i \times \vec{V}_s) \cdot \vec{n}_h \right] \tag{19}$$

Now, substituting (19) into (17),

$$(f_n, \psi_i) = -\rho \iint_H \bar{\phi}_n \frac{\partial H_i}{\partial n_h} dS - \rho \iint_H \left\{ \vec{\alpha}_i (\vec{V}_s \cdot \nabla\bar{\phi}_n) + \bar{\phi}_n \vec{\nabla} \times (\vec{\alpha}_i \times \vec{V}_s) \right\} \cdot \vec{n}_h \, dS$$

In the second integral the terms can be rewritten using vector identities,

$$\vec{\alpha}_i (\vec{V}_s \cdot \vec{\nabla}\bar{\phi}_n) = \vec{\nabla}\bar{\phi}_n \times (\vec{\alpha}_i \times \vec{V}_s) + (\vec{\nabla}\bar{\phi}_n \cdot \vec{\alpha}_i) \vec{V}_s$$

$$\bar{\phi}_n \vec{\nabla} \times (\vec{\alpha}_i \times \vec{V}_s) = \vec{\nabla} \times [\bar{\phi}_n (\vec{\alpha}_i \times \vec{V}_s)] - \vec{\nabla}\bar{\phi}_n \times (\vec{\alpha}_i \times \vec{V}_s)$$

so that

$$(f_n, \psi_i) = -\rho \iint_H \bar{\phi}_n \frac{\partial H_i}{\partial n_h} dS - \rho \iint_H (\vec{\nabla}\bar{\phi}_n \cdot \vec{\alpha}_i)(\vec{V}_s \cdot \vec{n}_h) dS$$

$$- \rho \iint_H \vec{\nabla} \times [\bar{\phi}_n (\vec{\alpha}_i \times \vec{V}_s)] \cdot \vec{n}_h dS \qquad (20)$$

$\vec{V}_s = U\vec{i} + \vec{\nabla}\phi_u + \vec{\nabla}\tilde{\phi}$ is the total steady fluid velocity vector. With respect to the coordinate system fixed in the hull

$$(\vec{V}_s \cdot \vec{n}_h) = 0 \qquad \text{on } H$$

Thus the second integral in (20) vanishes. The third integral can be expressed as a line integral using Stokes' theorem so that the generalized force becomes,

$$(f_n, \psi_i) = -\rho \iint_H \bar{\phi}_n \frac{\partial H_i}{\partial n_h} dS - \rho \oint \bar{\phi}_n (\vec{\alpha}_i \times \vec{V}_s) \cdot d\vec{\ell}$$

The line integral is over the intersection of the hull and the water surface, i.e., around the water-line.

Assume that at the water-line the ship sides are vertical so that the vectors $\vec{V}_s$ and $d\vec{\ell}$ define a vertical plane, as shown on Figure 2. For vertical and torsional vibration $\vec{\alpha}_i$ at the water-line (z = 0) also lies in the vertical plane, so that,

$$(\vec{\alpha}_i \times \vec{V}_s) \cdot d\vec{\ell} = 0$$

For athwartship and axial vibration assume that the vectors $\vec{V}_s$ and $d\vec{\ell}$ are colinear. For the small perturbation assumed with the linear theory, $\vec{V}_s$ is dominated by the uniform stream component, $U\vec{i}$, and wave distortion of the water-line is small. Then, $\vec{V}_s \times d\vec{\ell} = 0$ so that the integrand of the line integral is zero for all cases. Therefore,

$$(f_n, \psi_i) = -\rho \iint_H \overline{\Phi}_n \frac{\partial H_i}{\partial n_h} \, dS \qquad (21)$$

This is equivalent to the result obtained by Newman in reference 17 where his "generalized force" is a wave force exciting rigid body motion of the ship. In that case the function $\overline{\Phi}_n$ corresponds to the potential of the incident and diffracted waves.

The preparation has now been completed for the crucial step. The right hand side of (21) is one of the terms in Green's Second Identity for the functions $\overline{\Phi}_n$ and H_i :

$$\iiint_V (H_i \nabla^2 \overline{\Phi}_n - \overline{\Phi}_n \nabla^2 H_i) \, dV = \iint_{H+P+FS+S_1+S_2+S_3} \left(H_i \frac{\partial \overline{\Phi}_n}{\partial n} - \overline{\Phi}_n \frac{\partial H_i}{\partial n} \right) dS \qquad (22)$$

V is the water volume enclosed by the bounding surfaces, Figure 3. $\vec{n}$ is the unit normal vector to the respective surfaces. From (21) and (22),

$$(f_n, \psi_i) = \rho \iiint_V (H_i \nabla^2 \overline{\Phi}_n - \overline{\Phi}_n \nabla^2 H_i) \, dV - \rho \iint_{P+FS+S_1+S_2+S_3} \left(H_i \frac{\partial \overline{\Phi}_n}{\partial n} - \overline{\Phi}_n \frac{\partial H_i}{\partial n} \right) dS$$

$$- \rho \iint_H H_i \frac{\partial \overline{\Phi}_n}{\partial n_h} \, dS \qquad (23)$$

The remainder of the derivation for the generalized force is concerned with manipulating the terms in equation (23).

First consider the volume integral:

$$\iiint_V (H_i \nabla^2 \overline{\Phi}_n - \overline{\Phi}_n \nabla^2 H_i) \, dV$$

As a function defined in V , H_i will be required to satisfy some differential equation in V and boundary conditions on all of the bounding surfaces. So far, H_i has been specified only as to the value of its normal derivative on the surface of the hull, equation (18). Now require that H_i satisfy the Laplace equation in V ,

$$\nabla^2 H_i = 0 \qquad \text{in } V \tag{24}$$

The volume integral becomes,

$$\iiint\limits_V H_i \nabla^2 \overline{\Phi}_n \, dV \tag{25}$$

As previously stated, the translating hull and propeller, Figure 3, can be adequately described for this problem by a potential flow except in the propeller slip-stream. In Appendix A the differential equation governing the fluid disturbance of a lightly loaded propeller operating in open-water is derived. This equation, a Poisson equation, (32A) in Appendix A, is,

$$\nabla^2 \overline{\Phi}_{no} = - \frac{S\mu_o}{\rho U} \int_{x_1}^{\infty} e^{-inN\frac{\omega}{U}(\xi - x_1)} (\vec{n}_p \cdot \vec{\nabla}) \, \delta(x - \xi) \, \delta(y - y_1) \, \delta(z - z_1) \, d\xi - S\sigma_o \, \delta(x - x_1) \, \delta(y - y_1) \, \delta(z - z_1) \tag{26}$$

The "o" subscript has been used to denote "open-water." Reference can be made to Appendix A for a complete description of the variables in (26). With the aid of Figure 4, the notation is, in brief;

δ is the Dirac-delta function.

x_1, y_1, z_1 is the position of a point on the propeller blade at some time, t .

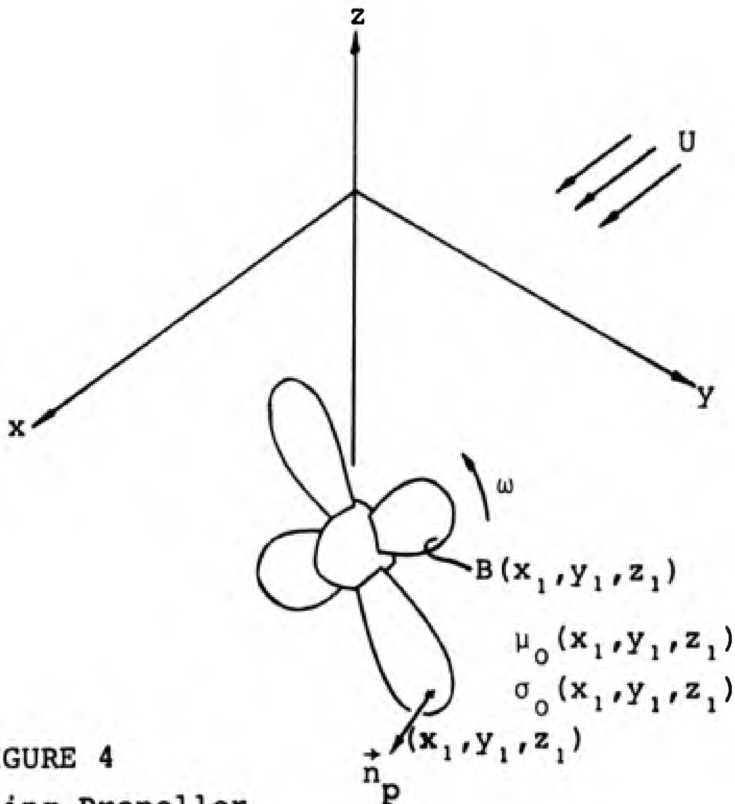

FIGURE 4

Free-Running Propeller Configuration

$\vec{n}_p$ is the unit normal vector to the blade at x_1, y_1, z_1 ($\vec{n}_p$ positive with axial component down-stream).

μ_o is the pressure jump across the blade at x_1, y_1, z_1.

σ_o is a source intensity representing the thickness effect of the blade at x_1, y_1, z_1.

S is the integral operator, $\frac{N\omega}{\pi} \int_{-\frac{\pi}{N\omega}}^{\pi/N\omega} e^{-inN\omega t}\, dt \iint_{B(x_1,y_1,z_1)} dS$

B is the blade surface

What (26) means is that the propeller blades and slip-stream have been replaced by singularities, sources of strength σ_o on the blades, representing the displacing effect of the blade thickness, and dipoles of strength related to μ_o on the blades and in the slip-stream representing the lifting effects. y_1 and z_1 are functions of time due to rotation of the propeller; hence, the time integral is needed to produce the n-th harmonic.

As stated, equation (26) is the differential equation governing a propeller operating in an infinite fluid, in the absence of a hull or free-surface. It is subject only to a radiation condition at large $|y|$ and $|z|$ and large negative x . It will now be assumed that the equation governing the operation of the propeller in the presence of the hull and free-surface has the same form as equation (26). This is, of course, not to say that the potential $\overline{\Phi}_n$ is assumed to be of the same form as the open-water potential $\overline{\Phi}_{no}$, only that the same differential equation governs both situations, much as the Laplace equation can govern two entirely different potential flow problems. The equation on $\overline{\Phi}_n$ will be subject to additional boundary conditions on the hull and free-surface. With this assumption,

$$\nabla^2\overline{\Phi}_n = - \frac{S\mu}{\rho U} \int_{x_1}^{\infty} e^{-inN\frac{\omega}{U}(\xi - x_1)} (\vec{n}_p \cdot \vec{\nabla})\ \delta(x-\xi)\ \delta(y-y_1)$$

$$\delta(z-z_1)\ d\xi - S\sigma\ \delta(x-x_1)\ \delta(y-y_1)\ \delta(z-z_1) \tag{27}$$

In this equation $\mu_0(\vec{R}_1)$ and $\sigma_0(\vec{R}_1)$ have been replaced by $\mu(\vec{R}_1, t)$ and $\sigma(\vec{R}_1, t)$ to account for the wake non-uniformity through which the propeller rotates. It will further be assumed with regard to (27) that μ and σ are known functions. μ is the same function needed in the analysis of bearing forces and hence could be determined from a procedure such as that developed by Tsakonas in reference 4. It has been found that σ can be adequately constructed using thin body theory [18.19]. Since propeller geometry and operating conditions are also presumed known, the right hand side of (27) is completely determined in terms of known functions.

Returning to the volume integral (25) with $\nabla^2\overline{\phi}_n$ in hand, we obtain the following:

$$\iiint_V H_i(\vec{R})\nabla^2\overline{\phi}_n(\vec{R},\ \vec{R}_1)\ dV = \frac{S\mu}{\rho U}\int_{x_1}^{\infty} e^{-inN\frac{\omega}{U}(\xi-x_1)}(\vec{n}_p\cdot\vec{\nabla})$$

$$H_i(\xi,y_1,z_1)\ d\xi - S\sigma\ H_i(x_1,y_1,\ z_1)$$

and $(f_n,\ \psi_i)$, equation (23), is re-expressed as,

$$(f_n,\ \psi_i) = \frac{S\mu}{U}\int_{x_1}^{\infty} e^{-inN\frac{\omega}{U}(\xi-x_1)}(\vec{n}_p\cdot\vec{\nabla})\ H_i(\xi,\ y_1,\ z_1)\ d\xi$$

$$-S\sigma\rho H_i(x_1,\ y_1,\ z_1) - \rho\iint_{FS+S_1+S_2+S_3}\left(H_i\ \frac{\partial\overline{\phi}_n}{\partial n} - \overline{\phi}_n\ \frac{\partial H_i}{\partial n}\right) dS \qquad (28)$$

$$-\rho\iint_H H_i\ \frac{\partial\overline{\phi}_n}{\partial n_h}\ dS$$

Note that the surface integral over the propeller blades has been dropped. This is because the propeller blades have been replaced by singularities and included within the fluid volume V , by Appendix A and equation (27).

On the hull surface $\frac{\partial\overline{\phi}_n}{\partial n_h} = 0$ so that the surface integral over H in (28) is zero.

So far the differential equation for H_i , (24), and hull boundary condition, (18), have been specified. It remains to specify conditions on the free-surface and the bounding surfaces at infinity. Referring still to Figure 3, let H_i satisfy a radiation condition for large $\vec{R}$,

$$\lim_{|\vec{R}|\to\infty} H_i = 0 \qquad (29)$$

$\overline{\phi}_n$ also satisfies a similar radiation condition for large $|y|$, $|z|$, and large negative x. In view of possible solutions for H_i and $\overline{\phi}_n$ it is obvious that,

$$\iint_{S_1+S_3} \left(H_i \frac{\partial \overline{\phi}_n}{\partial n} - \overline{\phi}_n \frac{\partial H_i}{\partial n} \right) dS = 0$$

It is not quite so obvious that the integral over S_2 vanishes; the singularities representing the propeller slipstream pass through this surface. It can be shown that the potential in the case of the open-water propeller behaves, for large x, like the modified Bessel functions $K_0(\overline{r})$, $K_1(\overline{r})$; $\overline{r} = \sqrt{y^2+z^2} = 0$ corresponds to the location of a singularity in the bounding vertical plane at large x. The singularities retain the same basic character in the presence of the free-surface and far down-stream of the hull. Therefore, the integral over S_2 is well defined at the singularity and converges for large $\overline{r}$ (y and z). For S_2 located at $x \to \infty$ the value of the integral is zero.

The remaining surface integral in (28) is the integral over the free surface. Therefore, as the last remaining condition on H_i, require that,

$$H_i \frac{\partial \overline{\phi}_n}{\partial n} - \overline{\phi}_n \frac{\partial H_i}{\partial n} = 0 \qquad \text{on FS} \tag{30}$$

This leads to the completion of the basic derivation of the formula expressing the generalized force. It is, from (28),

$$(f_n, \psi_i) = \frac{N\omega}{\pi} \int_{-\frac{\pi}{N\omega}}^{\frac{\pi}{N\omega}} e^{-inN\omega t}\, dt \iint_{B(R_1)} \left\{ \frac{\mu}{U} \int_{x_1}^{\infty} e^{-inN\frac{\omega}{U}(\xi - x_1)} (\vec{n}_p \cdot \vec{\nabla}) H_i(\xi, y_1, z_1)\, d\xi - \sigma\rho\, H_i(x_1, y_1, z_1) \right\} dS \tag{31}$$

Here S has been replaced by the integral operation which it represents. All variables in (31) are presumed known except the function H_i . H_i satisfies a boundary value problem formulated by equations (18), (24), (29), and (30):

$$\nabla^2 H_i = 0 \qquad \text{in } V \qquad (24)$$

$$\frac{\partial H_i}{\partial n_h} = \left[inN\omega\vec{\alpha}_i - \vec{\nabla} \times (\vec{\alpha}_i \times \vec{V}_s) \right] \cdot \vec{n}_h \qquad \text{on } H \qquad (18)$$

$$H_i \frac{\partial \bar{\phi}_n}{\partial z} - \bar{\phi}_n \frac{\partial H_i}{\partial z} = 0 \qquad \text{on } z=0 \qquad (30a)$$

$$\lim_{|\vec{R}|\to\infty} H_i = 0 \qquad (29)$$

(30) has been rewritten as (30a) since the linearized condition is satisfied on $z = 0$ and the normal direction is the negative z direction.

The formulation of the H_i problem is, at first glance, very disheartening. The hull and free-surface boundary conditions are written in terms of components of the hull-propeller interaction potential ϕ , as defined by equation (15); $\bar{\phi}_n$ appears in (30a) and $\tilde{\phi}$, by way of $\vec{V}_s$, appears in equation (18). Avoidance of the interaction potential was the motivation for this whole approach; it seems that just another unsolvable problem has been introduced. However, the situation is really much better this time. The hull and free-surface boundary conditions on H_i can both be legitimately replaced by conditions which do not involve the hull-propeller interaction potential, ϕ . This is as follows:

First consider the free-surface condition,

$$H_i \frac{\partial \bar{\phi}_n}{\partial z} - \bar{\phi}_n \frac{\partial H_i}{\partial z} = 0 \qquad \text{on } z=0$$

$\overline{\Phi}_n$ is the amplitude of the potential corresponding to the n-th order propeller disturbance in the presence of the translating ship. As such, it must satisfy the well known linearized free-surface boundary condition [17],

$$g \frac{\partial \overline{\phi}_n}{\partial z} - N^2 n^2 \omega^2 \overline{\phi}_n + 2inN\omega U \frac{\partial \overline{\phi}_n}{\partial x} + U^2 \frac{\partial^2 \overline{\phi}_n}{\partial x^2} = 0 \qquad \text{on } z=0 \quad (32)$$

However, in agreement with others' experience with the ship vibration problem [2,8], this condition is dominated by the second term and can be replaced by,

$$\overline{\phi}_n = 0 \qquad \text{on } z=0 \quad (33)$$

A rotating propeller is known not to create any significant far-field free-surface disturbance. This is because, for a propeller blade Froude number based on chord length and relative velocity at some radius, the Froude number range of the blades is very high. High blade Froude number or, equivalently, high excitation frequency, $nN\omega$, implies a zero-potential free-surface condition. With $\overline{\phi}_n = 0$ on $z = 0$ and, since $\frac{\partial \overline{\phi}_n}{\partial z} \neq 0$ on $z = 0$, (30a) and (33) imply that,

$$H_i = 0 \qquad \text{on } z=0 \quad (34)$$

With regard to the hull boundary condition on H_i,

$$\frac{\partial H_i}{\partial n_h} = \left[inN\omega\vec{\alpha}_i - \vec{\nabla} \times (\vec{\alpha}_i \times \vec{V}_s) \right] \cdot \vec{n}_h \qquad \text{on } H \quad (18)$$

recall that $\vec{V}_s$ defines the distribution of steady water velocity about the ship,

$$\vec{V}_s = U\vec{i} + \vec{\nabla}\phi_u + \vec{\nabla}\tilde{\phi}$$

ϕ_u and $\tilde{\phi}$ are perturbation potentials; $\vec{\nabla}\phi_u + \vec{\nabla}\tilde{\phi}$ represents the steady perturbation velocity field due to the hull and propeller. The perturbation must be small as required by the linear theory. Physically this means that the ship should be slender (so that ϕ_u is small) and that the propeller should have thin, lightly loaded blades (so that $\tilde{\phi}$ is small); the latter has already been explicitly assumed. Therefore, recognizing additionally the restriction to a slender ship, we assume that the ship is slender enough so that the terms in the boundary condition involving $\vec{\nabla}\phi_u$ and $\vec{\nabla}\tilde{\phi}$ are insignificant compared to the terms involving U . Therefore, (18) reduces to,

$$\frac{\partial H_i}{\partial n_h} = \left[inN\omega\vec{\alpha}_i - \vec{\nabla} \times (\vec{\alpha}_i \times U\vec{i})\right] \cdot \vec{n}_h \qquad \text{on } H \qquad (35)$$

H_i now defines a boundary value problem which is easy to solve relative to the scattering problem, ϕ . The formulation of H_i is now,

$$\nabla^2 H_i = 0 \qquad \text{in } V \qquad (24)$$

$$\frac{\partial H_i}{\partial n_h} = \left[inN\omega\vec{\alpha}_i - \vec{\nabla} \times (\vec{\alpha}_i \times U\vec{i})\right] \cdot \vec{n}_h \qquad \text{on } H \qquad (35) \qquad (36)$$

$$H_i = 0 \qquad \text{on } z=0 \qquad (34)$$

$$\lim_{|\vec{R}|\to\infty} H_i = 0 \qquad (29)$$

Recall that no attempt has ever been made to interpret the function H_i in any physical way. It is still not necessary to do so. H_i is just a function involved in the solution for the generalized force (31) which can be determined by solving a relatively simple boundary value problem. It is interesting, however, that H_i, as defined by the formulation (36), is the amplitude of the unsteady part of the velocity potential corresponding to the slender bare hull vibrating (with high frequency) in normal mode i and translating backwards with speed U across the surface of the water.

2. Application of the Generalized Force Formula

The amplitude of the generalized exciting force on a ship's hull due to propeller-generated pressure fluctuations on the hull surface can be calculated from equation (31) along with equations (36) from the previous section:

$$(f_n, \psi_i) = \frac{N\omega}{\pi} \int_{-\frac{\pi}{N\omega}}^{\frac{\pi}{N\omega}} e^{-inN\omega t} \, dt \iint_{B(\vec{R}_1)} \left\{ \frac{\mu}{U} \int_{x_1}^{\infty} e^{-inN\frac{\omega}{U}(\xi - x_1)} (\vec{n}_p \cdot \vec{\nabla}) \, H_i(\xi, y_1, z_1) \, d\xi - \sigma\rho \, H_i(x_1, y_1, z_1) \right\} dS \qquad (31)$$

$$\nabla^2 H_i = 0 \qquad \text{in } V$$

$$\frac{\partial H_i}{\partial n_h} = \left[inN\omega\vec{\alpha}_i - \vec{\nabla} \times (\vec{\alpha}_i \times U\vec{i}) \right] \cdot \vec{n}_h \qquad \text{on } H \qquad (36)$$

$$H_i = 0 \qquad \text{on } z=0$$

$$\lim_{|\vec{R}| \to \infty} H_i = 0$$

Note that the most important variables influencing the magnitude of propeller induced hull-surface excitation are accounted for in (31). The propeller configuration as to blade number, geometry, and loading is provided by N, μ, and σ. Hull geometry, such as the shape of counter and skeg, is supplied by H_i. H_i, along with the blade coordinates

x_1, y_1, z_1, also provides for the effect of hull-propeller clearances. The hull wake may appear in μ and σ. The willingness of the hull to respond to the propeller excitation is decided largely by $\vec{\alpha}_i$, through its influence on H_i.

The calculation of (f_n, ψ_i) from (31) and (36) is simple in concept. For a given hull mode shape the H_i problem (36) is first solved. H_i and $(\vec{n}_p \cdot \vec{\nabla})H_i$ are then evaluated at the positions of the propeller singularities, multiplied by the respective singularity strengths, and then summed over all of the singularities. A harmonic analysis is then performed to produce the n-th harmonic. Therefore, once all data pertaining to the design of the hull and propeller has been specified, along with the vibratory direction and mode shape, the evaluation of (f_n, ψ_i) is reduced essentially to three tasks: 1) solution of the H_i problem, 2) determination of μ and σ, and 3) integration.

a) <u>Propeller singularity strengths</u>

References 4 and 18, both by Breslin and Tsakonas, can be referred to for information on the functions μ and σ. As defined, $\mu(\vec{R}_1, t)$ is the surface distribution of net load on a blade of the propeller. Reference 4 determines this function as the solution of an integral equation whose kernel is constructed using unsteady lifting surface theory. The equation is solved numerically in terms of measured wake data and the procedure appears to give accurate results on the basis of experimental comparisons provided in [4]. The computing time listed suggests a practical limitation for this application, since the determination of μ will be only a part of the total computing effort required. In most cases less accurate estimates of μ would probably suffice. For example, precise knowledge of the distribution of propeller blade load should not be required in a trade-off study where the loading distribution is not varied. Burrill's procedure [3], a quasi-steady approach using two-dimensional air foil data and a measured wake distribution might be used. In the limit, μ could even be estimated from the mean effective wake, and would just be equivalent to the blade load corresponding to steady thrust and torque.

With regard to the blade thickness effect, which is surprisingly important in the hull excitation problem, the simple linearized thin body approach gives good results. The blades are replaced by rotating source-sink sheets, $\sigma(\vec{R}_1, t)$, so that the zero stream surface of the resultant flow pattern defines a closed body whose thickness distribution is approximately equivalent to that of the propeller blade; see Appendix A. Actually, previous work with regard to σ seems to have been limited to the case of uniform inflow, with σ consequently being independent of time. This is the case considered by Breslin in [18] and also in Kerwin's propeller design procedure, reference 20. A time dependent source distribution to account for the non-uniform wake could be approximated using the same procedure. Reference 19 compares experimental and calculated fluid pressures arising from blade thickness effect in the case of free-running propellers. The comparisons confirm that 1st order thin body theory is quite adequate for representing the effect of blade thickness.

b) Numerical integration

The integration required by (31) will, of course, have to be carried out numerically. Assume for the moment that H_i and $\vec{\nabla}H_i$, evaluated at positions of the singularities, are available as some readily accessible array. Then, with the remaining data known, four integrals must be performed; a single improper integral in x over the propeller slip stream, a double integral over the propeller blades, and another single integral in time needed to produce the n-th harmonic component. Note that integration over the hull is not required.

In the case of the slip-stream integral, the function H_i and its gradient decay at least as fast as $1/x$ astern of the ship. Therefore, the infinite upper limit will be replaced by some moderate value for the numerical integration. The size of the integration interval in x will depend strongly on the period of the alternating exponential. High frequency excitation will require somewhat finer resolution of the slip stream than a lower frequency situation. The lower harmonics will usually be of most interest.

A good example of the type of propeller blade surface integration required is provided by reference 21. In [21] Huse computes field pressures induced by a propeller operating in an infinite fluid. The integration is carried out chord-wise and then radially over the blade singularity sheets for both loading and thickness effects. Actually, with the H_i values supplied, the integrand in (31) is not necessarily more difficult than that handled by Huse. The computing times listed in [21] suggest that blade surface integration would not be a limiting factor in practical uses of the generalized force formula.

The time integral in (31) can probably be handled best as a harmonic analysis on values of the integrand computed at successive angular positions of the blade within the cycle. Care should be taken, however, not to interpret this integration as a harmonic analysis of some general time dependent function. The integration is valid only for one value of n, namely, the value of n used in the boundary condition of the H_i problem and appearing in the x exponential. The result has no meaning until the time integration is performed.

c. <u>Determination of H_i functions</u>

The formula developed for predicting the hull generalized exciting force, equation (31), has been discussed with regard to the propeller singularities μ and σ and the numerical integration. The true complexity of (31) and its suitability for practical applications depends most strongly, however, on the effort required to determine the functions H_i and $\vec{\nabla} H_i$.

The problem to be solved is that defined by equations (36). As stated, the problem physically describes the amplitude of the unsteady fluid disturbance created by the bare hull, H, vibrating in some i-th normal mode, and traveling astern with speed U across the water surface.

The general approach to solving such a potential flow problem is to place sources or other singularities which satisfy the Laplace equation on or interior to the hull surface, with negative images of the same singularities above the $z=0$ plane, so that the hull and free-surface boundary conditions are satisfied. The Laplace equation and radiation condition will then be satisfied identically in the fluid region. Numerical schemes, like the Hess and Smith algorithm, [22], could be adapted to this problem, or mapping techniques, using slender body approximations, might also be applied to obtain a solution.

In the special case of vertical vibration, which is usually considered to be the most important case, a possibility for approximating the H_i solution in a really easy way appears. For vertical vibration, from equation (11),

$$\vec{\alpha}_i = \psi_i^{(3)} \vec{\nu}^{(3)} + \psi_i^{(5)} \vec{\nu}^{(5)}$$

$\psi_i^{(5)} = \dfrac{d\psi_i^{(3)}}{dx}$ and the $\vec{\nu}$ are given by (9). $\vec{\alpha}_i$ is then,

$$\vec{\alpha}_i = \psi_i \vec{k} + z \frac{d\psi_i}{dx} \vec{i} \tag{37}$$

where ψ_i is now the mode shape corresponding to i-th mode vertical displacement of the hull girder. (37) shows that vertical flexure of the hull is a superposition of a shearing type displacement with sections moving vertically, ψ_i , and a section rotation producing axial displacement of the hull, $z \frac{d\psi_i}{dx}$. The contribution of section rotation will be small and, consequently, the second term in (37) is discarded. This is the same assumption made by Lewis in reference 2. Therefore, for vertical vibration,

$$\vec{\alpha}_i = \psi_i \vec{k} \qquad (38)$$

The formulation (36) now describes a simpler problem than before: the H_i problem corresponds to the double-hull vibrating vertically and traveling backwards through an infinite volume of water. Here, $z=0$ is the symmetry plane of the double-hull and, for the unsteady vertical motion, the potential is zero on the symmetry plane. Introducing (38) into (36), the boundary value problem which must be solved for the case of vertical vibration is,

$$\nabla^2 H_i = 0 \qquad \text{in } V'$$

$$\frac{\partial H_i}{\partial n_h} = V(x)\,(\vec{n}_h \cdot \vec{k}) \qquad \text{on } H' \qquad (39)$$

$$\lim_{|\vec{R}| \to \infty} H_i = 0$$

H' is the total surface of the double-hull, V' is the volume of the whole space exterior to H' , and $V(x) = inN\omega\psi_i - U \frac{d\psi_i}{dx}$ is a known function.

A time consuming numerical treatment of an integral equation can be used to solve this problem. Such an approach is, without doubt, capable of producing accurate solutions. For practical purposes, however, the degree of accuracy obtainable with these types of solutions, for this application, is probably not worth the effort required. Simple formulas, capable of representing the flow in a slightly less accurate way, would usually be preferred.

Simple formulas for H_i and $\vec{\nabla}H_i$ can be derived using slender body theory. Slender-body strip theories are known to give accurate results in ship-motions analysis [23]. However, the usual strip formulas obtained from a cartesian coordinate formulation of slender-body theory do not represent the flow accurately at the ends of the body. In the present problem, H_i and $\vec{\nabla}H_i$ must be evaluated on the propeller blades and in the propeller slip stream; the propeller is located at the end of the ship.

In reference 24, Tuck finds that difficulty with slender body theory at the ends of a translating slender body can be alleviated by formulating the theory in terms of prolate spheroid coordinates. In Appendix B, Tuck's ideas have been extended to the case of a slender body moving laterally, as described by equations (39). Strip theory formulas for H_i and the velocity components are derived in three coordinate systems in Appendix B: cylindrical, spheroidal, and paraboloidal. The spheroidal and paraboloidal coordinates have similar characteristics. The formulas are derived for bodies of revolution but in each case a two dimensional Laplace equation governs, so that the advantage of conformal mapping is available for treating conventional hull sections.

The accuracy of the strip formulas is evaluated by comparison with the corresponding exact solutions for heaving spheroids of beam-length ratios .10 and .15. As expected, this comparison shows that the cylindrical coordinate formulas are not adequate for this application. Conversely, the strip formulas derived in spheroidal and paraboloidal coordinates give results which are within approximately five per cent of the exact values off the end of the spheroid. The paraboloidal formulas, equations (24B) and (25B) in Appendix B, are slightly simpler than the spheroidal and, therefore, appear to be best for this application.

The reason that the spheroidal and paraboloidal strip theories succeed where the cylindrical fails is an obvious one. In slender body theory, the general formulation of a problem is simplified by discarding terms on the basis of small changes with respect to the length-wise coordinate. In the cylindrical coordinate formulation the simplifications are made on the basis of derivatives in x being small. This is fine near mid-ship, but it is grossly invalid in the region of high curvature near the end of the body. For the spheroidal and paraboloidal formulations the length-wise differentiation is with respect to a curvilinear coordinate in whose direction the velocity gradients are uniformly small over the entire body, including the end. This becomes clear on studying Appendix B.

In summary, for the case of vertical vibration, the formulas in Appendix B can be easily used to produce the array of H_i and $\vec{\nabla}H_i$ values needed to carry out the numerical integration. The only additional information needed is the inverse coordinate transformations, also given in Appendix B, and appropriate conformal mapping functions; Lewis forms [2], should be adequate for most ship sections. For the other modes, athwartship, torsional, and axial, the functions can also be evaluated and the integrations can also be carried out. The difference is that the H_i evaluations for these modes cannot, or so it appears at this time, be performed in a manner as equally advantageous for this application.

IV SUMMARY

The product of this work is a formula, equation (31), for estimating the hull surface component of the propeller generated force which excites ship hull vibration. Equation (31) can be used for computing the set of generalized forces associated with the normal modes of hull girder free vibration.

In a mathematical sense, generalized force, as used here, is the forcing function in the modal expansion of the hull vibratory displacement amplitude (see equations (5), (6), and (7)). It can also be defined as a coefficient in the Fourier series expansion of the hull surface force, the expansion being with respect to the length-wise coordinate and in terms of the hull normal mode shapes. Physically, the generalized force associated with some normal mode can be interpreted as the total hull surface force generated by the propeller but weighted according to its effectiveness in exciting the mode. Also in a physical sense, generalized force can be related to an excitation energy per unit of hull displacement amplitude. Excitation energy is proposed as a design criterion by Nowacki in [25].

In order to use equation (31) to compute the generalized force exciting some hull normal mode, the following information must be available:

a) Data pertaining to the hull and propeller design. These data, including hull and propeller geometry, ship speed, and propeller rpm, usually become available during the early stages of ship design.
b) An estimate of the propeller blade loading distribution. Determination of the propeller blade loading function is discussed in section III - 2.a; several methods are available.
c) The mode shape corresponding to the hull vibratory normal mode for which the generalized force is to be calculated.
d) The fluid velocity potential and velocity components corresponding to vibration of the bare hull in the normal mode for which the generalized force is to be calculated. This velocity potential, denoted as H_i , is treated in section III - 2.c and in Appendix B.

The solution of a hull-propeller interaction boundary-value problem is not required with regard to equation (31). The need for solving such a problem has been a major impediment to the success of previous efforts in estimating hull surface excitation force. The necessity for obtaining the solution to the interaction problem has been avoided here by the use of the Chertock reciprocity theorem. Chertock's theorem, which is a special application of Green's theorem, uses the formulation of the interaction problem, but it does not require its solution in order to establish the interaction force, equation (31). The formulation of the interaction boundary value problem is composed of partial differential equation and boundary conditions on the ship hull and water surface. The forms of the boundary conditions are well established; the governing differential equation is derived in Appendix A. A free-running propeller is used as a model for the derivation of the differential equation. The mathematical treatment in Appendix A produces a Poisson equation, (26), whose right hand side represents a propeller slip stream composed of helical sheets of dipoles streaming off the propeller blade trailing edges. This slip-stream model may not be extremely precise for the case of a propeller operating in a hull wake and in front of a rudder. It does not appear to be possible to develop the equivalent model for a wake-operating propeller in any rigorous way. The wake non-uniformity is, however, accounted for in (27) by way of the blade loading distribution, which must be supplied (see b) above). The loading distribution function, denoted as μ , can be estimated for the wake operating propeller.

Elimination of the need for solving the interaction problem is not accomplished without some expense. Chertock's theorem produces not only equation (31), but also a different boundary value problem, whose solution must be obtained in order to use (31). This new problem corresponds to the unsteady fluid behavior due to the bare hull traveling astern at constant speed across the water surface and vibrating in the normal mode for which the generalized force is to be calculated. Fortunately, the solution of this problem is not so extremely difficult to obtain as that of the interaction problem which it has replaced. The bare hull vibratory problem which must be solved is formulated in terms of a velocity potential, H_i , as equations (36).

For the special case of hull vertical vibration, the hull vibratory problem, (36), reduces to the problem of a double-hull vibrating and translating astern in an infinite fluid. Appendix B is devoted to a study of this case. In Appendix B solutions are developed for the particular case of bodies of revolution. Slender body theory is applied using three different coordinate systems: cylindrical, paraboloidal, and spheroidal. The curvilinear paraboloidal and spheroidal coordinate systems are found to relieve the end effect problems inherent in the usual cylindrical coordinate slender body solutions. Numerical comparisons are made between the three sets of slender body solutions and the corresponding exact solutions for a heaving spheroid. As expected, the cylindrical solutions do not compare well in the fluid region of interest off the ends of the spheroid. On the other hand, the paraboloidal and spheroidal solutions are within approximately five per cent, based on body heave velocity, of the exact solutions.

It can be concluded from the results contained in Appendix B that a special formulation of slender body theory, using either paraboloidal or spheroidal coordinates, can be used to calculate H_i and $\vec{\nabla} H_i$ for the case of vertical vibration; accurate results would be expected. Furthermore, the first order slender body solutions obtained in Appendix B for the body of revolution are 2-D "strip" formulas. A Laplace equation governs in each case. Therefore, conformal mapping can be used to apply the solutions in studying realistic hull sections. Equations (24B) and (25B), in paraboloidal coordinates, are recommended in this regard.

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APPENDIX A

A DIFFERENTIAL EQUATION GOVERNING THE DISTURBANCE CREATED BY A FREE-RUNNING PROPELLER

Consider a propeller mounted on the axis of a circular cylinder of infinite volume V , Figure 1A,

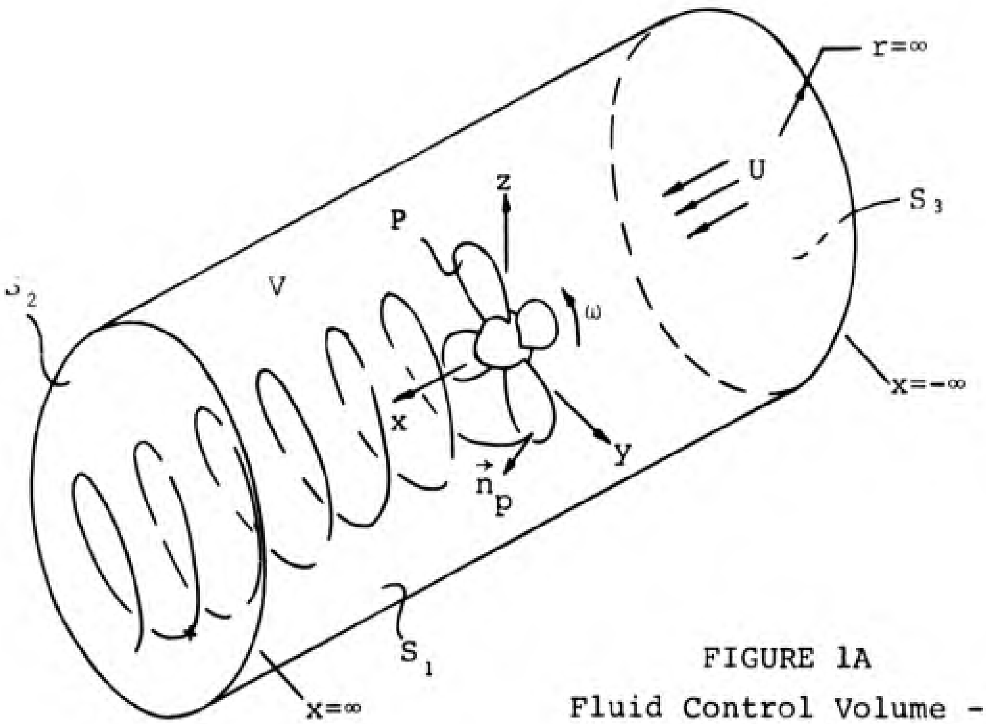


FIGURE 1A
Fluid Control Volume - Free-Running Propeller

The coordinate system is fixed in the fluid so that the position vector to a point on a rotating propeller blade is $\vec{R}_1(t) = x_1\vec{i} + y_1(t)\vec{j} + z_1(t)\vec{k}$. The position vector to a point in the fluid is $\vec{R} = x\vec{i} + y\vec{j} + z\vec{k}$. The surface of a propeller blade is denoted as $P(\vec{R}_1)$ and the blade thickness is denoted as $\bar{t}(\vec{R}_1)$.

The propeller rotates with angular velocity ω and far upstream of the propeller the flow is uniform with velocity U in the x direction. The camber surfaces of the propeller blades are approximately the helical surfaces, B , (see Figure 2A) defined by the unit normal vector $\vec{n}_p$,

$$\vec{n}_p(\vec{R}_1) = \vec{V} \times \vec{R}_1 \qquad (1A)$$

where

$$|\vec{V}| = \sqrt{U^2 + \omega^2(y_1{}^2 + z_1{}^2)}$$

so that the blades are lightly loaded with a known net pressure distribution $\mu(\vec{R}_1) = |p^+| + |p^-|$, by Figure 2A.

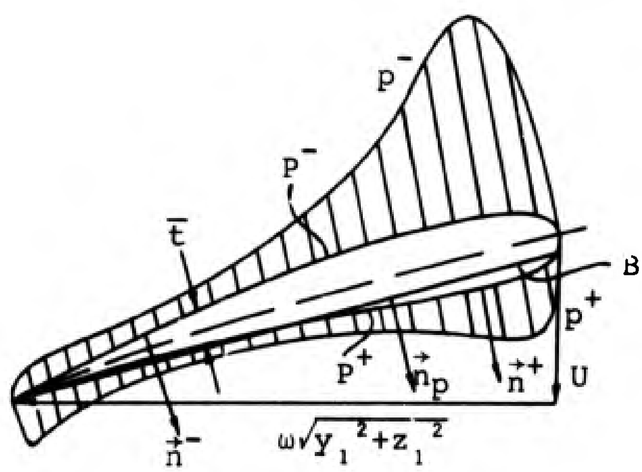


FIGURE 2A
Propeller Blade Section

The propeller action which perturbs the water from its state of uniform flow, assuming a linear theory, can be regarded as the superposition of two effects: 1) the effect of the blade pressure loading, $\mu(\vec{R}_1)$, and 2) the fluid-displacing effect of the blade thickness, $\bar{t}(\vec{R}_1)$. A partial differential equation governing the perturbation velocity potential will be derived by considering the blade loading and thickness effects separately and superimposing the results.

1. Blade Loading Effect

The fluid disturbance resulting from the propeller loading effect is representable using potential flow theory. The fluid perturbation pressure, p, can be expressed in terms of a velocity potential by the Bernoulli equation. For a fluid region free of singularities the velocity potential, Φ, satisfies a Laplace equation: $\nabla^2\Phi = 0$. If all products of the perturbation components of Φ appearing in the Bernoulli equation are discarded, then it can be shown that the pressure also satisfies the Laplace equation in the fluid volume.

In the case of the fluid region surrounding the propeller (V-P in Figure 1A) the velocity potential does not satisfy a Laplace equation. The propeller slip stream is known to be singular; the slip stream consists of vortex sheets shed from the trailing edges of the propeller blades. However, the fluid pressure must be continuous across the trailing vortex sheets; the pressure is not singular in the slip stream. Consequently, for a linearized potential flow situation outside the slip stream, which is assumed to exist, the pressure is harmonic everywhere in the fluid:

$$\nabla^2 p(\vec{R}, t) = 0 \quad \text{in the fluid volume V-P} \tag{2A}$$

and decays to zero at far distances from the propeller,

$$\lim_{|\vec{R}|\to\infty} p(\vec{R}, t) = 0 \tag{3A}$$

The solution of (2A) is, from Green's theorem,

$$p(\vec{R}, t) = \iint\limits_{S_1+S_2+S_3+P} \left(p \frac{\partial G}{\partial n} - G \frac{\partial p}{\partial n}\right) dS \tag{4A}$$

where S_1 , S_2 , S_3 are the surfaces bounding V at infinity, P is the bounding propeller blade surfaces (Figures 1A and 2A), and G satisfies the conditions,

$$\nabla^2 G = \delta(x - \xi)\ \delta(y - \eta)\ \delta(z - \zeta) \tag{5A}$$

$$\lim_{|\vec{R}| \to \infty} G = 0 \tag{6A}$$

The surface integrals over S_1 , S_2 , and S_3 in (4A) vanish so that,

$$p(\vec{R},\ t) = \iint\limits_{P(\vec{R}_1)} \left(p \frac{\partial G}{\partial n} - G \frac{\partial p}{\partial n}\right) dS \tag{7A}$$

Referring to Figure 2A, with the back of the blade denoted by "-" and the face by "+" , (7A) can be expressed as,

$$p(\vec{R},t) = \iint\limits_{P^+} \left(p^+ \frac{\partial G}{\partial n^+} - G \frac{\partial p^+}{\partial n^+}\right) dS - \iint\limits_{P^-} \left(p^- \frac{\partial G}{\partial n^-} - G \frac{\partial p^-}{\partial n^-}\right) dS$$

For a thin blade at small angle of attack with respect to the relative stream,

$$P^+ \simeq B \text{ and } \vec{n}^+ \simeq \vec{n}_p \quad \text{for } \vec{R}_1 = \vec{R}_1{}^+$$

$$P^- \simeq B \text{ and } \vec{n}^- \simeq \vec{n}_p \quad \text{for } \vec{R}_1 = \vec{R}_1{}^-$$

where B is the helical surface aligned with the relative stream whose unit normal is $\vec{n}_p$, (1A). Therefore $p(\vec{R},\ t)$ may be approximated as,

$$p(\vec{R},\ t) = \iint\limits_{B(\vec{R}_1)} (p^+ - p^-) \frac{\partial G}{\partial n_p} dS - \iint\limits_{B(\vec{R}_1)} \left(\frac{\partial p^+}{\partial n_p} - \frac{\partial p^-}{\partial n_p}\right) G\ dS \tag{8A}$$

where $\vec{R}_1$ is now the position vector to a point on the helical surface $B(\vec{R}_1)$. For an infinitesimally thin blade, the pressure and normal pressure gradient near the blade surface behave as depicted in Figure 3A.

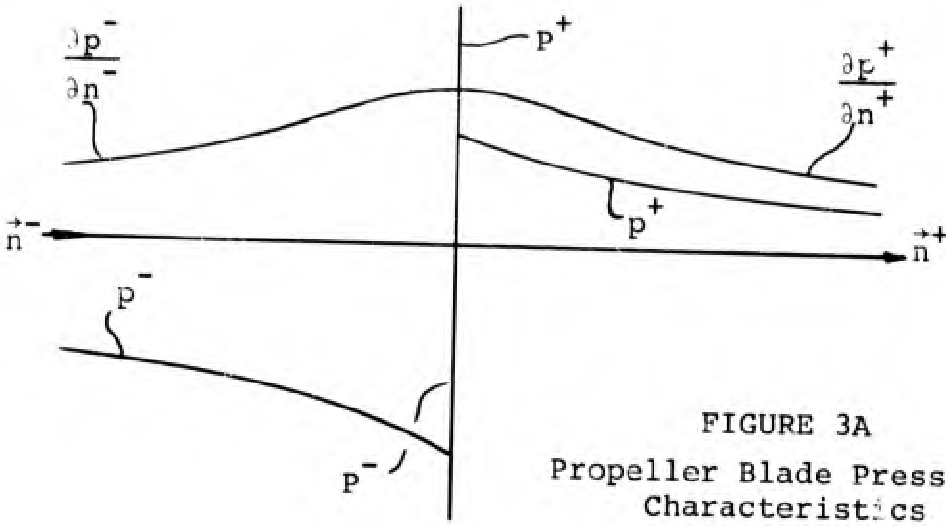


FIGURE 3A
Propeller Blade Pressure Characteristics

so that for a thin blade, $\frac{\partial p^+}{\partial n^+} \simeq \frac{\partial p^-}{\partial n^-}$ on P. Consequently, $\frac{\partial p^+}{\partial n_p} \simeq \frac{\partial p^-}{\partial n_p}$ on B and (8A) becomes,

$$p(\vec{R}, t) = \iint\limits_{B(\vec{R}_1)} (p^+ - p^-) \frac{\partial G}{\partial n_p} dS \tag{9A}$$

But $p^+ - p^- = |p^+| + |p^-|$ is the net blade load which has been defined as μ, a function presumed known. Therefore,

$$p(\vec{R}, t) = \iint\limits_{B(\vec{R}_1)} \mu(\vec{R}_1)(\vec{n}_p \cdot \vec{\nabla})\ G(\vec{R}; \vec{R}_1)\ dS \tag{10A}$$

This is the same form as stated by Tsakonas in reference 4; a Green's function satisfying (5A) and (6A) is just the source function $1/|\vec{R} - \vec{R}_1|$. Hence, the lightly loaded propeller blades are equivalent to helical sheets of pressure dipoles of strength density $\mu(\vec{R}_1)$.

As discussed, p may be related to the fluid velocity by means of a velocity potential function and the linearized Bernoulli equation. Define a potential function with respect to the fixed coordinate system (Figure 1A) as,

$$\Phi = Ux + \phi(\vec{R}, t) \tag{11A}$$

where Ux is the uniform stream potential and $\phi(R, t)$ is the potential corresponding to the perturbation of the uniform stream by the propeller. Then, by the Bernoulli equation,

$$p(\vec{R}, t) = -\rho \left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right) \phi(\vec{R}, t) \tag{12A}$$

For an N bladed propeller with the rotational speed ω , the pressure fluctuation at a point in the fluid can be decomposed into all multiples of blade rate frequency, $nN\omega$, $n = 1, 2, \ldots$. Therefore, p is representable as

$$p(\vec{R}, t) = \mathrm{Re} \sum_{n=0}^{\infty} p_n(\vec{R})\, e^{inN\omega t} \tag{13A}$$

Expand ϕ in the same way,

$$\phi(\vec{R}, t) = \mathrm{Re} \sum_{n=0}^{\infty} \bar{\phi}_n(\vec{R})\, e^{inN\omega t} \tag{14A}$$

where $\bar{\phi}_n$ can be complex. The amplitude of the n-th pressure harmonic can be expressed using (12A), (13A), and (14A) as,

$$p_n(\vec{R}) = -\rho \left(inN\omega + U \frac{\partial}{\partial x}\right) \bar{\phi}_n(\vec{R}) \tag{15A}$$

But from equation (10A) the n-th pressure harmonic can also be expressed as,

$$p_n(\vec{R}) = \frac{N\omega}{\pi} \int_{-\pi/N\omega}^{\pi/N\omega} e^{-inN\omega t} \iint_{B(\vec{R}_1)} \mu(\vec{R}_1)(\vec{n}_p \cdot \vec{\nabla})\, G(\vec{R};\, \vec{R}_1)\, dS\, dt \qquad (16A)$$

Now define S as the integral operator,

$$S = \frac{N\omega}{\pi} \int_{\frac{-\pi}{N\omega}}^{\frac{\pi}{N\omega}} e^{-inN\omega t} \iint_{B(\vec{R}_1)} dS\, dt$$

and equate (15A) and (16A),

$$\left(\frac{inN\omega}{U} + \frac{\partial}{\partial x}\right) \overline{\Phi}_n(\vec{R}) = -\, S\, \frac{\mu(R_1)}{\rho U} (\vec{n}_p \cdot \vec{\nabla})\, G(\vec{R};\, \vec{R}_1) \qquad (17A)$$

Take the Laplacian of (17A),

$$\left(\frac{inN\omega}{U} + \frac{\partial}{\partial x}\right) \nabla^2 \overline{\Phi}_n = -\, \frac{S\mu}{\rho U} (\vec{n}_p \cdot \vec{\nabla})\, \nabla^2 G \qquad (18A)$$

From (5A), $\nabla^2 G = \delta(\vec{R} - \vec{R}_1)$ so that

$$\left(\frac{inN\omega}{U} + \frac{\partial}{\partial x}\right) \nabla^2 \overline{\Phi}_n = -\, \frac{S\mu}{\rho U} (\vec{n}_p \cdot \vec{\nabla})\, \delta(\vec{R} - \vec{R}_1) \qquad (19A)$$

(19A) is a first order differential equation whose solution is,

$$\nabla^2 \overline{\Phi}_n = -\, \frac{S\mu}{\rho U} \int_{-\infty}^{x} e^{-inN\frac{\omega}{U}(x-\xi)} (\vec{n}_p \cdot \vec{\nabla})\, \delta(\xi - x_1)\, \delta(y - y_1)\, \delta(z - z_1)\, d\xi \qquad (20A)$$

(20A) is a Poisson equation for $\overline{\Phi}_n$, which applies in the whole space V. (20A) can be expressed in a more convenient form by making the variable change: $\xi = x + x_1 - \xi'$. Then,

$$\nabla^2 \Phi_n = -\frac{S\mu}{\rho U} \int_{x_1}^{\infty} e^{-inN\frac{\omega}{U}(\xi'-x_1)} (\vec{n}_P \cdot \vec{\nabla})\ \delta(x-\xi')\ \delta(y-y_1)\ \delta(z-z_1)\ d\xi' \tag{21A}$$

(21A) is the desired result for the blade loading contribution. This is just the propeller lifting system: the bound and trailing vortex system expressed as an integral over discrete singularities. The right hand side of (21A) represents the loaded propeller blade surfaces plus a rotational slip-stream consisting of helical surfaces of dipoles originating on the blades and being carried to infinity downstream.

2. Blade Thickness Effect

For the blade thickness effect, the thickness distribution, $\bar{t}(\vec{R}_1)$ (Figure 2A) is placed symmetrically about the helical surface B, as shown in Figure 4A,

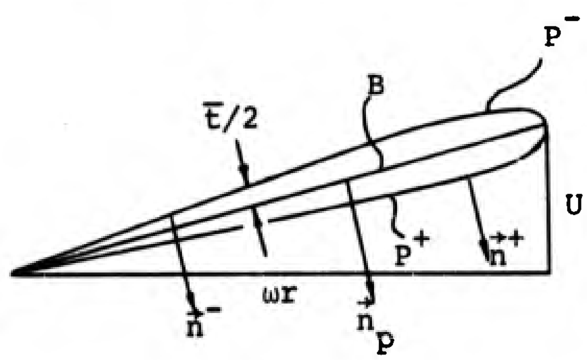


FIGURE 4A

Equivalent Blade Section - Thickness Effect

In this case the fluid region V-P is free of singularities so that,

$$\nabla^2 \phi(\vec{R}, t) = 0 \qquad \text{in the fluid volume V-P} \tag{22A}$$

Again using Green's theorem, ϕ can be expressed in terms of an integral over P,

$$\phi(\vec{R}, t) = \iint_P \left(\phi \frac{\partial G}{\partial n} - G \frac{\partial \phi}{\partial n}\right) ds \tag{23A}$$

with the Green's function satisfying, $\nabla^2 G = \delta(\vec{R} - \vec{R}_1)$ (24A)

and $\lim_{|\vec{R}| \to \infty} G = 0$ (25A)

With only the requirements of (24A) and (25A), G is the same as in the blade loading case; G is the source function $1/|\vec{R} - \vec{R}_1|$. (23A) can be written,

$$\phi(\vec{R}, t) = \iint_{P^+} \left(\phi^+ \frac{\partial G}{\partial n^+} - G \frac{\partial \phi^+}{\partial n^+}\right) dS - \iint_{P^-} \left(\phi^- \frac{\partial G}{\partial n^-} - G \frac{\partial \phi^-}{\partial n^-}\right) dS$$

Again for a thin blade,

$$P^+ \simeq B \text{ and } \vec{n}^+ \simeq \vec{n}_p \quad \text{for } \vec{R}_1 = \vec{R}_1^{\,+}$$

$$P^- \simeq B \text{ and } \vec{n}^- \simeq \vec{n}_p \quad \text{for } \vec{R}_1 = \vec{R}_1^{\,-}$$

Therefore,

$$\phi(\vec{R}, t) = \iint_{B(\vec{R}_1)} (\phi^+ - \phi^-) \frac{\partial G}{\partial n_p} dS - \iint_{B(\vec{R}_1)} \left(\frac{\partial \phi^+}{\partial n_p} - \frac{\partial \phi^-}{\partial n_p}\right) G \, dS$$

In this case, for the symmetrical section,

$$\phi^+ \simeq \phi^- \quad \text{and} \quad \frac{\partial \phi^-}{\partial n_p} \simeq -\frac{\partial \phi^+}{\partial n_p} \quad \text{as} \quad P^\pm \to B$$

Define

$$\sigma(\vec{R}_1) = 2 \frac{\partial \phi^+}{\partial n_p} \qquad (27A)$$

This produces the perturbation potential of a symmetric thin body in the well known form,

$$\phi(\vec{R}, t) = -\iint_{B(\vec{R}_1)} \sigma(\vec{R}_1) \, G(\vec{R}; \vec{R}_1) \, dS \qquad (28A)$$

Therefore, for the thickness effect, the thin propeller blades are equivalent to helical sheets of sources of strength density $\sigma(\vec{R}_1)$. σ can be obtained in the usual way using thin body theory [18].

As with the pressure, at a fixed position in the water the fluctuating fluid velocity is composed of harmonics corresponding to multiples of propeller blade-rate frequency; ϕ is expressible in the series,

$$\phi(\vec{R}, t) = \text{Re} \sum_{n=0}^{\infty} \overline{\Phi}_n e^{inN\omega t} \tag{29A}$$

The n-th harmonic of ϕ , from (28A) and (29A) is,

$$\overline{\Phi}_n(\vec{R}) = -\frac{N\omega}{\pi} \int_{\frac{\pi}{N\omega}}^{\frac{\pi}{N\omega}} e^{-inN\omega t} \iint_{B(\vec{R}_1)} \sigma(\vec{R}_1)\, G(\vec{R};\, \vec{R}_1)\, dS\, dt$$

The integrations have been previously expressed by the integral operator S . Using the same notation,

$$\overline{\Phi}_n(\vec{R}) = -S\sigma(\vec{R}_1)\, G(\vec{R};\, \vec{R}_1) \tag{30A}$$

Taking the Laplacian,

$$\nabla^2\overline{\Phi}_n(\vec{R}) = -S\sigma\, \delta(x - x_1)\, \delta(y - y_1)\, \delta(z - z_1) \tag{31A}$$

(31A) is valid in the whole space V .

(31A) is the differential equation governing the fluid disturbance potential due to the blade thickness effect, the equivalent of (21A) for loading effect.

Superimposing (21A) and (31A) the final result is obtained,

$$\nabla^2\overline{\Phi}_n = -\frac{S\mu}{\rho U} \int_{x_1}^{\infty} e^{-inN\frac{\omega}{U}(\xi - x_1)} (\vec{n}_p \cdot \vec{\nabla})\, \delta(x - \xi)\, \delta(y - y_1)\, \delta(z - z_1)\, d\xi - S\sigma\, \delta(x - x_1)\, \delta(y - y_1)\, \delta(z - z_1) \tag{32A}$$

APPENDIX B

APPROXIMATE SOLUTIONS FOR A SLENDER BODY EXECUTING LATERAL PERIODIC MOTION IN AN INFINITE FLUID

Consider a slender body translating through an ideal incompressible fluid of infinite expanse while executing periodic, vertical, small amplitude motion, as shown in Figure 1B.

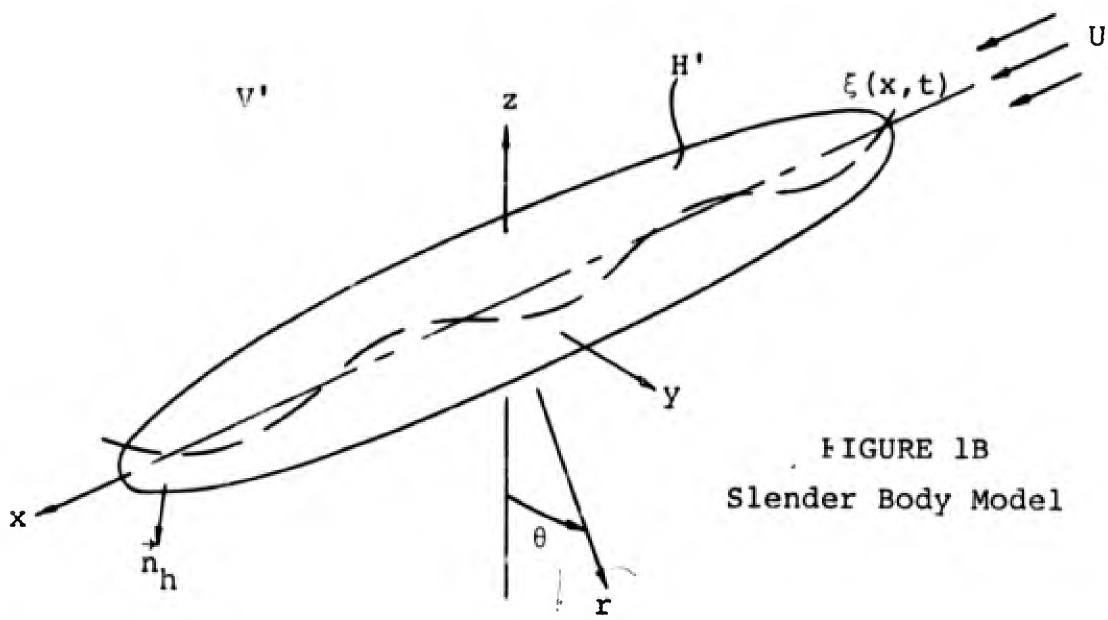


FIGURE 1B
Slender Body Model

Assuming a potential flow situation, the amplitude of the periodic fluid response can be defined by a potential function, denoted as H_i, which is the solution to the following boundary value problem:

$$\nabla^2 H_i = 0 \qquad \text{in the fluid volume, } V'$$

$$\frac{\partial H_i}{\partial n_h} = V(x)(\vec{n}_h \cdot \vec{k}) \qquad \text{on the mean position of the body surface, } H' \tag{1B}$$

$$\lim_{|\vec{R}| \to \infty} H_i = 0$$

This is the formulation (39) from the text. $V(x)$ is the amplitude of the unsteady vertical component of body surface velocity. $V(x)$ depends on the body motion, both translation and vibration (see text), and it is assumed to be known.

The fluid behavior in the region of the body ends is of primary interest. The potential function H_i and the associated velocity components can be evaluated in the end region using slender body theory. First order slender body theory formulated in terms of rectangular or cylindrical coordinates produces "strip" formulas which apply in the two-dimensional planes of the hull sections. Such strip formulas are accurate in some ship hydrodynamic applications; strip theory is an accepted tool in ship motions analysis. However, conventional strip theories cannot predict the flow near the ends of a slender body at all accurately. In reference 24 Tuck generalizes the "strip" theory idea by formulating the problem of a slender body of revolution traveling through an infinite fluid in terms of prolate spheroidal coordinates. A "strip" theory is obtained in the sense that the first order slender body problem is two-dimensional in terms of the curvilinear coordinates. In [24] Tuck shows that the body end effect problem inherent in the cylindrical strip theory is eliminated by the switch to spheroidal coordinates. This suggests that a special coordinate formulation of slender body theory might provide a simple means of obtaining the solution of (1B) while at the same time establishing an accuracy level, near the body ends, which is far above that obtainable with conventional slender body strip solutions.

1. Procedure

The applicability of slender body theory for predicting H_i and $\vec{\nabla} H_i$ in the stern region of the ship hull will be evaluated by studying a slender body of revolution vibrating laterally in an infinite fluid. The boundary value problem for the body of revolution, with ϕ denoting the potential function, is essentially the same as (1B),

$$\nabla^2\phi = 0 \qquad \text{in the fluid}$$

$$\frac{\partial\phi}{\partial n} = V(x)(\vec{n}\cdot\vec{k}) \qquad \text{on the mean position of the body surface} \qquad (1B')$$

$$\lim_{|\vec{R}|\to\infty} \phi = 0$$

where $\vec{n}$ is the unit normal vector to the surface of the body of revolution.

The procedure to be followed in the evaluation will be:

a) Derive the "strip" formulas for ϕ and $\vec{\nabla}\phi$, equations (1B'), in three different coordinate systems:

   i) cylindrical
   ii) spheroidal
   iii) paraboloidal

   The cylindrical system is not expected to be useful.

b) Compare numerical evaluations of the formulas with corresponding exact solutions for a spheroid.

It should be kept in mind that if a Laplace equation governs the two-dimensional "strip" problem formulation, then a great deal of flexibility is provided by the existence of a conformal transformation. In such a case the formulas developed for the body of revolution can be mapped from the circular sections to general irregular hull-type sections by way of conformal mapping.

2. Derivations

The following is a derivation of the formulas for ϕ and $\vec{\nabla}\phi$ using slender body theory formulated in three different coordinate systems: cylindrical, spheroidal, and paraboloidal.

Slender body solutions are obtained in a formal way by the method of matched asymptotic expansions, as outlined, for example, by Ogilvie in reference 26. A generally non-unique near-field solution is developed by viewing the body at very close range. This solution is then made unique by closing the field with a condition obtained from a far-field description of the problem. In the case of lateral motion, the closing condition, to the first order, is effectively just a radiation condition. In the following derivation a radiation condition is carried with the near-field development; the far field solutions are not shown.

The derivations in the three coordinate systems are as follows:

1) Cylindrical Coordinates

The coordinates are (r, θ, x) , as defined in Figure 1B, and the body surface is defined by the function, $r-r_o(x) = 0$. With $y = r \sin \theta$, and $z = -r \cos \theta$, the problem formulation, equations (1B'), in cylindrical coordinates, becomes,

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial x^2} = 0 \qquad \text{in the fluid}$$

$$\frac{\partial \phi}{\partial r} - \frac{\partial \phi}{\partial x}\frac{dr_o}{dx} = -V(x) \cos \theta \qquad \text{on } r = r_o(x) \qquad (2B)$$

$$\lim_{|\vec{R}| \to \infty} \phi = 0$$

For the slender body, derivatives with respect to x are assumed to be small. Therefore, to first order,

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0 \qquad r > r_o(x)$$

$$\frac{\partial \phi}{\partial r} = -V(x) \cos \theta \qquad r = r_o(x) \tag{3B}$$

$$\lim_{r \to \infty} \phi = 0$$

Note that the differential equation is the Laplace equation in the 2-D $(r - \theta)$ plane. This system, (3B), gives the solution,

$$\phi = \frac{V(x)\ r_o^{\,2}(x)}{r} \cdot \cos \theta \tag{4B}$$

The radial, axial, and tangential velocity components of the vector $\vec{\nabla}\phi$ are then,

$$u_r = -\frac{V(x)\ r_o^{\,2}(x)}{r^2} \cos \theta$$

$$u_x = \frac{\cos \theta}{r} \frac{d}{dx} [V(x)\ r_o^{\,2}(x)] \tag{5B}$$

$$u_\theta = -\frac{V(x) r_o^{\,2}(x)}{r^2} \sin \theta$$

The solution (4B) and (5B) will later be referred to as the "Cyl" solution.

2) Spheroidal Coordinates

Transform the problem to the prolate spheroidal coordinate system (ζ, μ, θ) by,

$$x = \ell\mu \sqrt{\zeta^2 + 1}\ , \quad r = \ell\zeta \sqrt{1 - \mu^2}\ , \quad \theta = \theta \tag{6B}$$

Referring to Figure 2B, surfaces defined by ζ = constant are spheroids with foci at $x = \pm\ell$. $\zeta = 0$ is the x-axis between $\pm\ell$, and $\zeta = \infty$ is the spherical surface at ∞ . Surfaces μ = constant are hyperboloids with foci at $x = \pm\ell$. $\mu = 1$ corresponds to the x-axis from $x = \ell$ to $x = \infty$.

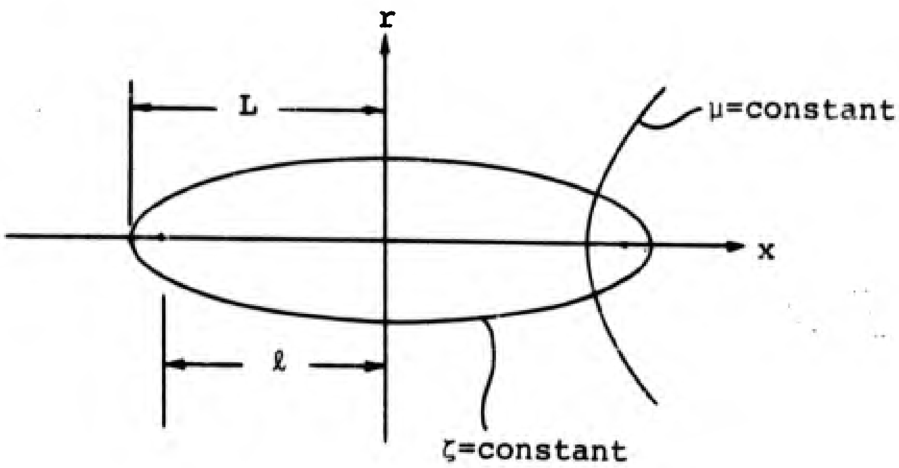


FIGURE 2B
Spheroidal Coordinate System

$\mu = -1$ corresponds to the x-axis from $x = -\ell$ to $x = -\infty$. $\mu = 0$ is the plane $x = 0$. The ranges of the variables are,

$$0 \leq \zeta < \infty \ , \quad -1 \leq \mu \leq 1 \ , \quad 0 \leq \theta \leq 2\pi$$

(ζ, μ, θ) is an orthogonal system so that constant ζ and constant μ surfaces intersect normally.

On a spheroidal surface $\zeta = \zeta_0$, $x = \ell\mu\sqrt{\zeta_0^{\,2} + 1}$ by (6B). Define the half-length of the spheroid as L , Figure 2B. Then, at $x = L$, $\mu = 1$ so that,

$$L = \ell\sqrt{\zeta_0^{\,2} + 1} \ , \qquad \ell = \frac{L}{\sqrt{\zeta_0^{\,2} + 1}}$$

Therefore, on the surface $\zeta = \zeta_0$, $\mu = x/L$. Then,

$$r_0 = \zeta_0 L \sqrt{\frac{1 - \mu^2}{\zeta_0^{\,2} + 1}} \qquad (7B)$$

At $\mu = 0$, $r_0 = B/2$, where B is the beam. For ζ_0 small, by (7B), $\zeta_0 \sim B/2L$, the beam-length ratio.

Inverse relationships for ζ and μ in terms of r and x are:

$$\zeta = \left\{\frac{1}{2}\left[\frac{1}{\ell^2}(r^2 + x^2) - 1 + \sqrt{\left(1 - \frac{r^2}{\ell^2} - \frac{x^2}{\ell^2}\right)^2 + 4\frac{r^2}{\ell^2}}\right]\right\}^{\frac{1}{2}}$$

$$\mu = \frac{x}{\ell}\left\{\frac{1}{2}\left[\frac{1}{\ell^2}(r^2 + x^2) + 1 + \sqrt{\left(1 - \frac{r^2}{\ell^2} - \frac{x^2}{\ell^2}\right)^2 + 4\frac{r^2}{\ell^2}}\right]\right\}^{-\frac{1}{2}} \qquad (8B)$$

In the (ζ, μ, θ) system, the Laplace equation (1B') takes the form,

$$\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial\phi}{\partial\mu}\right] + (\zeta^2+1)\frac{\partial^2\phi}{\partial\zeta^2} + \frac{1}{\zeta}(2\zeta^2+1)\frac{\partial\phi}{\partial\zeta} + \frac{\zeta^2+(1-\mu^2)}{\zeta^2(1-\mu^2)}\frac{\partial^2\phi}{\partial\theta^2} = 0 \qquad (9B)$$

and the body boundary condition (1B') is,

$$\frac{\partial\phi}{\partial\zeta} - \Gamma\frac{d\zeta_0}{d\mu}\frac{\partial\phi}{\partial\mu} = -V(\mu)\ell\left(\sqrt{1-\mu^2} - \Gamma\frac{d\zeta_0}{d\mu}\frac{\zeta\mu}{\sqrt{1-\mu^2}}\right)\cos\theta \quad \text{on } \zeta=\zeta_0(\mu) \qquad (10B)$$

where $$\Gamma = \frac{\dfrac{\mu^2\zeta^2}{\zeta^2+1} + 1 - \mu^2}{\dfrac{\mu^2\zeta^2}{1-\mu^2} + 1 + \zeta^2}$$

Again assuming a slender body where derivatives with respect to μ are considered to be small, (9B) and (10B) become,

$$(\zeta^2+1)\frac{\partial^2\phi}{\partial\zeta^2} + \frac{1}{\zeta}(2\zeta^2+1)\frac{\partial\phi}{\partial\zeta} + \frac{\zeta^2+(1-\mu^2)}{\zeta^2(1-\mu^2)}\frac{\partial^2\phi}{\partial\theta^2} = 0 \qquad \zeta>\zeta_0(\mu)$$

$$\frac{\partial\phi}{\partial\zeta} = -V(\mu)\ell\sqrt{1-\mu^2}\cos\theta \qquad \zeta=\zeta_0(\mu)$$

Since the fluid behavior near the body is of interest, ζ can also be considered small, giving to first order,

$$\frac{\partial^2 \phi}{\partial \zeta^2} + \frac{1}{\zeta}\frac{\partial \phi}{\partial \zeta} + \frac{1}{\zeta^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{1-\mu^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0 \qquad \zeta > \zeta_o(\mu) \qquad (11B)$$

$$\frac{\partial \phi}{\partial \zeta} = -V(\mu)\ell\sqrt{1-\mu^2}\cos\theta \qquad \zeta = \zeta_o(\mu) \qquad (12B)$$

$$\lim_{\zeta \to \infty} \phi = 0 \qquad (13B)$$

(11B) is a 2-D partial differential equation in the $\zeta - \theta$ plane; it is not a 2-D Laplace equation. The first three terms in (11B), however, are the 2-D Laplacian in the polar variables $\zeta - \theta$, just like equation (3B), which is in $r - \theta$. For small μ , near midship, the first three terms in (11B) dominate. But interest is in the fluid behavior near the end of the body. At the end, $\mu = 1$, and the last term could become quite large, depending on the behavior of $\frac{\partial^2 \phi}{\partial \theta^2}$ as $\mu \to 1$. Although seemingly unjustifiable in a mathematical sense, it is tempting to just discard the last term in (11B) to give the Laplace equation needed to allow the conformal transformation; this will be done.

The solution of (11B), involving Bessel functions, will also be developed. It would be expected that the solution of the Laplace equation would approach the solution of (11B) near midship but would be less accurate near the ends.

i) Bessel Solution (11B)

$$\frac{\partial^2 \phi}{\partial \zeta^2} + \frac{1}{\zeta}\frac{\partial \phi}{\partial \zeta} + \frac{1}{\zeta^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{1-\mu^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0 \qquad \zeta > \zeta_o(\mu) \qquad (11B)$$

$$\frac{\partial \phi}{\partial \zeta} = -V(\mu)\ell\sqrt{1-\mu^2}\cos\theta \qquad \zeta = \zeta_o(\mu) \qquad (12B)$$

$$\lim_{\zeta \to \infty} \phi = 0 \qquad (13B)$$

This problem has the solution,

$$\phi = \frac{2V(\mu)\ell\ (1 - \mu^2)\ K_1\left(\zeta/\sqrt{1 - \mu^2}\right)}{K_o\left(\zeta_o/\sqrt{1 - \mu^2}\right) + K_2\left(\zeta_o/\sqrt{1 - \mu^2}\right)} \cos\theta \qquad (14B)$$

where the K are the modified Bessel functions of the second kind.

The velocity components, $\vec{\nabla}\phi$ are:

$$u_r = \frac{\sqrt{1 - \mu^2}}{\ell(\zeta^2 + 1 - \mu^2)} \left[(\zeta^2 + 1)\frac{\partial\phi}{\partial\zeta} - \mu\zeta\,\frac{\partial\phi}{\partial\mu}\right]$$

$$u_x = \frac{\sqrt{\zeta^2 + 1}}{\ell(\zeta^2 + 1 - \mu^2)} \left[\mu\zeta\,\frac{\partial\phi}{\partial\zeta} + (1 - \mu^2)\,\frac{\partial\phi}{\partial\mu}\right] \qquad (15B)$$

$$u_\theta = \frac{1}{\ell\zeta\sqrt{1 - \mu^2}}\,\frac{\partial\phi}{\partial\theta}$$

where,

$$\frac{\partial\phi}{\partial\zeta} = -V(\mu)\ell\ \sqrt{1 - \mu^2}\ \frac{K_o(\gamma) + K_2(\gamma)}{K_o(\gamma_o) + K_2(\gamma_o)}\ \cos\theta$$

$$\frac{\partial\phi}{\partial\mu} = -2\ell\ \cos\theta \left[\frac{V(\mu)(1 - \mu^2)}{2\left[K_o(\gamma_o) + K_2(\gamma_o)\right]^2} \left\{[K_o(\gamma_o) + K_2(\gamma_o)]\right.\right.$$

$$\left.[K_o(\gamma) + K_2(\gamma)]\frac{d\gamma}{d\mu} - K_1(\gamma)\,[3K_1(\gamma_o) + K_3(\gamma_o)]\,\frac{d\gamma_o}{d\mu}\right\}$$

$$\left. + \frac{K_1(\gamma)}{[K_o(\gamma_o) + K_2(\gamma_o)]} \left\{2\mu V(\mu) - (1 - \mu^2)\,\frac{dV(\mu)}{d\mu}\right\}\right]$$

$$\frac{\partial\phi}{\partial\theta} = -\ \frac{2V(\mu)\ell\ (1 - \mu^2)\ K_1(\gamma)}{K_o(\gamma_o) + K_2(\gamma_o)}\ \sin\theta$$

and $\gamma = \frac{\zeta}{\sqrt{1 - \mu^2}}$, $\gamma_o = \frac{\zeta_o(\mu)}{\sqrt{1 - \mu^2}}$

This solution, (14B) and (15B), will later be referred to as the "Spher B" solution.

ii) Laplace Solution

Here the last term in the first order equation, (11B) is dropped, giving,

$$\frac{\partial^2\phi}{\partial\zeta^2} + \frac{1}{\zeta}\frac{\partial\phi}{\partial\zeta} + \frac{1}{\zeta^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 \qquad \zeta > \zeta_o(\mu) \qquad (11B')$$

$$\frac{\partial\phi}{\partial\zeta} = -V(\mu)\ell\sqrt{1 - \mu^2}\cos\theta \qquad \zeta = \zeta_o(\mu) \qquad (12B)$$

$$\lim_{\zeta\to\infty} \phi = 0 \qquad (13B)$$

whose solution is,

$$\phi = \frac{V(\mu)\ell\,\zeta_o^{\,2}(\mu)\sqrt{1 - \mu^2}}{\zeta}\cos\theta \qquad (16B)$$

with velocity components,

$$u_r = \frac{\sqrt{1 - \mu^2}}{\ell(\zeta^2 + 1 - \mu^2)}\left[(\zeta^2 + 1)\frac{\partial\phi}{\partial\zeta} - \mu\zeta\frac{\partial\phi}{\partial\mu}\right]$$

$$u_x = \frac{\sqrt{\zeta^2 + 1}}{\ell(\zeta^2 + 1 - \mu^2)}\left[\mu\zeta\frac{\partial\phi}{\partial\zeta} + (1 - \mu^2)\frac{\partial\phi}{\partial\mu}\right] \qquad (17B)$$

$$u_\theta = \frac{1}{\ell\zeta\sqrt{1 - \mu^2}}\frac{\partial\phi}{\partial\theta}$$

as before. Now,

$$\frac{\partial\phi}{\partial\zeta} = -\frac{V(\mu)\ell\,\zeta_o^{\,2}(\mu)\sqrt{1 - \mu^2}}{\zeta^2}\cos\theta$$

$$\frac{\partial\phi}{\partial\mu} = \frac{\ell \cos\theta}{\zeta}\frac{d}{d\mu}\left[V(\mu)\ \zeta_o^{\,2}(\mu)\ \sqrt{1-\mu^2}\right]$$

$$\frac{\partial\phi}{\partial\theta} = -\frac{V(\mu)\,\ell\ \zeta_o^{\,2}(\mu)\ \sqrt{1-\mu^2}}{\zeta}\sin\theta$$

This solution, (16B) and (17B), will later be referred to as the "Spher L" solution.

3) Paraboloidal Coordinates

Transform the (r, x, θ) problem to the paraboloidal coordinate system (ξ, η, θ) as,

$$x = a(\xi^2 - \eta^2)\ , \qquad r = 2a\xi\eta\ ,\ \theta = \theta$$

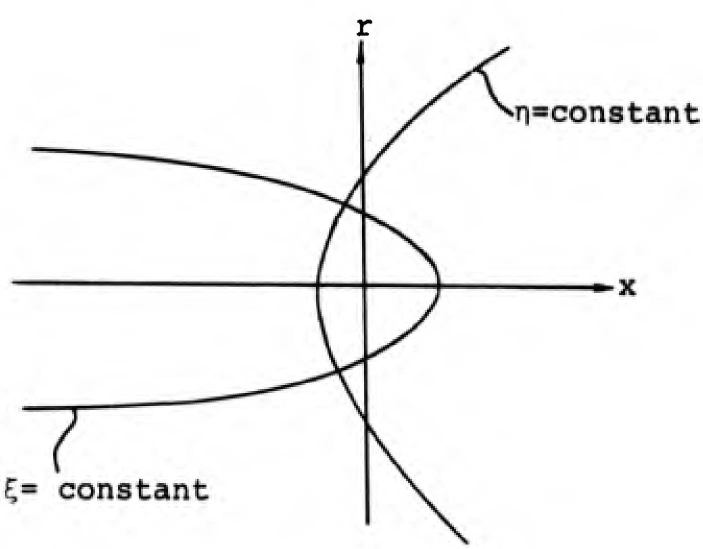


FIGURE 3B
Paraboloidal Coordinate System

Surfaces defined by ξ = constant and η = constant are both paraboloids with common focus at the origin, as indicated in Figure 3B. $0 \leq \xi < \infty$ and $0 \leq \eta < \infty$ so that $\xi = 0$ and

$\eta = 0$ correspond to the negative and positive x-axes, respectively. $\xi = \infty$ is the plane $x = \infty$ and $\eta = \infty$ is the plane $x = -\infty$. (ξ, η, θ) is an orthogonal system so that ξ and η surfaces intersect normally.

The inverse relationships for ξ and η in terms of r and x are:

$$\xi = \sqrt{\frac{1}{2a}\left[x + (x^2 + r^2)^{1/2}\right]}$$

$$\eta = \frac{r}{\sqrt{2a\left[x + (x^2 + r^2)^{1/2}\right]}} \tag{18B}$$

In this system the Laplace equation (1B') is,

$$\frac{\partial^2\phi}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial\phi}{\partial\xi} + \frac{\partial^2\phi}{\partial\eta^2} + \frac{1}{\eta}\frac{\partial\phi}{\partial\eta} + \frac{\xi^2 + \eta^2}{\xi^2\eta^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 \quad \text{in the fluid} \tag{19B}$$

and the body boundary condition (1B') is,

$$\frac{\partial\phi}{\partial\xi} - \frac{\partial\phi}{\partial\eta}\frac{d\xi_o}{d\eta} = -2V(\eta)a\left(\eta - \xi\frac{d\xi_o}{d\eta}\right)\cos\theta \qquad \xi = \xi_o(\eta) \tag{20B}$$

Again for the slender body, the equation and boundary condition, plus radiation condition, are approximately,

$$\frac{\partial^2\phi}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial\phi}{\partial\xi} + \frac{1}{\xi^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{1}{\eta^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 \qquad \xi > \xi_o(\eta) \tag{21B}$$

$$\frac{\partial\phi}{\partial\xi} = -\,2V(\eta)\ a\eta\cos\theta \qquad \xi = \xi_o(\eta) \tag{22B}$$

$$\lim_{\xi\to\infty}\phi = 0 \tag{23B}$$

As with the spheroidal formulation the slender body assumptions produce a two-dimensional problem in the $\xi - \theta$ plane. The behavior with regard to the last term in the differential equation (21B) is the same as before; it is higher order for large η, away from the nose of a paraboloid, but appears to gain in importance near the nose, $\eta = 0$, where the coefficient is singular. Therefore, as before, two solutions can be written down, a solution in terms of Bessel functions from (21B) and a solution of a Laplace equation, obtained by discarding the last term in (21B). The Laplace solution of the system (21B), (22B), and (23B) is,

$$\phi = \frac{2aV(\eta)\ \eta\xi_o^{\,2}(\eta)}{\xi} \cos\theta \qquad (24B)$$

with velocity components,

$$u_r = \frac{1}{2a(\xi^2+\eta^2)}\left[\eta\frac{\partial\phi}{\partial\xi} + \xi\frac{\partial\phi}{\partial\eta}\right]$$

$$u_x = \frac{1}{2a(\xi^2+\eta^2)}\left[\xi\frac{\partial\phi}{\partial\xi} - \eta\frac{\partial\phi}{\partial\eta}\right] \qquad (25B)$$

$$u_\theta = \frac{1}{2a\xi\eta}\frac{\partial\phi}{\partial\theta}$$

where,

$$\frac{\partial\phi}{\partial\xi} = -\frac{2aV(\eta)\ \eta\xi_o^{\,2}(\eta)}{\xi^2}\cos\theta$$

$$\frac{\partial\phi}{\partial\eta} = \frac{2a\cos\theta}{\xi}\frac{d}{d\eta}\left[V(\eta)\ \eta\xi_o^{\,2}(\eta)\right]$$

$$\frac{\partial\phi}{\partial\theta} = -\frac{2aV(\eta)\ \eta\xi_o^{\,2}(\eta)}{\xi}\sin\theta$$

This solution, (24B) and (25B), will later be referred to as the "Para L" solution. The Bessel solution of (21B) is not developed in paraboloidal coordinates.

3. Discussion

Formulas for predicting the fluid behavior resulting from a slender body of revolution vibrating vertically in an infinite fluid are developed in the preceding section. The formulas, giving the fluid velocity potential and velocity components, are developed in three different coordinate systems: cylindrical, spheroidal, and paraboloidal.

The value of these formulas should be judged by comparison with some exact solution. The formulas developed are valid for any vertical mode of body motion for any slender body of revolution. One specialization of the formulas is to a heaving spheroid. For the case of a heaving spheroid $\zeta_o(\mu) = \zeta_o$ = constant and $V(\mu) = V$ = constant. The solution to this problem is written down in Lamb [27]. It is,

$$\frac{\phi}{V\ell \cos\theta} = -\zeta c\sqrt{1-\mu^2}\left(\frac{1}{2}\ln\frac{\alpha+1}{\alpha-1} - \frac{\alpha}{\zeta^2}\right) \qquad (26B)$$

where, $$\frac{1}{c} = \frac{1}{2}\ln\frac{\alpha_o+1}{\alpha_o-1} - \frac{\zeta_o^2-1}{\zeta_o^2\alpha_o}$$

and, $$\alpha = \sqrt{\zeta^2+1}\,, \qquad \alpha_o = \sqrt{\zeta_o^2+1}$$

The velocity components are:

$$\frac{u_r}{V\cos\theta} = \frac{c}{\zeta^2+1-\mu^2}\left\{\left[\frac{\alpha}{\zeta^2} - \frac{1}{2}\ln\frac{\alpha+1}{\alpha-1}\right]\left[(1-\mu^2)\,\alpha^2 + \zeta^2\mu^2\right] + \frac{2\alpha(1-\mu^2)(\zeta^2-1)}{\zeta^2}\right\}$$

$$\frac{u_x}{V\cos\theta} = \frac{2c\mu\,(\zeta^2-1)\sqrt{1-\mu^2}}{\zeta^2+1-\mu^2} \qquad (27B)$$

$$\frac{u_\theta}{V \sin \theta} = c \left(\frac{1}{2} \ln \frac{\alpha + 1}{\alpha - 1} - \frac{\alpha}{\zeta^2}\right)$$

In the same non-dimensional form the approximate solutions are:

1) Cylindrical, (4B) and (5B)

$$\frac{\phi}{V\ell \cos \theta} = \frac{r_o^2(x)}{\ell r}$$

$$\frac{u_r}{V \cos \theta} = - \frac{r_o^2(x)}{r^2}$$

$$\frac{u_x}{V \cos \theta} = \frac{2r_o(x)\ r_o'(x)}{r} \qquad (28B)$$

$$\frac{u_\theta}{V \sin \theta} = - \frac{r_o^2(x)}{r^2}$$

2) Spheroidal

i) Spher B, (14B) and (15B)

$$\frac{\phi}{V\ell \cos \theta} = \frac{2(1 - \mu^2)\ K_1(\zeta/\sqrt{1 - \mu^2})}{K_o(\zeta_o/\sqrt{1 - \mu^2}) + K_2(\zeta_o/\sqrt{1 - \mu^2})}$$

$$\frac{u_r}{V \cos \theta} = \frac{\sqrt{1 - \mu^2}}{\zeta^2 + 1 - \mu^2} \left[(\zeta^2 + 1) \frac{\partial \phi}{\partial \zeta} - \mu\zeta \frac{\partial \phi}{\partial \mu}\right]$$

$$\frac{u_x}{V \cos \theta} = \frac{\sqrt{\zeta^2 + 1}}{\zeta^2 + 1 - \mu^2} \left[\mu\zeta \frac{\partial \phi}{\partial \zeta} + (1 - \mu^2) \frac{\partial \phi}{\partial \mu}\right] \qquad (29B)$$

$$\frac{u_\theta}{V \sin \theta} = - \frac{2\sqrt{1 - \mu^2}}{\zeta} \frac{K_1(\zeta/\sqrt{1 - \mu^2})}{K_o(\zeta_o/\sqrt{1 - \mu^2}) + K_2(\zeta_o/\sqrt{1 - \mu^2})}$$

where,

$$\frac{\partial \bar{\phi}}{\partial \zeta} = \sqrt{1 - \mu^2} \; \frac{K_o(\gamma) + K_2(\gamma)}{K_o(\gamma_o) + K_o(\gamma_o)}$$

$$\frac{\partial \bar{\phi}}{\partial \mu} = - \frac{(1 - \mu^2)}{[K_o(\gamma_o) + K_2(\gamma_o)]^2} \Big\{ [K_o(\gamma_o) + K_2(\gamma_o)] [K_o(\gamma) + K_2(\gamma)] \frac{d\gamma}{d\mu}$$

$$- K_1(\gamma) [3K_1(\gamma_o) + K_3(\gamma_o)] \frac{d\gamma_o}{d\mu} \Big\} - \frac{4\mu \; K_1(\gamma)}{K_o(\gamma_o) + K_2(\gamma_o)}$$

$$\gamma = \frac{\zeta}{\sqrt{1 - \mu^2}} \; , \qquad \gamma_o = \frac{\zeta_o}{\sqrt{1 - \mu^2}}$$

ii) Spher L, (16B) and (17B)

$$\frac{\phi}{V\ell \cos\theta} = \frac{\zeta_o^{\,2}\sqrt{1 - \mu^2}}{\zeta}$$

$$\frac{u_r}{V \cos\theta} = - \frac{\zeta_o^{\,2}}{\zeta^2} \; \frac{1}{\zeta^2 + 1 - \mu^2} \; [\zeta^2(1 - 2\mu^2) + (1 - \mu^2)]$$

(30B)

$$\frac{u_x}{V \cos\theta} = - \frac{2\mu \; \zeta_o^{\,2}\sqrt{1 - \mu^2}}{\zeta} \; \frac{\sqrt{\zeta^2 + 1}}{\zeta^2 + 1 - \mu^2}$$

$$\frac{u_\theta}{V \sin\theta} = - \frac{\zeta_o^{\,2}}{\zeta^2}$$

3) Paraboloidal

The question arises here as to how the paraboloidal formulas are to be applied to the spheroid. This is an important question because to apply either the spheroidal or paraboloidal solutions, a general body must be describable in terms of the coordinates.

Consider applying the paraboloidal formulas at a station $x = x_0$ on some body of revolution (Figure 4B).

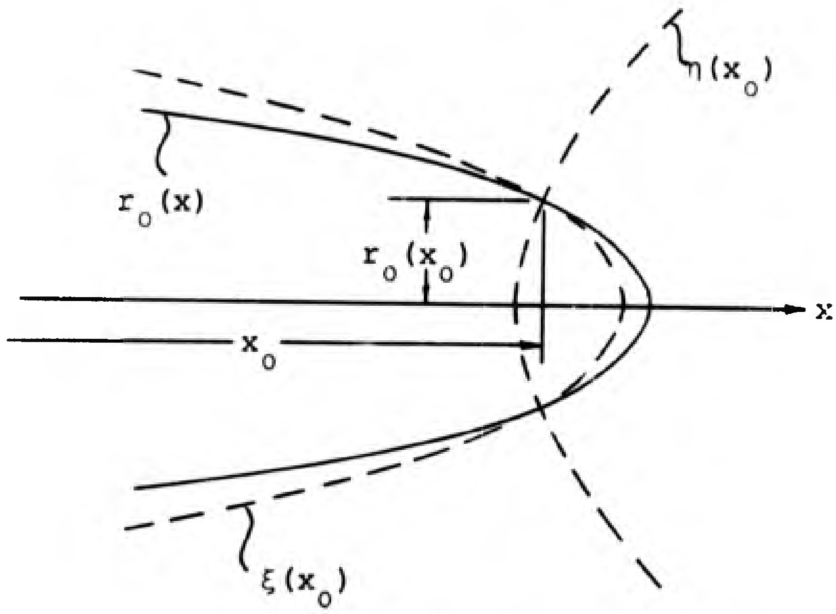


FIGURE 4B
Coordinate Fitting

If it is required that the paraboloid match both the radius and slope of the body at $x = x_0$, then,

$$\xi_0(x_0) = \sqrt{\frac{r_0(x_0)\,|r_0{}'(x_0)|}{2a}}$$

$$\eta(x_0) = \sqrt{\frac{r_0(x_0)}{2a\,|r_0{}'(x_0)|}} \tag{31B}$$

Selecting the curvilinear body coordinates on the basis of equivalent radius and slope is somewhat arbitrary. Body curvature and higher order derivatives would not, in general, be retained with this procedure. However, the effects of body curvature and higher order shape do not appear in the first order solutions for ϕ and $\vec{\nabla}\phi$ anyway. The precision of equations (31B), or the equivalent for the spheroidal coordinates, should be consistent with the level of approximation of the overall development. For the heaving spheroid, with $r_o = \zeta_o \ell \sqrt{1 - \mu^2}$, equations (31B) give,

$$\xi_o(\mu) = \zeta_o \sqrt{\frac{\mu\ell}{2a \sqrt{\zeta_o^2 + 1}}}$$

$$\eta(\mu) = \sqrt{\frac{(1 - \mu^2)\ \ell}{2a\mu} \sqrt{\zeta_o^2 + 1}}$$

Then, the Para L solutions, (24B) and (25B), become, for the heaving spheroid,

$$\frac{\phi}{V\ell \cos\theta} = \frac{2a\eta\ \xi_o^2}{\xi}$$

$$\frac{u_r}{V \cos\theta} = -\frac{\xi_o^2}{\eta^2 + \xi^2} \left(\frac{\eta^2}{\xi^2} - 1\right)$$

$$\frac{u_x}{V \cos\theta} = -\frac{2\eta\ \xi_o^2}{\xi\ (\eta^2 + \xi^2)} \qquad (32B)$$

$$\frac{u_\theta}{V \sin\theta} = -\frac{\xi_o^2}{\xi^2}$$

In summary, for the heaving spheroid, the exact solutions are given by (26B) and (27B). The first order slender body approximations are the Cyl, (28B), the Spher B, (29B), the Spher L, (30B), and the Para L, (32B).

The potentials on the surfaces of spheroids $\zeta_o = .10$ and $\zeta_o = .15$ have been calculated from the exact solution and from the slender body formulas; these surface potentials are shown on Figure 5B for $.75 \leq \mu \leq 1.0$. Figure 6B shows the corresponding velocity components on the surface of the $\zeta_o = .15$ spheroid (ζ_o is approximately the beam-length ratio).

Potentials and velocity components computed from the exact formulas and from the Spher L and Para L formulas are compared at off-body points for the $\zeta_0 = .15$ spheroid on Figures 7B and 10B. These Figures show the values versus radius in planes at $x/L = .90$, $.95$, 1.0, and 1.05.

4. Results

The adequacy of a strip theory formulation for studying the fluid behavior near the ends of a vertically vibrating slender body can be readily evaluated by studying Figures 5B through 10B.

Referring to Figure 5B, the three Laplace solutions for the potential Cyl, Spher L, and Para L, are identical on the body surface. These solutions over-estimate the potential but are best near the ends of the spheroid. The behavior of the Spher B solution relative to the Spher L is very strange. The Spher L solution was believed to have been obtained as an approximation to the Spher B solution due to the discarding of a term in the first order near-field differential equation, (11B). It was expected that discarding this term would produce solutions, the Spher L solutions, inferior to the Spher B solutions near the body ends, with improved agreement near mid-ship. Figure 5B, pertaining to the surface potential, shows the opposite trend. Although the mid-region of the spheroid is not covered by Figure 5B, the Spher B solution is uniformly better than the Spher L solution everywhere on the spheroid except in the region of interest, near the end. Near the end, the Spher L and also the Para L and Cyl solutions become quite accurate. This is fine; the L solutions were obtained by solving a 2-D Laplace equation and can be mapped to the real plane. The disparity with the mathematical reasoning is confusing, however.

The observation with regard to the Bessel and Laplace solutions from Figure 5B is supported by Figure 6B. The

Spher B solution is as good as or better than the other approximate solutions for the radial and axial surface velocities. The tangential velocity component, however, is predicted very poorly near the end of the spheroid. Additional study is in order with regard to the reasons for the relationship between the Spher B and Spher L solutions. The Spher L and Para L velocity components are identical on the surface of the spheroid. These velocity components generally overestimate but are correct in character over the entire body surface. Figure 6B shows that the Cyl solution is inadequate for predicting velocity components near the body end, as would be expected. The radial velocity component has large error and the axial velocity component blows-up at $\mu = 1$. From the standpoint of end behavior, the Cyl and Spher B solutions cannot contend. These are not considered in subsequent evaluations.

Figures 7B, 8B, 9B, and 10B compare the Spher L and Para L estimates with the exact solutions in radial planes on and just off the end of the spheroid. Note that in all cases the strip theories follow identically the character of the exact solutions but continue to over-estimate by a consistent, and reasonably small, amount.

From all of the figures it can be noted that the difference between the Spher L and Para L results is almost undetectable. This is important. It gives confidence that the procedure described for matching the paraboloids to the spheroid, equations (31B), can be used with the same accuracy in applying either the paraboloidal or spheroidal formulas to general slender bodies.

With regard to a measure of the accuracy of the strip theory formulas, the following table is presented; maximum and average estimates of per cent error for ϕ and the velocity components are shown. The extent of the region considered is that covered by Figures 7B to 10B, that is,

$.9 \leq x/L \leq 1.05$, $0 \leq r/L \leq .10$. The error estimate is based on a percentage of the spheroid velocity V .

	Max. % Error	Location x/L	r/L	Mean % Error
ϕ	4	.9	.10	3
u_r	7	1.0	0	2
u_x	4	1.0	.015	3
u_θ	7	on body surface		4

5. Conclusions

It can be concluded from the comparisons displayed on Figures 5B through 10B that a special formulation of slender body theory, using either paraboloidal or spheroidal coordinates, can be used to predict the fluid behavior near the ends of a vertically vibrating slender body; accurate results can be expected.

The slender body solutions developed in this appendix apply for a body of revolution. However, the solutions are in the forms of two-dimensional strip formulas obtained by solving a boundary-value problem governed by a 2-D Laplace equation. The presence of the 2-D Laplace equation is a fortunate circumstance, for it guarantees the existence of a conformal transformation. In the problem of calculating H_i and $\vec{\nabla} H_i$ for the vibrating ship double-hull, the 2-D solutions developed for the body of revolution can be mapped to the planes of the irregular hull sections by conformal mapping. Mapping of these strip solutions would be the best approach where hull sections are not extremely irregular; the simple Lewis form transformations [2] can be quite accurate. The paraboloidal formulas, equations (24B) and (25B), are recommended where the mapping approach is to be applied, due to their extreme simplicity.

FIGURE 5B

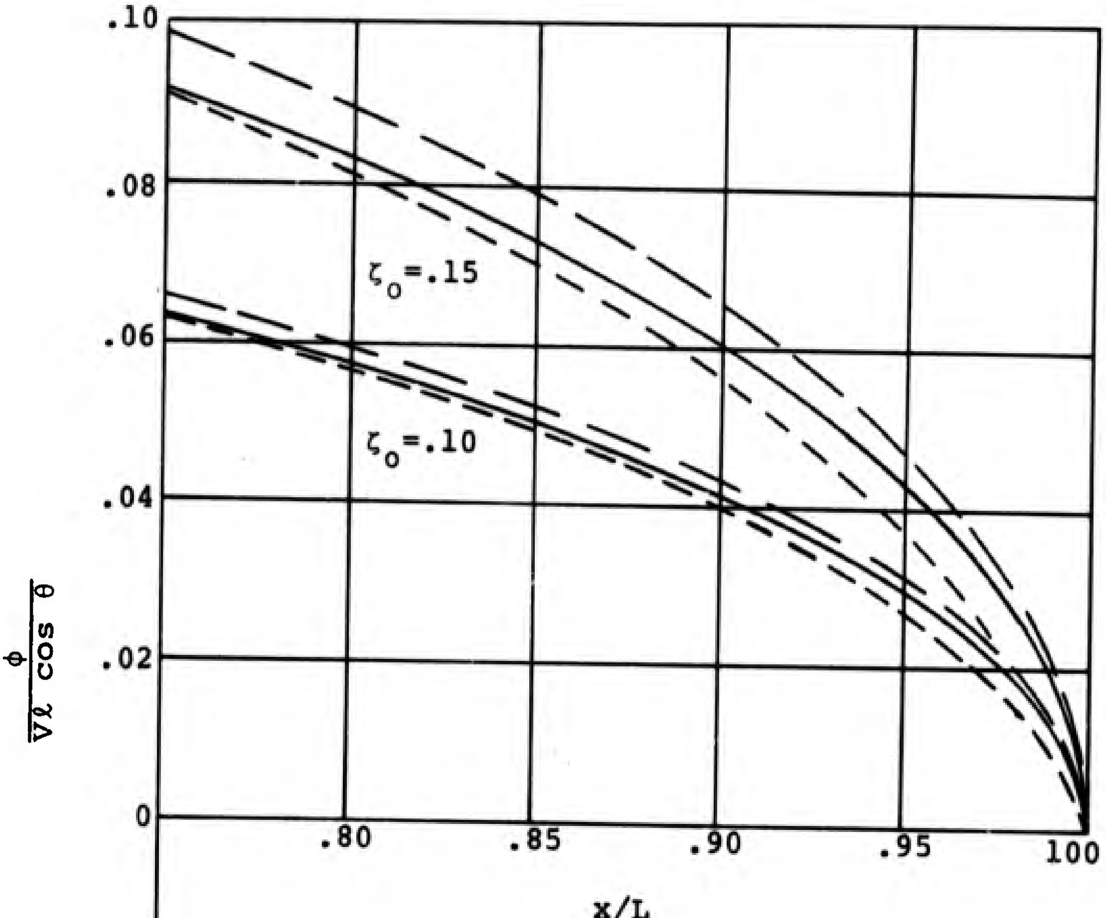
Surface Velocity Potential on Heaving Spheroids

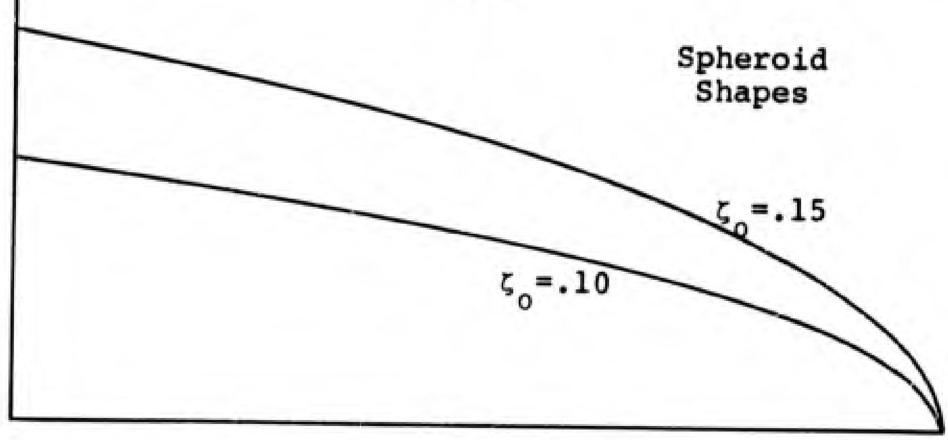
$\zeta_o = .10$ and $\zeta_o = .15$

——— Exact

— — {Cyl, Spher L, Para L}

- - - - Spher B





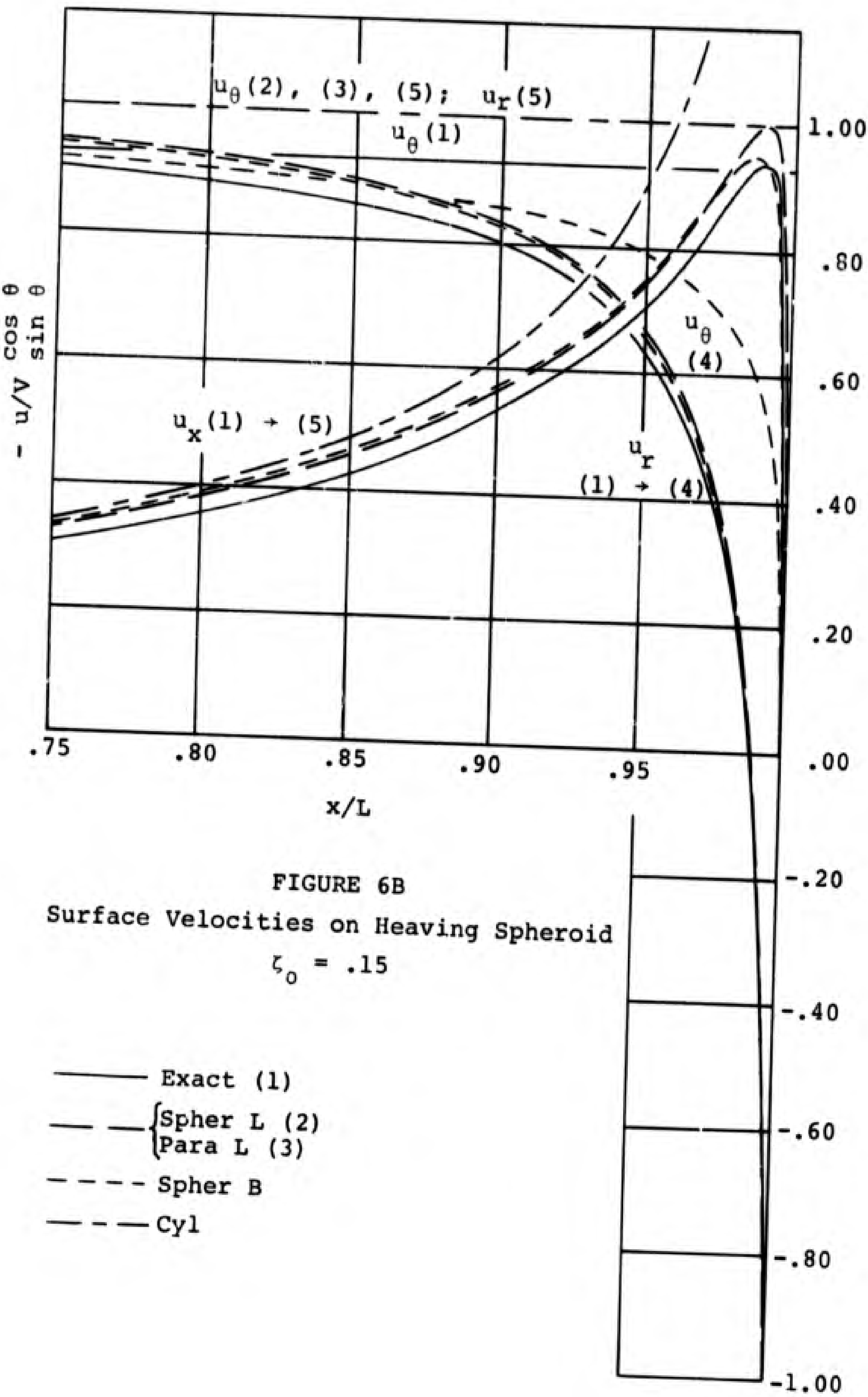
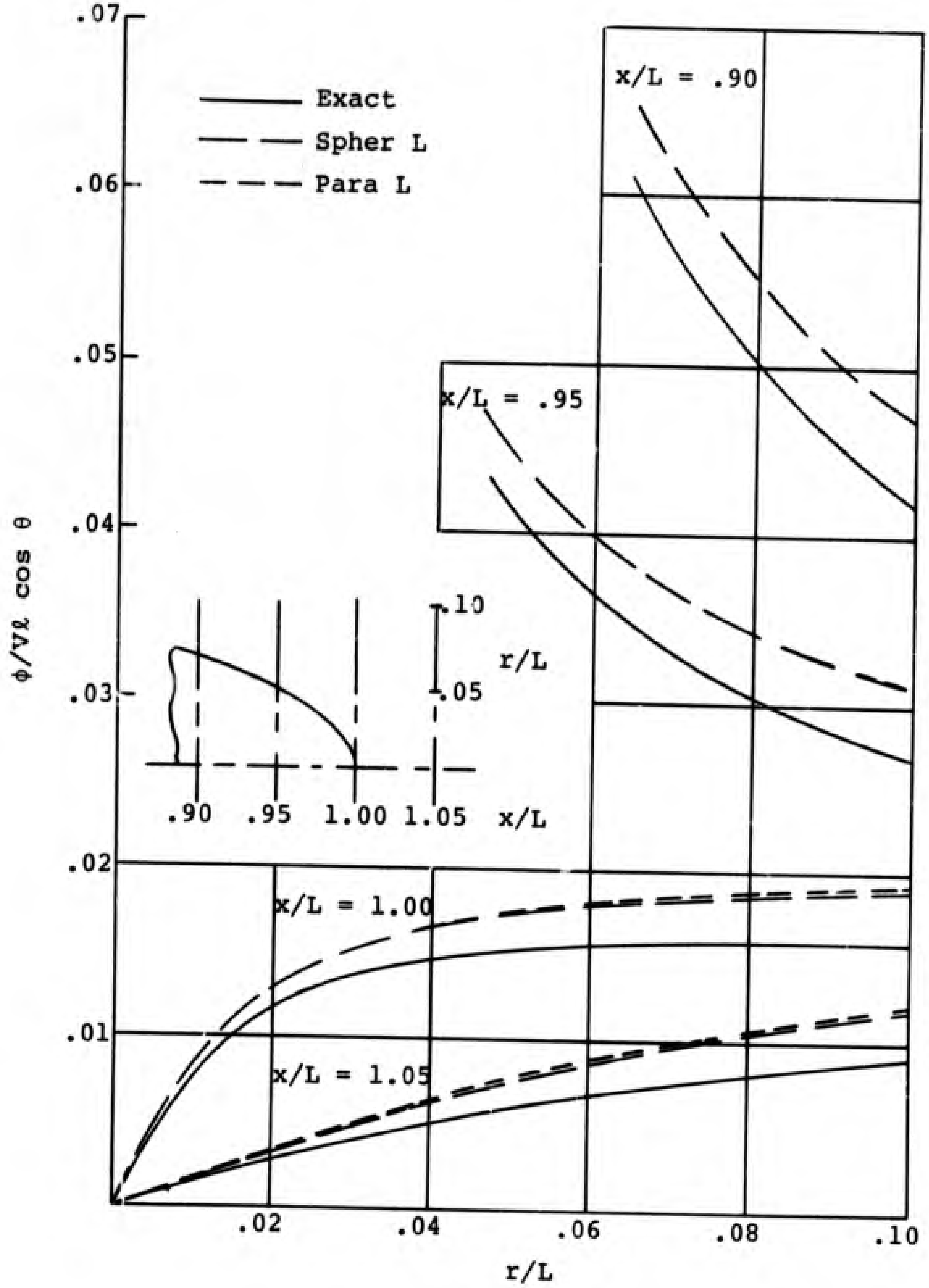


FIGURE 6B
Surface Velocities on Heaving Spheroid
$\zeta_o = .15$

——— Exact (1)
— — {Spher L (2) / Para L (3)}
- - - - Spher B
— - - — Cyl

FIGURE 7B

Fluid Velocity Potential Induced by Heaving Spheroid $\zeta_0 = .15$



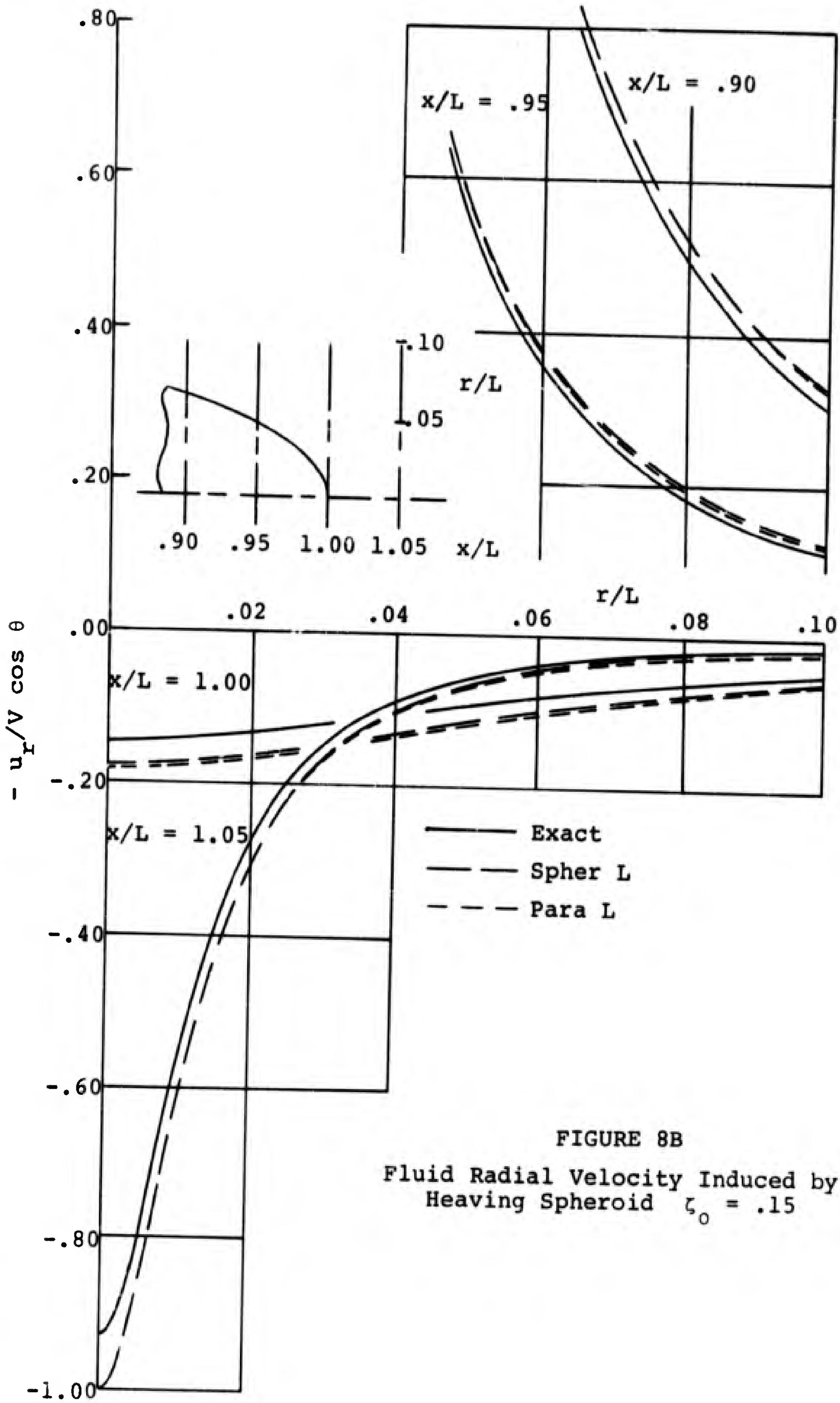


FIGURE 8B

Fluid Radial Velocity Induced by Heaving Spheroid ζ_o = .15

FIGURE 9B

Fluid Axial Velocity Induced by Heaving Spheroid $\zeta_o = .15$

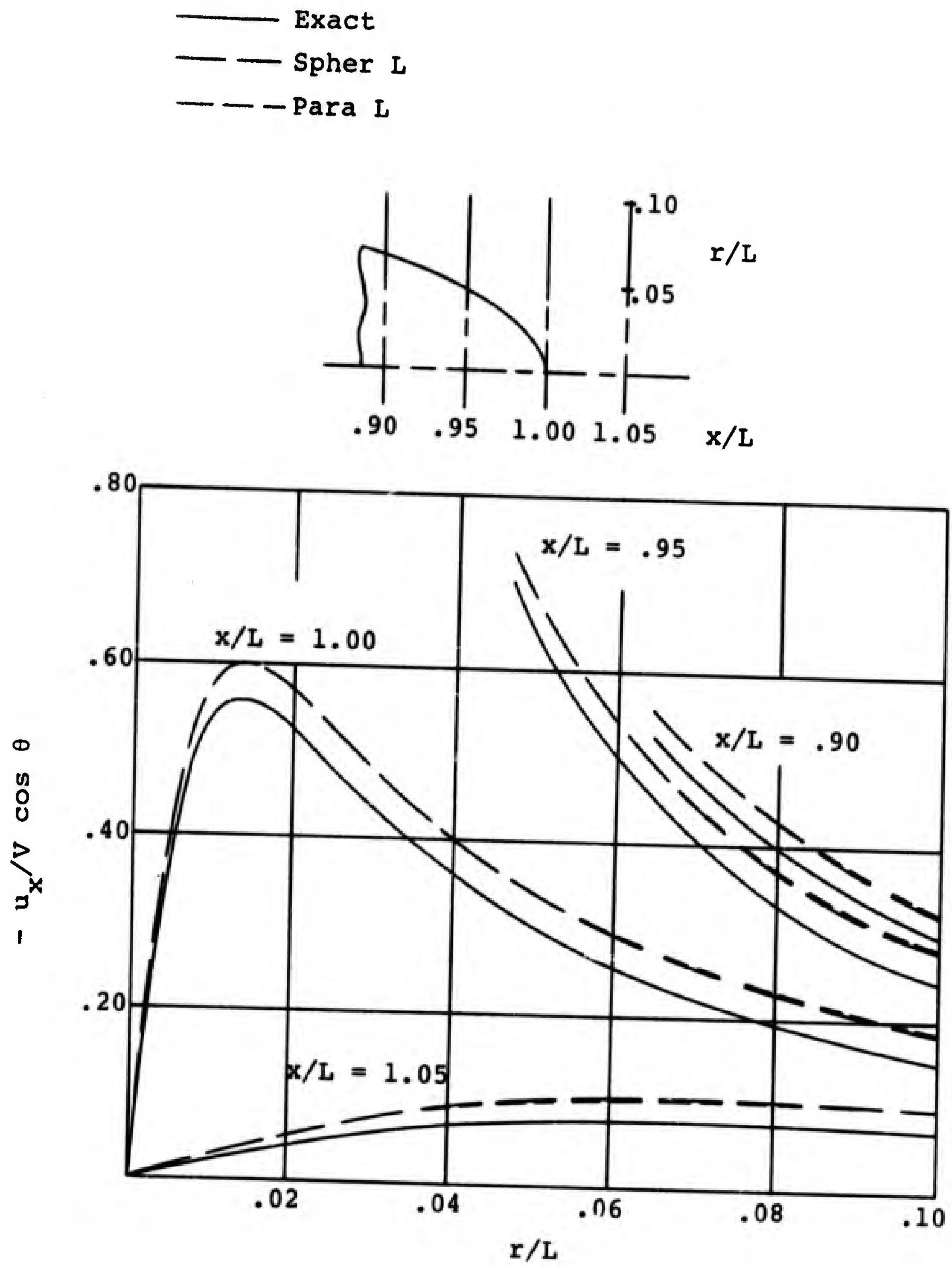


FIGURE 10B

Fluid Tangential Velocity Induced by Heaving Spheroid $\zeta_0 = .15$