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Stability and Control of Translunar Earth Orbits

R. H. Frick

A Report prepared for

UNITED STATES AIR FORCE PROJECT RAND

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10. ABSTRACT

Investigation of the problem of keeping a satellite in an orbit around the earth at translunar distances from the earth. The study concentrates on distances between 300,000 and 500,000 n mi, since in this range the earth's gravitational attraction is the dominant factor in producing acceleration of a satellite vehicle relative to the earth. The approach to the problem involves formulation of the general nonlinear equations of motion for an object under the gravitational attraction of the earth, sun, and moon. The solution of these equations is in terms of the variation of the resulting motion from a nominal unperturbed circular orbit, and is obtained by numerical integration of the equations of motion. Although cases are found in which the radial variations from a reference orbital radius of 300,000 n mi remain less than 50,000 n mi for as long as 5 years, it appears to be desirable to have an orbital control capability.

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PREFACE

This study of the motion of earth satellite vehicles in translunar space was undertaken at the request of the Directorate of Space, DCS/Research and Development. While it is sometimes possible to establish a satellite about the earth in a translunar orbit that will remain reasonably circular for as long as five years, it is desirable to have an orbital correction capability to reduce orbital variations caused by the sun and the moon. In this report, a method is described for determining orbital injection conditions that minimize the deviations of the orbit from circularity and a method of orbital correction is presented that can be used to reduce variations from circularity if they develop. The results obtained and the methodology developed herein should be of use in predicting and controlling the orbital motion of the satellite vehicles in regions where the gravitational perturbation due to the moon and sun are significant in comparison with the earth's gravitational attraction.

SUMMARY

This report investigates the problem of keeping a satellite in an orbit around the earth at translunar distances from the earth. The study concentrates on distances between 300,000 and 500,000 n mi, since in this range the earth's gravitational attraction is the dominant factor in producing acceleration of a satellite vehicle relative to the earth. The approach to the problem involves formulation of the general nonlinear equations of motion for an object under the gravitational attraction of the earth, the sun, and the moon. The solution of these equations is in terms of the variation of the resulting motion from a nominal unperturbed circular orbit, and is obtained by numerical integration of the equations of motion. The results indicate no instability in the motion of the orbital plane, which is characterized by a slow regression of the line of nodes in the ecliptic, while the inclination to the ecliptic remains essentially constant. The investigation of the motion in the orbital plane is restricted to orbits perpendicular to the ecliptic, since this tends to reduce the likelihood and duration of near approaches of the satellite to the moon, thus decreasing the resulting gravitational disturbance. Nevertheless, the in-plane motion appears to be characterized by an instability that develops as a divergent oscillation in the radial displacement of the satellite from the reference orbit. The onset of this oscillation is found to be dependent on the initial geometry of sun, moon, and satellite relative to the earth as well as on the orbital injection conditions. A method is developed for computing these injection conditions in terms of the initial geometry so that the development of instability is delayed for two to three years.

Although cases are found in which the radial variations from a reference orbital radius of 300,000 n mi remain less than 50,000 n mi for as long as five years, it appears to be desirable to have an orbital control capability. This is done by a modification of the method of determining orbital injection conditions. By means of this method a satellite can be kept within a radial displacement of 26,000 n mi of the reference orbit for a period of five years at an expenditure of less than 70 ft/sec per year in velocity correction.

CONTENTS

SYMBOLS

A_E = magnitude of the satellite acceleration due to earth

A_{Sr} = magnitude of the maximum disturbing acceleration due to the sun

A_{mr} = magnitude of the maximum disturbing acceleration due to the moon

A_o = contribution of the driven solution to the steady-state value of $\delta r/r_o$

A_1 = contribution of the driven solution to the coefficient of the $\cos\theta$ term in $\delta r/r_o$

A_2 = contribution of the driven solution to the coefficient of the $\sin\theta$ term in $\delta r/r_o$

a_E = semimajor axis of the earth's orbit

a_{Sr} = normalized radial component of the sun's disturbing acceleration

$a_{S\theta}$ = normalized horizontal component of the sun's disturbing acceleration

a_{ij} = amplitude of the oscillatory term in $\delta r/r_o$ with frequency ω_{ij}

a_m = semimajor axis of the moon's orbit

a_{mr} = normalized radial component of the moon's disturbing acceleration

$a_{m\theta}$ = normalized horizontal component of the moon's disturbing acceleration

B_{nij} = amplitude of the oscillatory term with frequency ω_{ij} from the n^{th} term of the binomial expansion

B_o = contribution of the driven solution to the steady-state value of $d\delta\theta/d\theta_o$

b_{ij} = amplitude of the oscillatory term in $\delta r/r_o$ with frequency Ω_{ij}

b_{ni} = coefficient of $\cos i\theta$ in the Fourier expansion of $\cos^n\theta$

C_{ij} = coefficient of the term with frequency ω_{ij} in the expansion of $(a_m/r_m)^3$

c_{ij} = amplitude of the oscillatory term in $d\delta\theta/d\theta_o$ with frequency ω_{ij}

d_{ij} = amplitude of the oscillatory term in $d\delta\theta/d\theta_o$ with frequency Ω_{ij}

G = universal gravitational constant

i = satellite frequency harmonic number

$\bar{i},\bar{j},\bar{k}$ = unit vectors of the x,y,z coordinate system

$\bar{i}_m$ = unit vector along $\bar{\rho}$

$\bar{i}_o,\bar{j}_o,\bar{k}_o$ = unit vectors of the x_o,y_o,z_o coordinate system

j = lunar frequency harmonic number

K = parameter in the binomial expansion of $(a_m/r_m)^3$

M_E = mass of the earth

M_S = mass of the sun

M_m = mass of the moon

N = maximum number of terms in binomial expansion of $(a_m/r_m)^3$

n = term number in the binomial expansion

n_o = maximum value for both i and j

P_{ij} = coefficient of the term in a_{Sr} with frequency Ω_{ij}

p_{ij} = coefficient of the term in a_{mr} with frequency ω_{ij}

Q_{ij} = coefficient of the term in $a_{S\theta}$ with frequency Ω_{ij}

q_{ij} = coefficient of the term in $a_{m\theta}$ with frequency ω_{ij}

$\bar{R}$ = vector from the earth-moon barycenter to the center of the sun

$\bar{R}_E$ = vector from the center of the sun to the center of the earth

$\bar{r}$ = vector from the center of the earth to the satellite

$\bar{r}_S$ = vector from the center of the sun to the satellite

$\bar{r}_m$ = vector from the center of the moon to the satellite

r_o = radius of the circular reference orbit

$\bar{r}_1$ = unit vector along $\bar{R}$

t = time

t_1 = correction time

$\bar{V}$ = orbital injection velocity

V_H = horizontal component of $\bar{V}$

V_R = radial component of $\bar{V}$

V_o = reference orbital velocity

x,y,z = orbital coordinate system

x_o,y_o,z_o = nonrotating earth-centered coordinate system

α = orbital inclination to the ecliptic plane

α_G = orbital inclination to the equatorial plane

$\alpha(0)$ = orbital inclination of the reference orbit to the ecliptic plane

γ = angle between the x_o and $\bar{r}$ when α_o is zero

γ_p = orbital injection path angle

$\overline{\Delta V}$ = orbital correction velocity

ΔV_H = horizontal component of $\overline{\Delta V}$

ΔV_R = radial component of $\overline{\Delta V}$

δr = radial displacement of the satellite from the reference orbit

δr_D = desired value of δr at time t_1

$\delta r'_D$ = desired value of $\delta r'$ at time t_1

δr_{SS} = steady-state value of δr

δr_o = initial value of δr

$\delta r'_o$ = initial derivative of δr with respect to θ_o

δr_{osc} = amplitude of the orbital frequency component of δr

δr_1 = current value of δr at time t_1

$\delta r'_1$ = current value of $\delta r'$ at time t_1

$\delta\Theta$ = angular variation of the earth from its mean motion

$\delta\alpha$ = variation of orbital inclination from α_o

$\delta\gamma$ = variation of γ from its reference orbit value

$\delta\theta$ = angular variation of the satellite from the reference satellite

$\delta\theta'_D$ = desired value of $\delta\theta'$ at time t_1

$\delta\dot{\theta}_{SS}$ = steady-state value of $\delta\dot{\theta}$

$\delta\theta'_c$ = corrected value of $\delta\theta'$ at time t_1

$\delta\theta_m$ = variation of the moon from the mean motion

$\delta\theta'_o$ = initial value of $d\delta\theta/d\theta_o$

$\delta\dot{\theta}_{osc}$ = amplitude of the orbital frequency component of $\delta\dot{\theta}$

$\delta\theta_1$ = value of $\delta\theta$ at time t_1

$\delta\theta'_1$ = current value of $\delta\theta'$ at time t_1

$\delta\Psi$ = variation of Ψ from that for the reference orbit, Ψ_o

$\delta\omega_z$ = variation of ω_z from its reference value, $\dot{\theta}_o$

ε_E = eccentricity of the earth's orbit

ε_m = eccentricity of the moon's orbit

Θ = solar longitude

Θ_P = longitude of solar perigee

$\Theta(0)$ = initial value of Θ

$\dot{\Theta}_o$ = mean orbital angular rate of the earth

θ = satellite orbital angle measured from the ascending node in the ecliptic

$\theta(0)$ = initial value of θ

θ_m = longitude of the moon

$\theta_m(0)$ = initial value of θ_m

$\dot{\theta}_{mo}$ = mean orbital angular rate of the moon

θ_{mp} = longitude of lunar perigee

$\theta_{mp}(0)$ = initial value of θ_{mp}

θ_o = angular displacement of reference satellite in time t

$\dot{\theta}_o$ = orbital angular rate of reference orbit

θ_{sm} = angle between r and ρ

θ_1 = value of θ_o at time t_1

λ = inclination of the equatorial plane to the ecliptic

μ = mass ratio $(M_E + M_m)/M_m$

$\bar{\rho}$ = vector from the center of the earth to the center of the moon

Ψ = regression angle measured from the x_o axis to the ascending node in the ecliptic

$\Psi(0)$ = initial value of Ψ

Ω_{ij} = normalized solar driving frequency

ω_{ij} = normalized lunar driving frequency

$\bar{\omega}_o$ = angular velocity of the x,y,z coordinate system relative to inertial space

ω_z = component of $\bar{\omega}_o$ along the z axis

NUMERICAL CONSTANTS

$a_E = 8.0726 \times 10^7$ n mi

$a_m = 207{,}428$ n mi

$M_E/M_m = 81.27$

$M_S/M_E = 332{,}958$

$\Theta_p = 281°33'53''$

$\dot{\Theta}_o = 0.0172$ rad/solar day

$\epsilon_E = 0.01674$

$\epsilon_m = 0.0549$

$\dot{\theta}_{mo} = 0.22998$ rad/solar day

$\dot{\theta}_{mp} = 0.001944$ rad/solar day

$\lambda = 23°27'$

I. INTRODUCTION

If the total acceleration of a satellite vehicle relative to the earth is due to a mutual inverse square law gravitational attraction between the two bodies, the motion of the satellite relative to the earth is said to be Keplerian. The resulting path is an ellipse with one focus at the earth's center and a fixed orientation relative to inertial space. As the satellite traverses this path, its orbital angular rate varies inversely as the square of its distance from the earth's center, thus maintaining a constant orbital angular momentum.

An actual earth satellite deviates from this idealized Keplerian motion, since its acceleration relative to the earth is due not only to the earth's inverse square law gravitational attraction but also to perturbing accelerations caused by residual atmospheric drag, solar radiation pressure, the nonspherical mass distribution of the earth, and the gravitational attraction of the sun and the moon. For altitudes up to about 100,000 n mi, all of these perturbing accelerations are small compared to the inverse square term in the earth's attraction; however, their relative importance varies considerably with altitude. While residual atmospheric drag can result in orbital decay and reentry for low-altitude satellites, its effect can be neglected at altitudes above 300 n mi. The magnitude of the acceleration due to solar radiation pressure is invariant with altitude, depending only on the area-to-mass ratio of the vehicle. For low values of this ratio the radiation pressure effect can also be neglected.

The acceleration due to the nonspherical mass distribution of the earth varies inversely as the fourth or higher powers of the distance from the earth's center and is negligible at distances greater than 50,000 n mi. At such distances from the earth the only significant perturbing accelerations are those due to the sun and moon. However, their magnitudes, even at a distance of 100,000 n mi from the earth, are less than one percent of the earth's gravitational attraction. A perturbing acceleration of this magnitude does not seem likely to result in orbital instability, and an examination of Ref. 1 indicates a number of examples of satellites at altitudes of the order of

70,000 n mi that have been in orbit for 8 to 12 years with no apparent orbital instability.

As the orbital radius increases and becomes comparable with that of the moon, the perturbing acceleration due to the moon may become as large as or greater than the earth's gravitational acceleration at times of close approach of the satellite to the moon. Under these conditions the vehicle can no longer be regarded as an earth satellite. However, it can be shown that at translunar distances of the order of 300,000 to 500,000 n mi, the perturbing accelerations due to the sun and the moon are again small compared to the earth's gravitational attraction.

This report presents the results of a study of the motion of a satellite in translunar space to see whether a stable earth orbit can be established at these distances. In view of the scarcity of experimental data on translunar earth-orbiting vehicles, the study is of necessity analytical in nature. In Section II the general equations are developed describing the motion relative to the earth of an object in translunar space under the influence of the gravitational attractions of the earth, the sun, and the moon. In Section III these equations are used to determine the variations of the satellite motion from a nominal unperturbed circular orbit. A method is developed for the determination of orbital injection conditions that minimize these variations and produce acceptable, stable, circular, orbital motion for several years. In the event that the variations from circularity exceed permissible tolerances, a method is also developed to compute the required orbital velocity correction and its time of application. Section IV presents the specific conclusions reached in this study in regard to the feasibility of maintaining translunar orbits either passively or actively, while some of the details of the analytical development of the orbital injection and correction technique are relegated to Appendices A and B. The computing programs used are included as Appendix C, together with a definition of terms and a description of their operation.

II. METHOD OF ANALYSIS

STATEMENT OF THE PROBLEM

The problem investigated in this report can be stated as follows: Can an artificial earth satellite be established in a stable translunar earth orbit without active orbital control, and, if not, what is the propulsion requirement to maintain such an orbit?

In the formulation of this problem certain simplifications and assumptions can be made without seriously affecting the accuracy of the results. The acceleration of the satellite relative to the earth can be expressed as the vector sum of the following elements: (1) the acceleration of the satellite due to the earth's gravitational attraction, which can be represented as a simple inverse square law variation at the assumed translunar distances from the earth; (2) the differential acceleration of the satellite and the earth caused by the moon's gravitational attraction; and (3) the differential acceleration of the satellite and the earth caused by the sun's gravitational attraction.

The motion of the earth relative to the sun is assumed to be an elliptical orbit in the plane of the ecliptic with an eccentricity, ε_E, of 0.01674, a mean orbital angular rate, $\dot{\theta}_o$, of 0.0172 rad/solar day, and a longitude of solar perigee, θ_p, of 281°33′53″. Similarly, the motion of the moon relative to the earth is also assumed to be an elliptical orbit in the plane of the ecliptic* with an eccentricity, ε_m, of 0.0549, a mean angular rate, $\dot{\theta}_{mo}$, of 0.22998 rad/solar day, and an advance of perigee, $\dot{\theta}_{mp}$, of 0.001944 rad/solar day.

It should be noted that in this representation the initial positions of the earth and the moon in their orbits, as well as the initial position of lunar perigee, are independently adjustable rather than being interrelated through an ephemeris. This facilitates the study of the effect of the initial geometry of the sun, moon, and earth on the satellite so that the results need not depend on the particular geometry that may exist on a specific launch date.

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*This ignores the 5°33′ inclination of the moon's orbit to the plane of the ecliptic, since its effect on the satellite orbit is small enough that the additional computation involved did not seem warranted.

COORDINATE SYSTEMS

In Fig. 1 the vector relationships of the sun, earth, moon, and satellite are shown schematically. As indicated previously, the sun, moon, and earth are assumed to lie in the plane of the ecliptic; however, there is no such restriction on the satellite.

The coordinate systems used in the formulation of the equations of motion are shown in Fig. 2 where the x_o, y_o, z_o axes define an earth-centered coordinate system with no rotation relative to inertial space. The $x_o y_o$ plane is the plane of the ecliptic and the x_o axis is in the direction of the first point of Aries. In this coordinate system the sun and moon are in the direction of the lines OS and OM in the $x_o y_o$ plane making angles of Θ and θ_m with the x_o axis. The x,y,z axes define an earth-centered coordinate system which moves in such a way that its x axis is always along the radius vector $\bar{r}$ between the earth's center and the satellite, while the xy plane is the instantaneous orbit plane. The orientation of this system relative to the x_o, y_o, z_o system is specified by three angles, Ψ, α, and θ. The angle Ψ is the orbital regression angle measured from the x_o axis to the line of nodes ON at the intersection of the xy and $x_o y_o$ planes, while α is the orbital inclination angle between the xy and $x_o y_o$ planes and θ is the angle between the x axis and ON measured in the xy plane.

The nine direction cosines relating the unit vectors $\bar{i}, \bar{j}, \bar{k}$ of the orbital system to the unit vectors $\bar{i}_o, \bar{j}_o, \bar{k}_o$ of the nonrotating system are given in Table 1 in terms of the angles Ψ, α, and θ.

In addition, it is convenient to define two unit vectors $\bar{r}_1$ and $\bar{i}_m$ in the direction of $\bar{R}$ and $\bar{\rho}$, respectively. These can be expressed as

$$\bar{r}_1 = \bar{i}_o \cos \Theta + \bar{j}_o \sin \Theta \qquad (1)^*$$

$$\bar{i}_m = \bar{i}_o \cos \theta_m + \bar{j}_o \sin \theta_m \qquad (2)$$

where the angles Θ and θ_m are the solar and lunar longitudes.

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*It is assumed that $\bar{R}$ and $\bar{R}_E$ are parallel.

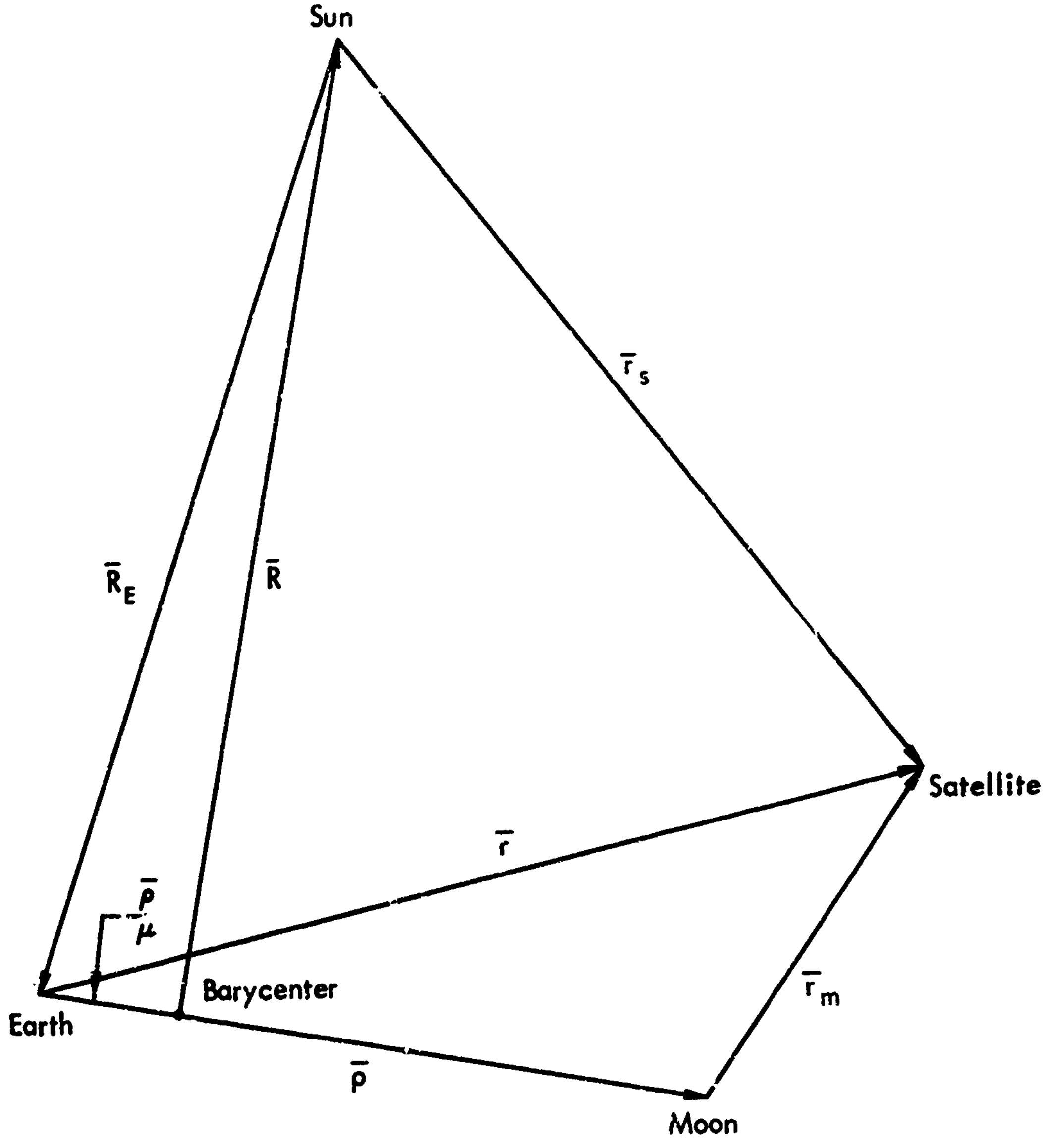


Fig. 1—Position vectors

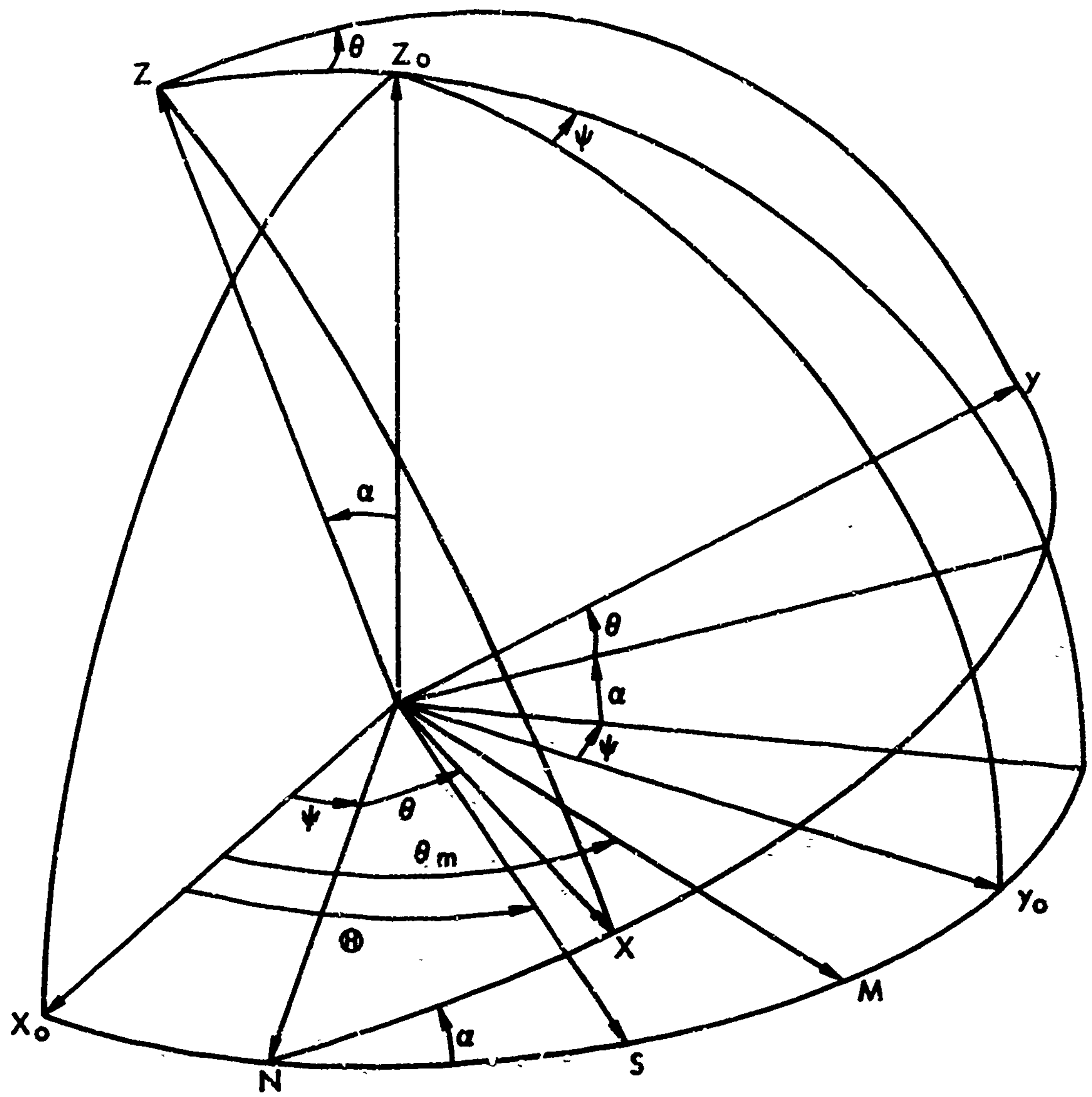


Fig. 2—Coordinate systems

Table 1

DIRECTION COSINES

	$\bar{i}_o$	$\bar{j}_o$	$\bar{k}_o$
$\bar{i}$	$\cos\theta\cos\Psi - \sin\theta\cos\alpha\sin\Psi$	$\cos\theta\sin\Psi + \sin\theta\cos\alpha\cos\Psi$	$\sin\theta\sin\alpha$
$\bar{j}$	$-\sin\theta\cos\Psi - \cos\theta\cos\alpha\sin\Psi$	$-\sin\theta\sin\Psi + \cos\theta\cos\alpha\cos\Psi$	$\cos\theta\sin\alpha$
$\bar{k}$	$\sin\alpha\sin\Psi$	$-\sin\alpha\cos\Psi$	$\cos\alpha$

FORMULATION OF THE EQUATIONS OF MOTION

General Equations

The acceleration of the satellite relative to the earth can be expressed in vector form using the approach of Ref. 2 as follows:

$$\ddot{\bar{r}} = -\frac{GM_E}{r^2}\,\bar{i} - r\dot{\theta}_o^2\left(\frac{a_E}{R}\right)^3\left[\bar{i} - 3(\bar{i}\cdot\bar{r}_1)\bar{r}_1\right]$$

$$- r\,\frac{\dot{\theta}_{mo}^2}{\mu}\left(\frac{a_m}{\rho}\right)^3\left[\frac{\rho^3}{r_m^3}\,\bar{i} + \left(1 - \frac{\rho^3}{r_m^3}\right)\frac{\rho}{r}\,\bar{i}_m\right] \qquad (3)$$

where G = universal gravitational constant

M_E = mass of the earth

M_m = mass of the moon

$\mu = (M_E + M_m)/M_m$

a_E = semimajor axis of the earth's orbit

a_m = semimajor axis of the moon's orbit

The principal difference between Eq. (3) and the corresponding result given in Ref. 2 as Eq. (B-37) is that the last term in Eq. (3) is an exact expression for the differential acceleration of the satellite and the earth due to the moon, since the linearization of this term used in Ref. 2 is not valid for the geometry of the present problem. However, the linearization of the corresponding solar term is still valid even for translunar orbits.

In addition, the inclusion of the eccentricity of the orbits of the earth and the moon introduces the factors a_E/R and a_m/ρ which can be determined as a function of time by the following equations:

$$\frac{a_E}{R} = 1 + \epsilon_E \cos(\Theta - \Theta_p) \quad (4)$$

$$\frac{a_m}{\rho} = 1 + \epsilon_m \cos(\theta_m - \theta_{mp}) \quad (5)$$

where

$$\dot{\Theta} = \frac{\dot{\Theta}_o}{\left(1 - \epsilon_E^2\right)^{3/2}} \left[1 + \epsilon_E \cos(\Theta - \Theta_p)\right]^2 \quad (6)$$

$$\dot{\theta}_m = \frac{\dot{\theta}_{mo}}{\left(1 - \epsilon_m^2\right)^{3/2}} \left[1 + \epsilon_m \cos(\theta_m - \theta_{mp})\right]^2 \quad (7)$$

Θ_p and θ_{mp} being the longitudes of solar and lunar perigee.

In addition the ratio ρ/r_m is given by the relation

$$\frac{\rho}{r_m} = \left[1 + \frac{r^2}{\rho^2} - \frac{2r}{\rho}(\bar{i}\cdot\bar{i}_m)\right]^{-1/2} \quad (8)$$

The satellite acceleration can also be expressed as

$$\begin{aligned} \ddot{\bar{r}} = & \left[\frac{d^2r}{dt^2} - r(\bar{\omega}_o\cdot\bar{j})^2 - r(\bar{\omega}_o\cdot\bar{k})^2\right]\bar{i} \\ & + \left\{\frac{1}{r}\frac{d}{dt}\left[r^2(\bar{\omega}_o\cdot\bar{k})\right] + r(\bar{\omega}_o\cdot\bar{i})(\bar{\omega}_o\cdot\bar{j})\right\}\bar{j} \\ & + \left\{-\frac{1}{r}\frac{d}{dt}\left[r^2(\bar{\omega}_o\cdot\bar{j})\right] + r(\bar{\omega}_o\cdot\bar{i})(\bar{\omega}_o\cdot\bar{k})\right\}\bar{k} \end{aligned} \quad (9)$$

where $\bar{\omega}_o$ is the angular velocity of the x,y,z system relative to inertial space. The components of $\bar{\omega}_o$ that appear in Eq. (9) are as follows:

$$(\bar{\omega}_o \cdot \bar{i}) = \dot{\alpha} \cos \theta + \dot{\gamma} \sin \alpha \sin \theta \tag{10}$$

$$(\bar{\omega}_o \cdot \bar{j}) = -\dot{\alpha} \sin \theta + \dot{\gamma} \sin \alpha \cos \theta \tag{11}$$

$$(\bar{\omega}_o \cdot \bar{k}) = \dot{\theta} + \dot{\gamma} \cos \alpha \tag{12}$$

However, it can be shown that if the xy plane is the instantaneous orbital plane, then $(\bar{\omega}_o \cdot \bar{j})$ must be identically zero. Thus, Eq. (9) becomes:

$$\ddot{\bar{r}} = \left[\frac{d^2 r}{dt^2} - r(\bar{\omega}_o \cdot \bar{k})^2\right]\bar{i} + \frac{1}{r}\frac{d}{dt}\left[r^2(\bar{\omega}_o \cdot \bar{k})\right]\bar{j} + r(\bar{\omega}_o \cdot \bar{i})(\bar{\omega}_o \cdot \bar{k})\bar{k} \tag{13}$$

Combination of Eqs. (10) and (11) with the three equations obtained by equating components of Eqs. (3) and (13) results in the following equations of motion for the satellite

$$\frac{d^2 r}{dt^2} - r\omega_z^2 = -\frac{GM_E}{r^2} - r\dot{\theta}_o^2\left(\frac{a_E}{R}\right)^3\left[1 - 3(\bar{r}_1 \cdot \bar{i})^2\right]$$

$$- r\frac{\dot{\theta}_{mo}^2}{\mu}\left(\frac{a_m}{\rho}\right)^3\left[\frac{\rho^3}{r_m^3} + \left(1 - \frac{\rho^3}{r_m^3}\right)\frac{\rho}{r}(\bar{i}_m \cdot \bar{i})\right] \tag{14}$$

$$\frac{d}{dt}(r^2\omega_z) = 3r^2\dot{\theta}_o^2\left(\frac{a_E}{R}\right)^3(\bar{r}_1 \cdot \bar{i})(\bar{r}_1 \cdot \bar{j}) - \frac{r^2\dot{\theta}_{mo}^2}{\mu}\left(\frac{a_m}{\rho}\right)^3\left(1 - \frac{\rho^3}{r_m^3}\right)\frac{\rho}{r}(\bar{i}_m \cdot \bar{j}) \tag{15}$$

$$\dot{\gamma}\omega_z \sin \alpha = \sin \theta \left[3\dot{\theta}_o^2\left(\frac{a_E}{R}\right)^3(\bar{r}_1 \cdot \bar{i})(\bar{r}_1 \cdot \bar{k}) - \frac{\dot{\theta}_{mo}^2}{\mu}\left(\frac{a_m}{\rho}\right)^3\left(1 - \frac{\rho^3}{r_m^3}\right)\frac{\rho}{r}(\bar{i}_m \cdot \bar{k})\right] \tag{16}$$

$$\dot{\alpha}\omega_z = \cos\theta \left[3\dot{\theta}_o^2 \left(\frac{a_E}{R}\right)^3 (\bar{r}_1 \cdot \bar{i})(\bar{r}_1 \cdot \bar{k}) - \frac{\dot{\theta}_{mo}^2}{\mu} \left(\frac{a_m}{\rho}\right)^3 \left(1 - \frac{\rho^3}{r_m^3}\right) \frac{\rho}{r} (\bar{i}_m \cdot \bar{k}) \right] \quad (17)$$

$$\dot{\theta} = \omega_z - \dot{\Psi} \cos\alpha \quad (18)$$

Variational Equations

A considerable improvement in computational accuracy can be achieved if the equations of motion are expressed in terms of the variations of r, ω_z, θ, Ψ, and α from their corresponding values along an unperturbed Keplerian reference orbit. Since there is no apparent advantage in selecting an eccentric reference orbit, it can be specified as follows:

$$r = r_o \quad (19)$$

$$\omega_z = \dot{\theta}_o \quad (20)$$

$$\dot{\theta} = \dot{\theta}_o \quad (21)$$

$$\dot{\Psi} = 0 \quad (22)$$

$$\dot{\alpha} = 0 \quad (23)$$

where the mean orbital angular rate, $\dot{\theta}_o$, is given by

$$\dot{\theta}_o = \sqrt{\frac{GM_E}{r_o^3}} = \dot{\theta}_{mo} \left[\frac{a_m^3}{r_o^3}\left(\frac{\mu - 1}{\mu}\right)\right]^{1/2} \quad (24)$$

In addition, it is also convenient to use θ_o rather than the time, t, as the independent variable where θ_o is given by

$$\theta_o = \dot{\theta}_o t \quad (25)$$

and represents the change in central angle along the reference orbit during the time, t.

The dependent variables r, ω_z, θ, Ψ, and α, can be expressed in terms of their variations from their reference orbit values by the following relations:

$$r = r_o + \delta r = r_o\left(1 + \frac{\delta r}{r_o}\right) \tag{26}$$

$$\omega_z = \dot{\theta}_o + \delta\omega_z = \dot{\theta}_o\left(1 + \frac{\delta\omega_z}{\dot{\theta}_o}\right) \tag{27}$$

$$\dot{\theta} = \dot{\theta}_o + \delta\dot{\theta} = \dot{\theta}_o\left(1 + \frac{d\delta\theta}{d\theta_o}\right) \tag{28}$$

$$\dot{\Psi} = \delta\dot{\Psi} = \dot{\theta}_o \frac{d\delta\Psi}{d\theta_o} \tag{29}$$

$$\dot{\alpha} = \delta\dot{\alpha} = \dot{\theta}_o \frac{d\delta\alpha}{d\theta_o} \tag{30}$$

$$\theta = \theta_o + \theta(0) + \delta\theta \tag{31}$$

$$\Psi = \Psi(0) + \delta\Psi \tag{32}$$

$$\alpha = \alpha(0) + \delta\alpha \tag{33}$$

where θ(0), Ψ(0), and α(0) are the initial values of θ, Ψ, and α when t (and θ_o) are zero.

Similarly the angles Θ, θ_m, and θ_{mp} can be expressed in terms of the independent variable θ_o by the relations

$$\Theta = \frac{\dot{\Theta}_o}{\dot{\theta}_o}\theta_o + \Theta(0) + \delta\Theta \tag{34}$$

$$\theta_m = \frac{\dot{\theta}_{mo}}{\dot{\theta}_o}\theta_o + \theta_m(0) + \delta\theta_m \tag{35}$$

$$\theta_{mp} = \frac{\dot{\theta}_{mp}}{\dot{\theta}_o}\theta_o + \theta_{mp}(0) \tag{36}$$

where $\Theta(0)$, $\theta_m(0)$, and $\theta_{mp}(0)$ are the initial values of Θ, θ_m, and θ_{mp}. The quantities $\delta\Theta$ and $\delta\theta_m$ are the angular variations of the sun and the moon from constant angular rate motion relative to the earth.

Substitution of Eqs. (26) through (36) into Eqs. (14) through (18) results in the following variational form of the equations of motion.

$$\frac{d^2}{d\theta_o^2}\left(\frac{\delta r}{r_o}\right) = \frac{1}{\left(1+\frac{\delta r}{r_o}\right)^2}\left[3\frac{\delta r}{r_o} + 3\left(\frac{\delta r}{r_o}\right)^2 + \left(\frac{\delta r}{r_o}\right)^3\right] + \left(1+\frac{\delta r}{r_o}\right)\left\{2\frac{\delta\omega_z}{\dot{\theta}_o} + \left(\frac{\delta\omega_z}{\dot{\theta}_o}\right)^2\right.$$

$$\left. \frac{\dot{\Theta}_o^2}{\dot{\theta}_o^2}\left(\frac{a_E}{R}\right)^3\left[1 - 3(\bar{r}_1\cdot\bar{i})^2\right] - \frac{\dot{\theta}_{mc}^2}{\mu\dot{\theta}_o^2}\left(\frac{a_m}{\rho}\right)^3\left[\frac{\rho^3}{r_m^3} + \left(1-\frac{\rho^3}{r_m^3}\right)\frac{\rho}{r}(\bar{i}_m\cdot\bar{i})\right]\right\} \tag{37}$$

$$\frac{d}{d\theta_o}\left(\frac{\delta\omega_z}{\dot{\theta}_o}\right) = -\frac{2}{\left(1+\frac{\delta r}{r_o}\right)}\left(1+\frac{\delta\omega_z}{\dot{\theta}_o}\right)\frac{d}{d\theta_o}\left(\frac{\delta r}{r_o}\right) + \frac{3\dot{\Theta}_o^2}{\dot{\theta}_o^2}\left(\frac{a_E}{R}\right)^3(\bar{r}_1\cdot\bar{i})(\bar{r}_1\cdot\bar{j})$$

$$-\frac{\dot{\theta}_{mo}^2}{\mu\dot{\theta}_o^2}\left(\frac{a_m}{\rho}\right)^3\left(1-\frac{\rho^3}{r_m^3}\right)\frac{\rho}{r}(\bar{i}_m\cdot\bar{j}) \tag{38}$$

$$\frac{d\delta\theta}{d\theta_o} = \frac{\delta\omega_z}{\dot{\theta}_o} - \frac{\sin\theta\cos\alpha}{\left(1+\frac{\delta\omega_z}{\dot{\theta}_o}\right)\sin\alpha}$$

$$\times\left[\frac{3\dot{\Theta}_o^2}{\dot{\theta}_o^2}\left(\frac{a_E}{R}\right)^3(\bar{r}_1\cdot\bar{i})(\bar{r}_1\cdot\bar{k}) - \frac{\dot{\theta}_{mo}^2}{\mu\dot{\theta}_o^2}\left(\frac{a_m}{\rho}\right)^3\left(1-\frac{\rho^3}{r_m^3}\right)\frac{\rho}{r}(\bar{i}_m\cdot\bar{k})\right] \tag{39}$$

$$\frac{d\delta\Psi}{d\theta_o} = \frac{\sin\theta}{\left(1 + \frac{\delta\omega_z}{\dot{\theta}_o}\right)\sin\alpha}$$

$$\times\left[\frac{3\dot{\Theta}_o^2}{\dot{\theta}_o^2}\left(\frac{a_E}{R}\right)^3(\bar{r}_1\cdot\bar{i})(\bar{r}_1\cdot\bar{k}) - \frac{\dot{\theta}_{mo}^2}{\mu\dot{\theta}_o^2}\left(\frac{a_m}{\rho}\right)^3\left(1 - \frac{\rho^3}{r_m^3}\right)\frac{\rho}{r}\,(\bar{i}_m\cdot\bar{k})\right] \tag{40}$$

$$\frac{d\delta\alpha}{d\theta_o} = \frac{\cos\theta}{\left(1 + \frac{\delta\omega_z}{\dot{\theta}_o}\right)}$$

$$\times\left[\frac{3\dot{\Theta}_o^2}{\dot{\theta}_o^2}\left(\frac{a_E}{R}\right)^3(\bar{r}_1\cdot\bar{i})(\bar{r}_1\cdot\bar{k}) - \frac{\theta_{mo}^2}{\mu\dot{\theta}_o^2}\left(\frac{a_m}{\rho}\right)^3\left(1 - \frac{\rho^3}{r_m^3}\right)\frac{\rho}{r}\,(\bar{i}_m\cdot\bar{k})\right] \tag{41}$$

where the dot products are given by the expressions

$$(\bar{i}_m\cdot\bar{i}) = \cos\theta\cos(\Psi - \theta_m) - \cos\alpha\sin\theta\sin(\Psi - \theta_m) \tag{42}$$

$$(\bar{i}_m\cdot\bar{j}) = -\sin\theta\cos(\Psi - \theta_m) - \cos\alpha\cos\theta\sin(\Psi - \theta_m) \tag{43}$$

$$(\bar{i}_m\cdot\bar{k}) = \sin\alpha\sin(\Psi - \theta_m) \tag{44}$$

$$(\bar{r}_1\cdot\bar{i}) = \cos\theta\cos(\Psi - \Theta) - \cos\alpha\sin\theta\sin(\Psi - \Theta) \tag{45}$$

$$(\bar{r}_1\cdot\bar{j}) = -\sin\theta\cos(\Psi - \Theta) - \cos\alpha\cos\theta\sin(\Psi - \Theta) \tag{46}$$

$$(\bar{r}_1\cdot\bar{k}) = \sin\alpha\sin(\Psi - \Theta) \tag{47}$$

Finally, the values of $\delta\Theta$ and $\delta\theta_m$ can be obtained from Eqs. (6) and (7) after substitution of Eqs. (34) through (36) to give

$$\frac{d\delta\Theta}{d\theta_o} = \frac{\dot{\Theta}_o}{\dot{\theta}_o}\left\{\frac{\left[1 + \varepsilon_E \cos\,(\Theta - \Theta_p)\right]^2}{\left(1 - \varepsilon_E^2\right)^{3/2}} - 1\right\} \tag{48}$$

$$\frac{d\delta\theta_m}{d\theta_o} = \frac{\dot{\theta}_{mo}}{\dot{\theta}_o}\left\{\frac{\left[1 + \varepsilon_m \cos\,(\theta_m - \theta_{mp})\right]^2}{\left(1 - \varepsilon_m^2\right)^{3/2}} - 1\right\} \tag{49}$$

Equations (37) through (41) represent the complete equations of motion applicable to satellites in translunar orbits. They constitute a set of five coupled nonlinear differential equations in the variables δr, $\delta\omega_z$, $\delta\theta$, $\delta\Psi$, and $\delta\alpha$ as functions of the independent variable θ_o. It should be emphasized that this formulation puts no limitation on the magnitude of any of the five dependent variables.

Determination of the orbital motion, in view of the nonlinearity of the equations, requires simultaneous numerical integration not only of the five equations of motion but also of Eqs. (48) and (49) to determine the values of $\delta\Theta$ and $\delta\theta_m$ as a function of the independent variable θ_o. While this can be done, it is found that certain approximations can be made that markedly reduce the solution time without any significant change in the nature of the results.

Uncoupled Equations

Equations (40) and (41) describe the part of the motion that results in a change in the orientation of the orbital plane, while Eqs. (37) through (39) determine the motion in the orbital plane that results in changes of size and shape of the orbit. While these two types of motion are not completely uncoupled, the coupling is weak enough so that the motion of the orbital plane can be treated separately by ignoring the effects of δr, $\delta\theta$, $\delta\omega_z$, and $\delta\alpha$ on the right side of Eqs. (40) and (41). With the additional assumption that ε_m and ε_E are zero, these equations become

$$\frac{d\delta\Psi}{d\theta_o} = \sin\theta\left[\frac{3\dot{\Theta}_o^2}{\dot{\theta}_o^2}(\bar{r}_1\cdot\bar{I})\sin(\Psi-\Theta) - \frac{\dot{\theta}_{mo}^2}{\mu\dot{\theta}_o^2}\left(1-\frac{a_m^3}{r_m^3}\right)\frac{a_m}{r_o}\sin(\Psi-\theta_m)\right] \quad (50)$$

$$\frac{d\delta\alpha}{d\theta_o} = \cos\theta\sin\alpha(0)\left[\frac{3\dot{\Theta}_o^2}{\dot{\theta}_o^2}(\bar{r}_1\cdot\bar{I})\sin(\Psi-\Theta) - \frac{\dot{\theta}_{mo}^2}{\mu\dot{\theta}_o^2}\left(1-\frac{a_m^3}{r_m^3}\right)\frac{a_m}{r_o}\sin(\Psi-\theta_m)\right] \quad (51)$$

These equations can be integrated numerically along with Eqs. (48) and (49) to determine the variations in the orientation of the orbital plane as given by $\delta\Psi$ and $\delta\alpha$. It is found, as will be shown in the section on results, that the solution for $\delta\Psi$ consists of a secular term with small oscillatory variations superposed, while $\delta\alpha$ can be represented by a small negative bias term with small oscillatory terms superposed.

In a similar manner, the variation of the in-plane motion can also be determined independently by replacing α with its reference value, $\alpha(0)$, and Ψ with its secular variation in Eqs. (37) through (39). These equations can also be integrated numerically along with Eqs. (48) and (49) to give the in-plane variations of the motion as specified by δr, $\delta\theta$, and $\delta\omega_z$.

The in-plane motion can be further simplified for two special cases that are of interest, namely when the orbital inclination to the ecliptic is either 0 or 90 deg.

If $\alpha(0)$ is equal to 0 deg, the orbit lies in the plane of the ecliptic and the line of nodes, ON, in Fig. 2 is indeterminate. As a result, the angles θ and Ψ, as well as their variations $\delta\theta$ and $\delta\Psi$, are also indeterminate. However, the sum of θ and Ψ is determinate and represents the angular displacement of the satellite about the earth's center measured from the x_o axis. If his angle is defined as γ, then by Eq. (18) $\dot{\gamma}$ is equal to ω_z and Eq. (39) reduces to an identity, while Eqs. (37) and (38) become

$$\frac{d^2}{d\theta_o^2}\left(\frac{\delta r}{r_o}\right) = \frac{1}{\left(1 + \frac{\delta r}{r_o}\right)^2}\left[3\,\frac{\delta r}{r_o} + 3\left(\frac{\delta r}{r_o}\right)^2 + \left(\frac{\delta r}{r_o}\right)^3\right] + \left(1 + \frac{\delta r}{r_o}\right)$$

$$\times \left\{2\,\frac{d\delta\gamma}{d\theta_o} + \left(\frac{d\delta\gamma}{d\theta_o}\right)^2 - \frac{\dot{\Theta}_o^2}{\dot{\theta}_o^2}\left(\frac{a_E}{R}\right)^3\left[1 - 3\cos^2(\gamma - \Theta)\right]\right.$$

$$\left. - \frac{\dot{\theta}_{mo}^2}{\mu\dot{\theta}_o^2}\left(\frac{a_m}{\rho}\right)^3\left[\frac{\rho^3}{r_m^3} + \left(1 - \frac{\rho^3}{r_m^3}\right)\frac{\rho}{r}\cos(\gamma - \theta_m)\right]\right\} \tag{52}$$

$$\frac{d^2\delta\gamma}{d\theta_o^2} = -\frac{2}{\left(1 + \frac{\delta r}{r_o}\right)}\left(1 + \frac{d\delta\gamma}{d\theta_o}\right)\frac{d}{d\theta_o}\left(\frac{\delta r}{r_o}\right) - \frac{3\dot{\Theta}_o^2}{2\dot{\theta}_o^2}\left(\frac{a_E}{R}\right)^3 \sin 2(\gamma - \Theta)$$

$$+ \frac{\dot{\theta}_{mo}^2}{\mu\dot{\theta}_o^2}\left(\frac{a_m}{\rho}\right)^3\left(1 - \frac{\rho^3}{r_m^3}\right)\frac{\rho}{r}\sin(\gamma - \theta_m) \tag{53}$$

Thus the in-plane motion is completely described by solving these equations for δr and $\delta\gamma$, since θ and Ψ no longer occur separately in the equations.

If α_o is 90 deg, then by Eq. (18) $\dot{\theta}$ is equal to ω_z, and Eq. (39) again reduces to an identity, while Eqs. (37) and (38) take the form

$$\frac{d^2}{d\theta_o^2}\left(\frac{\delta r}{r_o}\right) = \frac{1}{\left(1 + \frac{\delta r}{r_o}\right)^2}\left[3\,\frac{\delta r}{r_o} + 3\left(\frac{\delta r}{r_o}\right)^2 + \left(\frac{\delta r}{r_o}\right)^3\right] + \left(1 + \frac{\delta r}{r_o}\right)$$

$$\times \left\{2\,\frac{d\delta\theta}{d\theta_o} + \left(\frac{d\delta\theta}{d\theta_o}\right)^2 - \frac{\dot{\Theta}_o^2}{\dot{\theta}_o^2}\left(\frac{a_E}{R}\right)^3\left(1 - 3\cos^2\theta\cos^2\Theta\right)\right.$$

$$\left. - \frac{\dot{\theta}_{mo}^2}{\mu\dot{\theta}_o^2}\left(\frac{a_m}{\rho}\right)^3\left[\frac{\rho^3}{r_m^3} + \left(1 - \frac{\rho^3}{r_m^3}\right)\frac{\rho}{r}\cos\theta\cos\theta_m\right]\right\} \tag{54}$$

$$\frac{d^2\delta\theta}{d\theta_o} = -\frac{2}{\left(1+\frac{\delta r}{r_o}\right)}\left(1+\frac{d\delta\theta}{d\theta_o}\right)\frac{d}{d\theta_o}\left(\frac{\delta r}{r_o}\right) - \frac{3\dot{\Theta}_o^2}{2\dot{\theta}_o^2}\left(\frac{a_E}{R}\right)^3 \sin 2\theta \cos^2 \Theta$$

$$+ \frac{\dot{\theta}_{mo}^2}{\mu\dot{\theta}_o^2}\left(\frac{a_m}{\rho}\right)^3\left(1-\frac{\rho^3}{r_m^3}\right)\frac{\rho}{r}\sin\theta\cos\theta_m \qquad (55)$$

Thus the in-plane motion can again be described by solving Eqs. (54) and (55) for the variations δr and $\delta\theta$.

III. RESULTS AND DISCUSSION

SELECTION OF A REFERENCE ORBIT

If the orbit of a satellite about the earth is to be stable, it is necessary that its acceleration due to the earth's gravitational attraction be large compared with the differential accelerations of the satellite and the earth caused by either the sun or the moon. Thus an examination of the magnitudes of these accelerations, as a function of distance from the earth, should give some basis for the selection of the orbital radius of the reference orbit from which the orbital variations are to be measured. The maximum differential acceleration of the satellite and the earth due to the moon occurs when the satellite and the moon have a common direction from the earth's center ($\bar{i} = \bar{i}_m$). An examination of Eq. (3) shows that this maximum acceleration is along $\bar{r}$ and has a magnitude, A_{mr}, given by

$$A_{mr} = -r\frac{\dot{\theta}^2_{mo}}{\mu}\left[\frac{a_m^3}{r_m^3} + \left(1 - \frac{a_m^3}{r_m^3}\right)\frac{a_m}{r}\right] \tag{56}$$

where the eccentricity of the moon's orbit is neglected.

The acceleration of the satellite by the earth, A_E, is given by the expression

$$A_E = -\frac{GM_E}{r^2} \tag{57}$$

Thus the relative importance of the lunar effect can be determined as the ratio of A_{mr} to A_E as follows:

$$\frac{A_{mr}}{A_E} = \frac{r^3\dot{\theta}^2_{mo}}{\mu GM_E}\left[\frac{a_m^3}{r_m^3} + \left(1 - \frac{a_m^3}{r_m^3}\right)\frac{a_m}{r}\right] \tag{58}$$

Since $\dot{\theta}_{mo}$ is given by the relation

$$\dot{\theta}_{mo}^2 = \frac{\mu G M_m}{a_m^3} \tag{59}$$

Eq. (58) becomes

$$\frac{A_{mr}}{A_E} = \frac{M_m}{M_E}\left[\frac{r^2(r - a_m)}{r_m^3} + \frac{r^2}{a_m^2}\right] \tag{60}$$

For $\bar{i}$ equal to $\bar{i}_m$, r_m is equal to the absolute value of $(r - a_m)$ and Eq. (60) can be written as

$$\frac{A_{mr}}{A_E} = \frac{M_m}{M_E} \frac{r^2}{a_m^2} \pm \frac{r^2}{(r - a_m)^2} \tag{61}$$

where the plus sign applies if r is greater than a_m and the minus sign for r less than a_m.

Similarly, the maximum differential acceleration of the satellite and the earth due to the sun occurs when the directions of the satellite and the sun, relative to the center of the earth, are either the same or exactly opposite ($\bar{i} = \pm\bar{r}_1$). In either case Eq. (3) shows that this maximum acceleration is along $\bar{r}$ and has a magnitude, A_{Sr}, given by

$$A_{Sr} = 2r\dot{\theta}_o^2 \tag{62}$$

where the eccentricity of the earth's orbit is also neglected. Since $\dot{\theta}_o$ is given by the relation

$$\dot{\theta}_o^2 = \frac{GM_S}{a_E^3} \tag{63}$$

the ratio of A_{Sr} to A_E can be expressed in the form

$$\frac{A_{Sr}}{A_E} = \frac{2M_S}{M_E}\left(\frac{r}{a_E}\right)^3 \tag{64}$$

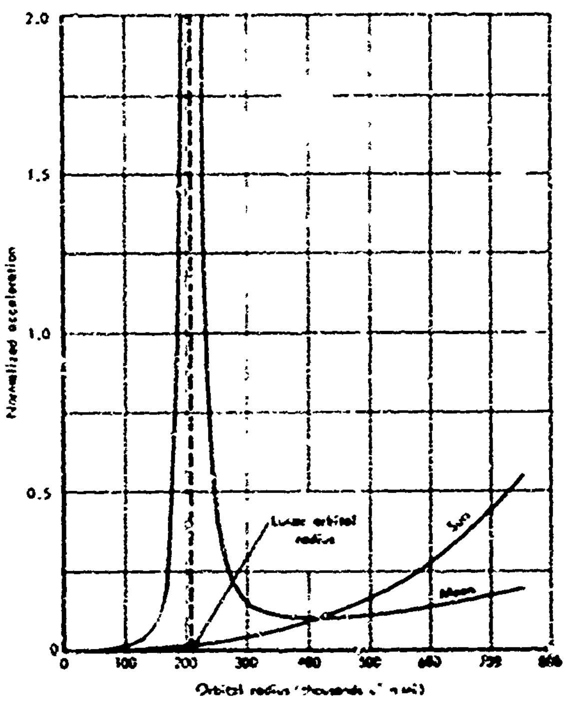


Fig. 3—Normalized solar and lunar perturbing accelerations

Figure 3 is a plot of the absolute values of the two ratios given by Eqs. (61) and (64) as functions of the distance from the center of the earth. It is seen that at distances from the earth up to 100,000 n mi the solar and lunar effects are less than one percent of the earth's gravitational attraction. In the vicinity of the moon ($r = 207,428$ n mi) the lunar effect becomes dominant and can cause drastic changes in the nature of the satellite's motion relative to the earth. However, at translunar distances the lunar effect passes through a minimum at 400,000 n mi, while the solar effect continues to increase as the cube of the distance from the earth. Consequently, there is a range between 300,000 and 500,000 n mi within which both the solar and lunar effects are less than 15 percent of the earth's gravitational attraction. Beyond 500,000 n mi both the solar and the lunar effects continue to increase in importance with the solar effect becoming dominant at a distance of about 1,000,000 n mi from the earth.

As a result, if stable translunar orbits are possible, their orbital radii should be between 300,000 and 500,000 n mi if the solar and lunar disturbances are to be minimized. While orbits with various radii within this range were examined, most of the data obtained is for a nominal 300,000-n mi orbital radius. The selection of 300,000 n mi is predicated on the fact that although the maximum lunar disturbance at this distance is 15 percent of the earth's attraction, this occurs only during close approaches of the satellite to the moon. During the rest of the orbit the average lunar effect is of the order of 6 percent or about twice the average solar effect. Thus if the frequency of occurrence and the duration of these close approaches of the satellite to the moon can be appreciably reduced by a suitable choice of the orbital orientation, then the average disturbance due to the moon can be reduced and the possibility of a stable orbit enhanced.

Thus the reference orbit used for most of this study is a circular orbit with a radius of 300,000 n mi and a period of 47.8 days. The variation of this orbital period with orbital radius in the translunar region is shown in Fig. 4.

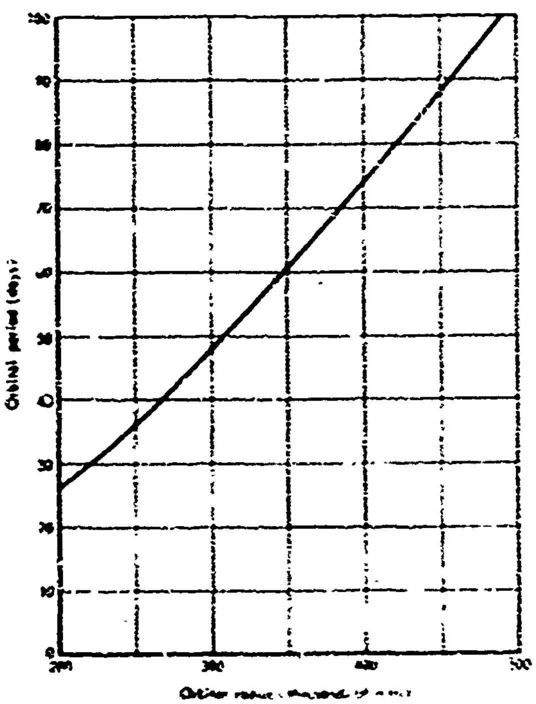


Fig. 4—Orbital period as a function of orbital radius

COMPUTATIONAL PROCEDURE

In the previous section, it was shown that a range of translunar distances exists in which the lunar and solar effects are minimized relative to the earth's attraction. However, it is still necessary to determine whether, under the influence of these residual disturbances, a suitable stable orbit can exist. This involves the solution of the equations of motion derived in Section II. In view of the nonlinearity of these equations, it is obvious that the solution must be obtained by numerical integration. This is accomplished by programming the equations for solution on the JOSS* computer using a fourth order Runge-Kutta

*JOSS is the trademark of the remote console, time sharing computer system developed at The Rand Corporation.

integration procedure. Initially, the completely general equations as represented by Eqs (37) through (41) were programmed. However, this involves the simultaneous integration of these five equations as well as Eqs. (48) and (49) for the motion of the sun and the moon, and the solution time is found to be excessive. If the uncoupled form of the equations of motion is used, then the determination of the motion of the orbital plane involves the integration of Eqs. (50) and (51) to obtain $\delta\Psi$ and $\delta\alpha$, as well as Eqs. (48) and (49) to obtain $\delta\Theta$ and $\delta\theta_m$. The in-plane motion is determined independently by integrating the uncoupled forms of Eqs. (37) through (39) to obtain δr, $\delta\omega_z$, and $\delta\theta$. Comparison of the results obtained from the uncoupled equations with those from the more general form shows no significant difference in the nature of the orbital motion. Thus, the uncoupled form of the equations is used in all cases.

MOTION OF THE ORBITAL PLANE

Motion Relative to the Ecliptic Plane

An examination of Eqs. (50) and (51) shows that the right-hand side of Eq. (50) averaged over θ_o has a nonzero value, while a similar average of the right side of Eq. (51) is zero. Thus, when the two equations are integrated, $\delta\Psi$ exhibits a secular variation with small oscillatory variations superposed, while $\delta\alpha$ undergoes small oscillations about a small negative bias. Thus, to a good approximation, the motion of the orbital plane can be described as a slow regression of its line of nodes in the plane of the ecliptic, while its inclination to the ecliptic remains constant. Figure 5 is a plot of the regression period as a function of orbital inclination to the ecliptic for a 300,000-n mi orbital radius. It is seen that the period increases monotonically with inclination having a value of about five years at 0 deg and becoming infinite at 90-deg inclination to the ecliptic. Thus for a 90-deg inclination the line of nodes simply executes small angular oscillations about a fixed direction in the plane of the ecliptic. To give some idea of the variation of regression period with altitude, several points are shown in Fig. 5 for an orbital radius

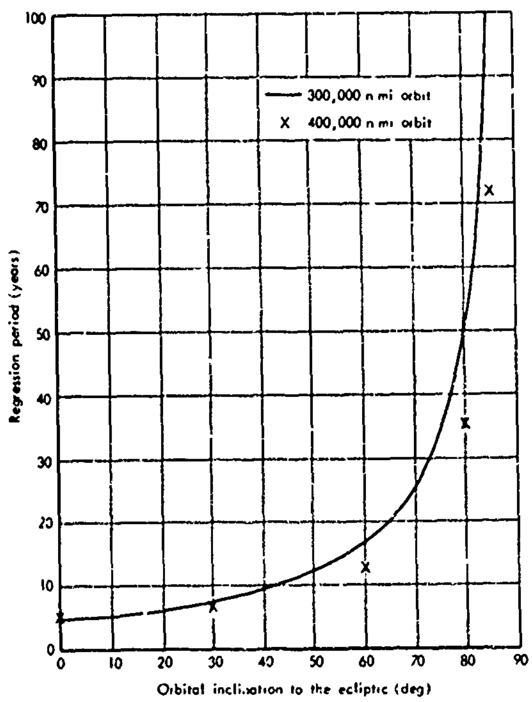


Fig. 5—Regression period as a function of orbital inclination

of 400,000 n mi. It is seen that for a given inclination the regression period decreases with altitude, particularly at higher values of orbital inclination.

In Fig. 6 a plot of the bias of the orbital inclination angle is shown as a function of the nominal inclination, $\alpha(0)$. The bias also varies monotonically from a value of ' deg when $\alpha(0)$ is zero, on t a value of about -2.5 deg when $\alpha(0)$ is equal to 90 deg. As in Fig. 5, several points are shown for an orbital radius of 400,000 n mi and it is seen that the magnitude of the bias increases with altitude. Since the amplitudes of the oscillatory terms in both $\delta\Psi$ and $\delta\alpha$, as well as the steady-state regression rate appear to be bounded, the resulting motion of the orbital plane does not constitute an orbital instability.

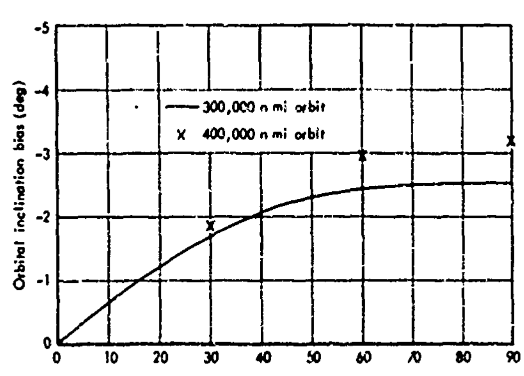


Fig. 6—Orbital inclination bias as a function of orbital inclination

Motion Relative to the Equatorial Plane

In the case of earth satellites, the earth's equatorial plane, rather than the ecliptic, is ordinarily used as a reference relative to which the motion of the orbital plane is determined. If this is done in the case of the motion described in the previous section, it is found that neither the orbital inclination nor the regression rate of the node remain constant in this equatorial reference system. As an example, an orbit that is initially in the equatorial plane is inclined at an angle of 23°27′ to the ecliptic. From Fig. 5 the regression period of the node in the ecliptic is 6.7 years; thus after only 3.35 years, the orbital inclination relative to the equatorial plane would increase from 0 deg to 46°54′. The orbital inclination to equatorial plane can remain constant only if its orientation relative to inertial space remains constant. This condition is satisfied if the orbital inclination to the ecliptic is either 0 or 90 deg. If the inclination to the ecliptic is 0 deg, the corresponding inclination to the equator is uniquely determined as 23°27′. On the other hand, if the inclination to the ecliptic is 90 deg, then the corresponding inclination to the equatorial plane, α_G, is given by the relation

$$\cos \alpha_G = \cos \Psi(0) \sin \lambda \tag{65}$$

where $\Psi(0)$ = steady-state position of the line of nodes in the ecliptic

λ = angle between the equatorial plane and the ecliptic.

Thus, with a 90-deg inclination to the ecliptic, the constant equatorial inclination may have a value anywhere from 66°33′ up to a 90-deg polar orbit.

Although it may not be necessary to hold a fixed orbital inclination to the equator, it does appear that an inclination of 90 deg to the ecliptic has certain advantages in minimizing the variations of the in-plane motion resulting from close approaches of the satellite to the moon. This is discussed further in the next section.

Preferred Orbital Inclination

If a satellite is in a nominal 300,000-n mi orbit, Fig. 3 shows that the maximum perturbing acceleration due to the moon is 15. percent of the acceleration due to the earth. This maximum acceleration occurs whenever the satellite and the moon are in the same direction relative to the earth. At this time the separation of the satellite and the moon is equal to the difference of their orbital radii, 92,572 n mi, if the eccentricity of the moon's orbit is neglected. If the satellite's orbit and the moon's orbit lie in the same plane, then one of these close approaches occurs each time the moon gains one complete revolution relative to the satellite. This occurs with a periodicity of 63.47 days corresponding to the difference of the two orbital angular rates. The variation of the radial component of the moon's perturbing acceleration during one of these close approaches is shown in Fig. 7, which is a plot of the lunar term of Eq. (52) as a function of time, where zero time occurs at closest approach.

If the satellite's orbital plane has an inclination of 90 deg to that of the moon, a close approach to the moon can only occur at the two points of intersection of the satellite orbit with the plane of the moon's orbit. If the orbital periods of the moon and the satellite are incommensurate, the probability of a close approach is much

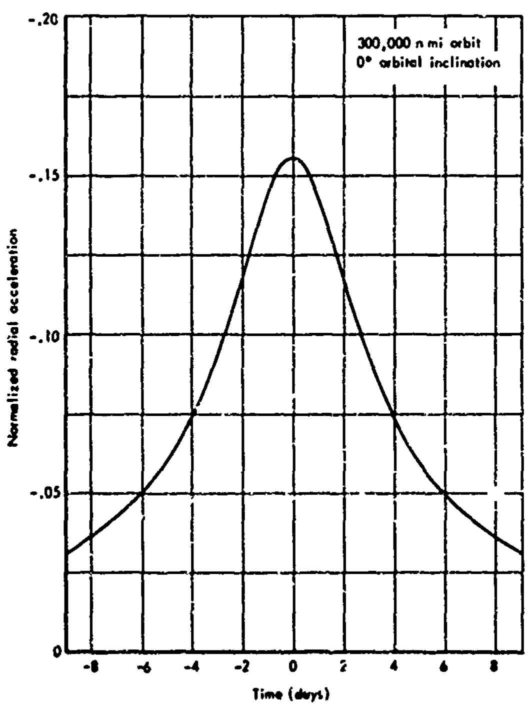


Fig. 7

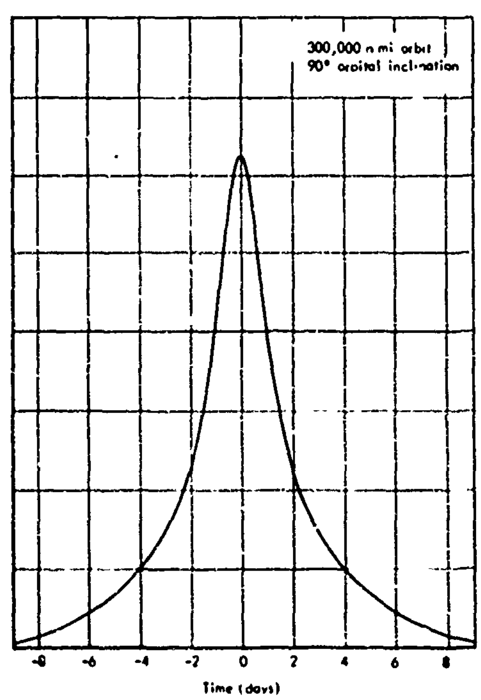


Fig. 8

Radial component of lunar perturbing acceleration as a function of time less than in the previous case where it could occur at any point along the satellite orbit.

However, if both the satellite and the moon do arrive at the line of intersection of the two orbital planes simultaneously, then the variation of the radial component of the perturbing acceleration with time is shown in Fig. 8, which is a plot of the lunar term in Eq. (54). It is seen that although the peak acceleration is the same as that in Fig. 7, the width of the peak is considerably less. Thus, the total impulse as measured by the area under the curve is less. In addition, Fig. 9 shows the effect on this acceleration peak, if the arrival of the moon at the intersection differs from that of the satellite. If the moon arrives earlier than the satellite, the peak occurs slightly before the satellite reaches the intersection and the height of the peak is reduced. The height of the peak is similarly reduced for a late arrival of the moon, but it occurs after the satellite has passed the intersection.

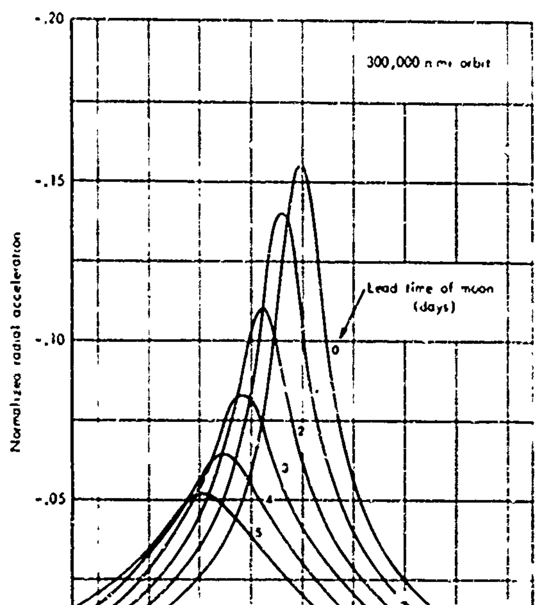


Fig. 9—Variation of lunar acceleration with difference in arrival time (90° orbital inclination)

On the basis of a comparison of Figs. 7, 8, and 9, it is evident that the perturbing acceleration due to the moon is less severe for the 90-deg inclination. Thus, in the investigation of the in-plane motion described in the next section, the principal emphasis is on orbits with this inclination to the ecliptic.

MOTION IN THE ORBITAL PLANE

Preliminary Results

The undisturbed in-plane motion of the satellite along the 300,000-n mi radius reference orbit has an orbital velocity of 2774.4 ft/sec, which results in an orbital angular rate of 0.13142 rad/day and an orbital period of 47.81 days. The orbital inclination of 90 deg to the ecliptic precludes any significant change in orientation of the plane as shown previously. The actual motion is determined relative to the reference satellite by numerical integration of Eqs. (54) and (55) for δr and $\delta\theta$, which represent the radial and tangential displacements of the actual satellite from its reference. At the same time, integration of Eqs. (48) and (49) determines the motion of the sun and the moon relative to the earth. The input data required include the initial values of θ, Θ, θ_m, δr, $d\delta r/d\theta_o$, and $d\delta\theta/d\theta_o$, the initial values of $\delta\theta$, $\delta\Theta$, and $\delta\theta_m$ being equal to zero. The equations were integrated with a 0.5-day integration step size, while the result is printed out at one-day intervals. The basic output of the program gives, as a function of time, the quantities $\delta r/r_o$, $\delta\theta$, and their rates of change with respect to θ_o as well as the variations $\delta\Theta$ and $\delta\theta_m$. These quantities, together with characteristics of the reference orbit, are then sufficient to completely describe the motion of the actual satellite relative to the earth. The description of the program is given in Appendix C.

The in-plane motion of a satellite can be regarded as stable if the magnitude of δr never exceeds some preassigned value, thus maintaining a reasonable approximation to a circular orbit. However, the cost of demonstrating any such absolute stability of an orbit by machine solution becomes exorbitant. For the purpose of this report the in-plane motion is consequently regarded as stable if the magnitude of δr remains less than 50,000 n mi for a period of five years.

As a first step in the study of this stability, it is of interest to determine whether the resulting motion is sensitive to the initial geometry of the satellite, the sun, and the moon relative to the earth as specified by the initial values of the angles θ, Θ, and θ_m. This is investigated by varying the initial value of θ in steps of 10 deg for the same initial positions of the sun and the moon. The initial values of δr, $d\delta r/d\theta_o$, and $d\delta\theta/d\theta_o$ are set equal to zero with the result that in all cases the orbital injection conditions are those of the reference orbit.

An examination of the resulting orbital motion shows a significant variation depending on the position of orbital injection as specified by the initial value of θ. In Fig. 10 the variation of the radial deviation, δr, from the reference orbit is shown as a function of time for two cases in which the only difference is an initial value of 190 deg for θ in the upper curve and a value of 20 deg in the lower curve. These two curves are roughly the extremes of the types of motion that result. In the upper curve representing the worst case, an oscillation in δr develops, reaching in amplitude in excess of 50,000 n mi within 250 days, while the frequency of the oscillation is approximately that of the reference orbit. In the lower curve of Fig. 10, which represents

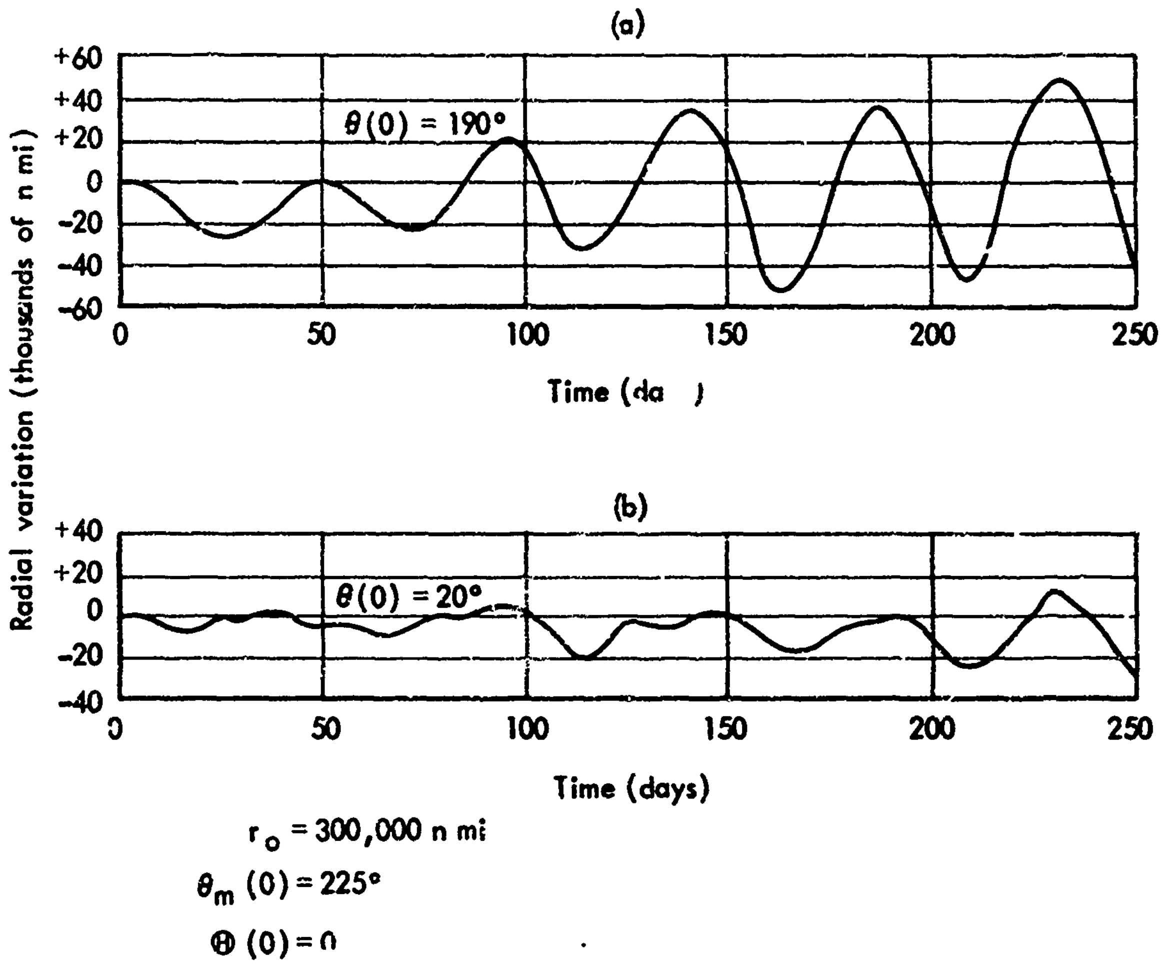


Fig. 10—Effect of initial geometry on orbital stability

the best case, the amplitude of the oscillation in δr does not increase as rapidly and its periodicity is more complex, involving oscillations at frequencies higher than that of the reference orbit.

In addition to the radial variations shown in Fig. 10 there are also tangential variations, which in both cases are characterized by a slow dirft of the satellite ahead of the corresponding position on the reference orbit. This constitutes a steady-state increment in the orbital angular rate, which can be explained qualitatively by the fact that the average radial acceleration of the satellite relative to the earth is increased by the presence of the sun and the moon. However, this change in orbital angular rate is not regarded as an orbital instability, since it does not alter the shape of the orbit. On the other hand, the radial variations shown in the upper curve of Fig. 10 have exceeded the 50,000-n mi stability limit in a little over 160 days.

If a value of -50,000 n mi for δr occurs at the time of closest approach of the satellite and the moon, the satellite is about 42,000 n mi from the moon and 250,000 n mi from the earth. An examination of Fig. 3 shows that under these conditions the lunar gravitational disturbance is about 45 percent of that due to the earth as compared with 15 percent at 300,000 n mi. Such an increase in the disturbing acceleration can result in even greater divergence in the oscillation of δr.

From this discussion it is seen that the resulting orbital motion is a function of the initial geometry of the satellite, the sun, and the moon relative to the earth. While it is evident that instability can occur, it is not necessarily inevitable. Hopefully, by a suitable choice of initial geometry and orbital injection conditions, a long duration stable orbit may be possible.

Long Duration Orbit

If the solution for case (b) shown in Fig. 10 is extended beyond the 250 days, the resulting radial variations are shown in the positive time region in Fig. 11. It is again seen that an oscillatory

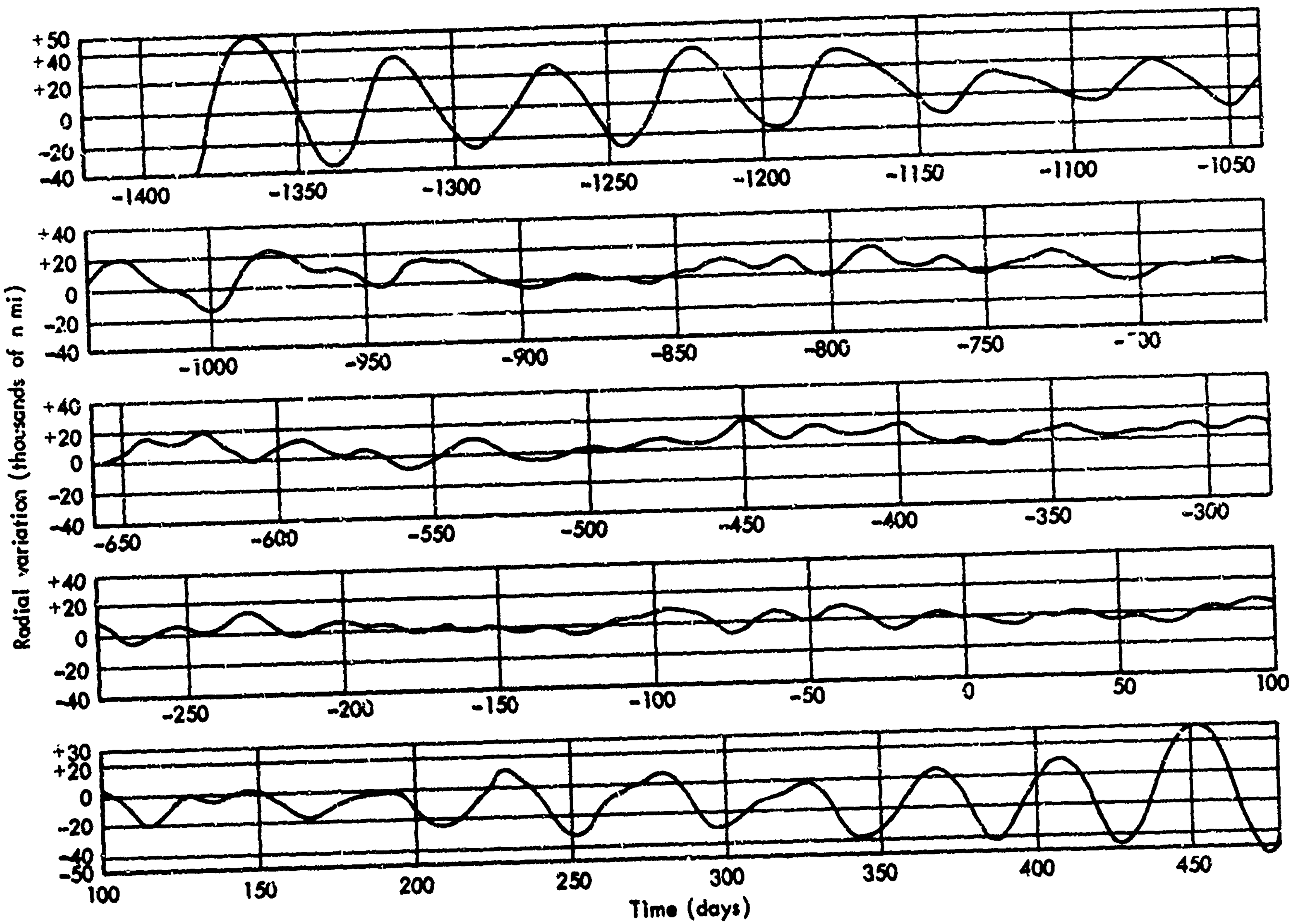


Fig. 11—Five year duration translunar orbit
$(\theta(0) = 20°,\ \theta_m(0) = 225°,\ \Theta(0) = 66°)$

divergence develops at orbital frequency, and its amplitude exceeds 50,000 n mi after 471 days. Since the computer has no compunction about running time backward, it is possible, by using a negative computing step size, to determine the variation of δr that would have occurred prior to zero time in order to achieve the specified initial conditions at zero time. The resulting solution is shown along the negative time axis in Fig. 11. The magnitude of δr remains less than 50,000 n mi for 1362 days in the negative time direction. The values of δr, $d\delta r/d\theta_o$, $\delta\theta$, $d\delta\theta/d\theta_o$, $\delta\Theta$, and $\delta\theta_m$ corresponding to any point on this curve constitute a set of initial conditions for orbital injection on that day which will generate the curve in Fig. 11 to the right of the injection point. Thus, if a satellite had been injected into orbit at time -1362 days with the initial conditions corresponding to that point in Fig. 11, the resulting δr variation would have a magnitude less than 50,000 n mi until +471 days, a total interval of 1833 days or 5.02 years, thus satisfying the previous definition of orbital stability. This procedure of determining initial conditions requires an accurate ephemeris of the motion of the moon and the sun relative to the earth for at least five years in advance of the proposed launch date. In addition, a considerable amount of computing time might be required to find a time in the future from which a stable orbit could be established over a period including the five-year interval starting from the desired launch time. Finally, even after the initial conditions are determined the precision required in realizing these conditions so that the orbit would follow the computer version for five years might be difficult to realize.

Determination of Initial Conditions

It is found that by making small changes in the orbital injection conditions from those corresponding to the reference orbit, considerable improvement in the resulting orbital motion can be achieved. However, the use of an empirical cut and try method to improve the injection conditions is rather uncertain and very time consuming.

While the specification of the optimum initial conditions for the complete nonlinear equations of motion is very difficult, it is possible to make use of the linearized form of these equations for such a determination. If the higher order terms in Eqs. (54) and (55) are neglected, the result is the following set of linearized equations:

$$\frac{d^2}{d\theta_o^2}\left(\frac{\delta r}{r_o}\right) - 3\,\frac{\delta r}{r_o} - 2\,\frac{d\delta\theta}{d\theta_o} = -\frac{\dot{\theta}_o}{\dot{\theta}_o^2}\left(1 - 3\cos^2\theta\cos^2\theta\right)$$

$$-\frac{\dot{\theta}_{mo}^2}{\mu\dot{\theta}_o^2}\left[\frac{a_m^3}{r_m^3} + \left(1 - \frac{a_m^3}{r_m^3}\right)\frac{a_m}{r_o}\cos\theta\cos\theta_m\right] \tag{66}$$

$$2\,\frac{d}{d\theta_o}\left(\frac{\delta r}{r_o}\right) + \frac{d^2\delta\theta}{d\theta_o^2} = -\frac{3\dot{\theta}_o^2}{2\dot{\theta}_o^2}\sin 2\theta\cos^2\theta + \frac{\dot{\theta}_{mo}^2}{\mu\dot{\theta}_o^2}\left(1 - \frac{a_m^3}{r_m^3}\right)\frac{a_m}{r_o}\sin\theta\cos\theta_m \tag{67}$$

where the eccentricity of the orbits of the earth and the moon are ignored. In addition, it is assumed $\delta\theta$ remains small so that it can be neglected on the right side of Eqs. (66) and (67).

Appendix A shows that the terms on the right side of Eqs. (66) and (67) can be expressed by a series expansion as follows:

$$-\frac{\dot{\theta}_{mo}^2}{\mu\dot{\theta}_o^2}\left[\frac{a_m^3}{r_m^3} + \left(1 - \frac{a_m^3}{r_m^3}\right)\frac{a_m}{r_o}\cos\theta\cos\theta_m\right] = \sum_{i=0}^{n_o+1}\ \sum_{j=-(n_o+1)}^{n_o+1} p_{ij}\cos(i\theta + j\theta_m) \tag{68}$$

$$\frac{\dot{\theta}_{mo}^2}{\mu\dot{\theta}_o^2}\left(1 - \frac{a_m^3}{r_m^3}\right)\frac{a_m}{r_o}\sin\theta\sin\theta_m = \sum_{i=0}^{n_o+1}\ \sum_{j=-(n_o+1)}^{n_o+1} q_{ij}\sin(i\theta + j\theta_m) \tag{69}$$

$$-\frac{\dot{\theta}_o^2}{\dot{\theta}_o^2}\left(1 - 3\cos^2\theta\cos^2\theta\right) = \sum_{i=0}^{2}\ \sum_{j=-2}^{2} P_{ij}\cos(i\theta + j\theta) \tag{70}$$

$$- \frac{3\dot{\theta}_o^2}{2\dot{\theta}_o^2} \sin 2\theta \cos^2 \Theta = \sum_{i=0}^{2} \sum_{j=-2}^{2} Q_{ij} \sin (i\theta + j\Theta) \qquad (71)$$

where the convergence of the series in Eqs. (68) and (69) is excellent for a value of 20 for n_o.

Substitution for θ, Θ, and θ_m in terms of θ_o by means of Eqs. (31), (34), and (35) transforms Eqs. (66) and (67) to

$$\frac{d^2}{d\theta_o^2}\left(\frac{\delta r}{r_o}\right) - 3\,\frac{\delta r}{r_o} - 2\,\frac{d\delta\theta}{d\theta_o} = \sum_{i=0}^{2} \sum_{j=-2}^{2} P_{ij} \cos \left[\Omega_{ij}\theta_o + i\theta(0) + j\Theta(0)\right]$$

$$+ \sum_{i=0}^{n_o+1} \sum_{j=-(n_o+1)}^{n_o+1} p_{ij} \cos \left[\omega_{ij}\theta_o + i\theta(0) + j\theta_m(0)\right] \qquad (72)$$

$$2\,\frac{d}{d\theta_o}\left(\frac{\delta r}{r_o}\right) + \frac{d^2\delta\theta}{d\theta_o^2} = \sum_{i=0}^{2} \sum_{j=-2}^{2} Q_{ij} \sin \left[\Omega_{ij}\theta_o + i\theta(0) + j\Theta(0)\right]$$

$$+ \sum_{i=0}^{n_o+1} \sum_{j=-(n_o+1)}^{n_o+1} q_{ij} \sin \left[\omega_{ij}\theta_o + i\theta(0) + j\theta_m(0)\right] \qquad (73)$$

where

$$\Omega_{ij} = \left(1 + \frac{\dot{\Theta}_o}{\dot{\theta}_o}\right) \qquad (74)$$

$$\omega_{ij} = \left(1 + \frac{\dot{\theta}_{mo}}{\dot{\theta}_o}\right) \qquad (75)$$

Thus, the driving functions of the equations of motion are expressed as summations of sinusoidal terms whose arguments are functions of θ_o.

By superposition, the solution of Eqs. (72) and (73) can be expressed as the summation of the solutions for the individual driving terms in the form

$$\frac{\delta r}{r_o} = \left[2\left(2\frac{\delta r_o}{r_o} + \delta\theta'_o\right) + A_o\right] + \left[-\left(\frac{3\delta r_o}{r_o} + 2\delta\theta'_o\right) + A_1\right]\cos\theta_o$$

$$+ \left[\frac{\delta r'_o}{r_o} + A_2\right]\sin\theta_o + \sum_{\substack{i=0 \\ i^2+j^2\neq 0}}^{n_o+1}\sum_{j=-(n_o+1)}^{n_o+1} a_{ij}\cos\left[\omega_{ij}\theta_o + i\theta(0) + j\theta_m(0)\right]$$

$$+ \sum_{\substack{i=0 \\ i^2+j^2\neq 0}}^{2}\sum_{j=-2}^{2} b_{ij}\cos\left[\Omega_{ij}\theta_o + i\theta(0) + j\Theta(0)\right] \qquad (76)$$

$$\frac{d\delta\theta}{d\theta_o} = \left[-3\left(2\frac{\delta r_o}{r_o} + \delta\theta'_o\right) + B_o\right] - 2\left[-\left(3\frac{\delta r_o}{r_o} + 2\delta\theta'_o\right) + A_1\right]\cos\theta_o$$

$$- 2\left[\frac{\delta r'_o}{r_o} + A_2\right]\sin\theta_o$$

$$+ \sum_{\substack{i=0 \\ i^2+j^2\neq 0}}^{n_o+1}\sum_{j=-(n_o+1)}^{n_o+1} c_{ij}\cos\left[\omega_{ij}\theta_o + i\theta(0) + j\theta_m(0)\right]$$

$$+ \sum_{\substack{i=0 \\ i^2+j^2\neq 0}}^{2}\sum_{j=-2}^{2} d_{ij}\cos\left[\Omega_{ij}\theta_o + i\theta(0) + j\Theta(0)\right] \qquad (77)$$

where

$$A_o = p_{oo} + 2 \sum_{\substack{i=0 \\ i^2 + j^2 \neq 0}}^{n_o+1} \sum_{j=-(n_o+1)}^{n_o+1} \frac{q_{ij} \cos\,[i\theta(0) + j\theta_m(0)]}{\omega_{ij}}$$

$$+ P_{oo} + 2 \sum_{\substack{i=0 \\ i^2 + j^2 \neq 0}}^{2} \sum_{j=-2}^{2} \frac{Q_{ij} \cos\,[i\theta(0) + j\Theta(0)]}{\Omega_{ij}} \tag{78}$$

$$A_1 = \sum_{i=0}^{n_o+1} \sum_{j=-(n_o+1)}^{n_o+1} \frac{(p_{ij} - 2\omega_{ij}q_{ij})}{\omega_{ij}^2 - 1} \cos \left[i\theta(0) + j\theta_m(0)\right]$$

$$+ \sum_{i=0}^{2} \sum_{j=-2}^{2} \frac{(P_{ij} - 2\Omega_{ij}Q_{ij})}{\Omega_{ij}^2 - 1} \cos \left[i\theta(0) + j\Theta(0)\right] \tag{79}$$

$$A_2 = \sum_{i=0}^{n_o+1} \sum_{j=-(n_o+1)}^{n_o+1} \frac{(2q_{ij} - \omega_{ij}p_{ij})}{\omega_{ij}^2 - 1} \sin \left[i\theta(0) + j\theta_m(0)\right]$$

$$+ \sum_{i=0}^{2} \sum_{j=-2}^{2} \frac{(2Q_{ij} - \Omega_{ij}P_{ij})}{\Omega_{ij}^2 - 1} \sin \left[i\theta(0) + j\Theta(0)\right] \tag{80}$$

$$B_o = -2p_{oo} - 3 \sum_{\substack{i=0 \\ i^2 + j^2 \neq 0}}^{n_o+1} \sum_{j=-(n_o+1)}^{n_o+1} \frac{q_{ij} \cos\,[i\theta(0) + j\theta_m(0)]}{\omega_{ij}}$$

$$-2P_{oo} - 3 \sum_{\substack{i=0 \\ i^2 + j^2 \neq 0}}^{2} \sum_{j=-2}^{2} \frac{Q_{ij} \cos\,[i\theta(0) + j\Theta(0)]}{\Omega_{ij}} \tag{81}$$

(In Eqs. (76), (77), (78), and (81) the summations do not include the term for $i = j = 0$ if either ω_{ij} or Ω_{ij} occurs as a factor in the denominator.)

$$a_{ij} = \frac{2q_{ij} - p_{ij}\omega_{ij}}{\omega_{ij}(\omega_{ij}^2 - 1)} \tag{82}$$

$$b_{ij} = \frac{2Q_{ij} - P_{ij}\Omega_{ij}}{\Omega_{ij}(\Omega_{ij}^2 - 1)} \tag{83}$$

$$c_{ij} = -\frac{(q_{ij}\omega_{ij}^2 - 2p_{ij}\omega_{ij} + 3q_{ij})}{\omega_{ij}(\omega_{ij}^2 - 1)} \tag{84}$$

$$d_{ij} = -\frac{(Q_{ij}\Omega_{ij}^2 - 2P_{ij}\Omega_{ij} + 3Q_{ij})}{\Omega_{ij}(\Omega_{ij}^2 - 1)} \tag{85}$$

An examination of the linearized solutions for $\delta r/r_o$ and $d\delta\theta/d\theta_o$ given by Eqs. (76) and (77) shows that the choice of the initial conditions* δr_o, $\delta r_o'$, and $\delta\theta_o'$ appear in the constant terms and the coefficients of $\sin\theta_o$ and $\cos\theta_o$ but not in the coefficients of the driven solution a_{ij}, b_{ij}, c_{ij}, and d_{ij}.

If these initial conditions are set equal to zero, the orbital injection conditions are those for the reference orbit. Under these conditions, δr_{SS}, the constant bias term in δr, is equal to $r_o A_o$, while B_o is the steady-state rate term in $d\delta\theta/d\theta_o$. The amplitude of the orbital frequency oscillatory term in $\delta r/r_o$ is given by

$$\frac{\delta r_{osc}}{r_o} = \sqrt{A_1^2 + A_2^2} \tag{86}$$

---

*$\delta r_o'$ and $\delta\theta_o'$ are the initial values of $d\delta r/d\theta_o$ and $d(\delta\theta)/d\theta_o$.

while the corresponding amplitude in $d\delta\theta/d\theta_o$ is twice this value. For $\theta(0)$ equal to zero the value of this oscillatory amplitude in $\delta r/r_o$ due to the lunar terms is plotted in Fig. 12a as a function of the initial lunar longitude $\theta_m(0)$; Fig. 12b is the amplitude due to the solar terms plotted as a function of the initial solar longitude $\Theta(0)$. The curves are plotted only up to 180 deg, since both are even functions of the respective latitudes. The reason for the observed sensitivity of the orbital motion to the initial geometry of the sun, moon, and satellite is evident from Figs. 12a and 12b.

However, considerable improvement in the orbital motion can be achieved by an appropriate choice of initial conditions. Since the development of orbital instability seems to be characterized by the buildup of a divergent oscillation in δr at orbital frequency, it seems desirable to reduce the initial amplitude of this oscillation to zero. This can be accomplished in both Eqs. (76) and (77) by satisfying the relations

$$3\frac{\delta r_o}{r_o} + 2\delta\theta'_o = A_1 \qquad (87)$$

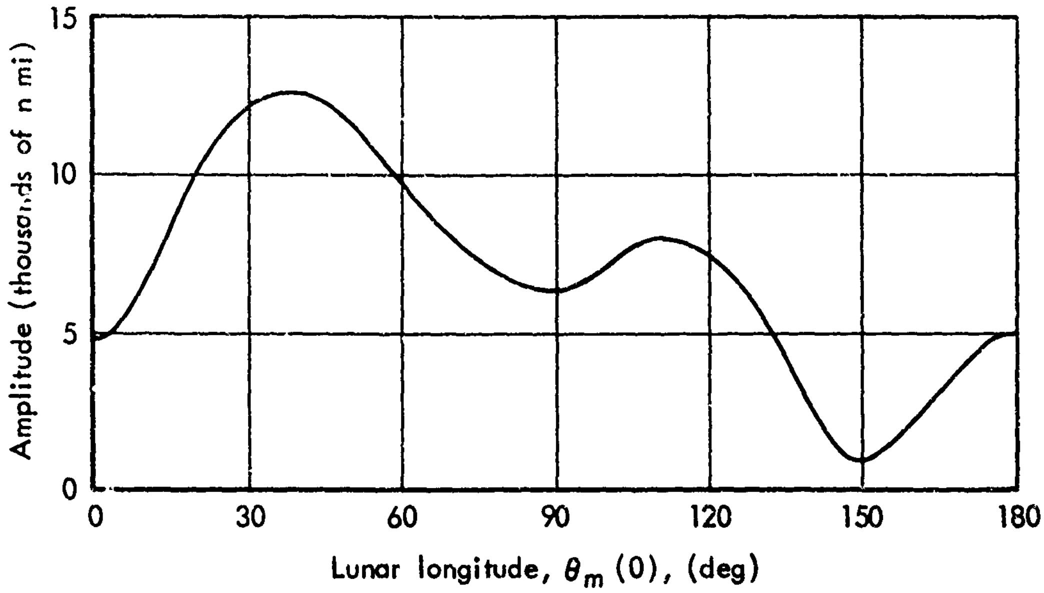
$$\frac{\delta r'_o}{r_o} = -A_2 \qquad (88)$$

In addition, the assumption that $\delta\theta$ remains small cannot be satisfied if there is a steady-state term in Eq. (77). This can also be made zero by satisfying the relation

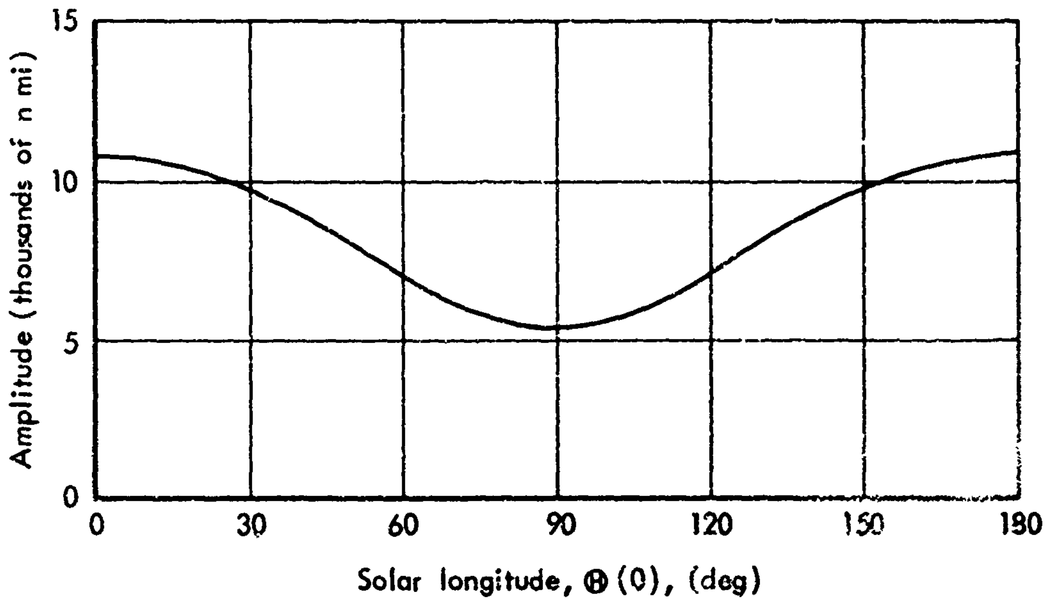
$$2\frac{\delta r_o}{r_o} + \delta\theta'_o = \frac{B_o}{3} \qquad (89)$$

Equations (87) through (89) can be solved to give the desired initial conditions in the form

$$\frac{\delta r_o}{r_o} = \frac{2B_o - 3A_1}{3} \qquad (90)$$



(a) Lunar effect



(b) Solar effect

Fig. 12—Amplitude of the orbital frequency component of δr due to the sun and the moon as functions of the initial values of Θ and θ_m

$$\frac{\delta r_o'}{r_o} = -A_2 \tag{91}$$

$$\delta\theta_o' = 2A_1 - B_o \tag{92}$$

With these initial conditions, Eqs. (76) and (77) can be reduced to

$$\frac{\delta r}{r_o} = -\frac{1}{3}(p_{oo} + P_{oo})$$

$$+ \sum_{\substack{i=0 \\ i^2 + j^2 \neq 0}}^{n_o+1} \sum_{j=-(n_o+1)}^{n_o+1} a_{ij} \cos\left[\omega_{ij}\theta_o + i\theta(0) + j\theta_m(0)\right]$$

$$+ \sum_{\substack{i=0 \\ i^2 + j^2 \neq 0}}^{2} \sum_{j=-2}^{2} b_{ij} \cos\left[\Omega_{ij}\theta_o + i\theta(0) + j\Theta(0)\right] \tag{93}$$

and

$$\frac{d\delta\theta}{d\theta_o} = \sum_{\substack{i=0 \\ i^2 + j^2 \neq 0}}^{n_o+1} \sum_{j=-(n_o+1)}^{n_o+1} c_{ij} \cos\left[\omega_{ij}\theta_o + i\theta(0) + j\theta_m(0)\right]$$

$$+ \sum_{i=0}^{2} \sum_{j=-2}^{2} d_{ij} \cos\left[\Omega_{ij}\theta_o + i\theta(0) + j\Theta(0)\right] \tag{94}$$

where the terms of the homogeneous solution present in Eqs. (76) and (77) are eliminated with the exception of a small bias term in δr.

It must be emphasized, however, that these linearized solutions are used only as a basis for determining initial conditions and not as a substitute for the numerical integration of the actual nonlinear equations.

Equations (90) through (92) express the desired initial conditions in terms of A_1, A_2, and B_o, which are all functions of the initial values of θ, θ_m, and Θ. An examination of Eqs. (79) through (81) shows that A_1, A_2, and B_o are each summations of about 480 terms; these expresions can be programmed for evaluation by JOSS so that δr_o, $\delta r'_o$, and $\delta\theta'_o$ can be determined.

As a check on the foregoing analysis the following case is considered:

Reference Orbit

Orbital radius, r_o = 303,000 n mi
Orbital period, T_o = 48.53 days
Orbital velocity, V_o = 2761 ft/sec

Initial Geometry

Orbital angle, $\theta(0)$ = 0 deg
Lunar longitude, $\theta_m(0)$ = 40 deg
Solar longitude, $\Theta(0)$ = 0 deg

Table 2 presents a comparison of the reference orbital injection conditions and those computed from Eqs. (90) through (92) as well as the values of δr_{osc} and δr_{SS}, which should result in each case.

Table 2

ORBITAL INJECTION CONDITIONS
(Case #1)

Parameter	Reference	Computed
Radial increment, δr_o (n mi)	0	21
Horizontal veloctiy (ft/sec)	2761	2860
Vertical velocity (ft/sec)	0	-61
Oscilatory amplitude, δr_{osc} (n mi)	22,849	0
Steady-state bias, δr_{SS} (n mi)	-20,359	1522

Figure 13a shows the resulting orbital variation when the reference orbital injection conditions are used. It is seen that during the initial part of the orbit the bias and oscillatory amplitude are in good agreement with the values tabulated for δr_{SS} and δr_{osc}; however, as time goes on, the amplitude increases and exceeds the 50,000-n mi stability limit after 187 days. In comparison, Fig. 13b shows the corresponding orbital motion when the computed injection conditions are used. The resulting magnitude of δr remains less than 17,000 n mi for 250 days with no indication of an orbital frequency oscillation. From Table 2 it is seen that this marked improvement

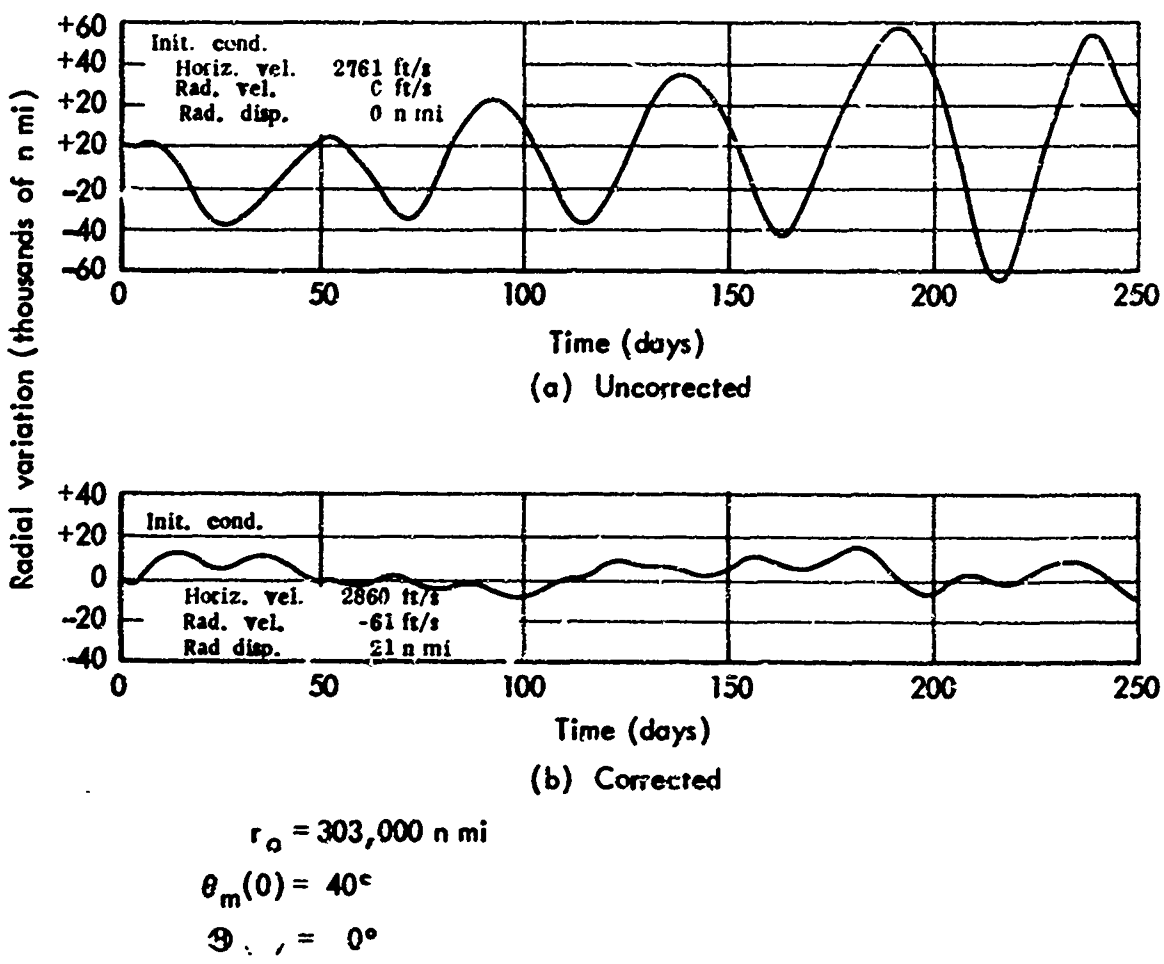


Fig. 13 — Improvement in orbital stability by alteration of initial conditions

is achieved for an increase of about 100 ft/sec in injection velocity and 21 n mi in injection altitude.

In the application of this method of determining injection conditions, there are certain restrictions on the selection of the reference orbital radius. An examination of Eqs. (78) through (85) shows that values of ω_{ij} or Ω_{ij} nearly equal to either 0 or ±1 can result in very large terms in the summations. Thus it is necessary that r_o be selected so that the corresponding orbital angular rate $\dot{\theta}_o$ is incommensurate with $\dot{\theta}_{mo}$ and $\dot{\Theta}_o$. This selection is discussed in more detail in Appendix B.

<u>Orbital Correction Procedure</u>

It must be recognized that the initialization method described above is based on a linearization of the equations of motion and its usefulness in establishing a stable orbit depends on how rapidly the nonlinear effects begin to appear. It is found that the development of a divergent oscillation at orbital frequency does not ordinarily occur until at least a year after orbital injection, and stability in the sense of not exceeding 50,000 n mi for δr may last for 2 to 3 years. However, it appears that even with the initialization procedure, instability can occur in less than five years unless further corrective action is taken.

Orbital correction can be achieved by a modification of the method used to determine the orbital injection condition at time t equal to zero. If t_1 is the time at which the correction is to be made, then the linearized solution in the vicinity of t_1 is given by the expressions

$$\frac{\delta r}{r_o} = \left[2\left(2\frac{\delta r_1}{r_o} + \delta\theta_1'\right) + A_o\right]_{t=t_1} + \left[-\left(\frac{3\delta r_1}{r_o} + \delta\theta_1'\right) + A_1\right]_{t=t_1} \cos(\theta_o - \theta_1)$$

$$+ \left(\frac{\delta r_1'}{r_o} + A_2\right)_{t=t_1} \sin(\theta_o - \theta_1) + \text{driven solutions} \qquad (95)$$

$$\frac{d\delta\theta}{d\theta_o} = \left[-3\left(2\frac{\delta r_1}{r_o} + \delta\theta_1'\right) + B_o\right]_{t=t_1} - 2\left[-\left(3\frac{\delta r_1}{r_o} + 2\delta\theta_1'\right) + A_1\right]_{t=t_1}$$

$$\times \cos(\theta_o - \theta_1) - 2\left(\frac{\delta r_1'}{r_o} + A_2\right)_{t=t_1} \sin(\theta_o - \theta_1)$$

$$+ \text{ driven solutions} \qquad (96)$$

where

$$\theta_1 = \dot{\theta}_o t_1 \qquad (97)$$

while δr_1, $\delta r_1'$, and $\delta\theta_1'$ are the variations existing at time t_1, and the values of A_o, A_1, A_2, and B_o are determined by Eqs. (78) through (81), also evaluated at time t_1. The driven solutions are the same as those appearing in Eqs. (76) and (77), since they are independent of the value of t_1.

Since orbital instability is characterized by a buildup of an oscillation in both δr and $d\delta\theta/d\theta_o$ at orbital frequency, the current amplitude of this oscillation can be used as a criterion for initiating an orbital correction. This amplitude at time, t_1, is given by

$$\delta r_{osc} = \left\{\left[-\left(\frac{3\delta r_1}{r_o} + 2\delta\theta_1'\right) + A_1\right]^2 + \left[\frac{\delta r_1'}{r_o} + A_2\right]^2\right\}^{\frac{1}{2}}_{(t=t_1)} \qquad (98)$$

When this amplitude exceeds some preassigned limit the necessity of an orbital correction is indicated.

Application of Eqs. (90) through (92) at time t_1 gives the following set of desired initial conditions:

$$\frac{\delta r_D}{r_o} = \left(\frac{2B_o - 3A_1}{3}\right)_{t=t_1} \qquad (99)$$

$$\frac{\delta r'_D}{r_o} = (-A_2)_{t=t_1} \tag{100}$$

$$\delta\theta'_D = (2A_1 - B_o)_{t=t_1} \tag{101}$$

As before, if these values are substituted for $\delta r_1/r_o$, $\delta r'_1/r_o$, and $\delta\theta'_1$ in Eqs. (95) and (96) the amplitude of the orbital frequency oscillations and the steady-state term in Eq. (96) vanish.

Unfortunately, this procedure is only possible when the current value of δr_1 happens to be equal to the computed value of δr_D, since step changes in δr_1 are not possible. However, an impulsive velocity change can produce step changes in both $\delta r'_1$ and $\delta\theta'_1$. By changing these two quantities, it is possible to reduce the coefficient of sin $(\theta_o - \theta_1)$ in Eqs. (95) and (96) to zero as well as the constant term in Eq. (95). The corrected values of $\delta r'_1$ and $\delta\theta'_1$ are given by

$$\frac{\delta r'_c}{r_o} = (-A_2)_{t=t_1} \tag{102}$$

$$\delta\theta'_c = \left(\frac{B_o}{3} - 2\,\frac{\delta r_1}{r_o}\right)_{t=t_1} \tag{103}$$

Substitution of Eqs. (99) through (103) into Eqs. (95) and (96) results in the following form of the linearized solution in the vicinity of t_1:

$$\frac{\delta r}{r_o} = -\left(\frac{P_{oo} + P_{oc}}{3}\right) + \left(\frac{\delta r_1}{r_o} - \frac{\delta r_D}{r_o}\right)\cos\,(\theta_o - \theta_1) + \text{driven solutions} \tag{104}$$

$$\frac{d\delta\theta}{d\theta_o} = -2\left(\frac{\delta r_1}{r_o} + \frac{\delta r_D}{r_o}\right)\cos\,(\theta - \theta_o) + \text{driven solutions} \tag{105}$$

Examination of Eqs. (104) and (105) shows that if the current value δr_1 equals the desired value δr_D computed from Eq. (99), then the linearized solutions are similar to those given previously in Eqs. (93) and (94) with no orbital frequency oscillation and no constant term in $d\delta\theta/d\theta_o$. If δr_1 and δr_D are not equal, the resulting value of δr_{osc} is equal to their difference.

The horizontal and radial velocity changes required to accomplish this correction are given by the relations

$$\Delta V_H = r_o\dot{\theta}_o\left(1 + \frac{\delta r_1}{r_o}\right)\left[2\left(\frac{\delta r_D}{r_o} - \frac{\delta r_1}{r_o}\right) + (\delta\theta'_D - \delta\theta'_1)\right] \tag{106}$$

$$\Delta V_R = r_o\dot{\theta}_o\left(\frac{\delta r'_D}{r_o} - \frac{\delta r'_1}{r_o}\right) \tag{107}$$

The correction time t_1 can be determined by computing the values of δr_1 and δr_D as functions of time for several days in advance, thus permitting the selection of a value of t_1 such that δr_1 and δr_D are equal.

This procedure is applied using values of 303,444 n mi for r_o, 158.8 deg for $\theta_m(0)$, 0 deg for $\theta(0)$, and 10,000 n mi for the threshold value of δr_{osc}. The resulting variation of δr as a function of time is shown in Fig. 14, where the positions at which corrections are applied are indicated by a vertical line. After each correction the solid curve represents the corrected motion, while the dotted curve shows the motion that results if no correction is made.

The initial conditions at orbital injection, as well as those for the reference orbit, are shown in Table 3, while Table 4 shows the details of the four corrections applied including the time of application, the horizontal and vertical components of the velocity increment, its total magnitude and the accumulated magnitude of all of the increments.

From Fig. 14 and Table 4 it is seen that for a total velocity correction of 342.6 ft/sec, the magnitude of δr remains less than

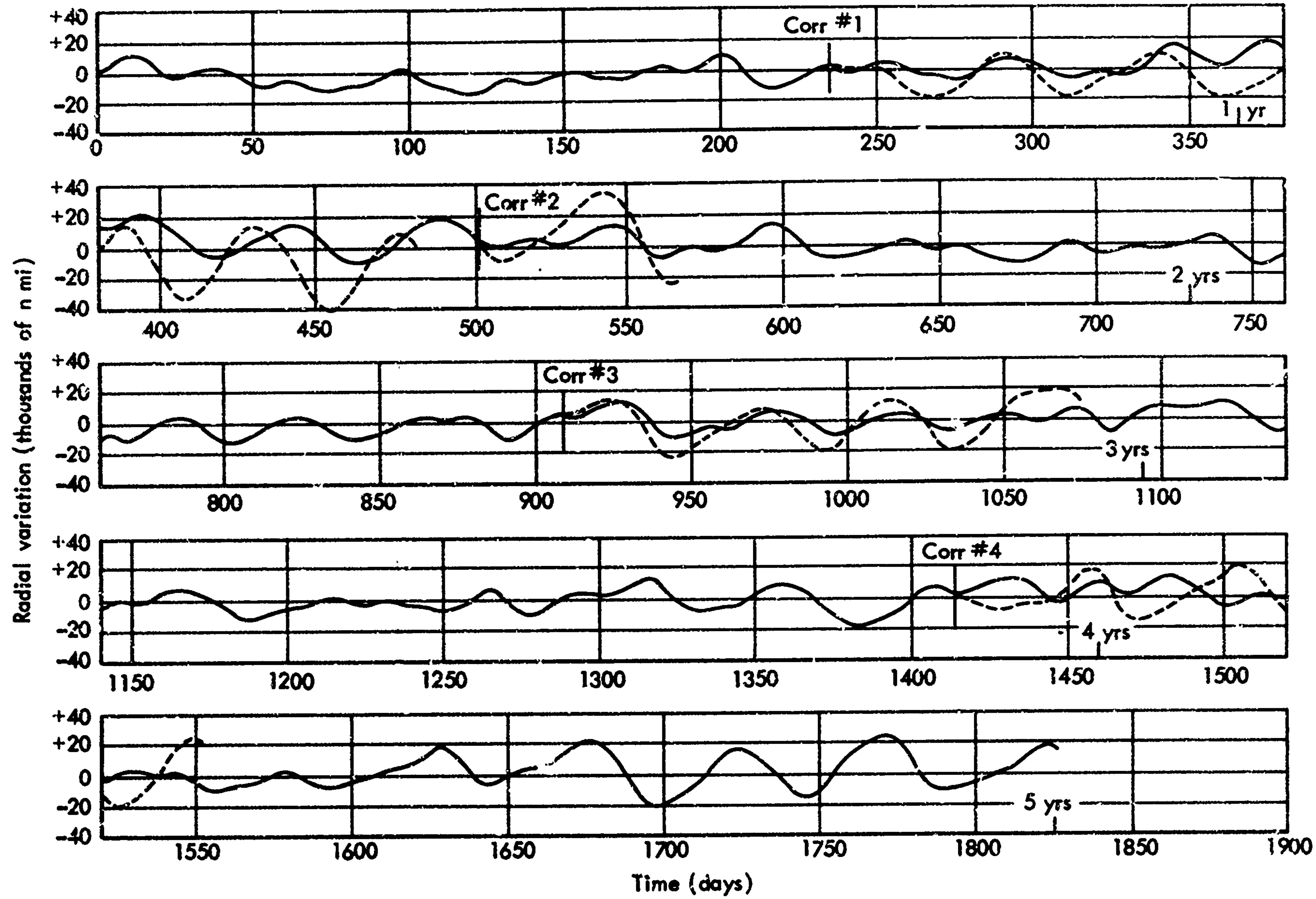


Fig. 14—Five year duration orbit using orbital control

Table 3

ORBITAL INJECTION CONDITIONS
(Case #2)

Parameter	Computed	Reference	Correction
Orbital radius (n mi)	305,762.4	303,444.0	2318.4
Horizontal velocity (ft/sec)	2790.09	2758.59	31.50
Vertical velocity (ft/sec)	9.80	0	9.80

Table 4

ORBITAL CORRECTIONS
(Case #2)

		Velocity Increment			
Number	Time (days)	Horizontal (ft/sec)	Vertical (ft/sec)	Total (ft/sec)	Accumulated (ft/sec)
1	235.5	32.2	-56.4	65.3	65.3
2	501.5	-18.8	115.8	117.3	182.6
3	909.0	19.2	-67.3	69.9	252.5
4	1413.5	25.4	86.5	90.1	342.6

25,000 n mi for a period of five years. This represents an expenditure of 68.5 ft/sec per year.

The choice of a threshold value of 10,000 n mi for the value of δr_{osc} at which orbital correction is initiated may seem low in view of the 50,000-n mi stability limit on δr specified earlier. However, an examination of Table 4 shows that the major part of the correction is involved in reducing the magnitude of the radial velocity. If the oscillation at orbital frequency reaches an amplitude of 50,000 n mi, the maximum radial velocity is of the order of 450 ft/sec and occurs when δr is near zero. Since the correction also occurs when δr is small, a considerable fraction of this 450 ft/sec is required as a corrective velocity impulse to reduce this radial velocity. This is more than the accumulated value of 342.6 ft/sec shown in Table 4 for the four corrections applied in that case. This magnitude of corrective impulse in itself is not important, if the buildup of the oscillation is linear. Under such conditions, it is immaterial whether the orbital correction is achieved by small velocity impulses at short intervals or large impulses at longer intervals, since the accumulated impulse is the same. However, it is found that the buildup is not linear, and thus it is more economical to correct more frequently. In addition, this procedure results in maximum values for δr considerably less than the 50,000-n mi limit originally specified.

IV. CONCLUSIONS

On the basis of the foregoing results and discussion, the following specific conclusions can be stated regarding the motion of earth satellites at translunar distances from the earth.

- The existence of stable earth orbits in translunar space seems more likely at distances between 300,000 to 500,000 n mi from the earth, since in this range the maximum gravitational disturbance due to the sun and the moon is minimized relative to the earth's attraction.
- A value of orbital radius closer to the 300,000-n mi limit appears to be preferable, since the average gravitational disturbance due to the moon is considerably less than its maximum during a close approach of the satellite to the moon.
- The steady-state motion of the orbital plane of a translunar earth satellite is a slow regression at constant inclination to the plane of the ecliptic with small oscillatory variations in regression rate and inclination superposed. Since the orbital planes of near-earth satellites are subject to similar motions, this can hardly be regarded as orbital instability.
- The period of orbital regression increases monotonically with inclination to the ecliptic, ranging from five years at 0 deg inclination to infinity at 90 deg and being relatively insensitive to the value of orbital radius in the 300,000- to 500,000-n mi range.
- The orbital inclination to the equatorial plane is not necessarily constant and may vary as much as 45 deg in intervals as short as three or more years.
- The only conditions under which the orbital inclination to the equator can remain constant are those for an orbital plane in the ecliptic or perpendicular to it.
- An orbit in the ecliptic maintains a constant equatorial inclination of 23°27′, while the corresponding constant inclination for an orbit perpendicular to the ecliptic may have any

value between 66°33' and 90 deg depending on the fixed orientation of its line of nodes in the ecliptic.

- o The choice of an orbital plane perpendicular to the ecliptic tends to reduce the magnitude of the disturbance due to the moon both by decreasing the probability of a near approach to the moon and by reducing its duration if it does occur. This should in turn improve the characteristics of the motion of the satellite in the orbital plane.
- o Orbital instability can develop in connection with the in-plane motion of the satellite and is characterized by a divergent orbital frequency oscillation in the radial displacement from a nominal circular orbit.
- o By a cut and try approach, initial conditions can sometimes be found so that the amplitude of the radial oscillations remains less than 50,000 n mi for periods as long as five years.
- o The orbital injection conditions can be determined in terms of the initial positions of the satellite, the sun, and the moon relative to the earth so that the development of the oscillatory instability is delayed for two to three years.
- o It is also possible, if instability begins to develop, to determine the magnitude and time of application of an impulsive velocity correction to reduce the oscillation.
- o Application of this method of orbital correction indicates that it is possible to keep the amplitude of the radial oscillation below 25,000 n mi for a period of five years at an expenditure of 68.5 ft/sec per year in velocity impulse.

To summarize these conclusions, it appears that although orbits can be determined that have satisfactory stability over a five-year interval, the actual realization of such an orbit may be highly sensitive to the accuracy with which the orbital injection conditions are achieved. As a result, it seems advisable to provide an orbital correction capability to insure the desired performance.

Appendix A

SERIES EXPANSION OF THE SOLAR AND LUNAR DISTURBING ACCELERATIONS

EQUATIONS OF MOTION

A linearized form of the equations for the in-plane motion of a satellite with a 90-deg orbital inclination to the ecliptic is developed in the body of this report as Eqs. (66) and (67) in the form

$$\frac{d^2}{d\theta_o^2}\left(\frac{\delta r}{r_o}\right) - 3\,\frac{\delta r}{r_o} - 2\,\frac{d\delta\theta}{d\theta_o} = a_{Sr} + a_{mr} \qquad \text{(A-1)}$$

$$2\,\frac{d}{d\theta_o}\left(\frac{\delta r}{r_o}\right) + \frac{d^2\delta\theta}{d\theta_o} = a_{S\theta} + a_{m\theta} \qquad \text{(A-2)}$$

where a_{Sr}, a_{mr}, $a_{S\theta}$, $a_{m\theta}$ are the normalized radial and tangential disturbing accelerations due to the sun and the moon. These quantities are expressed as

$$a_{Sr} \approx -\frac{\dot{\Theta}_o^2}{\dot{\theta}_o^2}\left(1 - 3\cos^2\theta\cos^2\Theta\right) \qquad \text{(A-3)}$$

$$a_{S\theta} = -\frac{3\dot{\Theta}_o^2}{2\dot{\theta}_o^2}\sin 2\theta\cos^2\Theta \qquad \text{(A-4)}$$

$$a_{mr} = -\frac{\dot{\theta}_{mo}^2}{\mu\dot{\theta}_o^2}\left[\frac{a_m^3}{r_m^3} - \left(\frac{a_m^3}{r_m^3} - 1\right)\frac{a_m}{r_o}\cos\theta\cos\theta_m\right] \qquad \text{(A-5)}$$

$$a_{m\theta} = -\frac{\dot{\theta}_{mo}^2}{\mu\dot{\theta}_o^2}\left(\frac{a_m^3}{r_m^3} - 1\right)\frac{a_m}{r_o}\sin\theta\cos\theta_m \qquad \text{(A-6)}$$

For use in Eqs. (A-1) and (A-2) it is more convenient to express these normalized accelerations as a summation of sinusoids in terms of the independent variable θ_o. This is done in the following subsections.

SOLAR ACCELERATION

The desired expansion of the solar terms is as follows:

$$a_{Sr} = -\frac{\dot{\Theta}_o^2}{\dot{\theta}_o^2}\left[1 - \frac{3}{4}(1 + \cos 2\theta)(1 + \cos 2\Theta)\right]$$

$$= -\frac{\dot{\Theta}_o^2}{4\dot{\theta}_o^2}\left(1 - 3\cos 2\theta - 3\cos 2\Theta - 3\cos 2\theta \cos 2\Theta\right)$$

$$= -\frac{\dot{\Theta}_o^2}{4\dot{\theta}_o^2}\left[1 - 3\cos 2\theta - 3\cos 2\Theta - \frac{3}{2}\cos 2(\theta - \Theta) - \frac{3}{2}\cos 2(\theta + \Theta)\right] \qquad \text{(A-7)}$$

which can be written as

$$a_{Sr} = \sum_{i=0}^{2}\sum_{j=-2}^{2} P_{ij}\cos(i\theta + j\Theta) \qquad \text{(A-8)}$$

where the nonzero values of P_{ij} are

$$P_{ij} = \begin{cases} -\dfrac{\dot{\Theta}_o^2}{4\dot{\theta}_o^2} & \text{if} \quad i = 0 \text{ and } j = 0 \\[2ex] \dfrac{3\dot{\Theta}_o^2}{4\dot{\theta}_o^2} & \text{if} \quad i + j = 2 \text{ and } i \neq j \\[2ex] \dfrac{3\dot{\Theta}_o^2}{8\dot{\theta}_o^2} & \text{if} \quad i = 2 \text{ and } |j| = 2 \end{cases} \qquad \text{(A-9)}$$

Similarly,

$$a_{S\theta} = -\frac{3\dot{\theta}_o^2}{4\dot{\theta}_o^2} \sin 2\theta \, (1 + \cos 2\Theta)$$

$$= -\frac{3\dot{\theta}_o^2}{4\dot{\theta}_o^2} (\sin 2\theta + \sin 2\theta \cos 2\Theta)$$

$$= -\frac{3\dot{\theta}_o^2}{4\dot{\theta}_o^2} \left[\sin 2\theta + \frac{1}{2} \sin 2(\theta + \Theta) + \frac{1}{2} \sin 2(\theta - \Theta)\right] \qquad \text{(A-10)}$$

which can be written as

$$a_{S\theta} = \sum_{i=0}^{2} \sum_{j=-2}^{2} Q_{ij} \sin (i\theta + j\Theta) \qquad \text{(A-11)}$$

where the nonzero values of Q_{ij} are

$$Q_{ij} = \begin{cases} -\dfrac{3\dot{\theta}_o^2}{4\dot{\theta}_o^2} & \text{if} \quad i = 2 \text{ and } j = 0 \\ \\ -\dfrac{3\dot{\theta}_o^2}{8\dot{\theta}_o^2} & \text{if} \quad i = 2 \text{ and } |j| = 2 \end{cases} \qquad \text{(A-12)}$$

LUNAR ACCELERATION

The corresponding expansion for the lunar terms is considerably more complicated due to the large variations of the quantity (a_m/r_m) which appears cubed in both the radial and tangential components. This ratio is given by the relation

$$\left(\frac{a_m}{r_m}\right)^3 = a_m^3 \left(r_o^2 + a_m^2 - 2r_o a_m \cos\theta \cos\theta_m\right)^{-3/2} \qquad \text{(A-13)}$$

which becomes

$$\left(\frac{a_m}{r_m}\right)^3 = \left(\frac{Ka_m}{2r_o}\right)^{3/2} \left(1 - K\cos\theta\cos\theta_m\right)^{-3/2} \qquad \text{(A-14)}$$

where

$$K = \frac{2a_m r_o}{a_m^2 + r_o^2} \qquad \text{(A-15)}$$

Equation (A-14) can be expressed as a binomial expansion in the form of an infinite series as

$$\left(\frac{a_m}{r_m}\right)^3 = \left(\frac{Ka_m}{2r_o}\right)^{3/2} \sum_{n=0}^{\infty} \frac{(2n+1)!}{2^{2n}(n!)^2} K^n \cos^n\theta \cos^n\theta_m \qquad \text{(A-16)}$$

which can be approximated by truncating the series after the first N terms as

$$\left(\frac{a_m}{r_m}\right)^3 = \left(\frac{Ka_m}{2r_o}\right)^{3/2} \sum_{n=0}^{N} \frac{(2n+1)!}{2^{2n}(n!)^2} K^n \cos^n\theta \cos^n\theta_m \qquad \text{(A-17)}$$

Since the value of K for a 300,000-n mi orbit is 0.9356, the convergence of the series is not rapid, particularly if θ and θ_m are both equal to either 0 or 180 deg. As a result, a value of at least 100 for N is required to give a satisfactory approximation to the actual function.

Equation (A-17) can be further expanded by making use of the Fourier expansion of $\cos^n\theta$ and $\cos^n\theta_m$ in the form

$$\cos^n \theta = \sum_{i=0}^{n} b_{ni} \cos i\theta \qquad \text{(A-18)}$$

where

$$b_{no} = \frac{1}{2\pi} \int_{o}^{2\pi} \cos^n \theta \, d\theta$$

$$= \begin{cases} 0 & \text{if} \quad n \text{ is odd} \\ \dfrac{n!}{2^n \left(\frac{n}{2}!\right)^2} & \text{if} \quad n \text{ is even} \end{cases} \qquad \text{(A-19)}$$

$$b_{ni} = \frac{1}{\pi} \int_{o}^{2\pi} \cos^n \theta \cos i\theta \, d\theta$$

$$= \begin{cases} 0 & \text{if} \quad i > n \text{ or } n - i \text{ is odd} \\ \dfrac{2n!}{2^n \left(\frac{n-i}{2}!\right)\left(\frac{n+i}{2}!\right)} & \text{if} \quad n - i \text{ is even} \end{cases} \qquad \text{(A-20)}$$

These coefficients are related by the following recursion formulas:

$$b_{no} = \begin{cases} 1 & \text{if} \quad n = 0 \\ 0 & \text{if} \quad n \text{ is odd} \\ b_{n-2,0} \left(\frac{n-1}{n}\right) & \text{if} \quad n \neq 0 \text{ and is even} \end{cases} \qquad \text{(A-21)}$$

$$b_{n1} = \begin{cases} 1 & \text{if} \quad n = 1 \\ 0 & \text{if} \quad n \text{ is even} \\ b_{n-2,1}\left(\dfrac{n}{n+1}\right) & \text{if} \quad n \neq 1 \text{ and is odd} \end{cases} \tag{A-22}$$

$$b_{ni} = \begin{cases} \dfrac{1}{2^{n-1}} & \text{if} \quad i = n \\ b_{n,i-2}\left(\dfrac{n-i+2}{n+i}\right) & \text{if} \quad i \neq 2 \\ b_{n,o}\left(\dfrac{2n}{n+2}\right) & \text{if} \quad i = 2 \end{cases} \tag{A-23}$$

Equation (A-17) can now become

$$\left(\frac{a_m}{r_m}\right)^3 = \left(\frac{a_m K}{2r_o}\right)^{3/2} \sum_{n=0}^{N} \frac{(2n+1)!\,K^n}{2^{2n}(n!)^2} \left(\sum_{i=0}^{n} b_{ni} \cos i\theta\right)\left(\sum_{j=0}^{n} b_{nj} \cos j\theta_m\right) \tag{A-24}$$

The product of the two inner summations can be expressed in terms of sums and differences of the angles $i\theta$ and $j\theta_m$ so that Eq. (A-24) becomes

$$\left(\frac{a_m}{r_m}\right)^3 = \left(\frac{a_m K}{2r_o}\right)^{3/2} \sum_{n=0}^{N} \frac{(2n+1)!}{2^{2n}(n!)^2} K^n \sum_{i=0}^{n} \sum_{j=-n}^{n} B_{nij} \cos(i\theta + j\theta_m) \tag{A-25}$$

where

$$B_{nij} = \begin{cases} \dfrac{b_{ni} b_{nj}}{2} & \text{if} \quad j \neq 0 \\ b_{ni} b_{no} & \text{if} \quad j = 0 \end{cases} \tag{A-26}$$

By reversing the order of the summations in Eq. (A-25) it can be rewritten as

$$\left(\frac{a_m}{r_m}\right)^3 = \sum_{i=0}^{n_o} \sum_{j=-n_o}^{n_o} C_{ij} \cos (i\theta + j\theta_m) \qquad \text{(A-27)}$$

where

$$C_{ij} = \left(\frac{a_m K}{2r_o}\right)^{3/2} \sum_{n=0}^{N} \frac{(2n+1)!}{2^{2n}(n!)^2} K^n B_{nij} \qquad \text{(A-28)}$$

and n_o is the highest harmonic of the θ and θ_m motion necessary to give an accurate expansion of the function $(a_m/r_m)^3$. It is found that a value of 20 for n_o is satisfactory.

Equation (A-27) is the desired expansion for the first term of a_{mr} as given in Eq. (A-5), and can be used to obtain a similar expansion of the second term as follows:

$$\frac{a_m}{r_o}\left(\frac{a_m}{r_m}\right)^3 \cos\theta \cos\theta_m = \frac{a_m}{r_o} \sum_{r=0}^{n_o} \sum_{s=-n_o}^{n_o} C_{rs} \cos (r\theta + s\theta_m)$$

$$\times \left[\frac{\cos(\theta + \theta_m) + \cos(\theta - \theta_m)}{2}\right]$$

$$= \frac{a_m}{4r_o} \sum_{r=0}^{n_o} \sum_{s=-n_o}^{n_o} C_{rs} \Big\{ \cos\left[(r+1)\theta + (s+1)\theta_m\right]$$

$$+ \cos\left[(r-1)\theta + (s-1)\theta_m\right]$$

$$+ \cos\left[(r+1)\theta + (s-1)\theta_m\right]$$

$$+ \cos\left[(r-1)\theta + (s+1)\theta_m\right] \Big\}$$

$$= \frac{a_m}{4r_o}\left[\sum_{i=1}^{n_o+1}\sum_{j=-(n_o-1)}^{n_o+1} C_{i-1,j-1}\right.$$

$$+ \sum_{i=-1}^{n_o-1}\sum_{j=-(n_o+1)}^{n_o-1} C_{i+1,j+1}$$

$$+ \sum_{i=1}^{n_o+1}\sum_{j=-(n_o+1)}^{n_o-1} C_{i-1,j+1}$$

$$\left.+ \sum_{i=-1}^{n_o-1}\sum_{j=-(n_o-1)}^{n_o+1} C_{i+1,j-1}\right] \cos\,(i\theta + j\theta_m) \qquad \text{(A-29)}$$

Since C_{ij} can have a nonzero value only if $0 \leq i \leq m_o$ and $-n_o \leq j \leq n_o$, Eq. (A-29) can be rewritten as

$$\frac{a_m}{r_o}\left(\frac{a_m}{r_m}\right)^3 \cos\theta\,\cos\theta_m = \frac{a_m}{4r_o}\left[\sum_{i=0}^{n_o+1}\sum_{j=-(n_o+1)}^{n_o+1}\left(C_{i-1,j-1} + C_{i+1,j+1}\right.\right.$$

$$\left.+ C_{i-1,j+1} + C_{i+1,j-1}\right) \cos\,(i\theta + j\theta_m)$$

$$\left.+ \sum_{j=-(n_o+1)}^{n_o+1}\left(C_{o,j+1} + C_{o,j-1}\right) \cos\,(\theta - j\theta_m)\right]$$

$$= \frac{a_m}{4r_o}\sum_{i=0}^{n_o+1}\sum_{j=-(n_o+1)}^{n_o+1} D_{ij} \cos\,(i\theta + j\theta_m) \qquad \text{(A-30)}$$

where

$$D_{ij} = \begin{cases} C_{i-1,j-1} + C_{i+1,j+1} + C_{i-1,j+1} + C_{i+1,j-1} & \text{if} \quad i \neq 1 \\ 2C_{o,j-1} + C_{2,j+1} + 2C_{o,j+1} + C_{2,j-1} & \text{if} \quad i = 1 \end{cases} \tag{A-31}$$

Substitution of Eqs. (A-25) and (A-30) in Eq. (A-5) gives

$$a_{mr} = -\frac{\dot{\theta}_{mo}^2}{\mu\dot{\theta}_o^2}\left\{\sum_{i=0}^{n_o+1}\sum_{j=-(n_o+1)}^{n_o+1}\left(C_{ij} - \frac{a_m}{4r_o}D_{ij}\right)\cos(i\theta + j\theta_m) + \frac{a_m}{2r_o}\left[\cos(\theta + \theta_m) + \cos(\theta - \theta_m)\right]\right\}$$

$$= \sum_{i=0}^{n_o+1}\sum_{j=-(n_o+1)}^{n_o+1} P_{ij}\cos(i\theta + j\theta_m) \tag{A-32}$$

where

$$P_{ij} = \begin{cases} \frac{\dot{\theta}_{mo}^2}{\mu\dot{\theta}_o^2}\left(-C_{ij} + \frac{a_m}{4r_o}D_{ij}\right) & \text{if} \quad |ij| \neq 1 \\ \frac{\dot{\theta}_{mo}^2}{\mu\dot{\theta}_o^2}\left[-C_{ij} + \frac{a_m}{4r_o}(D_{ij} - 2)\right] & \text{if} \quad |ij| = 1 \end{cases} \tag{A-33}$$

Equation (A-32) is the desired expansion of a_{mr} as a summation of sinusoidal terms with arguments that are sums and differences of the first n_o harmonics of θ and θ_m.

The quantity $a_{m\theta}$ can be expanded in a similar manner, its first term from Eq. (A-6) being given by

$$\frac{a_m}{r_o}\left(\frac{a_m}{r_m}\right)^3 \sin\theta\cos\theta_m = \frac{a_m}{2r_o}\left[\sum_{r=0}^{n_o}\sum_{r=-n_o}^{n_o} C_{rs}\cos\left(r\theta + s\theta_m\right)\right]$$

$$\times\left[\sin\left(\theta+\theta_m\right) + \sin\left(\theta-\theta_m\right)\right] \qquad \text{(A-34)}$$

which can be transformed by the same methods used on Eq. (A-20) to the following expression:

$$\frac{a_m}{r_o}\left(\frac{a_m}{r_m}\right)^3 \sin\theta\cos\theta_m = \frac{\rho_o}{4r_o}\left[\sum_{i=0}^{n_o+1}\sum_{j=-(n_o+1)}^{n_o+1}\left(C_{i-1,j-1} - C_{i+1,j+1}\right.\right.$$

$$\left.+ C_{i-1,j+1} - C_{i+1,j-1}\right)\sin\left(i\theta + j\theta_m\right)$$

$$\left.+\sum_{j=-(n_o+1)}^{n_o+1}\left(C_{o,j+1} + C_{o,j-1}\right)\sin\left(\theta - j\theta_m\right)\right]$$

$$= \frac{\rho_o}{4r_o}\sum_{i=0}^{n_o+1}\sum_{j=-(n_o+1)}^{n_o+1} E_{ij}\sin\left(i\theta + j\theta_m\right) \qquad \text{(A-35)}$$

where

$$E_{ij} = \begin{cases} C_{i-1,j-1} - C_{i+1,j+1} + C_{i-1,j+1} - C_{i+1,j-1} & \text{if} \quad i \neq 1 \\ 2C_{o,j-1} - C_{2,j+1} + 2C_{o,j+1} - C_{2,j-1} & \text{if} \quad i = 1 \end{cases} \qquad \text{(A-36)}$$

Substitution of Eq. (A-35) in Eq. (A-6) gives the following expression for $a_{m\theta}$

$$a_{m\theta} = \frac{\dot{\theta}_{mo}^2}{\mu\dot{\theta}_o^2} \left\{ -\frac{a_m}{4r_o} \sum_{i=0}^{n_o+1} \sum_{j=-(n_o+1)}^{n_o+1} E_{ij} \sin(i\theta + j\theta_m) + \frac{a_m}{2r_o}\left[\sin(\theta + \theta_m) + \sin(\theta - \theta_m)\right] \right\}$$

$$= \sum_{i=0}^{n_o+1} \sum_{j=-(n_o+1)}^{n_o+1} q_{ij} \sin(i\theta + j\theta_m) \qquad \text{(A-37)}$$

where

$$c_{ij} = \begin{cases} -\dfrac{\dot{\theta}_{mo}^2}{\mu\dot{\theta}_o^2}\left(\dfrac{a_m}{4r_o}\right) E_{ij} & \text{if } |ij| \neq 1 \\[2ex] -\dfrac{\dot{\theta}_{mo}^2}{\mu\dot{\theta}_o^2}\left(\dfrac{a_m}{4r_o}\right) (E_{ij} - 2) & \text{if } |ij| = 1 \end{cases} \qquad \text{(A-38)}$$

ACCURACY

In the development of the series expansion for a_{mr} and $a_{m\theta}$ above, the binomial expansion in Eq. (A-17) is terminated after 100 terms. In addition, harmonics of θ and θ_m above the twenty-first are neglected in the Fourier series for each of the terms of the binomial expansion. As a check on the accuracy of this truncation, Table 5 gives a comparison between the values of a_{mr} computed from the series of Eq. (A-32) and from the exact expression given by Eq. (A-5). Table 6 represents a similar comparison between Eqs. (A-6) and (A-37) for $a_{m\theta}$. These evaluations are made for a 300,000-n mi orbit in the vicinity of a close approach where θ_o and θ_m are both zero, and even under this extreme condition, the series expansion is in error by less than 0.5 percent.

Table 5

ACCURACY OF a_{mr}

θ (deg)	θ_m (deg)	a_{mr} Series	a_{mr} Exact
0	0	-0.154226	-0.154966
1	1.75	-0.152744	-0.153405
2	3.50	-0.148458	-0.148919
3	5.25	-0.141819	-0.142047
4	7.00	-0.133474	-0.133523
5	8.75	-0.124143	-0.124107
6	10.50	-0.114499	-0.114461
7	12.25	-0.105080	-0.105080
8	14.00	-0.096248	-0.096284
9	15.75	-0.088197	-0.088242
10	17.50	-0.080983	-0.081015
15	26.25	-0.055594	-0.055585
20	35.00	-0.041447	-0.041450
25	43.75	-0.032389	-0.032388
30	52.50	-0.025632	-0.025631

Table 6

ACCURACY OF $a_{m\theta}$

θ (deg)	θ_m (deg)	$a_{m\theta}$ Series	$a_{m\theta}$ Exact
0	0.00	0.000000	-0.000000
1	1.75	-0.004509	-0.004535
2	3.50	-0.008634	-0.008670
3	5.25	-0.012064	-0.012091
4	7.00	-0.014609	-0.014617
5	8.75	-0.016217	-0.016211
6	10.50	-0.016955	-0.016947
7	12.25	-0.016965	-0.016966
8	14.00	-0.016420	-0.016429
9	15.75	-0.015483	-0.015497
10	17.50	-0.014292	-0.014303
15	26.25	-0.007352	-0.007349
20	35.00	-0.001749	-0.001750
25	43.75	+0.001865	+0.001865
30	52.50	+0.003839	+0.003839

Appendix B

SELECTION OF A REFERENCE ORBITAL RADIUS

In the body of the report, a method is described for the determination of the orbital injection conditions δr_o, $\delta r_o'$, and $\delta\theta_o'$ in terms of three summations A_1, A_2 and B_0 given by Eqs. (79) through (81). An examination of these equations shows that if any of the values of ω_{ij} or Ω_{ij} are near zero the corresponding term in A_o tends toward infinity. Similarly, if ω_{ij} or Ω_{ij} approach either +1 or -1, the corresponding terms in A_1 and A_2 can become infinite. In either case the injection conditions computed from Eqs. (90) through (92) are not valid. The reason for this behavior can be seen by examining the driven solution for a single frequency, ω_{ij}, where

$$\frac{\delta r}{r_o} = \frac{2q_{ij}}{\omega_{ij}} \cos \phi_{ij} + \frac{(p_{ij} - 2\omega_{ij}q_{ij})}{\omega_{ij}^2 - 1} \cos \phi_{ij} \cos \theta_o + \frac{2q_{ij} - p_{ij}\omega_{ij}}{\omega_{ij}^2 - 1} \sin \phi_{ij} \sin \theta_o + \frac{2q_{ij} - p_{ij}\omega_{ij}}{\omega_{ij}(\omega_{ij}^2 - 1)} \cos (\omega_{ij}\theta_o + \phi_{ij}) \tag{B-1}$$

and

$$\frac{d\delta\theta}{d\theta_o} = -\frac{3q_{ij}}{\omega_{ij}} \cos \phi_{ij} + 2\left(\frac{2\omega_{ij}q_{ij} - p_{ij}}{\omega_{ij}^2 - 1}\right) \cos \phi_{ij} \cos \theta_o - 2\left(\frac{2q_{ij} - \omega_{ij}p_{ij}}{\omega_{ij}^2 - 1}\right) \sin \phi_{ij} \sin \theta_o - \frac{(q_{ij}\omega_{ij}^2 - 2p_{ij}\omega_{ij} + q_{ij})}{\omega_{ij}(\omega_{ij}^2 - 1)} \cos (\omega_{ij}\theta_o + \phi_{ij}) \tag{B-2}$$

where

$$\phi_{ij} = i\theta(0) + j\theta_m(0) \tag{B-3}$$

Ordinarily the first term in Eq. (B-1) is included in the summation for A_o, while the coefficients of cos θ_o and sin ´$_o$ are included in A_1 and A_2, respectively. Likewise, the first term of Eq. (B-2) is part of the summation for B_o. However, as ω_{ij} approaches 0, the above solution becomes indeterminate and Eqs. (B-1) and (B-2) approach the forms

$$\frac{\delta r}{r_o} = q_{ij}\omega_{ij}\theta^2 \cos \phi_{ij} \tag{B-4}$$

and

$$\frac{d\delta\theta}{d\theta_o} = \frac{3}{2} q_{ij}\omega_{ij}\theta^2 \cos \phi_{ij} \tag{B-5}$$

Similarly, if ω_{ij} approaches either +1 or -1, the solutions become

$$\frac{\delta r}{r_o} = \frac{(p_{ij} \mp 2q_{ij})}{2} \theta \sin (\theta \pm \phi_{ij}) \tag{B-6}$$

and

$$\frac{d\delta\theta}{d\theta_o} = -(p_{ij} \mp 2q_{ij})\theta \sin (\theta \mp \phi_{ij}) \tag{B-7}$$

Since the values of p_{ij} and q_{ij} are much less than 1, the resulting solutions for $\delta r/r_o$ and $d\delta\theta/d\theta_o$ in Eqs. (B-4) through (B-7) diverge from 0 very slowly even though in each case the solutions are the difference of two large quantities. As a result, it does not seem reasonable to include one of these large terms in A_0, A_1, A_2, or B_0 to be cancelled by the choice of initial conditions when this leaves a nearly equal term uncompensated in the driven solution.

However, these resonance conditions can be avoided by a suitable choice of the reference orbital radius. From Eqs. (24), (74) and (75) the dependence of ω_{ij} and Ω_{ij} on r_o is given as

$$\omega_{ij} = i + j \frac{\dot\theta_{mo}}{\dot\theta_o} \tag{B-8}$$

$$\Omega_{ij} = i + j\frac{\dot{\theta}_o}{\theta_o} \qquad \text{(B-9)}$$

where

$$\dot{\theta}_o = \dot{\theta}_{mo}\left[\frac{r_o^3\mu}{a_m^3(\mu - 1)}\right]^{\frac{1}{2}} \qquad \text{(B-10)}$$

For the range of values of r_o considered in this report and the values of i and j involved, Ω_{ij} never becomes equal to either 0 or ± 1.

This is not the case with ω_{ij} as can be seen from Fig. 15 where the values of i are plotted horizontally and those of -j vertically. The points on the i,j plane represent the combination of integral values of i and j for which an ω_{ij} exists. The three parallel lines originating at -1, 0, and +1 on the i axis represent the contours in the i,j plane along which ω_{ij} is equal to -1, 0, and +1, respectively, and are given by Eq. (B-8) as

$$i = -j\frac{\dot{\theta}_{mo}}{\theta_o} - 1 \qquad \text{(B-11)}$$

$$i = -j\frac{\dot{\theta}_{mo}}{\theta_o} \qquad \text{(B-12)}$$

$$i = -j\frac{\dot{\theta}_{mo}}{\theta_o} + 1 \qquad \text{(B-13)}$$

where the slope of these lines is a function of r_o through Eq. (B-10). To avoid the resonance conditions it is necessary to select a value of r_o so that none of the three parallel lines pass through any of the frequency points in the i,j plane. A simplified version of this determination is shown in Fig. 16 where all of the integer combinations of i and j are indicated as frequency points instead of just those for which i and j are both even or both odd. Thus, if a straight line from the origin misses all of these frequencies, then Eqs. (B-11) through B-13) are satisfied and the intersection of this line with the vertical r_o scale determines an acceptable reference orbital radius.

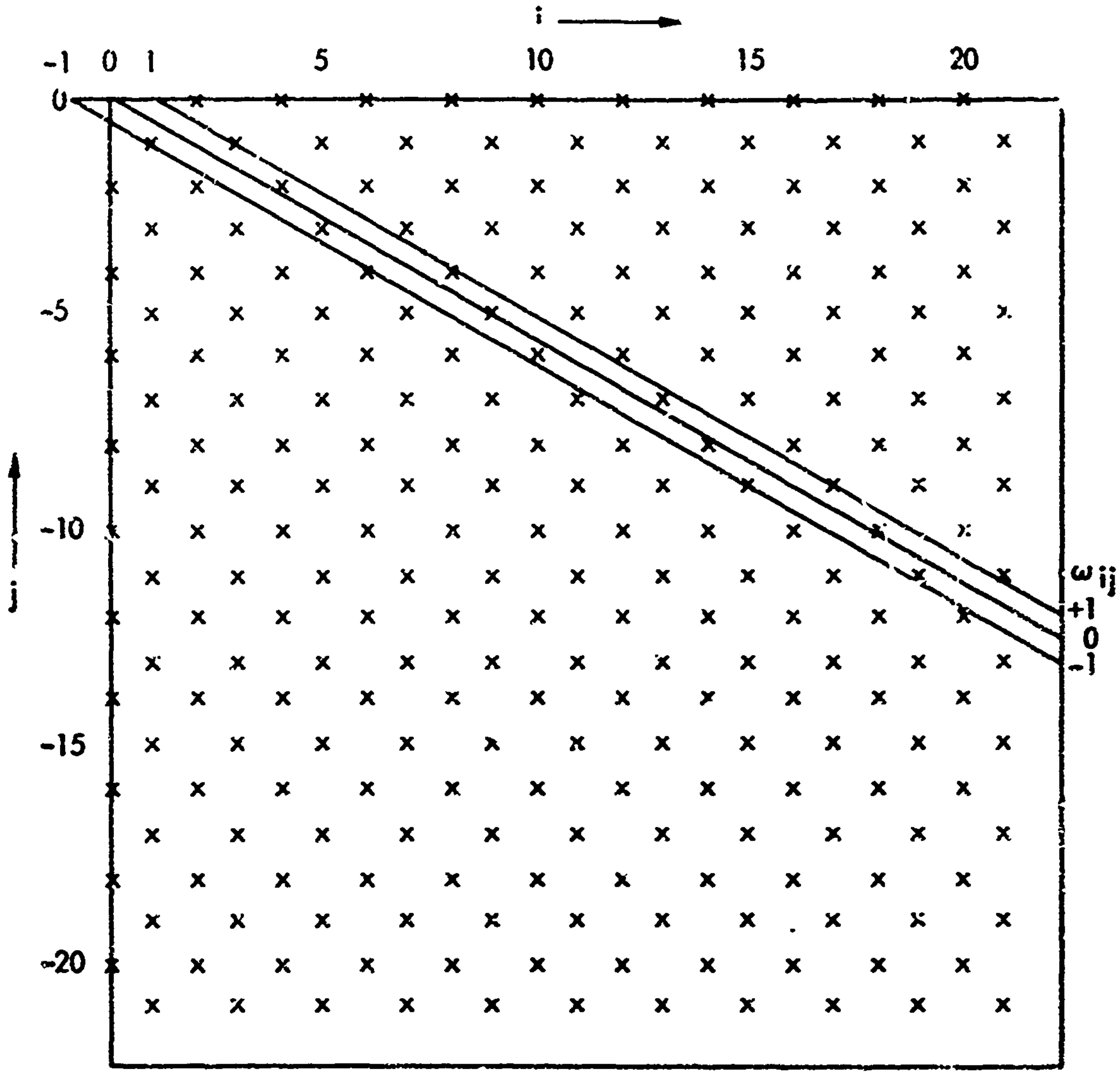


Fig. 15—Possible resonant frequencies in the i,j plane

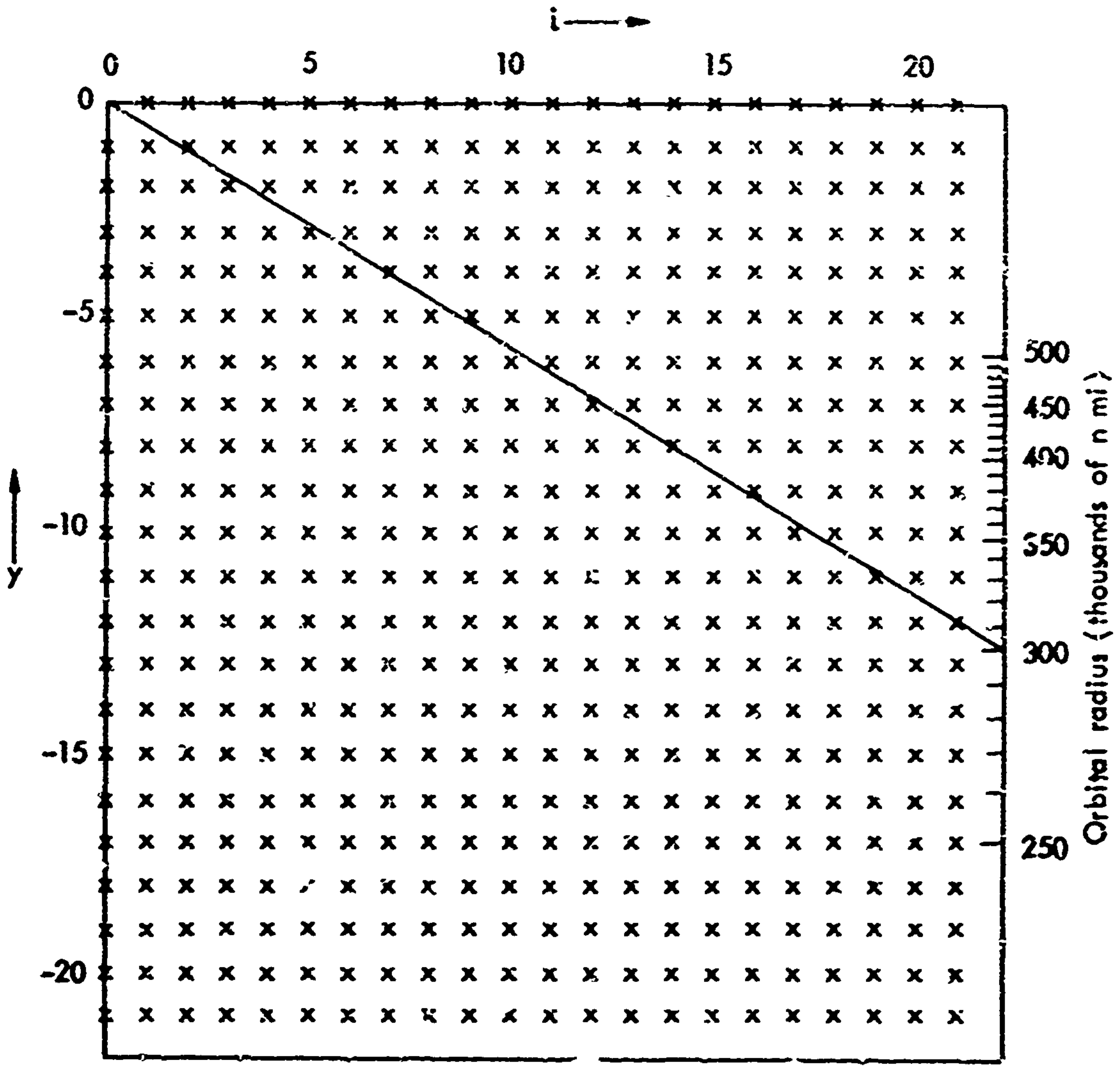


Fig. 16—Determination of reference radius to avoid resonance

At first glance it may appear that it is impossible to avoid coming close to some of these frequency points. However, the values of p_{ij} and q_{ij}, which occur in the numerators of the terms in Eqs. (B-1) and (B-2), are much less than unity and the resulting resonance is very sharp. As an example, the value 300,000 n mi for r_o turns out to be a poor choice, since the corresponding ratio of $\dot{\theta}_{mo}/\dot{\theta}_o$ is equal to 1.74998782, which is a good approximation to 7/4. An examination of Fig. 16 shows that a line from the origin to the 300,000-n mi position on the orbital radius scale passes very close to the points (i = 7, j = -4), (i = 14, j = -8), and (i = 21, j = -12). This indicates that the following resonances are possible

$$\omega_{8,\ -4} = 1$$

$$\omega_{6,\ -4} = -1$$

$$\omega_{14,\ -8} = 0$$

$$\omega_{20,\ -12} = -1$$

In Table 7, the amplitudes a_{ij} and c_{ij} of the last terms in Eqs. (B-1) and (B-2) are tabulated for all terms with a value of $|\omega_{ij}|$ less than 1.5 for an orbital radius of 300,000 n mi. It is seen that all four of the predicted resonances ... t, but the resulting amplitudes are much smaller for the higher values of i and j. This is primarily due to the fact that the quantities p_{ij} and q_{ij}, which appear in the numerators of a_{ij} and c_{ij}, descrease rapidly as i and j become larger. In comparison, Table 8 gives the corresponding amplitudes when the value of r_o is shifted from 300,000 to 303,000 n mi and shows no significant resonance.

Thus, by a relatively small shift in the reference orbital radius, the resonances can be avoided and the method described for computing orbital injection conditions can be used.

Table 7

RESONANCE EFFECTS
(r_o = 300,000 n mi)

i	j	$\|\omega_{ij}\|$	a_{ij}	c_{ij}
1	-1	0.74999	0.00612	0.00052
2	-2	1.49998	0.00594	0.01358
3	-1	1.25001	0.00149	0.00507
4	-2	0.50002	0.00583	0.00847
5	-3	0.24996	0.00934	0.01490
6	-4	0.99995	18.68356	37.36655
8	-4	1.00005	2.62238	5.24508
9	-5	0.25006	0.00135	0.00195
10	-6	0.49993	0.00077	0.00131
11	-7	1.24991	0.00033	0.00071
13	-7	0.75009	0.00013	0.00020
14	-8	0.00010	0.44030	0.66043
15	-9	0.74989	0.00011	0.00021
16	-10	1.49988	0.00002	0.00004
17	-9	1.25011	0.00001	0.00002
18	-10	0.50012	0.00002	0.00002
19	-11	0.24987	0.00002	0.00004
20	-12	0.99985	0.03023	0.06046

Table 8

RESONANCE EFFECTS
(r_o = 303,000 n mi)

i	j	$\|\omega_{ij}\|$	a_{ij}	c_{ij}
1	-1	0.77630	0.00617	0.00065
3	-1	1.22370	0.00181	0.00569
4	-2	0.44739	0.00615	0.00888
5	-3	0.32891	0.00738	0.01204
6	-4	1.10521	0.00725	0.01497
8	-4	0.89479	0.00151	0.00271
9	-5	0.11848	0.00268	0.00394
10	-6	0.65782	0.00074	0.00133
11	-7	1.43412	0.00014	0.00033
12	-6	1.34218	0.00002	0.00008
13	-7	0.56588	0.00011	0.00016
14	-8	0.21043	0.00021	0.00032
15	-9	0.98673	0.00131	0.00261
17	-9	1.01327	0.00019	0.00038
18	-10	0.23697	0.00003	0.00004
19	-11	0.53933	0.00001	0.00002
20	-12	1.31564	0.00001	0.00002
21	-11	1.46067	0.00000	0.00001

Appendix C

JOSS* COMPUTING PROGRAMS

This appendix includes the programs used for the Runge-Kutta Integration of the Orbital Equations and for the computation of Orbital Injection and Correction. Following each program is a sample of the output, an identification of JOSS terms, and a description of the individual parts of the program.

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*JOSS is the trademark and service mark of The Rand Corporation for its computer program and services using that program.

Runge-Kutta Integration of Orbital Equations

```
1.001 *Initiation (Do part 1.)
1.03 Demand A(0) as "Reference Orbital Radius (n mi)".
1.04 Demand A(1) as "Starting Time (days)".
1.05 Demand A(2) as "Terminal Time (days)".
1.06 Demand A(3) as "Orbital Inclination to Ecliptic (deg)".
1.07 Demand A(4) as "Computing Step Size (days)".
1.08 Demand A(5) as "Printout Step Size (days)".
1.09 Demand A(61) as "Initial Orbital Central Angle (deg)".
1.10 Set A(62)=A(61)·arg(-1,0)/180.
1.11 Demand A(6) as "Initial Lunar Longitude (deg)".
1.12 Demand A(7) as "Initial Solar Longitude (deg)".
1.13 Demand A(52) as "Initial Longitude of Lunar Perigee (deg)".
1.14 Demand z(1,0) as "dr/R".
1.15 Demand z(1,1) as "d(dr/R)/dQ".
1.16 Demand z(2,0) as "dq (rad)" if A(1)≠0.
1.17 Set z(2,0)=0 if A(1)=0.
1.18 Demand z(2,1) as "d(dq)/dQ".
1.19 Demand z(3,0) as "dq(M) (rad)" if A(1)≠0.
1.20 Set z(3,0)=0 if A(1)=0.
1.21 Demand z(4,0) as "dq(S) (rad)" if A(1)≠0.
1.22 Set z(4,0)=0 if A(1)=0.
1.23 Do part 2.
1.24 Do part 3.
1.25 Do part 5.

2.001 *Evaluation of Parameters
2.01 Set A(8)=82.27.
2.02 Set A(9)=arg(-1,0).
2.03 Set A(10)=A(9)/180.
2.04 Set A(11)=238857·5280/6080.
2.05 Set A(12)=9.3·10*7·5280/6080.
2.06 Set A(13)=.0549.
2.07 Set A(14)=.01674.
2.08 Set A(15)=.0019437.
2.09 Set A(16)=[281+33/60+53/3600]·A(10).
2.10 Set A(17)=A(0)·6080.
2.11 Set A(18)=.22998·sqrt[(A(8)-1)/A(8)·A(11)*3/A(0)*3].
2.12 Set A(19)=A(18)/86400.
2.13 Set A(20)=A(17)·A(19).
2.14 Set A(21)=A(17)·A(19)*2.
2.15 Set A(22)=2·A(9)/A(18).
2.16 Set A(23)=.22998.
2.17 Set A(24)=.0172.
2.18 Set A(25)=(A(23)/A(18))*2/A(8).
2.19 Set A(26)=(A(24)/A(18))*2.
2.20 Set t=A(1).
2.21 Set q=A(1)·A(18).
2.22 Set A(30)=0.
2.23 Set A(31)=A(23)/A(18).
2.24 Set A(32)=A(24)/A(18).
```

```
2.25 Set A(33)=A(15)/A(18).
2.26 Set A(34)=A(31)-A(33).
2.27 Set A(53)=A(52)·A(10).

3.001 *Format
3.01 Page.
3.02 Type A(0) in form 1.
3.03 Type A(3) in form 2.
3.04 Type A(18) in form 3.
3.05 Type A(22) in form 4.
3.06 Type A(21) in form 5.
3.07 Type A(4) in form 6.
3.08 Type [A(61)<0:A(61)+360;A(61)] in form 18.
3.09 Type [A(6)<0:A(6)+360;A(6)] in form 22.
3.10 Type [A(7)<0:A(7)+360;A(7)] in form 23.
3.11 Type A(52) in form 24.
3.12 Line.
3.13 Do part 4 if t=A(1).
3.14 Type form 30, form 7,form 8,form 31.

4.001 *Initial Condition Printout
4.01 Type t in form 14.
4.02 Line.
4.03 Type z(1,0),z(1,0)·A(0) in form 10.
4.04 Type z(1,1),z(1,1)·A(20) in form 11.
4.05 Type z(2,0),z(2,0)/A(10) in form 12.
4.06 Type z(2,1),(z(2,1)+1)·r(z)·A(20) in form 13.
4.07 Type z(3,0) in form 20.
4.08 Type z(4,0) in form 21.
4.09 Line.
4.10 Done if A(30)=1.
4.11 Do part 3 if [t-A(2)]·sgn(A(4))<0 and t=A(1).

5.001 *Control of Computation and Printout
5.01 Do part 6 if fp[t/A(5)]=0.
5.02 Set A(29)=disj[(t-A(2))·sgn(A(4))≥0,conj[A(30)=1,fp[t/A(5)]=0]].
5.03 Type form 31,_ if A(29) or $=43.
5.04 Do part 4 if Ā(29) or $=45.
5.05 Done if A(29).
5.06 Do part 7 for i=9.01(.01)9.08.
5.07 Set t=t+A(4).
5.08 Set q=q+h.
5.09 To step 5.01.

6.001 *Printout
6.01 Set i(1)=z(1,0)·A(0).
6.02 Set i(2)=z(2,0)/A(10).
6.03 Set i(3)=M(q,z)/A(10)+tv(M(q,z)<0)·360.
6.04 Set i(4)=S(q,z)/A(10)+tv(S(q,z)<0)·360.
6.05 Set i(5)=arg[x(M,q,z),sqrt(1-x(M,q,z)*2)]/A(10).
6.06 Set i(6)=z(1,1)·A(20).
6.07 Set i(7)=b(q,z).
6.08 Set i(8)=r(z)·[N(x,q,z)-A(25)·d(q)·p(q,z)-A(26)·e(q)].
6.09 Set i(9)=(z(2,1)+1)·r(z)·A(20).
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6.10 Set j(1)=D(q,z).
6.11 Set j(2)=N(y,q,z).
6.12 Type t,i(1),i(6),i(7),i(8),i(9),j(1),j(2),i(2),i(3),i(4),i(5) in form 15.

7.001 *Variable Selection
7.01 Do part 8 for k=1(1)4.

8.001 *Derivative Selection
8.01 Do step i for j=0,1 if k<3.
8.02 Do step i for j=0 if k≥3.

9.001 *Runge-Kutta Integration
9.01 Set K(1,k,j)=h·J(q,z).
9.02 Set Z(k,j)=z(k,j)+K(1,k,j)/2.
9.03 Set K(2,k,j)=h·J(q+h/2,Z).
9.04 Set Z(k,j)=z(k,j)+K(2,k,j)/2.
9.05 Set K(3,k,j)=h·J(q+h/2,Z).
9.06 Set Z(k,j)=z(k,j)+K(3,k,j).
9.07 Set K(4,k,j)=h·J(q+h,Z).
9.08 Set z(k,j)=z(k,j)+[K(1,k,j)+2·K(2,k,j)+2·K(3,k,j)+K(4,k,j)]/6.

      B(q,z): [j=1:z(1,j+1);b(q,z)]
      C(q,z): [j=1:z(2,j+1);D(q,z)]
      D(q,z): -2·(1+z(2,1))/r(z)·z(1,1)+N(y,q,z)
      F(q,z): A(31)·[[1+A(13)·c(M(q,z)-A(33)·q-A(53))]*2/(1-A(13)*2)*1.5-1]
      J(q,z): [k=1:B(q,z);k=2:C(q,z);k=3:F(q,z);k=4:L(q,z)]
      L(q,z): A(32)·[[1+A(14)·c(S(q,z)-A(16))]*2/(1-A(14)*2)*1.5-1]
      M(q,z): fp[(A(31)·q+A(6)·A(10)+z(3,0))/2/A(9)]·2·A(9)
    N(y,q,z): A(25)·d(q)·[p(q,z)-1]/R(q,z)·y(M,q,z)+O(y,q,z)
    O(y,q,z): 3·A(26)·e(q)·x(S,q,z)·y(S,q,z)
      Q(q,z): fp[(A(62)+q+z(2,0))/2/A(9)]·2·A(9)
      R(q,z): A(0)·r(z)/A(11)·d(q)*(1/3)
      S(q,z): fp[(A(32)·q+A(7)·A(10)+z(4,0))/2/A(9)]·2·A(9)
      T(q,z): 2·z(2,1)+z(2,1)*2-A(25)·d(q)·p(q,z)+N(x,q,z)-A(26)·e(q)
           a: A(3)·A(10)
      b(q,z): z(1,0)/r(z)*2·[3+3·z(1,0)+z(1,0)*2]+r(z)·T(q,z)
        c(x): cos(x)
        d(q): [[1+A(13)·c(M(q,z)-A(33)·q-A(53))]/[1-A(13)*2]]*3
        e(q): [[1+A(14)·c(S(q,z)-A(16))]/[1-A(14)*2]]*3
           h: A(4)·A(18)
      p(q,z): [1+R(q,z)*2-2·R(q,z)·x(M,q,z)]*(-1.5)
        r(z): 1+z(1,0)
        s(x): sin(x)
    x(m,q,z): c(Q(q,z))·c(m(q,z))+s(Q(q,z))·s(m(q,z))·c(a)
    y(m,q,z): -s(Q(q,z))·c(m(q,z))+c(Q(q,z))·s(m(q,z))·c(a)
```

Form 1:
Orbital Radius ____________ n mi

Form 2:
Orbital Inclination to the Ecliptic ___.__ deg.

Form 3:
Mean Orbital Rate _._____ rad/solar day

Form 4:
Orbital Period _____.___ days

Form 5:
Normal Orbital Acceleration .______ ft/sec/sec

Form 6:
Computing Step Size _____.___ days

Form 7:
| t | dr |V(r)| a(r) |F(r)/m|V(q)| a(q) |F(q)/m| dq |Q(M)|Q(S)|Q(sm)|

Form 8:
| days | n mi |ft/s| | |ft/s| | | deg| deg| deg| deg |

Form 10:
dr/R = ... dr = ________ n mi

Form 11:
d(dr/R)/dQ = ... V(r) = _____._ ft/s

Form 12:
dq (rad) = ... dq = ____.__ deg

Form 13:
d(dq)/dQ = ... V(q) = _____._ ft/s

Form 14:
Initial Conditions at t= ____._ days.

Form 15:
|_____|______|___|_.___|_.____|____|_.____|_.____|____|____|____|____|_____|

Form 18:
Initial Orbital Central Angle _____._ deg

Form 20:
dq(M) = ...

Form 21:
dq(S) = ...

Form 22:
Initial Lunar Longitude _____._ deg

Form 23:
Initial Solar Longitude ____.__ deg

Form 24:
Initial Lunar Perigee Longitude ____.__ deg

Form 30:

____

Form 31:
|____|____|___|____|____|___|____|____|___|___|___|___|___|

Orbital Radius	300000 n mi
Orbital Inclination to the Ecliptic	90.00 deg.
Mean Orbital Rate	.13142 rad/solar day
Orbital Period	47.811 days
Normal Orbital Acceleration	.004220 ft/sec/sec
Computing Step Size	.500 days
Initial Orbital Central Angle	20.0 deg
Initial Lunar Longitude	225.0 deg
Initial Solar Longitude	66.0 deg
Initial Lunar Perigee Longitude	180.0 deg

Initial Conditions at t= .0 days.

dr/R	= 0	dr	= 0 n mi
d(dr/R)/dQ	= 0	V(r)	= .0 ft/s
dq (rad)	= 0	dq	= .00 deg
d(dq)/dQ	= 0	V(q)	= 2774.4 ft/s
dq(M)	= 0		
dq(S)	= 0		

t days	dr n mi	V(r) ft/s	a(r)	F(r)/m	V(q) ft/s	a(q)	F(q)/m	dq deg	Q(M) deg	Q(S) deg	Q(sm) deg
0	0	0	.0043	.0043	2774	-.0088	-.0088	0	225	66	132
1	3	0	-.0059	-.0035	2771	-.0087	-.0089	0	239	67	117
2	-25	-4	-.0158	-.0114	2768	-.0041	-.0072	0	253	68	104
3	-134	-12	-.0240	-.0185	2767	.0042	-.0041	0	266	69	93
4	-366	-21	-.0295	-.0241	2769	.0151	-.0003	0	280	70	84
5	-750	-33	-.0316	-.0279	2773	.0272	.0035	0	292	71	78
6	-1296	-44	-.0300	-.0295	2780	.0393	.0072	0	305	72	76
7	-1996	-56	-.0248	-.0290	2790	.0506	.0108	0	317	73	77
8	-2822	-62	-.0164	-.0269	2802	.0602	.0144	0	329	74	82
9	-3733	-66	-.0056	-.0236	2816	.0667	.0174	0	341	75	88
10	-4672	-66	.0065	-.0199	2832	.0691	.0193	1	353	76	96
11	-5577	-61	.0190	-.0162	2848	.0667	.0199	1	5	77	104
12	-6383	-52	.0307	-.0129	2862	.0593	.0190	1	17	77	111
13	-7032	-39	.0405	-.0106	2075	.0472	.0168	2	29	78	116
14	-7472	-23	.0475	-.0096	2885	.0315	.0136	2	41	79	117
15	-7667	-5	.0509	-.0102	2891	.0134	.0098	3	53	80	115
16	-7599	14	.0504	-.0127	2893	-.0052	.0059	3	66	81	109
17	-7273	32	.0458	-.0170	2891	-.0225	.0024	4	78	82	100

Initial Conditions at t= 17.0 days.

dr/R	= -2.42423402-02	dr	= -7273 n mi
d(dr/R)/dQ	= 1.14110033-02	V(r)	= 31.7 ft/s
dq (rad)	= 6.56216631-02	dq	= 3.76 deg
d(dq)/dQ	= 6.80574022-02	V(q)	= 2891.4 ft/s
dq(M)	= -1.84920476-01		
dq(S)	= -8.47662480-03		

Orbital Radius	300000 n mi
Orbital Inclination to the Ecliptic	90.00 deg.
Mean Orbital Rate	.13142 rad/solar day
Orbital Period	47.811 days
Normal Orbital Acceleration	.004220 ft/sec/sec
Computing Step Size	.500 days
Initial Orbital Central Angle	20.0 deg
Initial Lunar Longitude	225.0 deg
Initial Solar Longitude	66.0 deg
Initial Lunar Perigee Longitude	180.0 deg

t days	dr n mi	V(r) ft/s	a(r)	F(r)/m	V(q) ft/s	a(q)	F(q)/m	dq deg	Q(M) deg	Q(S) deg	Q(sm) deg
18	-6711	47	.0376	-.0232	2886	-.0368	.0000	4	91	83	89
19	-5955	59	.0265	-.0309	2879	-.0467	-.0010	5	105	84	75
20	-5063	66	.0132	-.0403	2870	-.0514	-.0005	5	119	85	61
21	-4104	68	-.0022	-.0518	2860	-.0518	.0003	6	133	86	47
22	-3157	64	-.0210	-.0668	2851	-.0506	-.0021	6	147	87	35
23	-2320	53	-.0409	-.0817	2841	-.0508	-.0114	6	162	88	26
24	-1687	36	-.0454	-.0801	2829	-.0414	-.0145	6	176	89	27
25	-1282	21	-.0347	-.0653	2822	-.0199	-.0041	6	191	90	36
26	-1057	11	-.0220	-.0520	2820	-.0027	.0054	7	206	91	48
27	-947	5	-.0103	-.0420	2822	.0059	.0097	7	220	92	61
28	-891	3	.0000	-.0342	2825	.0072	.0097	7	234	93	72
29	-836	5	.0080	-.0283	2828	.0034	.0070	7	248	94	81
30	-740	9	.0128	-.0245	2829	-.0038	.0027	7	262	95	88
31	-580	14	.0141	-.0227	2827	-.0122	-.0021	7	275	96	91
32	-349	19	.0119	-.0228	2824	-.0202	-.0065	8	288	97	90
33	-57	22	.0068	-.0246	2818	-.0261	-.0099	8	301	98	87
34	270	23	-.0005	-.0278	2811	-.0288	-.0118	8	314	98	81
35	593	22	-.0089	-.0318	2803	-.0275	-.0118	8	326	99	72
36	871	17	-.0175	-.0366	2797	-.0216	-.0095	8	338	100	63
37	1058	9	-.0254	-.0419	2792	-.0113	-.0048	8	350	101	54
38	1114	-1	-.0317	-.0476	2791	.0027	.0017	8	2	102	46
39	1007	-14	-.0350	-.0525	2794	.0169	.0070	8	13	103	40
40	722	-26	-.0330	-.0535	2799	.0258	.0068	9	25	104	39
41	267	-37	-.0256	-.0490	2805	.0296	.0026	8	37	105	43
42	-319	-45	-.0161	-.0418	2811	.032[illegible]	.0000	8	49	106	51
43	-989	-49	-.0062	-.0343	2817	.0357	-.0003	8	62	107	62
44	-1690	-49	.0038	-.0268	2824	.0366	.0000	9	74	108	74

Initial Conditions at t= 44.0 days.

dr/R	= -5.63352144-03	dr	= -1690 n mi
d(dr/R)/dQ	= -1.77619235-02	V(r)	= -49.3 ft/s
dq (rad)	= 1.48446267-01	dq	= 8.51 deg
d(dq)/dQ	= 2.36641115-02	V(q)	= 2824.0 ft/s
dq(M)	= -1.81743453-01		
dq(S)	= -2.34823162-02		

Orbital Radius	300000 n mi
Orbital Inclination to the Ecliptic	90.00 deg.
Mean Orbital Rate	.13142 rad/solar day
Orbital Period	47.811 days
Normal Orbital Acceleration	.004220 ft/sec/sec
Computing Step Size	.500 days
Initial Orbital Central Angle	20.0 deg
Initial Lunar Longitude	225.0 deg
Initial Solar Longitude	66.0 deg
Initial Lunar Perigee Longitude	180.0 deg

t days	dr n mi	V(r) ft/s	a(r)	F(r)/m	V(q) ft/s	a(q)	F(q)/m	dq deg	Q(M) deg	Q(S) deg	Q(sm) deg
45	-2372	-46	.0137	-.0195	2830	.0339	-.0005	9	87	109	87
46	-2983	-39	.0225	-.0126	2836	.0271	-.0025	9	100	110	100
47	-3479	-30	.0292	-.0066	2839	.0165	-.0061	9	114	111	112
48	-3825	-19	.0328	-.0021	2839	.0031	-.0109	9	128	112	122
49	-4002	-7	.0322	.0004	2836	-.0114	-.0163	10	142	113	128
50	-4015	4	.0272	.0004	2829	-.0248	-.0215	10	156	114	129

Initial Conditions at t= 50.0 days.

dr/R	= -1.33824949-02	dr	= -4015 n mi
d(dr/R)/dQ	= 1.60698345-03	V(r)	= 4.5 ft/s
dq (rad)	= 1.74472449-01	dq	= 10.00 deg
d(dq)/dQ	= 3.36623597-02	V(q)	= 2829.4 ft/s
dq(M)	= -1.28559426-01		
dq(S)	= -2.68203919-02		

IDENTIFICATION OF JOSS TERMS

The following list defines the terms appearing in the preceding program in terms of the variables used in the body of the report wherever possible.

JOSS	Definition
q	θ_0 (rad)
t	t (days)
A(0)	r_0 (n mi)
A(1)	starting value of t
A(2)	terminal value of t
A(3)	σ_0(deg)
A(4)	computing step size (days)
A(5)	printout step size (days)
A(6)	$\theta_m(0)$
A(7)	$\theta(0)$ (deg)
A(8)	μ
A(9)	π
A(10)	$\pi/180$ (rad/day)
A(11)	a_m(n mi)
A(12)	a_E(n mi)
A(13)	ϵ_m
A(14)	ϵ_E
A(15)	$\dot{\theta}_{mp}$ (rad/deg)
A(16)	θ_p (rad)
A(17)	r_o (ft)
A(18)	$\dot{\theta}_o$ (rad/day)
A(19)	$\dot{\theta}_c$ (rad/sec)
A(20)	unperturbed orbital velocity (ft/sec)
A(21)	unperturbed orbital acceleration (ft/sec^2)
A(22)	unperturbed orbital period (days)
A(23)	$\dot{\theta}_{mo}$ (rad/day)
A(24)	$\dot{\theta}_o$ (rad/day)
A(25)	$\dot{\theta}_{mo}^2/\mu\dot{\theta}_o^2$
A(26)	$\dot{\theta}_o^2/\dot{\theta}_o^2$

JOSS	Definition
A(29)	printout control parameter
A(30)	termination control parameter
A(31)	$\dot{\theta}_{mo}/\dot{\theta}_o$
A(32)	$\dot{\Theta}_o/\dot{\theta}_o$
A(33)	$\dot{\theta}_{mp}/\dot{\theta}_o$
A(34)	$(\dot{\theta}_{mo} - \dot{\theta}_{mp})/\dot{\theta}_o$
A(52)	$\theta_{mp}(0)$ (deg)
A(53)	$\theta_{mp}(0)$ (rad)
A(61)	$\theta(0)$ (deg)
A(62)	$\theta(0)$ (rad)
i(1)	δr (n mi)
i(2)	$\delta\theta$ (deg)
i(3)	θ_m (deg)
i(4)	Θ (deg)
i(5)	$\cos^{-1}(\bar{i} \cdot \bar{i}_m)$
i(6)	$d(\delta r)/dt$ (ft/s)
i(7)	$d^2(\delta r/r_o)/d\theta_o^2$
i(8)	$a_{sr} + a_{mr}$
i(9)	$r\dot{\theta}$
j(1)	$d^2(\delta\theta)/d\theta_o^2$
j(2)	$a_{s\theta} + a_{m\theta}$
z(1, 0)	$\delta r/r_o$
z(1, 1)	$d(\delta r/r_o)/d\theta_o$
z(2, 0)	$\delta\theta$ (rad)
z(2, 1)	$d(\delta\theta)/d\theta_o$
z(3, 0)	$\delta\theta_m$ (rad)
z(4, 0)	$\delta\Theta$ (rad)

Functions

D(q, z)	right side of Eq. (55)
F(q, z)	right side of Eq. (49)
L(q, z)	right side of Eq. (48)
M(q, z)	θ_m (rad)
Q(q, z)	θ (rad)

JOSS	Definition
R(q, z)	r/ρ
S(q, z)	Θ (rad)
a	α_o (rad)
b(q, z)	right side of Eq. (54)
d(q)	$(a_m/\rho)^3$
e(q)	$(a_E/R)^3$
h	computing step (rad)
p(q, z)	$(\rho/r_m)^3$
r(z)	r/r_o
x(M, q, z)	$(\bar{i} \cdot \bar{i}_m)$
x(S, q, z)	$(\bar{i} \cdot \bar{r}_1)$
y(M, q, z)	$(\bar{j} \cdot \bar{i}_m)$
y(S, q, z)	$(\bar{j} \cdot \bar{r}_1)$

It should be emphasized that the above definitions apply only to the program for Runge-Kutta Integration of Orbital Equations.

PROGRAM DESCRIPTION

Part 1

The input data required in the execution of this part specify the following aspects of the problem:

- o Reference orbit
- o Starting time, t_1
- o Initial geometry at $t = 0$ (injection)
- o Starting values of $\delta r/r_o$, $\delta\theta$, $\delta r'/r_o$, $\delta\theta'$, $\delta\theta_m$, $\delta\Theta$

Part 2

In this part, the parameters necessary in the solution are evaluated.

Part 3

This part controls the printout of the material from Orbital Radius down to the first line of output data (see sample printout).

Part 4

This part results in the printout of the values of the dependent variables $\delta r/r_o$, $\delta\theta$, $\delta r'/r_o$, $\delta\theta'$, $\delta\theta_m$, and $\delta\Theta$ at the current time, t, as well as δr(n, mi), $\delta\theta$(deg), V_R (ft/s) and V_H (ft/s).

Part 5

This part controls the integration procedure, the printout of the data at the end of each printout interval, and finally, advances the independent variables q and t, after which the procedure is repeated until the terminal time is reached.

Part 6

This part computes the output quantities and prints them as one line in the output table in the following order: t (days), δr (n mi), V_R (ft/s), $d^2(\delta r/r_o)/d\theta_c^2$, $a_{mr} + a_{sr}$, V_H (ft/s), $\delta\theta$ (deg), θ_m (deg), Θ (deg), θ_{sm} (deg).

Part 7

This part specifies the dependent variable to be evaluated in the integration as follows:

k	Variable
1	$\delta r/r_o$
2	$\delta\theta$
3	$\delta\theta_m$
4	$\delta\Theta$

Part 8

This part specifies the order of the derivative of the dependent variable by the value of j.

Part 9

Each step of this part is executed for $\delta r/r_o$, $\delta\theta$, $\delta r'/r_o$, $\delta\theta'$, $\delta\theta_m$, and $\delta\Theta$ before advancing to the next step. Step 9.08 completes the Runge-Kutta Integration over one computing interval for the six quantities listed in the previous sentence.

Orbital Injection and Correction Program

****

Main Control Program

****

```
1.001 *Initiation (Do part 1.)
1.02 Demand A(0) as "Reference Orbital Radius (n mi)".
1.03 Demand A(8) as "Initial Orbital Central Angle (deg)".
1.04 Demand A(6) as "Initial Lunar Longitude (deg)".
1.05 Demand A(7) as "Initial Solar Longitude (deg)".
1.06 Demand t as "Current Time (days)".
1.07 Demand Z(1,0) as "Current dr/R".
1.08 Demand Z(1,1) as "Current d(dr/R)/dQ".
1.09 Demand Z(2,0) as "Current dq (rad)".
1.10 Demand Z(2,1) as "Current d(dq)/dQ".
1.11 Demand Z(3,0) as "Current dq(M) (rad)".
1.12 Demand Z(4,0) as "Current dq(S) (rad)".
1.13 Type "The computation can be shortened if the reference radius".
1.14 Type "is the same as in the previous case.".
1.15 Line.
1.16 Demand B as "The radius is the same as before. (true or false)".
1.17 Recall item 12 (initA) if not B.
1.18 Recall item 13 (initB) if B.
1.19 To step 1.25 if B.
1.20 Do part 2.
1.21 Do part 3.
1.22 Do part 4 for n=0(1)100.
1.23 Do part 16 for i=2(1)7.
1.24 Recall item 13 (initB) if not B.
1.25 Do part 8.
1.26 Page.
1.27 Do part 13.
1.28 Do part 16 for i=8(1)11,13(1)15.
1.29 Delete all forms,all formulas.

16.01 Delete part i.
```

Item 12 (initA)

Computation of Coefficients in Series Expansion

****

```
2.001 *Evaluation of Parameters
2.01 Set m=82.27.
2.02 Set o=arg(-1,0)/180.
2.03 Set A(1)=238857*5280/6080.
2.04 Set A(2)=1/sqrt[(m-1)/m*A(1)*3/A(0)*3].
2.05 Set A(4)=.22998/A(2).
2.06 Set A(5)=.0172/A(4).
```

```
2.07 Set K=2·A(0)·A(1)/[A(0)*2+A(1)*2].
2.08 Set V=A(4)·A(0)·6080/86400.
2.09 Set k=[A(1)·K/2/A(0)]*1.5.
2.10 Set C(0,0)=0.
2.11 Delete C.
2.12 Set C(0,0)=0.
2.13 Let C be sparse.
2.14 Set b(0)=0.
2.15 Let b be sparse.

3.001 *Determination of P(i,j) and Q(i,j)
3.01 Set P(0,0)=-A(5)*2/4.
3.02 Set P(2,0)=-3·P(0,0).
3.03 Set P(0,2)=P(2,0).
3.04 Set P(2,2)=P(2,0)/2.
3.05 Set P(2,-2)=P(2,2).
3.06 Set Q(2,0)=-P(2,0).
3.07 Set Q(2,2)=-P(2,2).
3.08 Set Q(2,-2)=Q(2,2).
3.09 Let P be sparse.
3.10 Let Q be sparse.

4.001 *Determination of b(n,i)
4.002 Type n if fp(n/10)=0.
4.01 Set b(0)=[n=0:1;b(0)·(n-1)/n] if fp(n/2)=0.
4.02 Set b(1)=[n=1:1;b(1)·n/(n+1)] if fp(n/2)≠0.
4.03 Do part 5 if n=0 or n=1.
4.04 Done if n=0 or n=1.
4.05 Set i=[fp(n/2)=0:2;3].
4.06 Set b(i)=[i=n:1/2*(n-1);b(i-2)·(n-i+2)/(n+i)·[i=2:2;1]].
4.07 Set A(9)=disj[i=20,i=19,i=n].
4.08 Do part 5 if A(9).
4.09 Done if A(9).
4.10 Set i=i+2.
4.11 To step 4.06.

5.001 *Determination of C(i,j) (Parts 5,6,7)
5.01 Set A(10)=[n=0:1;A(10)·(2·n+1)·K/2/n].
5.02 Set A(11)=[fp(n/2)=0:0;1].
5.03 Set A(12)=[fp(n/2)=0:20;19].
5.04 Do part 6 for i=A(11)(2)A(12).

6.01 Do part 7 for j=A(11)(2)A(12).

7.01 Set C(i,j)=C(i,j)+k·A(10)·b(i)·b(j)·[j=0:1;.5].
```

Item 13 (initB)

Computation of Orbital Injection or Correction
****

```
8.001 *Initialization of Summations
8.01 Set A(16)=fp[(A(6)+.22998·t/o)/360]·360.
8.02 Set A(17)=fp[(A(7)+.0172·t/o)/360]·360.
8.03 Set A(18)=fp[(A(8)+A(4)·t/o+Z(2,0)/o)/360]·360.
8.04 Set A(21)=0.
8.05 Set A(22)=0.
8.06 Set A(23)=0.
8.07 Do part 9 for i=0(1)21.

9.001 *Range of j in Summations
9.01 Type i if  fp(i/5)=0.
9.02 Set A(13)=[fp(i/2)=0:20;21].
9.03 Do part 10 for j=-A(13)(2)A(13).

10.001 *Lunar Terms in Summations
10.01 Set A(21)=A(21)+F(p,q,w,u).
10.02 Set A(22)=A(22)+G(p,q,w,u).
10.03 Set A(23)=A(23)+H(p,q,w,s).
10.04 Do part 11 if disj[conj(i=0,0≤j≤2),conj(i=2,-2≤j≤2)].
10.05 Done if disj[i≠21,j≠21].
10.051 *Determination of Injection Conditions
10.06 Set z(1,0)=2·A(21)-A(22).
10.07 Set z(1,1)=A(23).
10.08 Set z(2,0)=Z(2,0).
10.09 Set z(2,1)=-3·A(21)+2·A(22).
10.10 Set z(3,0)=Z(3,0).
10.11 Set z(4,0)=Z(4,0).
10.12 Set A(24)=-[p(0,0)+P(0,0)]·A(0)/3.
10.13 Set A(29)=z(2,1)-2·(Z(1,0)-z(1,0)).
10.14 Set A(30)=(1+Z(1,0))·(1+Z(2,1))·V.
10.15 Set A(31)=(1+Z(1,0))·(1+A(29))·V.
10.16 Set A(32)=2·A(0)·[2·(Z(1,0)-z(1,0))+Z(2,1)-z(2,1)]+A(24).
10.17 Set A(33)=-3·[2·Z(1,0)+Z(2,1)-A(21)]·A(4).
10.18 Set A(34)=[3·Z(1,0)+2·Z(2,1)-A(22)]·A(0).
10.19 Set A(35)=[Z(1,1)-A(23)]·A(0).
10.20 Set A(36)=A(31)-A(30).
10.21 Set A(37)=(z(1,1)-Z(1,1))·V.
10.22 Set A(38)=sqrt[A(36)*2+A(37)*2].
10.23 Set A(39)=sqrt[A(34)*2+A(35)*2].
10.24 Set A(40)=2·A(39)/A(0)·A(4).
10.25 Set A(41)=(1+z(1,0))·(1+z(2,1))·V.
10.26 Set A(42)=z(1,1)·V.
10.27 Set A(43)=sqrt[A(41)*2+z(1,1)*2·V*2].
10.28 Set A(44)=arg[A(41),z(1,1)·V]/o.

11.001 *Solar Terms in Summations
11.01 Set A(21)=A(21)+F(P,Q,W,U).
11.02 Set A(22)=A(22)+G(P,Q,W,U).
```

```
11.03 Set A(23)=A(23)+H(P,Q,W,S).

13.001 *Printout
13.01 Type "Orbital Injection" if t=0.
13.02 Type "Orbital Correction" if t≠0.
13.03 Type _,form 1,_.
13.04 Type A(0) in form 2.
13.05 Type form 27.
13.06 Type V in form 3.
13.07 Type A(6) in form 5.
13.08 Type A(7) in form 6.
13.09 Type A(8) in form 4.
13.10 Line.
13.11 Type t in form 34 +tv(t≠0).
13.12 Do part 14 if t=0.
13.13 Do part 15 if t≠0.

14.001 *Printout of Injection Conditions
14.01 Type _,form 7,form 24,_.
14.02 Type Z(1,0)•A(0),z(1,0)•A(0) in form 8.
14.03 Type A(30),A(41) in form 9.
14.04 Type Z(1,1)•V,z(1,1)•V in form 10.
14.05 Type A(32),A(24) in form 11.
14.06 Type A(33),0 in form 12.
14.07 Type A(39),0 in form 13.
14.08 Type A(40),0 in form 14.
14.09 Line.
14.10 Type A(43) in form 30.
14.11 Type A(44) in form 31.
14.12 Type _,_,form 25,form 26,_.
14.13 Type Z(1,0),z(1,0) in form 18.
14.14 Type Z(1,1),z(1,1) in form 19.
14.15 Type Z(2,0),z(2,0) in form 20.
14.16 Type Z(2,1),z(2,1) in form 21.
14.17 Type Z(3,0),z(3,0) in form 22.
14.18 Type Z(4,0),z(4,0) in form 23.

15.001 *Printout of Correction Conditions
15.01 Type _,form 7,form 24,_.
15.02 Type Z(1,0)•A(0),Z(1,0)•A(0) in form 8.
15.03 Type A(30),A(31) in form 9.
15.04 Type Z(1,1)•V,z(1,1)•V in form 10.
15.05 Type A(32),A(24) in form 11.
15.06 Type A(33),0 in form 12.
15.07 Type A(39),(Z(1,0)-z(1,0))•A(0) in form 13.
15.08 Type A(40),2•(Z(1,0)-z(1,0))•A(4) in form 14.
15.09 Line.
15.10 Type A(36) in form 15.
15.11 Type A(37) in form 16.
15.12 Type A(38) in form 17.
15.13 Type _,_,form 25,form 26,_.
15.14 Type Z(1,0),Z(1,0) in form 18.
15.15 Type Z(1,1),z(1,1) in form 19.
15.16 Type Z(2,0),Z(2,0) in form 20.
```

15.17 Type Z(2,1),A(29) in form 21.
15.18 Type Z(3,0),Z(3,0) in form 22.
15.19 Type Z(4,0),Z(4,0) in form 23.

```
F(p,q,w,u):  -1/3·[|i|+|j|=0:2·p(0,0);3·q(i,j)·u(i,j)/w(i,j)]
G(p,q,w,u):  (p(i,j)-2·q(i,j)·w(i,j))/(w(i,j)*2-1)·u(i,j)
H(p,q,w,u):  (p(i,j)·w(i,j)-2·q(i,j))·u(i,j)/(w(i,j)*2-1)
     S(i,j):  sin(h(i,j))
     U(i,j):  cos(h(i,j))
     W(i,j):  i+j·A(5)
     c(i,j):  C(i,|j|)
     d(i,j):  -A(1)/A(0)/4·[sum[r=|j|+1,|j|-1:c(i-1,r)-c(i+1,r)]-l(i,j)]
     e(i,j):  A(1)/A(0)/4·[sum[r=i+1,i-1:sum[s=|j|+1,|j|-1:c(r,s)]]-l(i,j)]
     g(i,j):  fp((i·A(18)+j·A(16))/360)·360·o
     h(i,j):  fp((i·A(18)+j·A(17))/360)·360·o
     l(i,j):  [|i·j|=1:2;0]-[i=1:c(0,|j|+1)+c(0,|j|-1);0]
     p(i,j):  [-c(i,j)+e(i,j)]·A(2)*2/m
     q(i,j):  d(i,j)·A(2)*2/n
     s(i,j):  sin(g(i,j))
     u(i,j):  cos(g(i,j))
     w(i,j):  i+j·A(2)
```

Form 1:
Reference Condition

Form 2:
Reference Orbital Radius (n mi) ________

Form 3:
Reference Orbital Velocity (ft/s) ______.__

Form 4:
Initial Orbital Central Angle (deg) ______.__

Form 5:
Initial Lunar Longitude (deg) ______.__

Form 6:
Initial Solar Longitude (deg) ______.__

Form 7:

	Current	Corrected
Form 8: Increment in Orbital Radius (n mi)	______.__	______.__
Form 9: Horizontal Velocity (ft/s)	______.__	______.__
Form 10: Radial Velocity (ft/s)	______.__	______.__

Form 11:
Average Radial Increment (n mi) ______.__ ______.__

Form 12:
Average Orbital Rate Increment (rad/day) _.______ _.______

Form 13:
Oscillatory Amplitude in dr (n mi) ______.__ ______.__

Form 14:
Oscillatory Amplitude in d(dq)/dt (rad/day) _.______ _.______

Form 15:
Horizontal Velocity Correction (ft/s) ______.__

Form 16:
Radial Velocity Correction (ft/s) ______.__

Form 17:
Total Velocity Correction (ft/s) ______.__

Form 18:
dr/R •••••••••••••• ••••••••••••••

Form 19:
d(dr/R)/dQ •••••••••••••• ••••••••••••••

Form 20:
dq •••••••••••••• ••••••••••••••

Form 21:
d(dq)/dQ •••••••••••••• ••••••••••••••

Form 22:
dq(M) •••••••••••••• ••••••••••••••

Form 23:
dq(S) •••••••••••••• ••••••••••••••

Form 24:
Value Value

Form 25:
Current Corrected

Form 26:
Value Value

Form 27:
Orbital Inclination to Ecliptic (deg) 90

Form 30:
Injection Velocity (ft/s) ______.__

Form 31:
Injection Path Angle (deg) ___.___

Form 34:
Injection at _ days

Form 35:
Correction at ____._ days

Orbital Injection

Reference Condition

Reference Orbital Radius (n mi)	303444
Orbital Inclination to Ecliptic (deg)	90
Reference Orbital Velocity (ft/s)	2758.59
Initial Lunar Longitude (deg)	150.80
Initial Solar Longitude (deg)	.00
Initial Orbital Central Angle (deg)	.00

Injection at 0 days

	Current Value	Corrected Value
Increment in Orbital Radius (n mi)	.00	2318.40
Horizontal Velocity (ft/s)	2758.59	2790.09
Radial Velocity (ft/s)	.00	9.80
Average Radial Increment (n mi)	-10021.86	1526.93
Average Orbital Rate Increment (rad/day)	.0073751	.0000000
Oscillatory Amplitude in dr (n mi)	9293.12	.00
Oscillatory Amplitude in d(dq)/dt (rad/day)	.0079128	.0000000
Injection Velocity (ft/s)		2790.11
Injection Path Angle (deg)		.201

	Current Value	Corrected Value
dr/R	0	7.64029580-03
d(dr/R)/dQ	0	3.55243699-03
dq	0	0
d(dq)/dQ	0	3.74893350-03
dq(M)	0	0
dq(S)	0	0

Orbital Correction

Reference Condition

Reference Orbital Radius (n mi)	303444
Orbital Inclination to Ecliptic (deg)	90
Reference Orbital Velocity (ft/s)	2758.59
Initial Lunar Longitude (deg)	158.80
Initial Solar Longitude (deg)	.00
Initial Orbital Central Angle (deg)	.00

Correction at 235.5 days

	Current Value	Corrected Value
Increment in Orbital Radius (n mi)	1986.31	1986.31
Horizontal Velocity (ft/s)	2771.34	2803.57
Radial Velocity (ft/s)	-18.74	-75.48
Average Radial Increment (n mi)	-5516.00	1526.93
Average Orbital Rate Increment (rad/day)	.0044976	.0000000
Oscillatory Amplitude in dr (n mi)	9425.58	20.24
Oscillatory Amplitude in d(dq)/dt (rad/day)	.0080256	.0000172
Horizontal Velocity Correction (ft/s)		32.22
Radial Velocity Correction (ft/s)		-56.74
Total Velocity Correction (ft/s)		65.25

	Current Value	Corrected Value
dr/R	6.54590009-03	6.54590009-03
d(dr/R)/dQ	-6.79425278-03	-2.73623055-02
dq	6.22404691-01	6.22404691-01
d(dq)/dQ	-1.91080992-03	9.69417132-03
dq(N)	7.18840936-02	7.18840936-02
dq(S)	-5.92090254-02	-5.92090254-02

IDENTIFICATION OF JOSS TERMS

The following list defines the terms appearing in the preceding program in terms of the variables used in the body of the report wherever possible.

JOSS	Definition
K	K
V	V_o
k	$\left(\frac{K a_m}{2 r_o}\right)^{3/2}$
m	μ
n	n
o	$\pi/180$ rad/deg
t	t_1 (days)
A(0)	r_o (n mi)
A(1)	a_m (n mi)
A(2)	$\dot{\theta}_{mo}/\dot{\theta}_o$
A(4)	$\dot{\theta}_o$ (rad/day)
A(5)	$\dot{\Theta}_o/\dot{\theta}_o$
A(6)	$\theta_m(0)$ (deg)
A(7)	$\Theta(0)$ (deg)
A(8)	$\theta(0)$ (deg)
A(16)	$\theta_m(t_1)$ (deg)
A(17)	$\Theta(t_1)$ (deg)
A(18)	$\theta(t_1)$ (deg)
A(21)	$B_o/3$
A(22)	A_1
A(23)	$-A_2$
A(24)	δr_{ss} (corrected) (n mi)
A(29)	$\delta\theta_c'$
A(30)	V_H (current) (ft/sec)
A(31)	V_H (corrected) (ft/sec)
A(32)	δr_{ss} (current) (n mi)
A(33)	$\delta\dot{\theta}_{ss}$ (current) (rad/day)
A(36)	ΔV_H (ft/sec)
A(37)	ΔV_R (ft/sec)

JOSS	Definition
A(38)	ΔV (ft/sec)
A(39)	$\delta \dot{r}_{osc}$ (current) (n mi)
A(40)	$\delta\theta_{osc}$ (current) (rad/day)
A(41)	V_H (injection) (ft/sec)
A(42)	V_R (injection) (ft/sec)
A(43)	V (injection) (ft/sec)
A(44)	γ_p (injection) (deg)
b(i)	b_{ni}
C(i, j)	C_{ij}
P(i, j)	P_{ij}
Q(i, j)	Q_{ij}
Z(1, 0)	$\delta r_1/r_o$
Z(1, 1)	$\delta r_1'/r_o$
Z(2, 0)	$\delta\theta_1$ (rad)
Z(2, 1)	$\delta\theta_1'$
Z(3, 0)	$\delta\theta_m$ (rad)
Z(4, 0)	$\delta\Theta$ (rad)
z(1, 0)	$\delta r_D/r_o$
z(1, 1)	$\delta r_D'/r_o$
z(2, 0)	$\delta\theta_D$ (rad)
z(2, 1)	$\delta\theta_D'$
z(3, 0)	$\delta\theta_m$ (rad)
z(4, 0)	$\delta\Theta$ (rad)

Functions

F(p,q,w,u)	ij increment to $B_o/3$
G(p,q,w,u)	ij increment to A_1
H(p,q,w,s)	ij increment to $-A_2$
S(i,j)	sin $(i\theta(t_1) + j\Theta(t_1))$
U(i,j)	cos $(i\theta(t_1) + j\Theta(t_1))$
W(i,j)	Ω_{ij}
c(i,j)	C_{ij}
g(i,j)	$i\theta(t_1) + j\theta_m(t_1)$ (rad)
h(i,j)	$i\theta(t_1) + j\Theta(t_1)$ (rad)

JOSS	Definition
p(i,j)	p_{ij}
q(i,j)	q_{ij}
s(i,j)	$\sin(i\theta(t_1) + j\theta_m(t_1))$
u(i,j)	$\cos(i\theta(t_1) + j\theta_m(t_1))$
w(i,j)	ω_{ij}

As before, these definitions apply only to the Orbital Injection and Correction Program.

PROGRAM DESCRIPTION

Part 1

The input data required in the execution of this part specify the following aspects of the problem:

- Reference orbit
- Initial geometry (t = 0) (injection)
- Current time, t_1
- Current values of the dependent variables, $\delta r_1/r_o$, $\delta\theta_1$, $\delta r_1'/r_o$, $\delta\theta_1'$, $\delta\theta_m$, and $\delta\Theta$

Next, Item 12 (initA), which includes parts 2 through 7, is executed if the values of C_{ij} corresponding to this referen. . orbit are not already available in storage; otherwise the control is transferred directly to Item 13 (initB), which includes parts 8 through 18.

Item 12 (initA)

This subroutine evaluates the coefficients C_{ij} in the series expansion of $(a_m/r_o)^3$.

Part 2

In this part, the parameters necessary in the solution are evaluated.

Part 3

This part evaluated the coefficients P_{ij} and Q_{ij} in the expansion of the solar perturbing functions a_{sr} and $a_{s\theta}$, respectively.

Part 4

This part evaluates the coefficients b_{ni} of the Fourier expansion of

$$\cos^n \theta = \sum_{i=0}^{i_{max}} b_{ni} \cos i\theta$$

where n takes on all integer values from 0 to 100, while i_{max} is equal to 20 or n, whichever is smaller. After this part is completed for a given value of n, parts 5, 6, and 7 are executed before advancing to the next value of n.

Part 5

This part determines the values of i considered in the evaluation of C_{ij}.

Part 6

This part determines the values of j considered in the evaluation of C_{ij}.

Part 7

This part evaluates the contribution of the nth term of the binomial expansion of $(a_m/r_o)^3$ to each of C_{ij} elements.

Item 13 (initB)

This subroutine determines the orbital injection conditions if $t = 0$, and the orbital correction conditions if $t \neq 0$.

Part 8

This part determines the current geometry at time t_1 by evaluating $\theta(t_1)$, $\theta_m(t_1)$ and $\Theta(t_1)$ and initiates the summations involved in determining A_1, A_2 and B_o.

Part 9

This part specifies the values of j included in the summations for A_1, A_2, B_o.

Part 10

In this part, the values of A_1, A_2, and B_o are computed and then used to determine the quantities $\delta r_D/r_o$, $\delta r'_D/r_o$, and $\delta\theta'_D$ after which various other output quantities are computed.

Part 11

This part computes the contributions of the solar terms to A_1, A_2, and B_o.

Part 13

This part controls the printout through the value of the Initial Orbital Central Angle (see sample printout), after which control is transferred to part 14 if orbital injection is being determined ($t_1 = 0$), or to part 15 if an orbital correction is required ($t_1 \neq 0$).

Part 14

This part controls the printout of the orbital injection conditions as follows:

- The current and corrected values of δr (n mi), V_H (ft/sec), V_R (ft/sec), δr_{ss} (n mi), $\delta\dot{\theta}_{ss}$ (rad/day), δr_{osc} (n mi), and $\delta\dot{\theta}_{osc}$ (rad/day).
- The orbital injection velocity, V (ft/sec), and the orbital injection path angle, γ_p (deg).
- The current and corrected values of $\delta r/r_o$, $\delta r'/r_o$, $\delta\theta$, $\delta\theta'$, $\delta\theta_m$, and $\delta\theta$ in the form needed as input to the Runge-Kutta integration program.

Part 15

This part controls the printout of the orbital correction conditions. It differs from part 14 in printing the horizontal and radial components of the correction velocity, ΔV_H and ΔV_R, as well as its total magnitude ΔV, all in ft/sec rather than V and γ_p.

REFERENCES

1. *TRW Space Log*, TRW Systems Group, TRW, Inc., Vol. 8, No. 4, Winter 1968-69.
2. Frick, R. H., *Orbital Regression of Synchronous Satellites Due to the Combined Gravitational Effects of the Sun, the Moon and the Oblate Earth*, The Rand Corporation, R-454-NASA, August 1967.